

# P1.T1. Valuation & Risk Models

# Bionic Turtle FRM 2013 Study Notes

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# **Allen, Boudoukh, and Saunders, Chapter 2: Quantifying Volatility in VaR Models**

**Learning Outcomes:**

﻿

**Explain** how asset return distributions tend to deviate from the normal distribution.

**Explain** potential reasons for the existence of fat tails in a return distribution and discuss the implications fat tails have on analysis of return distributions.

**Distinguish** between conditional and unconditional distributions.

**Describe** the implications regime switching has on quantifying volatility.

**Explain** the various approaches for estimating VaR.

**Compare, contrast and calculate** parametric and non-parametric approaches for estimating conditional volatility, including:

* Historical standard deviation
* Exponential smoothing
* GARCH approach
* Historic simulation
* Multivariate density estimation
* Hybrid methods

**Explain** the process of return aggregation in the context of volatility forecasting methods.

**Describe** implied volatility as a predictor of future volatility and its shortcomings.

**Explain** long horizon volatility/VaR and the process of mean reversion according to an AR(1) model.

Key terms

Risk varies over time. Models often assume a normal (Gaussian) distribution (“normality”) with constant volatility from period to period. But actual returns are non-normal and volatility varies over time (volatility is “time-varying” or “non-constant”). Therefore, it is hard to use parametric approaches to random returns; in technical terms, it is hard to find robust “distributional assumptions for stochastic asset returns”

Conditional parameter (e.g., conditional volatility): a parameter such as variance that depends on (is conditional on) circumstances or prior information. A conditional parameter, by definition, changes over time.

Persistence: In EWMA, the lambda parameter (λ). In GARCH (1,1), the sum of the alpha (α) and beta (β) parameters. High persistence implies slow decay toward to the long-run average variance.

Autoregressive: Recursive. A parameter (today’s variance) is a function of itself (yesterday’s variance).

Heteroskedastic: Variance changes over time (homoskedastic = constant variance).

Leptokurtosis: a fat-tailed distribution where relatively more observations are near the middle and in the “fat tails (kurtosis > 3)

Stochastic behavior of returns

Risk measurement (VaR) concerns the tail of a distribution, where losses occur. We want to impose a mathematical curve (a “distributional assumption”) on asset returns so we can estimate losses. The parametric approach uses parameters (i.e., a formula with parameters) to make a distributional assumption but actual returns rarely conform to the distribution curve. A parametric distribution plots a curve (e.g., the normal bell-shaped curve) that approximates a range of outcomes but actual returns are not so well behaved: they rarely “cooperate.”

Value at Risk (VaR) – 2 asset, relative vs. absolute

Know how to compute two-asset portfolio variance & scale portfolio volatility to derive VaR:

|  |  |  |  |
| --- | --- | --- | --- |
| Inputs (per annum) |  |  |  |
| Trading days /year | 252 |  |  |
| Initial portfolio value (W) | $100 |  |  |
| VaR Time horizon (days) (h) | 10 |  |  |
| VaR confidence interval | 95% |  |  |
| Asset A |  |  |  |
| Volatility (per year) | 10.0% |  |  |
| Expected Return (per year) | 12.0% |  |  |
| Portfolio Weight (w) | 50% |  |  |
| Asset B |  |  |  |
| Volatility | 20.0% |  |  |
| Expected Return (per year) | 25.0% |  |  |
| Portfolio Weight (1-w) | 50% |  |  |
| Correlation (A,B) | 0.30 |  |  |
| Autocorrelation (h-1, h) | 0.25 |  | Independent, = 0. Mean reverting = negative |
|  |  |  |  |
| Outputs |  |  |  |
| Annual |  |  |  |
| Covariance (A,B) | 0.0060 0.0060 |  | COV = (correlation A,B)(volatility A)(volatility B) |
| Portfolio variance | 0.0155 0.0155 |  |  |
| Exp Portfolio return | 18.5% |  |  |
| Portfolio volatility (per year) | 12.4% |  |  |
| Period (h days) |  |  |  |
| Exp periodic return (u) | 0.73% |  |  |
| Std deviation (h), i.i.d | 2.48% |  |  |
| Scaling factor | 15.78 |  | Don’t need to know this, used for AR(1) |
| Std deviation (h), Autocorrelation | 3.12% |  | Standard deviation if auto-correlation. |
| Normal deviate (critical z value) | 1.64 |  | Normal deviate |
| Expected future value | 100.73 100.73 |  |  |
| **Relative VaR, i.i.d** | **$4.08** |  | Doesn’t include the mean return |
| **Absolute VaR, i.i.d** | **$3.35** |  | Includes return; i.e., loss from zero |
| Relative VaR, AR(1) | $5.12 |  | The corresponding VaRs, if autocorrelation incorporated. Note VaR is higher! |
| Absolute VaR, AR(1) | $4.39 |  |

Relative VaR, iid = $100 value \* 2.48% 10-day sigma \* 1.645 normal deviate

Absolute VaR, iid = $100 \* (-0.73% + 2.48% \* 1.645)

Relative VaR, AR(1) = $100 value \* 3.12% 10-day AR sigma \* 1.645 normal deviate

Absolute VaR, AR(1) = $100 \* (-0.73% + 3.12% \* 1.645)

Explain how asset return distributions tend to deviate from the normal distribution.  
  
Fat tails

Compared to a normal (bell-shaped) distribution, actual asset returns tend to be:

Fat-tailed (a.k.a., heavy tailed): A fat-tailed distribution is characterized by having more probability weight (observations) in its tails relative to the normal distribution.

Skewness

Skewed: A skewed distribution refers—in this context of financial returns—to the observation that declines in asset prices are more severe than increases. This is in contrast to the symmetry that is built into the normal distribution.

Time-varying/Stable

Unstable: the parameters (e.g., mean, volatility) vary over time due to variability in market conditions.

|  |  |
| --- | --- |
| NORMAL RETURNS | ACTUAL FINANCIAL RETURNS |
| Symmetrical distribution | Skewed |
| “Normal” Tails | Fat-tailed (leptokurtosis) |
| Stable distribution | Time-varying parameters |

Interest rate distributions are not constant over time

10 years of interest rate data are collected (1982 – 1993). The distribution plots the daily change in the three-month treasury rate. The average change is approximately zero, but the “probability mass” is greater at both tails. It is also greater at the mean; i.e., the actual mean occurs more frequently than predicted by the normal distribution.

## Explain potential reasons for the existence of fat tails in a return distribution and discuss the implications fat tails have on analysis of return distributions.

A distribution is unconditional if tomorrow’s distribution is the same as today’s distribution. But fat tails could be explained by a conditional distribution: a distribution that changes over time. Two things can change in a normal distribution: mean and volatility. Therefore, we can explain fat tails in two ways:

Conditional mean is time varying; but this is unlikely given the assumption that markets are efficient

Conditional volatility is time varying; Allen says this is the more likely explanation!

**If fat tails, expected VaR loss is understated!**

Explain how outliers can really be indications that the volatility varies with time.

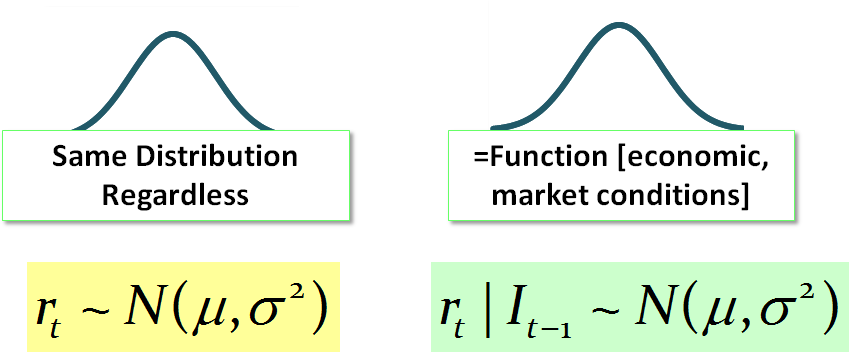
We observe that actual financial returns tend to exhibit fat-tails. Jorion (like Allen et al) offers two possible explanations:

1. The true distribution is stationary. Therefore, fat-tails reflect the true distribution but the normal distribution is not appropriate
2. The true distribution changes over time (it is “time-varying”). In this case, outliers can in reality reflect a time-varying volatility.

## Distinguish between conditional and unconditional distributions.

An unconditional distribution is the same regardless of market or economic conditions; for this reason, it is likely to be unrealistic.

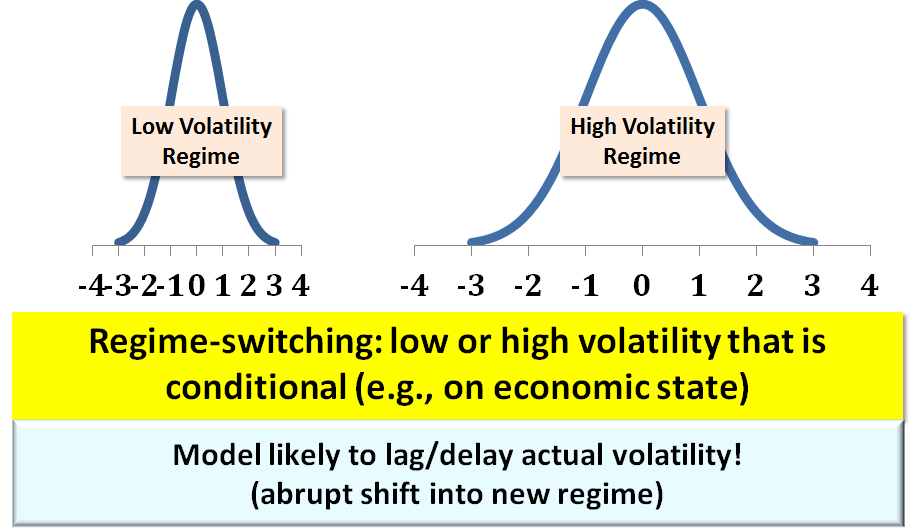
A conditional distribution in not always the same: it is different, or conditional on, some economic or market or other state. It is measured by parameters such as its conditional mean, conditional standard deviation (conditional volatility), and conditional skew, and conditional kurtosis.



## Describe the implications regime switching has on quantifying volatility.

A typical distribution is a regime-switching volatility model: the regime (state) switches from low to high volatility, but is never in between. A distribution is “regime-switching” if it changes from high to low volatility.

The problem: a risk manager may assume (and measure) an unconditional volatility but the distribution is actually regime switching. In this case, the distribution is conditional (i.e., it depends on conditions) and might be normal but regime switching; e.g., volatility is 10% during a low-volatility regime and 20% during a high-volatility regime but during both regimes, the distribution may be normal. However, the risk manager may incorrectly assume a single 15% unconditional volatility. But in this case, the unconditional volatility is likely to exhibit fat tails because it does not account for the regime switching.



## Explain the various approaches for estimating VaR.

Volatility versus Value at Risk (VaR)

Volatility is an input into our (parametric) value at risk (VaR):



Linda Allen’s Historical-based approaches

The common attribute to all the approaches within this class is their use of historical time series data in order to determine the shape of the conditional distribution.

Parametric approach

The parametric approach imposes a specific distributional assumption on conditional asset returns. A representative member of this class of models is the conditional (log) normal case with time-varying volatility, where volatility is estimated from recent past data.

Nonparametric approach

This approach uses historical data directly, without imposing a specific set of distributional assumptions. Historical simulation is the simplest and most prominent representative of this class of models.

Hybrid approach

An example of a popular hybrid approach is Filtered Historical Simulation (FHS). Filtered historical simulation “updates” the volatility by fitting a model such as GARCH to the time-series. The historical data is then used, in conjunction with the GARCH volatility, where the volatility “updates” the volatility of the time series, such that in a low-volatility ‘regime’ the

Implied volatility based approach

This approach uses derivative pricing models and current derivative prices in order to impute an implied volatility without having to resort to historical data. The use of implied volatility obtained from the Black–Scholes option pricing model as a predictor of future volatility is the most prominent representative of this class of models.

Jorion’s Value at Risk (VaR) typology

Please note that Jorion’s taxonomy approaches from the perspective of local versus full valuation. In that approach, local valuation tends to associate with parametric approaches:

Value at Risk (VaR)

|  |  |
| --- | --- |
| Parametric  Delta normal  Non parametric  Historical Simulation  Bootstrap  Monte Carlo  Hybrid (semi-p)  HS + EWMA  EVT  POT (GPD)  Block maxima (GEV) |  |

Volatility

|  |  |
| --- | --- |
| Implied Volatility  Equally weighted returns or un-weighted (STDEV)  More weight to recent returns  GARCH(1,1)  EWMA  MDE (more weight to similar states!) |  |

Historical approaches

A historical-based approach can be non-parametric, parametric or hybrid (both). Non-parametric directly uses a historical dataset (historical simulation, HS, is the most common). Parametric imposes a specific distributional assumption (this includes historical standard deviation and exponential smoothing)

## Compare, contrast and calculate parametric and non-parametric approaches for estimating conditional volatility, including: HISTORICAL STANDARD DEVIATION

Historical standard deviation

Historical standard deviation is the simplest and most common way to estimate or predict future volatility. Given a history of an asset’s continuously compounded rate of returns we take a specific window of the K most recent returns.

This standard deviation is called a moving average (MA) by Jorion. The estimate requires a window of fixed length; e.g., 30 or 60 trading days. If we observe returns (rt) over M days, the volatility estimate is constructed from a moving average (MA):



Each day, the forecast is updated by adding the most recent day and dropping the furthest day. In a simple moving average, all weights on past returns are equal and set to (1/M). Note raw returns are used instead of returns around the mean (i.e., the expected mean is assumed zero). This is common in short time intervals, where it makes little difference on the volatility estimate.

For example, assume the previous four daily returns for a stock are 6% (n-1), 5% (m-2), 4% (n-3) and 3% (n-4). What is a current volatility estimate, applying the moving average, given that our short trailing window is only four days (m=14)? If we square each return, the series is 0.0036, 0.0025, 0.0016 and 0.0009. If we sum this series of squared returns, we get 0.0086. Divide by 4 (since m=4) and we get 0.00215. That’s the moving average variance, such that the moving average volatility is about 4.64%.

The above example illustrates a key weakness of the moving average (MA): since all returns weigh equally, the trend does not matter. In the example above, notice that volatilty is trending down, but MA does not reflect in any way this trend. We could reverse the order of the historical series and the MA estimation would produce the same result.

The moving average (MA) series is simple but has two drawbacks

1. The MA series ignores the order of the observations. Older observations may no longer be relevant, but they receive the same weight.
2. The MA series has a so-called ghosting feature: data points are dropped arbitrarily due to length of the window.

## Compare, contrast and calculate parametric and non-parametric approaches for estimating conditional volatility, including: GARCH APPROACH, EXPONENTIAL SMOOTHING (EWMA), and Exponential smoothing (conditional parametric)

Modern methods place more weight on recent information. Both EWMA and GARCH place more weight on recent information. Further, as EWMA is a special case of GARCH, both EWMA and GARCH employ exponential smoothing.

GARCH (p, q) and in particular GARCH (1, 1)

GARCH (p, q) is a general autoregressive conditional heteroskedastic model:

Autoregressive (AR): tomorrow’s variance (or volatility) is a regressed function of today’s variance—it regresses on itself

Conditional (C): tomorrow’s variance depends—is conditional on—the most recent variance. An unconditional variance would not depend on today’s variance

Heteroskedastic (H): variances are not constant they flux over time

GARCH regresses on “lagged” or historical terms. The lagged terms are either variance or squared returns. The generic GARCH (p, q) model regresses on (p) squared returns and (q) variances. Therefore, GARCH (1, 1) “lags” or regresses on last period’s squared return (i.e., just 1 return) and last period’s variance (i.e., just 1 variance).

GARCH (1, 1) is given by the following equation.

|  |  |
| --- | --- |
|  |  |

Persistence is a feature embedded in the GARCH model.

In the above formulas, persistence is = (b + c) or (alpha-1+ beta). Persistence refers to how quickly (or slowly) the variance reverts or “decays” toward its long-run average. High persistence equates to slow decay and slow “regression toward the mean;” low persistence equates to rapid decay and quick “reversion to the mean.”

A persistence of 1.0 implies no mean reversion. A persistence of less than 1.0 implies “reversion to the mean,” where a lower persistence implies greater reversion to the mean.

As above, the sum of the weights assigned to the lagged variance and lagged squared return is persistence (b + c = persistence). A high persistence (greater than zero but less than one) implies slow reversion to the mean.

But if the weights assigned to the lagged variance and lagged squared returns are greater than one, the model is *non-stationary*. If (b + c) > 1 the model is non-stationary and, according to Hull, unstable. In which case, EWMA is preferred.

**Linda Allen says about GARCH (1, 1):**

GARCH is both “compact” (i.e., relatively simple) and remarkably accurate. GARCH models predominate in scholarly research. Many variations of the GARCH model have been attempted, but few have improved on the original.

The drawback of the GARCH model is its nonlinearity.

**For example: Solve for long-run variance in GARCH (1,1)**

Consider the GARCH (1, 1) equation below:



Assume that:

the alpha parameter = 0.2,

the beta parameter = 0.7, and

Note that omega is 0.2 but don’t mistake omega (0.2) for the long-run variance. Omega is the product of gamma and the long-run variance. So, if alpha + beta = 0.9, then gamma must be 0.1. Given that omega is 0.2, we know that the long-run variance must be 2.0 (0.2 ÷ 0.1 = 2.0).

EWMA

EWMA is a special case of GARCH (1,1). Here is how we get from GARCH (1,1) to EWMA:



Then we let a = 0 and (b + c) =1, such that the above equation simplifies to:



This is now equivalent to the formula for exponentially weighted moving average (EWMA):



In EWMA, the lambda parameter now determines the “decay:” a lambda that is close to one (high lambda) exhibits slow decay.

RiskMetrics™ Approach

RiskMetrics™ is a branded form of the exponentially weighted moving average (EWMA) approach. The optimal (theoretical) lambda varies by asset class, but the overall optimal parameter used by RiskMetrics™ has been 0.94. In practice, RiskMetrics™ only uses one decay factor for all series:

0.94 for daily data

0.97 for monthly data (month defined as 25 trading days)

Technically, the daily and monthly models are in­consistent. However, they are both easy to use, they ap­proximate the behavior of actual data quite well, and they are robust to misspecification.

Each of GARCH (1, 1), EWMA and RiskMetrics™ are each parametric and recursive.

Advantages and Disadvantages of MA (i.e., STDEV) vs. GARCH

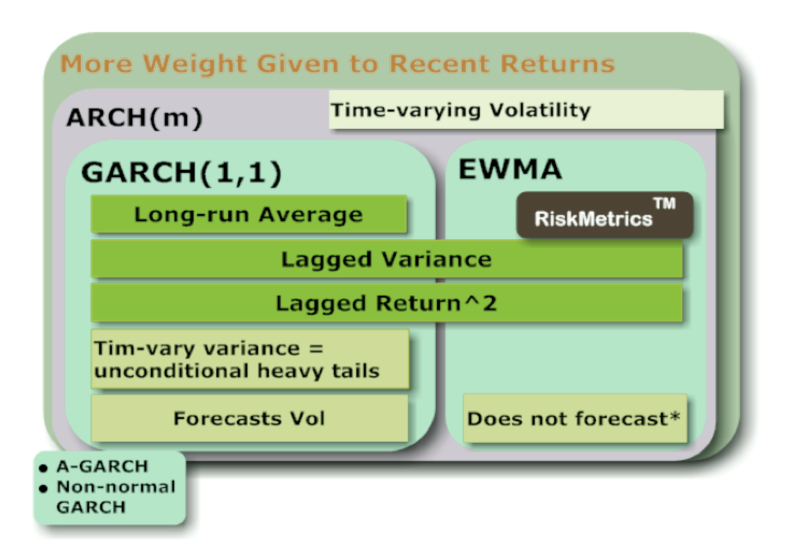
GARCH estimations can provide estimations that are more accurate than MA

|  |  |
| --- | --- |
| Jorion’s Moving Average = Allen’s STDEV | GARCH |
| Ghosting feature | More recent data assigned greater weights |
| Trend information is not incorporated | A term added to incorporate mean reversion |

Except Linda Allen warns: GARCH (1,1) needs more parameters and may pose greater MODEL RISK (“chases a moving target”) when forecasting out-of-sample

Graphical summary of the parametric methods that assign more weight to recent returns

(GARCH & EWMA)

Summary Tips:

GARCH (1, 1) is generalized RiskMetrics; and, conversely, RiskMetrics is restricted case of GARCH (1,1) where a = 0 and (b + c) =1. GARCH (1, 1) is given by:



The three parameters are weights and therefore must sum to one:



Be careful about the first term in the GARCH (1, 1) equation: omega (ω) = gamma(λ) \* (average long-run variance). If you are asked for the variance, you may need to divide out the weight in order to compute the average variance.

Determine when and whether a GARCH or EWMA model should be used in volatility estimation

In practice, variance rates tend to be mean reverting; therefore, the GARCH (1, 1) model is theoretically superior (“more appealing than”) to the EWMA model. Remember, that’s the big difference: GARCH adds the parameter that weights the long-run average and therefore it incorporates mean reversion.

GARCH (1, 1) is preferred unless the first parameter is negative (which is implied if alpha + beta > 1). In this case, GARCH (1,1) is unstable and EWMA is preferred.

Explain how the GARCH estimations can provide forecasts that are more accurate

The moving average computes variance based on a trailing window of observations; e.g., the previous ten days, the previous 100 days.

There are two problems with moving average (MA):

1. Ghosting feature: volatility shocks (sudden increases) are abruptly incorporated into the MA metric and then, when the trailing window passes, they are abruptly dropped from the calculation. Due to this the MA metric will shift in relation to the chosen window length
2. Trend information is not incorporated

GARCH estimates improve upon these weaknesses in two ways:

1. More recent observations are assigned greater weights. This overcomes ghosting because a volatility shock will immediately impact the estimate but its influence will fade gradually as time passes
2. A term is added to incorporate reversion to the mean

Explain how persistence is related to the reversion to the mean

Given the GARCH (1, 1) equation:



GARCH (1, 1) is unstable if the persistence > 1. A persistence of 1.0 indicates no mean reversion. A low persistence (e.g., 0.6) indicates rapid decay and high reversion to the mean.

GARCH (1, 1) has three weights assigned to three factors. Persistence is the sum of the weights assigned to both the lagged variance and lagged squared return. The other weight is assigned to the long-run variance.

If P = persistence and G = weight assigned to long-run variance, then P+G = 1.

Therefore, if P (persistence) is high, then G (mean reversion) is low: the persistent series is not strongly mean reverting; it exhibits “slow decay” toward the mean.

If P is low, then G must be high: the impersistent series does strongly mean revert; it exhibits “rapid decay” toward the mean.

The average, unconditional variance in the GARCH (1, 1) model is given by:



## Compare, contrast and calculate parametric and non-parametric approaches for estimating conditional volatility, including: HISTORIC SIMULATION

Historical simulation is easy: we only need to determine the “lookback window.” The problem is that, for small samples, the extreme percentiles (e.g., the worst one percent) are less precise. Historical simulation effectively throws out useful information.

“The most prominent and easiest to implement methodology within the class of nonparametric methods is historical simulation (HS). HS uses the data directly. The only thing we need to determine up front is the lookback window. Once the window length is determined, we order returns in descending order, and go directly to the tail of this ordered vector. For an estimation window of 100 observations, for example, the fifth lowest return in a rolling window of the most recent 100 returns is the fifth percentile. The lowest observation is the first percentile. If we wanted, instead, to use a 250 observations window, the fifth percentile would be somewhere between the 12th and the 13th lowest observations (a detailed discussion follows), and the first percentile would be somewhere between the second and third lowest returns.” –Linda Allen

Compare and contrast the use of historic simulation, multivariate density estimation, and hybrid methods for volatility forecasting.

Nonparametric Volatility Forecasting

|  |  |  |
| --- | --- | --- |
|  | Advantages | Disadvantages |
| Historical Simulation | Easiest to implement (simple, convenient) | Uses data inefficiently (much data is not used) |
| Multivariate density estimation | Very flexible: weights are function of state (e.g., economic context such as interest rates) not constant | Onerous model: weighting scheme; conditioning variables; number of observations  Data intensive |
| Hybrid approach | Unlike the HS approach, better incorporates more recent information | Requires model assumptions; e.g., number of observations |

## Compare, contrast and calculate parametric and non-parametric approaches for estimating conditional volatility, including: MULTIVARIATE DENSITY ESTIMATION

Multivariate Density Estimation (MDE)

The key feature of multivariate density estimation is that the weights (assigned to historical square returns) are not a constant function of time. Rather, the current state—as parameterized by a state vector—is compared to the historical state: the more similar the states (current versus historical period), the greater the assigned weight. The relative weighting is determined by the kernel function:



**Vector describing economic state at time t-i**

**Kernel function**

**Instead of weighting returns^2 by time,   
Weighting by proximity to current state**

Compare EWMA to MDE:

Both assign weights to historical squared returns (squared returns = variance approximation);

Where EWMA assigns the weight as an exponentially declining function of time (i.e., the nearer to today, the greater the weight), MDE assigns the weight based on the nature of the historical period (i.e., the more similar to the historical state, the greater the weight)

## Compare, contrast and calculate parametric and non-parametric approaches for estimating conditional volatility, including: HYBRID METHODS

The hybrid approach is a variation on historical simulation (HS). Consider the ten (10) illustrative returns below. In simple HS, the returns are sorted from best-to-worst (or worst-to-best) and the quantile determines the VaR. Simple HS amounts to giving equal weight to each returns (last column). Given 10 returns, the worst return (-31.8%) earns a 10% weight under simple HS.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Sorted | Periods | Hybrid | Cum'l Hybrid | Compare |
| Return | Ago | Weight | Weight | to HS |
| -31.8% | 7 | 8.16% | 8.16% | 10% |
| -28.8% | 9 | 6.61% | 14.77% | 20% |
| -25.5% | 6 | 9.07% | 23.83% | 30% |
| -22.3% | 10 | 5.95% | 29.78% | 40% |
| 5.7% | 1 | 15.35% | 45.14% | 50% |
| 6.1% | 2 | 13.82% | 58.95% | 60% |
| 6.5% | 3 | 12.44% | 71.39% | 70% |
| 6.9% | 4 | 11.19% | 82.58% | 80% |
| 12.1% | 5 | 10.07% | 92.66% | 90% |
| 60.6% | 8 | 7.34% | 100.00% | 100% |

However, under the hybrid approach, the EWMA weighting scheme is instead applied. Since the worst return happened seven (7) periods ago, the weight applied is given by the following, assuming a lambda of 0.9 (90%):

Weight (7 periods prior) = 90%^(7-1)\*(1-90%)/(1-90%^10) = 8.16%

Note that because the return happened further in the past, the weight is below the 10% that is assigned under simple HS.

Hybrid methods using Google stock’s prices and returns:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
| Google (GOOG) | | Period |  |  |  | Number of | Cumulative Weight | | |
| Date | Close | Return |  | Sorted |  | days ago | HS |  | Hybrid |
| 6/24/2009 | 409.29 | 0.89% | 1 | -5.90% |  | 76 | 1.0% | 0.2% | 0.2% |
| 6/23/2009 | 405.68 | -0.41% | 2 | -5.50% |  | 94 | 2.0% | 0.1% | 0.3% |
| 6/22/2009 | 407.35 | -3.08% | 3 | -4.85% |  | 86 | 3.0% | 0.1% | 0.4% |
| 6/19/2009 | 420.09 | 1.45% | 4 | -4.29% |  | 90 | 4.0% | 0.1% | 0.5% |
| 6/18/2009 | 414.06 | -0.27% | 5 | -4.25% |  | 78 | 5.0% | 0.2% | 0.7% |
| 6/17/2009 | 415.16 | -0.20% | 6 | -3.35% |  | 47 | 6.0% | 0.6% | 1.3% |
| 6/16/2009 | 416 | -0.18% | 7 | -3.26% |  | 81 | 7.0% | 0.2% | 1.4% |
| 6/15/2009 | 416.77 | -1.92% | 8 | -3.08% |  | 3 | 8.0% | 3.7% | 5.1% |
| 6/12/2009 | 424.84 | -0.97% | 9 | -3.01% |  | 88 | 9.0% | 0.1% | 5.2% |
| 6/11/2009 | 429 | -0.84% | 10 | -2.64% |  | 55 | 10.0% | 0.4% | 5.7% |

In this case:

* Sample includes 100 returns (n=100)
* We are solving for the 95th percentile (95%) value at risk (VaR)
* For the hybrid approach, lambda = 0.96
* Sorted returns are shown in the purple column

The HS 95% VaR = ~ 4.25% because it is the fifth-worst return (actually, the quantile can be determined in more than one way)

However, the hybrid approach returns a 95% VaR of 3.08% because the “worst returns” that inform the dataset tend to be further in the past (i.e., days ago = 76, 94, 86, 90…). Due to this, the individual weights are generally less than 1%.

## Explain the process of return aggregation in the context of volatility forecasting methods.

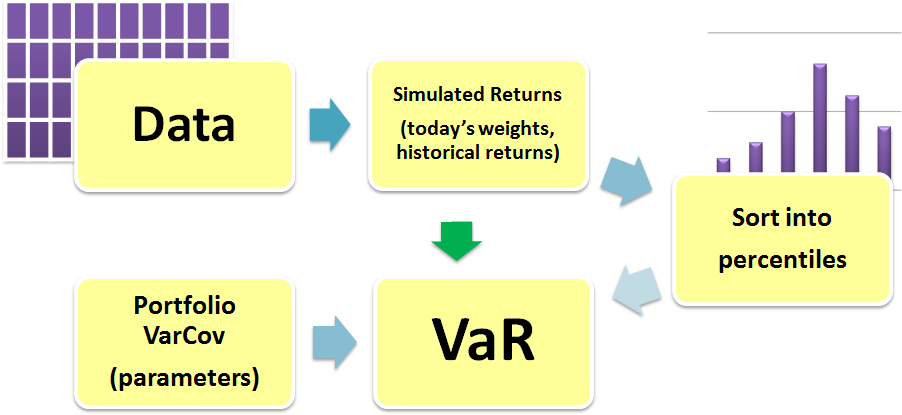
The question is: how do we compute VAR for a portfolio that consists of several positions.

The first approach is the **variance-covariance approach**: if we make (parametric) assumptions about the covariances between each position, then we extend the parametric approach to the entire portfolio. The problem with this approach is that correlations tend to increase (or change) during stressful market events; portfolio VAR may underestimate VAR in such circumstances.

The second approach is to **extend the historical simulation (HS) approach** to the portfolio: apply today’s weights to yesterday’s returns. In other words, “what would have happened if we held this portfolio in the past?”

The third approach is to **combine these two approaches**: aggregate the simulated returns and then apply a parametric (normal) distributional assumption to the aggregated portfolio.

The first approach (variance-covariance) requires the dubious assumption of normality—for the positions “inside” the portfolio. The text says the third approach is gaining in popularity and is justified by the law of large numbers: even if the components (positions) in the portfolio are not normally distributed, the aggregated portfolio will converge toward normality.



Explain how implied volatility can be used to predict future volatility

To impute volatility is to derivate volatility (to reverse-engineer it, really) from the observed market price of the asset. A typical example uses the Black-Scholes option pricing model to compute the implied volatility of a stock option; i.e., option traders will average at-the-money implied volatility from traded puts and calls.

|  |  |
| --- | --- |
| The advantages of implied volatility are: | The shortcomings (or disadvantages) of implied volatility include: |
| Truly predictive (reflects market’s forward-looking consensus) | Model-dependent |
| Does not require, nor is restrained by, historical distribution patterns | Options on the same underlying asset may trade at different implied volatilities; e.g., volatility smile/smirk |
|  | Stochastic volatility; i.e., the model assumes constant volatility, but volatility tends to change over time |
|  | Limited availability because it requires traded (set by market) price |

Explain how to use option prices to derive forecasts of volatilities

This requires that a market mechanism (e.g., an exchange) can provide a market price for the option. If a market price can be observed, then instead of solving for the price of an option, we use an option pricing model (OPM) to reveal the implied (implicit) volatility. We solve (“goal seek”) for the volatility that produces a model price equal to the market price:



Where the implied standard deviation (ISD) is the volatility input into an option pricing model (OPM). Similarly, implied correlations can also be “recovered” (reverse-engineered) from options on multiple assets. According to Jorion, ISD is a superior approach to volatility estimation. He says, “Whenever possible, VAR should use implied parameters” [i.e., ISD or market implied volatility].

## Describe implied volatility as a predictor of future volatility and its shortcomings.

Many risk managers describe the application of historical volatility as similar to “driving by looking in the rear-view mirror.” Another flaw is the assumption of stationarity; i.e., the assumption that the past is indicative of the future.

Implied volatility, “an intriguing alternative,” can be imputed from derivative prices using a specific derivative pricing model. The simplest example is the Black–Scholes implied volatility imputed from equity option prices.

In the presence of multiple implied volatilities for various option maturities and exercise prices, it is common to take the at-the-money (ATM) implied volatility from puts and calls and extrapolate an average implied; this implied is derived from the most liquid (ATM) options

The advantage of implied volatility is that it is a forward-looking, predictive measure.

“A particularly strong example of the advantage obtained by using implied volatility (in contrast to historical volatility) as a predictor of future volatility is the GBP currency crisis of 1992. During the summer of 1992, the GBP came under pressure as a result of the expectation that it should be devalued relative to the European Currency Unit (ECU) components, the deutschmark (DM) in particular (at the time the strongest currency within the ECU). During the weeks preceding the final drama of the GBP devaluation, many signals were present in the public domain … This was the case many times prior to this event, especially with the Italian lira’s many devaluations. Therefore, the market was prepared for a crisis in the GBP during the summer of 1992. Observing the thick solid line depicting option-implied volatility, the growing pressure on the GBP manifests itself in options prices and volatilities. Historical volatility is trailing, “unaware” of the pressure. In this case, the situation is particularly problematic since historical volatility happens to decline as implied volatility rises. The fall in historical volatility is dueto the fact that movements close to the intervention band are bound to be smaller by the fact of the intervention bands’ existence and the nature of intervention, thereby dampening the historical measure of volatility just at the time that a more predictive measure shows increases in volatility.” – Linda Allen

Is implied volatility a superior predictor of future volatility?

“It would seem as if the answer must be affirmative, since implied volatility can react immediately to market conditions. As a predictor of future volatility this is certainly an important feature.”

Why does implied volatility tend to be greater than historical volatility?

According to Linda Allen, “empirical results indicate, strongly and consistently, that implied volatility is, on average, greater than realized volatility.” There are two common explanations.

Market inefficiency due to supply and demand forces.

Rational markets: implied volatility is greater than realized volatility due to stochastic volatility. “Consider the following facts: (i) volatility is stochastic; (ii) volatility is a priced source of risk; and (iii) the underlying model (e.g., the Black–Scholes model) is, hence, misspecified, assuming constant volatility. The result is that the premium required by the market for stochastic volatility will manifest itself in the forms we saw above – implied volatility would be, on average, greater than realized volatility.”

But implied volatility has shortcomings.

Implied volatility is model-dependent. A misspecified model can result in an erroneous forecast.

“Consider the Black–Scholes option-pricing model. This model hinges on a few assumptions, one of which is that the underlying asset follows a continuous time lognormal diffusion process. The underlying assumption is that the volatility parameter is constant from the present time to the maturity of the contract. The implied volatility is supposedly this parameter. In reality, volatility is not constant over the life of the options contract. Implied volatility varies through time. Oddly, traders trade options in “vol” terms, the volatility of the underlying, fully aware that (i) this vol is implied from a constant volatility model, and (ii) that this very same option will trade tomorrow at a different vol, which will also be assumed to be constant over the remaining life of the contract.” –Linda Allen

At any given point in time, options on the same underlying may trade at different vols. An example is the [volatility] smile effect – deep out of the money (especially) and deep in the money (to a lesser extent) options trade at a higher volatility than at the money options.

