UNIVERSITY COLLEGE LONDON
Department of Physics and Astronomy

Personal Study Notes for

Phas0040
Nuclear and Particle Physics

Unofficial notes compiled by a student

Academic Year 2024−2025
Version: 4 Last Update: February 16, 2026

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Notes for the Reader

These lecture notes were prepared with the following goals in mind:

- These notes aim to cover all important course material while removing unnecessary text from the original notes. The goal was to create clear and complete notes that are easier to read and study.
- Historical information (such as when particles were discovered and details about specific experiments) has been removed to keep the notes focused on the main concepts. This was done to make the notes more useful for studying.

About Figures and Diagrams: Most plots and diagrams in these notes are the same as those in the professor's original materials. Some figures could not be found because they lacked proper references in the source materials.

All images have been saved and uploaded in the best quality available from online sources rather than taking screenshots. If any images appear blurry, this is because higher quality versions were not available online. Some screenshots were used in Chapter 10 because similar diagrams could not be found online.

These notes are designed to help with studying and serve as a reference guide alongside the original course materials and lectures.

0.1 Modifications Log

Version 2 (July 07, 2025):

- Page 19 - Fixed diagrams for branch 1 and 2, there was an extra ν_e
- Section "Cross sections and luminosity" and "Different cross sections" (in chapter 1) have been moved at the begging of chapter 3 to group all the related topics together.
- Rewritten section 3 in chapter 3.

Version 3 (August 02, 2025):

- Full review of all chapters (except 1-2-3), with additional clarifications, and with minor error corrections.
- Page 32 Remove square of q in first equation.
- Changes notation in chapter 6: K_1 and K_2 are now K_+ and K_-.

Version 4 (August 11, 2025):

- p.10 - Application of the Heisenberg time-energy uncertainty principle.
- p.53 - changed μ to e^- in the ZZ box diagram (lower left vertex).
- p.55 - New explicit equation for computing decay rates for H boson.
- p.56 - Added why it is more difficult to discover H from decaying $\tau^+\tau^-$ than decaying $\mu^+\mu^-$.
- New section in chapter 7 on particles discovery strategy. Key theory for Q3 exam 2021-2022.
- Improved section 10.3.3 on stellar fusion, PPI chain.

Chapter 1

Basic Ideas of Particle Physics

1.1 The Standard Model

Fermions (matter particles)

The fundamental matter particles are all fermions (half-integer spin), arranged into three generations with increasing mass. Antiparticles have opposite electric charges (other properties like mass and spin are the same as the corresponding particle).

Table 1.1: Elementary Particles

Charge		1st generation	2nd generation	3rd generation
QUARKS	+2/3	u up	c charm	t top
	-1/3	d down	s strange	b bottom
LEPTONS	0	ν_e electron-neutrino	ν_μ muon-neutrino	ν_τ tau-neutrino
	-1	e^- electron	μ muon	τ tau

The force carrying bosons

Interactions between fermions are governed by the exchange of bosons (integer spin) Table 1.2.

Table 1.2: Interaction types and their mediators

Interaction	**Particles Affected**	**Mediator (Gauge Boson)**
Weak	All particles	$W^\pm$, Z^0 (massive)
Electromagnetic	All charged particles	Photon, γ (massless)
Strong	Quarks only	Gluons, g (massless)

Higgs boson

A neutral boson with zero spin, that interacts with all particles with mass. The strength of the interaction is dependent on the mass. All bosons are summarized in Table 1.3

1.2 Units and scales

A common distance unit is the Fermi: 1 fm = 10^{-15} m.
For cross sections, the unit is the Barn: 1 b = 10^{-28} m^2.

Table 1.3: The Standard Model bosons

Boson	Electrical Charge	Spin
gluon	0	1
photon	0	1
W^+	+1	1
W^-	-1	1
Z	0	1
Higgs	0	0

1.2.1 Natural units

Natural units are based on fundamental constants: GeV, c, $\hbar$.

- 1 GeV $= 10^9$ eV $= 10^9 \times 1.6 \times 10^{-19}$ J $= 1.6 \times 10^{-10}$ J. (1 eV is the energy gained by the charge of a single electron moved across an electric potential difference of 1V).
- $c = 3 \times 10^8$ ms^{-1}
- $\hbar = 1.055 \times 10^{-34}$ Js

We often choose to **set** $c = \hbar = 1$.

Energy: Given in **GeV**. To convert to Joules, multiply by 1.6×10^{-10} J/GeV.

Momentum: Given in **GeV/c**. For a massless particle, $E = pc$. For example, a massless particle with $E = 6.5$ TeV has momentum $p = 6.5$ TeV/c. To convert to SI units: $p = \frac{6.5\times10^3\times1.6\times10^{-10}\text{ J}}{3\times10^8\text{ ms}^{-1}} = 3.5 \times 10^{-15}$ kg ms^{-1}.

Mass: Given in **GeV/c^2**. Recall $E = mc^2$. For example, the electron has a mass $m_e = 0.511$ MeV/c^2 $= 0.511 \times 10^{-3}$ GeV/c^2. To convert to SI units: $m_e = \frac{0.511\times10^{-3}\times1.6\times10^{-10}\text{ J}}{(3\times10^8\text{ ms}^{-1})^2} = 9.1 \times 10^{-31}$ kg.

Length: In natural units, $\hbar c$/GeV. In SI units (m): $\frac{\hbar[\text{Js}]\times c[\text{ms}^{-1}]}{\text{GeV[J]}} = [\text{m}]$. Example: radius of a proton is $r = 4.1$ GeV^{-1} (where $c = \hbar = 1$). To convert to SI, reinsert $\hbar$ and c: $r = \frac{4.1\times1.055\times10^{-34}\times3\times10^8}{1.6\times10^{-10}} = 0.8 \times 10^{-15}$ m $= 0.8$ fm.

Time: In natural units, $\hbar$/GeV. In SI units (s): $\frac{\hbar[\text{Js}]}{\text{GeV[J]}} = [\text{s}]$.

1.3 Particle physics is high energy physics

We need high-energy accelerators or cosmic rays for two reasons:

1. To resolve the smallest distances associated with the fundamental constituents of matter, we need the highest possible momenta, and hence energies, according to the de Broglie wavelength $\lambda = h/p$.
2. Since $E = mc^2$, the factor c^2 is very large, therefore creating massive particles requires large energies.

1.4 Particle interactions

1.4.1 Decay

Most particles in the Standard Model are unstable. There are often different allowed decay modes, and the rate of each decay mode depends on the strength of the interaction.

- The **strong force** is the strongest, therefore the **decays** will be the **fastest**.

- The **weak force** is the weakest, therefore the **decays** will be the **slowest**.
- Electron, positron, the lightest neutrino, and the photon are **stable particles**, as they cannot decay due to conservation laws.
- Quarks and gluons are never found in isolation but are bound into states known as hadrons.

The following conservation rules ensure that these particles are stable:

- **Conservation of energy and momentum:** A particle must always decay to two or more particles, and the mass of the decaying particle must be greater than the sum of the masses of the particles produced in the decay.
- **Conservation of electric charge:** The sum of electric charges before and after an interaction must be the same.

1.4.2 Scattering

Particles interact in two possible ways, and they interact via the exchange of bosons:

- Via **elastic scattering**, where the initial and final state particles are the same (e.g., $e^+ + e^- \to e^+ + e^-$).
- Via **inelastic scattering**, where the initial and final state particles are different (e.g., $e^+ + e^- \to \mu^+ + \mu^-$).

1.4.3 Feynman diagrams

A way to represent particle decay and scattering interactions.

- **Convention:** Time runs from left to right.
  Note: *Some authors use bottom to top*
- Arrow directions denote particles flowing in the direction of time and antiparticles flowing against the direction of time.
- Each vertex must obey conservation laws.
- Each vertex represents the coupling of a photon to the charge of the electron/positron. The coupling is proportional to $e \propto \sqrt{\alpha}$ (where α is the fine-structure constant).
- The probability of the process occurring is proportional to $(\sqrt{\alpha})^{2n}$, where n is the number of vertices in the diagram.

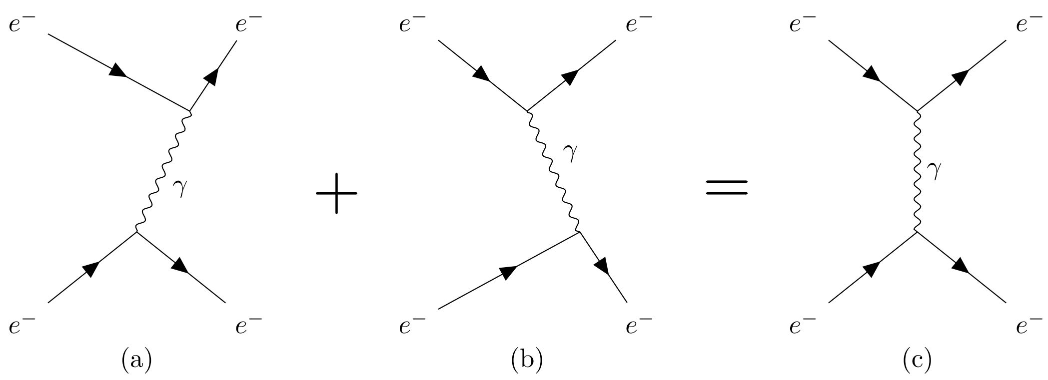
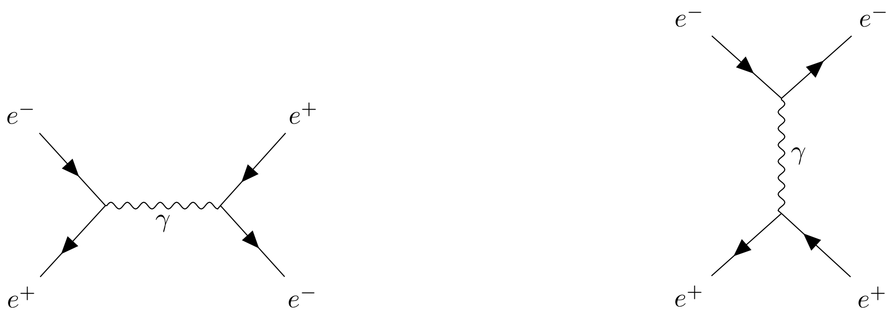


Figure 1.1: Feynman diagram for e^-e^- elastic scattering. a) The bottom electron emits a photon which is absorbed by the top electron. This exchange causes the momenta of the electrons to change. b) The photon is emitted from the top electron and absorbed by the bottom one. c) The sum of (a) and (b) is used to represent both time-orderings.



(a) s-channel (b) t-channel

Figure 1.2: Feynman diagrams for $e^+ + e^-$ elastic scattering.

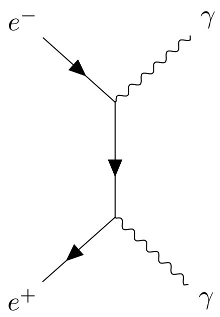


Figure 1.3: Inelastic scattering process $e^+ + e^- \to \gamma\gamma$.

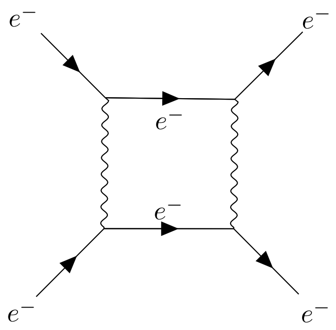


Figure 1.4: A higher-order Feynman diagram for e^-e^- elastic scattering, also known as **box diagram**.

1.4.4 Four-vectors and invariants

- Time-space four-vector is $(ct, \vec{x}) = (ct, x, y, z)$.
- Energy-momentum four-vector (or **four-momentum**) is $P = (E, c\vec{p}) = (E, cp_x, cp_y, cp_z)$.
- The **inner product** of a four-vector A is defined as:

$$A^2 = A \cdot A = A^0 A^0 - \vec{A} \cdot \vec{A}$$

Mass invariance

- For four-momentum: $P^2 = P \cdot P = E^2 - \vec{p}^{\,2} c^2$, which is invariant under Lorentz transformations.

- When a particle is at rest,$\vec{p} = 0$, $E^2 = m^2c^4$, therefore:

$$P^2 = E^2 - \vec{p}^{\,2}c^2 = m^2c^4$$

  This equation is true in any frame and is the **Einstein energy-momentum relationship**.

- The invariant mass is defined as $W = P/c^2$.
- For a system of N particles, the total four-momentum is the sum of the individual four-momenta:

$$\boxed{W^2c^4 = \left(\sum_i^N E_i\right)^2 - \left(c\sum_i^N \vec{p}_i\right)^2} \tag{1.1}$$

  This is an invariant and conserved quantity.

1.4.5 Centre-of-mass frame

- The centre-of-mass (or centre-of-momentum) frame is the frame in which $\sum_i \vec{p_i}$ is zero.
- The energy in the centre-of-mass frame (E_{CM}) gives the maximum energy available to create a massive particle.
- The square of E_{CM}, is equal to the invariant mass squared in the CoM frame. $\boxed{E_{CM} = W_{CM}}$

Example 1: Two beams colliding with equal energy in opposite directions. The CoM frame is the same as the lab frame. Letting $c = 1$, Eq. 1.1 becomes: $W^2 = (2 \times E_{beam})^2 - (0)^2 = 4E_{beam}^2$. Therefore, $E_{CM} = 2E_{beam}$.

Example 2: Fixed target experiment: where a moving beam of particles collides with a stationary target. The invariant mass in the lab-frame is:

$$W^2 = (E_{beam} + m_{target})^2 - (\vec{p}_{beam})^2 = E_{beam}^2 + m_{target}^2 + 2E_{beam}m_{target} - \vec{p}_{beam}^{\,2}$$

Substituting $E_{beam}^2 = \vec{p}_{beam}^{\,2} + m_{beam}^2$ (with c=1):

$$W^2 = m_{beam}^2 + m_{target}^2 + 2E_{beam}m_{target}$$

Assume m_{beam}^2 and $m_{target}^2 << E_{beam}$:

$$W^2 = 2E_{beam}m_{target}$$

Therefore,

$$E_{CM} = \sqrt{W^2} = \sqrt{2E_{beam}m_{target}}$$

1.5 Range of forces

Consider two particles A and B scattering via the exchange of particle X.

- In the rest frame of particle A ($\vec{p}_A = 0$), the four-momentum before interaction is $(M_Ac^2, 0)$. After the interaction, it's $(E_A, \vec{p}) + (E_X, -\vec{p})$.
- The energy after interaction is $E_A + E_X = \sqrt{M_A^2c^4 + p^2c^2} + \sqrt{M_X^2c^4 + p^2c^2}$, which is larger than the energy before interaction (M_Ac^2).
- This apparent energy violation is allowed by Heisenberg's uncertainty principle: $\Delta t \approx \frac{\hbar}{\Delta E}$. **Important:** A direct application of this principle is the following relation:

$$\Delta t = \hbar/\Delta E \quad \rightarrow \quad \boxed{\tau = \hbar/\Gamma}$$

  where τ is mean lifetime and Γ decay rate of the particle.

- The energy difference ΔE is always $\geq M_X c^2$. Therefore, $\Delta t \leq \frac{\hbar}{M_X c^2}$. The maximum distance the particle can travel in this time is $r \leq \Delta t \times c$, hence $r \leq \frac{\hbar c}{M_X c^2}$.
- This defines the **maximum range** R_X of the **force mediated by particle X** (Table 1.4):

$$\boxed{R_X = \frac{\hbar}{M_X c}}$$

Table 1.4: Range of Forces

Force	**Carrier**	**Range**
EM	γ (photon)	∞
Weak	W/Z bosons	$\sim \hbar/M_{W/Z}c$
Strong	Gluon, $M_g = 0$	Theoretically ∞, actually short range since gluons self-interact.

1.5.1 Virtual particles

- Exchanged particles are known as **virtual particles** and exist only for a short time.
  - Do not satisfy the Einstein energy-momentum relation: $E^2 - \vec{p}^{\,2}c^2 = m^2c^4$.
  - The four-momentum squared is still defined as $P^2 = E^2 - \vec{p}^2c^2$. P^2 can be negative.
  - Often indicated with an asterisk, e.g., γ^* for a virtual photon.
- Particles that exist for long periods are known as **real particle**s.

1.6 Yukawa potential

Derivation of the Klein-Gordon equation

The wavefunction of a force-carrying particle satisfies the Klein-Gordon equation, a relativistic wave equation. Starting with $E^2 = p^2c^2 + M_X^2c^4$ and using the quantum operators $\hat{E} = i\hbar\frac{\partial}{\partial t}$ and $\hat{p} = -i\hbar\nabla$, we get the Klein-Gordon equation:

$$-\hbar^2\frac{\partial^2}{\partial t^2}\phi(\vec{x},t) = -\hbar^2c^2\nabla^2\phi(\vec{x},t) + M_X^2c^4\phi(\vec{x},t) \tag{1.2}$$

Static solution

For a static solution, the time derivative is zero. **Interpreting $\phi(\vec{x},t)$ as a potential** $V(r)$, for a spherically symmetric potential, Eq. 1.2 becomes:

$$\nabla^2V(r) = \frac{M_X^2c^2}{\hbar^2}V(r)$$

For electromagnetism, the exchanged particle (photon) has $M_X = 0$, so $\nabla^2V(r) = 0$. A solution is the Coulomb potential $V(r) = C/r$. For EM, $C = -e^2/(4\pi\epsilon_0)$, so the strength of the interaction is given by the coupling $-e^2/(4\pi\epsilon_0)$.