## Explain long horizon volatility/VaR and the process of mean reversion according to an AR(1) model.

Explain the implications of mean reversion in returns and return volatility

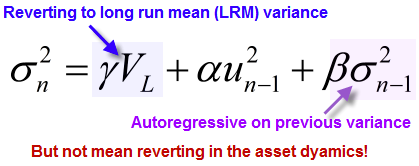
The key idea refers to the application of the square root rule (S.R.R. says that variance scales directly with time such that the volatility scales directly with the square root of time). The square root rule, while mathematically convenient, doesn’t really work in practice because it requires that normally distributed returns are independent and identically distributed (i.i.d.). What I mean is, we use it on the exam, but in practice, when applying the square root rule to scaling delta normal VaR/volatility, we should be sensitive to the likely error introduced.

Allen gives two scenarios that each illustrate “violations” in the use of the square root rule to scale volatility over time:

|  |  |
| --- | --- |
| If mean reversion… | Then square root rule |
| In returns | Overstates long run volatility |
| In return volatility | If current vol. > long run volatility, overstates  If current vol. < long run volatility, understates |

For FRM purposes, three definitions of mean reversion are used:

1. Mean reversion in the asset dynamics. The price/return tends towards a long-run level; e.g., interest rate reverts to 5%, equity log return reverts to +8%
2. Mean reversion in variance. Variance reverts toward a long-run level; e.g., volatility reverts to a long-run average of 20%. We can also refer to this as negative autocorrelation, but it's a little trickier. Negative autocorrelation refers to the fact that a high variance is likely to be followed in time by a low variance. The reason it's tricky is due to short/long timeframes: the current volatility may be high relative to the long run mean, but it may be "sticky" or cluster in the short-term (positive autocorrelation) yet, in the longer term it may revert to the long run mean. So, there can be a mix of (short-term) positive and negative autocorrelation on the way being pulled toward the long run mean.
3. Autoregression in the time series. The current estimate (variance) is informed by (a function of) the previous value; e.g., in GARCH (1,1) and exponentially weighted moving average (EWMA), the variance is a function of the previous variance.



Square root rule

The simplest approach to extending the horizon is to use the “square root rule”

|  |  |  |
| --- | --- | --- |
|  |  |  |

For example, if the 1-period VAR is $10, then the 2-period VAR is $14.14 ($10 x square root of 2) and the 5-period VAR is $22.36 ($10 x square root of 5).

The square-root-rule: under the two assumptions below, VaR scales with the square root of time. Extend one-period VaR to J-period VAR by multiplying by the square root of J.

The square root rule (i.e., variance is linear with time) only applies under restrictive i.i.d. assumptions.

The square-root rule for extending the time horizon requires two key assumptions:

1. Random-walk (acceptable)
2. Constant volatility (unlikely)

## Chapter Summary

Empirically, the return distribution of most assets is not normal. We observe such as *skewness*, *fat-tails*, and *time-varying parameters* in the returns distribution. This poses some problems as normality is often assumed, and normality allows for simple and elegant solutions to risk-measurement. However, when the normality assumption is breached and assets returns are not normal this can distort our risk measurement. In particular, fat-tails causes us to *underestimate* the risk of severe losses if we are working based on an assumption of normality.

We can explain fat-tails by a distribution that changes over time (conditional distribution). There are two things that can change in a normal distribution: the mean and volatility. Therefore, we can explain fat tails in two ways: a time-varying conditional mean; but this is unlikely given the assumption that markets are efficient. The second explanation is that conditional volatility is time varying; according to this reading’s author the latter is the more likely explanation.

VaR can be estimated using a variety of approaches. Full-valuation models tend to be non-parametric, whilst local-valuation tends to be parametric. To be specific, the main approaches include: the *parametric* approach, where distributional assumptions are imposed, *non-parametric* estimation, where no distributional assumptions are imposed but it is implied that history will inform the future; a *hybrid approach*, where historical data is combined with e.g. a weighting scheme; and the *implied volatility* approach where volatility is inferred from pricing models such as, e.g., Black-Scholes.

The main methods for volatility modeling include EWMA, GARCH, MA and MDE. The Moving Average is the simplest of these, but suffers from a ghosting feature, whereby one extreme observation dominates the data until it’s dropped. EWMA is a popular and widespread method, both because it has the more realistic assumption of assigning greater weight to recent observations, its ease of implementation, and its relatively non-technical approach making it easy to explain to management. Multivariate density estimation has the attractive feature of having its weights informed by the current economic conditions, where the scenarios with similar economic conditions are given a larger weight. GARCH has grown in popularity and is generally superior to EWMA. Indeed EWMA is a special case of GARCH. GARCH has the advantage of being able to forecast volatility, whereas EWMA simply gives you the current volatility estimate. GARCH must not be used if the weights assigned to the lagged variance and lagged squared returns are greater than one as this implies that the model is *non-stationary*. This will results in unreliable estimates.

The square-root-rule, states that the process may be scaled, provided it follows a random walk and the volatility is constant. This can be use to extend a one-period VaR to a J-period VAR by multiplying by the square root of J. Unfortunately, volatility is generally not constant so this rule must be applied with caution: in industry, it is generally considered acceptable to scale up to a week, however beyond that the model error makes the estimate too uncertain.

## Questions and Answers

Questions

18.1 For which approach is the ratio of *weight (t-1)* to *weight (t),* i.e., the ratio of consecutive weights, a constant?

1. Moving average (MA; aka, equally weighted)
2. EWMA
3. GARCH(1,1)
4. At least two of the above, or all of the above

18.2 What does the (1,1) refer to in GARCH(1,1) as an instance of GARCH(p, q)?

1. 1\*1 covariance matrix is employed
2. One unconditional variance (p=1) and one gamma weight (q =1)
3. One lagged variance (p=1) and one lagged squared return (q =1 innovation)
4. 1,1 connotes i.i.d. and permits the time scaling

18.3 If w is a column vector of portfolio weights, w(T) is the transposed row vector of the same weights and Z is a covariance matrix, which of the following is LEAST likely to suggest a violation of the consistency condition?

1. w(T)Zw < 0
2. We compute a negative portfolio variance
3. The diagonal of the covariance matrix contains 1.0 in each cell
4. The covariance matrix is not positive semi-definite

18.4 Let volatility(t) be the current estimate of today’s volatility, and let volatility(t+10) be the projected estimate for 1-day volatility ten days forward (one day volatility estimate but +10 days). For which model is the 10-day forward forecast (t+10) of one-day volatility equal to the current volatility estimate (t)?

1. Moving average (MA; aka, equally weighted)
2. EWMA
3. GARCH(1,1)
4. At least two of the above, or all of the above

Answers

18.1 D.

MA, EWMA and GARCH(1,1)  
In EWMA, lambda is the constant ratio of consecutive, declining weights.  
In GARCH(1,1), beta (decay rate) is the ratio between consecutive weights (analogous to lambda); both lambda in EWMA and beta in GARCH are “exponential” in the sense they are less than one: EWMA lambda < 1.0 and GARCH beta < 1.0.  
In MA, since all weights are equal, the ratio of t/(t-1) = 1.0. So MA weights are also constant ratio.

18.2 C.

One lagged variance (p=1) and one lagged squared return (q =1 innovation)e.g., GARCH(2,2) would give weights to both variance(t-1) and variance(t-2) plus weights to both return^2(t-1) and return^2(t-2).

18.3 C.  
In regard to (A) and (B), these are the same: w(T)Zw is the portfolio variance and consistency ensures the variance is positive. In regard to (D), the matrix generally needs to be positive semi-definite. In regard to (C), 1.0s in the diagonal do suggest a correlation rather than a covariance matrix, but this can still be viable without violating consistency condition.

18.4 D.

MA and EWMAOnly the GARCH(1,1) mean reverts. The EW(MA) approaches can only project the current volatility into the future.

# Linda Allen, Chapter 3: Putting VaR to Work

**Learning Outcomes:**

﻿

**Explain** and give examples of linear and non‐linear derivatives.

**Explain** how to calculate VaR for linear derivatives.

**Describe** the delta‐normal approach to calculating VaR for non‐linear derivatives.

**Describe** the limitations of the delta‐normal method.

**Explain** the full revaluation method for computing VaR.

**Compare** delta‐normal and full revaluation approaches.

**Explain** structural Monte Carlo, stress testing and scenario analysis methods for computing VaR, identifying strengths and weaknesses of each approach.

**Describe** the implications of correlation breakdown for scenario analysis.

**Describe** worst-case scenario analysis.

## Explain and give examples of linear and non‐linear derivatives.

A linear derivative is when the relationship between the derivative and the underlying pricing factor(s) is linear. It does not need to be one-for-one but the “transmission parameter” (delta) needs to be constant for all levels of the underlying factor. A non-linear derivative has a delta that is not constant.

Linear derivative: Price of derivative = linear function of underlying asset. For example, a futures contract on S&P 500 index is approximately linear. The key is that the transmission parameter (delta) is constant.

Non-linear derivative. Price of derivative = non-linear function of underlying asset. For example, a stock option is non-linear

**All assets are locally linear.** For example, an equity option is the classic example of a non-linear derivative: the option is convex in the value of the underlying. But maybe a better perspective is that its delta is not constant. The option delta is the slope of the tangent line.

But for tiny (infinitesimal) changes in the underling, the delta is approximately constant. So, we consider delta to be an approximation; i.e., the relationship is “locally linear.”

## Explain how to calculate VaR for linear derivatives.

By definition, the transmission parameter is constant. Therefore, in the case of a linear derivative, VaR scales directly with the underlying risk factor.



**For example (typical FRM question):**

2012 Practice Exam Part 1, Question 1: You have been asked to estimate the VaR of in investment in Big Pharma Inc. The company’s stock is trading at USD 23 and the stock has a daily volatility of 1.5%. Using the delta-normal method, the VaR at the 95% confidence level of a long position in an at-the-money put on this stock with a delta of -0.5 over a 1-day holding period is closest to which of the following choices?

Answer: VaR = |delta| \* 1.645 \* sigma \* S(0) = 0.5 \* 1.645 \* 0.015 \* 23 = 0.28.

**A classic FRM questions:**   
First, see how it is satisfied to apply the linear approximation using only option delta. Second, we do realize this is an estimate: the accurate relationship is non-linear such that we are omitting the curvature (option gamma).

## Describe the delta‐normal approach to calculating VaR for non‐linear derivatives.

In the delta-normal approach, the linear approximation is assumed (i.e., as if the derivative were linear) and the underlying factor is assumed to follow a normal distribution. We use delta-normal, for example, when relying on both option delta and bond duration to estimate underlying price changes—based respectively on asset price and yield changes (the risk factors). Both are first derivatives (or functions of the first derivative, in the case of duration). In the case of an option, the underlying factor is the stock price and we’d assume the stock price is normally distributed. In the case of a bond, we’d assume the yield is normally distributed.

Taylor Series Approximation

The Taylor Series approximation says that any "well behaved" function[[1]](#footnote-1) can be approximated by a polynomial of order two (i.e., a quadratic) as follows:



For example: application of Taylor Series to option price

First (step 1), we price an option as a function of its six inputs (yellow): stock = strike = $30, volatility = 30%, risk-free rate = 4%, option term = 1.0 years, and no dividend yield. The price of this option, per the Black-Scholes-Merton model, is $6.88.

Next (step 2), we increase only the stock price and the volatility: stock price increases $1 to $51.00 and volatility increases + 2% to 32.0%. Then, we re-price the option (step 3) and the value is $7.88. This is simple, but it’s an example of “full revaluation;” i.e., although only two inputs changed, we ran the entire pricing model again. So, due to full revaluation we know that an increase in the stock price of $1.00 and volatility of 2.0% implies and increase of $1.00 in the option price (from $6.88 to $7.88):

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Six Inputs | | | Initial | Adjust |  | Major steps: |
| 1 | Stock (S) | | $50.00 | $51.00 |  | 1. Give six inputs into the Black-Scholes model |
| 2 | Strike (K) | | $50.00 | $50.00 |  |
| 3 | Volatility | | 30.0% | 32.0% |  | 2. Imagine stock price jumps +$1 and volatility jumps +2%; i.e., we “shock" two risk factors. The value of the derivative (the call option) has a non-linear relationship with the underlying "risk factors" |
|  | Variance | | 9.00% | 10.24% |  |
| 4 | Risk-free (r) | | 4.00% | 4.00% |  |
| 5 | Term (T) | | 1.00 | 1.00 |  |
| 6 | Div Yield | | 0.00% | 0.00% |  |
| Pricing both (initial & adjusted) | | | | |  |  |
|  | d1 |  | 0.28 | 0.35 |  | 3. The second column (“Adjust”) is a full re-pricing. The first column is the option value under initial assumptions.  The second column uses Black-Scholes to re-price the option under the "shocked" assumptions. |
|  | N(d1) | | 0.61 | 0.64 |  |
|  | d2 |  | -0.02 | 0.03 |  |
|  | N(d2) | | 0.49 | 0.51 |  |
|  | Call Price | | $6.88 | $7.88 |  |

Now we compare the former full revaluation to a Taylor Series approximation. For the Taylor Series, we need the option Greeks as inputs (step 4).

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Greeks (approximations) | | | |  | 4. Compare to a Taylor Series approximation where we use the derivatives. Gamma is second-order approximation (this is the convexity - it is essentially the same thing as convexity in the bond price/yield curve). Vega is sensitivity to change in volatility (what is the small change in call option value given small change in volatility) |
|  | N'(d1) | 0.383 | 0.376 |  |
|  | Delta | 0.612 | 0.636 |  |
|  | Gamma | 0.0255 | 0.0230 |  |
|  | Vega | 19.16 | 19.16 |  |

Finally (step 5), we apply the Taylor Series. In this case, as price and volatility changes, we can use delta, gamma and vega:

Estimated change = Delta \* Change in Stock Price + 0.5 \* Gamma \* Change in Stock Price^2 + Vega \* Change in Volatility = 0.60 \* $1.00 + 0.5 \* 0.0255 \* $1.00^2 + 19.16\*0.02 = $1.008. Without the effort of a full-revaluation, the Taylor approximation gives an estimate ($1.01) by using only the risk factors that change:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Re-price with Greek approximations | | | | |
| *Let's change stock price and volatility:* | | | | |
|  | Stock price change | $1.00 |  |  |
|  | New Stock Price | $51.00 |  |  |
|  | Volatility change | 2.00% |  |  |
|  | New Volatility | 32.00% |  |  |
| *Estimate change in price with delta/gamma(& vega) approximation* | | | | |
|  | Change in Stock | 1.00 |  | 5. Instead of a (full) re-pricing, the Taylor series approximation says that the estimated price change is $1.01.  Note this gives a new option price ($7.884) that is nearly the same as the full re-pricing.  Instead of re-pricing the option, we “merely” estimated the new price based on delta and vega. |
|  | Delta | 0.61 |  |
|  | Gamma | 0.03 |  |
|  | Change in Volatility | 0.02 |  |
|  | Vega | 19.16 |  |
|  | Truncated Taylor Series: | 1.01 |  |
|  | Estimated new call price: | $7.884 |  |

## Describe the limitations of the delta‐normal method.

Although the delta-normal method analytically tractable, it is only an approximation

It is not good for “derivatives with extreme nonlinearities” (e.g., MBS). The Taylor approximation is not helpful when the derivative exhibits extreme non-linearities. This includes mortgage-backed securities (MBS) and fixed income securities with embedded options.

In the case of “delta-normal,” we are assuming the underlying risk factors are normally distributed.

## Explain the full revaluation method for computing VaR.

Full revaluation is the full re-pricing of the portfolio under the assumption that the underlying risk factor(s) are “shocked” to experience a loss. Effectively, full revaluation shocks the risk factors according to VaR; i.e., what is the worst expected change in the risk factor, given some confidence and time horizon. Then, full revaluation prices the portfolio under the changed risk factors. Full revaluation considers portfolio value for a wide range of price levels. New values can be generated by:

* Historical simulation,
* Bootstrap (simulation), or
* Monte Carlo simulation



## Compare delta‐normal and full revaluation approaches.

Full Revaluation

Every security in the portfolio is re-priced. Full revaluation is accurate but computationally burdensome.

Delta-Normal

A linear approximation is created. This linear approximation is an imperfect proxy for the portfolio. This approach is computationally easy but may be less accurate. The delta-normal approach (generally) does not work for portfolios of nonlinear securities.

“There are two primary approaches to the measurement of the risk of nonlinear securities.

The first is the most straightforward approach: the full revaluation approach. This approach has the great advantage of accuracy. It does not involve any approximations. However, this approach can be computationally very burdensome. Specifically, we may be able to reprice a bond or an option easily, but repricing a portfolio of complex derivatives of MBSs, swaptions, exotic options and so on can require many computations. In particular, as we will see later on, we may want to evaluate thousands of different scenarios. Thousands of revaluations of a portfolio consisting of hundreds of exotic securities using simulations or binomial trees may require computing power that takes days to generate the results, thereby rendering them useless.

The alternative is the approach known as the "delta-normal" approach, which involves the delta (linear) approximation, or the delta-gamma (Taylor Series) approximation. The approach is known as "delta-normal" because the linear approximation shown in equation (3.1) is often used in conjunction with a normality assumption for the distribution of fluctuations in the underlying factor value. The approach can be implemented relatively simply. This approach is extremely inexpensive computationally. Calculating the risk of a complex security can be almost "free" as far as computational time in concerned.” –Linda Allen

## Explain structural Monte Carlo, stress testing and scenario analysis methods for computing VaR, identifying strengths and weaknesses of each approach.

Structured Monte Carlo

The main advantage of the use of structured Monte Carlo (SMC) simulation is that we can generate correlated scenarios based on a statistical distribution.

|  |  |  |
| --- | --- | --- |
|  | Advantage | Disadvantage |
| Structured Monte Carlo | Able to generate correlated scenarios based on a statistical distribution  By design, models multiple risk factors | Generated scenarios may not be relevant going forward |

Scenario analysis (3.2.3) and Stress Testing (3.2.3.3)

Please note that Jorion has scenario analysis as a sub-class of stress testing; i.e., stress testing includes scenario analysis as one tool. But Linda Allen, on the other hand, essentially classifies stress testing as a type of scenario analysis. The key advantage of scenario analysis is that it gives us a means to explicitly incorporate scenarios (e.g., correlations spiking to one during a crisis) that would not necessarily be accessible by historical or simulated means.

|  |  |  |
| --- | --- | --- |
|  | Advantage | Disadvantage |
| Stress Testing | Can illuminate riskiness of portfolio to risk factors  By design, models multiple risk factors  Can specifically focus on the tails (extreme losses)  Complements VaR | May generate unwarranted red flags  Highly subjective (can be hard to imagine catastrophes) |

“The main advantage of the use of structured Monte Carlo (SMC) simulation is that we can generate correlated scenarios based on a statistical distribution. To see this advantage one needs to compare this approach to the standard scenario analysis approach, of, say, revaluing the portfolio given a 100bp rise in rates. Analyzing the effect of a parallel shift of 100bp on the portfolio's value tells us something about its interest rate risk, but nothing about the overall risk of the portfolio. The SMC approach to portfolio risk measurement addresses most of the relevant issues.” –Linda Allen

The snapshot on the next page (from a learning spreadsheet) illustrates a structured Monte Carlo simulation.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| 1a. INPUT: Time horizon & confidence level | |  |  |  |  |  |
| 1b. INPUT: Returns: Expected excess return; Volatilities; Factor Exposure | | | | | | |
| 1c. INPUT: Correlation Matrix |  |  |  |  |  |  |
| 2. [Next sheet] Cholesky Decomposition returns correlated matrix | | | | | | |
| 3. Per INVERSE TRANSFORM, random standard normals generated: 5 factors, 100 Trials | | | | | | |
| 4a. Random standard normals multiplied by matrix (A') returns: correlated volatilities | | | | | | |
| 4b. Add expected excess return = CORRELATED RETURNS | | | | | | |
| 5. Portfolio Return = Sum of [Factor Exposures]\*Return | | | | | | |
| 6. VaR |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| 1a. Time horizon (T days) |  | 10 |  |  |  |  |
| 1a. Confidence Level |  | 99% |  |  |  |  |
|  |  | Five Risk Factors | |  |  |  |
| 1b. Returns |  | 1 | 2 | 3 | 4 | 5 |
| Expected Excess Return (Annual) |  | 4.0% | 5.0% | 2.0% | 0.0% | -1.0% |
| Risk Factor Volatility (Annual) |  | 20.0% | 30.0% | 15.0% | 10.0% | 40.0% |
| Risk Factor Exposure |  | 0.75 | 0.50 | 0.25 | 0.10 | -0.05 |
| Expected Excess T-day Return |  | 0.16% | 0.20% | 0.08% | 0.00% | -0.04% |
|  |  |  |  |  |  |  |
|  |  | 1 | 2 | 3 | 4 | 5 |
| 1c. Correlation Matrix (Unitless) | 1 | 1.0 | 0.8 | 0.5 | 0.3 | (0.1) |
|  | 2 | 0.8 | 1.0 | 0.3 | 0.4 | (0.3) |
|  | 3 | 0.5 | 0.3 | 1.0 | 0.5 | (0.5) |
|  | 4 | 0.3 | 0.4 | 0.5 | 1.0 | 0.1 |
|  | 5 | (0.1) | (0.3) | (0.5) | 0.1 | 1.0 |
|  |  |  |  |  |  |  |
|  |  | 1 | 2 | 3 | 4 | 5 |
| 2. Cholesky Decomposition (A') | 1 | 0.0400 | 0.0450 | 0.0150 | 0.0050 | -0.0040 |
| Scaled to T days | 2 | 0.0000 | 0.0397 | -0.0057 | 0.0049 | -0.0257 |
| i.e., this represents a correlated, | 3 | 0.0000 | 0.0000 | 0.0254 | 0.0100 | -0.0507 |
| time-scaled matrix (Sigma) that | 4 | 0.0000 | 0.0000 | 0.0000 | 0.0159 | 0.0462 |
| can be multiplied by the normal Zs | 5 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0319 |
|  |  |  |  |  |  |  |
| 6. Monte Carlo Value at Risk (VaR): 11.79% | | | | | | |

## Describe the implications of correlation breakdown for scenario analysis.

The problem with the SMC approach is that the covariance matrix is meant to be “typical”

But severe stress events wreak havoc on the correlation matrix. That’s correlation breakdown.

Scenarios can attempt to incorporate correlation breakdowns. One approach is to stress test (simulate) the correlation matrix. This is easier said than done; e.g., the variance-covariance matrix needs to be invertible.

Diversification benefits diminish under stress events, which is the worst time for them: “These two events [i.e., Brady Bonds issued by Bulgaria and the Philippines; yield change series of US government bonds and Japanese Government Bonds] of breakdown in historic correlation matrices occur when the investor needs the diversification benefit the most. In particular, note that the increase in volatility that occurs during such crises would require an even stronger diversification effect in order not to generate extreme returns. The opposite is true in the data. Spikes in volatility occur at the same time that correlations approach one because of an observed contagion effect. A rise in volatility and correlation generates an entirely different return generating process.” –Linda Allen

## Describe worst case scenario analysis

The worst-case scenario measure asks, what is the worst loss that can happen over a period of time?

Compare this to VAR, which asks, what is the worst expected loss with 95% or 99% confidence? The probability of a “worst loss” is certain (100%); the issue is its location

As an extension to VAR, there are three points regarding the WCS:

The WCS assumes the firm increases its level of investment when gains are realized; i.e., that the firm is “capital efficient.”

The effects of time-varying volatility are ignored

There is still the extreme tail issue: it is still possible to underestimate the likelihood of extreme left-tail losses

## Chapter Summary

A linear derivative is when the relationship between the derivative and the underlying pricing factor is linear: the delta needs to be constant for all levels of the underlying factor. A non-linear derivative has a delta that is not constant. We saw that a futures contract on S&P 500 index is approximately linear, whereas the price of a stock option is non-linear.

For a linear derivative the delta, is constant. Therefore, in the case of a linear derivative, VaR scales directly with the underlying risk factor. VaR for a linear derivative can be calculated as: VaR = |delta| \* z \* sigma \* S(0). Where sigma is the volatility, z represents the confidence level in terms of the number of standard deviations from the mean, and S(0) is the price of the underlying

When using the delta-normal method we assume that the derivative is *linear* and that the underlying factor follows a *normal distribution*. These are two strong assumptions. However, the delta-normal method can be used to approximate the price of a derivative. Alternatively we can use the delta-gamma (Taylor Series) approximation, which accounts for convexity. Both of these approaches are easy to implement and the major benefit is that it is computationally inexpensive, and time can often be a critical factor. That being said, the Taylor approximation is not helpful when the derivative exhibits extreme non-linearity, such as a fixed income security with embedded options. In those cases a precise valuation may only be obtained using the full valuation method. The important point to take away from this is that there is a *trade-off* between *computational complexity* and *estimation error*.

Structured Monte-Carlo simulation (SMC) has the great advantage of being able to generate correlated scenarios based on a statistical distribution and thus models multiple risk factors simultaneously. Thus the SMC approach can help understand the portfolio exposure to risk factors. Moreover, we can specifically focus on the tails extreme loss scenarios, and SMC can thus both be used to calculate VaR as well as to complement it. Jorion and Allen notes that drawbacks of SMC include that it may generate unwarranted red flags, that it is highly subjective and that generated scenarios may not be relevant going forward. It is important to add one drawback the authors ignore which is general model risk of which the prior examples are just special cases.

Correlation breakdown can pose problems for the SMC approach: severe stress events may radically alter the correlation matrix. Scenarios can attempt to incorporate correlation breakdowns and one approach is to simulate the correlation matrix. Allen says that this is easier said than done, as the variance-covariance matrix needs to be invertible. However, you should be aware that this is no longer a problem as efficient numerical procedures to ensure that the correlation matrix is positive definite now exist.

The worst-case scenario analysis (WCS) is a useful complement to VaR, since the probability of a “worst loss” is 100%. WCS focuses on the magnitude and distribution of the loss during the worst trading period, over a given horizon, rather than the number of times we can expect to lose “some” amount (greater than or equal to VaR)

## Questions and Answers

Questions

18.2.1 A trader tells you that the Taylor series approximation can be used to price derivatives. Is the trader correct?

1. The trader is incorrect. The Taylor series approximation is only used in sensitivity analysis.
2. The Taylor series approximation is only useful for pricing more complex instruments that exhibit non-linearity, such as MBS.
3. The trader is partially correct. A Taylor series approximation can be used to approximate the price of derivatives.
4. The trader is partially correct. A Taylor series approximation can be used to approximate the price of linear derivatives.

Answers

1. 18.2.1 C.  
   The trader is partially correct. A Taylor series approximation can be used to approximate the price of derivatives. The derivatives do not necessarily have to be linear as we can add additional terms, such as the second term in the Taylor series expansion to account for curvature of the function. However, the Taylor series approximation should not be used in the case where the security exhibits extreme non-linearity, as is the case with MBS.

# **Hull, Chapter 12: Binomial Trees**

**Learning Outcomes:**

**Calculate** the value of a European call or put option using the one‐step and two‐step binomial model.

**Calculate** the value of an American call or put option using a two‐step binomial model.

**Describe** how volatility is captured in the binomial model.

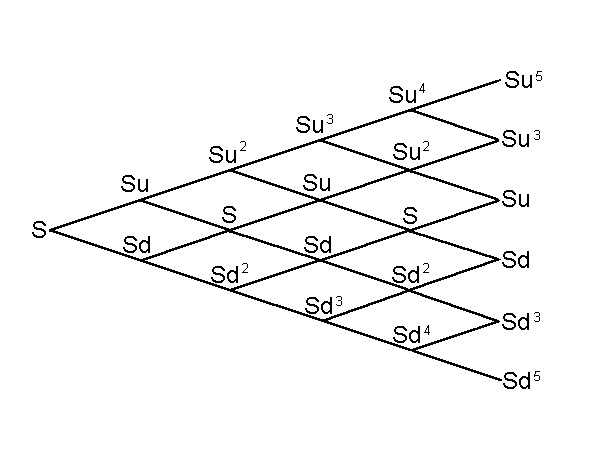
**Describe** how the binomial model value converges as time periods are added.

**Explain** how the binomial model can be altered to price options on: stocks with dividends, stock indices, currencies, and futures.

**Describe** how volatility is captured in the binomial model.

Two basic approaches to option valuation

The two basic approaches to option valuation are Black-Scholes (analytical or closed-form) and Binomial (simulation or “open” lattice)





**Black-Scholes  
(continuous time)  
(closed form)**

**Binomial  
(discrete time)  
(lattice)**

## Calculate the value of a European call or put option using the one‐step and two‐step binomial model.

We need the following notation:



The **risk-neutral probability of an “up jump”** (up movement) is denoted by (p) and given by:



This probability (p) then plugs into the equation that solves for the option price:

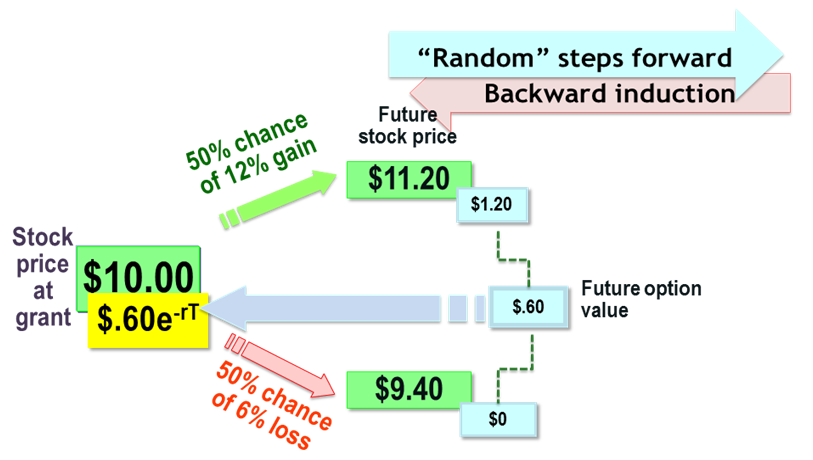


Risk neutral Valuation

In a risk–neutral world all individuals are indifferent to risk, and investors would require no compensation for risk. The expected return on a stock would be the risk free rate:



The principle of risk–neutral valuation says that we can generalize: when pricing an option under the risk–neutral assumption, our result will be accurate in the “real world” (i.e., where individuals are not indifferent to risk). Keep in mind there are two basic steps in the binomial pricing model: (i) building the paths forward and (ii) backward induction



Two step Binomial Trees

Here is the two-step binomial for a European call option on a stock index (Asset = $800, Strike = $800, Time = 0.25 years, Volatility = 20%, Riskless rate = 5%, and Dividend Yield = 2%)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Asset | $810.00 |  | Solved: |  |  |  |  |
| Strike | $800.00 |  | u | 1.1052 | *<< Magnitude of up jump* | | |
| Time (Yrs.) | 0.25 |  | d | 0.9048 | *<< Magnitude of down jump* | | |
| Volatility | 20% |  | a | 1.0075 |  |  |  |
| Riskless | 5.0% |  | p | 0.5126 | *<< Probability of up jump* | | |
| Div. Yield | 2.0% |  | 1-p | 0.4874 | *<< Probability of down jump* | | |
|  |  |  |  |  |  |  |  |
|  |  |  | Time Node (two steps @ three months = six months) | | | | |
|  |  |  | **0.0** |  | **0.25** |  | **0.50** |
|  |  |  |  |  |  |  | 989.34 |
|  |  |  |  |  |  |  | 189.34 |
|  |  |  |  |  | 895.19 |  |  |
|  |  |  |  |  | 100.66 |  |  |
|  | Stock |  | 810.00 |  |  |  | 810.00 |
|  | Option |  | 53.39 |  |  |  | 10.00 |
|  |  |  |  |  | 732.92 |  |  |
|  |  |  |  |  | 5.06 |  |  |
|  |  |  |  |  |  |  | 663.17 |
|  |  |  |  |  |  |  | 0 |

Here is the two-step binomial for a **European put option** (Asset = $50, Strike = $52, Time = 1.0 year, Volatility = 30%, Riskless rate = 5%, and Dividend Yield = 0%)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Asset | $50.00 |  | Solved: |  |  |  |  |
| Strike | $52.00 |  | u | 1.3499 | *<< magnitude of up jump* | | |
| Time (Yrs) | 1.0 |  | d | 0.7408 | *<< magnitude of down jump* | | |
| Volatility | 30% |  | a | 1.0513 |  |  |  |
| Riskless | 5.0% |  | p | 0.5097 | *<< probability of up jump* | | |
| Div Yield | 0.0% |  | 1-p | 0.4903 | *<< probability of down jump* | | |
|  |  |  |  |  |  |  |  |
|  |  |  | Time Node (two steps @ 1 year = 2 years) | | | | |
|  |  |  | **0.0** |  | **1.00** |  | **2.00** |
|  |  |  |  |  |  |  | 91.11 |
|  |  |  |  |  |  |  | - |
|  |  |  |  |  | 67.49 |  |  |
|  |  |  |  |  | 0.93 |  |  |
|  | Stock |  | 50.00 |  |  |  | 50.00 |
|  | Option |  | 6.25 |  |  |  | 2.00 |
|  |  |  |  |  | 37.04 |  |  |
|  |  |  |  |  | 12.42 |  |  |
|  |  |  |  |  |  |  | 27.44 |
|  |  |  |  |  |  |  | 24.56 |

Here is the two-step binomial for a **European call option** (Asset = $20, Strike = $21, Time = six months, Volatility = 19%, Riskless rate = 12%, and Dividend Yield = 0%)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 1=call, 0=put | **1** |  | **Call Option** | |  |  |  |
| **Asset** | **20** |  | **Solved:** |  |  |  |  |
| **Strike** | **21** |  | **u** | 1.100 | *<< magnitude of up jump* | | |
| **Time (yrs)** | **0.25** |  | **d** | 0.900 | *<< magnitude of down jump* | | |
| **Volatility** | **19%** |  | **a** | 1.030 |  |  |  |
| **Riskless** | **12.0%** |  | **p** | 0.652 | *<< probability of up jump* | | |
| **Div Yield** | **0.0%** |  | **1-p** | 0.348 | *<< probability of down jump* | | |
|  |  |  |  |  |  |  |  |
|  | **Time Node** | | **0.0** |  | **0.25** |  | **0.50** |
|  |  |  |  |  |  |  | **24.20** |
|  |  |  |  |  |  |  | **3.200** |
|  |  |  |  |  | **22.00** |  |  |
|  |  |  |  |  | **2.0256** |  |  |
|  | **Stock** |  | **20.00** |  |  |  | **19.80** |
|  | **Option** |  | **1.2822** |  |  |  | **-** |
|  |  |  |  |  | **18.00** |  |  |
|  |  |  |  |  | **-** |  |  |
|  |  |  |  |  |  |  | **16.20** |
|  |  |  |  |  |  |  | **0.0** |

## Calculate value of an American call or put option using a two‐step binomial model

The key difference is that each node is a MAXIMUM function of [intrinsic value if option were exercised, discounted value of two subsequent nodes]

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 1=call, 0=put | 0 |  | PUT |  |  |  |  |
| Asset | $50.00 |  | Solved: |  |  |  |  |
| Strike | $52.00 |  | u | 1.2000 | *<< magnitude of up jump* | | |
| Time (yrs) | 1.0 |  | d | 0.8000 | *<< magnitude of down jump* | | |
| Volatility | 30% |  | a | 1.0513 |  |  |  |
| Riskless | 5.0% |  | p | 0.6282 | *<< probability of up jump* | | |
| Div Yield | 0.0% |  | 1-p | 0.3718 | *<< probability of down jump* | | |
|  |  |  |  |  |  |  |  |
|  |  |  | **Time Node** | |  |  |  |
|  |  |  | **0.0** |  | **1.00** |  | **2.00** |
|  |  |  |  |  |  |  | 72.00 |
|  |  |  |  |  |  |  | - |
| **Weighted PV of future option nodes** |  |  |  |  | 60.00 |  |  |
|  |  |  |  |  | 1.41 |  |  |
|  | Stock |  | 50.00 |  |  |  | 48.00 |
|  | Option |  | 5.090 |  |  |  | 4.00 |
|  |  |  |  |  | 40.00 |  |  |
|  |  |  |  |  | 12.00 |  |  |
| **Intrinsic value = $40 - $52** |  |  |  |  |  |  | 32.00 |
|  |  |  |  |  |  |  | 20.00 |

## Describe how volatility is captured in the binomial model

In this case, volatility informs the magnitude of up (u) and down (d) jumps, and therefore also the probability (p) of an up jump. Let:

p = probability of up movement; 1-p probability of down movement

u = magnitude of up movement,

d = magnitude of down movement

**Up movement (u) and down movement (d)**



**Probability of up (p)**



The probability (p) is based on the discounted expected value of the option given by:



For example

Make sure you can do these calculations. These calculations allow you to calculate u, d, and p from the volatility (σ) and time (t).

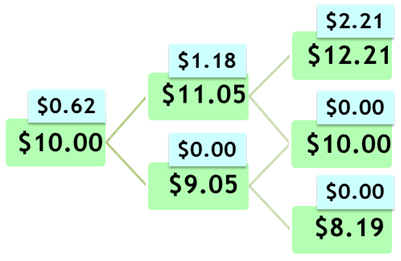
Scenario #1, Assume:

* Stock = Strike = $10
* Time (t) = 3 months (0.25)
* Volatility (σ) = 20%
* Riskless rate = 5%



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Scenario #1 | | | | |
| Stock | $10.00 |  | u | 1.11 |
| Strike | $10.00 |  | d | 0.9 |
| Time (yrs) | 0.25 |  | a | 1.01 |
| Volatility | 0.20 |  | p | 0.54 |
| Riskless | 0.05 |  | 1-p | 0.46 |





Scenario #2, Assume:

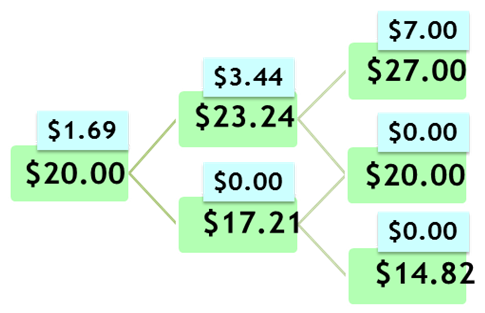
* Stock = Strike = $20
* Time (t) = 3 months (0.25)
* Volatility (σ) = 30%
* Riskless rate = 4%

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Scenario #2 | | | | |
| Stock | 20 |  | u | 1.16 |
| Strike | 20 |  | d | 0.86 |
| Time (yrs) | 0.25 |  | A | 1.01 |
| Volatility | 0.30 |  | p | 0.5 |
| Riskless | 0.04 |  | 1-p | 0.5 |



## Describe how the binomial model value converges as time periods are added

The accuracy of the binomial is partly a function of the number of time periods modeled. Generally, when less than forty or fifty time periods are used, the value produced by the binomial fluctuates (i.e., up then down, then up, then down). As the number of time periods is increased, the value converges toward a value. The option is said to converge or stabilize.



## Explain how the binomial model can be altered to price options on: stocks with dividends, stock indices, currencies, and futures.

Stocks with dividends

Consider a stock paying a known dividend yield at rate (q). The total return from dividends and capital gains in a risk-neutral world is (r). The dividends provide a return of (q). Capital gains must therefore provide a return of (r – q). If the stock starts at S(0), its expected value after one time step of length must be S(0)\*exp[(r-q)\*T]. This implies:



Stock indices

The valuation of an option on a stock index is very similar to the valuation of an option on a stock paying a known dividend yield.

Currencies

A foreign currency can be regarded as an asset providing a yield at the foreign risk-free rate of interest, r(f). By analogy with the stock index case, we let a = exp([r-r(f)]\*T) such that:



Futures

Since it costs nothing to take a long or short position in a futures contract, in a risk-neutral world the futures price has an expected growth rate of zero. In this case, we can use:



## Describe how volatility is captured in the binomial model.

## Chapter Summary

This chapter looked at how we can value options using a Binomial model, more specifically Binomial trees. Binomial trees are relatively easy to solve for, and we do so using a method known as *backwards induction*, that is, working our way from the end of the tree, back to the very first node. A risk-less portfolio was created using a portfolio of stocks and a short position in a call option. This was achieved by setting the portfolio in the upstate, equal to the value of the portfolio in the downstate, where the number of stocks in our portfolio was equal to delta ( Solving for delta, gives us the number of stocks we need for our portfolio, of stocks and 1 short call option, to be risk-free. This was achieved using only no-arbitrage arguments.

The same result can be deduced using *risk-neutral* valuation. A key result for the purpose of valuing an option is that we can assume that we are risk-neutral. This is due to the fact that our risk preferences are, in a sense, captured by the price of the underlying instrument, e.g., a stock. This is the reason why the probability of up and down moves in the underlying does not affect the price of an option: the option is calculated in terms of the price of the underlying stock.

It is important to note that the objective, or real world probabilities for up and down moves for the option are, in general, not the same as the probabilities in a risk-free world.

Valuing American options using binomial trees includes an extra step. We still value the option using backwards induction, discounting to the previous node. However, with an American option, we also need to consider whether it is optimal to exercise the option at each node.

When using a binomial tree to represent the movement in the price of the underlying, the parameters u and d are chosen so as to match the volatility of the stock price. It turns out that the volatility is the same in both the real world and the risk-neutral world. So even though the probabilities of up and down moves differ, and the expected return on the stock depends on whether we are in a real world or risk-neutral world, the volatility does not.

This reading contains several formulas you should be comfortable with, as you will undoubtedly need some of them for the exam. In particular you should know how to derive (or at the very least memorize) the formulas for:

* (option delta)
* The value of an option f, and how you can apply the formula repeatedly in a tree with multiple time periods
* p and (1-p) – the probabilities of up and down moves
* The values for u and d
* How to apply the value of an option f when the option is American
* The equation for valuing an option using a two-step tree (this is useful as it is a typical questions and knowing the formula to get the result directly rather than using repeated application of the general formula can save you valuable time).

## Questions and Answers

Questions

12.1 The current price of a stock is $10, and it is known that at the end of three (3) months the stock's price will be either $13 or $7. The risk-free rate is 4% per annum. What is the implied no-arbitrage price of a three-month (T = 0.25) European call option on the stock with a strike price of $10? (note: this does not include an assumption about the stock's volatility).

1. $0.97
2. $1.28
3. $1.53
4. $1.55

12.2 A stock with a (continuous) dividend yield of 1.0% has a current price of $30 and volatility of 22%. We use a two-step binomial model to value a two-year European style call option on the stock; i.e., each time step is one year. The risk-free rate is 3.0%. In the binomial tree, what is the stock price at the node with the lowest stock price?

1. $14.78
2. $19.32
3. $22.49
4. $25.25

12.3 A non-dividend-paying stock has a current price of $10 and a volatility of 12% per annum. The risk-free rate is 4.0%. We use a twelve-step binomial model to value a one-year European-style put option on the stock; i.e., each step is one month. What is the second-largest stock price among all of the nodes on the binomial tree?

1. $14.64
2. $19.68
3. $23.29
4. $97.15

12.4 Given a binomial model used to price a European-style call option on a non-dividend paying stock, if we hold everything constant (for a given stock price, strike price, volatility, term to expiration and risk-free rate), which of the following necessarily increases as we increase the NUMBER of steps in the binomial model?

1. Magnitude of an up movement (u)
2. Magnitude of a down movement (d)
3. Risk-neutral probability of an up movement (p)

Price at the lowest node on the tree

Answers

12.1 C. $1.53

Following Hull, a riskless portfolio consists of long delta (d) shares + short one option.

If the stock moves up, value of the riskless portfolio = $13\*delta - $3 loss on the written call option; and if the stock moves down, value of the riskless portfolio = $7\*delta. Setting them equal (i.e., riskless payoff): $13\*d - $3 = $7\*d, and 6d =3, so d = 0.5.

If delta (d) = 0.5, then value of portfolio today is: $10\*0.5 - f = 5 -f = $3.5\*exp(-1%), such that

f = 5 - 3.5\*exp(-1%) = $1.53483

12.2 B. $19.32

A two-step binomial has six nodes; we know the lower price occurs at S(0)\*d\*d, in the lower right.

d = exp[-volatility \* SQRT(time step)] = exp[-22% \*SQRT(1)] = 0.8025;

The lowest node = $30\*exp(-22%)^2 = $19.321

12.3 A. $14.64

The largest value is the top-most node at the end of the year: S(0)\*u^12. The second largest must be the one month's prior node, S(0)\*u^11, as it must be higher than the second-highest node at maturity which is S(0)\*u^11\*d. Keep in mind we assume a recombining tree, and in a recombining tree the communicative property applies; e.g., S(0)\*u\*d = S(0)\*d\*u.

As u = exp[volatility \* SQRT(time step)] = exp[12% \* SQRT(1/12)] = 1.0352, this node is given by:

S(0)\*u^11 = 10\*exp[12% \* SQRT(1/12)]^11 = $14.638

12.4 B. Magnitude of a down movement (d).

To increase the number of steps, ceteris paribus, is to decrease the length of each step.

d = exp[-volatility \* SQRT(time step)] = 1/ exp[volatility \* SQRT(time step)], which is an increasing function as the time step decreases

In regard to (A), u will decrease necessarily;

In regard to (C), p is ambiguous as numerator and denominator each decrease;

In regard to (D), the lowest value must decrease due to the effect of the square root of time; think of the tree dispersing as the number of steps increases, with higher highs and lower lows.

# Hull, Chapter 14: The Black-Scholes-Merton Model

**Learning Outcomes:**

**Explain** the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return.

**Compute** the realized return and historical volatility of a stock.

**List and describe** the assumptions underlying the Black‐Scholes‐Merton option pricing model.

**Compute** the value of a European option using the Black‐Scholes‐Merton model on a non‐dividend‐paying stock.

**Identify** the complications involving the valuation of warrants.

**Define** implied volatilities and describe how to compute implied volatilities from market prices of options using the Black‐Scholes‐Merton model.

**Explain** how dividends affect the early decision for American call and put options.

**Compute** the value of a European option using the Black‐Scholes‐Merton model on a dividend‐paying stock.

**Use Black's Approximation to compute** the value of an American call option on a dividend-paying stock.

The Black-Scholes Merton (BSM) Model

The way I like to memorize the BSM is to start with the formula for minimum value; minimum value is the present value of the option if the stock grows at the risk-free rate. So, minimum value is value without volatility. Then “wrap in” the N() functions which effectively increase the option value to account for volatility:

**Black-Scholes =**

**Minimum value + Volatility**



**N() is the cumulative standard normal distribution function**

## Explain the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return

Under GBM (a Wiener process), period log returns are normally distributed,



which implies that price levels (or the ratio of price levels—wealth ratios) are log-normally distributed



An Ito process is a generalized Weiner process (a stochastic process) where the change in the variable during a short interval is normally distributed. The mean and variance of the distribution are proportional to δt. In an Ito process, the parameters are a function of the variables x and t.



Let ST equal the stock price at future time T. The expected value of ST [i.e., E(ST)] is given by:



The expected percentage change in the stock price is assumed to be normally distributed. However, the expected stock price at future time T, as shown in the formula above is lognormally distributed.

We can assume asset returns are normally distributed: from day to day, the stock can go up (+) or down (-). But the future stock price, say in ten days, is lognormally distributed: it cannot be nonzero.

The variance of ST is given by:



Distribution of the Rate of Return

The continuously compounded rate of return per annum is normally distributed. The distribution of this rate of return is given by the following:



The Expected Return: Arithmetic vs. Geometric

The phrase “expected return” has two common meanings: arithmetic and geometric.

|  |  |
| --- | --- |
|  |  |

The continuously compounded return realized over T years is given by:



## Compute the realized return and historical volatility of a stock

Start with the variable (ui) that is the natural log of the ratio between a stock price at time (i) and the previous stock price at time (i-1):



An unbiased estimate of the variance is given by:



Important: the equation above is the variance. The volatility is the standard deviation and, therefore, is given by:



For purposes of calculating VAR—and often for volatility calculations in general—a few simplifying assumptions are applied to this volatility formula. Specifically:

1. Instead of the natural log of the ratio [Si/Si-1], we can substitute a simple percentage change in price: %S = [(Si-Si-1)/Si-1]
2. Assume the average price change is zero
3. Replace the denominator (m-1) with (m)

With these three simplifications, an alternative volatility calculation is based on the following simplified variance:



The third simplification above can be confusing. This is when (m-1) is replaced with (m) in the denominator. In technical terms, (m-1) is called “unbiased” and is appropriate for calculating the variance/standard deviation of a sample (i.e., when the data series is only part of the entire population but not the entire population). If you use Excel, you may notice there are two functions that measure standard deviation: = STDEV() and = STDEVP(). The only difference is that the first function assumes a sample and contains (m-1) in the denominator; the second assumes a population and contains (m) in the denominator.

One way to remember this is: the unbiased estimator (m-1) creates a larger variance/standard deviation (because it makes the denominator smaller), which is a “safer” result in the case of a sample statistic because we are not measuring the entire population. Given this, (m-1) is technically correct, but nevertheless, as a practical matter, it is fine to use (m) especially if the returns are daily. Put differently, if your data is statistically sensitive as to whether you divide by m or m-1, your dataset is likely to be either too short or not robust.

Realized Return

Realized return of $100 growing, with volatility, to $179.40 over five periods:

|  |  |  |
| --- | --- | --- |
| Initial | $100.00 | Period Return |
|  | $115.00 | 15% |
|  | $138.00 | 20% |
|  | $179.40 | 30% |
|  | $143.52 | -20% |
| Final | $179.40 | 25% |
|  |  |  |
| Arithmetic Avg. | | 14.00% |
| Geometric Avg | | 12.40% |
| Realized (continuous) | | 11.69% |

Realized Return



Realized return of stock growing from $40 over two years with expected return of 15%



|  |  |
| --- | --- |
| **ADBE Stock Price** | $40.00 |
| **Sample Std. Deviation** | 2.60% |
| **Annualized Volatility** | 41.1% |
| **Expected Return** | 15.0% |
|  |  |
| Time: | 2.0 |
| Mean (arithmetic) Price | $53.99 |
|  |  |
| Geometric return | 6.55% |
| Median Price | $45.60 |

Historical volatility (120 days)



List and describe the assumptions underlying the Black-Scholes-Merton pricing model

The assumptions used to derive the Black–Scholes–Merton differential equation include:

Stock price follows a Weiner process (itself a particular Markov stochastic process) with a constant volatility

* Short selling is allowed
* No transaction costs and no taxes; securities are perfectly divisible
* Dividends are not paid
* There are no (risk-less) arbitrage opportunities
* Security trading is continuous
* The risk-free rate of interest is constant and the same for all maturities

The stock price process is described by the following formula:



The Black–Scholes–Merton Differential Equation is given by:



## Compute the value of a European option using the Black‐Scholes‐Merton (BSM) model on a non‐dividend‐paying stock

*“The best model in all of economics is the Black-Scholes Model for valuing options[[2]](#footnote-2)”*

In the case of a European option, the BSM model gives for a call (c) and a put (p):

|  |  |
| --- | --- |
|  |  |

Where d1 and d2 are given by:

|  |  |
| --- | --- |
|  |  |

To illustrate, assume a call option with a strike price of $10 (K = $10) on a stock with a current price of $10 (S = $10). If the risk-free rate is 5%, then the value of the call is given by:



In this case, N (d1) = 0.64 and N (d2) = 0.56, such that



To understand the Black–Scholes, it helps to start with its resemblance to the simple – put–call– parity formula: c=S Ke-rT. See how – put–call– parity is embedded inside the formula? This part is just the stock price minus the discounted exercise price! The Black–Scholes “augments” the – put–call parity formula by adding N(d1) and N(d2). The “introduction” of N(d1) and N(d2) into the formula have the net effect of increasing the value of the call—the higher the volatility, the greater these increase effected by these terms.

A key feature of the Black–Scholes is that is has no place for (it does not require) the expected return of the stock (nor does it require an assumption about the probability distribution of returns). This is counterintuitive but it is a feature of no arbitrage put–call parity and it extends to the Black–Scholes that is derived from the no arbitrage premise.

Keep in mind that the option values given by the model apply under the assumptions. The model is a proof—it follows from a no-arbitrage scenario—given the assumptions. In other words, if the assumptions are true, the value must be true or there would be an arbitrage opportunity. Many criticisms of the model stem from the observation that one or more assumptions are not true in real-life. For example, application to ESOs is criticized because several of the assumptions are not true for ESOs.

For Example (Black-Scholes-Merton)

* Stock price (S) is $10



* Strike (K) is $9
* Volatility (σ) is 20%
* Term (t) is six months (0.5)
* Riskless rate is 5%

For Example (Black-Scholes-Merton)

|  |  |  |
| --- | --- | --- |
|  | **Black-Scholes Inputs** | |
| 1 | Stock (S) | **$40.00** |
| 2 | Strike (K) | **$60.00** |
| 3 | Volatity | **30.0%** |
|  | Variance | 9.00% |
| 4 | Riskfree rate (r) | **3.00%** |
| 5 | Term (T) | **5.00** |
| 6 | Div Yield | **0.00%** |

|  |  |
| --- | --- |
| **Call option** |  |
| d1 | -0.0027 |
| N(d1) | 0.4989 |
| d2 | -0.6735 |
| N(d2) | 0.2503 |
| **Call Price** | **$7.030** |
| **Put option** |  |
| -d1 | 0.0027 |
| N(-d1) | 0.5011 |
| -d2 | 0.6735 |
| N(-d2) | 0.7497 |
| **Put Price** | **$18.673** |
| **Call + Disc. Strike** | **58.67** |
| **Put + Stock** | **58.67** |

## Identify the complications involving the valuation of warrants

Assume that V(T) equals the value of the company’s equity and N equals the number of outstanding shares. Further, assume that a company will issue (M) number of warrants with a strike price equal to K. S(T) equals the stock price at time T. The (adjusted) stock price, after we account for the dilution effect of the issued warrants, is:



The Black–Scholes can be used to value a warrant; however, three adjustments are required

1. The stock price (S0) is replaced by an “adjusted” stock price
2. The volatility input is calculated on equity (common equity + warrants) not stock price
3. A multiplier reduces the calculation. The multiplier captures dilution and is also called a “haircut.” The haircut is given by:



The primary complication is circularity: the warrant depends on the stock price, but the stock price is a function of (diluted by) the issuance of warrants. This is also called an iterative solution. The exercise of warrants and employee stock options (ESOs) is dilutive because strike



|  |  |
| --- | --- |
| Warrant dilution |  |
| # of Shares | 1,000,000 |
| # of warrants | 200,000 |
| Value each warrant | 5.86 |
| Warrant issue cost | 1,171,732 |
| Reduced market cap | 38,828,268 |
| Implied Share price | 38.83 |

Explain the risk-neutral evaluation framework

The risk-neutral valuation framework says that the risk preferences of the investor (e.g., is the investor risk-averse or risk-seeking?) are not incorporated—and therefore not required—to value the option. The three-step procedure includes the following steps:

1. Assume the expected return from the underlying asset is the risk-free interest rate
2. Calculate the expected payoff from the option at its maturity
3. Discount the expected payoff at the risk-free interest rate

Discuss how cash flows affect the pricing of an option

The original (generic) Black-Scholes option-pricing model assumes that the underlying asset does not pay dividends (i.e., the underlying asset generates no interim cash flows). If dividends are paid, the treatment is the following:

European options: the stock price input is reduced by the present value of expected dividends.

American options: Black’s approximation can be used. This approach sets the American call option price equal to the greater of two European call option prices. The first European call option expires when the American call expires; the second expires immediately prior to the final ex-dividend date.

Identify the methods for estimating future volatility

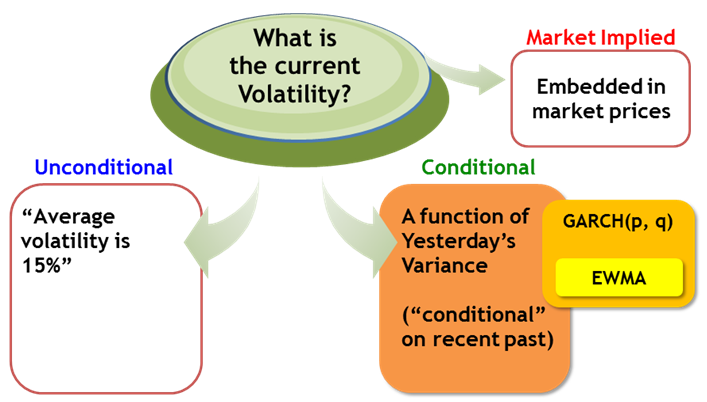
The methods are discussed in great detail in the Quantitative Module. In summary, the methods include:

* Implied volatility: The use of an option pricing model (e.g., Black-Scholes) to solve for (iterative “goal seek” required) the volatility that produces a model estimate equal to the market price of the option
* Un-weighted moving average (MA)
* Exponentially weighted moving average (EWMA)
* GARCH(p, q): Lagged variances and lagged squared returns

## Define implied volatilities and describe how to compute implied volatilities from market prices of options using the Black‐Scholes‐Merton model

Volatility is the only parameter in the Black–Scholes pricing model that cannot be directly observed. The implied volatility is simply the result of “reverse engineering” the Black–Scholes: if we are given a market price for the option, we solve for the volatility that makes the equation work. Implied volatility is the volatility the produces a model price equal to the (observed) market price.

The solution requires an iterative search procedure since we cannot solve directly for the volatility (σ)



**For example, assume:**

* Stock price (S) is $10
* Strike (K) is $10
* Term (t) is six months (0.5)
* Riskless rate is 5%
* Call price is $1.25

$1.25 = Black-Scholes [$10, $10, t=0.5 years, r = 5%, σ]

Solve for the implied volatility: σ ≅ **.405**

**Implied volatility cannot be inverted, it requires an iterative solution (“goal seek”)**

**Another example (Implied volatility)**

Assume the (observed) market price of the call option is $2.00. An implied volatility of 46% produces an option model price of $2.00. Note also: the “error” is the same for the call and put (implied by put-call parity).

|  |  |  |  |
| --- | --- | --- | --- |
|  | BS | Implied |  |
|  | Model | Volatility | Error |
| Stock | $10 | $10 |  |
| Strike | $10 | $10 |  |
| Volatility | 50% | **46%** |  |
| Riskless rate | 4.0% | 4.0% |  |
| Term | 1.00 | 1.00 |  |
| Dividend Yield | 0% | 0% |  |
| d1 | 0.33 | 0.32 |  |
| d2 | (0.17) | (0.15) |  |
| Call (c) | $2.14 | $2.00 | **$0.14** |
| Put (p) | $1.75 | $1.61 | **$0.14** |
|  |  |  |  |
| Call + Discounted Strike | $11.75 | $11.61 |  |
| Put + Stock | $11.75 | $11.61 |  |

## Explain how dividends affect the early decision for American call and put options

European options

Black-Scholes formula can be used provided the stock price is reduced by the present value of all dividends during the life of options

American options

The most likely time for the early exercise of an American call is immediately before the final ex-dividend date

Black’s Approximation: a procedure for taking account of early exercise in call options.

Compute the value of a European option using the Black‐Scholes‐Merton model on a dividend‐paying stock

A European option on a dividend-paying stock can be analyzed as the sum of two components:

1. A riskless component = known dividends during the life of the option, plus
2. A risk component

A dividend yield effectively reduces the stock price (the option holder forgoes dividends).

Operationally, this amounts to reducing the stock price by the present value of all the dividends during the life of the option. If (q) represents the annual continuous dividend-yield on a stock (or stock index), the adjusted Black-Scholes-Merton for a European call option is given by:



|  |  |
| --- | --- |
|  |  |

## Use Black's Approximation to compute the value of an American call option on a dividend-paying stock.

Black suggests an approximate procedure for taking account of early exercise in call options. This involves calculating the prices of European options that mature at times (T) and t(n) and then setting the American price equal to the greater of the two. This approximation seems to work well in most cases.

## Chapter Summary

This chapter presents you with material that is probably a step up in difficulty. However, later chapters, as well as Part 2 of the FRM will assume knowledge of the material presented herein. This chapter covers important material: the calculation of volatility from historical data, the Black–Scholes–Merton differential equation, risk-neutral valuation, the Black–Scholes–Merton option pricing formulas, implied volatilities, and the impact of dividends. Make sure that these are terms you are comfortable with.

Stock *prices* are distributed according to the *lognormal* distribution. It thus follows that the stock *returns* are *normally* distributed. The expected rate of return in a short period of time is equal to . However, the expected continuously compounded rate of return over a longer period of time is equal to

The price of European options can be derived by either solving the Black-Scholes-Merton partial differential equation, or by using risk-neutral valuation. However, note that it is important that you understand that we are not assuming risk neutrality. It just happens that the value of a derivative security is independent of risk preferences.

Volatility is the only parameter in the Black–Scholes pricing model that cannot be directly observed. If we are given a market price for an option, we solve for the volatility that makes the equation work. Implied volatility is the volatility the produces a model price equal to the observed market price. The solution requires an iterative search procedure since we cannot solve directly for the volatility.

The Black–Scholes is quite flexible and robust. Indeed it can be extended to the case of options on an underlying stock paying dividends, by discounting the stock price by the value of the dividends. Furthermore, the B-S model can be used to value a warrant however, some adjustments need to be made; perhaps the most important [to understand] is the “hair-cut”.

For American options on a dividend paying stock Fisher Black suggested an approximate procedure for taking account of early exercise in call options. This involves calculating the prices of a European option that mature at times T and t(n), where the expiration times represent the expiration of the American option, and expiration immediately prior to the ex-dividend date, respectively. The (American) option price is then set equal to the greater of the two. Black’s approximation is used extensively in industry, and has proved surprisingly accurate. Therefore, traders often use it, rather than more computationally intensive models when time of execution is important.

## Questions and Answers

Questions

14.1 Let r(0,t) = LN[S(t)/S(0)]; i.e., the continuously compounded daily (log) return. Our key assumption is that r(0,t) is normally distributed. Consider the following statements, which are true?:

1. S(t) and S(t)/S(0) must have a lognormal distribution
2. The sum of r(0,1) + r(1,2) + ... \* r(n-1,n) is normal
3. The product of r(0,1) \* r(1,2) \* ... \* r(n-1,n) is normal
4. The sum of S(1)/S(0) + S(2)/(1) + ... + S(n)/S(n-1) is lognormal
5. The product of S(1)/S(0) \* S(2)/(1) \* ... \* S(n)/S(n-1) is lognormal
6. I. only
7. I., II. and V.
8. I., III. and IV.
9. All are true

14.2 Consider a stock with an expected return of 10% per annum and volatility of 24% per annum. If the stock prices exhibits the lognormal property, what is, respectively, (i) the mean rate of return (continuously compounded) per annum and (ii) the standard deviation of the rate of return over an entire ten (10) year period?

1. 7.12% mean and 2.40% standard deviation
2. 7.12% mean and 7.59% standard deviation
3. 10.0% mean and 7.59% standard
4. 10.0% mean and 24.0% standard

14.3 Each of the following is an underlying assumption of the basic Black-Scholes option-pricing model EXCEPT:

1. The stock price follows a geometric Brownian motion (GBM) which is a continuous process without jumps
2. The continuously compounded rate of return on the stock is normally distributed, such that the distribution of the future stock price is lognormal
3. The expected rate of return on the stock (mu) and volatility (sigma) are constant
4. The expected real-world (risky) rate of return on the stock is known and the value of the option is an increasing function of this rate of return

14.4 In the absence of dividends, Hull shows that an American-style call option should never be exercised early. However, if the American-style call option instead does pay dividends, which of the following is true?

1. It is still never optimal to early exercise an American call option
2. It may be optimal to early exercise an American call option immediately after the ex-dividend date
3. It may be optimal to early exercise an American call option immediately before the ex-dividend date
4. It is always optimal to early exercise an American call option immediately before the ex-dividend date

Answers

14.1 B. I., II. and V.

In regard to (I), this is the essential property. If x is normal, then y = exp(x) is lognormal; in this case r(0,t) = LN[S(t)/S(0)] is normal such that exp(LN[S(t)/S(0)]) = S(t)/S(0) is lognormal.

In regard to the others: the sum of normals is normal (summation stability!) but the product of normals is not;

and the product of lognormals is lognormal but the sum is not.

14.2 B. 7.12% mean and 7.59% standard deviation

Mean rate of return (mu) = 10% - 24%^2/2 = 7.12% per annum;

Standard deviation = 24%/SQRT(10) = 7.59% per annum

14.3 D. False.

While the drift rate (%) is assumed constant, per the risk-neutral valuation, we let drift rate equal the riskless rate. The real-world rate of return is not required, is not an input in the Black-Scholes, and as Hull explains, is not an increasing function of the option (as a higher implied discount rate offsets the higher expected growth rate).

In regard to (A), (B) and (C), EACH is TRUE as a key assumption underlying the Black-Scholes OPM.

14.4C.

It may be optimal to early exercise an American call option immediately before the ex-dividend date

It may be optimal if the dividend is large, but it will only be optimal immediately before the ex-dividend date.

(Note: the ex dividend date is the first date following the declaration of a dividend on which the buyer of a stock is not entitled to receive the next dividend payment)

# Hull, Chapter 18: The Greek Letters

**Learning Outcomes:**

**Describe and assess** the risks associated with naked and covered option positions.

**Explain** how naked and covered option positions generate a stop‐loss trading strategy.

**Describe** delta hedging for an option, forward, and futures contracts.

**Compute** delta for an option.

**Describe** the dynamic aspects of delta hedging.

**Define** the delta of a portfolio.

**Define and describe** theta, gamma, vega, and rho for option positions.

**Explain** how to implement and maintain a gamma‐neutral position.

**Describe** the relationship between delta, theta, and gamma.

**Describe** how hedging activities take place in practice, and discuss how scenario analysis can be used to formulate expected gains and losses with option positions.

**Describe** how portfolio insurance can be created through option instruments and stock index futures.

## Describe and assess the risks associated with naked and covered option positions

If you sell a call option without owning the underlying asset, you hold a naked position (i.e., you have no hedge whatsoever). If you sell a call option while owning the option, you have a covered position. Neither does it provide a good hedge. If you hold a naked position, you lose because the call is exercised. If you hold a covered position, you lose if the stock drops.

## Explain how naked and covered option positions generate a stop‐loss trading strategy

The strategy is to hold a naked position when the option is out-of-the-money and a covered position when the option is in-the-money. Although “superficially attractive,” the strategy becomes too expensive if the stock price crosses the strike price level many times.

Stop-loss Strategy

* Write call option with strike price = K
* When stock prices rises above K (when option is in-the-money)
* Buy one share (“Cover the position”)
* When stock price falls below K (when option is out-of-the-money)
* Sell the share (“Go naked”)

“Superficially attractive” but too expensive if the stock price crosses the strike price level many times (transaction costs).

Stop-loss is also considered an inferior hedging scheme.

## Describe delta hedging for an option, forward, and futures contracts

Delta is the rate of change of the option price with respect to the price of the underlying asset:

|  |  |
| --- | --- |
|  |  |

Delta of European Stock Options



Delta of Forward Contracts

The delta of a forward contract on one share of stock is **1.0.**

Delta of Futures Contract



Compute delta for an option

The left-hand chart below plots delta as a function of stock price (left-hand side) and the right-hand chart plots delta as a function of time to maturity:



In precise terms, the delta changes every time the stock price changes, even if by a small amount. Therefore, in order to maintain a delta-neutral position, rebalancing must be done on a continual basis. That, is holding a genuine delta-neutral position requires dynamic (continual) rebalancing. In reality this is impossible due to the prohibitively large transaction costs this would entail.

**For example**

* Stock = Strike = $10



|  |  |  |
| --- | --- | --- |
| Stock Price | Option Price | N (d1) |
| $8.00 | **$0.46** |  |
| 9.00 | **0.87** |  |
| 10.00 | **1.42** | **0.62** |
| 11.00 | **2.11** |  |
| 12.00 | **2.89** |  |

* Term = 1 year
* Volatility = 30%
* Riskless rate = 5%



**Delta of European call (non-dividend) = N(d1)**

**Delta of European put (non-dividend) = N(d1) - 1**

## Describe the dynamic aspects of delta hedging

Delta hedging aims to maintain an unchanged value of the net position. A position with a delta of zero is called a delta neutral position.

Simulation of delta hedge: option closes in the money

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Strike |  | **$50.00** | |  |  |  |  |  |  |  |
| RF Rate |  | **5%** | |  |  |  |  |  |  |  |
| Time |  | **20.00** | |  |  |  |  |  |  |  |
| Volatility | | **20%** | |  |  |  |  |  |  |  |
| weeks/year | | **52** | |  |  |  |  |  |  |  |
| # of options | | **100,000** | |  |  |  |  |  |  |  |
|  |  |  | |  |  |  |  |  |  |  |
|  | **Stock** | | **Weeks** |  | **Delta** | **Position** | **Shares** |  | **Cum’l.** | **Interest** |
| **Week** | **Price** | | **Left** | **d1** | **N(d1)** | **Delta** | **Purch.** | **Cost** | **Cost** | **Cost** |
| 0 | $49.00 | | 20 | 0.05418 | 0.522 | (52,160) | 52,160 | 2,555,863 | 2,555,863 | 2,458 |
| 1 | $48.12 | | 19 | -0.10545 | 0.458 | (45,801) | (6,359) | (306,019) | 2,252,301 | 2,166 |
| 2 | $47.37 | | 18 | -0.25328 | 0.400 | (40,003) | (5,798) | (274,665) | 1,979,802 | 1,904 |
| 3 | $50.25 | | 17 | 0.24373 | 0.596 | (59,628) | 19,626 | 986,182 | 2,967,888 2,967,888 | 2,854 |
| 4 | $51.75 | | 16 | 0.50424 | 0.693 | (69,295) | 9,667 | 500,268 | 3,471,009 | 3,338 |
| … | … | | … | … | … | … | … | … | … | … |
| 19 | $55.87 | | 1 | 4.05086 | 1.000 | (99,997) | 1,010 | 56,405 | 5,258,007 | 5,056 |
| 20 | $57.25 | | 0 | 488.20885 | 1.000 | (100,000) | 3 | 146 | 5,263,209 | - |

Because delta changes, a position is delta neutral only instantaneously (for a very short period of time). To maintain a delta neutral position, the trader must re-balance the portfolio. Rebalancing is required more frequently if delta changes more rapidly; if gamma is higher (such as the case when the options are at-the-money), rebalancing more frequently is required.

On the other hand, when gamma is lower (such is the case when options are deeply in the money or deeply out of the money), rebalancing is required less frequently.