General solution

The general solution for $M_X > 0$ is the **Yukawa potential**:

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-r/R_X}}{r}$$

where $R_X = \hbar/(M_X c)$ is the range of the interaction and g **is the coupling constant**. For $r \leq R_X$, it becomes Coulomb-like.
Important dimensionless couplings (divided by $\hbar c$ to make them dimensionless):

- **Fine-structure constant** (EM): $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \sim \frac{1}{137}$
- **Strong coupling constant**: $\alpha_s = \frac{g^2}{4\pi\hbar c}$

1.7 The scattering amplitude

The **probability for a particle to be scattered by a potential** $V(r)$ **is proportional to** $|M|^2$, where **M** is the **scattering amplitude**. For small coupling constants ($g^2 \ll 4\pi\hbar c$), the amplitude is given by the Fourier transform of the potential:

$$M = \int e^{i\vec{p}_i\cdot\vec{x}/\hbar} V(x) e^{-i\vec{p}_f\cdot\vec{x}/\hbar} d^3\vec{x}$$

For the Yukawa potential, this integral yields (for full derivation look at end of chapter Section 1.9):

$$M(\vec{p}\,^2) = \frac{-g^2\hbar^2}{|\vec{p}|^2 + M_X^2 c^2}$$

A full relativistic calculation gives the **propagator**:

$$\boxed{M(q^2) = \frac{g^2\hbar^2 c^2}{q^2 - M_X^2 c^4}} \tag{1.3}$$

where $q^2 = (E_f - E_i)^2 - (\vec{p}_f c - \vec{p}_i c)^2$ is the four-momentum exchange squared.

- If $M_X \gg q$: $M \sim -\frac{g^2\hbar^2}{M_X^2 c^2}$
- If $M_X \ll q$: $M \sim \frac{g^2\hbar^2 c^2}{q^2}$

Note: *Section "Cross sections and luminosity" and "Different cross sections" have been moved at the begging of chapter 3 to group all the related topics together.*

Note: *The section from the original notes "1.9.2 Differential cross sections: predicting the rate" is missing since it is non-examinable.*

1.8 Unstable particles and the Breit-Wigner formula

The relativistic scattering amplitude (Eq.1.3) diverges for $q^2 = M_X^2 c^4$. This is because the formula **does not account for the particle being unstable**.
For an **unstable particle**, the **probability density should obey exponential decay**: $\psi\psi^* \propto e^{-t/\tau} = e^{-t\Gamma}$, where τ is the mean lifetime and Γ is the **decay rate constant**. This is achieved by making the change $M_X c^2 \to M_X c^2 - i\Gamma\hbar/2$ in the scattering amplitude, which gives (neglect Γ^2 term as $\Gamma << M_X$):

$$M(q^2) = \frac{g^2\hbar^2 c^2}{q^2 - M_X^2 c^4 + iM_X c^2 \Gamma\hbar} \tag{1.4}$$

The cross section, σ, is proportional to $|M|^2$:

$$\sigma \propto \frac{g^4\hbar^4c^4}{(q^2 - M_X^2c^4)^2 + M_X^2c^4\Gamma^2\hbar^2}$$

This is known as the **Breit-Wigner resonance**. Plotting σ versus center-of-mass energy (Fig. 1.5) gives a distribution that **peaks at the mass of the decaying particle**, with a width that depends on its decay rate (Γ).

The total decay rate is the sum of individual decay rates of the final states f: $\Gamma = \sum_f \Gamma_f$

Branching ratio of the final state f: $B_f \equiv \frac{\Gamma_f}{\Gamma}$

The number of decay modes can be extracted by measuring the width of the distribution. Γ depends on the type of interaction (Table 1.5).

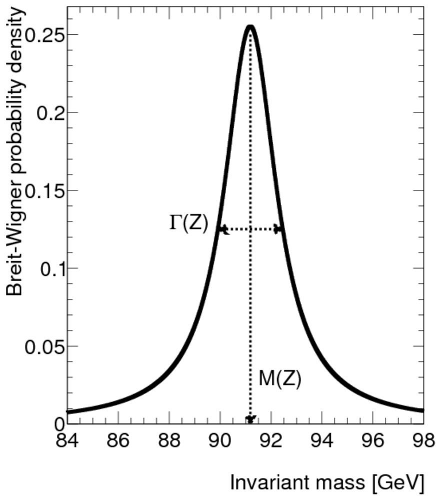


Figure 1.5: Breit-Wigner probability density. Width is the decay rate.

Table 1.5: Decay Lifetimes by Interaction

Interaction	**Lifetime [seconds]**
Strong	$10^{-25} - 10^{-20}$
Electromagnetic	$10^{-20} - 10^{-14}$
Weak	$10^{-14} - 10^{5}$

1.9 Scattering amplitude using Yukawa potential: Full derivation

The following was an exercise for problem sheet 1.

Yukawa potential:

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-r/R}}{r}$$

Scattering amplitude:

$$M = \int e^{i\bar{p_i}\cdot\bar{x}/\hbar}V(x)e^{-ip_f\cdot\bar{x}/\hbar}d^3x \tag{1.5}$$

Use spherical coord.:

$$d^3x = r^2\sin\theta dr d\theta d\phi$$

Apply $\bar{p} = \bar{p}_f - \bar{p}_i$, $\bar{p}\cdot\bar{x} = |\bar{p}||\bar{x}|\cos\theta = |p|r\cos\theta$ (since $|\bar{x}| = r$).

Eq. 1.5 becomes:

$$M = -\frac{g^2}{4\pi}\int e^{-i(p_f - \bar{p_i})\cdot\frac{\bar{x}}{\hbar}}\frac{e^{-r/R}}{r}r^2\sin\theta dr d\theta d\phi$$

$$\begin{aligned} M &= -\frac{g^2}{4\pi}\int_0^{\infty} e^{-i(\bar{p}\cdot\bar{x})/\hbar}e^{-r/R}rdr\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi \\ &= -\frac{g^2}{2}\int_0^{\pi}\int_0^{\infty}e^{-i(pr\cos\theta)/\hbar}\sin\theta e^{-r/R}rd\theta dr \end{aligned} \tag{1.6}$$

Consider only the θ term: $=\int_0^{\pi}e^{-ipr\cos\theta/\hbar}\sin\theta d\theta$

Let $x=\cos\theta$, $dx=-\sin\theta d\theta$

$$\begin{aligned} &= \int_{-1}^{1}e^{-iprx/\hbar}dx = -\frac{\hbar}{ipr}e^{-iprx/\hbar}\Big|_{x=-1}^{1} = -\frac{\hbar}{ipr}(e^{-ipr/\hbar}-e^{ipr/\hbar}) \\ &= -\frac{\hbar}{ipr}(-i\sin\frac{pr}{\hbar}-i\sin\frac{pr}{\hbar}) = \frac{2\hbar}{pr}\sin\left(\frac{pr}{\hbar}\right) \end{aligned}$$

1.6 becomes:

$$\begin{aligned} M &= -\frac{g^2}{2}\frac{2\hbar}{p}\int_0^{\infty}\sin\left(\frac{pr}{\hbar}\right)e^{-r/R}dr \\ &= -\frac{g^2\hbar}{p}\int_0^{\infty}\sin\left(\frac{pr}{\hbar}\right)e^{-r/R}dr \end{aligned} \tag{1.7}$$

Use standard integral $\int\sin(ax)e^{bx}dx=\frac{e^{bx}}{a^2+b^2}(b\sin(ax)-a\cos(ax))$ where $a=\frac{p}{\hbar}$, $b=-\frac{1}{R}$

1.7 becomes:

$$= -\frac{g^2\hbar}{p}\frac{e^{-r/R}}{(\frac{p}{\hbar})^2+(\frac{1}{R})^2}\left[-\frac{1}{R}\sin\left(\frac{p}{\hbar}r\right)-\frac{p}{\hbar}\cos\left(\frac{p}{\hbar}r\right)\right]_0^{\infty}$$

For $r=\infty$, $e^{-r/R}\to 0$.
At $r=0$:

$$\begin{aligned} M(p^2) &= -\frac{g^2\hbar}{p}\frac{1}{\frac{|p|^2}{\hbar^2}+\frac{M_x^2c^2}{\hbar^2}}\left(+\frac{p}{\hbar}\right) \\ &= \boxed{-\frac{g^2\hbar^2}{|p|^2+M_x^2c^2}} \end{aligned}$$

Chapter 2

Leptons and Quarks

2.1 Leptons

- Leptons are fundamental fermions that **do not interact through the strong force.**
- They come in three generations, where each generation consists of:

  ⇒electrically charged lepton l^-:

  - Interacting through the electromagnetic and weak forces.

  ⇒A corresponding neutrino ν_l.:

  - Interacting only through the weak force.

2.2 Lepton number

There are 3 kinds of Lepton numbers, summaries in table 2.1

Table 2.1: Lepton Number Assignments

Lepton Family	**Particles** ($N = +1$)	**Antiparticles** ($N = -1$)
Electron Lepton Number $N(e)$	e^-, ν_e	$e^+, \bar{\nu}_e$
Muon Lepton Number $N(\mu)$	μ^-, ν_μ	$\mu^+, \bar{\nu}_\mu$
Tau Lepton Number $N(\tau)$	τ^-, ν_τ	$\tau^+, \bar{\nu}_\tau$

Individual lepton number (L_e, L_μ, L_τ) must be **conserved in all interactions**.

2.2.1 Example

$\nu_\mu + n \to \mu^- + p \quad (L_\mu = 1) \to (L_\mu = 1)$ Allowed
$\nu_\mu + n \to e^- + p \quad (L_\mu = 1, L_e = 0) \to (L_\mu = 0, L_e = 1)$ Forbidden
$\nu_\mu + n \to \mu^+ + p \quad (L_\mu = 1) \to (L_\mu = -1)$ Forbidden
$\mu^+ + e^- \to \nu_\mu + \bar{\nu_e} \quad (L_\mu = -1, L_e = 1) \to (L_\mu = 1, L_e = -1)$ Forbidden

2.3 Lepton decay

- **Neutrinos** are the lightest leptons → **Stable**
  - Any decay would violate lepton #no. conservation
- **Electrons** are the lightest charged leptons → **Stable**
  - Any decay would violate charge conservation.
- Heavier leptons can decay through the exchange of a weak W boson.
- **Lepton universality:** the coupling of the W boson to any lepton doublet (charged lepton and neutrino) is the same for all three generations.

2.3.1 Decay rate

- **Decay rates** will **depend** on the 'phase space' e.g. **available energy** (hence mass).
- Decay width has units of [mass].
- Decay amplitude (Eq. 1.4) contains a propagator $\frac{1}{m_W^2}$. This squared gives decay rate, which has units $[\text{mass}]^{-4}$. To get [mass] units for Γ we need to multiply by $[\text{mass}]^5$.
- Decay rate of leptons:

$$\frac{1}{\tau_\mu} = \Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) \sim \left| \frac{1}{m_W^2} \right|^2 (m_\mu)^5 \tag{2.1}$$

- Compare the decay rate of τ and μ:
  - Take into account that the heavier τ has more options to decay into.
  - Compare the rate of a single, equivalent decay channel for both particles.
  - To do this: correct with the branching fraction $Br(\tau^- \to e^- \nu_\tau \bar{\nu}_e)$.

$$\frac{\Gamma(\tau^- \to e^- \nu_\tau \bar{\nu}_e)}{\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e)} = \frac{\Gamma(\tau) Br(\tau^- \to e \nu_\tau \bar{\nu}_e)}{\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e)} \tag{2.2}$$

- Estimate the value of $Br(\tau^- \to e \nu_\tau \bar{\nu}_e)$ by considering the possible decay modes for τ.
  - $\tau^- \to e^- \nu_\tau \bar{\nu}_e$
  - $\tau^- \to \mu^- \nu_\tau \bar{\nu}_\mu$
  - $\tau^- \to \nu_\tau \bar{u} d$ (this can appear in 3 different colours)

  Estimate of Br is: $1/5 = 0.2$. Measured value of $Br(\tau^- \to e \nu_\tau \bar{\nu}_e)$ is 0.17.
- Use Eq. 2.1 into 2.2 (m_W^2 cancels out):

$$\frac{\Gamma(\tau^-) \times 0.17}{\Gamma(\mu^-)} = \left(\frac{m_\tau}{m_\mu} \right)^5 \tag{2.3}$$

- Hence ratio of decay rates are:
  - Using lifetimes: $\frac{\Gamma(\tau)}{\Gamma(\mu^-)} = \left(\frac{\tau_\mu}{\tau_\tau} \right) \sim \frac{2.2 \mu s}{2.9 \times 10^{-13} s} \sim 7.58 \times 10^6$
  - Using Eq.2.3: $\frac{1}{0.17} \left(\frac{m_\tau}{m_\mu} \right)^5 \sim \frac{1}{0.17} \left(\frac{1.777 \text{ GeV}}{0.106 \text{ GeV}} \right)^5 \approx 7.79 \times 10^6$
    Hence τ decays 10^6 faster than μ.

2.3.2 Neutrino oscillations

- For ν there are two sets of basis functions, which are not the same.
- **Flavour eigenstates** ν_e, ν_μ, ν_τ are linear combinations of **mass eigenstates** ν_1, ν_2, ν_3.
- For simplicity consider only 2 flavour mixing:

$$|\nu_e\rangle = |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta$$
$$\underbrace{|\nu_\mu\rangle}_{\text{flavour basis}} = -|\nu_1\rangle \sin\theta + \underbrace{|\nu_2\rangle}_{\text{mass basis}} \cos\theta$$

This corresponds to a rotation between two sets of basis:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{2.4}$$

- Due to the different time evolution this leads to an oscillation between neutrino states.

$$|\nu, t=0\rangle = |\nu_e\rangle = |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta$$

- Each mass state has a different mass, hence different E.
- The flavour state at time t:

$$|\nu, t\rangle = |\nu_1\rangle e^{-iE_1 t/\hbar} \cos\theta + |\nu_2\rangle e^{-iE_2 t/\hbar} \sin\theta \tag{2.5}$$

This means that a neutrino that started out as e.g. a ν_e can at a later time be identified as a ν_μ.

- Use Eq. 2.5 to calculate the time-dependent transition probability:

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu | \nu, t\rangle|^2 \\ &= |-\sin\theta\cos\theta e^{-iE_1 t/\hbar} + \sin\theta\cos\theta e^{-iE_2 t/\hbar}|^2 \\ &= \sin^2\theta\cos^2\theta |e^{-iE_2 t/\hbar} - e^{-iE_1 t/\hbar}|^2 \\ &= \frac{\sin^2 2\theta}{4}(e^{-iE_2 t/\hbar} - e^{-iE_1 t/\hbar})(e^{iE_2 t/\hbar} - e^{iE_1 t/\hbar}) \\ &= \frac{\sin^2 2\theta}{2}(1 - \cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)) \\ &= \sin^2 2\theta \sin^2\left[\frac{(E_1 - E_2)t}{2\hbar}\right] \end{aligned}$$

Another convenient form is derived by using Binomial expansion approx. (last equality)

$$E = \sqrt{p^2c^2 + m^2c^4} = pc\sqrt{1 + \frac{m^2c^4}{p^2c^2}} \sim pc\left(1 + \frac{1}{2}\frac{m^2c^2}{p^2}\right) \tag{2.6}$$

- Replace time t with flight distance L, assuming speed of light: $t = L/c$, and define: $\Delta m^2 = (m_\alpha^2 - m_\beta^2)$ [eV^2]

$$P(\nu_\alpha \to \nu_\beta) = \sin^2(2\theta)\sin^2\left[1.27\frac{\Delta m^2 c^4}{E}L\right] \qquad \left[\frac{\text{GeV}}{\text{eV}^2 \cdot \text{km}}\right] \tag{2.7}$$

where $sin^2(2\theta)$ gives the **maximum of the oscillation**. Remember to use radiants in calculations.

- Extend to 3 neutrino families:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{2.8}$$

- We can define:
  - **Survival probability**: $P(\nu_\mu \to \nu_\mu) = 1 - P(\nu_\mu \to \nu_e) - P(\nu_\mu \to \nu_\tau)$
  - **Oscillation probability**: $P(\nu_\mu \to \nu_e)$
- Neutrino mixing matrix is known as the PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix.

2.3.3 Absolute neutrino mass scale

- Neutrino oscillation measurements only tell us mass difference between the neutrino species, they **do not provide a measurement of an absolute mass scale**.
- Normal hierarchy $m_1 < m_2 < m_3$.
- There are 3 ways to measure the absolute mass scale:
  - End point of the β-decay spectrum (using Kurie plot, section 9.4.1)
  - Structure formation in the early universe:
    Measurements of density fluctuations allow limits to be placed on the sum of neutrino masses.
  - Neutrino-less double beta decay experiments
- **Particles** with a **distinct antiparticle** are called **Dirac particles**.
- **Neutrinos** carry no charge, they can be **their own antiparticle**, such particles are called **Majorana particles**.