Simulation of delta hedge: option closes out of the money

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Strike |  | $50.00 |  |  |  |  |  |  |  |
| RF |  | 5% |  |  |  |  |  |  |  |
| Time |  | 20.00 |  |  |  |  |  |  |  |
| Volatility | | 20% |  |  |  |  |  |  |  |
| weeks/year | | 52 |  |  |  |  |  |  |  |
| # of options | | 100,000 |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |
|  | Stock | Weeks |  | Delta | Position | Shares |  | Cum’l. | Interest |
| Week | Price | Left | d1 | N(d1) | Delta | Purch. | Cost | Cost | Cost |
| 0 | $49.00 | 20 | 0.05418 | 0.522 | (52,160) | 52,160 | 2,555,863 | 2,555,863 | 2,458 |
| 1 | $49.75 | 19 | 0.17010 | 0.568 | (56,754) | 4,593 | 228,504 | 2,786,825 | 2,680 |
| 2 | $52.00 | 18 | 0.53923 | 0.705 | (70,514) | 13,760 | 715,531 | 3,505,035 | 3,370 |
| 3 | $50.00 | 17 | 0.20012 | 0.579 | (57,931) | (12,583) | (629,153) | 2,879,253 | 2,769 |
| 4 | $48.38 | 16 | -.10274 | 0.459 | (45,908) | (12,022) | (581,635) | 2,300,386 | 2,212 |
| **…** |  |  |  |  |  |  |  |  |  |
| 19 | $46.63 | 1 | -2.4 | 0.007 | (681) | (17,641) | (822,619) | 288,990 | 278 |
| 20 | $48.12 | 0 | -138.1 | 0.000 | - | (681) | (32,747) | 256,521 | - |

## Define the delta of a portfolio

The delta of a portfolio is simply the summation of the product of each option position and its delta:



Delta is change in option price given (with respect to) change in underlying stock price



These are plots for the delta of a European call option

## Define and describe theta, gamma, vega, and rho for option positions

Theta

Theta is also called time decay. Theta is the rate of change of the value of the portfolio with respect to the passage of time (keeping all other things equal). On the left below, we chart theta as a function of the stock price; on the right, we chart theta as a function of time to maturity:

These are plots for the theta of a European call option

Theta is the one Greek here that is “deterministic:” the change in maturity (the time dimension) is known and can be predicted. Unlike, say, volatility or the asset price dynamics that is stochastic (random).

Gamma

Gamma is the rate of change of the portfolio’s delta with respect to the underlying asset; it is therefore a second partial derivative of the portfolio:

|  |  |
| --- | --- |
|  |  |

On the left below, we chart gamma as a function of stock price. On the right, we chart gamma as a function of time to maturity:

These are plots for the gamma of a European call option

Vega

Vega is the rate of change of the value of a portfolio (of derivatives) with respect to the volatility of the underlying asset:



On the left-hand side chart below, we plot vega as function of stock price. On the right-hand side chart, we plot vega as a function of time to maturity:

|  |  |
| --- | --- |
|  |  |

Rho

Rho is the rate of change of the value of a portfolio (of derivatives) with respect to the interest rate (or, as in the Black–-Scholes, the risk-free interest rate):



On the charts below, we chart rho versus stock price (left-hand) and maturity (right-hand):

|  |  |
| --- | --- |
|  |  |

Explain how to implement and maintain a gamma‐neutral position

Assume that a delta-neutral portfolio has a gamma of (Γ) and a traded option has a gamma of (ΓT). The position in the traded option necessary to make the portfolio and gamma neutral is given by:



**For example:**

Using Hull’s example, assume a portfolio is delta neutral with position gamma of -3,000. Greek (per option) delta is 0.62 and Greek gamma is 1.5.

To neutralize the -3,000 position gamma, we take a long position in 2,000 call options because this adds +3,000 position gamma (+2,000 \* 1.5). However, these options add +1,240 position delta (+2,000 \* 0.62).

To neutralize this 1,240 position delta, we short 1,240 shares (a share has delta = 1.0 and zero Greek gamma!). Now both delta and gamma are neutralized.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  | **Hull p 357** |
| **Position Delta**  **sdfd** | | | **0** |
| **Position Gamma** | | | **-3,000** |
|  | |  |  |
| Per Option | |  |  |
|  | **delta** | | **0.62** |
|  | **gamma** | | **1.5** |
|  |  |  |  |
| **Try to make portfolio gamma** neutral | | |  |
| 1. Add options (+ is long, - is short) | | | |
|  | Number of options | | **2,000** |
|  |  | Gamma | **3,000** |
|  |  | Additional delta | **1,240** |
| 2. Correct the delta | | |  |
|  | Number of shares (- sells) | | **(1,240)** |

## Describe the relationship between delta, theta, and gamma

The risk-free rate multiplied by the portfolio (i.e., a fractional share of the portfolio) is directly related to a linear function of theta, delta and gamma:

|  |  |
| --- | --- |
|  |  |

If theta is large and positive then gamma tends to be large and negative. Delta is zero by definition in a “delta-neutral” portfolio, in which case the formula simplifies to:



**For example:**

|  |  |
| --- | --- |
| Six Inputs |  |
| Stock (S) | **$100.00** |
| Strike (K) | **$80.00** |
| Volatility | **30.0%** |
| Variance | **9.00%** |
| Riskfree rate (r) | **4.00%** |
| Term (T) | **1.00** |
| Dividend Yield | **0.00%** |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **0.5\*σ^2\*** |  |  |  |
| **Term** | **Theta** | **r\*S\*delta** | **S^2\*d2c/dS2** | **Term** |  | **r\*Value** |
| **1** | **(5.887)** | **3.391** | **3.531** | **1.035** |  | **1.0349** |
| **2** | **(4.799)** | **3.292** | **2.754** | **1.247** |  | **1.2467** |
| **3** | **(4.123)** | **3.285** | **2.262** | **1.424** |  | **1.4243** |
| **4** | **(3.651)** | **3.304** | **1.926** | **1.579** |  | **1.5793** |
| **5** | **(3.292)** | **3.332** | **1.678** | **1.718** |  | **1.7178** |
| **6** | **(3.005)** | **3.363** | **1.485** | **1.844** |  | **1.8436** |
| **7** | **(2.765)** | **3.395** | **1.330** | **1.959** |  | **1.9589** |
| **8** | **(2.561)** | **3.426** | **1.201** | **2.065** |  | **2.0653** |
| **9** | **(2.383)** | **3.456** | **1.092** | **2.164** |  | **2.1641** |
| **10** | **(2.226)** | **3.484** | **0.998** | **2.256** |  | **2.2562** |

Theta can (to some extent) be regarded as a proxy for gamma in a delta-neutral portfolio

## Describe how hedging activities take place in practice, and discuss how scenario analysis can be used to formulate expected gains and losses with option positions

Although it may be theoretically desirable, in practice institutions do not constantly rebalance in order to achieve zero-delta, zero-gamma, and zero-vega portfolios. In practice institutions tend to zero out delta at least once a day and then monitor gamma and vega. A trader can enjoy economies of scale: it is probably too expensive to maintain delta neutrality on an individual position, but it is realistic for a large portfolio of options.

In addition to monitoring delta, gamma, and vega, traders also carry out scenario analysis: calculating gains/losses on portfolio over specified period under variety of scenarios

Time period likely depends on liquidity of instruments

Scenarios can be chosen by management or generated by model

Scenarios can be selected by management or model-generated.

## Describe how portfolio insurance can be created through option instruments and stock index futures

The typical approach to creating portfolio insurance is to acquire a put option. For example, a portfolio manager may buy a put on the S&P 500 index. The alternative is to create the option synthetically by shorting a futures contract.

## Chapter Summary

## Questions and Answers

Questions

18.1 In sequence FROM LOWEST to highest value of option delta, what is the correct order of the following four options: in-the-money (ITM) call option, out-of-the-money (OTM) call option, in-the-money (ITM) put option, and out-of-the-money (OTM) put option?

1. OTM put, OTM call, ITM call, ITM put
2. OTM call, ITM call, ITM put, OTM put
3. ITM call, ITM put, OTM put, OTM call
4. ITM put, OTM put, OTM call, ITM call

18.2 A market maker today writes 100 at-the-money (ATM) call option contracts (i.e., short 10,000 options) and immediately starts a dynamic delta hedge by purchasing the underlying non-dividend-paying shares, but due to transaction costs will only re-balance weekly. Next week the underlying share price, volatility and riskfree rate are unchanged. What is the next week's dynamic delta hedge trade?

1. Sell some amount of shares (reduced long position in shares)
2. No transaction (maintain long position in shares)
3. Buy some amount of shares (increase long position in shares)
4. Not enough information (we need the option delta)

18.3 If at-the-money (ATM) options are otherwise identical, which of the following will have the LOWEST value of rho?

1. Put with distant time to expiration
2. Put near to expiration
3. Call near to expiration
4. Call with distant time to expiration

18.4 A delta-neutral option portfolio has a large and positive position theta. Which of the following trades is most likely to neutralize the portfolio's gamma?

1. Buy call options
2. Write put options
3. Sell shares
4. None of the above: theta must be negative!

Answers

**18.1 D. ITM put, OTM put, OTM call, ITM call**

ITM put has lowest value as deep ITM put approaches -1.0

OTM put has next highest value as deep OTM put approaching zero but always negative

OTM call has next highest value as deep OTM call approaching zero but always positive, and

ITM call has highest value as deep ITM call approaching 1.0

**18.2 A. Sell some amount of shares (reduced long position in shares)**

Next week, the percentage delta due to a slightly shorter maturity, ceteris paribus, must be slightly lower. The position delta on the written calls must therefore INCREASE from a negative to a slightly higher negative. For example, if today's percentage delta is 0.57 and, due only to a maturity of one week less, next week's percentage delta is 0.56, then the position delta increases from -5,700 to -5,600. As today's delta hedge is long 5,700 shares, next week 100 shares must be sold to maintain delta neutrality.

**18.3 A. Put with distant time to expiration**

Rho is positive for calls and negative for puts, with interest rate having the most impact when expiration is distant.

Rho (put) = -K\*T\*exp(-rT)\*N(-d2), such that Rho(put) is a decreasing function with increasing (T).

**18.4 A. Buy call options**

Percentage theta is typically negative. A SHORT option position, in either calls or puts, therefore typically has positive position theta. Nevertheless, per Hull 17.4, in a delta-neutral portfolio, a large and positive position theta implies a large and negative position gamma. To neutralize gamma, the trade needs to add positive position gamma; i.e., either buy calls or puts.

In regard to (B), writing puts adds negative position gamma.

In regard to (C), shares have zero gamma (i.e., constant delta = 1.0).

# Tuckman, Chapter 1: Prices, Discount Factors, and Arbitrage

Learning Outcomes:

**Define** discount factor and use a discount function to compute present and future values.

**Define** the “law of one price”, support it using an arbitrage argument, and describe how it can be applied to bond pricing.

**Identify** the components of a U.S. Treasury coupon bond, and compare and contrast the structure to Treasury STRIPS, including the difference between P‐STRIPS and C‐STRIPS.

**Construct** a replicating portfolio using multiple fixed-income securities in order to match the cash flows of a single given fixed income security.

**Identify** arbitrage opportunities for fixed income securities with certain cash flows.

**Differentiate** between “clean” and “dirty” bond pricing and explain the implications of accrued interest with respect to bond pricing.

**Describe** the common day-count conventions used in bond pricing.

## Define discount factor and use a discount function to compute present and future values.

The discount factor, d(t), for a term of (t) years, gives the present value of one unit of currency ($1) to be received at the end of that term.

If d(.5)=.97557, the present value of $1 to be received in six months is 97.557 cents

Assume A pays $105 in six months. Given the same discount factor of 0.97557, $105 to be received in six months is worth .97557 x $105 = $102.43



The discount function is simply the series of discount factors that correspond to a series of times to maturity (t). For example, a discount function is the series of discount factors: d(0.5), d(1.0), d(1.5), d(2.0).

7 7/8 bond due in six months has market price of 101 12 ¾

|  |  |  |  |
| --- | --- | --- | --- |
| Face (par) value | | | $100.00 |
| Years to Maturity | | | 0.5 |
| Bond price (PV) | | | $101.40 |
| Coupon rate | | | 7.875% |
| Discount factor | | | 0.97557 |
| Spot rates | | | 5.008% |
|  | | |  |
| FV Cash flows | | |  |
| Years | | 0.5 | $103.94 |
|  |  | DF | 0.9756 |



|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Par value | | | $100.00 |  |  |  |  |
| Yrs. to Maturity | | | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
| Bond price (PV) | | | $101.40 | $108.98 | $102.16 | $102.57 | $100.84 |
| Coupon rate | | | 7.875% | 14.250% | 6.375% | 6.250% | 5.250% |
| Discount F() | | | 0.97557 | 0.95247 | 0.93045 | 0.90796 | 0.88630 |
| Spot rates | | | 5.008% | 4.929% | 4.864% | 4.886% | 4.887% |
| Forward rates | | |  | 4.851% | 4.734% | 4.953% | 4.888% |
| FV Cash flows | | |  |  |  |  |  |
| Years | | 0.5 | $103.94 | $7.13 | $3.19 | $3.13 | $2.63 |
|  |  | DF | 0.9756 | 0.9756 | 0.9756 | 0.9756 | 0.9756 |
|  |  |  |  |  |  |  |  |
| Years | | 1.0 |  | $107.13 | $3.19 | $3.13 | $2.63 |
|  |  | DF |  | 0.9525 | 0.9525 | 0.9525 | 0.9525 |
|  |  |  |  |  |  |  |  |
| Years | | 1.5 |  |  | $103.19 | $3.13 | $2.63 |
|  |  | DF |  |  | 0.9304 | 0.9304 | 0.9304 |
|  |  |  |  |  |  |  |  |
| Years | | 2.0 |  |  |  | $103.13 | $2.63 |
|  |  | DF |  |  |  | 0.9080 | 0.9080 |
|  |  |  |  |  |  |  |  |
| Years | | 2.5 |  |  |  |  | $102.63 |
|  |  | DF |  |  |  |  | 0.8863 |

## Define the “law of one price”, support it using an arbitrage argument, and describe how it can be applied to bond pricing.

Law of one price: absent confounding factors (e.g., liquidity, special financing rates, taxes, credit risk), two securities with exactly the same cash flows should sell for the same price.

The value of $1 dollar to be received in T (e.g., six months) does not depend on where dollar comes from.

Fixing a set of cash flows to be received on any set of dates, an investor should not care about how those cash flows were assembled from traded securities.

## Identify the components of a U.S. Treasury coupon bond, and compare and contrast the structure to Treasury STRIPS, including the difference between P‐STRIPS and C‐STRIPS.

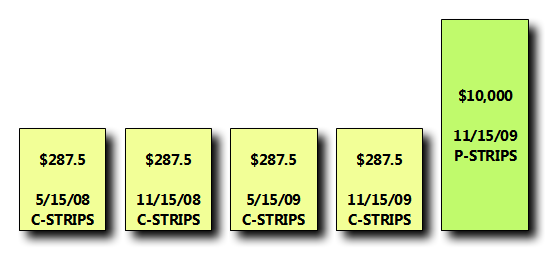
An investor typically pays for a Treasury bond on the business day following his/her purchase.

The seller typically must deliver the bond on the following business day. Delivery or settlement one day after a transaction is known as “T+l settle.” Prices are expressed as a percent of face value and the numbers after the hyphens denote 32nds, often called ticks.

Unlike coupon bonds, which make payments every six months, zero coupon bonds do not pay until maturity. Zero coupon bonds issued by the U.S. Treasury are called STRIPS. STRIPS are created when someone delivers a particular bond to the Treasury and asks for it to be “stripped” into its principal and coupon components:

The coupon or interest STRIPS are called C-STRIPS (a.k.a., TINTs or INTs)

Principal STRIPS are called P-STRIPS (a.k.a., TP or P)

****

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Par | $10,000 |  |  |  |  |
| coupon | 5.75% |  |  |  |  |
| yield | 5.75% |  |  |  |  |
|  |  |  |  |  |  |
|  | Cash Flow |  |  | Price | Price |
| 6 mos | coupon | Face |  | C-STRIP | P-STRIP |
| 0.5 | $287.5 |  |  | $279.5 |  |
| 1.0 | $287.5 |  |  | $271.7 |  |
| 1.5 | $287.5 |  |  | $264.1 |  |
| 2.0 | $287.5 |  |  | $256.7 |  |
| 2.5 | $287.5 | $10,000.0 |  | $249.5 | $8,678.6 |
|  |  |  |  |  |  |
|  |  |  |  |  | $10,000.0 |

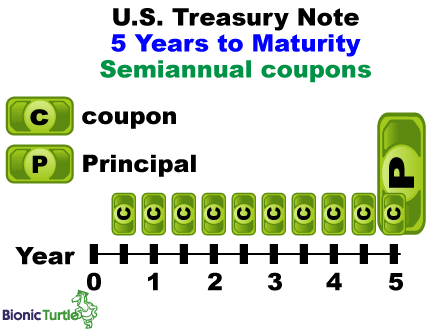
The Treasury creates and retires STRIPS. For example, an investor can deliver the set of STRIPS and ask the Treasury to reconstitute the face amount. C-STRIPS are fungible (i.e., when reconstituting a bond, any C-STRIPS maturing on a particular coupon payment date may be used as that bond’s coupon payment) but P-STRIPS are not: P-STRIPS are identified with particular bonds so they inherit the cheapness/richness of the bonds from which they are derived.

Investors like zero coupon bonds (i.e., STRIPS) for at least two reasons.

They can be combined or re-constructed into any required sequence of cash flows;

They are more sensitive to interest rates (interest-rate sensitive) than coupon-bearing bonds (all other things being equal).

|  |  |
| --- | --- |
| Advantages of STRIPS (why investors like them) | Disadvantages of STRIPS |
| They can be combined or re-constructed into any required sequence of cash flows | Can be illiquid |
| More interest-rate sensitive) than coupon bearing bonds (all other things being equal). | Short-term (long-term) C-STRIPS often trade rich (cheap) |



|  |  |
| --- | --- |
| p-strips.gif | c-strips.gif |

## Construct a replicating portfolio using multiple fixed-income securities in order to match the cash flows of a single given fixed income security.

The law of one price can be used to infer cash flows.

**For example (GARP Sample Question 18, 2010):**

Question: The following table gives the prices of two out of three US Treasury notes for settlement on August 30, 2008. All three notes will mature exactly one year later on August 30, 2009. Assume annual coupon payments and that all three bonds have the same coupon payment date.

Coupon Price

2 7/8 98.40

4 1/2 ?

6 1/4 101.30

What is (should be) the price of the 4 1/2 US Treasury note?

Answer: $99.80

2.875% \* x + 6.25% \*(1 - x) = 4.5% X = 52% . The portfolio that has cash flows identical to the 4 1/2 bond consists of 52% of the 2 7/8 and 48% of the 6 1/4 bonds. As this portfolio has cash flows identical to the 4 1/2 bond, precluding arbitrage, the price of the portfolio should equal to 52% \* 98.4 + 48% \* 101.30 or 99.80

## Identify arbitrage opportunities for fixed income securities with certain cash flows.

The law of one price: absent confounding factors (e.g., liquidity, special financing rates, taxes, credit risk), two securities (or portfolios of securities) with exactly the same cash flows should sell for the same price.

A violation of the “law of one price” implies the existence of an arbitrage opportunity.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Face (par) value | |  | $100.00 |  |  |  |  |  |
| Yrs. to Maturity | |  | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 2.5 |
|  |  |  | (A) |  |  | (B) | (C) | (D) |
| Bond price (PV) | |  | $104.080 |  |  | $110.938 | $102.020 | $114.375 |
| Coupon rate | |  | 13.375% |  |  | 10.750% | 5.750% | 11.125% |
| Discount function | | | 0.97557 | 0.95247 | 0.93045 | 0.90796 | 0.88630 | 0.88630 |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  | Predicted Price | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 2.5 |
|  |  |  | $106.688 |  |  |  |  |  |
| Bond (A) | cheap | $104.081 | $104.081 |  |  |  |  |  |
|  |  |  | $5.375 | $5.375 | $5.375 | $105.375 |  |  |
| Bond (B) | cheap | $111.041 | $5.244 | $5.120 | $5.001 | $95.677 |  |  |
|  |  |  | $2.875 | $2.875 | $2.875 | $2.875 | $102.875 |  |
| Bond (C) | rich | $102.007 | $2.805 | $2.738 | $2.675 | $2.610 | $91.178 |  |
|  |  |  | $5.563 | $5.563 | $5.563 | $5.563 |  | $105.563 |
| Bond (D) | cheap | $114.511 | $5.427 | $5.298 | $5.176 | $5.051 |  | $93.560 |

## Differentiate between “clean” and “dirty” bond pricing and explain the implications of accrued interest with respect to bond pricing.

The dirty price, also known as the full price, represents the present value of a bond, which includes accrued interest. The clean price, also known as the quoted price, represents the present value of the bond, less accrued interest.

If two traders agree to a bond transaction, the buyer gets to keep the full coupon payment at the next coupon date, even though the seller may have held the bond for most of the accrual period. In order for the seller to agree to the buyer receiving the entire coupon payment, she must be compensated for the accrued interest foregone on the bond, which she would otherwise receive.

The reason traders negotiate based on the quoted, or clean price, rather than the dirty, or full price, is exactly due to the accrued interest component. The dirty price will experience large swings in the price as interest is accrued, and then a sudden drop as a coupon payment is made. The quoted price on the other hand will be much more stable over time, gradually tending to its par value at maturity.NEED CONTENT

## Describe the common day-count conventions used in bond pricing.

NEED CONTENT BUT SIMILAR TO HULL AND FABOZZI

As we already saw in Hull’s reading in part 3, there are a number of day-count conventions commonly used.

Three important conventions are:

1. Actual/Actual – used for Government bonds
2. Actual/360 – commonly used in money markets. Tuckman mentions this is also a common convention for the floating leg of an interest rate swap, however, that is not an actual convention, just a general tendency.
3. 30/360 – Typically used for corporate bonds and the fixed leg of an interest rate swap. Rather than using the actual number of days, this convention ‘defines’ every month to be 30 days.

It is important that you remember the key day-count conventions, as questions regarding many financial instruments rely on you applying the correct day-count. This will typically appear as part of a calculation on the FRM exam.

It is important to recognize that the day-count convention does not matter. As previously stated, the value of a bond is equal to its quoted price and accrued interest. That is PV = p +AI. If the day-count convention is favorable or unfavorable to either party of a transaction, this difference is made up in the quoted price such that the sum always equals the present value of the cash flows.

## Questions and Answers

Questions

20.1.1 Assume that a U.S. Treasury bill will pay $1,000 in one year and the security is default free (there is absolutely no credit risk). The price of this bill today is given by P(0). Which of the following statements, according to Tuckman, is most true about individual versus market expressions of the theory of the time value of money?

1. Rational, well-informed individuals are each willing to pay a DIFFERENT price, P(0); and therefore the market should exhibit various (DIFFERENT) fair prices for the security
2. Rational, well-informed individuals should arrive at the SAME willingness-to-pay price, P(0); and therefore the market should reflect this (SAME) price upon which all participants agree
3. Rational, well-informed individuals are each willing to pay a DIFFERENT price, P(0); but the market should reflect only one (SAME) fair price
4. Rational, well-informed individuals should arrive at the SAME willingness-to-pay price, P(0); but the market should reflect various (DIFFERENT) prices

20.1.2 A $10 million Treasury bond (note) with a 10-year maturity pays semi-annual coupons at a coupon rate of 4.0% per annum. If the bond is fully "stripped" such that STRIPS are created, each of the following is TRUE except:

1. The stripping creates 21 zero-coupon bonds
2. Each of the C-STRIPS and the P-STRIP implies an exact spot (a.k.a., zero) interest rate
3. The duration of the P-STRIP is greater than the duration of the original Treasury bond
4. The C-STRIPS each have durations near to zero

20.1.3 Which of the following would be the most likely reason for a C- or P-STRIP to "trade rich" or "trade cheap?"

1. Arbitrageurs
2. Technical (non-fundamental) factors; e.g., liquidity, supply/demand
3. A shift in the spot rates which changes discount rate(s) abruptly
4. Individual investors have different views (preferences) with respect to the time value of money

Answers

20.1.1 C.

Rational, well-informed individuals are each willing to pay a DIFFERENT price, P(0); but the market should reflect only one (SAME) fair price

Please keep in mind this very narrowly refers to a risk-free asset: this is the condition that allows us to say this is only about the "time value of money."

The "law of one price" is a theoretical function of the market, not different participants or individuals: its violation creates arbitrage opportunities.

Please also keep in mind this is a theory about fundamentals; it does not preclude technical factors from interfering.

Individuals are expected to express DIFFERENT preferences toward the time value of money. Tuckman's point is that the market price reflects a SINGLE CONSENSUS:

"While the three people in the examples are willing to pay different amounts for $1,000 next year, there exists only one market price for this $1,000. If that price turns out to be $950 then the first person will pay $950 today to fund a $1,000 party in a year. The second person would be indifferent between buying the $950 stereo system today and putting away $950 to purchase the $1,000 stereo system next year. Finally, the third person would refuse to pay $950 for $1,000 in a year because the business can transform $940 today into $1,000 over the year. In fact, it is the collection of these individual decisions that determines the market price for $1,000 next year in the first place."

20.1.2 D.

The C-STRIPS are not floating-rate notes, they are zero-coupon bonds corresponding to the respective coupons.

So, in this case, the twenty C-STRIPS have Macaulay durations of: 0.5, 1.0, 1.5, ..., 20.

In regard to (A), (B), and (C), each is TRUE about the STRIPS.

In regard to (A), the bonds twenty coupons (10 years \* 2 coupons/year) create 20 C-STRIPS plus the principal repayment creates a single P-STRIP.

20.1.3 B. Technical (non-fundamental) factors; e.g., liquidity, supply/demand

The essence of Tuckman's explanation for the difference between market and model (predicted) prices is technical factors; i.e., factors not included in the fundamental pricing model.

In regard to (A), arbitrage tends to reduce the trading premium/discount.

In regard to (D), this is a valid but inferior answer because Tuckman already incorporates varying individual views and cites the discount rates as a consensus view; i.e., that individual preferences vary is already "built-into" the theory of the law of one price. (of course, different preferences inform supply/demand, so this is indirectly correct).

# Tuckman, Chapter 2: Spot, Forward and Par Rates

**Learning Outcomes:**

**Calculate** **and describe** the impact of different compounding frequencies on a bond’s value.

**Calculate** discount factors given interest rate swap rates.

**Compute** spot rates given discount factors.

**Define and interpret** the forward rate, and compute forward rates given spot rates.

**Define** par rate and describe the equation for the par rate of a bond.

**Interpret** the relationship between spot, forward and par rates.

**Assess** the impact of maturity on the price of a bond and the returns generated by bonds.

**Define** the “flattening” and “steepening” of rate curves and construct a hypothetical trade to reflect expectations that a curve will flatten or steepen.

## Calculate and describe the impact of different compounding frequencies on a bond’s value.

Investing (x) at an annual rate of (r) compounded semiannually for (T) years produces a terminal wealth (w) of:



Discount factor

Let d(t) equal the discounted value of one unit of currency. Assuming the one unit of currency is discounted for (t) years at the semiannual compound rate r(t), then the discount rate d(t) is given by:



The relationship between continuous compounding and discrete compounding (semi-annual compounding is discrete compounding where the number of periods per year is equal to 2) is given by:



|  |  |
| --- | --- |
| The **continuous rate of return** as function of the discrete rate of return (where m is the number of periods per year) is given by: | The **discrete rate of return** as a function of the continuous rate of return is given by: |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| EAY, r | |  |  | 10.0000% |  |
| Initial wealth (A) | | |  | $100.00 |  |
| Number of years (n) | | |  | 1.00 |  |
|  |  | |  |  |  |
|  |  | |  | Equivalent |  |
| Compound | Terminal | |  | Periodic | Continuous |
| Frequency | Value | |  | Return | Return |
| 1 | $110.000 | |  | 10.0000% | 9.531% |
| 2 | $110.250 | |  | 9.7618% | 9.531% |
| 4 | $110.381 | |  | 9.6455% | 9.531% |
| 12 | $110.471 | |  | 9.5690% | 9.531% |
| 360 | $110.516 | |  | 9.5323% | 9.531% |
| Cont. | $110.517 | |  | 9.5310% | 9.531% |

Compute semi‐annual compounded rate of return for a C‐Strip

If the price of one unit of currency maturing in t years is given by d(t), the semiannual compounded return, is given by:



The relationship between spot rates and maturity/term is called the term structure of spot rates. When spot rates increase with maturity, the term structure is said to be upward sloping. When spot rates decrease with maturity, the term structure is said to be downward sloping or inverted.

If a 10-year C-STRIP is quoted at 58.779, then the semi-annual compounded rate of return is given by:



## Calculate discount factors given interest rate swap rates.

NEEDS CONTENT BUT SIMILAR TO HULL

## Compute spot rates given discount factors.

Given a t-period discount factor d(t), the semiannual compounded return is given by:



The relationship between spot rates and maturity/term is called the term structure of spot rates. When spot rates increase with maturity, the term structure is said to be upward sloping. When spot rates decrease with maturity, the term structure is said to be downward sloping or inverted.

Compute semi‐annual compounded rate of return for a C‐Strip

10-year C-STRIP quoted at 58.779

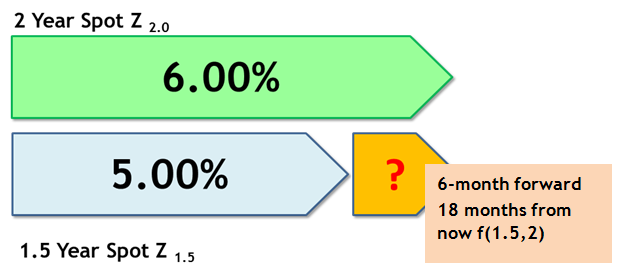


## Define and interpret the forward rate, and compute forward rates given spot rates.

A spot rate is given by S(T). For example, we can indicate a one-year spot rate with S(1). (Note: it is also common to use ST). The forward rate can be given by f(t,T) where little “t” indicates when the forward contract is created and big “T” indicates when the forward expires. For example, f(0.5, 1.0) is a forward created in six months that expires in one year.

There is a natural relationship between forward rates and a series of spot rates. The one-year spot rate must equal the six-month spot rate multiplied by the six-month forward rate:





For example, assume the two-year spot-rate is 6% and the eighteen-month spot-rate is 5%. What is the six-month forward rate, f(1.5,2.0)? We can solve for the by re-arranging:



That produces a semi-annual forward rate. Don’t forget to multiply by two. Therefore, the implied forward rate f(1.5,2.0) is about 9%.

The equality is based on a no-arbitrage principle. Imagine you have a choice. Consider the case of a one-year investment horizon. You can invest in one-year CD at the one-year spot rate, S(1). You ought to be indifferent between this and, alternatively, investing in a six-month CD (at the six-month spot rate) that automatically “rolls-over” into another six-month CD. Today, the best you know about the future CD rate is the six-month forward rate, f(0.5,1.0). You may say, “but it’s better to have access to the investment in six months (liquidity).” And you’d be right, except that a normal term structure would incorporate this by baking a bit of premium into the longer term.

No-arbitrage “indifference:” investing at 2.0 year spot should have same expected return as investing @ 1.5 year spot and “roll over into” 0.5 year forward

Calculate the price of a bond using discount factors, spot rates, or forward rates

Assume a 1-year treasury bond that pays a 6% semi-annual coupon.

|  |  |  |  |
| --- | --- | --- | --- |
| Maturity | Spot rate (%) | Discount Factor | 6 month Forward  Rate (%) |
| 0.50 | 1.50 | 0.992556 | 1.50 |
| 1.00 | 2.00 | 0.980296 | 2.50 |
| 1.50 | 2.25 | 0.966995 | 2.75 |
| 2.00 | 2.50 | 0.951524 | 3.25 |
| 2.50 | 2.75 | 0.933997 | 3.75 |

Calculate the bond price using discount factors:

Price = ($3 × 0.992556) + ($103 × 0.980296) = $103.95

The discount factor is simply $1 “discounted” to its present value. For example, if you are discounting a face value bond of $100, the present value (PV) is $100/[(1+r)T], which is $100  1/[(1+r)T]. The discount factor is just the second term, without the face value: 1/[(1+r)T].

Calculate the bond price using spot rates:

This is just a series of present value calculations:



Don’t forget that the final cash flow probably includes the principal. In this example, the final cash flow includes a coupon ($3) plus the principal repayment ($100).

Calculate the bond price using forward rates:

We still start with the cash flows. But instead of spot rates, we discount will forward rates. The key here is to keep your “raise to powers” consistent.



If you would like a better understanding, on the member page you can access a simple spreadsheet that compares these rates side-by-side (row by row) for a common bond. The worksheet is copied below. Given spot rates as an input, you can study the calculation of the discount factors and the forward. Note this bond has a yield (yield to maturity) of 2.72%; the yield is the a single rate that that discounts all cash flows to the price so it is essentially a flat line while the spot rate is a curve.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| $100 Par, 6% Coupon | | |  |  | YTM: | 2.72% |
|  |  |  |  |  |  |  |
| Years to Maturity | | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
| Cash flows | | $3.0 | $3.0 | $3.0 | $3.0 | $103.0 |
| Spot rates | | 1.50% | 2.00% | 2.25% | 2.50% | 2.75% |
| Discount function | | 0.993 | 0.980 | 0.967 | 0.952 | 0.934 |
| 6 mo. forward | |  | 2.50% | 2.75% | 3.25% | 3.75% |
|  |  |  |  |  |  |  |
| Discounted (spot) | | $2.98 | $2.94 | $2.90 | $2.85 | $96.20 |
| Disc. (function) | | $2.98 | $2.94 | $2.90 | $2.85 | $96.20 |
| Disc. (forward) | | $2.98 | $2.94 | $2.90 | $2.85 | $96.20 |
| Bond Price (in all cases) = | | $107.88 |  |  |  |  |

## Define par rate and describe the equation for the par rate of a bond.

NEED CONTENT, but similar to HULL

## Interpret the relationship between spot, forward and par rates.

NEED CONTENT BUT SIMILAR TO HULL

## Assess the impact of maturity on the price of a bond and the returns generated by bonds.

Impact of maturity on bond price

More generally, price increases with maturity whenever the coupon rate exceeds the forward rate over the period of maturity extension. Price decreases as maturity increases whenever the coupon rate is less than the relevant forward rate.