2.3.4 Double β-decay

- Double β-decay is allowed in the Standard Model (SM).
- β-decay transforms a neutron into a proton and emits an e^- and an anti-electron neutrino ($\bar{\nu}_e$).
- In double β-decay two such transitions occur simultaneously (fig. 2.1). For a nucleus of mass A and charge Z:

$$(A, Z) \to (A, Z + 2) + 2e^- + 2\bar{\nu}_e \tag{2.9}$$

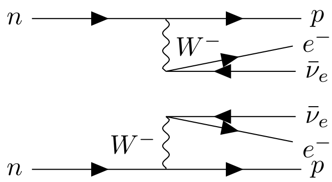
- If neutrinos are Majorana type, they can annihilate and **neutrino-less double β-decay** can occur:

$$(A, Z) \to (A, Z + 2) + 2e^- \tag{2.10}$$

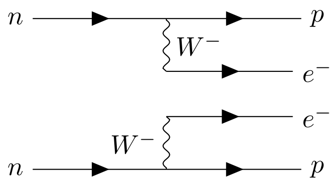
  This **violates lepton number**, neutrino-less double β-decay is forbidden in the SM.

2.3.5 Charged lepton number violation

- **Neutrino oscillation violates lepton number** on a macroscopic scale.
- **Neutrino oscillations occur over km**, the **length scale of interactions is much smaller**.
- Consider μ decay:

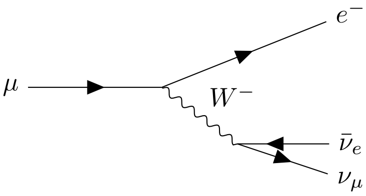


Conventional Double Beta Decay

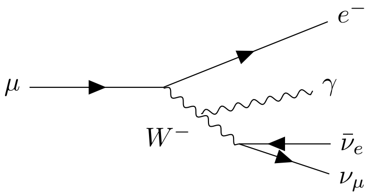


Neutrinoless Double Beta Decay

Figure 2.1: Feynman diagrams for double beta decay processes



Branch 1: $\mu \to e^- \bar{\nu}_e \nu_\mu$



Branch 2: $\mu \to e^- \bar{\nu}_e \nu_\mu \gamma$

- The probability of emitting a photon is given by the EM coupling constant $\alpha = 1/137$: Hence

$$Br(2) = \alpha Br(1)$$

$$Br(\mu \to e^- \bar{\nu}_e \nu_\mu \gamma) = \alpha Br(\mu \to e^- \bar{\nu}_e \nu_\mu)$$

- Consider ν_μ, ν_e oscillations in Branch 3. This is a lepton number violation decay $\mu \to e\gamma$.

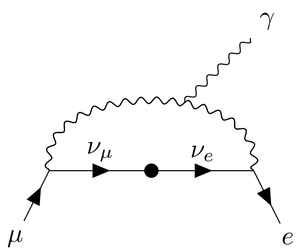


Figure 2.2: Branch 3: ν_μ is converted into a ν_e

- Probability of violation decay:

$$Br(\mu \to e\gamma) = \alpha Br(\mu \to e\bar{\nu}_e \nu_\mu) P(\nu_\mu \to \nu_e)$$

The branching ratio of the main decay $Br(\mu \to e\bar{\nu}_e \nu_\mu) \sim 1.0$.

$$= \frac{1}{137} \times 1.0 \times P(\nu_\mu \to \nu_e) \tag{2.11}$$

Use Eq.2.7:

$$= \frac{1}{137} \sin^2(2\theta_{12}) \sin^2 \left[1.27 \frac{\Delta m_{12}^2 L}{E_\nu} \right] \tag{2.12}$$

From experiments $\sin^2(2\theta_{12}) \sim 0.9$, $\Delta m_{12}^2 \sim 7 \times 10^{-5}$.

$$= \frac{0.9}{137} \sin^2 \left[1.27 \frac{7 \times 10^{-5} L}{E_\nu} \right] \tag{2.13}$$

Small angle approx $\sin^2\theta \approx \theta^2$

$$= 10^{-3}\left[1.27\frac{7\times 10^{-5}L}{E_\nu}\right]^2 \tag{2.14}$$

- L can be estimated using Heisenberg's uncertainty relation for a W boson (the exchanged virtual particle): $\Delta t \sim \frac{\hbar}{\Delta E}$

$$L = c\Delta t = \frac{c\hbar}{\Delta E} = \frac{c\hbar}{M_W} \sim \frac{197 \text{ MeV fm}}{80 \text{ GeV}} \sim 2.5\times 10^{-3}\times 10^{-15}\text{m} \tag{2.15}$$

This is the range of the weak force.

- Eq. 2.14 leads to a **probability of** $< 10^{-48}$. This is **negligible**.
- Hence **lepton number remains conserved in decays and interactions**.

2.4 Quarks

- Fundamental fermions, all have spin 1/2.
- Quarks interact via the strong force, with gluons being the carrier. There are 8 different gluons.
- Form doublets, occur in 3 generations.

Table 2.2: Quark Properties

Name	**Symbol**	**Mass [GeV]**	**Q**
Down	d	0.002	-1/3
Up	u	0.005	+2/3
Strange	s	0.1	-1/3
Charm	c	1.3	+2/3
Bottom	b	4.5	-1/3
Top	t	173	+2/3

- Anti-quarks have same properties but opposite charge.
- **Quark Confinement:** Quarks cannot be observed freely.
- Carry colour charge: Red, Blue, Green.

2.4.1 Hadrons

- Hadrons are particles made up of quarks.
- **Particles have to be colourless**. Two possibilities to form colourless objects:
  - combine a **colour** with its **anti-colour** (Blue and anti-Blue) $\rightarrow$ **Mesons** ($q\bar{q}$)
  - **Combine 3 colours**, red + blue + green $\rightarrow$ **Baryons** (qqq) and **Antibaryons** ($\bar{q}\bar{q}\bar{q}$)

2.4.2 Quark Numbers

- Number of quarks $N_q = N(q) - N(\bar{q})$ is conserved.
- Baryons have baryon number B=1.
- Antibaryons have baryon number B=-1.
- Mesons have baryon number B=0.
- **Strong** and **EM** (but not weak) **interactions conserve individual quark flavour** number $N_f = N(f) - N(\bar{f})$ where $f = u, d, s, c, b, t$.

2.4.3 Mesons

- Lightest mesons are the π mesons. $\pi^-(d\bar{u})$, $\pi^0(u\bar{u})$, $\pi^+(u\bar{d})$.
- Realistically π^0 is a **superposition**: $\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$.
- Further examples: $K^+(u\bar{s})$ (mass: 494 MeV), $K^{*+}(u\bar{s})$ (mass: 892 MeV), $D^-(d\bar{c})$, $B^-(b\bar{u})$.
  NB: For a given quark-antiquark pair, the lowest energy configuration is given when spins are anti-aligned, hence with spin zero.
  In higher energy configuration, also known as excited state, the spins are aligned, spin 1.
  Example: In K^+ spins are anti-aligned and in $K^{\star +}$ are aligned.

2.4.4 Baryons

$$p(uud) \quad n(udd) \quad \Lambda^0(uds) \quad \Delta^{++}(uuu) \quad \Omega^-(sss) \quad \Lambda_c^+(udc)$$

2.4.5 Hadron decay

- **Proton** is the lightest baryon hence **stable**.
- Neutrons can decay, like all other hadrons.
- **Lifetime depends** on the allowed **decay mechanism**.
  - Decays through the strong interactions lead to a very short lifetime.
  - Particles that can only decay weakly have longer lifetimes.

  Examples:
  - $\rho^0(u\bar{u} + d\bar{d}) \to \pi\pi$ decays strongly via gluon emission.
  - $\pi^0(u\bar{u} + d\bar{d}) \to \gamma\gamma$. There are no lighter hadrons it could decay into, so it has to decay EM.
  - Charged pion π^+, only decays via weak force, with W boson exchange: $\pi^+(u\bar{d}) \to \mu^+\nu_\mu$.
- **Strong** and **EM interactions conserve individual quark flavour** number $N_f = N(f) - N(\bar{f})$ where $f = u, d, s, c, b, t$.
- **Weak interaction changes flavour**.
- **Lifetime** also **depends** on the **available energy** states. $n \to p + e^- + \bar{\nu}_e$ has a low Q-value, $Q = m_n - m_p - m_e$, hence has a long lifetime.
- Neutral Kaons can decay into:

  $$K^0 \to \pi\pi \qquad \text{or} \qquad K^0 \to \pi\pi\pi$$

  Both are flavour changing, weak decays ($s \to d$). $K^0 \to \pi\pi\pi$ is suppressed since it has a smaller Q-value ($Q_{3\pi} < Q_{2\pi}$), hence longer lifetime.

Chapter 3

Experimental Methods

3.1 Sources of high energetic particles

Cosmic rays || Particle accelerators || Nuclear reactors
Natural radiations || Secondary beams

3.2 Accelerators

- All employ the Coulomb force to increase particle energy.
- Divided into **Linear** and **circular machines**:
- Circular accelerators use B fields to bend the charged particles into an orbit.
  - Advantage: Energy can be increased in several revolutions, hence **reducing the machine size required** to achieve the same E compared to linear.
  - Disadvantage:**Energy losses due to synchrotron radiation** (any accelerating charged particle radiates energy).

3.2.1 Center-of-mass energy

Experiments at accelerators divide into two categories: Collider experiments and Fixed target experiments.

3.2.1.1 Collider experiments

Here we get the same result as section 1.4.5 with different procedure.

- Two particle beams are collided. Usually the two beams have the same energy ($E_1 = E_2 = E$).
- Initial 4-vectors: $P_1 = (E, \bar{p})$, $P_2 = (E, -\bar{p})$.
- Center-of-mass energy squared s:

$$E_{CM}^2 = (P_1 + P_2)^2 = P_1^2 + 2P_1 \cdot P_2 + P_2^2$$

- Neglect masses: $|\bar{p}| \approx E$, $P_1^2 = P_2^2 = 0$. Center-of-mass energy becomes:

$$E_{CM} = \sqrt{2P_1 \cdot P_2} = \sqrt{2(EE - \bar{p} \cdot (\bar{p}))} = \sqrt{2(E^2 - |\bar{p}||\bar{p}|cos\theta)}$$

  For a head-on collision, $cos\theta \to -1$, use $\bar{p} \approx E$:

$$E_{CM} = \sqrt{4E^2} = 2E$$

3.2.1.2 Fixed target experiments

- A beam of energy E hits a fixed target of mass m.
- Initial 4-vectors: $P_1 = (E, \bar{p})$, $P_2 = (m, 0)$.
- Neglect mass of beam particles: $P_1^2 = 0$, $P_2^2 = m^2$. Center-of-mass energy becomes:

$$E_{CM} = \sqrt{P_1^2 + 2P_1 \cdot P_2 + P_2^2} = \sqrt{2P_1 \cdot P_2 + m^2} = \sqrt{2Em + m^2}$$

For large E, neglect target mass term: $E_{CM} \sim \sqrt{2Em}$

3.2.2 Cross sections and luminosities

Note: *the following two sub-sections come from ch1 of the original notes.*
The rate of a certain process occurring is given by:

$$R = \frac{dN}{dt} = \sigma \mathcal{L} \tag{3.1}$$

- $\mathcal{L}$ is the **instantaneous luminosity**, which is the number of incident particles per unit area per unit time (units of $m^{-2}s^{-1}$).

The **integrated luminosity**, $L = \int \mathcal{L} dt$, has units of inverse area (inverse barns, b^{-1}). The number of events of a certain type is given by $N = \sigma L$.
For a beam colliding with a stationary target: $\mathcal{L} = n_b \times v_i \times N$, where

- n_b is the number density of particles in the beam,
- v_i is the velocity of the beam
- N is the number of illuminated particles in the target

When two particles interact, many different processes can occur. The total cross section is the sum of the cross sections for all possibilities.

3.2.3 Differential cross sections: what we measure

This gives the dependence of the cross section on some kinematic variable. The direction of a particle is specified by a polar angle, θ, and an azimuthal angle, ϕ. The measured rate for a particle to be emitted into an element of solid angle $d\Omega = d(\cos\theta)d\phi$ is:

$$dR = \mathcal{L} \times \frac{d\sigma(\theta, \phi)}{d\Omega} d\Omega$$

To obtain the total cross section, one integrates over all solid angles:

$$\sigma = \int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos\theta) \frac{d\sigma(\theta, \phi)}{d\Omega}$$

3.2.4 Cross Section and Luminosity

The rate at which a specific particle interaction occurs, R, is given by:

$$R = \frac{dN}{dt} = \sigma \mathcal{L} \tag{3.2}$$

where:

- σ is the **cross section**, representing the effective area for an interaction to occur. It's a measure of probability with units of area (e.g., barns, where 1 $b = 10^{-28} m^2$). It depends on the interaction's energy but is independent of the accelerator's beam parameters.
- $\mathcal{L}$ is the **instantaneous luminosity**, a measure of the accelerator's performance. It represents the number of incident particles per unit area per unit time (units of $m^{-2}s^{-1}$).

3.2.4.1 Differential Cross Section

Often, we measure the interaction probability as a function of kinematic variables, like the scattering angle. The **differential cross section**, $\frac{d\sigma}{d\Omega}$, gives this angular dependence. The rate of particles scattered into an element of solid angle $d\Omega = d(\cos\theta)d\phi$ is:

$$dR = \mathcal{L} \times \frac{d\sigma(\theta,\phi)}{d\Omega} d\Omega$$

The **total cross section**, σ, which is the sum of cross sections for all possibilities, can be recovered by integrating over all solid angles:

$$\sigma = \int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos\theta) \frac{d\sigma(\theta,\phi)}{d\Omega}$$

3.2.4.2 Integrated Luminosity and Event Calculation

The total number of events, **N**, produced over a period of time is calculated using the **integrated luminosity**, $L = \int \mathcal{L} dt$. It has units of inverse area (e.g., inverse barns, b^{-1}).

$$N = \sigma L$$

For a beam with particle flux $\phi_1 = n_b v_i$ colliding with a stationary target containing N_{target} particles, the instantaneous luminosity is given by

$$\mathcal{L} = \phi_1 N_{target} = n_b v_i N_{target} \tag{3.3}$$

Collision rate is related to incoming flux, number and size of the target:

$$\frac{dN}{dt} = \phi_1 N_{target} \sigma \tag{3.4}$$

where number of targets is given by $N_{target} = N_2 = n_2 A l$.
Equate 3.2 and 3.4: $\mathcal{L} = \phi_1 N_2$.
Sub. in flux $\phi_1 = n_b v_i$, consider density $n = N_1/Al$:

$$\mathcal{L} = n_1 v_1 N_2 = \frac{N_1}{Al} v_1 N_2$$

For a **circular collider**, the luminosity is given by:

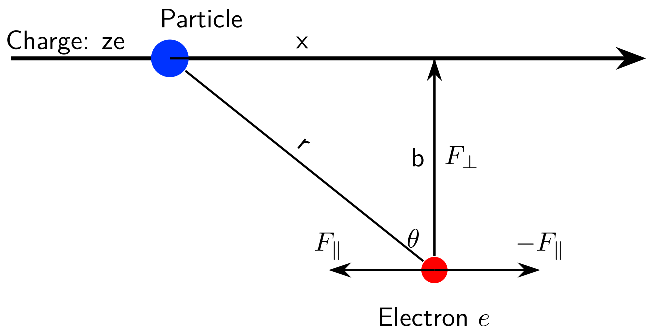
$$\mathcal{L} = \frac{n N_1 N_2 f}{A}$$

where:

- n: number of colliding bunches
- $N_{1,2}$: number of particles per bunch for each beam
- f: revolution frequency
- A: cross-sectional overlap area of the beams ($A = 4\pi\sigma_x\sigma_y$)

3.3 Ionisation energy loss: Bethe-Bloch formula

- Particles heavier than e^- inside a material, will lose energy through EM interactions with electrons of the atom.
- Consider a simple model of a particle moving along a straight line in the E field of an electron.
- The force between particles is given by Coulomb's law.



- From geometry we know that:

$$x = b\ tan\theta \qquad\qquad b = r\ cos\theta \tag{3.5}$$

$$\Delta t = \frac{\Delta x}{v} \qquad\qquad dx = d(b\ tan\theta) = \frac{b}{cos^2\theta}d\theta \tag{3.6}$$

- Perpendicular component of force:

$$F_\perp = F\cos\theta = \frac{ze^2}{r^2}\cos\theta = \frac{ze^2}{(b/\cos\theta)^2}\cos\theta = \frac{ze^2\cos^3\theta}{b^2}$$

- Find momentum transferred by integrating $F = dp/dt$:

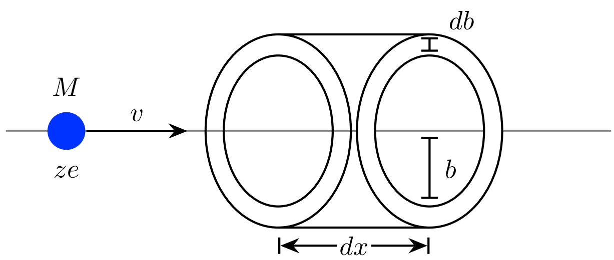
$$\Delta p = \int F_\perp dt = \int F_\perp \frac{dx}{v} = \int_{-\pi/2}^{\pi/2} \frac{ze^2\cos^3\theta}{vb^2}\frac{b}{cos^2\theta}d\theta = \frac{2ze^2}{vb} \tag{3.7}$$

- Hence energy loss of the particle to one electron is:

$$\Delta E = \frac{(\Delta p)^2}{2m_e} = \frac{2z^2e^4}{m_e v^2 b^2} \tag{3.8}$$

  Hence **slower particles lose more energy**.

- Consider an open cylinder geometry:



- The total number of interactions with electrons dN_e in a volume element dV:

$$dN_e = n_e dV = n_e 2\pi b\ db\ dx$$

- Assume each atom has Ze^- and the electron density $n_e = Z\frac{N_A\rho}{A}$:

$$dN_e = Z\frac{N_A\rho}{A}2\pi b\ db\ dx \tag{3.9}$$

  where: N_A: Avogadro's number (number of atoms), A Atomic weight, Z atomic number (number of e^- per atom), ρ mass density, b impact parameter.

- Total energy loss:

$$-dE = \Delta E dN_e = \frac{2z^2e^4}{m_e v^2 b^2} Z \frac{N_A \rho}{A} 2\pi b\, db\, dx = \frac{4\pi z^2 e^4}{m_e v^2} Z \frac{N_A \rho}{A} \frac{db}{b} dx$$

- Energy loss per length:

$$-\frac{dE}{dx} = \frac{4\pi z^2 e^4}{m_e v^2} Z \frac{N_A \rho}{A} \int_{b_{min}}^{b_{max}} \frac{db}{b} = \frac{4\pi z^2 e^4}{m_e v^2} N_A \rho \frac{Z}{A} \ln \frac{b_{max}}{b_{min}}$$

- Max momentum transfer is $\Delta p_{max} = 2m_e v$ (analogue to a ball bouncing off a wall).
- Let I be the Ionisation potential: the min energy required to kick out an e^- from the atom.
- Use the two statements above and rearrange formulas for the impact parameter:

$$\text{from eq.3.7} \qquad b_{min} = \frac{2ze^2}{v \Delta p_{max}} = \frac{ze^2}{v m_e v} \tag{3.10}$$

$$\text{from eq.3.8} \qquad b_{max} = \sqrt{\frac{2z^2e^4}{m_e v^2 \Delta E_{min}}} = \frac{ze^2}{v} \sqrt{\frac{2}{I m_e}} \tag{3.11}$$

- Sub. this into the equation for dE/dx:

$$\frac{dE}{dx} = \frac{2\pi z^2 e^4}{m_e v^2} n_e \ln \left(\frac{2m_e v^2}{I} \right)$$

- When relativistic effects and other corrections are taken into account we get the Bethe-Bloch formula:

$$-\left\langle \frac{dE}{dx} \right\rangle = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \left(\frac{2m_e c^2 \beta^2 \gamma^2 T_{max}}{I^2} \right) - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right]$$

  - A **particle loses most energy when it is slowest**
  - The **particle will deposit most of its energy** at the **end of its path** (**Bragg Peak**).
  - In fig.3.1 it is shown the **mass stopping power**, which is the energy loss per distance divided by the material's density.
    Note: *At first glance, the plot may appear counter-intuitive. For a given $\beta\gamma$, the curve for a high-Z material like lead (Pb) is lower than that for a low-Z material like liquid hydrogen (H_2), suggesting less energy loss. However, it is crucial to note that the y-axis represents the mass stopping power, $(-dE/dx)/\rho$, where the energy loss per unit length is normalized by the material's density, ρ. The density of lead is substantially greater than that of liquid hydrogen. Therefore, while the absolute energy loss per unit distance ($-dE/dx$) is much higher in lead, the energy loss per unit of mass thickness is lower.*

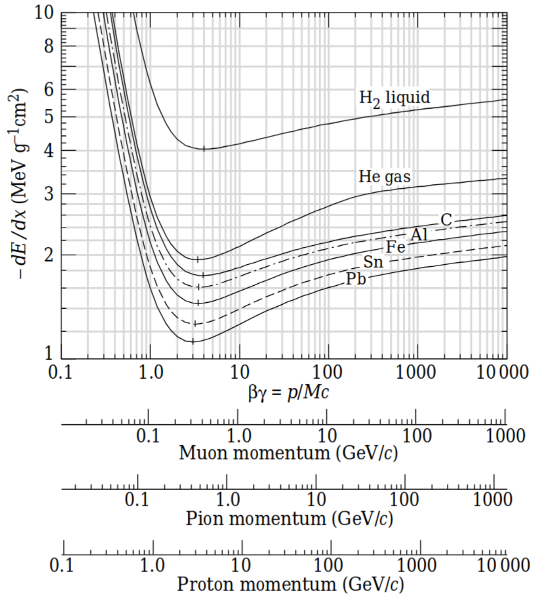


Figure 3.1: Bethe-Bloch Curve. Energy loss per distance divided by the material's density. Taken from: https://physics.stackexchange.com/questions/418251/bethe-bloch-equation-plotting-incorrectly-at-high-energies

3.3.1 Interaction of photons and electrons

Photons

Photons can lose energy through 3 mechanisms (Fig. 3.2):

- **Photoelectric effect:** γ hits an e^- in the orbit around the nucleus, liberating it from its shell, while the γ gets absorbed in the process.
  - Probability for occurrence $\sim Z^5/E_\gamma^{3/2}$.
  - Large loss at low energy and for material with large Z.
- **Compton scattering:** γ scatters off an e^-, gets deflected and transfers some of its energy.
  - Probability $\sim Z \ln(E)/E$.
- **Pair production:** dominant method of energy loss at high energies, once γs have enough energy ($E_\gamma > 2m_e$) to produce an e^+e^- pair.
  - Probability (above threshold) $\sim Z^2$.
  - For this mechanism to conserve energy and momentum, the electric field of a nucleus is needed.

Photon intensity is reduced exponetially with depth in a material:

$$I = I_0 e^{-\mu x}$$

where μ is the attenuation coefficient.

Electrons

- **Bremsstrahlung effect:** Electrons are deflected by E field of the nucleus. In this process it will loose energy,
- Feynman diagrams for pair creation and bremsstrahlung have the same mechanism (fig. 3.3).
- The quantity that describes energy loss through these two mechanisms is the **radiation length $\mathbf{X}_0$** or L_R.

$$\frac{1}{X_0} = \frac{1}{L_R} = 4N_A \frac{\alpha^3}{m_e^2} \frac{Z^2}{A} \rho \ln\left(\frac{183}{Z^{1/3}}\right)$$

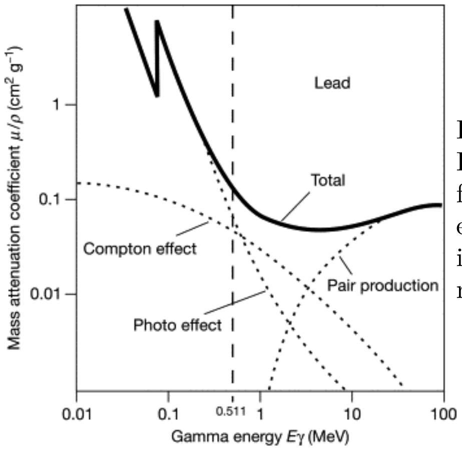
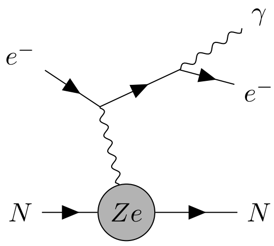


Figure 3.2: Photon Energy Loss Mechanisms. Taken from: https://www.sciencedirect.com/topics/immunology-and-microbiology/photon



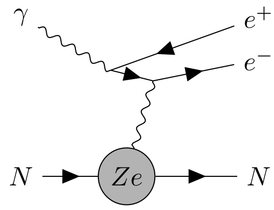


Figure 3.3: Bremsstrahlung (left) and pair production (right)

where α is the fine structure constant, ρ is the density, Z is the atomic number of material, and A is the mass number of material.

- Energy loss is given by:

$$\frac{dE}{dx} = -\frac{E}{X_0} \qquad \Rightarrow \qquad \boxed{E = E_0 e^{-x/X_0}}$$

3.3.2 Electromagnetic shower

- When a high energy e^- or γ enters a material, an EM shower develops through **continuous repetition** of **bremsstrahlung** and **pair production**.
- Consider the following simplifications:
  - Interactions occur once a particle has travelled one X_0.
  - Energy is split equally among resulting particles.
- After t steps (where each step is one radiation length $t = x/X_0$), the #NO of particles is $N(t) = 2^t$ and their energy is $E(t) = E_0/N = E_0/2^t$.
- The shower process stops once bremsstrahlung is no longer the dominant process.
- For e^-, this is at the critical energy E_C at which the energy loss contributions from ionisation and bremsstrahlung are equal.

$$N_{max} = \frac{E_0}{E_C} = 2^{t_{max}}$$

3.3.3 Hadronic showers

- In addition to ionisation, hadrons undergo strong interactions with nuclei. In these interactions new hadrons are produced.
- Produces showers similar to the EM showers.

- Nuclear absorption length:

$$\lambda_a = \frac{1}{n\sigma_{INEL}} = \frac{A}{N_A \rho \sigma_{INEL}}$$

  where σ_{inel} is the inelastic cross section.

- In these showers π^0 are produced, which decay to 2 photons leading to an **EM component**.

3.3.3.1 Cosmic ray air shower

- Primary cosmic rays hitting atomic nuclei in the top of the atmosphere create hadronic showers, termed cosmic ray showers.
- Many of the produced particles $(\pi^0, \pi^\pm, K^\pm)$ can decay.
- These decays produce EM showers from:

$$\pi^0 \to \gamma\gamma \qquad \pi^\pm \to \mu^\pm \nu_\mu \qquad K^\pm \to \pi^0 \mu^\pm \nu_\mu$$

- At ground level, mainly $\mu^\pm$ survive.

3.4 Particle detectors

3.4.1 Position measurement

3.4.1.1 Scintillators

- Indicate the passing of a charged particle by emitting light.
- Passing particles excite atoms and molecules; light is emitted during the decay of these excited states.
- Divided into:
  - **Inorganic crystal scintillators**: Better energy resolution, Slow: 200-2000 ns.
  - **Organic scintillators**: Fast: 1-10 ns.
- To readout, electronically couple to a light-sensitive detector. Examples: photomultiplier tube (PMT) or SiPMT.
- PMT consists of a photocathode, where light frees e^- via photoelectric effect. e^- are then accelerated with help of an applied high voltage towards a dynode, the impact frees more e^-. This cascade of dynodes creates an amplified electric pulse.

3.4.1.2 Gas detectors

- Basic counters that record particle passing by.
- Consist of a wire inside a gas volume. High voltage is applied to create a potential difference between the wire and the outer wall.
- Then e^- are accelerated towards the wire.
- Different gas detectors can be distinguished. The main difference is the strength of the applied electric field.
  - **Ionisation counters:** low voltage. Only charge carriers initially created in the ionisation process drift to the anode and cathode.
  - **Proportional counters:** Moderate voltages. Initial ionization process e^- are accelerated enough that they themselves can ionise the medium. Resulting avalanche leads to gas multiplication. Initial pulse is amplified.

  - **Geiger-Müller counters:** High voltage. The gas multiplication saturates and the created pulse becomes independent of the initial ionisation.

3.4.2 Tracking detectors

- Usually **operate inside a magnetic field**, so that charged particles exhibit curved trajectories in the B field that depends on their momentum.
- Deflection is given by the Lorentz force (equate with centrifugal force):

$$F = qv \times B = \frac{mv^2}{R} \implies R = \frac{mv}{qB} = \frac{p}{qB}$$

- Useful conversion formula: $R[\mathrm{m}] = \frac{p[\mathrm{GeV}]}{0.3 \times B[\mathrm{Tesla}]}$.
- These detectors use different position measurement methods:
  - Multi-wire proportional chamber (**MWPC**): array of wires inside a gas volume.
  - **Drift chambers:** measures the time between the passing of the particle and the arrival of the drifting ionisation e^-.
  - **Pixel detectors:** Similar to digital camera.
  - **Scintillating fibre tracker:** fibres made of plastic scintillator grouped together in bundles.

3.4.3 Calorimeters: Energy measurement

- Calorimeters stop particles and measure their energy.
- There are two kinds: **Electromagnetic** (ECAL), **Hadronic** (HCAL).
- Calorimeters that actively sample parts of the shower are called sampling or sandwich calorimeters.
- They are an alternation of dense absorber layer and an active layer (scintillator).
- Measured energy is proportional to the number of final particles recorded.
- Number of particles N is governed by Poisson statistics, it has uncertainty $\sqrt{N}$.
- Energy resolution is given by:

$$\frac{\sigma(E)}{E} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \sim \frac{1}{\sqrt{E}}$$

- Usual ordering of calorimeters:
  1. **ECAL:** Requires small radiation length X_0, so choose large Z material. Stops e^- and γ. Other charged particles lose energy through ionisation.
  2. **HCAL:** Requires large nuclear absorption length λ_a, choose high density ρ materials. Stops Hadrons.

3.5 Particle identification (PID)

- In normal detectors it is impossible to distinguish the different types of charged hadrons (pions, Kaons and protons). See table 3.1.
- Also it is hard to distinguish pions and e^- (due to $\pi^0 \to \gamma\gamma$ producing EM showers in ECAL).

Table 3.1: Particle Signatures in a Generic Detector. ⋆ not interacting since not charged. ■ Proton and Kaon don't produce EM showers themselves. Their strong interaction can produce π^0, which decay via $\pi^0 \to \gamma\gamma$

	TRACKING SYSTEM	**ECAL**	**HCAL**	**MUON SYSTEM**
Photon γ	⋆	X		
$e^{\pm}$	X	X		
Muon	X	X	X	X
Proton Kaon Pions	X	X ■	X	
Neutron K^0	⋆	X	X	

- General principle is to find mass by measuring the momentum and velocity simultaneously:

$$p = \beta\gamma m = \frac{m\beta}{\sqrt{1-\beta^2}}$$

- This can be done using:
  - **Time of flight:** Find velocity $\beta = v/c$ by measuring the time Δt a particle takes to cross a known distance L: $v = L/\Delta t$. Often used to separate Kaons, pions, protons.
  - **dE/dx:** Use Bethe-Bloch formula.
  - **Cherenkov radiation:** Emitted if the speed of a charged particle passing through a medium (refractive index n) is greater than the speed of light in that medium, threshold $v > c/n$.
    This leads to threshold Cherenkov detectors. Cherenkov light is emitted in a cone with an opening angle $\cos\theta_c = 1/(n\beta)$ around the direction of the particle. This is exploited in ring imaging Cherenkov detectors (RICH).
  - **Transition radiation:** Emitted when particles above a certain velocity pass through the boundary of two materials with different dielectric constants. Helps identify e^-.

3.5.1 Particle identification summary

Different methods work in different $\gamma\beta = p/m$ regimes:

Table 3.2: PID Methods vs. Momentum Range

For	**Methods**
$\gamma\beta < 3$	Time of Flight and dE/dx
$3 < \gamma\beta < 14$	Threshold Cherenkov
$14 < \gamma\beta < 140$	Ring-Imaging Cherenkov
$100 < \gamma\beta < 1000$	dE/dx relativistic rise
$1000 < \gamma\beta$	Transition radiation

3.5.2 Trigger

Purpose is to signal an interesting event. At LHC a fast trigger system is essential to cope with the data rate. Events with large deposits in the calorimeter or with large missing energy are selected.

Chapter 4

The Strong Interaction

This chapter introduces the strong interaction, one of the four fundamental forces, which is responsible for binding atomic nuclei together and is described by the theory of Quantum Chromodynamics (QCD).

4.1 QCD versus QED

Table 4.1: Comparison of QCD and quantum Electrodynamics (QED)

Property	**QCD (Strong Force)**	**QED (Electromagnetic Force)**
Mediating Particle	Gluon (spin-1, massless, neutral)	Photon (spin-1, massless, neutral)
Source of Force	Colour Charge	Electric Charge
Self-Interaction	Yes, gluons carry colour charge	No, photons are electrically neutral
Coupling Strength between mediator and source of the force	Large	Small

4.2 Deep Inelastic Scattering (DIS)

Deep Inelastic Scattering (DIS) experiments, which scatter high-energy leptons (electrons, muons, neutrinos) off nucleons (protons or neutrons), allowed physicists to probe the internal structure of nucleons at very small distance scales.

Terms explanation:

- **Deep**: refers to the high momentum transfer, probing distances much smaller than a nucleon's size (0.8 fm)
- **Inelastic**: means the nucleon breaks up, and final-state particles are different from initial ones.

4.2.1 Evidence for Quarks

DIS experiments consists of firing a electron beam at stationary protons. The observation of more high-Q^2 scattering events than anticipated for a diffuse proton indicated that the proton is composed of smaller, point-like constituents, initially called **partons**.