Exhibit from Tuckman comparing price to maturity:

|  |  |  |
| --- | --- | --- |
| Maturity | Price | Forward |
| 0.5 | $99.935 | 5.008% |
| 1.0 | $99.947 | 4.851% |
| 1.5 | $100.012 | 4.734% |
| 2.0 | $99.977 | 4.953% |
| 2.5 | $99.971 | 4.888% |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Years to Maturity | | | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |  |
| Bond price (PV) | | | $101.40 | $108.98 | $102.16 | $102.57 | $100.84 |  |
| Coupon rate | | | 7.875% | 14.250% | 6.375% | 6.250% | 5.250% |  |
| Discount function | | | 0.97557 | 0.95247 | 0.93045 | 0.90796 | 0.88630 |  |
|  |  |  | |  |  |  |  |  |
|  |  | Face (par) value | | $100.00 |  |  |  |  |
|  |  | Coupon |  | 4.875% |  |  |  |  |
|  |  | Cash Flows | Time: 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |  |
|  |  | Bond #1 | $102.44 |  |  |  |  |  |
|  |  | Bond #2 | $2.44 | $102.44 |  |  |  |  |
|  |  | Bond #3 | $2.44 | $2.44 | $102.44 |  |  |  |
|  |  | Bond #4 | $2.44 | $2.44 | $2.44 | $102.44 |  |  |
|  |  | Bond #5 | $2.44 | $2.44 | $2.44 | $2.44 | $102.44 |  |
|  |  |  |  |  |  |  |  |  |
|  |  | Present Values | |  |  |  |  | Price |
|  |  | Bond #1 | $99.935 |  |  |  |  | $99.935 |
|  |  | Bond #2 | $2.378 | $97.569 |  |  |  | $99.947 |
|  |  | Bond #3 | $2.378 | $2.322 | $95.313 |  |  | $100.012 |
|  |  | Bond #4 | $2.378 | $2.322 | $2.268 | $93.009 |  | $99.977 |
|  |  | Bond #5 | $2.378 | $2.322 | $2.268 | $2.213 | $90.791 | $99.971 |

Impact of maturity on returns

See below (replicated Tuckman). We start with the comparison of two scenarios. The first scenario invests $10,000 and “rolls over” six-month investments; here we assume the six month rates realized equal the initial forward rates. Under the second scenario, a 2.5 year bond is purchased and the coupons are reinvested. They produce the same terminal value only under the unlikely scenario that realized short-term rates match the forward rates built into the bond price:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| "Maturity and Bond Return" | | | |  |  |  |  |
| Forward Rate: | | | 5.008% | 4.851% | 4.734% | 4.953% | 4.888% |
|  |  |  |  |  |  |  |  |
| Invest: | | $10,000 | 10,250 | 10,499 | 10,748 | 11,014 | $11,283 |
|  |  |  |  |  |  |  |  |
| Invest: | | $10,000 | Price | $100.84 |  | Coupon | 5.25% |
|  |  | $9,916 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
|  |  |  | $260.30 | $266.62 | $272.93 | $279.69 | $286.52 |
|  |  |  |  | $260.30 | $266.46 | $273.06 | $279.74 |
|  |  |  |  |  | $260.30 | $266.75 | $273.27 |
|  |  |  |  |  |  | $260.30 | $266.67 |
|  |  |  |  |  |  |  | $260.30 |
|  |  |  |  |  |  |  | $9,916.33 |
|  |  |  |  |  |  |  | $11,283 |

Investors who roll over short-term investments do better than investors in longer-term bonds when the realized short-term rates exceed the forward rates built into bond prices. Investors in bonds do better when the realized short-term rates fall below these forward rates.

## Define the “flattening” and “steepening” of rate curves and construct a hypothetical trade to reflect expectations that a curve will flatten or steepen.

NEED CONTENT. AGAIN, similar to HULL… expectation here similar to in the case of hedging scenarios.

## Questions and Answers

Questions

20.2.1 If the spot rate term structure is flat, what is true of the discount function (i.e., the set of discount factors) as function of maturity?

1. Flat
2. Increasing with maturity
3. Decreasing with maturity
4. Insufficient information: we need the yield (YTM) to answer

20.2.2. The following discount function contains semi-annual discount factors out to two years: d(0.5) = 0.9970, d(1.0) = 0.9911, d(1.5) = 0.9809, d(2.0) = 0.9706. What is the implied eighteen-month (1.5 year) spot rate (aka, 1.5 year zero rate)?

1. 0.600%
2. 1.176%
3. 1.290%
4. 1.505%

20.2.3 The price of a six-month zero-coupon bond (bill) is $99.90 and the price of a one-year zero-coupon bond is $98.56. What is the implied six-month forward rate, under semi-annual compounding?

1. 1.30%
2. 2.95%
3. 2.73%
4. 3.08%

20.2.4 The six-month and one-year discount factors are, respectively, d(0.5) = 0.9920 and d(1.0) = 0.9760. What is the implied six-month forward rate, under semi-annual compounding?

1. 2.34%
2. 3.28%
3. 3.95%
4. 4.01%

Answers

20.2.1 C.

Decreasing with maturity

Greater maturity requires more discounting. For example if the spot rate term structure is flat at 5%, then semi-annual discount function is: d(0.5) = 0.9756, d(1.0) = 0.9518, d(1.5) = 0.9286 ...

In regard to (D), please note that a flat spot/zero term structure is the special case where the yield must match; e.g., flat spot rates at 5% imply yield must also be 5%.

20.2.2 C.

1.290%

As r(t) = 2\*[(1/d(t))^(1/2t) - 1], r(1.5) = 2\*[(1/0.9809)^(1/3) - 1] = 1.2898%

20.2.3 C. 2.73%

(99.90/98.56-1)\*2 = 2.728%

20.2.4 B. 3.28%

(0.9920/0.9760-1)\*2 = 3.279%

# Tuckman, Chapter 3: Returns, Spreads and Yields

Learning Outcomes:

Distinguish between gross and net realized returns, and calculate the realized return for a bond over a holding period including reinvestments.

Define and interpret the spread of a bond, and explain how a spread is derived from a bond price and a term structure of rates.

Define, interpret, and apply a bond’s yield‐to‐maturity (YTM) to bond pricing.

Compute a bond's YTM given a bond structure and price.

Calculate the price of an annuity and a perpetuity.

Explain the relationship between spot rates and YTM.

Define the coupon effect and explain the relationship between coupon rate, YTM, and bond prices.

Explain the decomposition of P&L for a bond into separate factors including carry roll-down, rate change and spread change effects.

Identify the most common assumptions in carry roll-down scenarios, including realized forwards, unchanged term structure, and unchanged yields.

## Distinguish between gross and net realized returns, and calculate the realized return for a bond over a holding period including reinvestments.

NEED CONTENT

## Define and interpret the spread of a bond, and explain how a spread is derived from a bond price and a term structure of rates.

NEED CONTENT

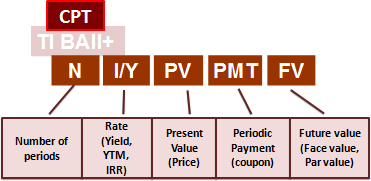
## Define, interpret, and apply a bond’s yield‐to‐maturity (YTM) to bond pricing.

Yield-to-maturity (YTM), sometimes just yield, is the single rate that, when used to discount a bond’s cash flows, produces the bond’s market price. Given an annual coupon of c (and therefore a semi-annual coupon of c/2), a final principal payment of F, a market price of P(T) with T years to maturity, the yield to maturity (YTM) is given by (y) is the following equation:



Note that there are 2T terms being added together through the summation sign since a T-year bond makes 2T semiannual coupon payments. This sum equals the present value of all the coupon payments, while the final term equals the present value of the principal payment.

## Compute a bond's YTM given a bond structure and price.

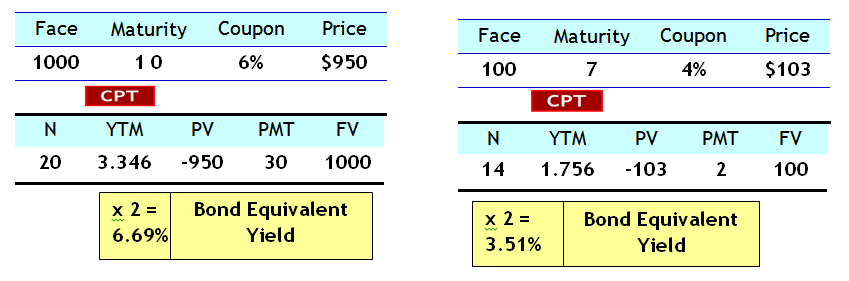


When using the Texas Instruments BA II Plus calculator to price a plain vanilla bond-related parameter, the thing to remember is that the bond is described with five parameters: N=number of periods; I/Y = interest rate; PV = present value; PMT=payment and FV = future (or face) value.

For most problems, you identify the four inputs four and solve (CPT) for the fifth. Typically, the problem entails solving for the rate (I/Y) or the bond price (PV).

The five bond parameters are the following:

|  |  |
| --- | --- |
| Button | Description |
| N | Number of payments |
| I/Y | Interest rate (or YTM) |
| PV | Present Value |
| PMT | Payment |
| FV | Future value |
| CPT | Compute (Solve the calculation) |



**Example #1: Calculate YTM**

For example, assume we want to compute the YTM of the following bond:

10 Year bond with a face value of $1,000—just issued

4% coupon pays semiannually

Current price is $982

To solve, we enter the following as inputs into the calculator:

N = 20 (10 years x 2 semiannual periods per year)

PV = -$982

PMT = $20 (4% x $1,000 x 0.5 for semiannual period)

FV = $1,000 (face value)

Then we compute (CPT) the interest rate: CPT I/Y, which equals 2.11%.

CPT I/Y = 2.11% and (2.11 \* 2) = 4.22% YTM. The yield to maturity is therefore 4.22%

**Example #2: Calculate YTM**

Now let’s move forward in time 3 years, but we will assume that nothing else has changed except for two things: the price is now $1,070; and the bond has only seven years left to maturity:

N = 14 (7 years x 2 semiannual periods per year)

PV = -$1,070

PMT = $20 (4% x $1,000 x 0.5 for semiannual period)

FV = $1,000 (face value)

To solve, CPT I/Y = 1.44% and (1.44 \* 2) = 2.88% YTM. In this case, the YTM is 2.88%.

Remember you need to use both a positive and (+) and (-) as inputs. The easiest thing to do is make the PV negative because you would spend money to buy the bond (i.e., cash outflow = minus). In this case, you receive payments and the principal back (positive signs). To summarize: PV = (-), PMT = (+), FV = (+).

## Calculate the price of an annuity and a perpetuity.

Annuity: makes semiannual payments of c/2 ever six months for T years but never makes a final payment. Price is given by:



An annuity with semiannual payments is a security that makes a payment c/2 every six months for T years but never makes a final “principal” payment (i.e., FV=0). The price of an annuity, A(T), is given by:



A perpetuity bond is a bond that pays coupons forever. The price of a perpetuity is simply the coupon divided by the yield (i.e., the price of a perpetuity = c/y).

The valuation of a perpetuity bond is a classic idea in finance: we “capitalize” a constant income stream by dividing by the discount rate (a.k.a., cap rate).

## Explain the relationship between spot rates and YTM.

There are several spot rates and a single yield (YMT). The YTM is a summary of all the spot rates that enter into the bond pricing equation. Consider three patterns:

A flat term structure of spot rates (i.e., all of the spot rates are equal): the yield must equal the one-year spot rate level as well.

A term structure where spot rates are upward sloping over a two-year period: the two-year bond yield is below the two-year spot rate.

A term structure where spot rates are downward sloping over a two-year period: the two-year bond yield is above the two-year spot rate.

If a bond’s YTM over a six-month period remains unchanged, then the annual total return of the bond over that period equals its YTM.

YTM is a single factor and implies assumption of (unrealistic) flat term structure

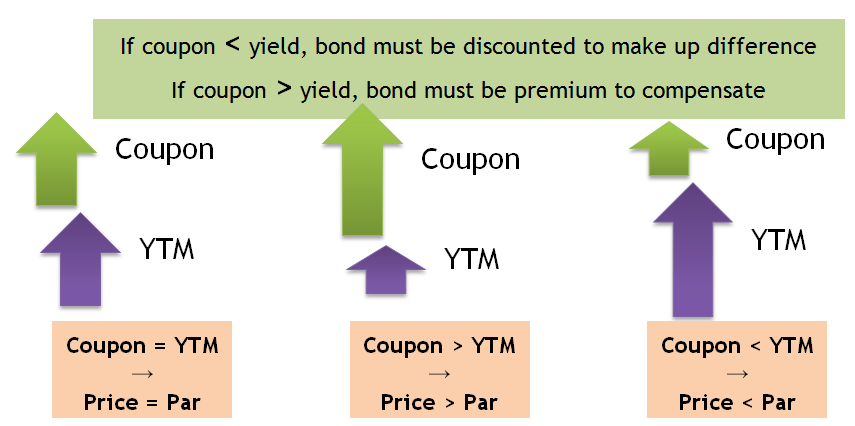
Calculate the price of an annuity and a perpetuity using a calculator with time value functions

## Define the coupon effect and explain the relationship between coupon rate, YTM, and bond prices.

When the coupon rate equals the YTM (c = 100y), bond price equals face value, or par.

If the coupon rate exceeds the yield (c>100y), then the bond sells at a premium to par, that is, for more than face value.

If the coupon rate is less than the yield (c<100y) then the bond sells at a discount to par, that is, for less than face value.



## Explain the decomposition of P&L for a bond into separate factors including carry roll-down, rate change and spread change effects.

NEED CONTENT

## Identify the most common assumptions in carry roll-down scenarios, including realized forwards, unchanged term structure, and unchanged yields.

NEED CONTENT

## Questions and Answers

Questions

Answers

# Tuckman, Chapter 4: One‐Factor Risk Metrics and Hedges

**Learning Outcomes:**

**Describe** an interest rate factor and name common examples of interest rate factors.

**Define and compute** the DV01 of a fixed income security given a change in yield and the resulting change in price.

**Calculate** the face amount of bonds required to hedge an option position given the DV01 of each.

**Define, compute, and interpret** the effective duration of a fixed income security given a change in yield and the resulting change in price.

**Compare and contrast** DV01 and effective duration as measures of price sensitivity.

**Define, compute, and interpret** the convexity of a fixed income security given a change in yield and the resulting change in price.

**Explain** the process of calculating the effective duration and convexity of a portfolio of fixed income security.

**Explain** the impact of negative convexity on the hedging of fixed income securities.

**Construct** a barbell portfolio to match the cost and duration of a given bullet investment, and explain the advantages and disadvantages of bullet versus barbell portfolios.

## Describe an interest rate factor and name common examples of interest rate factors.

An interest rate factor is a random variable that impacts interest in some way. The simplest formulations assume that there is only one factor driving all interest rates and that the factor is itself an interest rate. The interest rate factor, is typically:

Spot rate,

Forward rate, or

Yield (yield-to-maturity)

One-factor measures of sensitivity

DV01 = dollar value of an ’01

a.k.a., PV01, price value of an ’01

Gives the dollar value change of a fixed income security for a one-basis point decline in rates.

Modified duration

Percentage change in value of security for a one unit change (10,000 basis points)

A key relationship is between DV01, price (P) and modified duration (D\_mod):

**KEY FORMULA**



## Define and compute the DV01 of a fixed income security given a change in yield and the resulting change in price.

DV01 is an acronym for “dollar value of a 01” (.01%). DV01 gives the change in the value of a fixed income security for a one-basis point decline:



Importantly, the DV01 is related to modified duration:



Consider a zero-coupon bond with 30 years to maturity. Given a yield of 4%, the price is $30.12. Then re-price the bond with a yield of 3.99%; i.e., 4% minus one basis point (1 bps). The price difference is about $0.09. This is the DV01.

|  |  |
| --- | --- |
| Face | $100 |
| Maturity | 30.0 |
|  |  |
|  |  |
| Yield | 4.0% |
| Duration | -28.85 |
| Actual | -30.12 |
| Slope | (868.83) |
|  |  |
| Yield | Price |
| 4.00% | $30.12 |
| 3.99% | $30.21 |
| DV01 | $0.090 |

Further example:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Dollar value of an 01 (DV01; aka, price value of a basis point) | | | | |
|  | Par | | $100.00 | $100.00 |
|  | Coupon | | 5.00% | 5.00% |
|  | Maturity (yrs) | | 5 | 30 |
|  | Initial Yield | | 5.00% | 5.00% |
|  | Initial price | | $100.0000 | $100.0000 |
|  |  | |  |  |
|  | Shock up + 1 bps | |  |  |
|  |  | Yield | 4.99% | 4.99% |
|  |  | Price | $100.0438 | $100.1547 |
|  |  | DV01 | $0.0438 | $0.1547 |
|  |  |  |  |  |
|  | Shock down - 1 bps | | |  |
|  |  | Yield | 5.01% | 5.01% |
|  |  | Price | $99.9563 | $99.8456 |
|  |  | DV01 | $0.0437 | $0.1544 |

## Calculate the face amount of bonds required to hedge an option position given the DV01 of each.

|  |  |
| --- | --- |
| Zero-coupon bond: | |
| Face | $100 |
| Maturity | 42.4 |
|  |  |
| Yield | 5.0% |
| Duration | -40.38 |
| Actual | -12.00 |
| Slope | (484.70) |
|  |  |
| Yield | Price |
| 4.00% | $18.34 |
| 3.99% | $18.42 |
| DV01 | $0.0779 |

If DV01 is expressed in terms of a fixed face amount, hedging a position of FA face amount of security A requires a position of FB face amount of security where:



|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Call Option | |  |  | Option |  | Bond |
| Stock (S) | $100.00 |  |  | Face x DV01 | = | Face x DV01 |
| Strike (K) | $100.00 |  |  | $100,000,000 | = | $47,348,140 |
| Volatility | 40% |  |  | x $0.0369 |  | x $0.0779 |
| Term (T) | 5.0 |  |  |  |  |  |
|  |  |  |  | Write |  | Buy |
| Rate | Price |  |  | options |  | Bonds |
| 5.00% | $3.050 |  | Face | $100,000,000 |  | $47,348,140 |
| 5.01% | $3.087 |  | - 1 bps | ($0.037) |  | $0.078 |
| DV01 | $0.0369 |  |  | ($36,900) |  | $36,900 |

The hedge is based on the following equality (which simply serves to calibrate the face value of both instruments such that a one basis point change produces and approximately equivalent dollar value change):



Here is another example:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Call Option | |  | Option |  | Bond |
| Stock (S) | $100.00 |  | Face x DV01 | = | Face x DV01 |
| Strike (K) | $100.00 |  | $1,000,000 | = | $186,166 |
| Volatility | 40% |  | x $0.017 |  | x $0.090 |
| Term (T) | 5.0 |  |  |  |  |
|  |  |  | Write |  | Buy |
| Rate | Price |  | options |  | Bonds |
| 4.00% | $41.190 | Face | $1,000,000 |  | $186,166 |
| 3.99% | $41.207 | - 1 bps | ($0.017) |  | $0.090 |
| DV01 | $0.017 |  | ($168) |  | $168 |

## Define, compute, and interpret the effective duration of a fixed income security given a change in yield and the resulting change in price.

Dollar duration is the slope of the blue tangent line. Effective duration measures the percentage change in the value (price) of a security for a unit change in the interest rate.

Duration (D) is given by:



If we multiply both sides of equation**,** then we get the following key equation:



The above equation says: the percentage change in the price is equal to the modified duration multiplied by the change in the rate (the minus sign indicates they move in opposite directions; i.e., a positive yield change corresponds to a negative price change).

Duration can be calculated with the following formula:



The text refers to the above formula as “duration”—ultimately you will read about three “flavors” of duration: effective, modified, and Macaulay. The formula above—because it is general—could describe either effective or modified duration (but not Macaulay). It is okay to refer to the formula above as either effective duration or simply “duration.”

**For example:**

$1,000 par bond,

4% semi-annual coupon,

10 years to maturity

|  |  |  |  |
| --- | --- | --- | --- |
|  | | |  |
| Par value | | | $1,000.00 |
| Years to Maturity | | | 10 |
| Coupon, % | | | 4.0% |
| Yield | | | 6.0% |
| Semiannual equivalents: | | | |
| Coupon, % | | | 2.0% |
| coupon, $ | | | $20.00 |
| Periods | | | 20 |
| Semiannual Yield | | | 3.0% |
|  | |  |  |
| Bond Price (PV) | | | $851.23 |
|  | |  |  |
| Modified Duration | | |  |
| Shock, bps | | | 10 |
| Shock, % | | | 0.10% |
| Yield up | | | 6.10% |
| Price (Shock up) | | | $844.51 |
| Yield down | | | 5.90% |
| Price (Shock down) | | | $858.01 |
| Duration | | | 7.931 |



## Compare and contrast DV01 and effective duration as measures of price sensitivity.

Tuckman: “Duration tends to be more convenient than DV01 in the investing context”

However, in portfolio aggregation (e.g., hedging a long position with a short position), we need to use dollar duration (or DV01)

## Define, compute, and interpret the convexity of a fixed income security given a change in yield and the resulting change in price.

Convexity also measures interest rate sensitivity. Mathematically, convexity is given by the formula below where the term (d2P/dy2) is the second derivative of the price-rate function:



The common convexity formula is given by:



Where:

* V0 is the initial price of the bond
* V+ is the price of the bond if yields increase by Δy
* V- is the price of the bond if yields decrease by Δy
* Δy is a change in the yield (in decimal terms)

Applying the Convexity Measure

In order to estimate the percentage price change due to a bond’s convexity (i.e., the percentage price change not explained by duration), the convexity measure must by “translated” into a convexity adjustment:



The (1/2) in the formula above is called the “scaling factor.”

Without or without the scaling factor: either way is okay

Instead of using the “scaling factor” above, the (1/2) divider is sometimes found in the convexity measure. If the convexity measure includes a (2) in the denominator, then the (1/2) drops out of the convexity adjustment. So if we use:



Then the convexity adjustment is given by:



The end result is exactly the same and it doesn’t really matter because, unlike the convexity adjustment, the convexity measure has no intrinsic interpretation.

To summarize: the convexity measure is calculated in order to determine the convexity adjustment:



Dollar Convexity (aka, Value Convexity)

Dollar convexity: second partial derivative of change in price with respect to the change in yield. In this way, just as duration is -1/P\*dP/dy, convexity is 1/P\*d^2/dy^2.



For example:

* **$1,000 par bond**
* **4% semiannual coupon**
* **10 years to maturity**
* **- $851.23 @ 6% Yield**



➊



➋



Using Duration and Convexity Together

* **$1,000 par bond**
* **4% semiannual coupon**
* **10 years to maturity**
* **$851.23 @ 6% Yield**



Tuckman’s Convexity Calculations

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Tuckman's Table 5.3, Bond: 5s Treasury of February 15, 2011 | | | | | |
|  |  |  | Application @ 5% (only applies at 5%) | | |
| Face | 100 |  | If rates change (bps) | | 25 |
| Coupon | 5.0% |  | Duration term | | -1.95% |
| Maturity | 10.0 |  | Convexity term | | 0.02% |
|  |  |  | % Change in Bond Price | | -1.93% |
|  |  |  |  |  |  |
|  |  |  | 1st Deriv | 2nd Deriv |  |
| Rate | Bond | Mod. | (Dollar | (Dollar |  |
| Level | Price | Duration | Duration) | Convexity) | Convexity |
| 3.99% | $108.2615 |  |  |  |  |
| 4.00% | $108.1757 | 7.92 | -857.4290 | 8,164.29 | 75.47 |
| 4.01% | $108.0901 |  | -856.6126 |  |  |
| 4.99% | $100.0780 |  |  |  |  |
| 5.00% | $100.0000 | 7.79 | -779.8264 | 7,362.87 | 73.63 |
| 5.01% | $99.9221 |  | -779.0901 |  |  |
| 5.99% | $92.6322 |  |  |  |  |
| 6.00% | $92.5613 | 7.67 | -709.8187 | 6,644.55 | 71.79 |
| 6.01% | $92.4903 |  | -709.1542 |  |  |

Estimate, given the DV01, the duration and the convexity of a fixed income security, the price change of a security

Duration and convexity combine to produce an approximation of the price change of the bond. The price change is given by:



Note this is identical to the following equation because the Convexity Adjustment equals [convexity measure x yield change2]:



Let’s do an example. Assume a $100 par value 10-year bond with a 3% coupon that yields 5% (i.e., the yield to maturity is 5%). We compute the duration by “shocking” 50 basis points (Note: the amount of shock is somewhat arbitrary; we could shock with 20 basis points or 100 basis points).

First, we compute the price of the bond: The price of the bond is about $84.41 because N=20, I/Y=2.5, PMT=$1.5, FV=100 are the inputs that allow us to compute (CPT) a present value (PV) of $84.41.

Next, we “shock up” by re-pricing the same bond with a 50 bps rate increase: the higher yield produces a lower value (V-) of about $80.97 because the re-priced bond is given by N=20, I/Y=2.75, PMT=$1.5, FV=100 and CPT PV ≅ $80.97 (note the 50 basis point shock implies a 5.5% annual yield, which is 2.75% semi-annually).

Then we “shock down” by re-pricing the bond with a 50 bps rate decrease: the lower yield produces a higher value (V+) of about $88.03 because the re-priced bond is given by N=20, I/Y=2.25, PMT=$1.5, FV=100 and CPT PV ≅ $88.03 (note the 50 basis point shock implies a 4.5% yield, which is 2.25% semi-annually).

We can now solve for the duration equation. The Duration is given by (88.03 - 80.97) ÷ [(2)(84.41)(.005)] ≅ 8.36 (or 8.37 if you didn’t round anything along the way):



We solved for duration, now we need convexity. The Convexity Measure is given by [88.03 + 80.97 – (2) (84.41)] ÷ [(2) (84.41) (0.005)2] ≅ 42.6 (or 40.6 if you didn’t round along the way).



We have solved for the duration and the convexity measure. Now we apply these metrics to answer the question, “What is the percentage change in price for a given percentage change in yield?” We have been “shocking” by 50 basis points, and we could continue to use 50 basis points. However, we can also simply use 100 basis points (1%) for the final sensitivity.

The Convexity Measure no meaning by itself. We need to translate the Convexity Measure into a proper Convexity Adjustment. The convexity adjustment (CA) is given by (Convexity Measure)(0.01)^2 = 0.00426 or 0.426% .

The other way to get to the Convexity Adjustment is to compute a Convexity Measure given by [88.03 + 80.97 – (2) (84.41)] ÷ [(84.41) (0.005)2] ≅ 81.3. Then plug this Convexity Measure (81.3) into a Convexity Adjustment formula that is given by (1/2)(convexity measure)(0.01)^2 which gets to the same result.

Finally, we combine duration and convexity. We could use any shock, but now we will shock with 1% (100 basis points). The price impact due to the duration component is simply (D)(1%) or 0.0836. This the normal (linear) duration: our duration of 8.36 means that a 1% change in yield implies an 8.36% change in price.

The percentage change in the price of the security can now be determined as a function of both duration and convexity:

The % price change for a 1% yield increase (+100 bps)

= (-8.36)(1%) + 0.426% = -7.9%.

The % price change for a 1% yield decrease (-100 bps)

= (-8.36)(-1%) + 0.426% = +8.79%

Notice we plugged in the Convexity Adjustment. Alternatively, we could have plugged the Convexity Measure directly into the final equation:

% price Δ for a 1% yield increase (+100 bps)

= (-8.36)(1%)+(42.6)(1%)^2 ≅ -7.9%.

% price Δ for a 1% yield decrease (-100 bps)

= (-8.36)(-1%)+(42.6)(1%)^2 ≅ +8.79%.

Duration is a first-order linear (partial derivative) approximation of the sensitivity of the price to small changes in the yield – it is therefore flawed because the price-yield curve is not linear.

Convexity is the second-order approximation that we use to explain the change in price that is not explained by duration. The effect of convexity is additive for both yield increases and decreases.

Interpret and apply convexity in investment and asset‐liability management

The greater the convexity, then the less reliable is duration because duration is linear. For highly convex securities, duration is unsafe. Further, because the term (Δy2) is always positive, positive convexity increases returns for any given movement in interest rates.

Asset–liability managers can better hedge against interest rate changes by hedging both duration and convexity (i.e., instead of only duration).

Compute the duration of a portfolio

The duration of a portfolio equals a weighted sum of individual durations where each security’s weight is its value as a percentage of portfolio value:



## Explain the process of calculate the effective duration and convexity of a portfolio of fixed income security.

In regard to both modified (effective) duration and convexity, portfolio duration and convexity equal the weighted sum of individual (component), respectively, durations and convexities where each component’s (security’s) weight is its value as a percentage of portfolio value:



## Explain the impact of negative convexity on the hedging of fixed income securities.

A callable bond exhibits negative convexity at lower yields:

This negative convexity could also characterize a mortgage-backed security (MBS). In a later chapter on the negative convexity of an MBS, Tuckman makes three points about the characteristics of a mortgage pass-through:

As you’d expect, at low yields it shows negative convexity. At low yields, borrowers will exercise their prepayment option by refinancing. (a popular test question asks if all bonds are positively convex. Answer, yes, all plain vanilla bonds have positive convexity, but bonds with embedded options can have negative convexity).

At low yields the price tends to rise above the par a bit. That’s because refinancing isn’t perfectly responsive and rational; it tends to lag.

At higher yields, the pre-payable pass-through shows a higher price. This one’s a little tricky. It is because prepayments are not triggered only by lower rates. They are also triggered by housing turnover. Since there will be some prepayments are higher rates, that makes the pass-through a bit more valuable at higher rates-investor like prepayments when rates rise!

The striking aspect of [callable bond with negative convexity] is the positive convexity of the bond and the negative convexity of the callable bond combine to make the DV01 hedge quite unstable away from 5%.

Care must be exercised when mixing securities of positive and negative convexity because the resulting hedges or comparative return estimates are inherently unstable.

## Construct a barbell portfolio to match the cost and duration of a given bullet investment, and explain the advantages and disadvantages of bullet versus barbell portfolios.

NEEDS CONTENT

## Questions and Answers

Questions

20.4.1 The modified duration is 10.46 years of a bond with a current price of $716.38. What is the bond's DV01?

1. $0.40
2. $0.75
3. $1.25
4. Need more information (yield, maturity)

20.4.2 A 15-year zero-coupon bond has a price of $63.98 when the yield is 3.00%. At this 3.00% yield, the bond's dollar duration is -952.0; if the yield increases by 10 basis points to 3.10% the bond's dollar duration drops to -938.0. Recall that the dollar duration is the first derivative of the price-rate function, dP/dy (modified duration is -1/P multiplied by this dollar duration). What is the bond's convexity at 3.00%?

1. 28
2. 124
3. 219
4. 435

20.4.3 A market maker sells (writes) $100 million face value of call options on underlying bonds when the interest rate is 4.0%. The price of the call options is $3.0 million and their (modified) duration is 80.0 years. At the same 4.0% rate, as the underlying bonds pay a 4.0% coupon, the price of the underlying happens to equal $100 par with a duration of 7.0 years. What is the market maker's hedge transaction?

1. Short $12.9 million of underlying bond
2. Short $24.0 million of underlying bond
3. Long $24.0 million of underlying bond
4. Long $34.3 million of underlying bond

20.4.4 A fixed income manager owns a barbell portfolio with equal weights in two zero-coupon bonds: 50% of its value in a two-year zero-coupon bond and 50% in a 12-year zero-coupon bond. The manager is considered shifting the investment to a bullet portfolio with the same value but instead 100% invested in a zero-coupon bond with seven years to maturity. Each of the following is true about the portfolios EXCEPT (please assume semi-annual compounding):

1. At a yield of 4.0%, the bullet portfolio's Macaulay duration is 7.0 years
2. At all yields, the barbell's Macaulay duration is similar to bullet's Macaulay duration
3. At a yield of 4.0%, the barbell portfolio's convexity approximately 50 years^2
4. At a yield of 4.0%, the barbell portfolio's convexity is higher than the bullet's convexity

Answers

20.4.1 B. $0.75

DV01 = modified duration \* Price = 10.46 \* 716.38 / 10,000 = $0.74933

20.4.2 C. 219

C = [(952 - 938)/0.0010]/63.98 = 218.82; please note how near it is to 15 years ^ 2 = 225.

Per Tuckman, we can estimate the second derivative (dollar convexity) by dividing the change in the first derivative by the change in the rate.

In this case, the dollar convexity is given by (952 - 938)/0.001 = 14,000.

Convexity (C) = d^2P/dy^2 \* 1/P = dollar convexity \* 1/P = 14,000/63.98 = 218.82

20.4.3 D. Long $34.3 million of underlying bond

The DV01 of the written call options, DV01 = P\*D/10000 = 3 million\*80/10000 = $24,000 or $240 per 100 face.

To hedge, the market marker should buy P = DV01\*10,000/D = 240 \* 10,000 / 7 = $342,857 per 100 face or $34.285 million in the underlying bond.

20.4.4 C. False, while the convexity of the bullet is ~50 years^2, the convexity of the barbell ~75, such that (D) is true.

Under semi-annual compounding, convexity of a zero = T\*(T+0.5)/(1+y/2)^2.

In this case, the convexity of the 2-year = 2\*2.5/1.02^2 = 4.81; convexity of the 12-year = 12\*12.5/1.02^2 = 144.18; convexity of the 7-year = 7\*7.5/1.02^2 = 50.46.

Please note, in the case annual compounding, the convexities would be similar as C=T\*(T+1)/(1+y)^2

Convexity, like duration, is a weighted-average of portfolio components.

In the case of the barbell, portfolio convexity = 50%\*4.81 + 50%\*144.18 = 74.49.

In the case of the bullet, portfolio convexity = 100%\*50.46 = 50.46.

# Tuckman, Chapter 5: Multi-Factor Risk Metrics and Hedges

Learning Outcomes:

**Describe and assess** the major weakness attributable to single-factor approaches when hedging portfolios or implementing asset liability techniques.

**Define** key rate exposures and know the characteristics of key rate exposure factors including partial ‘01s and forward-bucket ‘01s.

**Describe** key-rate shift analysis.

**Define, calculate, and interpret** key rate ‘01 and key rate duration.

**Describe** the key rate exposure technique in multi-factor hedging applications and summarize its advantages and disadvantages.

**Calculate** the key rate exposures for a given security, and compute the appropriate hedging positions given a specific key rate exposure profile.

**Describe** the relationship between key rates, partial '01s and forward-bucket ‘01s, and calculate the forwardbucket ‘01 for a shift in rates in one or more buckets.

**Construct** an appropriate hedge for a position across its entire range of forward bucket exposures.

**Explain** how key rate and multi-factor analysis may be applied in estimating portfolio volatility.

## Describe and assess the major weakness attributable to single-factor approaches when hedging portfolios or implementing asset liability techniques.

## Define key rate exposures and know the characteristics of key rate exposure factors including partial ‘01s and forward-bucket ‘01s.

## Describe key-rate shift analysis.

## Define, calculate, and interpret key rate ‘01 and key rate duration.

## Describe the key rate exposure technique in multi-factor hedging applications and summarize its advantages and disadvantages.

## Calculate the key rate exposures for a given security, and compute the appropriate hedging positions given a specific key rate exposure profile.

## Describe the relationship between key rates, partial '01s and forward-bucket ‘01s, and calculate the forwardbucket ‘01 for a shift in rates in one or more buckets.

## Construct an appropriate hedge for a position across its entire range of forward bucket exposures.

## Explain how key rate and multi-factor analysis may be applied in estimating portfolio volatility.

## Questions and Answers

Questions

Answers

# Tuckman, Chapter 6: Empirical Approaches to Risk Metrics and Hedging

**Learning Outcomes:**

**Explain** the drawbacks to using a DV01-neutral hedge for a bond position.

**Describe** a regression hedge and explain how it improves on a standard DV01-neutral hedge.

**Calculate** the regression hedge adjustment factor, beta.

**Calculate** the face value of an offsetting position needed to carry out a regression hedge.

**Calculate** the face value of multiple offsetting swap positions needed to carry out a two-variable regression hedge.

**Compare and contrast** between level and change regressions.

**Describe** principal component analysis and explain how it is applied in constructing a hedging portfolio.

## Explain the drawbacks to using a DV01-neutral hedge for a bond position.

## Describe a regression hedge and explain how it improves on a standard DV01-neutral hedge.

## Calculate the regression hedge adjustment factor, beta.

## Calculate the face value of an offsetting position needed to carry out a regression hedge.

## Calculate the face value of multiple offsetting swap positions needed to carry out a two-variable regression hedge.

## Compare and contrast between level and change regressions.

## Describe principal component analysis and explain how it is applied in constructing a hedging portfolio.

## Questions and Answers

Questions

Answers

# Narayanan, Chap 23: Country Risk Models

Learning Outcomes:

**Define and differentiate** between country risk and transfer risk and discuss some of the factors that might lead to each.

**Describe** country risk in a historical context.

**Identify and describe** some of the major risk factors that are relevant for sovereign risk analysis.

**Compare and contrast** corporate and sovereign historical default rate patterns∙

**Explain a**pproaches for and challenges in assessing country risk.

**Describe** how country risk ratings are used in lending and investment decisions.

**Describe** some of the challenges in country risk analysis.

## Define and differentiate between country risk and transfer risk and discuss some of the factors that might lead to each.