How to measure Q^2? Let q be the 4-momentum transfer of the interaction.
$P = (E, c\bar{p})$ is the 4-momentum of the incoming e-.
$P' = (E', c\bar{p}')$ is the 4-momentum of the outgoing e-.
The 4-momentum transfer is

$$q = (E - E', \bar{p} - \bar{p}')$$

Using the definition of four-momentum:

$$q^2 = (E - E') \cdot (E - E') - (\bar{p}c - \bar{p}'c) \cdot (\bar{p}c - \bar{p}'c) \quad = E^2 + E'^2 - 2EE' - |\bar{p}c|^2 - |\bar{p}'c|^2 + 2\bar{p} \cdot \bar{p}'c^2$$

Neglect e- mass, hence

$$E^2 = p^2c^2 + m^2c^4 \approx p^2c^2$$

and use $\bar{p} \cdot \bar{p}' = |p||p'|cos\theta$

$$q^2 = 2EE'(\cos\theta - 1)$$

θ is the scattering angle, note it is always ≤ 0 .

We refer to this type of interactions as space-like. In experiments we measure:

$$\boxed{Q^2 = -q^2 = 2EE'(1 - \cos\theta)}$$

High- Q^2 interactions means larger θ .

4.2.2 Proton momentum fraction x

A crucial Lorentz invariant quantity defined in DIS is $x = -q^2/(2P \cdot q)$. This x is interpreted as the **fraction of the incoming proton's momentum carried by the struck parton** in a frame where the proton has a very large momentum. Detailed analysis showed these partons were consistent with spin-1/2 particles with fractional charges, supporting the quark model. *Note: The initial parton model described partons as quarks, but later partons also included neutral gluons within the proton.*

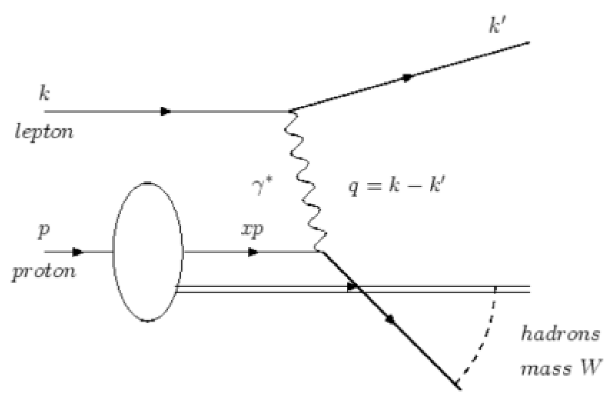


Figure 4.1: Deep inelastic scattering. $q = k - k'$ is the momentum transferred from lepton to the parton. Taken from lecture notes.

4.2.2.1 Bjorken Scaling

Let e- scatters off a massless parton carrying a fraction x of the proton's momentum.
4-momentum of proton:

$$P = (E_p, \bar{p}_p c)$$

4-momentum of parton before interaction:

$$xP = (xE_p, x\bar{p}_p c)$$

4-momentum is an invariant (Einstein equation $E^2 = (pc)^2 + (m_0c^2)^2$, Energy momentum equation: $P^2 = E^2 - (pc)^2$, pc term cancel out hence $P^2 = (m_0c^2)^2$) for the parton:

$$(xP)^2 = (m_{parton}c^2)^2$$

since parton is massless, $m_{parton} = 0$

$$(xP)^2 = 0$$

Assuming the parton remains massless after the interaction, the square of its new four-momentum is zero: $(xP + q)^2 = 0$

$$(xP)^2 + 2xp \cdot q + q^2 = 0$$

$$2xP \cdot q = -q^2$$

$$\boxed{x = \frac{-q^2}{2P \cdot q}}$$

gives the fraction of the proton's momentum carried by a parton.

4.2.3 Parton Distribution Functions (PDFs)

The electron-proton cross-section (σ_{ep}) in DIS can be described as:

$$\sigma_{ep}(x)dx = \sum_{\text{quark flavours i}} \sigma^i_{eq} \times f_i(x)dx$$

- σ^i_{eq} electron-quark cross section of each quark flavour i
- $f_i(x)$ are the PDF
- $f_i(x)dx$ is the number of partons of a certain type in the proton with momentum fraction between x and x+dx

PDFs are **universal**: they are measured experimentally and then used to predict cross-sections at other colliders. In experiments at HERA it is also measured the "proton structure functions", example of one:

$$F_2(x) = \sum_i e_i^2 x f_i(x) \tag{4.1}$$

where e_i is the quark charge

4.2.3.1 Scaling Violations

The simple quark model predicted that structure functions (like Eq. 4.1) should be independent of the four-momentum transfer Q^2. While this held for some x values, smaller x showed a clear **increase of structure functions with** Q^2 Fig 4.2. This 'scaling violation' indicated that the number of partons in the proton depends on how energetically it is probed.

What causes scaling violation?
The proton is not made up of only 3 valence quarks. Quarks within a hadron are continuously interacting by exchanging gluons, which can momentarily split into quark-antiquark pairs (quantum fluctuations) Fig 4.3. Larger Q^2 means that the structure is probed to a higher resolution, hence more sea quarks are seen.

[1]Taken from: Raj, Dharma & Kaur, Prabhdeep. (2020). Calculation of Structure Functions of Protons using Electron Proton Scattering Cross Sections with HERA Master of Science in Physics. 10.13140/RG.2.2.28573.13287.

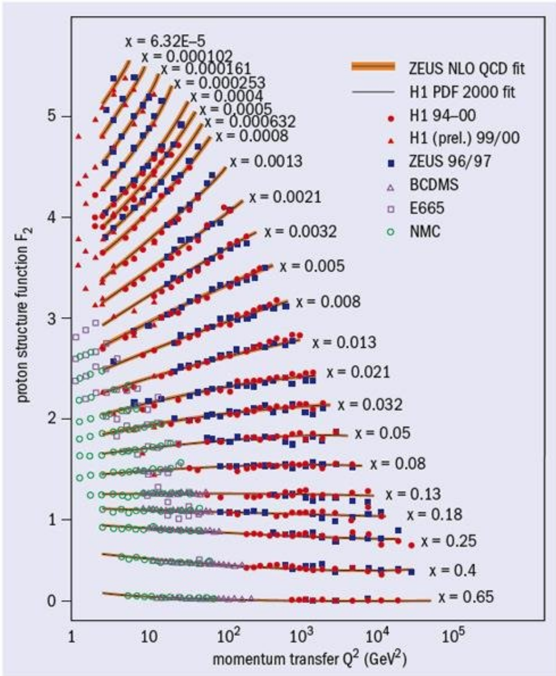


Figure 4.2: Proton structure function versus Q^2 for different x values.[1]

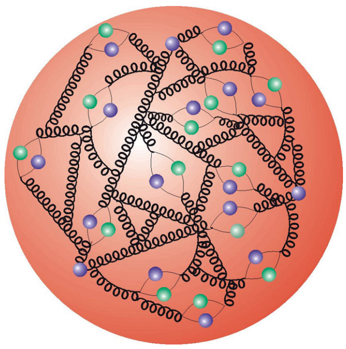
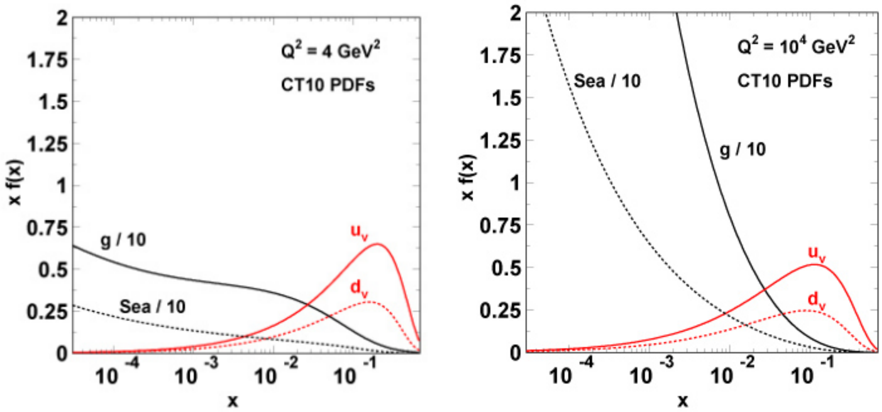


Figure 4.3: Parton content of a baryon. The unpaired quarks are called valence quarks, the paired ones represent the "sea of quarks".

4.2.3.2 PDF shape

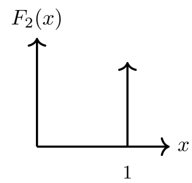
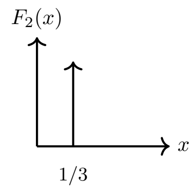
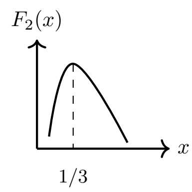
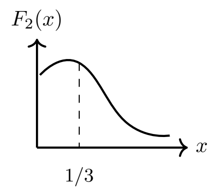
Integration of the combined quark and antiquark PDFs ($xQ(x)+x\bar{Q}(x)$) only accounts for roughly 50% of the proton's momentum. This indicates that the other **50% of the proton's momentum is carried by gluons**. In Fig. 4.4 up and down valence PDFs are shown (u_v and d_v) as well as the sea quark and gluon PDFs. The latter two are divided by 10 so that they can fit onto the plot. As Q^2 increases the contribution from the sea quarks and the gluons get very large at low x values. In table 4.2 are represented different hypothetical proton models.



(a) (b)

Figure 4.4: Proton PDFs at different Q^2 values.

Table 4.2: Hypothetical Proton Model and Predicted $F_2(x)$ Shape

Proton Model Assumption	Description of Partons	Predicted Shape of $F_2(x)$	Shape of $F_2(x)$
A single point-like particle	The proton acts as a single entity.	A sharp peak at $x = 1$, as the single particle carries all the momentum.	
3 independent valence quarks	The proton consists of three quarks that do not interact and share the momentum equally.	A sharp peak at $x = 1/3$.	
3 interacting valence quarks	The three valence quarks can exchange momentum among themselves via gluons.	A broader peak centered around $x = 1/3$, as the momentum is no longer equally distributed.	
Valence quarks, sea quarks, and gluons	The proton contains valence quarks plus a 'sea' of virtual quark-antiquark pairs and gluons that appear at low x.	A peak around $x = 1/3$ from valence quarks and a significant rise at low x values due to contributions from sea quarks and gluons.	

4.3 Colour

- **Colour Saves the Pauli Principle:** The Pauli exclusion principle states that identical fermions cannot occupy the exact same quantum state. The Ω^- baryon (spin 3/2), composed of three strange quarks with aligned spins, presented an apparent violation of this principle.

To resolve this, it was proposed that quarks possess an additional intrinsic property called **colour**. Quarks can exist in a baryon with the same spin if they have different colours.

- **Colour Charges:** Quarks exist in three colour states: r, g and b. These colour states are independent of quark flavour. Each state has two conserved quantum numbers: **colour isospin charge** (I_C^3) and **colour hypercharge** (Y^C), which are additive Table 4.3.

Table 4.3: Colour charges for quarks and antiquarks.

Colour State	I_C^3 (Colour Isospin)	Y_C (Colour Hypercharge)
Quarks		
red (r)	1/2	1/3
green (g)	−1/2	1/3
blue (b)	0	−2/3
Antiquarks		
antired ($\bar{r}$)	−1/2	−1/3
antigreen ($\bar{g}$)	1/2	−1/3
antiblue ($\bar{b}$)	0	2/3

Note: These values represent the two conserved colour charges that characterise each state. They are analogous to electric charge in QED.

- **Colour Singlets (Hadrons):** Hadrons are only observed as **colour singlets**, meaning their total colour charge is zero ($I_C^3 = Y^C = 0$). This is achieved by combining a colour with its anticolour (e.g., $r\bar{r}$) in **mesons** ($q\bar{q}$) or by combining red, green, and blue quarks in **baryons** (qqq). The colour wavefunction for a meson is given by the following superposition:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

For a baryon is given by:

$$\psi = \frac{1}{\sqrt{6}}(r_1g_2b_3 - g_1r_2b_3 - r_1b_2g_3 + g_1b_2r_3 + b_1r_2g_3 - b_1g_2r_3)$$

where r_1 indicates that quark 1 is in the red colour state, and so on.
<u>Note</u>*: By swapping the indices of any two of the quarks (e.g. 1 2) the wavefunction is antisymmetric*

Free quarks or combinations like qq or $qq\bar{q}$ are not allowed due to colour confinement. Exotic combinations like pentaquarks ($qqqq\bar{q}$) are colourless and have been observed.

- **Gluons and Colour Flow:** Gluons carry both a colour and an anti-colour, as required by colour charge conservation at quark-quark scattering vertices.

- **Evidence for $N_C = 3$:** The number of quark colour states (N_C), can be computed from rates of two different outcomes in electron-positron (e^+e^-) annihilation experiments. The process involves the e^+e^- pair annihilating into a virtual photon, which then decays into a pair of fermions. The two possible outcomes are:
  - The production of hadrons: $e^+e^- \to \text{hadrons}(q\bar{q})$.
  - The production of a muon-antimuon pair: $e^+e^- \to \mu^+\mu^-$.

Define a ratio, R, of the cross-section for producing hadrons to the cross-section for producing muons.

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

- The rate (cross-section) of each decay/process channel $X \to f\bar{f}$ is proportional to the square of the final state fermion electrical charge (Q_f^2) and its number of colour state (N_c)
  - Leptons have colour state: $N_c = 1$

  The accessible quark flavours at a centre-of-mass energy of around 30 GeV (specifically chosen to neglect fermion masses) are the quarks: u, d, s, c, b except the top quark, which is too massive to be produced. R becomes:

$$R = \frac{\sigma(u\bar{u}) + \sigma(d\bar{d}) + \sigma(c\bar{c}) + \sigma(s\bar{s}) + \sigma(b\bar{b})}{\sigma(\mu^+\mu^-)} = \frac{N_C \times \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]}{(-1)^2}$$

  This simplifies to:

$$\boxed{R = N_C \times \frac{11}{9}}$$

  The experimentally measured R ratio is approximately 33/9, which implies there are 3 colours ($N_C = 3$).

4.4 Asymptotic Freedom

Asymptotic freedom states that the strong interaction becomes **weaker at high energies** (and therefore short distances).

- **Coupling Constants:** The strength of an interaction is defined in terms of the coupling of the force carrying particles to the charge of the interaction:
  - photon to electrical charge in QED: α coupling
  - gluon to colour charge in QCD: α_s coupling

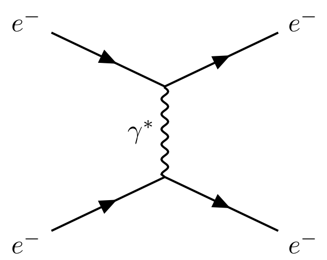


Figure 4.5: Electron-electron scattering via virtual photon exchange. The strength of the interaction for each vertex is given by

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$$

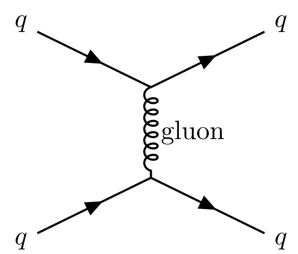


Figure 4.6: Strong interaction scattering of two quarks via gluon exchange. The strength of the interaction for each vertex is given by

$$\alpha_s \sim 0.1$$

  These 'constants' are not truly constant but **depend on the four-momentum transfer** (Q^2) of the measurement. This phenomenon is known as the **running** of the coupling constant.

- **QED Coupling (α):** In Quantum Electrodynamics (QED), the "bare" charge of an electron is shielded by a phenomenon known as **photon vacuum polarization**. This process describes how the vacuum around the electron is filled with virtual electron-positron pairs that constantly fluctuate into existence. The virtual positrons are attracted toward the central electron, while the virtual electrons are repelled.

This polarization creates a cloud that has a **screening effect**, reducing the electron's observable charge when measured from a distance. When probed at higher energies (shorter distances), this screening is penetrated, revealing more of the bare charge. Consequently, the effective charge and thus the coupling strength appears to increase. This effect is represented by diagrams containing a fermion loop, as shown in Fig 4.7a.

- **QCD Coupling (α_s):** Two competing effects are at play.
  1. **Screening**: Similar to QED, virtual quark-antiquark loops screen the color charge. This effect, on its own, would cause α_s to increase at high energy. Fig 4.7b
  2. **Antiscreening**: Gluons carry color charge themselves and can form self-interacting loops, Fig 4.7c. This has the opposite effect of antiscreening the color charge, causing the coupling to grow weaker at high energies. The antiscreening from gluon loops is the dominant effect in QCD. As a result, α_s decreases as energy increases (Q^2 increases), leading to **asymptotic freedom**.

  The strong coupling constant is given by combination of these two types of effects and is given approximately by:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + B\alpha_s(\mu^2)\ln\left(\frac{Q^2}{\mu^2}\right)} \tag{4.2}$$

  where $B = \frac{33-2N_f}{12\pi}$, N_f is the quark flavours, μ is some reference four-momentum value at which α_S is measured. Has to be experimentally measured. At $\mu^2 = M_Z^2$, $\alpha_S(M_Z^2) = 0.118$.

  Since the number of quark flavours N_f is 6, the term $33 - 2N_f$ is positive, ensuring that α_s decreases with increasing Q^2.