Country risk:

* Broadest and most inclusive level of credit risk.
* Risk that the full and timely servicing of obligations may be adversely affected by the normal, ambient country-speciﬁc economic factors, and also by transfer risk.

Transfer risk:

Transfer risk arises when credit and counterparty obligations are extended across national borders and involve different currencies, different legal systems and different sovereign governments.

## Describe country risk in a historical context.

Over the years, there have been incidences of sovereign default and a long history of cross-border lending. Sovereign defaults were high in the periods:

* 1820-1990.
* Asian countries and Russia defaults in the late 1990s.
* Argentina defaulted in 2001.

However, over the past several years net capital flows have been increasing substantially year over year, and the cost and terms of borrowing declined before the 2007 financial crisis. Following the crisis it is unclear how this trend will continue. Generally speaking, 90% of developing countries have accessed the syndicated loan market and 40% of developing countries have accessed the bond markets. Developing countries have also taken pre-emptive action to strengthen themselves financially according to the World Bank.

## Identify and describe some of the major risk factors that are relevant for sovereign risk analysis.

Political Risk

* Stability and legitimacy of political institutions
* Popular participation in political processes
* Orderliness of leadership succession
* Transparency in economic policy decisions and objectives
* Public security
* Geopolitical risk

Income and Economic Structure

* Prosperity, diversity, and degree to which economy is market oriented
* Effectiveness of financial sector in intermediating funds; availability of credit
* Competitiveness and profitability of nonfinancial market sector
* Efficiency of public sector
* Protectionism & other nonmarket inﬂuences
* Labor ﬂexibility

Economic Growth Prospects

* Size and composition of savings and investment
* Rate and pattern of economic growth

Fiscal Flexibility

* General government revenue, expenditure, and surplus/ deficit trends
* Timeliness, coverage, and transparency in reporting

General Government Debt Burden

* General government gross and net (of assets) debt
* Share of revenue devoted to interest
* Currency composition and maturity proﬁle

Offshore and Contingent Liabilities

* Size and health of NFPEs
* Robustness of financial sector

Monetary Flexibility

* Price behavior in economic cycles
* Compatibility of exchange-rate regime and monetary goals
* Institutional factors, such as central bank independence
* Range and efficiency of monetary policy tools

External Liquidity

* Impact of fiscal and monetary policies on external accounts
* Structure of the current account
* Composition of capital flows
* Reserve adequacy

External Debt Burden

* Gross & net external debt, incl. deposits & structured debt
* Maturity profile, currency composition, & interest rate sensitivity
* Access to concessional funding

## Compare and contrast corporate and sovereign historical default rate patterns∙

Moody’s and S&P report “similar in most cases” but small number of sovereign defaults esp. in higher grades

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| TABLE 23.2 | Sovereign & Corporate Default Rate Comparison (S&P) | | | | | | |  |
| Percent of |  |  |  |  |  |  |  |  |
| Rated | One-year | |  | Three-year | |  | Five-year | |
| Issuers | Sov. (S) | Corp (C) |  | S | C |  | S | C |
| AAA | 0.0 | 0.0 |  | 0.0 | 0.1 |  | 0.0 | 0.3 |
| AA | 0.0 | 0.0 |  | 0.0 | 0.1 |  | 0.0 | 0.3 |
| A | 0.0 | 0.1 |  | 0.0 | 0.3 |  | 0.0 | 0.7 |
| BBB | 0.0 | 0.2 |  | 2.0 | 1.2 |  | 5.1 | 2.6 |
| BB | 1.0 | 1.1 |  | 5.0 | 5.6 |  | 8.7 | 10.1 |
| B | 1.9 | 5.0 |  | 8.5 | 15.9 |  | 16.8 | 22.6 |
| CCC/CC | 41.2 | 26.3 |  | 58.8 | 40.0 |  | 58.8 | 46.2 |

## Explain approaches for and challenges in assessing country risk.

Assessing country risk can be challenging, as the data is limited: defaults have occurred on a consistent basis, however the number of countries that have defaulted over the years is small in a statistical sense. Moreover, using macroeconomics analysis alone to analyze country risk is not sufficient. As with companies, there are principal-agent problems that can affect political outcomes. In the context of public policy, this is a relatively new and active field of research in economics called *public choice theory*, which describes the rent-seeking behavior and self-interest of those who hold office. Public choice theory is likely to become increasingly important when assessing country risk.

Furthermore, problems such as model risk in econometrics and the fact that data is only available after a significant time lag make it hard to construct models with predictive power. There is also the notion that analysts do not want to raise criticism when times seem to be good so as not to sound alarmist.

## Describe how country risk ratings are used in lending and investment decisions.

* Once a country risk rating approach has been established, often using a combination of rating sources,
* Lenders use the risk rating to create a scale to limit their country risk and transfer risk exposures. Typically an overall country
* The size of the overall limits is proportional to the lending institution’s risk appetite and to the ratings. Low ratings would permit only small exposures with short tenors

## Describe some of the challenges in country risk analysis.

Small number of actual defaulting countries

Interdependencies of variables in country analysis can be so complex that it is difficult, if not impossible, to model or anticipate outcomes.

Can be difficult to get data

Country-level information available only after a significant delay

Geopolitical factors can shape/shift business environment abruptly and dramatically

## Chapter Summary

Country risk is the broadest and most inclusive level of credit risk. It involves the risk that the full and timely servicing of obligations may be adversely affected by the normal, ambient country-speciﬁc economic factors, and also by transfer risk. Transfer risk arises when credit and counterparty obligations are extended across national borders and involve different currencies, different legal systems and different sovereign governments.

Over the years, there have been incidences of sovereign default and a long history of cross-border lending. Sovereign defaults were high in the periods: 1820-1990; Asian countries and Russia defaulted in the late 1990s; and Argentina defaulted in 2001. However, over the past several years net capital flows have been increasing substantially year over year, and the cost and terms of borrowing declined before the 2007 financial crisis. Following the crisis it is unclear how this trend will continue.

Assessing country risk can be challenging, as the data is limited: defaults have occurred on a consistent basis, however the number of countries that have defaulted over the years is small in a statistical sense. Moreover, using macroeconomics analysis alone to analyze country risk is not sufficient. As with companies, there are principal-agent problems that can affect political outcomes. In the context of public policy, this is a relatively new and active field of research in economics called *public choice theory*, which describes the rent-seeking behavior and self-interest of those who hold office.

# de Servigny, Chapter 2: External and Internal Ratings

**Learning Outcomes:**

**Describe** external rating scales, the rating process, and the link between ratings and default.

**Describe** the impact of time horizon, economic cycle, industry, and geography on external ratings.

**Review** the results and explanation of the impact of ratings changes on bond and stock prices.

**Compare** external and internal ratings approaches.

**Explain and compare** the through‐the‐cycle and at‐the‐point approaches to score a company.

**Define and explain** a ratings transition matrix and its elements.

**Describe** the process for and issues with building, calibrating and back testing an internal rating system.

**Identify and describe** the biases that may affect a rating system.

Traditional Credit Analysis

Classic credit analysis is an “expert system” that relies on the subjective judgment of trained, experienced professionals. In such an expert system, knowledge tends to warehouse among tenured professionals:

The senior lender understands the institution’s “culture credit” and the boundaries that are shaped by tradition.

The senior lender is a source of “rules of thumb.”

Banks rely on their senior lending officers to zero in on the most important issues under which the bank will make the loan.

Emphasis Shifted from Balance Sheet toward Cash Flow

Historically, banks made loans against the borrower’s inventories and receivables—collateral that could be liquidated. But more recently, banks expanded to financing fixed assets and collateral that does not have liquid markets. This shift renders collateralization **less relevant** to the credit process.

Since debt must be repaid in cash, banks need to focus on the borrower’s cash flow. Cash flow from operations (CFO) defines the company’s liquidity; i.e., is it able to generate sufficient cash from internal operations to service its debt. Cash flow lending replaced secured lending as the principal activity of a commercial bank.

Traditional credit analysis is associated with the “three Cs”: character (the borrower’s willingness to repay), capacity (the borrower’s ability to repay) and capital (either the borrower’s capital cushion or their avenues to alternative forms of repayment).

Major Rating Agencies

When the assigned reading was published, there were five agencies designated by the SEC as “nationally recognized statistical rating organizations” (NRSROs). Now there are at least nine

NRSROs:

A.M. Best Company, Inc.

Dominion Bond Rating Service Limited

Fitch, Inc.—Independent

Moody’s Investors Service—Subsidiary of Dun and Bradstreet

Standard & Poor’s (S&P )—Division of the McGraw Hill

Japan Credit Rating Agency, Ltd

R&I, Inc.

Egan-Jones Ratings Company

LACE Financial

In June 2007, the SEC promulgated rules (Oversight of Credit Rating Agencies Registered as Nationally Recognized Statistical Rating Organizations) that enacted provisions of the Credit Rating Agency Reform Act.

Delegated Monitoring Function

Rating agencies are “delegated monitors”: they issue an independent credit opinion based on the application of consistent criteria. The ratings are not recommendations to buy or sell, but rather opinions. A rating is a broad indicator or broad bucket of the probability of default.

Style Differences among Firms

Style varies by firm:

Standard & Poor’s (S&P) tends to refer to the likelihood of default. S&P assigns greater weight to industry risk; i.e., the strength and stability of the industry in which the firm operates.

Moody’s tends to refer to expected loss. Moody’s ratings place weight on “business fundamentals such as demand-supply characteristics, market leadership, and cost positions.” Expected loss (EL) = probability of default  loss severity

Duff and Phelps prioritizes the company’s presentation to their rating committee.

An S&P rating tends to correspond to a probability of default (PD), while a Moody’s rating tends to refer to expected loss (EL) where EL = PD \* LGD.

## Describe external rating scales, the rating process, and the link between ratings and default.

Rating Scales

Long and short-term instruments

Issue-specific or issuer-specific

Opinions into categories, not buy/sell recommendations

S&P tends to reflect view on probability of default (PD) - Moody’s tends to reflect view on expected loss

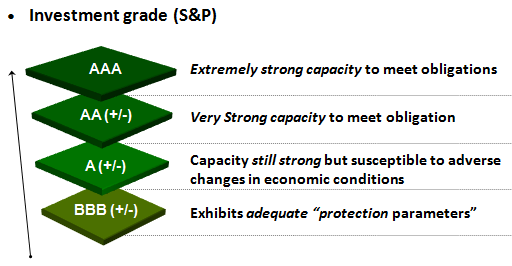
Rating scales can be short-term or long-term and they can apply to specific obligations or instruments (i.e., issue-specific credit ratings) or to an issuer’s general creditworthiness (i.e., issuer credit ratings). The bond universe is broadly divided into investment grade and non-investment grade bonds.

In rating long-term debt, each agency uses a system of letter grades that locate an issuer or issue on a spectrum of credit quality from the very highest (triple-A) to the very lowest (D).

Investment grade = rated Baa3 or BBB- and above

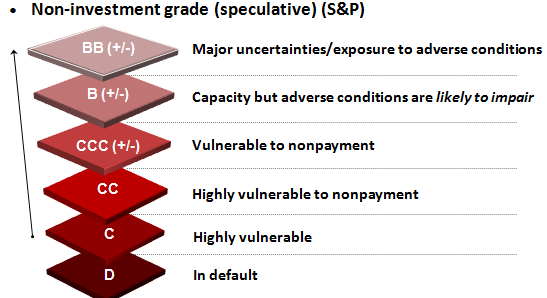
Speculative or non-investment grade = issues rated Ba1 or BB + and below

Investment grade (S&P)



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Investment Grade Ratings | | | | |
| Interpretation (abridged) | **S&P**  **Rating** |  | **Moody’s**  **Rating** | **Interpretation (abridged)** |
| Highest rating. Extremely strong capacity to meet obligations | **AAA** |  | **Aaa** | **Highest quality, with minimal credit risk** |
|  | AA+ |  | Aa1 |  |
| Capacity to meet its financial obligation is very strong | **AA** |  | **Aa2** | **High quality and subject to very low credit risk** |
|  | AA- |  | Aa3 |  |
|  | A+ |  | A1 |  |
| Capacity to meet obligation still strong but susceptible to adverse changes in economic conditions | **A** |  | **A2** | **Considered upper-medium grade and subject to low credit risk** |
|  | A- |  | A3 |  |
|  | BBB+ |  | Baa1 |  |
| Exhibits adequate “protection parameters” | **BBB** |  | **Baa2** | **Subject to moderate credit risk** |
|  | BBB- |  | Baa3 |  |

Non-investment grade (speculative) (S&P)



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Speculative Grade Ratings** | | | | |
| **Interpretation (abridged)** | **S&P** |  | **Moody’s** | |
|  | BB+ |  | Ba1 |  |
| Less vulnerable to nonpayment than other speculative issues but faces “major ongoing uncertainties” | BB |  | Ba2 | Judged to have speculative elements and are subject to substantial credit risk |
|  | BB- |  | Ba3 |  |
|  | B+ |  | B1 |  |
| More vulnerable to nonpayment than ‘BB’ but currently has the capacity to meet its financial obligation | B |  | B2 | Considered speculative and are subject to high credit risk |
|  | B- |  | B3 |  |
|  | CCC+ |  | Caa1 |  |
| Vulnerable to nonpayment | CCC |  | Caa2 | In poor standing |
|  | CCC- |  | Caa3 |  |
| Highly vulnerable to nonpayment. | CC |  |  |  |
| Highly vulnerable to nonpayment | C |  | Ca | Highly speculative and likely to default |
| **In payment default** | **D** |  | **C** | **Lowest rated bonds—typically in default, with little prospect for recovery** |

Note: Moody’s adds numerical modifiers (1, 2, and 3) to each generic rating classification from Aa through Caa, where ‘1’ indicates an obligation that ranks at the higher end of category and ‘3’ indicates the lower end of the category.

Rating Process

Business reviews and quantitative analysis

Company may appeal

Recently see the “outlook concept” (e.g., “trend is positive”)

Equity analysts focus on shareholders’ perspectives but rating agencies analyze from bondholders’ perspectives. Equity analysis is similar to credit analysis, but rating agencies have a longer time horizon. The rating process is divided into a business review (i.e., qualitative component) and a quantitative analysis. Recently, the “outlook concept” has emerged. The outlook concept refers to a positive or negative trend.

Corporate issuers typically pay rating agencies. Some believe this is a conflict, says the text. (And the recent credit crunch has surely heighted the controversy). The text says: agencies have a natural incentive to maintain their reputation; the agencies tend to act more like independent academic research centers than businesses (e.g., analysts do not discuss fees).

Link between ratings and default

The rating agencies do not specify a precise probability of default; instead, ratings are broad risk buckets. Still, in order to use ratings in a quantitative risk management system, we need to map the ratings to numbers—typically, we map the ratings to default probabilities.

Agencies publish cumulative default rates categorized by rating (i.e., the cumulative default rate per rating category) and transition matrices. Transition matrices plot the frequency of rating migrations over time; e.g., how many times did obligors rated “AA” migrate to “A?”

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | AAA | AA | A | BBB |  | D\* | Total |
| AAA | 89% | 6% | 3% | 1% |  | 0 | 100% |
| AA | 3% | 86% | 2.5% | 2.0% |  | .01% | 100% |
| A | 1.5% | 4% | 80% | 4% |  | .06% | 100% |
| BBB | 0.02% | 1% | 3% | 78% |  | .24% | 100% |

Note about the transition matrix:

Probabilities in a row must sum to 100% (1.0). Each row can be viewed as an empirical distribution.

The diagonals have the largest numbers as an obligor rating grade is most likely to remain, at least over one period.

The outcomes are directionally as we expect. It is an ordinal relationship: lower rated obligors tend to default more frequently.

Better (worse) ratings are associated with lower (higher) default rates

Ratings tend to have homogenous default rates across industries

There is a big disparity between investment-grade and speculative-grade categories

Overall, there is a clear link between default rates and rating categories

## Describe the impact of time horizon, economic cycle, industry, and geography on external ratings.

Time horizon

Agencies issue ratings with a long-term perspective—not a snapshot. However, the use of “through-the-cycle” ratings (see below) may overshoot or undershoot when economic conditions vary significantly from the average. As Moody’s has said, there is a necessary trade-off between rating accuracy and stability. A long time horizon will be more stable but less accurate instantaneously. On the other hand, a structural model like the Merton model may be accurate “in the moment” but will necessarily be more volatile.

As we expect, the cumulative probability of default increases with the time horizon (i.e., given more time, an obligor has a greater chance of defaulting). However, the cumulative probability of default increases more rapidly for non-investment grade obligors.

Economic cycle

Downgrades and probability of defaults (PDs) increase significantly during a recession. Migration volatility is higher during a recession and lower during a growth phase.

Industry

Agencies strive for consistency (ratings homogeneity), but there is less ratings consensus among financial firms—probably due to the opacity of these firms’ financials. In general, increased opacity (i.e., lack of transparency, greater ambiguity) is associated with higher ratings variability (i.e., less consensus or consistency).

Geography

Because methodologies were developed in the US, non-U.S. methodologies may be biased due to shorter timeframes

Summary of impacts (according to de Servigny):

* Use of through-the-cycle approach implies more stability but less (temporal) accuracy
* Longer time horizons imply higher cumulative default probability, but even more so for speculative ratings
* During a recession, probability of default increases and so does rating migration volatility
* Financial firms tend to exhibit less rating consensus (i.e., variability among agencies)
* Non-U.S. methods may be biased due to lack of methodological seasoning

## Review the results and explanation of the impact of ratings changes on bond and stock prices.

Ratings change impact bond prices

Downgrades have a negative impact on bond prices; upgrades have a positive impact

But the relationship is statistically stronger for upgrades than downgrades.

How does a rating change impact bond prices?

There are different reasons a rating change impacts bond price:

Supply and demand (policy asset allocation) policy: Some portfolios by policy may only hold investment grade bonds. For these, if a holding is downgraded to speculative (junk), the position must be sold. Therefore, a bond that is downgraded from investment-grade to speculative will experience less demand. A decrease in demand (or increase in supply) will lower the price.

Supply and demand (Basel I): Under Basel I, all corporate exposures are treated the same. Therefore, a bank must hold the same regulatory capital to cover a AAA-rated corporate bond as a BBB-rated corporate bond. Under this assumption, the bank is incentivized to hold the riskier bond because it gives greater yield for the same regulatory capital. Note: this is a point only about Basel I, de Servigny is more hopeful about Basel II.

Ratings triggers are covenants based on the rating; e.g., a rating trigger could cause a step-up bond to increase the coupon. Rating triggers (bond covenants) are very significant because they are self-fulfilling: a downgrade can trigger a covenant, which in turn can “trigger” other triggers.

Credit derivatives have (arguably) increased bond price volatility. For example, a buyer of a credit default swap (CDS) is buying protection and synthetically SHORTING the underlying bond. De Servigny argues these give market players more means to (synthetically) short bonds, so this contributes to price volatility.

**de Servigny argues that ratings changes are proof that**

**analysts bring new information**

How to determine price impact of a ratings change?

There are at least four ways to determine the price impact of a rating change:

1. Duration: Multiply the change in yield spread between the initial rating and the new rating by the modified duration (the percentage change in price associated with a 100 basis point move in interest rates) of the bond. This methodology utilizes either the average yield-to-maturity or the option (primarily call option) adjusted spread, by bond rating class.
2. Estimation of rating change: Estimate the possible rating change for the next period (e.g., one year). Then discount the remaining cash flows from that period to maturity using the forward zero coupon curve for the bond in the new rating class.
3. Direct observation: Direct observation of the price changes of a large sample of bonds of different rating classes. This is a type of so-called “event-study analysis.”
4. Spread decomposition: Decompose the observed market spreads of bonds in various rating classes so that you can isolate the impact of expected rating drift. Combined with historical rating drift patterns, these observed spreads can reveal the expected economic consequence of a change in rating.

## Compare external and internal ratings approaches.

Historically, bank credit produced black/white good/bad credit ratings; i.e., we will either loan to the borrower or not. But this evolved for at least two reasons:

External rating agency scales have become commonplace

Basel II rules encouraged refined rating scales

An Internal Rating System

A bank can try to mimic the external agencies. Typically, a template assigns weights to credit risk factors (a scorecard) in order to produce a weighted score.

When banks build an internal rating system, they have at least two objectives:

To assess creditworthiness of companies

To input into portfolio tools that determine the amount of needed economic or regulatory capital

Explain how internal ratings models may create a pro-cyclicality effect.

A key concern surrounds the use of internal ratings during (macro-) economic cycles (especially if banks tend to apply point-in-time metrics). Many argue that banks will tend to over-lend in up/strong cycles and under-lend in down/recessionary cycles (as credit rationing contributes to scarce capital).

The pro-cyclicality effect therefore refers to the tendency of internal ratings models to (indirectly) reinforce credit and business cycles. In particular:

INTERNAL APPROACHES UNDER BASEL II: In regard to Basel II, banks that use an internal system to determine their capital requirements may tend to over-lend in good times and under-lend in bad times

EXPECTED LOSS RATHER THAN EXPOSURE: Banks that set their internal credit limits in terms of expected loss rather than exposure may create even more pro-cyclicality: expected loss will be volatile due to the high volatility of PDs calculated using at-the-point-in-time methods.

AT-THE-POINT-IN-TIME: At-the-point-in-time measures of risk in economic capital calculations tend to underestimate risk during growth periods; conversely, they tend to over-estimate risk during recessions.

SHORT-TERM PROJECTS: At-the-point-in-time-measures are biased in favor of short-term projects; the selection of short-term projects can lead to suboptimal lending decisions.

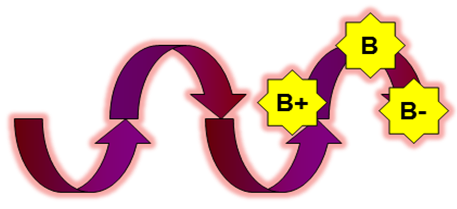
## Explain and compare the through‐the‐cycle and at‐the‐point approaches to score a company.

There are two broad ways to rate or score a company:

Point-in-time, or

Through-the-cycle

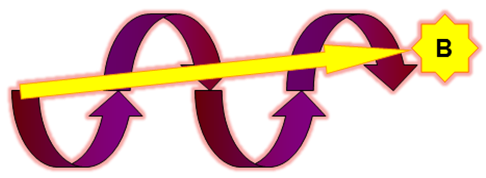
Point-in-Time



“At-the-point-in-time” assesses credit quality over the near term; i.e., a few months or one year. This approach is widely used by banks that employ quantitative scoring systems.

**Includes structural models (e.g., KMV Credit Monitor)**

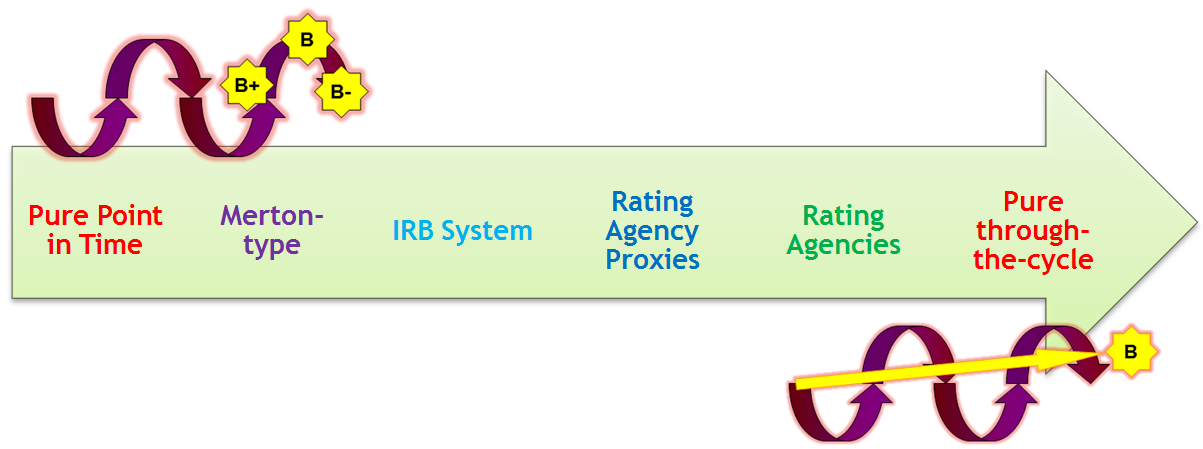
Through-the-cycle



But agencies try to incorporate business cycles. Ratings are therefore typically considered “through-the-cycle.” Through-the-cycle ratings try to “filter out” cycle fluctuations. Because they incorporate an average, when economic conditions vary from the average, through-the-cycle may over- or under-estimate credit quality. Through-the-cycle ratings are more stable.

**Agency ratings tend to be through-the-cycle**

Note we can think of pure point-in-time (where a Merton-type model near is more point-in-time) and through-the-cycle as a continuum:



## Define and explain a ratings transition matrix and its elements.

Transition (a.k.a., migration) matrix gives probability of state change

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Moody's (2006): One-Year Average Rating Transition Matrix, 1983-2005 | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |
| Beginning | |  |  |  |  |  |  |  |  |
| of Year | End of Year Rating | | | | | | | | |
| Rating | Aaa | Aa | A | Baa | Ba | B | Caa-C | Default | WR |
| Aaa | **89.54** | 7.14 | 0.41 | 0.00 | 0.02 | 0.00 | 0.00 | 0.00 | 2.89 |
| Aa | 1.25 | **88.82** | 5.72 | 0.25 | 0.04 | 0.02 | 0.00 | 0.01 | 3.89 |
| A | 0.05 | 2.63 | **87.35** | 5.29 | 0.59 | 0.13 | 0.02 | 0.02 | 3.92 |
| Baa | 0.04 | 0.22 | 4.92 | **83.95** | 4.81 | 0.99 | 0.32 | 0.21 | 4.53 |
| Ba | 0.01 | 0.06 | 0.54 | 6.10 | **75.53** | 7.93 | 0.72 | 1.15 | 7.98 |
| B | 0.01 | 0.05 | 0.16 | 0.41 | 4.66 | **73.56** | 6.63 | 5.76 | 8.75 |
| Caa-C | 0.00 | 0.04 | 0.03 | 0.22 | 0.60 | 5.47 | **59.46** | 10.41 | 23.78 |

Nickell, Perraudin, & Varotto (2000) found that a single transition matrix is not time stationary. They found transition matrices to be stable within broad homogenous sectors and by geography, but variable across sectors. Further, matrices are less stable during recession and more stable during economic growth phases. Bangia, Diebold, Kronimus et al (2002) found a longer time horizon associates with a less stable (non-stationary) transition matrix.

## Describe the process for and issues with building, calibrating and back testing an internal rating system.

Rating templates allow banks to calibrate their internal rating process.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Weighting | Scoring  (0-100) | Weighted Score |
| 1. Industry characteristics |  |  |  |
| 2. Market position |  |  |  |
| 3. Management |  |  |  |
| Total for business profile |  |  |  |
|  |  |  |  |
| 4. Financial policy |  |  |  |
|  |  |  |  |
| Total for financial profile |  |  |  |

Banks need to link their internal rating scale to cumulative probabilities of default (PD)

Need a historical sample to backtest

De Servigny: at least (>) 10 years (based on S&P)

Carey and Hrycay: 11 to 18 year (based on Moody’s)

## Identify and describe the biases that may affect a rating system.

Pro-cyclicality: tendency of internal ratings models to reinforce credit and business cycles.

Basel II Internal Ratings-Based (IRB) approaches: linking capital requirements to PDs may induce banks to over(under)-lend in good (bad) times

Expected Loss (EL) Rather Than Exposure (EAD): if banks set their internal credit limits in terms of expected loss, rather than exposure, may create even more pro-cyclicality: expected loss will be volatile due to the high volatility of PDs calculated using at-the-point-in-time methods.

But at-the-point-in-time measures (e.g., structural models) are less stable

Economic capital calculations 🡪 underestimate risk during growth periods; over-estimate risk during recessions.

If banks set internal credit limits ~ f [expected loss] rather than exposure. Expected loss more volatile.

Biased in favor of short-term projects 🡪 suboptimal lending decisions

Key issue with at-the-point-in-time: Credit cycle tends to lag economic cycle (“always a step behind, not predictive”)

## Chapter Summary

Rating agencies issue an independent credit opinion based on the application of consistent criteria. The *ratings are not recommendations* to buy or sell, but rather opinions. Ratings are more focused on the likelihood of default of an issuer.

The methodological approach used by the rating agencies varies. S&P tends to refer to the likelihood of default. S&P tends to emphasize the strength of the (sub)-industry classification of the firm. Moody’s tends to emphasize business fundamentals and market leadership.

Expected loss (EL) = probability of default \* loss severity.

An S&P rating tends to correspond to a *probability of default* (PD), while a Moody’s rating tends to refer to *expected loss* (EL) where EL = PD \* LGD. The latter has had made its appearance in one form or another as a question on the FRM exam multiple times.

Ratings are divided into *investment grade* and *non-investment grade* issues. In rating long-term debt, each agency uses a system of letter grades that locate an issuer or issue on a spectrum of credit quality from the very highest (triple-A) to the very lowest (D). Traditionally, only long-term ratings have been the focus of the exam. Note that investment grade issues are those that are rated Baa3 or BBB- and *above*, whilst speculative, or non-investment grade, issues are rated Ba1 or BB + and *below*.

Ratings changes impact the price of bond issues. A downgrade has a negative price impact on bonds; whilst an upgrade has a positive impact. Do note that this relationship is asymmetrical: a ratings downgrade will tend to move prices more than an upgrade, on the other hand, the statistical results show that the data on the relationship between ratings changes is stronger for upgrades.

There is concern that internal ratings during economic cycles will lead to over-lending during good times and less lending during bad times, such as a recession. The pro-cyclicality effect therefore refers to the tendency of internal ratings models to (indirectly) reinforce credit and business cycles because they tend to measure credit with *point-in-time-rating*. The credit rating agencies, on the other hand, try to incorporate business cycles. Ratings are therefore typically considered *through-the-cycle*: they try to filter out cycle fluctuations. If economic conditions deviate from past averages, a through-the-cycle rating may over- or under-estimate credit quality, as through-the-cycle ratings are more stable.

A rating transition matrix gives the probability of a state change: usually there are one or two questions regarding transition matrixes. Be sure you understand how to calculate the probabilities of different events from a transition matrix.

A single transition matrix is not time stationary; they are stable within broad homogenous sectors and by geography, but variable *across sectors*. It’s important to note that matrices are *less stable during a recession*.

## Questions and Answers

Questions

Answers

# Ong, Chapter 4: Loan Portfolios and Expected Loss

**Learning Outcomes:**

**Describe** the objectives of measuring credit risk for a bank’s loan portfolio.

**Define, calculate and interpret** the expected loss for an individual credit instrument.

**Distinguish** between loan and bond portfolios.

**Explain** how a credit downgrade or loan default affects the return of a loan.

**Distinguish** between expected and unexpected loss.

**Define** exposures, adjusted exposures, commitments, covenants, and outstandings.

**Explain** how drawn and undrawn portions of a commitment affect exposure.

**Explain** how covenants impact exposures.

**Define** usage given default and how it impacts expected and unexpected loss.

**Explain** the concept of credit optionality.

**Describe** the process of parameterizing credit risk models and its challenges.

## Describe the objectives of measuring credit risk for a bank’s loan portfolio.

Despite innovations (e.g., securitization, credit derivatives), banks must inevitably put highly illiquid assets on the balance sheet

Ong: Illiquid assets are “loan portfolio”

Bank must have a means of quantifying unanticipated change in value of risk assets

Motivated by two main objectives:

Quantify measures of credit risk

Devise risk-adjusted return measures

## Define, calculate and interpret the expected loss for an individual credit instrument.

Expected loss = Assured payment at maturity time T x Loss Given Default (LGD) x Probability that default occurs before maturity T (PD)

However, “Assure payment at maturity time T” should be replaced with “Exposure.”

Therefore, the key formula is given by:

Expected loss = Exposure (at default, EAD) x Loss Given Default (LGD) x Probability of default (PD)

Or, equivalently:

Expected loss = Exposure at default (EAD) x Loss Given Default (LGD) x Expected Default Frequency (EDF)

**Probability of Default (Exp Default Freq.)**



**Expected Loss**

For example:

**Loss Given Default = 1 – Recovery Rate**

**Exposure At Default**

The expected loss is the product of the adjusted exposure (AE), the expected default frequency (EDF), and the loss given default (LGD). From Table 4.2:

|  |  |
| --- | --- |
| Commitment (COM) | $10,000,000 |
| Outstanding (OS) | $5,000,000 |
| Unused commitment | $5,000,000 |
| Rating equivalent | BBB |
| UGD | 65% |
| Adjusted Exposure (AE) | $8,250,000 |
| Prob. of default (PD) | 0.15% |
| Loss given default (LGD) | 50% |
| Expected Loss = (AE)(EDF)(LGD) | $6,188 |

Note about the expected loss (EL):

In the adjusted exposure, the entire $5 million outstanding is included

In the adjusted exposure, the portion of the unused (remaining) commitment that is included is based on UGD. UGD parameterizes the optionality” $3.25 million = (65%)($5 million)

Correlations are not entering into the formula for EL

EL is a linear combination of AE, EDF and LGD

## Distinguish between loan and bond portfolios.

Classifications of long-term debt

Senior Secured Loan

* Most are amortizing and floating rate.
* Often syndicated
* Typically high-yield companies

Senior Unsecured Loan

* Not guaranteed by assets
* Typically investment grade (S&P - BBB or higher)

Bond

* Sold to investors (traded)
* Not amortizing (usually)
* Includes floater & zeroes
* Embedded options (put call)

Medium Term Notes

* Sold in tranches according to maturity (flexible)
* Shelf registration

Convertible Debt

* Lower coupon but compensation is ability to convert into equity
* Typically a bond

Bonds are “trading instruments” where payments are often a stream of cash-flows that includes the promised repayment of principal at maturity. Bonds tend to have fewer complicated indentures, covenant structures and tax and accounting treatments.

In concept, however, “loans are really par bonds whose valuation depends on some loan forward curves imbued with upgrade or downgrade characteristics and a loan recovery rate to the amount of the principal in the event of default.”

In regard to bond portfolios, as they are traded, credit upgrades typically lead to price appreciation.

Summary comparison, loans versus bonds:

Instrument complexity: loans typically more complex (indentures, covenant structures, and tax and accounting) than bonds

Return distributions: loan returns are non-symmetric and with typically no mark-to-marketing valuation gains/losses due to credit quality upgrade/downgrade.

Liquidity: loans are typically less liquid; “because of liquidity constraints, most loans do not have current market prices.”

## Explain how a credit downgrade or loan default affects the return of a loan.

Loan returns are highly non-symmetric because there is no upside potential as with a stock. If the credit quality of a loan improves, the lending bank typically does not benefit from the improvement

If the credit quality of a loan deteriorates, the bank generally is not compensated for taking on the increased risk because the loan pricing does not change.

## Distinguish between expected and unexpected loss.

In regard to expected Loss (EL): A credit portfolio is not risk-free. The yield above the riskless rate is compensation for the risk. A risky asset, by definition has an expected loss (EL):

We expect defaults to be greater than zero, and

Among defaults we expect recovery > 100%

In regard to unexpected Loss (UL): This is the unanticipated loss due to unknown loss distribution. Uncertainty implies the existence of an unexpected loss.