- **Perturbation Theory Breakdown:** At lower energies (larger distances), the strong coupling constant α_s becomes very large. In this region, **perturbation theory**, which relies on the coupling being a small number for reliable calculations, **breaks down**. This means QCD is not predictive at low momenta, often requiring empirical models.

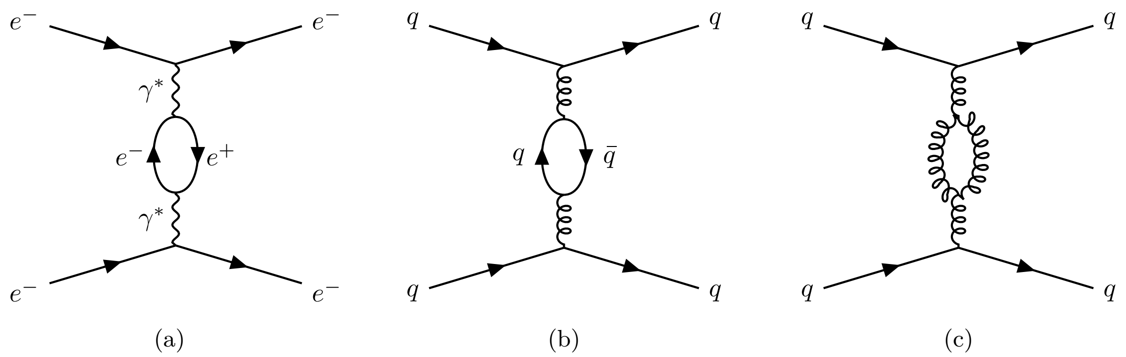


Figure 4.7: Higher-order corrections to scattering processes. Diagram (a) shows screening in QED. Diagrams (b) and (c) show screening and antiscreening in QCD, respectively.

4.5 Colour Confinement

Colour confinement is the postulate, supported by experiment, that **free coloured objects (quarks and gluons) are never observed in nature**. Instead, they are always confined within colour-singlet hadrons.

- This phenomenon arises because the strong interaction continues to get **stronger at larger distances**.

- The potential energy between two coloured objects **increases linearly with distance**:

$$V(r) = -\frac{\alpha_s \hbar c}{r} + \lambda r$$

  where λ is a constant measured experimentally.

- At short distances ($r \leq 0.1$ fm), the potential is Coulomb-like (one-gluon exchange).
- At larger distances, exchanged gluons are believed to be squeezed into a 'tube', and the stored energy in this tube is proportional to the distance.
- This linear increase in potential energy means an **infinite amount of energy would be required to separate coloured objects to infinity**, thus preventing them from being observed freely.

4.6 Jets

Quarks and gluons are never observed in isolation due to color confinement. In high-energy scattering processes, where quarks or gluons are produced, they instead manifest as collinear streams of hadrons known as **jets**.

- **Quark Jets:** In a process like $e^+e^- \to q\bar{q}$, a high-energy quark and antiquark are produced moving back-to-back. As they separate, the energy in the colour field between them increases linearly. It eventually becomes more energetically favourable to create new quark-antiquark pairs from this energy. This process, called **hadronization**, continues until the energy is dissipated, resulting in two back-to-back jets of hadrons.

  To a good approximation, the total momentum of the hadrons within a jet equals the momentum of the original quark.

- **Gluon Jets and Evidence for the Gluon** Occasionally, in e^+e^- collisions, three distinct jets are observed instead of two. These events occur when one of the initial quarks radiates a high-momentum gluon, which then independently forms its own non-collinear jet, Figure 4.8. The observation of these three-jet events was the first direct evidence for the existence of the gluon.

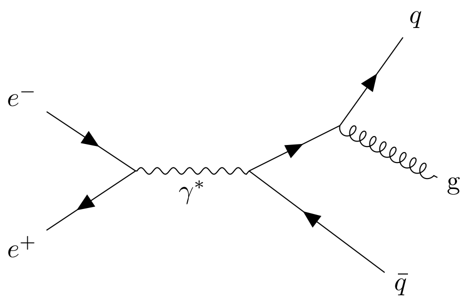


Figure 4.8: A Feynman diagram for a three-jet event, $e^+e^- \to q\bar{q}g$.

4.7 QCD Today

- Today, QCD is very successful in predicting interaction rates at colliders, especially at high momentum where it is 'incredibly successful' at predicting jet rates.
- However, at low momentum, where perturbation theory fails (as the coupling constant is too big), empirical models with parameters fitted to data are used.

Chapter 5

Electroweak Interaction: Part I

5.1 Introduction

This chapter delves into the electroweak interaction, which is a unified theory describing both electromagnetic and weak interactions. At low energies, these forces appear distinct, but the electroweak theory reveals their common origin. A central theme is the discovery of fundamental asymmetries in the weak force, namely parity (P) and charge conjugation (C) violation, which are explained by its unique spin structure.

5.2 Symmetries and Conservation Laws

According to Noether's theorem, every continuous symmetry of a physical system corresponds to a conserved quantity. For example:

- Spatial translation symmetry leads to the conservation of momentum.
- Rotational symmetry leads to the conservation of angular momentum.
- Time translation symmetry leads to the conservation of energy.

5.3 Parity (P)

Parity is a quantum number related to the symmetry of space-inversion, where spatial coordinates are inverted ($\vec{x} \to -\vec{x}$). Effect of space-inversion:

1. Position vector $\vec{r}$ becomes $-\vec{r}$
2. Momentum vector $\vec{p}$ becomes $-\vec{p}$
3. Angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ remains unchanged

Parity (P_a) is equal to the eigenvalue of any wavefunctions that is an eigenstate of the space-inversion transformation operator $\hat{P}$:

$$\hat{P}\psi_a = P_a\psi_a$$

Appling $\hat{P}$ twice has no effect on the eigenstate:

$$\hat{P}\hat{P}\psi_a = \hat{P}P_a\psi_a = P_a^2\psi_a = \psi_a$$

- **Fermion** and its **antiparticle** have **opposite parity**
- **Boson** and its **antiparticle** have the **same parity**

For a system of multiple particles with orbital angular momentum L, the total parity is multiplicative:

$$\boxed{P_{total} = P_a P_b \dots (-1)^L}$$

Conservation of parity in an interaction can be considered in one of two ways, which amount to the same question:

1. Is the interaction invariant under the space-inversion transformation?
2. Is the total parity before the interaction equal to the total parity after the interaction?

It was believed that all fundamental interactions conserved parity.

5.3.1 The $\tau - \theta$ Puzzle and Parity Violation

It is experimentally determined that $\boldsymbol{\pi^-}$ and $\boldsymbol{\pi^0}$ have **parity** of **-1**.
In the 1950s, two particles, the θ^+ and τ^+, were observed to have the same mass, charge, and spin, suggesting they were the same particle. However, they decayed into final states with opposite parities:

- $\theta^+ \to \pi^+\pi^0$: The final state has a parity of $P = (-1)^2(-1)^0 = +1$.
- $\tau^+ \to \pi^+\pi^0\pi^0$: The final state has a parity of $P = (-1)^3(-1)^0 = -1$.

If parity were conserved in weak decays, these could not be the same particle. This led to propose that the weak interaction might violate parity conservation.

5.3.2 Parity Violation

Chien-Shiung Wu's experiment on the β-decay of polarized (all spins are aligned) Cobalt-60 ($^{60}Co \to {}^{60}Ni^* + e^- + \bar{\nu}_e$) confirmed this hypothesis.

The experiment showed that the emitted electrons were preferentially directed opposite to the nuclear spin of the ^{60}Co nuclei. Under a parity transformation, the electron's emitted direction would reverse while the nuclear spin (an angular momentum) would not. If parity were conserved, the electron emission would be isotropic with respect to the spin direction, as **parity conservation demands that a physical process and its spatial inversion be indistinguishable.** The observed asymmetry was clear **proof of parity violation in the weak interaction.**

This resolved the puzzle: the $\boldsymbol{\theta^+}$ and $\boldsymbol{\tau^+}$ are indeed the **same particle**, now known as the charged kaon (K^+), which can decay into final states of opposite parity because the **weak force does not conserve parity.**

Note: Strong and electromagnetic interactions do conserve parity.

5.4 Charge Conjugation (C)

Charge conjugation ($\hat{C}$) is a discrete **symmetry that transforms a particle into its antiparticle**. Eigenvalues equation for it:

$$\hat{C}\psi_a = C_a\psi_a \tag{5.1}$$

ψ_a is only an eigenstate of $\hat{C}$ if the particle is its own antiparticle: $a = \bar{a}$, e.g. for the photon and the neutral pion. C_a known as the intrinsic C-parity of the particle.

Apply the operator twice:

$$\hat{C}\hat{C}\psi_a = \hat{C}C_a\psi_a = C_a^2\psi_a = \psi_a$$

hence $C_a = \pm 1$.

- **Conservation:** The total C-parity of an interaction is conserved if the interaction is symmetric under particle $\leftrightarrow$ antiparticle exchange.

- For particles that are not their own antiparticles, eigenstates of charge conjugation can only be formed by linear combinations of a particle−antiparticle pair, e.g. $\sqrt{\frac{1}{2}}(K^0 + \overline{K}^0)$ where $K^0 = d\overline{s}$ and $\overline{K}^0 = s\overline{d}$.

  Notes: *In the case that $a \neq \overline{a}$, the phase factor C_a in Eq 5.1 cannot be measured and can therefore be set to 1.*

- C-parity is a multiplicative quantum number.
- **Weak interaction is not invariant under $\hat{C}$.**
- **Electromagnetic and strong interactions are invariant under $\hat{C}$.**

5.4.1 Parity and charge conjugation violation in muon decay

- **C-symmetry violation** Evidence came from the decay of polarized muons and antimuons:

$$\mu^- \to e^- \bar{\nu}_e \nu_\mu \quad \text{and} \quad \mu^+ \to e^+ \nu_e \bar{\nu}_\mu$$

  The angular distribution of the decay products was found to be different for the particle and antiparticle decays, with a rate given by:

$$R_{e^\pm}(\theta) = \frac{1}{2}\Gamma_{\mu^\pm}(1 \pm \frac{1}{3}\cos\theta)$$

  - $R_{e^-}(\theta)$ is the rate of electrons from muon decay as a function of the angle between the muon spin and the electron direction.
  - $R_{e^+}(\theta)$ is the same thing for positrons in antimuon decay.
  - Γ_{μ^-} is the muon decay rate constant.
  - Γ_{μ^+} is the antimuon decay rate constant.

  The **opposite sign in the** $\cos\theta$ term for electrons versus positrons is a **direct violation of C-symmetry**. If charge conjugation symmetry were conserved, the law for decay would be identical. Since a charge conjugation transformation turns a muon (μ^-) decay into an antimuon (μ^+) decay, the mathematical formulas describing both processes would be expected to be identical.

  Combined CP transformation (charge conjugation followed by parity) restores the symmetry in this case, as the sign changes twice. For many years, it was thought that CP was a conserved symmetry.

- **Parity violation:** Parity transformation is equivalent to changing the angle from θ to $\pi - \theta$. Appling the change, $cos(\pi - \theta) = -cos\theta$, hence parity transform alters the formula for the decay rate.

5.5 The Spin Structure of Weak Interactions

The violation of P and C symmetries is fundamentally explained by the chiral nature of the weak interaction.

This is described by the concept of **helicity**, which is the projection of a particle's spin ($\vec{s}$) onto its direction of motion ($\vec{p}$):

$$H = \frac{\vec{s} \cdot \vec{p}}{|\vec{p}|}$$

Projection of a fermions spin can take two possible values: +1/2 or -1/2. Therefore fermions on can either have

- **Positive helicity** (**right-handed**, Fig 5.1): H = +1/2

- **Negative helicity** (**left-handed**, Fig 5.1): H = -1/2

Only **left-handed neutrinos** (ν_L) and **right-handed antineutrinos** ($\bar{\nu}_R$) **interact weakly**. This means that $\bar{\nu}_L$ **and** ν_R **do not exist**.

Important: Weak force preferentially interacts with left-handed particles and right-handed antiparticles.

For massive fermion, observer can travel faster and overtake the particle, and left-handed particle would then appear right-handed (as v would look like going in opposite direction). Therefore, for fermions other than neutrinos we cannot say that they are 100% left- or right-handed.

Figure 5.2 shows:

- Parity transformation on a ν_L creates a ν_R (the momentum changes direction and the spin remains unchanged). ν_L do not exist, demonstrating that **parity is violated**.
- Charge conjugation transformation on a ν_L turns it into a $\bar{\nu}_L$, which does not exist, demonstrating that **charge conjugation is violated**.
- Combined CP transformation change a ν_L into a $\bar{\nu}_R$, which does exist, so it appears **CP can be conserved**.

(b) Left-handed, Negative helicity.

(b) Right-handed, Positive helicity.

Figure 5.1: Diagram inspired form "Introduction to Elementary Particles" by David Griffiths, edition 2, page 138.

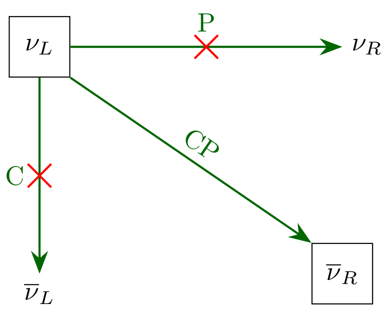


Figure 5.2: Effect of Parity (P), Charge Conjugation (C), and CP transformations on neutrino helicity states. Red cross indicates violation.

5.5.1 β decay revised

Consider the process from Section 5.3.2:

$$^{60}Co \to {}^{60}Ni^* + e^- + \bar{\nu}_e$$

here Co has spin 5 and Ni has spin 4.

- Conservation laws require:
  - e^- and $\bar{\nu}_e$ to have spins (both spin-1/2) aligned with the nuclear spin
  - Conservation of momentum: e^- and $\bar{\nu}_e$ must travel in opposite directions
- There are two possible resulting configurations for the particle helicities:

  - **Allowed:** A $\bar{\nu}_R$ and a left-handed electron [1]. This occurs when the electron is emitted opposite to the direction of the nuclear spin, this way the ν and e^- spins align to add up to 1 in ^{60}Ni direction.
  - **Forbidden:** A $\bar{\nu}_L$ (the forbidden state) and a right-handed electron. This would occur if the electron were emitted in the same direction as the nuclear spin.
- This explains the experimental result: electrons are preferentially emitted opposite to the nuclear spin, demonstrating parity violation.

5.5.2 Muon decay revised

- Consider the configuration with the highest possible electron energy, this requires the resulting neutrino and antineutrino to travel back-to-back with the electron (momentum conservation). Consider the following configurations:
  1. Electron travels opposite to the decaying muon's spin, angular momentum is conserved by the electron having negative helicity (hence having spin in the muon's spin direction).
  2. Electron travels in the same direction as the muon's spin, conservation of angular momentum requires the electron to have positive helicity.
- In the **relativistic limit**, the **positive-helicity** state for the electron is **suppressed** (since weak force favours negative helicity for particles, hence left-handed particles). The electron's mass is very small compared to the decaying muon, this is a good approximation.
- This suppression explains the experimental observation that electrons are preferentially emitted in the direction opposite to the muon's spin, hence 1 is the dominate configuration.

5.5.3 Pion decay

The decay:

$$\pi^+ \to l^+ + \nu_l$$

where l^+ is a positron or an antimuon.

The branching ratio to $\mu^+\nu_\mu$ is 99.9% and to $e^+\nu_e$ is 0.01%, why?

- **Conservation Laws:** The spin-0 pion (contains anti-aligned spins) must decay into two spin-1/2 particles with opposite spins. To conserve both linear and angular momentum, so two decay products must be emitted back-to-back with opposite spins.
- **Required Helicity:** Since the weak interaction only couples to left-handed neutrinos, the emitted neutrino (ν_l) **must be left-handed**. To conserve angular momentum, the corresponding charged lepton (l^+) must therefore also be left-handed.
- **Helicity Suppression:** The weak interaction preferentially produces right-handed antiparticles. A **left-handed antiparticle** (l^+) is a **"forbidden" helicity state**. The probability of producing this state is not zero but is suppressed by a factor proportional to:

$$\boxed{\left(\frac{m_l c^2}{E_l}\right)^2} \tag{5.2}$$

  where m_l is the lepton's mass and E_l is its energy.
- **Suppression factor ratio:** Considere rest frame of π
  1. Use energy and momentum conservation in the pion's rest frame.

$$m_\pi c^2 = E_l + E_\nu \quad \text{(Energy Conservation)}$$

$$|\vec{p}_l| = |\vec{p}_\nu| \quad \text{(Momentum Conservation)}$$

[1] In the massless limit

2. For a massless neutrino, $E_\nu = p_\nu c$. Combining this with momentum conservation gives:

$$E_\nu = p_\nu c = p_l c$$

3. Substitute this into the energy conservation equation:

$$m_\pi c^2 = E_l + p_l c$$

4. Rearrange for the lepton's momentum term and square it.

$$p_l^2 c^2 = (m_\pi c^2 - E_l)^2$$

5. Substitute this into the relativistic energy-momentum relation for the massive lepton ($E_l^2 = p_l^2 c^2 + m_l^2 c^4$).

$$E_l^2 = (m_\pi c^2 - E_l)^2 + m_l^2 c^4$$

6. Expand and rearrange:

$$E_l = \frac{(m_l^2 + m_\pi^2)c^2}{2m_\pi}$$

7. Sub. this into suppression factor Eq. 5.2

$$\text{Suppression} = \left(\frac{2m_l\, m_\pi}{m_l^2 + m_\pi^2}\right)^2$$

8. Ratio of the suppression:

$$\frac{\text{Suppression}(e)}{\text{Suppression}(\mu)} = \left(\frac{m_e(m_\mu^2 + m_\pi^2)}{m_\mu(m_e^2 + m_\pi^2)}\right)^2$$

- **Conclusion:** Because $m_e \ll m_\mu$, the above ratio is $\approx 5.7 \times 10^{-5}$, hence e^- branch is far more suppressed.

  This explains why the decay $\pi^+ \to e^+\nu_e$ is rarer than $\pi^+ \to \mu^+\nu_\mu$.

Chapter 6

Electroweak Interactions: Part II, Quark Mixing

6.1 Introduction

While the strong and electromagnetic interactions conserve quark flavour, the weak interaction can change it. These **flavour-changing interactions** occur via the exchange of a $W^{\pm}$ **boson** and predominantly couple an "up-type" quark to a "down-type" quark. Exchange of a **Z boson preserves flavour**.

Quark	1^{st} **Gen.**	2^{nd} **Gen.**	3^{rd} **Gen.**
Up-type	u	c	t
Down-type	d	s	b

Table 6.1: Type and generation classification of quarks

6.2 Quark Mixing

Hadron decays revealed that weak interaction transitions are not confined within quark generations. The decays

$$K^- \to \mu^- \bar{\nu}_\mu \qquad\qquad K^0 \to \pi^+ \pi^-$$

have quark transit from 2^{nd} to 1^{st} generation ($s \to u$ transition). However, these transitions are suppressed compared to those within a generation.

6.2.1 The Cabibbo Angle and GIM Mechanism

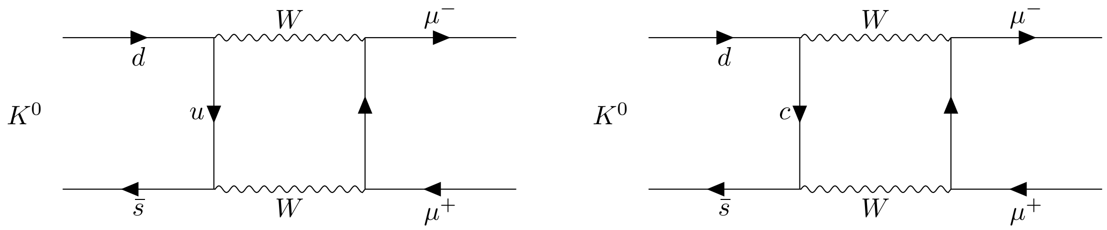
- Define weak eigenstate as the state of a particle as it participates in the weak interaction.
- Nicola Cabibbo proposed that the weak interaction eigenstates (d's') of the down-type quarks are a linear superposition of their mass eigenstates (ds).
- For the two known generations at the time, this was a rotation:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

  Here, θ_C is the **Cabibbo angle**, which parametrizes the mixing between the first two generations. ***Note:*** *Its convention that the mixing is between the "down-type" quarks.*

- This fixed a problem which was that certain flavour-changing neutral-current decays of kaons, which were expected to happen, were not seen in experiments [1]. The **GIM mechanism** explained that this was due to a **cancellation between two different decay pathways** (Fig. 6.1):
  - A) Pathway: Before the charm quark was proposed, the known decay path involved a transition with an intermediate up quark
  - B) Pathway (The GIM Proposal): The GIM mechanism proposed a second, competing pathway that involved a transition with the hypothetical charm quark.

  These two amplitudes are equal in magnitude but opposite in sign, they cancel each other out. This cancellation leads to the suppression of these specific kaon decays, explaining why they were not observed experimentally.



$A)\ M \sim \sin\theta_c \cos\theta_c$ $\qquad$ $B)\ M \sim -\sin\theta_c \cos\theta_c$

Figure 6.1: Two different K^0 decay pathways, this decay are suppressed since they have opposite mixing amplitudes.

6.2.2 The CKM Mixing Matrix

The Cabibbo-Kobayashi-Maskawa (V_{CKM}) matrix extend mixing formalism to a 3x3 unitary matrix. It relates the weak eigenstates (d', s', b') to the mass eigenstates (d, s, b) [2]:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

The matrix elements $|\mathbf{V_{ij}}|$ represent the strength of the **coupling between quark i** and **quark j**.
The matrix is unitary:

$$V_{CKM}V_{CKM}^{\dagger} = V_{CKM}^{\dagger}V_{CKM} = \begin{pmatrix} V_{ud}^* & V_{cd}^* & V_{td}^* \\ V_{us}^* & V_{cs}^* & V_{ts}^* \\ V_{ub}^* & V_{cb}^* & V_{tb}^* \end{pmatrix} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which leads to 9 constraints. The diagonals yields:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$
$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1$$
$$|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 1$$

For instance, bottom left zero constraint is:

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \tag{6.1}$$

[1] While the overall decay appears to be neutral current process, the diagram shows that the true mechanism. A simple favour changing neutral current is forbidden, so the decay must proceed through the charged current.

[2] As before this is just convention. Up-quarks eigenstates can be used as well.

This relation can be visualized as a triangle in the complex plane, known as the **Unitarity Triangle**, specifically for this relation (Fig. 6.2):

$$1 + z_1 + z_2 = 0 \qquad \text{where} \quad z_1 = \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*},\ z_2 = \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}$$

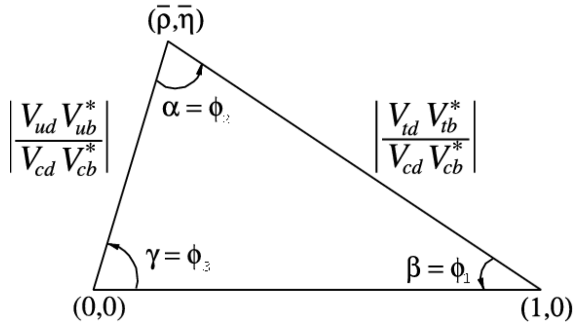


Figure 6.2: The unitarity triangle in the complex plane, visualising Eq. 6.1. The sides are labelled with ratios of CKM elements.[3]

Common parametrisation of the CKM matrix is given by Wolfenstein:

$$\begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

where $\lambda = sin\theta_C \approx 0.2$ and $A \approx 0.8$. The matrix elements empirically computed [4]:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 0.9742 & 0.2257 & 3.59\times10^{-3} \\ 0.2256 & 0.9733 & 41.5\times10^{-3} \\ 8.74\times10^{-3} & 40.7\times10^{-3} & 0.9991 \end{pmatrix}$$

the **coupling is largest along the diagonal**, so **mixing is most probable within the same generation**.

6.2.3 Kaon States and CP Violation

The neutral kaon system, consisting of the K^0 $(d\bar{s})$ and its antiparticle $\bar{K}^0$ $(s\bar{d})$, is a prime example of mixing and CP violation. Since they are not the antiparticle of each other $(a \neq \bar{a})$, $\mathbf{K}^0$ and $\bar{\mathbf{K}}^0$ are **not charge conjugate** nor CP eigenstates.

$$\hat{C}|K^0\rangle = -|\overline{K}^0\rangle; \qquad \hat{C}|\overline{K}^0\rangle = -|K^0\rangle \tag{6.2}$$

$$\hat{P}|K^0\rangle = -|K^0\rangle; \qquad \hat{P}|\overline{K}^0\rangle = -|\overline{K}^0\rangle \tag{6.3}$$

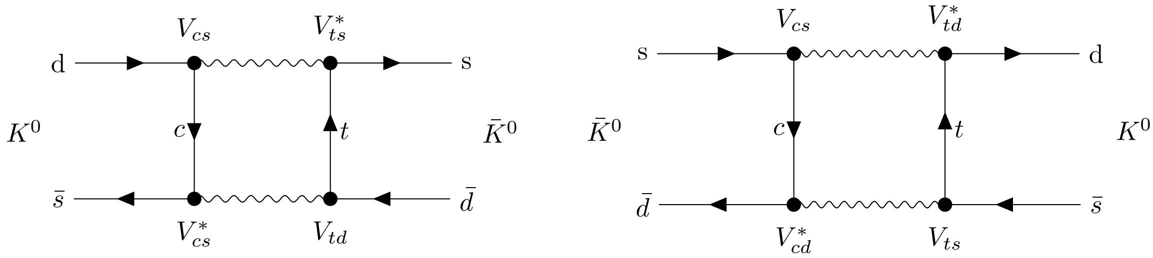
$$\hat{CP}|K^0\rangle = +|\overline{K}^0\rangle; \qquad \hat{CP}|\overline{K}^0\rangle = +|K^0\rangle \tag{6.4}$$

Note: *The negative sign when $\hat{C}$ is applied is just convention.*
The -1 once $\hat{P}$ is applied indicates that K^0 has -1 parity.

Weak interaction can turn K^0 to $\bar{K}^0$, via the so called box diagram Fig. 6.3. This means that the observed particles can be superpositions of K^0 and $\bar{K}^0$ states. We can chose these superpositions such that they are CP eigenstates.

$$|K^0_+\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \qquad |K^0_-\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \tag{6.5}$$

[3]Diagram taken from: arXiv:0911.3127v1 [hep-lat] 16 Nov 2009
[4]Data taken from arXiv:0911.3127v1 [hep-lat] 16 Nov 2009



$M_{fi} \propto A_{ct} V_{cd} V_{cs}^* V_{td} V_{ts}^*$ $\qquad$ $M'_{fi} \propto A_{ct} V_{cd}^* V_{cs} V_{td}^* V_{ts} = M_{fi}^*$

Figure 6.3: Exchange of two W bosons that convert a K^0 into a $\bar{K}^0$ and vice versa. Reminder: Probability amplitude is M_{fi}, decay rate is $\Gamma \propto |M|^2$.

Apply CP to both:

$$\hat{CP}|K_+^0\rangle = \frac{1}{\sqrt{2}}(\hat{CP}|K^0\rangle + \hat{CP}|\overline{K}^0\rangle) = \frac{1}{\sqrt{2}}(|\overline{K}^0\rangle + |K^0\rangle) = +|K_+^0\rangle \tag{6.6}$$

$$\hat{CP}|K_-^0\rangle = \frac{1}{\sqrt{2}}(\hat{CP}|K^0\rangle - \hat{CP}|\overline{K}^0\rangle) = \frac{1}{\sqrt{2}}(|\overline{K}^0\rangle - |K^0\rangle) = -|K_-^0\rangle \tag{6.7}$$

Possible decays

K^0 and $\bar{K}^0$ can decay into $\pi^+\pi^-$ and $\pi^0\pi^0$.

- $\boldsymbol{\pi^0\pi^0}$: are $\hat{CP}$ even, for K at rest, $L_\pi = 0$ due to momentum conservation:

$$CP(\pi^0\pi^0) = (P_{\pi^0})^2(-1)^L(C_{\pi^0})^2 = (-1)^2(-1)^0(1)^2 = +1 \tag{6.8}$$

- $\boldsymbol{\pi^+\pi^-}$: have parity of -1 and C interchanges π^- and π^+, hence $\hat{CP}$ leaves the $\pi^-\pi^+$ unchanged (hence even parity).

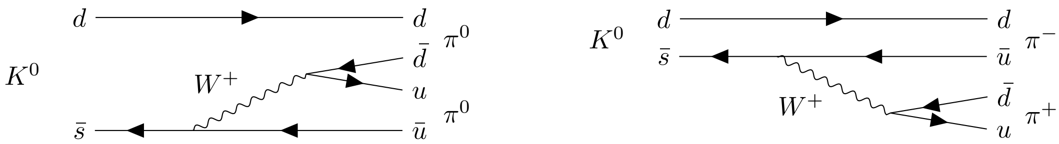


Figure 6.4: Feynman diagrams for the decay $K^0 \to \pi\pi$.

Assuming CP is conserved:

- The CP-even state $|K_+^0\rangle$ can decay quickly into two pions ($\pi\pi$, which is a CP-even state), so it is called the short-lived kaon, $K_{S(hort)}^0$.
- The CP-odd state $|K_-^0\rangle$ must decay into three pions ($\pi\pi\pi$, a CP-odd state), which has a much smaller phase space, resulting in a much longer lifetime; it is called the long-lived kaon, $K_{L(ong)}^0$.

Meson oscillation

- For a beam starting only as $|K^0\rangle$, which is a superposition of the state

$$|K^0\rangle = \frac{1}{\sqrt{2}}(|K_+^0\rangle + |K_-^0\rangle) \sim \frac{1}{\sqrt{2}}\left[|K_S^0\rangle(t) + |K_L^0\rangle(t)\right]$$

- $|K_S^0\rangle$ component of the mixture decays away very quickly, leaving the beam composed almost entirely of the long-lived $|K_L^0\rangle$ component.
- The surviving $|K_L^0\rangle$ state is itself a 50/50 superposition of the original flavor states, $|K^0\rangle$ and $|\overline{K}^0\rangle$.
- Therefore, a beam that began as pure $|K^0\rangle$ evolves into a state that contains a $|\overline{K}^0\rangle$ component. Hence the oscillation.
- Meson oscillation phenomena have been observed in many neutral meson (e.g. B^0, D^0)

Cronin and Fitch discovered that a small fraction of K_L^0 mesons decay into two pions, a direct violation of CP conservation ($-1 \to +1$). This violation arises from two sources:

- **Indirect CP violation**: The physical mass eigenstates K_L^0 and K_S^0 are not perfect CP eigenstates but contain a small contamination of the other, parameterized by ϵ.

$$|K_L^0\rangle = \frac{1}{\sqrt{1+\epsilon^2}}(\epsilon|K_+^0\rangle + |K_-^0\rangle)$$

$$|K_S^0\rangle = \frac{1}{\sqrt{1+\epsilon^2}}(|K_+^0\rangle - \epsilon|K_-^0\rangle)$$

[5] Both effects have been experimentally confirmed, but they are very small and insufficient to explain the large matter-antimatter asymmetry observed in the universe.

- **Direct CP violation**: The decay process itself violates CP symmetry, allowing a direct transition from a CP-odd state to a CP-even state.

Jarlskog Invariant and CP Violation

- CP violation arises because the CKM matrix contains a non-removable complex phase. This complex phase is the reason that the CKM matrix is not equal to its complex conjugate ($V \neq V^*$).
- This difference between V and V^* leads to different transition amplitudes M for a particle process and its corresponding antiparticle process, which is the physical manifestation of CP violation.
- The Jarlskog invariant, J, quantifies the magnitude of this difference (area of the CKM unitarity triangles), and therefore the magnitude of CP violation. A non-zero value for the Jarlskog invariant indicates that CP violation occurs.

[5] The negative sign in front of K_-^0 component is convention.

Chapter 7

The Electroweak Interaction: Part III

7.1 Problems with early Weak Theory

- Initial theories treated weak decays (like β-decay) as a four-point interaction (Fig. 7.1).

$$n \to pe^-\bar{\nu}_e$$

  This worked at low energies but led to unphysical, infinite cross sections at high energies.

- Introducing a massive W boson propagator solved this, but theories with multiple W bosons still produced infinities in calculations, indicating an incomplete model.

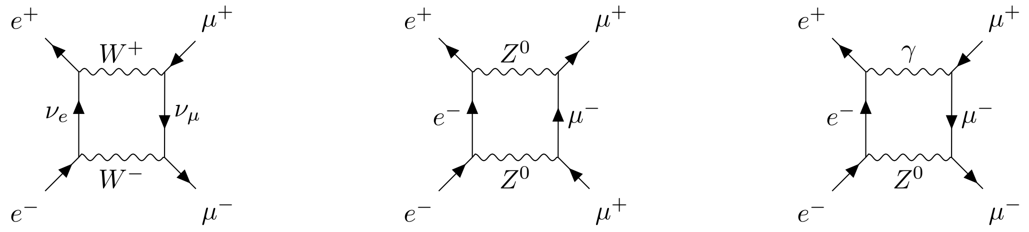
(a) Four-point interaction

(b) W^- boson exchange

Figure 7.1: Feynman diagrams for nuclear β-decay.

7.2 The Unified Electroweak Theory

- It unifies the electromagnetic and weak interactions.
- Predicted a new neutral boson (the Z boson), which led to an exact cancellation of the infinities.
- Consider the higher order correction to the $e^+e^- \to \mu^+\mu^-$ process involving 2 W bosons shown in Fig. 7.2. If only a) is considered, calculations lead to infinities, if b) and c) are included, the infinities cancel, leading to sensible results.
- At very high energies, the theory describes four massless bosons (W_1, W_2, W_3, B).
- **Spontaneous Symmetry Breaking:** As the universe cooled, this symmetry broke via the Higgs mechanism. The four massless bosons are mixed to form the physical massive bosons we observe today (W^+, W^-, Z) and one massless boson (the photon, γ).
- The **weak mixing angle** (Weinberg angle, θ_W) relates the couplings and masses of the bosons ($\cos\theta_W = M_W/M_Z$).