In Ong, unexpected loss (UL) = One standard deviation of the asset value at horizon

Where Ong’s UL is one (1) standard deviation, credit value at risk (Credit VaR) can be expressed as a multiple of standard deviations

**Unexpected Loss (UL) covered by Economic capital**

**Expected Loss (EL) priced into the yield   
(“cost of doing business”) and covered by the loan loss reserve provision**

## Define exposures, adjusted exposures, commitments, covenants, and outstanding.

Assume

* Value of bank asset = V
* Outstanding = OS
* Commitments = COM
* Then V = OS + COM

**Outstanding**: generic term referring to the portion of the bank asset which has already been extended to the borrowers and also to other receivables in the form of contractual payments which are due from customers. Examples of outstanding include term loans, credit cards, and receivables.

**Commitments**: An amount the bank has committed to lend, at the borrower’s request, up to the full amount of the commitment. An example of a commitment is a line of credit (LOC). A commitment consists of two portions:

1. Drawn, or
2. Undrawn

But the drawn commitment should be treated as part of the outstanding (i.e., the amount currently borrowed).

**Covenants**: terms or provisions that attach to the commitments. Covenants are either options the bank reserves to (for) itself or options granted to the obligor.

Convents include:

* A reduction of the maximum percentage of drawdown under the commitment.
* An increase in the seniority of the borrowing.
* An increase in the collateral requirement.
* Re-pricing of the loan

**Exposure**: The assured payment expected by the bank; the outstanding (OS).

Adjusted exposure: the portion of the totality of all exposures that a bank would not be able to recover in the event of default. The adjusted exposure includes the risky assets. Therefore, adjusted exposure includes both the outstanding and some portion of the commitment.

Adjusted exposure = Outstanding (OS) + Unused Commitment (COM)

## Explain how drawn and undrawn portions of a commitment affect exposure

The drawn portion of the commitment should be treated as part of the amount currently borrowed (i.e., the outstanding). Since it is exactly like a term loan, the entire drawn portion is subject to risk of loss on default. The undrawn portion of the commitment, however, has an embedded contingent claim (a call option) that the borrower can exercise at any time.

## Explain how covenants impact exposures.

Once the covenants are in place, the loss given default (LGD) for the exposure should be based strictly on the expected recovery rate assuming an increase in seniority or collateral requirement. This approach assumes the covenants will be initiated and used prior to default.

Distinguish between the influence of risky and risk‐free parts of an exposure

In Ong, the value of the asset at the horizon (V1) is divided into two components: a risky part and a riskless (risk-free) part:



Only the risky part is subject to loss (i.e., the undrawn commitment is not lost!). The adjusted exposure is the risky part of the asset: the outstanding plus the fraction of the commitment (COM) that is likely to be drawn (UGD).

## Define usage given default and how it impacts expected and unexpected loss.

The usage given default (UGD) is the fraction of the commitment that is likely to be drawn in the event of a default. The UGD is a contingent claim (a “credit option”) owned by the obligor; the “option premium” is the commitment fee paid by the obligor to the bank.

Because the expected loss is a direct function of the adjusted exposure, and the adjusted exposure includes the portion of the commitment that is likely to be drawn, an increase in the UGD increases the expected loss.

**COM**

**UGD**

Average UGD for borrowers with different ratings (Ong Table 4.1, source: Asarnow and Marker, 1995):

|  |  |
| --- | --- |
| Rating | Usage Given  Default |
| AAA | 69% |
| AA | 73% |
| A | 71% |
| BBB | 65% |
| BB | 52% |
| B | 48% |
| CCC | 44% |

## Explain the concept of credit optionality.

Credit optionality refers to the idea that the bank has extended a commitment to the borrower. This is a credit option: the borrower (obligor) pays a commitment fee, which is much like an option premium; in return for this fee, the borrower has the option (the right but not the obligation) to draw down on the commitment.

The usage given default (UGD) is analogous to an “exercise” of the option; i.e., the borrower exercises his/her right to “draw down” the commitment.

## Describe the process of parameterizing credit risk models and its challenges.

The necessary ingredients for estimating the expected loss of a single risky asset in a two-state default process are:

* Adjusted exposure: outstanding, commitments, usage given default;
* Loss given default: secured or unsecured
* Expected default frequency (EDF)
* Maturity
* Internal risk class rating

The parameterization of credit risk models can be problematic and cumbersome.

The only parameter needed for the calculation of adjusted exposure is the usage given default (UGD). The UGD, in turn, depends on the risk rating of the facility.

The loss given default (LGD) is equal to (1 – recovery rate). This is a *difficult parameter to estimate*. In practice, the LGD is dependent on the risky asset’s seniority in claim and the collateral guaranteed by the asset.

The probability of default (PD, or EDF) is crucially important. Both Moody’s and Standard and Poor’s publish estimates based on historical compilations, but they are based on publicly held firms.

## Chapter Summary

This chapter presents the concepts of Expected Loss (EL), and it is paramount to understand what it is, how it works and how one may calculate it.

1. Firstly, note that correlations *are not* taken into account when calculating EL.
2. Secondly; EL is a linear combination of AE, EDF and LGD.

This chapter presents a formula that you must understand and remember as it is all but guaranteed to make its appearance on the exam. Knowing this will get you one step closer to passing. It is usually presented in three equivalent forms:

1. Expected loss = Exposure at default (EAD) x Loss Given Default (LGD) x Probability of default (PD).
2. Expected loss = EAD x LGD x Expected Default Frequency (EDF).
3. Expected loss = EAD x (1 – Recovery rate) x EDF.

Note that the Loss Given Default is Equivalent to 1 – the recovery rate.

Expected Loss: A credit portfolio is not risk-free. The yield above the riskless rate is compensation for the risk. A risky asset, by definition has an Expected Loss. Contrast this to Unexpected Loss, which is the unanticipated loss due to unknown loss distribution. Uncertainty implies the existence of an unexpected loss. In Ong, unexpected loss (UL) = One standard deviation of the asset value at horizon.

Bonds are trading instruments. In regard to bond portfolios, as they are traded, credit upgrades typically lead to price appreciation. A typical loan is usually more complex than a bond is (indentures, covenant structures, and tax and accounting). Loan returns have a non-symmetric returns distribution and generally no mark-to-market valuation due to changes in credit quality. Generally speaking, loans are less liquid assets. They must be securitized in order for there to be price discovery as to what the market value of the loan is.

Loan returns are highly non-symmetric because there is no upside potential as with a stock. If the credit quality of a loan improves, the lending bank typically does not benefit from the improvement. However, if the credit quality of a loan deteriorates, the bank generally is not compensated for taking on the increased risk because the loan pricing does not change.

*Adjusted exposure* is the portion of the total of all exposures that a bank would not be able to recover in the event of default. The adjusted exposure includes the risky assets. Therefore, adjusted exposure includes both the outstanding and some portion of the commitment.

Adjusted exposure = Outstanding (OS) + Unused Commitment (COM)

The usage given default (UGD) is the fraction of the commitment that is likely to be drawn in the event of a default. The EL is a direct function of the adjusted exposure, and the adjusted exposure includes the portion of the commitment that is likely to be drawn, an increase in the UGD increases the expected loss.

The loss given default (LGD) is equal to (1 – recovery rate). This is a *difficult parameter to estimate*.

## Questions and Answers

Questions

23.4.1 A bank has a $10 million commitment (COM) of which $6 million is outstanding (OS) and the usage given default (UGD) assumption is 50.0%. The probability of default (PD) is 1.0% and the loss conditional on default (LGD) has a beta distribution with a mean of 70.0% and a standard deviation of 25.0%. The PD and LGD are not independent; rather, PD and LGD are positively correlated. What is the expected loss (EL) of the adjusted exposure (AE)?

1. Less than $56,000
2. $56,000
3. More than $56,000
4. $112,000

23.4.2 Assume a bank sets the contractually promised gross return on a loan (k) according to the following formula which equates the expected (net) return on the loan equal to the risk-free rate of return, per a risk-neutral assumption: PD\*RR + (1-PD)\*(1+k) = 1+Rf, where PD=probability of default, RR = recovery rate, k = promised gross return, and Rf = risk-free rate. If the risk-free rate (Rf) is 3.0%, the probability of default (PD) is 4.0%, and the loss given default (LGD) is 75.0%, what is the risk-neutral promised gross return on the loan (k)?

1. 3.00%
2. 3.75%
3. 5.75%
4. 6.25%

23.4.3 Among the variables or parameters necessary to the estimation of adjusted exposure (AE), which is the most difficult to parameterize?

1. Outstanding (OS)
2. Commitment (COM)
3. Usage given default (UGD)
4. Loss given default (LGD)

Answers

23.4.1 C. More than $56,000, as $56,000 is the EL under independence between PD and LGD.

Typically, we do assume independence between PD and LGD such that EL = PD\*LGD. In which case, the problem is straightforward:

Adjusted exposure (AE) = $6 million OS + ($4 million unused COM \* 50% UGD) = $8 million.

EL (assuming independence between PD & LGD) = $8 million \* 1.0% PD \* 70% LGD = $56,000.

However, with positive correlation the EL must be greater.

23.4.2 D. 6.25%

Since PD\*RR + (1-PD)\*(1+k) = 1+Rf,

k = [(1+Rf) - PD\*(1-LGD)]/(1-PD) - 1. In this case, k = [(1+3%) - 4%\*(1-75%)]/(1-4%) - 1 = 6.25%

23.4.3 C. AE = OS + (UGD \* COM), where OS and COM are observed but UGD, like LGD, is difficult to parameterize (and stochastic).

In regard to (D), LGD is not required for AE, but informs expected loss (EL).

# Ong, Chapter 5: Unexpected Loss

**Learning Outcomes:**

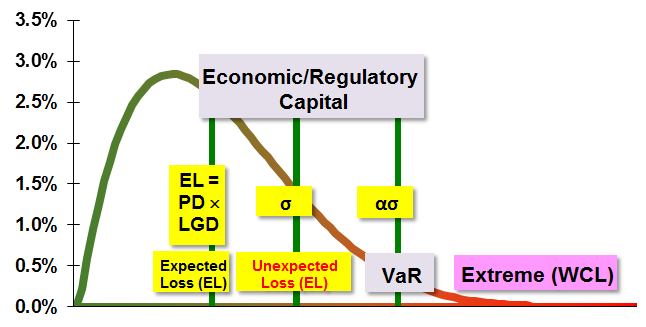
**Explain** the objective for quantifying both expected and unexpected loss.

**Describe** factors contributing to expected and unexpected loss.

**Define, calculate and interpret** the unexpected loss of an asset.

**Explain** the relationship between economic capital, expected loss and unexpected loss.

EC (α) = VaR (α) - EL



In the typical loss distribution pictured above:

Expected losses are the mean/median of the distribution. Expected losses are covered by loan loss reserves, not economic capital (although in Basel II, if reserves are insufficient, capital must make up the difference)

Ong’s unexpected loss is one standard deviation

Internal unexpected loss, as a VaR-type concept, will be “further to the right” (greater losses); i.e., some multiple of Ong’s unexpected loss

Basel’s IRB unexpected loss, similarly, also exceeds Ong’s unexpected loss because its confidence is higher.

All of the unexpected losses (e.g., Ong’s, internal, regulatory) share in common: they are, like value at risk (VaR), some multiple of standard deviation where the multiple is a function of confidence and time horizon

How capital relates to loss distribution:

* Economic capital absorbs unexpected losses, up to a certain point, depending on the desired confidence level.
* The confidence level is a policy decision that should be set by senior management and endorsed by the board.

* Economic capital is most relevant to shareholders.

* Reserves are set aside for expected losses; e.g., priced into higher yields.

* Economic capital does not cover expected losses; economic capital is meant to absorb unexpected losses.
* Regulatory capital is rule-based (e.g., BIS 88, BIS 98) with intent to ensure enough capital is in the banking system.

* Most banks hold more capital than required by regulators.

## Explain the objective for quantifying both expected and unexpected loss.

Because unexpected loss is the estimated volatility of potential loss in value of the asset (around its expected value), it is imperative that the bank put aside sufficient capital to sustain the uncertain loss

Required capital reserve acts as “a buffer against insolvency”

Assume EL is already covered; e.g., Basel II charges capital if EL is not covered

Expected loss

Banks expect to bear. Covered by loan loss reserve.

Unexpected loss

Uncertainty implies unanticipated losses. Two primary sources of unanticipated risk:

1. Default
2. Unexpected Credit Migration

## Describe factors contributing to expected and unexpected loss.

Factors contributing to expected loss:

* Banks expect to bear. Covered by loan loss reserve.

Factors contributing to unexpected loss:

* Uncertainty implies unanticipated losses
* Market conditions (business cycle)

Two primary sources of unanticipated risk: Default and Unexpected Credit Migration

## Define, calculate and interpret the unexpected loss of an asset.

Unexpected loss (UL) = Standard Deviation of unconditional value of the asset at horizon.

Unexpected loss (UL) is given by:



Where the variance of the default frequency (EDF) is given by:



Note: the variance of loss given default (LGD), unlike the variance of EDF, is non-trivial. You are not expected to know how to calculate the variance of LGD. Unexpected loss (UL) is average loss bank can expect (to lose on its asset) over the specified horizon. From Ong’s Table 5.1 (calculation of unexpected loss):

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Unexpected Loss: |  |  | Rating | UGD |
| Commitment (COM) | $10,000,000 |  | AAA | 69% |
| Outstanding (OS) | $5,000,000 |  | AA | 73% |
| Unused commitment | $5,000,000 |  | A | 71% |
| Rating equivalent | BBB |  | BBB | 65% |
| UGD | 65% |  | BB | 52% |
| Adjusted Exposure (AE) | $8,250,000 |  | B | 48% |
| Probability of default (EDF) | 0.15% |  | CCC | 44% |
| Standard Deviation of EDF | 3.87% |  |
| Loss given default (LGD) | 50% |  |
| Standard Deviation of LGD | 25% |  |
| Expected Loss = (AE)(EDF)(LGD) | $6,188 |  |
| Unexpected Loss | $178,511 |  |

Note the following:

* The standard deviation of EDF = SQRT[(EDF)(1-EDF)]
* The standard deviation of LGD is given as an input (not solved, being non-trivial)
* Unexpected loss = SQRT[(EDF)(variance of LGD) + (LGD^2)(Variance of EDF)

## Explain relationship between economic capital, expected loss and unexpected loss.

Economic capital is the capital reserve (buffer), which protects against insolvency. Unexpected loss is the estimated volatility of potential loss in the asset.

Economic capital = Function of [UL]

However, recall from earlier, because unexpected loss is a function of desired confidence levels, there are several (infinite) levels of economic capital:

Here in Ong, economic capital = Function [UL at one standard deviation]

Internal economic capital = Function [UL at internal confidence]

Regulatory capital under Basel II = Function [UL at 99%/99.9%]

Derive, mathematically, the unexpected loss on an asset.

*The mathematical derivation of unexpected loss will not be tested.*



## Chapter Summary

All of the *unexpected losses* (e.g., Ong’s, internal, regulatory) share one thing in common: they are, like Value at Risk, some multiple of the standard deviation, where the multiple is a function of confidence and the time horizon.

*Economic capital* absorbs unexpected losses, up to a certain point, depending on the desired confidence level, where the confidence level to a large degree is subjective. Separate reserves are set aside for *expected losses*, this can be in the form of higher yields from the vantage point of a counterparty. Economic capital does *not cover expected losses*; economic capital is *meant to absorb unexpected losses*.

Because unexpected loss is the estimated volatility of potential loss in value of the asset, banks must put aside sufficient capital to sustain the uncertain loss. Required capital reserve acts as “a buffer against insolvency,” though most bank have more than what the regulatory minimum stipulates. The required regulatory capital is rarely changed as it has deep implications on the transmission mechanism of the monetary system and the value of the currency. Uncertainty implies unanticipated losses. The primary sources of unanticipated risk are default and Unexpected Credit Migration

For the FRM exam, it is important that you know the formulas for both unexpected loss, and for the variance of the Expected Default Frequency, as it is a common question. Ong’s chapters on Expected and Unexpected Loss have been in the curriculum for a while now, and it would come as no surprise if you were asked to apply the formulas for expected and unexpected loss, as well as the variance of the EDF. The formula for Unexpected Loss is:



and the variance of the Expected Default Frequency is:



## Questions and Answers

Questions

23.5.1 An exposure has a default probability (PD) of 4.0% and loss given default of 50.0%. The standard deviation of the LGD is 25.0%. What is the ratio of the unexpected loss to the expected loss, UL/EL?

1. 1.33
2. 3.72
3. 5.50
4. 9.64

23.5.2 In the assigned reading on unexpected loss (Ong Chapter 5), unexpected loss (UL) is given as: UL = AE \* SQRT[EDF\*variance(LGD) + LGD^2\*variance(EDF)]. Each of the following is TRUE about this definition of unexpected loss (UL) EXCEPT:

1. It assumes independence (zero default correlation) between the default probability and loss given default (LGD)
2. Economic capital will necessarily equal this value of this unexpected loss, as defined; i.e., EC = UL
3. This unexpected loss (UL) is the standard deviation (volatility) of the unconditional value of the asset at the horizon
4. Whereas expected loss (EL) increases as a linear function of EDF and LGD, unexpected loss (UL) increases as a non-linear function of EDF and LGD

Answers

23.5.1 C. 5.50

Unexpected loss (%) = SQRT[EDF \* variance(LGD) + LGD^2 \* variance(EDF)] = SQRT[4%\*25%^2 + 50%^2\*4%\*96%] = 11.00%

Expected loss (%) = EDF\*LGD = 4%\*50% = 2.0%.

Ratio of UL/EL = 11.0%/2.0% = 5.50

23.5.2 B. This definition of UL is only ONE standard deviation (aka, volatility); as such, it corresponds to a relatively low confidence level. Economic capital, of any practical confidence, will need to be some multiple of this UL. In unassigned chapter 8, Ong offers EC = CM \* UL, where CM is a "capital multiplier."

In regard to (A), (C), and (D), each is TRUE.

In regard to (A), Ong: *"Assumptions: Explicit in the derivation of unexpected loss above is the assumption that the random risk factors contributing to an obligor's default (resulting in the default probability, EDF) are statistically independent of the severity of loss (as given by the loss given default, LGD). The reader is referred to Appendix A at the end of the chapter for clarification on this point. If the risk factors contributing to the expected default frequency and loss given default were not independent, the multiplier in equation (5.4b) would contain covariance cross-terms owing to the nature of the variance calculation. However, in practice it is not clear whether or not the assumption of statistical independence is well justified. Our conjecture is that statistical dependence would require only a small corrective modification to the expression for unexpected loss."*

In regard to (C), Ong: UNEXPECTED LOSS: “*The expected loss of an asset is the average loss the bank can expect to lose on its asset over the period up to a specified horizon; during that time the asset can fluctuate in value due to the two main sources of unanticipated risk mentioned above [i.e., DEFAULT and unexpected credit risk MIGRATION]. The risk at the horizon can be conveniently measured using the standard deviation of the value at the horizon. We shall call this quantity the unexpected loss. In other words, unexpected loss is the estimated volatility of the potential loss in value of the asset around its expected loss.*

*As defined, the unexpected loss, UL(H), of the asset value V(H) at the horizon t(H) is simply the standard deviation of the unconditional value of the asset at the horizon.”*

In regard to (D), non-linearity of UL can be inferred from the formula.

# Dowd, Chapter 2: Measures of Financial Risk

**Learning Outcomes:**

**Describe** the mean-variance framework and the efficient frontier.

**Explain** the limitations of the mean- variance framework with respect to assumptions about the return distributions.

**Define** the Value-at-risk (VaR) measure of risk, discuss assumptions about return distributions and holding period, and explain the limitations of VaR.

**Define** the properties of a coherent risk measure and explain the meaning of each property.

**Explain** why VaR is not a coherent risk measure.

**Explain and calculate** expected shortfall (ES), and compare and contrast VaR and ES.

**Describe** spectral risk measures and explain how VaR and ES are special cases of spectral risk measures.

**Describe** how the results of scenario analysis can be interpreted as coherent risk measures.

## Describe the mean-variance framework and the efficient frontier.

In the mean-variance framework, we model financial risk in terms of the mean and variance (or standard deviation, as the square root of the variance) of P/L (or returns). As a related convenience, we assume the daily profit and loss (P/L) or returns obey a normal distribution. Please note that, by specifying only the first two moments (mean and variance) we implicitly suggest a normal distribution; e.g., a normal does not require a third (skew) or fourth (kurtosis) moment specification.

As Dowd explains: “A related attraction of particular importance is that the normal distribution requires only two parameters – the mean and the standard deviation (or variance), and these parameters have ready financial interpretations: the mean is the expected return on a position, and the standard deviation can be interpreted as the risk associated with that position. This latter point is perhaps the key characteristic of the mean–variance framework: it tells us that we can use the standard deviation (or some function of it, such as the variance) as our measure of risk. And conversely, the use of the standard deviation as our risk measure indicates that we are buying into the assumptions normality or, more generally, elliptically–on which that framework is built.”

The efficient frontier refers either to the universe without the risk-free asset or with the risk-free asset. Before the introduction of the risk-free asset, the efficient frontier refers to the combination (allocation) of risky assets—which includes the market portfolio—that are “superior.” Specifically, the investor generally prefers higher returns and lower variance/standard deviation (i.e., the investor is, to some degree, risk averse). The efficient frontier is the set of points for which we cannot find an obvious improvement: a point is “efficient” if any increase in portfolio return implies an increase in risk (i.e., a trade-off). The points on the lower (red-ish) segment below are inefficient because they are vertically inferior to points with equivalent risk and higher return: we can improve the risk without sacrificing risk.

Then, if we add the risk-free asset, we can draw a line segment from the risk-free rate (on the y-axis) that is tangent to the curved “efficient” segment and contacts the formerly efficient frontier segment exactly at the market portfolio (and, because it is a tangency line, only overlaps at the market portfolio). The new, straight capital market line (CML) becomes the efficient frontier in the presence of the risk-free rate.

## Explain the limitations of the mean- variance framework with respect to assumptions about the return distributions.

The normality assumption (implied by mean-variance framework) is only appropriate if we are dealing with a symmetric (i.e., zero-skew) distribution that also has “normal” tails (i.e., kurtosis = 3).

If our distribution is skewed or has heavier tails – as is typically the case with financial returns – then the normality assumption is inappropriate and the mean–variance framework can produce misleading estimates of risk.

## Define the Value-at-risk (VaR) measure of risk, discuss assumptions about return distributions and holding period, and explain the limitations of VaR.

In order to be specific about our VaR, we need to specify (i) a confidence level (α), which indicates the likelihood that we will get an outcome no worse than our VaR, and which might be any value between 0 and 1; and (ii) a holding or horizon period, which is the period of time until we measure our portfolio profit or loss, and which might be a day, a week, a month, or whatever.

Given a confidence level (α) then p = 1 – α and if qp is the p-quantile for a portfolio’s prospective profit/loss (P/L) over some holding period, then the VaR of the portfolio at that confidence level and holding period is equal to:



VaR is unambiguously defined when dealing with a continuous P/L distribution. However, the VaR can be ambiguous when the P/L distribution is discontinuous.

VaR varies with the holding period, and the way it varies with the holding period depends significantly on the mean parameter (µ).

In regard to the holding period:

Dowd says “the usual holding periods are one day or one month”

The holding period can also depend on the liquidity of the market; “the ideal holding period appropriate in any given market is the length of time it takes to ensure orderly liquidation of positions in that market.”

The holding period might also be specified by regulation; e.g., 10 business days for BIS capital adequacy (market risk)

The choice of holding period depends on two other factors:

The assumption that the portfolio does not change over the holding period is more easily defended with a shorter holding period.

A short holding period is preferable for model validation or backtesting purposes: reliable validation requires a large dataset, and a large dataset requires a short holding period.

Limitations of VaR as a Risk Measure

VaR only tells us the most we can lose if a tail event does not occur (e.g., it tells us the most we can lose 95% of the time); if a tail event does occur, we can expect to lose more than the VaR, but the VaR itself gives us no indication of how much that might be. The failure of VaR to take account of the magnitude of losses in excess of itself implies that two positions can have the same VaR—and therefore appear to have the same risk if we use the VaR to measure risk—and yet have very different risk exposures. This can lead to an undesirable outcome: it can encourage high-return, high-risk trades when the higher loss does not impact the VaR.

If the VaR can lead an investor working on his/her own behalf to make perverse decisions, it creates even more scope for problems when there are principal–agent (or delegation) issues. This would be the case where decision-making is decentralized and traders or asset managers work to VaR-defined risk targets or remuneration packages. The classic example is where traders who face a VaR-defined risk target have an incentive to sell out-of-the-money options that lead to higher income in most states of the world and the occasional large hit when the firm is unlucky. If the options are suitably chosen, the bad outcomes will have probabilities low enough to ensure that there is no effect on the VaR, and the trader benefits from the higher income earned in “normal” times with the options expire out of the money.

Finally, VaR is not sub-additive (see below), which is a genuinely practical problem. Also, since VaR is not sub-additive, VaR is not coherent.

## Define the properties of a coherent risk measure and explain the meaning of each property.

The risk measure rho (.) is coherent if it satisfies the following four (4) properties:

Monotonicity: Y ≥ X ⇒ ρ(Y ) ≤ ρ(X). A random cash flow or future value Y that is always greater than X should have a lower risk: this makes sense, because it means that less has to be added to Y than to X to make it acceptable, and the amount to be added is the risk measure

Sub-additivity: ρ(X + Y ) ≤ ρ(X) + ρ(Y ).

Positive homogeneity: ρ(hX) = hρ(X)for h > 0. The risk of a position is proportional to its scale or size, and makes sense if we are dealing with liquid positions in marketable instruments.

Translational invariance: ρ(X + n) = ρ(X) − n for some certain amount n. Requires that the addition of a sure amount reduces *pari passu* the cash needed to make our position acceptable, and is obviously valid when one appreciates that the cash needed is our risk measure.

## Explain why VaR is not a coherent risk measure.

VaR is not coherent because we can identify circumstances under which the risk metric effectively penalizes risk; i.e., circumstances under which the risk of a portfolio is greater than the sum of the risks of constituent portfolios (a nonsensical outcome; at worst, we expect the portfolio risk to equal the sum of risks).

The following simple example employs Dowd’s illustration to demonstrate that VaR is not sub-additive. Consider three bonds, each with an identical probability of default (PD) equal to 2%. The 95% VaR for a single bond is zero; that is because the 5% tail has two sections, the 2% default tail and the other 3% in the no-default area, such that the 95% VaR starts “before” the default.

Now combine three of these identical bonds into a portfolio, and assume default independence. The 95% VaR is now $100 (i.e., one default out of three). Strangely, the combination of three bonds, each with VaR = zero, results in a portfolio with VaR = $100:

|  |  |  |
| --- | --- | --- |
| Bond either defaults or does not | | |
| (payoff = 1 or 0). Assume recovery rate = 0% | | |
|  |  |  |
|  | Default (PD) | No Default |
| Probability | 2% | 98% |
| Payoff | 0 | 1 |
| Face value: | $100 |  |
|  | Confidence | 95% |
| # Bonds | No. of | Value at |
| in Port | Defaults | Risk (VaR) |
| 1 | 0 | $0.00 |
| 2 | 0 | $0.00 |
| 3 | 1 | $100.00 |

Is VaR’s subadditive failure just academic, or is it a real problem?

It is a real problem, according to Dowd: “Non-subadditivity is treacherous because it suggests that diversification might be a bad thing, which would suggest the laughable conclusion that putting all your eggs into one basket might be good risk management practice!

Non-subadditive risk measures can tempt agents trading on an exchange to break up their accounts, with separate accounts for separate risks, in order to reduce margin requirements.

If regulators use non-subadditive risk measures to set capital requirements, then a financial ﬁrm might be tempted to break itself up to reduce its regulatory capital requirements.

If risks are subadditive, adding risks together would give us an overestimate of combined risk, and this means that we can use the sum of risks as a conservative estimate of combined risk. This facilitates decentralized decision-making within a firm, because a supervisor can always use the sum of the risks of the units reporting to him or her as a conservative back-of-the-envelope risk measure. But if risks are not subadditive, adding them together gives us an underestimate of combined risks, which makes the sum of risks treacherous and therefore effectively useless as a back-of-the-envelope measure.

## Explain and calculate expected shortfall (ES), and compare and contrast VaR and ES.

Expected shortfall (ES) is the average of the worst 100\*(1-α)% of losses. For discrete distribution:



For continuous distribution, ES is given by:



In the following example, we again assume that each bond has PD = 2%. The one-bond portfolio returns and 95% expected shortfall (ES) of 0.4 because, assuming default = 0 and no default = 1, [2% \* 1 + (5% - 2%) \* 0] / 5% = 0.4; i.e., conditional on the 5%, the expected value = 0.4.

Consider a two-bond portfolio (each PD = 2%, no default dependence). The expected shortfall (ES) is given by: (0 defaults \* 1.04% + 1 default \* 3.92% + 2 defaults \* 0.04%) / 5% = 0.8.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Expected Shortfall (ES), E[X | X > Q] | | |  |  |
| # Bonds in Port |  | ES @ 1 - 5.00% |  |  |
| 1 |  | 0.4000 |  |  |
| 2 |  | 0.8000 |  |  |
| 3 |  | 1.0238 |  |  |
| Getting the 2-bond ES | |  |  |  |
| # of Defaults | PDF | CDF |  | Worst % |
| 0 | 0.9604 | 0.9604 |  | 1.0400% |
| 1 | 0.0392 | 0.9996 |  | 3.9200% |
| 2 | 0.0004 | 1.0000 |  | 0.0400% |
|  |  |  |  | 5.00% |
| Getting the 3-bond ES | |  |  |  |
| # of Defaults | PDF | CDF |  | Worst % |
| 0 | 0.9412 | 0.9412 |  |  |
| 1 | 0.0576 | 0.9988 |  | 4.8816% |
| 2 | 0.0012 | 1.0000 |  | 0.1176% |
| 3 | 0.0000 | 1.0000 |  | 0.0008% |
|  |  |  |  | 5.00% |

Expected shortfall (ES) has many of the same uses as VaR. However, Dowd says ES is a better risk measure than VaR for several reasons:

Unlike VaR, ES gives expected loss in bad states

Unlike VaR, an ES-based risk-expected return decision rule is valid under general conditions

Unlike VaR, ES is coherent (which implies that it is sub-additive)

The sub-additivity of ES implies that the portfolio risk surface will be convex. Convexity ensures that portfolio optimization problems using ES measures, unlike VaR measures, will always have a unique well-behaved optimum

## Describe spectral risk measures and explain how VaR and ES are special cases of spectral risk measures.

Both VaR and expected shortfall (ES) are special cases of a spectral risk measures. A spectral risk measure is given by:



Where the weighting function is:

* Non-negative
* Normalized (i.e., weights sum to 1.0)
* Weakly increasing

Recalling that alpha (α) is our confidence level (only here in Dowd!), ES is a special case of the spectral measure above:



VaR is also a special case because VaR is a single quantile:



## Describe how the results of scenario analysis can be interpreted as coherent risk measures.

Suppose we consider a set of loss outcomes combined with a set of associated probabilities. The losses can be regarded as tail drawings from the relevant distribution function, and their expected (or average) value is the ES associated with this distribution function.

Since the ES is a coherent risk measure, this means that the outcomes of scenario analyses are also coherent risk measures.

## Questions and Answers

Questions

24.2.1 With respect to the mean-variance framework, Dowd asserts EACH of the following as true EXCEPT for:

1. The mean-variance framework assumes that standard deviation (or variance, the second central moment) is the primary risk measure
2. If a distribution has a mean of zero (0) and a variance of 1.0, it must be the standard normal distribution
3. The family of elliptical distributions includes the normal as a special case
4. Levy distributions are stable, and the normal is stable because it is a Levy (with alpha parameter = 2.0)

24.2.2 Your colleague reports a 95.0% one-day value-at-risk (VaR) of $1.4 million for a equities portfolio. If we assume 250 trading days in a year, each of the following is a valid conclusion EXCEPT which of the following is FALSE (cannot be concluded from the statement)?

1. If the VaR is accurate, we expect a daily loss in excess of $1.4 million to occur on about 12 or 13 days (12.5) during year
2. If the return distribution is normal, then we can assume the VaR is sub-additive
3. This is a parametric VaR and therefore cannot characterize a heavy-tailed distribution
4. If the returns are i.i.d. normal, we can scale to a 10-day VaR with $1.4\*SQRT(10) = $4.3 million 95% 10-day VaR

24.2.3 A bond with a face value of $10.0 million has a one-year probability of default (PD) of 1.0% and an expected recovery rate of 35.0%. What is the bond's one-year 99.0% expected shortfall?

1. $3.25 million
2. $6.5 million
3. $9.1 million
4. Not enough information: need the tail distribution

24.2.4 Each of the following is true about spectral risk measures EXCEPT for:

1. Both value at risk (VaR) and expected shortfall (ES) are spectral measures
2. Neither VaR nor ES are coherent
3. Expected shortfall gives all weights in the tail region an equal weight, which implies the user is risk-neutral between tail-region outcomes
4. According to Dowd, the best risk measure would reflect users' risk aversion and therefore employ a weighting function that assigns higher weights to greater losses, but neither VaR nor ES reflect this risk-aversion

Answers

24.2.1 B. False: if it is a normal distribution (i.i.f. is has skew = 0 and kurtosis 3.0) with mean of zero and variance of 1.0, then it is a standard normal. But Dowd illustrations are meant to show that a normal distribution can have the same mean and variance as non-normal distributions (e.g., skewed Gumbel in Figure 2.4 and heavy-tailed student's t distribution i Figure 2.5).

In regard to (A), (C) and (D), each is TRUE.

Dowd A2.8: "There are also other properties that we sometimes look for in the distributions we choose. One of these is the property of stability: a pdf is stable if the sum of two or more random variables drawn from that distribution is distributed according to the same distribution. It is clearly helpful in practice to have stability so we can avoid situations where adding random variables leads to a sum with a different distribution. However, the only distributions that are stable are Levy ones, for which reason the Levy distributions are sometimes also known as stable Levy distributions. Thus, the normal is stable because it is a Levy with a = 2, and a Cauchy (or Lorentzian) distribution is stable because it is a Levy with a = 1, but a distribution such as the 't' is not stable except in the special case where it has 1 degree of freedom and is then equivalent to a Cauchy."

24.2.2 C. The distributional assumption is not indicated. It may be empirical or parametric non-normal and therefore may characterize heavy-tails.

In regard to (A), (B) and (D), each is TRUE.

24.2.3 B. $6.5 million

As expected shortfall (ES) is the expected loss conditional on exceeding the VaR, and the VaR significance coincides with the PD, the ES is the expected (average) loss conditional on default, which is 1-recovery rate = 65% \* $10 million = $6.5 million.

24.2.4 B. Expected shortfall (ES) is coherent

In regard to (A), (C) and (D), each is true.

# Hull, Chapter 18: Operational Risk

**Learning Outcomes:**

**Calculate** the regulatory capital using the basic indicator approach and the standardized approach.

**Explain** the Basel Committee's requirements for the advanced measurement approach (AMA) and their seven categories of operational risk.

**Explain** how to get a loss distribution from the loss frequency distribution and the loss severity distribution using Monte Carlo simulations.

**Describe** the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions.

**Describe** how to use scenario analysis in instances when there is scarce data.

**Describe** how to identify causal relationships and how to use risk and control self assessment (RCSA) and key risk indicators (KRIs) to measure and manage operational risks.

**Describe** the allocation of operational risk capital and the use of scorecards.

**Explain** how to use the power law to measure operational risk.

**Explain** the risks of moral hazard and adverse selection when using insurance to mitigate operational risks.

## Calculate the regulatory capital using the basic indicator approach and the standardized approach.

In the Basic Indicator Approach (BIA), banks must hold capital for operational risk equal to a fixed percentage (currently 15%) of positive annual gross income (GI; GI = net interest income plus noninterest income) over the previous three years:



In the standardized approach (SA), activities divided into eight business lines:

|  |  |
| --- | --- |
| Corporate finance,  Trading and sales,  Retail banking,  Commercial banking | Payment and settlement,  Agency services,  Asset management, and  Retail brokerage |

Within each business line, gross income is a proxy for scale. Capital charge is gross income of business line multiplied by a factor (called beta).

The total capital charge is calculated as the three-year average of the simple summation of the regulatory capital charges across each of the business lines in each year.



## Explain the Basel Committee's requirements for the advanced measurement approach (AMA) and their seven categories of operational risk.

The Basel Committee has listed conditions that a bank must satisfy in order to use the standardized approach or the AMA approach. It expects large internationally active banks to move toward adopting the AMA approach through time. To use the standardized approach a bank must satisfy the following conditions:

The bank must have an operational risk management function that is responsible for identifying, assessing, monitoring, and controlling operational risk.

The bank must keep track of relevant losses by business line and must create incentives for the improvement of operational risk.

There must be regular reporting of operational risk losses throughout the bank.

The bank's operational risk management system must be well documented.

The bank's operational risk management processes and assessment system must be subject to regular independent reviews by internal auditors. It must also be subject to regular review by external auditors or supervisors or both.

To use the AMA approach, the bank must satisfy additional requirements

It must be able to estimate unexpected losses based on an analysis of relevant internal and external data, and scenario analyses.

The bank's system must be capable of allocating economic capital for operational risk across business lines in a way that creates incentives for the business lines to improve operational risk management.

The Committee identified seven categories of operational risk

Internal fraud: Acts of a type intended to defraud, misappropriate property or circumvent regulations, the law, or company policy (excluding diversity or discrimination events which involve at least one internal party). Examples include intentional misreporting of positions, employee theft, and insider trading on an employee's own account.

External fraud: Acts by third party of a type intended to defraud, misappropriate property or circumvent the law. Examples include robbery, forgery, check kiting, and damage from computer hacking.

Employment practices and workplace safety: Acts inconsistent with employment, health or safety laws or agreements, or which result in payment of personal injury claims, or claims relating to diversity or discrimination issues. Examples include workers compensation claims, violation of employee health and safety rules, organized labor activities, discrimination claims, and general liability (e.g., a customer slipping and falling at a branch office).

Clients, products, and business practices: Unintentional or negligent failure to meet a professional obligation to specific clients and the use of inappropriate products or business practices. Examples are fiduciary breaches, misuse of confidential customer information, improper trading activities on the bank's account, money laundering, and the sale of unauthorized products.

Damage to physical assets: Loss or damage to physical assets from natural disasters or other events. Examples include terrorism, vandalism, earthquakes, fires, and floods.

Business disruption and system failures: Disruption of business or system failures. Examples include hardware and software failures, telecommunication problems, and utility outage.

Execution, delivery, and process management: Failed transaction processing or process management, and disputes with trade counterparties and vendors. Examples include data entry errors, collateral management failures, incomplete legal documentation, unapproved access given to clients accounts, non-client counterparty misperformance, and vendor disputes.

Banks must estimate VaR for each of the 7 x 8 = 56 combinations of risk types and business lines.

## Explain how to get a loss distribution from the loss frequency distribution and the loss severity distribution using Monte Carlo simulations.

The loss frequency distribution is the distribution of the number of losses observed during the time horizon (typically one year). The loss severity distribution is the distribution of the size of a loss, given that a loss occurs. It is typically assumed that loss severity and loss frequency are independent

For loss frequency, a common probability distribution is the Poisson distribution:



For the loss severity distribution, a lognormal probability distribution is often uses.

The frequency and severity distributions must be combined; Monte Carlo simulation can be used for this purpose. For each simulation trial, we proceed as follows:

We sample from the frequency distribution to determine the number of loss events (=n)

We sample n times from the loss severity distribution to determine the loss experienced for each loss events (L1, L2, … Ln)

We determine the total loss experienced (= L1 + L2 + … Ln)

## Describe the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions.

The key data issue is the fact that relatively little data exist that is highly relevant.

According to Hull, the loss frequency distribution should be estimated from the bank’s own data as far as possible. In regard to the loss severity data, regulators encourage banks to use their own data in conjunction with external data. There are two sources of external data: data obtained through sharing arrangements between banks; and publicly available data collected by third-party vendors.

Both internal and external historical data must be adjusted for inflation.

## Describe how to use scenario analysis in instances when there is scarce data.

Relevant historical data is difficult to obtain, so regulators encourage banks to use scenario analysis, in addition to internal and external loss data. This involves managerial judgment to generate scenarios where large losses occur. Managers estimate the loss frequency parameter lambda (λ) associated with each scenario and the parameters of the loss severity distribution.

The advantage of scenario analysis is that it contemplates losses that the financial institution has never experienced, but in the judgment of management could occur.

Another advantage of scenario analysis is that it incents management to think actively and creatively about potential adverse events.

The key drawback of scenario analysis, says Hull, is that it requires a great deal of senior management time.

## Describe how to identify causal relationships and how to use risk and control self assessment (RCSA) and key risk indicators (KRIs) to measure and manage operational risks.

Risk control and self-assessment (RCSA) involves asking business unit managers to identify their operational risks. Sometimes questionnaires designed by senior managers are used.

Risk indicators are key tools in the management of operational risk. The most important indicators are prospective. They provide an early-warning system to track the level of operational risk in the organization. Examples of key risk indicators are staff turnover and number of failed transactions.

## Describe the allocation of operational risk capital and the use of scorecards.

Some banks use scorecard approaches to allocated operational risk capital. Experts identify the key determinants of each risk type and then formulate questions for business unit managers to enable the quantification of risk levels. Examples of such questions include: what is the number of sensitive positions filled by temps? What is the ratio of supervisors to staff?

Scores are assigned to the answers. The total score for a particular business unit indicates the amount of risk present in the business unit and can be uses as a basis for allocating capital to the business unit. The scores given by a scorecard approach should be validated by comparing scores with actual loss experience whenever possible.

## Explain how to use the power law to measure operational risk.

The power law stats that for a wide range of variables:



Where (v) is the value of the variable, (x) is the relatively large value of (v), and K and alpha (α) are constants. According to a study by De Fountnouvelle the power law holds well for large losses experience by banks. This makes the calculation of VaR with high degrees of confidence (e.g., 99%) possible. Internal or external loss data is used to estimate the power law parameters using the maximum likelihood estimation (MLE) approach.

When loss distributions are aggregated, the distribution with the heaviest tails tends to dominate. This means that the loss with the lowest alpha defines the extreme tails of the total loss distribution.

## Explain the risks of moral hazard and adverse selection when using insurance to mitigate operational risks.

Moral hazard is the risk that the existence of the insurance contract will cause the bank to behave differently than it otherwise would. This changed behavior increases the risks to the insurance company. Insurance companies typically deal with moral hazard in several ways:

* Deductible: bank is responsible for bearing the first portion of the loss
* Coinsurance provision: insurance company pays a predetermined percentage (< 100%) of losses in excess of the deductible.
* Policy limit: limit on the total liability of insurer

Adverse selection is when an insurance company cannot distinguish between good and bad risks; consequently, it offers the same price to everyone and inadvertently attracts more of the bad risks. For example, banks without good internal controls are more likely to enter into rogue trader insurance contracts; banks without good internal controls are more likely to buy insurance policies to protect themselves against external fraud.

To overcome adverse selection problem, an insurance company must try to understand the controls that exist within banks and the losses that have been experienced. As a result of such an assessment, it may not charge the same premium to all banks.

## Chapter Summary

In the Basic Indicator Approach (BIA), banks must hold capital for operational risk equal to a fixed percentage (currently 15%) of positive annual gross income (GI; GI = net interest income plus noninterest income) over the previous three years. In the standardized approach (SA), activities divided into eight business lines:

|  |  |
| --- | --- |
| Corporate finance,  Trading and sales,  Retail banking,  Commercial banking | Payment and settlement,  Agency services,  Asset management, and  Retail brokerage |

Within each business line, gross income is a proxy for scale. Capital charge is gross income of business line multiplied by a factor (called beta). The total capital charge is calculated as the three-year average of the simple summation of the regulatory capital charges across each of the business lines in each year. The Basel Committee has listed conditions that a bank must satisfy in order to use the standardized approach or the AMA approach. It expects large internationally active banks to move toward adopting the AMA approach through time. To use the standardized approach a bank must satisfy a number of criteria.

The loss frequency distribution is the distribution of the number of losses observed during the time horizon (typically one year). The loss severity distribution is the distribution of the size of a loss, given that a loss occurs. It is typically assumed that loss severity and loss frequency are independent. For loss frequency, a common probability distribution is the Poisson distribution, while the lognormal probability distribution is often used for the loss severity distribution.

Risk control and self-assessment (RCSA) involves asking business unit managers to identify their operational risks. Risk indicators are key tools in the management of operational risk. The most important indicators are prospective ones. Some banks use scorecard approaches to allocated operational risk capital. Experts identify the key determinants of each risk type and then formulate questions for business unit managers to enable the quantification of risk levels.

Moral hazard is the risk that the existence of the insurance contract will cause the bank to behave differently than it otherwise would. Adverse selection is when an insurance company cannot distinguish between good and bad risks; consequently, it offers the same price to everyone and inadvertently attracts more of the bad risks.

The key data issue is the fact that relatively little data exist that is highly relevant. As a result, regulators encourage banks to use scenario analysis, in addition to internal and external loss data. This involves generating scenarios where large losses occur.

The power law holds well for large losses experience by banks. This makes the calculation of VaR with high degrees of confidence (e.g., 99%) possible. Internal or external loss data is used to estimate the power law parameters using the maximum likelihood estimation (MLE) approach.

## Questions and Answers

Questions

Answers

# Jorion, Chapter 14: Stress Testing

**Learning Outcomes:**

**Describe** the purposes of stress testing and the process of implementing a stress-testing scenario.

**Contrast between** event‐driven scenarios and portfolio‐driven scenarios.

**Identify** common one‐variable sensitivity tests.

**Describe** the Standard Portfolio Analysis of Risk (SPAN®) system for measuring portfolio risk.

**Describe** the drawbacks to scenario analysis.

**Explain** the difference between unidimensional and multidimensional scenarios.

**Compare and contrast** various approaches to scenario analysis.

**Define and distinguish** between sensitivity analysis and stress testing model parameters.

**Explain** how the results of a stress test can be used to improve our risk analysis and risk management systems.

About Stress Testing

Generically, simple stress testing consists of three steps:

1. Create a set of extreme market scenarios (i.e., stressed scenarios)—often based on actual past events;
2. For each scenario, determine the price changes to individual instruments in the portfolio; sum the changes in order to determine change in portfolio value
3. Summarize the results: show estimated level of mark-to-market gains/losses for each stressed scenario; show where losses would be concentrated.

Stress testing, a process to identify and manage situations that could cause huge losses, includes:

Scenario analysis: evaluating the portfolio under various extreme but probable world states (typically includes large movements in key variables). Scenarios can be historical or prospective (a.k.a., hypothetical)

Stressing models volatilities, and correlations

Policy responses

Stress testing is a non-statistical approach to risk measurement. Stress testing is:

Required by Basel Committee as one of seven conditions required to satisfy use of internal models

Endorsed by Derivatives Policy Group and by Group to Thirty (G30)

Stress testing is one area where you do not need to “stress” about memorizing formulas for the exam. “Despite recent advances in approaches to stress testing, there is no standard way to stress test a portfolio, no standard set of scenarios to consider, and even no standard approach for generating scenarios”.

## Discuss how stress testing complements VaR

The goal of stress testing is to identify unusual scenarios that would not be covered by standard VaR models, including:

Simulating shocks that have never occurred (unanticipated by history)

Simulating shocks that reflect permanent structural breaks or changed statistical patterns

|  |  |
| --- | --- |
| VAR | Stress Testing |
| Gives no information on the size of losses in excess of (greater than) VaR | Captures the “magnitude effect” of large market moves. |
| Gives little/no information about the direction of exposure; e.g., is exposure related to price increase or market decline | Simulates changes in market rates and prices, in both directions |
| Says nothing about the risk due to omitted factors; e.g., due to lack of data or to maintain simplicity | Incorporates multiple factors and captures the effect of nonlinear instruments. |

A key weakness of VaR is that it tells us nothing about losses in excess of VaR. This is a weakness also addressed by extreme value theory (EVT), which attempts to fit a second, loss-specific distribution to extreme tail. Therefore, both EVT and stress testing compensate for this drawback in the VaR approach

Describe the benefits and drawbacks of stress testing

|  |  |
| --- | --- |
| Advantage | Disadvantage |
| Complements (does not replace) value at risk. A tool to be used in addition to VaR.  Simple and intuitive  Directly examines the tails (i.e., as opposed to measures of central tendency) | Highly subjective  Could generate false alarms: implausible scenarios  Could miss plausible scenarios  Difficult to interpret: can produce lots of unfiltered information |

Compare and contrast the use of unidimensional and multidimensional scenario analysis

Unidimensional scenarios focus “stressing” on key one variable at time; e.g., shift in the yield curve, change in swap spread. Scenarios consist of shocking one variable at a time. The key weakness of a unidimensional analysis is that scenarios cannot, by definition, account for correlations.

The multidimensional is more realistic and attempts to “stress” multiple variables and their relationships (correlations). Multidimensional scenario analysis consists of:

* First, posit a state of the world (high severity event)
* Then, infer movements in market variables

Multidimensional analysis includes:

* Factor push method: first, shock risk factors individually. Then, evaluate a worst-case scenario.
* Conditional scenario method: systematic approach

Describe the advantages and disadvantages of using prospective scenarios and historical scenarios

Prospective scenarios try to analyze the implications of hypothetical one-off surprises; e.g., a major bank failure, a geopolitical crisis.

Historical scenarios look to actual (past) events to identify scenarios that fall outside the VaR window. Events that are often used include:

* The one-month period in October 1987 (S&P 500 index fell by > 21%)
* Exchange rate crisis (1992) and U.S. dollar interest rates changes (spring of 1994)
* The 1995 Mexican crisis
* The East Asian crisis (summer of 1997)
* The Russian devaluation of August 1998 and the Brazilian devaluation of 1999

|  |  |  |  |
| --- | --- | --- | --- |
|  | Advantage | | Disadvantage |
| Prospective Scenarios in MDA | Relies on input of managers to frame scenario and therefore may be most realistic vis-à-vis actual extreme exposures | May not be well-suited to “large, complex” portfolios  FACTOR PUSH METHOD: ignores correlations | |
| Historical scenarios in MDA | Useful for measuring joint movements in financial variables | Typically, limited number of events to draw upon | |

Discuss an advantage and disadvantage of using the conditional scenario method as a means to generate a prospective scenario

|  |  |  |
| --- | --- | --- |
|  | Advantage | Disadvantage |
| Conditional Scenario Method | More realistically incorporates correlations across variables: allows us to predict certain variables conditional on movements in key variables | Relies on correlations derived from entire sample period. Highly subjective |

Discuss possible responses when scenario analysis reveals unacceptably large stress losses.

Although not every scenario requires a response, an institution should address relevant scenarios. The institution can:

* Set aside economic capital to absorb worst-case losses
* Purchase protection or insurance
* Modify the portfolio
* Restructure the business or product mix to enhance diversification
* Develop a corrective or contingency plan should a scenario occur
* Prepare alternative funding sources in anticipation of liquidity crunches

Discuss the implications of correlation breakdown for scenario analysis

The problem with the SMC approach is that the covariance matrix is meant to be “typical;” but severe stress events wreak havoc on the correlation matrix. That is correlation breakdown. Scenarios can attempt to incorporate correlation breakdowns. One approach is to stress test (simulate) the correlation matrix. This is easier said than done; e.g., the variance-covariance matrix needs to be invertible, however, new techniques such as spectral decomposition of the correlation matrix, and panic copulas are designed to handle such problems.

Describe the primary approaches to stress testing and the advantages and disadvantages of each approach

The common practice is to provide two independent sections to the risk report: (i) a VAR-based risk report and (ii) a stress testing-based risk report. The VAR-based analysis includes a detailed top-down identification of the relevant risk generators for the trading portfolio. The stress testing-based analysis typically proceeds in one of two ways: (i) it examines a series of historical stress events and (ii) it analyzes a list of predetermined stress scenarios.

In regard to stressing historical events, this can be informative about portfolio weaknesses. The analysis of predetermined (standard) scenarios can be good at highlighting weaknesses relative to standard risk factors (e.g., interest rate factors). However, the analyzing pre-prescribed scenarios may create false red flags.

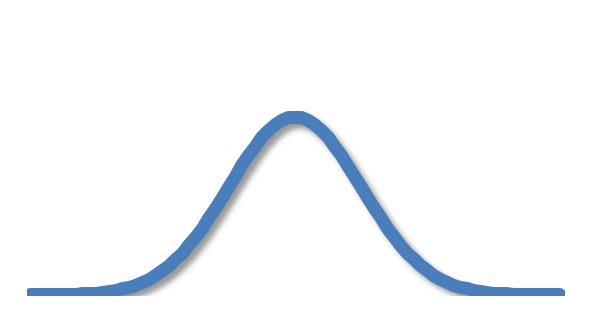
The problem with historical stress testing is that it could miss altogether important risk sources (i.e., because they happened not to arise in historical events).

|  |  |  |
| --- | --- | --- |
|  | Advantage | Disadvantage |
| Stress Testing | Can illuminate riskiness of portfolio to risk factors  Can specifically focus on the tails (extreme losses)  Complements VaR | May generate unwarranted red flags  Highly subjective (can be hard to imagine catastrophes) |

## Describe the purposes of stress testing and the process of implementing a stress-testing scenario.

The purpose of stress testing is to identify events that could greatly impact the bank but are not captured in VAR measures.

A major goal of stress testing is to “evaluate the capacity of the bank’s capital to absorb large potential losses.



**?**

**VaR does not give information about extreme tail losses (i.e., within the significance %)**

Stress testing includes:

* Scenario Analysis
* Moving key variables one at a time
* Using historical scenarios
* Creating prospective scenarios
* Stressing models, volatilities and correlations
* Developing policy responses

## Contrast between event‐driven scenarios and portfolio‐driven scenarios.

Event-driven

Scenario formulated from plausible events that generate  
movements in risk factors

Portfolio-driven

First, risk vulnerabilities in the current portfolio are identified. Second, translated into adverse movements in risk factors.

For example, pension funds invested in long-term bonds must consider upward shifts in yield curve

## Identify common one‐variable sensitivity tests.

* Derivatives Policy Group (DPG) unidimensional scenarios
* Parallel yield-curve shifting by +/- 100 bps
* Yield curve twisting by +/- 25 bps
* Each of four combinations or yield-curve shifts and twists
* Implied volatilities changing by +/- 20%
* Equity index values changing by +/- 10%
* Currencies moving by +/- 6% (major) or 20% (other)
* Swap spreads changing by 20 bps

## Describe the Standard Portfolio Analysis of Risk (SPAN®) system for measuring portfolio risk.

SPAN is a scenario-based method for measuring portfolio risk.

Calculates worst possible loss portfolio might incur over a specified time period  
 (one trading day).

Risk-array value = result of computation for each risk scenario. Risk array = a set of risk array values that indicate how a particular contract will gain or lose value under various conditions.

SPAN is a scenario-based approach with full valuation

## Discuss the drawbacks to scenario analysis.

* “Highly subjective”
* Bad or implausible scenarios will lead to irrelevant potential losses.
* Worse, plausible scenarios may not be considered.
* Stress test result presented without attached probabilities, making them difficult to interpret

Unlike VaR, can lead to large amount of unfiltered information (“too much information”)

Too many scenarios make it hard for management to figure out what to do (“anxiety of choices”)

## Explain the difference between unidimensional and multidimensional scenarios.

Unidimensional scenarios

Unidimensional scenarios focus “stressing” on key one variable at time; e.g., shift in the yield curve, change in swap spread. Scenarios consist of shocking one variable at a time.

The key weakness of a unidimensional analysis is that scenarios cannot, by definition, account for correlations.

Multidimensional scenarios

Multidimensional scenarios try to predict multiple variables and their correlations

Multidimensional scenario analysis is more realistic and attempts to “stress” multiple variables and their relationships (correlations).

Multidimensional scenario analysis consists of:

* First, posit a state of the world (high severity event)
* Then, infer movements in market variables

And includes:

* Factor push method: first, shock risk factors individually. Then, evaluate a worst-case scenario.
* Conditional scenario method: systematic approach

Example of multidimensional: Recent U.S. Bank Stress Tests

**2009 Treasury Guidelines for Bank Stress Tests (Supervisory2223Qa # Capital Assessment Program)**

* **Stress Assumptions**

|  |  |  |
| --- | --- | --- |
|  | **2009** | **2010** |
| **Real  GDP (% change)** | **(%)** | **(%)** |
| Average  Baseline2 | ‐2.0 | 2.1 |
| Consensus  Forecasts | ‐2.1 | 2.0 |
| Blue  Chip | ‐1.9 | 2.1 |
| Survey  of  Professional  Forecasters | ‐2.0 | 2.2 |
| Alternative  More  Adverse | ‐3.3 | 0.5 |
| **Civilian  unemployment  rate (%)** |  |  |
| Average  Baseline2 | 8.4 | 8.8 |
| Consensus  Forecasts | 8.4 | 9.0 |
| Blue  Chip | 8.3 | 8.7 |
| Survey  of  Professional  Forecasters | 8.4 | 8.8 |
| Alternative  More  Adverse | 8.9 | 10.3 |
| **House  prices (% change)** |  |  |
| Baseline | ‐14 | ‐4 |
| Alternative  More  Adverse | ‐22 | ‐7 |

## Compare and contrast various approaches to scenario analysis.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Advantage | | Disadvantage | |
| **Stress Testing** | Complements (does not replace) value at risk. A tool to be used in addition to VaR.  Simple and intuitive  Directly examines the tails (i.e., as opposed to measures of central tendency) | | Highly subjective  Could generate false alarms: implausible scenarios  Could miss plausible scenarios  Difficult to interpret: can produce lots of unfiltered information | |
|  | Advantage | Disadvantage | | |
| **Prospective Scenarios in MDA** | Relies on input of managers to frame scenario and therefore may be most realistic vis-à-vis actual extreme exposures | May not be well-suited to “large, complex” portfolios  FACTOR PUSH METHOD: ignores correlations | | |
| **Historical scenarios in MDA** | Useful for measuring joint movements in financial variables | Typically, limited number of events to draw upon | | |
|  | Advantage | | | Disadvantage |
| **Conditional Scenario Method** | More realistically incorporates correlations across variables: allows us to predict certain variables conditional on movements in key variables | | | Relies on correlations derived from entire sample period.  Highly subjective |

## Define and distinguish between sensitivity analysis and stress testing model parameters.

|  |  |
| --- | --- |
| Sensitivity Analysis | Model Parameters |
| Changes the functional form of the model | **Changes the values of the inputs into the model** |
| For example, bond portfolio  Single-factor Duration  Duration vector  Key rate shift | For example,  200 bps vs. 20 bps yield shock  Correlation matrix  **Change Input Value (param)** |
| **Change Model (Sensitivity)** | |

## Explain how the results of a stress test can be used to improve our risk analysis and risk management systems.

Not every scenario requires a response, but responses include:

* Set aside economic capital to absorb worst-case losses
* Purchase protection or insurance
* Modify the portfolio
* Restructure the business or product mix to enhance diversification
* Develop a corrective or contingency plan should a scenario occur
* Prepare alternative funding sources in anticipation of liquidity crunches

## Chapter Summary

Generically, simple stress testing consists creating a set of extreme market scenarios; for each scenario, determine the price changes to individual instruments in the portfolio; sum the changes in order to determine change in portfolio value. The estimated level of MTM P&L for each stressed scenario is then calculated. Consequently, this can help show where losses would likely be concentrated.

Stress testing is a process to identify and manage situations that could cause huge losses. Stress testing includes scenario analysis, which involves evaluating the portfolio under various extreme but probable states of the world where the scenarios can be based upon historical events or prospective simulated scenarios. Typically, factors such as the volatility assumptions, correlations, interest rate movements and price movements would be stressed to generate a range of possible scenarios to which one would assign a probability range.

The purpose of stress testing is to identify events that could greatly impact the bank but are not captured in VaR measures. A major goal of stress testing is to evaluate the capacity of the bank’s capital to absorb large potential losses.

Event-driven scenario analysis is formulated from plausible events that generate  
movements in risk factors. Portfolio-driven scenario analysis on the other hand, focuses on identifying risk vulnerabilities in the current portfolio. These are then translated into adverse movements in risk factors. SPAN is a scenario-based method for measuring portfolio risk. Calculates worst possible loss portfolio might incur over a specified time period. SPAN is a scenario-based approach with full valuation

Unidimensional scenarios focus on shocking one factor at the time such as a shift in the yield curve, or increased volatility. The key weakness of a unidimensional analysis is that scenarios cannot, by definition, account for correlations. Multidimensional scenario analysis is more realistic and attempts to shock multiple variables and their relationships, such as correlations and co-dependence.

Note that there is a difference between sensitivity analysis and stress testing. Sensitivity analysis involves **changing the functional form of the model**, such as a key rate shift (exogenous); whereas stress testing **changes the values of the inputs**, such as the correlation matrix, **in the model** (endogenous)**.**

Prospective Scenarios in MDA relies on input of managers to frame scenario and therefore may be most realistic vis-à-vis actual extreme exposures but may not be well suited to large complex portfolios. The conditional scenario method incorporates correlations across variables: allows us to predict certain variables conditional on movements in key variables. However, it relies on correlations derived from the sample period, and can be highly subjective. Historical scenarios in MDA are useful for measuring joint movements in financial variables but may have few samples to draw on. Stress testing complements VaR, it is simple and intuitive, and directly examines the tails, as opposed to measures of central tendency.

## Questions and Answers

Questions (Sourced from Jorion’s FRM Handbook)

26.14.1

Which of the following statements about stress testing are *true*?

1. Stress testing can complement VaR estimation in helping risk managers identify crucial vulnerabilities in a portfolio.
2. Stress testing allows users to include scenarios that did not occur in the lookback horizon of the VaR data but are nonetheless possible.
3. A drawback of stress testing is that it is highly subjective.
4. The inclusion of a large number of scenarios helps management better understand the risk exposure of a portfolio.
5. I and II only.
6. III and IV only.
7. I, II and III only
8. All of the above

26.14.2  
Which of the following about stress testing is true?

1. It is used to evaluate the potential impact on portfolio values of unlikely, although plausible, events or movements in a set of financial variables.
2. It is a risk management tool that directly compares predicted results to observed actual results. Predicted values are also compared with historical data.
3. Bot a) and b) are true.
4. None of the above is true.

26.14.3

Philippe Noiroj, the manager of a $150 million distressed bond portfolio, conducts stress tests on the portfolio. The portfolio’s annualized return is 12%, with a return volatility of 25%. In the past two years, the portfolio encountered several days when the daily value change of the portfolio was more than 3 standard deviations. If the portfolio would suffer a 4-sigma daily event, estimate the change in the value of this portfolio. Assume there are 252 trading days in a year.

1. $9.45 million
2. $23.70 million
3. $37.50 million
4. $150 million

Answers

26.14.1

c) All of the statements except IV are correct. IV is incorrect because too many scenarios will make it more difficult to interpret the risk exposure.

26.14.2

a) Stress testing is indeed used to evaluate the effect of extreme movements. Answer b) should be recognized as backtesting rather than stress testing.

26.14.3

a) First we transform the volatility into a daily volatility: = 1.57485%. Multiplying the $150 million portfolio with the volatility, 1.57% and the 4-sigma, we get:

# Principles for Sound Stress Testing Practices and Supervision, Jan 2009

**Learning Outcomes:**

**Describe** the rationale for the use of stress testing as a risk management tool.

**Describe** weaknesses identified and recommendations for improvement in:

* The use of stress testing and integration in risk governance
* Stress testing methodologies
* Stress testing scenarios
* Stress testing handling of specific risks and products.

**Describe** stress testing principles for banks within:

* Use of stress testing and integration in risk governance
* Stress testing methodology and scenario selection
* Principles for supervisors

## Describe the rationale for the use of stress testing as a risk management tool.

* Provides forward-looking assessments of risk
* Overcoming limitations of models and historical data
* Complements Value at Risk (VaR)
* Supports internal and external communication
* Feeds into capital and liquidity planning procedures
* Informs setting of a banks’ risk tolerance; and
* Facilitates development of risk mitigation or contingency plans across a range of stressed conditions.

## Describe weaknesses identified and recommendations for improvement in: The use of stress testing and integration in risk governance

Integral part of governance, with actionable results impacting strategic decisions

Promotes risk identification and control; provide a complementary risk perspective to other risk management tools (VaR, EC); improve capital and liquidity management; and enhances internal and external communication

* Stress testing programs should take account of views from across the organization and should cover a range of perspectives and techniques.
* Should be documented including assumptions and fundamental elements
* Bank should have suitably robust infrastructure, flexible to accommodate stress test to appropriate level of granularity
* Bank should maintain and update its stress-testing framework

## Stress testing methodologies and scenarios

Should cover (shock) a range of risks factors

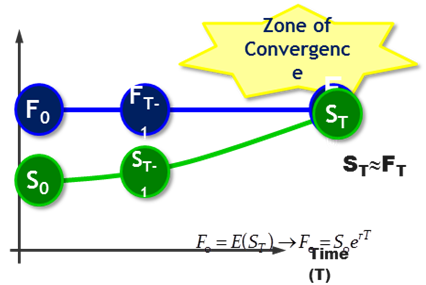
* Firm-wide, business-specific, product specific
* Typical metrics: Asset values, accounting/economic profit & loss (P&L), regulatory capital or RWA, economic capital, liquidity and funding gaps
* Should cover range of scenarios, including forward-looking
* Various time horizons
* Include judgments (“failure of imagination” leads to underestimation)
* Range of severities, including reverse stress testing (starting from a known outcome; e.g., breaching regulatory ratios)
* Should take account of simultaneous pressures in funding and asset markets, and the impact of a reduction in market liquidity on exposure valuation.

## Stress testing handling of the following specific risks:

Risks arising from the use of complex structured products

Should consider underlying assets, their exposure to systematic market factors, relevant contractual arrangements (embedded triggers), and impact of leverage (esp. subordination)

“Banks have mistakenly assessed the risk of some products (e.g., CDOs of ABS) by relying on external credit ratings or historically observed credit spreads related to (seemingly) similar products like corporate bonds with the same external rating. Such approaches cannot capture relevant risk characteristics of complex, structured products under severely stressed conditions.

Basis risk

The effectiveness of risk mitigation techniques should be systematically challenged.

Performance of risk mitigating techniques, like hedging, netting and the use of collateral, should be challenged…

Wrong‐way risk

Banks should stress test for highly leveraged counterparties, including potential for wrong-way risk

In case of severe market shocks, exposures may increase abruptly and potential cross-correlation of the creditworthiness of such counterparties with the risks of assets being hedged may emerge (i.e. wrong-way risk).

Pipeline risk

Should cover pipeline and warehousing risks

Many of the risks associated with pipeline and warehoused exposures emerge when a bank is unable to access the securitization market due to either bank specific or market stresses

Warehousing and pipeline risk refers to the event where originating banks are unable to off-load assets due to unexpected changes in market conditions. Involuntary holding of these assets expose the bank to losses due to declining values of these assets.

Contingent risk

Another weakness of the models was that they did not adequately capture contingent risks that arose either from legally binding credit and liquidity lines or from reputational concerns related, for example, to off-balance sheet vehicles.

Had stress tests adequately captured contractual and reputational risk associated with off-balance sheet exposures, concentrations in such exposures may have been avoided

Funding risk

With regard to funding liquidity, stress tests did not capture the systemic nature of the crisis or the magnitude and duration of the disruption to interbank markets. For a more in-depth discussion of the shortcomings of liquidity stress tests, see the Basel Committee’s Principles for Sound Liquidity Risk Management and Supervision (September 2008).

## Describe stress testing principles for banks within: Use of stress testing and integration in risk governance

Stress testing should form an integral part of the overall governance and risk management culture of the bank. Stress testing should be actionable, with the results from stress testing analyses impacting decision-making at the appropriate management level, including strategic business decisions of the board and senior management. Board and senior management involvement in the stress-testing program is essential for its effective operation.

A bank should operate a stress testing program that promotes risk identification and control; provides a complementary risk perspective to other risk management tools; improves capital and liquidity management; and enhances internal and external communication

Stress testing programs should take account of views from across the organization and should cover a range of perspectives and techniques.

A bank should have written policies and procedures governing the stress testing program. The operation of the program should be appropriately documented

A bank should have a suitably robust infrastructure in place, which is sufficiently flexible to accommodate different and possibly changing stress tests at an appropriate level of granularity

A bank should regularly maintain and update its stress-testing framework. The effectiveness of the stress testing program, as well as the robustness of major individual components, should be assessed regularly and independently.

## Stress testing methodology and scenario selection

Stress tests should cover a range of risks and business areas, including at the firm-wide level. A bank should be able to integrate effectively, in a meaningful fashion, across the range of its stress testing activities to deliver a complete picture of firm-wide risk.

Stress testing programs should cover a range of scenarios, including forward-looking scenarios, and aim to take into account system-wide interactions and feedback effects.

Stress tests should feature a range of severities, including events capable of generating the most damage whether through size of loss or through loss of reputation. A stress-testing program should also determine what scenarios could challenge the viability of the bank (reverse stress tests) and thereby uncover hidden risks and interactions among risks.

As part of an overall stress-testing program, a bank should aim to take account of simultaneous pressures in funding and asset markets, and the impact of a reduction in market liquidity on exposure valuation.

Specific areas of focus

The effectiveness of risk mitigation techniques should be systematically challenged.

The stress-testing program should explicitly cover complex and bespoke products such as securitized exposures. Stress tests for securitized assets should consider the underlying assets, their exposure to systematic market factors, relevant contractual arrangements and embedded triggers, and impact of leverage, particularly as it relates to the subordination level in the issue structure.

The stress-testing program should cover pipeline and warehousing risks. A bank should include such exposures in its stress tests regardless of their probability of being securitized

A bank should enhance its stress testing methodologies to capture the effect of reputational risk. The bank should integrate risks arising from off-balance sheet vehicles and other related entities in its stress-testing program.

A bank should enhance its stress testing approaches for highly leveraged counterparties in considering its vulnerability to specific asset categories or market movements and in assessing potential wrong-way risk related to risk mitigating techniques.

## Principles for supervisors

Supervisors should make regular and comprehensive assessments of a bank’s stress testing program.

Supervisors should require management to take corrective action if material deficiencies in the stress-testing program are identified or if the results of stress tests are not adequately taken into consideration in the decision-making process.

Supervisors should assess and if necessary challenge the scope and severity of firm-wide scenarios. Supervisors may ask banks to perform sensitivity analysis with respect to specific portfolios or parameters, use specific scenarios or to evaluate scenarios under which their viability is threatened (reverse stress testing scenarios).

Under Pillar 2 (supervisory review process) of the Basel II framework, supervisors should examine a bank’s stress testing results as part of a supervisory review of both the bank’s internal capital assessment and its liquidity risk management. In particular, supervisors should consider the results of forward-looking stress testing for assessing the adequacy of capital and liquidity.

Supervisors should consider implementing stress test exercises based on common scenarios.

Supervisors should engage in a constructive dialogue with other public authorities and the industry to identify systemic vulnerabilities. Supervisors should also ensure that they have the capacity and skills to assess a bank’s stress testing program.

## Questions and Answers

Questions

Answers

1. [↑](#footnote-ref-1)
2. Derman, E. (2011). *Models.Behaving.Badly*. New York: Free Press [↑](#footnote-ref-2)