(a) WW box (b) Z^0Z^0 box (c) γ-Z Box

Figure 7.2: Higher-order corrections to $e^+e^- \to \mu^+\mu^-$.

- **Low energy approximation:** The apparent weakness of the weak force at low energies is due to the large mass of the W and Z bosons, which suppresses the interaction amplitude (recall from Section 1.7: amplitude of an interaction has the form $M \sim -\frac{g_W^2\hbar^2}{q^2-M_W^2c^2}$). At high energies $q^2 \gg M_W^2$, hence $M \sim -\frac{g_W^2\hbar^2}{q^2}$, the weak and electromagnetic interactions have similar strengths.

7.2.1 Discovery of Neutral Currents

- In 1973, the Gargamelle experiment at CERN was observed that muon-neutrino scattering off nucleons in the atoms of the detector two possible interactions occured:
  - **Charged current** interactions, caused by the exchange of charged W boson (Fig. 7.3).
  - **Neutral current** interactions, caused by the exchange of neutral Z^0 boson (Fig. 7.4).
- The neutral current interactions, only possible if the exchange particle is neutral (photon couples to electric charge and so cannot be responsible for this interaction), hence this was a direct evidence of Z^0 existence.

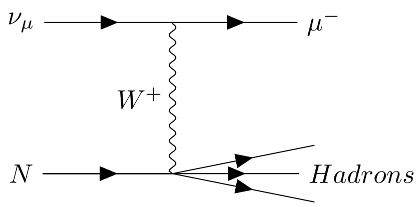


Figure 7.3: Charged Current interaction, ν_μ changes to μ^-

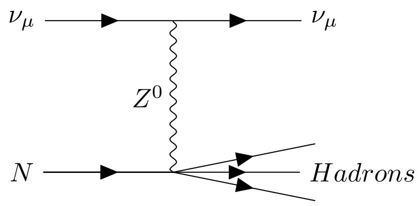


Figure 7.4: Neutral Current interaction, ν_μ is unchanged.

7.2.2 Discovery of W and Z Bosons

- Directly discovered in 1983 at CERN's SppS collider.
- They were produced in proton-antiproton collisions [1] and identified by searching for a peak (Breit-Wigner resonance) in the invariant mass distribution of their decay products.
- The Z boson was found via its decays to e^+e^- and $\mu^+\mu^-$.
- The W boson was found via its decays to $e\nu_e$ and $\mu\nu_\mu$, where the neutrino was inferred from missing transverse momentum.

[1] The actual fundamental interaction that produces W/Z is between q and $\bar{q}$.

7.3 The Higgs Boson and Gauge Invariance

- The Standard Model is a "gauge theory," requiring its equations to be invariant under local gauge transformations.
- To maintain this symmetry, the force-carrying gauge bosons must be massless. This created a paradox, as the W and Z bosons are known to be massive.

7.3.1 The Higgs Mechanism

- Solves the mass problem by introducing a **new field** (the **Higgs field**) that permeates the universe.
- The potential of this field is shaped like a "Mexican hat" fig. 7.5, with its minimum value (vacuum expectation value) being non-zero, all the other fields have zero expectation values in the vacuum.
- The universe "choosing" a point in this minimum spontaneously breaks the symmetry.
- **Particles acquire mass** through their **interaction with** this **non-zero vacuum field**. The theory's equations remain gauge invariant.
- The theory predicted a new particle: an **excitation of the Higgs field** known as the **Higgs boson** (just as photons are excitations of an EM field). This **boson couples to all particles with mass**, with a **strength proportional to their mass**.
- **Simply put**: the mass of boson is massless when on top at the local max, once the symmetry is broken (ball falling in the minima) the boson acquires mass. Stronger coupling (between particle and Higgs field) $\Rightarrow$ larger particle mass.

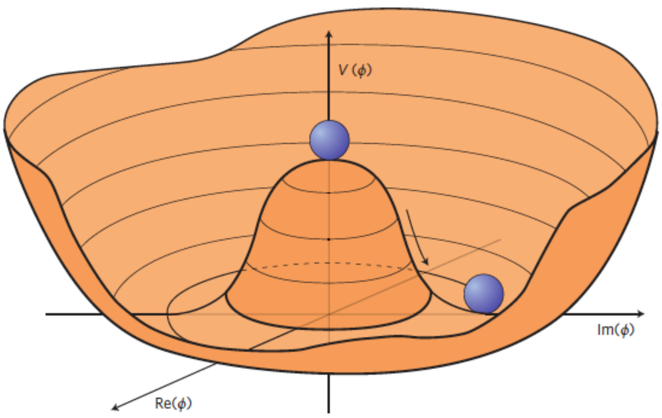


Figure 7.5: The potential of the Higgs field versus the real and imaginary components of the field. Diagram taken from: https://cds.cern.ch/record/2012465/plots

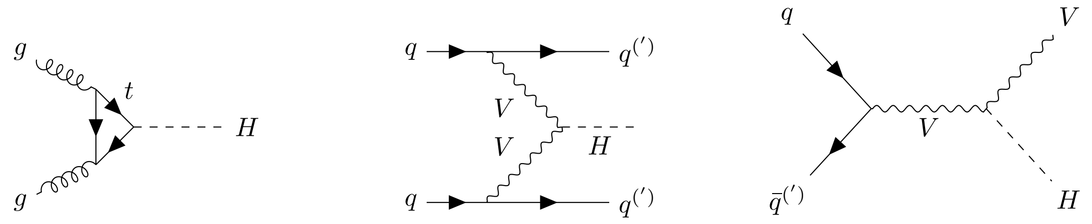
7.3.2 Discovery of the Higgs Boson

7.3.2.1 Production mechanisms

The main production mechanise are the following (from dominate to less probable):

1. **Gluon-gluon fusion:** Gluons are massless hence do not couple directly with the Higgs, process occurs via a quark loop (Fig. 7.6). The loop is usually made up of top quarks as it is the heaviest.
2. **Weak boson fusion:** Two quarks each radiate a $W^{\pm}$ or Z^0 boson, and this boson pair subsequently fuses to form a Higgs boson.

3. **$W^{\pm}$ or Z^0 associated production:** virtual $W^{\pm}$ or Z^0 is produced by quark-antiquark annihilation, and the boson radiates a Higgs boson.



(a) Gluon-gluon fusion

(b) Weak boson fusion

(c) Associated W/Z production

Figure 7.6: In these diagrams V indicates a $W^{\pm}$ or a Z boson.
Meaning of $q^{(\prime)}$:
If the process occurs via a Z boson, then the quark does not change flavour (q).
If it is via a $W^{\pm}$ boson, the outgoing quark is a different type (q').

7.3.3 Decay

- A Higgs boson will **decay** almost **instantaneously**.
- The Higgs coupling strength is proportional to a particle's mass, so it preferentially decays to the heaviest particles that are kinematically accessible [2]:

$$\Gamma(H \to f\bar{f}) \propto N_C \, m_f^2$$

- With a mass of $M_H = 125$ GeV/c^2, it cannot decay into two top quarks (each with $M_t = 175 GeV/c^2$).
- Theoretical branching fraction of the Higgs boson depending on the Higgs mass are illustrated in Fig. 7.7. The red line represents the mass of the Higgs boson. From the diagram:
  - The **dominant decay** channel is to a **bottom quark-antiquark pair** ($b\bar{b}$), as the bottom quark is the heaviest particle with less than half the Higgs mass.
  - Other important decays channels are W^+W^-, gg and $\tau^+\tau^-$ pairs.
- Another important decay is ($H \to \gamma\gamma$), even though photons are massless. This decay occurs via a loop of heavy virtual particles, most likely top quarks or W bosons.
- Before its discovery, the Higgs mass was unknown, and the branching fractions for its decays were highly dependent on this mass, which necessitated searching across many different decay channels.

7.3.4 The Challenge of the Higgs Discovery

- Higgs boson production is an extremely rare event (one in every 10 billion proton-proton collisions at the LHC).
- The primary challenge is to distinguish the rare Higgs signal from the immense number of other interactions (backgrounds) that can produce similar final-state particles.
- The discovery could not be based on identifying a single Higgs event, but rather on finding a statistically significant excess of events above the predicted background level.

[2] A question on this appears in 2018-2019 exam paper Q10, the solution neglect colour flavour.

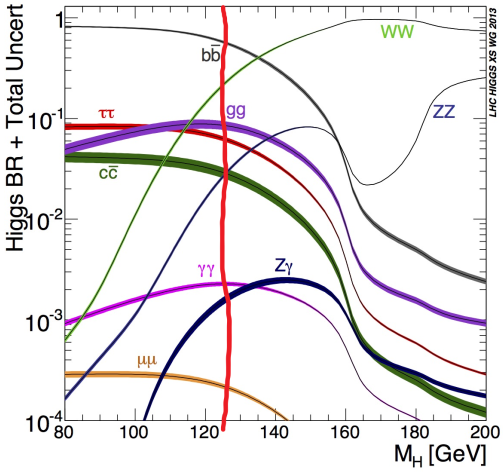


Figure 7.7: Theoretical branching fraction of the Higgs boson depending on the Higgs mass. Plot taken from `https://opendata.atlas.cern/docs/visualization/the-higgs-boson`

- The **search strategy** involves looking for a **peak in the invariant mass distribution of the decay products**. This peak is **expected to appear at the mass of the Higgs boson**.
- Excellent energy and momentum resolution in the detectors is critical. Poor resolution could smear out the peak, making it indistinguishable from the background continuum.
- Decay channels ending in electrons, muons, and photons are easier to analyse because these particles can be measured with very high resolution.
- Despite having the highest branching fraction (57%), the $H \to b\bar{b}$ decay was not a discovery channel. The Higgs was initially discovered through the much rarer $H \to ZZ$ and $H \to \gamma\gamma$ decay modes. See Fig. 7.8
- Why it is more difficult to discover H from decaying $\tau^+\tau^-$ than decaying $\mu^+\mu^-$, despite having larger branching fraction?
  - $\tau^+\tau^-$ decays much faster than μ, τ will decay before reaching detectors, only its decays products will reach.
  - τ decays:

$$\tau^- \to \nu_\tau + e^- + \bar{\nu_e} \qquad \tau^- \to \nu_\tau + \mu^- + \bar{\nu_\mu} \qquad \tau^- \to \nu_\tau + q\bar{q}$$

The neutrinos are undetected, hence we can't reconstruct the invariant mass.

(a) $H \to \gamma\gamma$ decay

(b) $H \to ZZ$ decay

Figure 7.8: Higgs boson discovery channels

$H \to ZZ$ discovery channel

The two Z will decays to a pair of fermions. The easiest to detect are e^+e^- and $\mu^+\mu^-$ pairs. So the combinations can be: $e^+e^-\mu^+\mu^-$, $e^+e^-e^+e^-$ or $\mu^+\mu^-\mu^+\mu^-$. See fig. 7.9 for background processes. Fig. 7.10 illustrates the boson peak from this process.

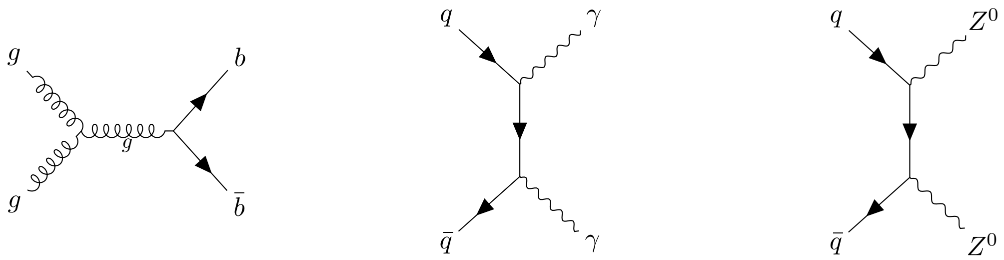


Figure 7.9: Leading order Feynman diagrams for important background processes in Higgs searches that lead to a similar signature.

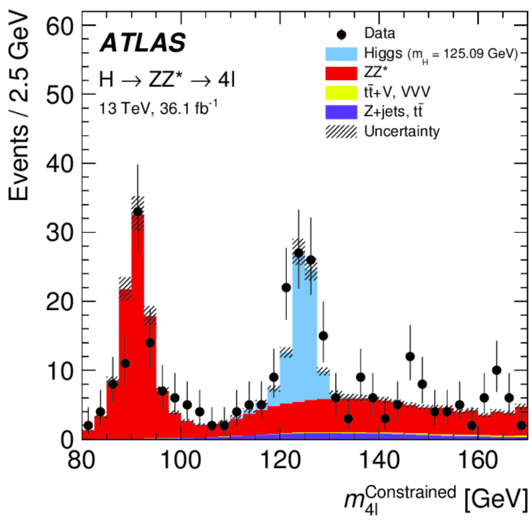


Figure 7.10: Invariant mass of 4 muons. The peak at 90 GeV/c^2 comes from $Z \to \mu^+\mu^-\mu^+\mu^-$ decays. Plot taken from DOI:10.1007/JHEP03(2018)095

$H \to \gamma\gamma$ discovery channel

Photons from this decay channel are identified in the calorimeters of the detectors. See Fig. 7.11.

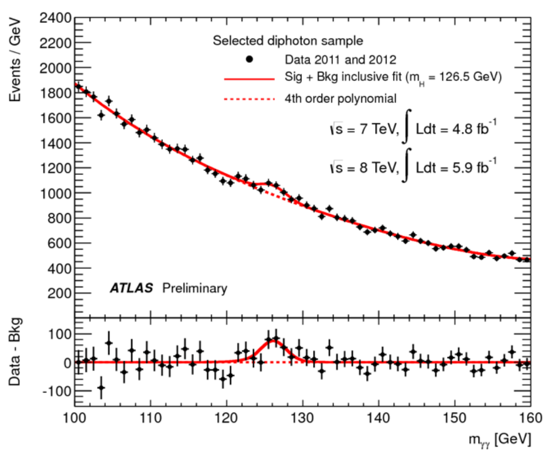


Figure 7.11: Invariant mass of two photons. The peak at 125 GeV/c^2 is due to $H \to \gamma\gamma$ decays.
Plot taken from: `https://atlas.cern/updates/feature/higgs-boson`

7.4 Particles discovery strategy

Look at question 3 exam 2021-2022. The question asks how to identify a hypothetical heavy particle (mass of 2.5 TeV/c^2) that decays via $X \to ZZ$ or $X \to WW$.

- **Origin of Particles:** The final state particles we detect (jets, leptons) originate from the subsequent decays of these W and Z bosons.
- **Strategic Goal:** The choice of final state is a compromise between maximizing the number of observable events (rate) and the ability to distinguish the signal from background (cleanliness).
- **Detected Particle Properties:**
  - **Leptons (e, μ):** Offer a very clean signature and precise momentum measurement. However, they come from low-probability (low branching ratio) decays.
  - **Quarks (Jets):** Result from high-probability decays, maximizing the event rate. Their signature is less clean and their energy is harder to measure precisely.
  - **Neutrinos:** Cannot be directly detected, leading to incomplete information and poor mass reconstruction.
- **Optimal Strategy:** Semi-leptonic final states:

  $$X \to W(\to l\nu) + W(\to jets) \qquad \text{or} \qquad X \to Z(\to l^+l^-) + Z(\to jets)$$

  This strategy uses the clean lepton signature as a trigger to identify a rare event, while relying on the more common jet decay to ensure the event rate is high enough to be practical.

7.5 Summary

- **Discovery of Z^0:** ν_μ scattering off nucleon can occur via:
  1. Charged current, ν_μ changes to μ^-, exchange of W^+.
  2. Neutral current, ν_μ is unchanged, exchange of Z^0
- Excitation of the Higgs field is a Higgs boson. This particle couples to all particles with mass, with strength proportional to the particle's mass.
- **Decay:** Higgs boson decays preferentially to the heaviest particles kinematically available. The dominant decay is $H \to b\bar{b}$, followed by $H \to W^+W^-$.
- **Discovery Channels:** Despite having small branching ratios, the $H \to ZZ \to 4l$ and $H \to \gamma\gamma$ channels were key to the discovery due to their clean experimental signatures and excellent mass resolution.
- **Other channels** like $H \to W^+W^-$, $H \to \tau^+\tau^-$, and $H \to b\bar{b}$ were confirmed later, as they suffer from larger backgrounds or poorer mass resolution.

Chapter 8

Nuclear Phenomenology

This chapter introduces the fundamental properties of atomic nuclei, including their composition, size, mass, and the concept of binding energy. We will explore the Semi-Empirical Mass Formula (SEMF) as a model to understand nuclear stability and the various modes of radioactive decay that unstable nuclei undergo to reach more stable configurations.

8.1 Introduction

The nucleus is composed of **protons** and **neutrons**, collectively known as **nucleons**.

- **Atomic Number (Z)**: The number of protons in a nucleus, which determines the chemical element.
- **Neutron Number (N)**: The number of neutrons in a nucleus.
- **Mass Number (A)**: The total number of nucleons, where $A = N + Z$.

A specific nuclear species, or **nuclide**, is denoted as ${}^{A}_{Z}X$, where X is the chemical symbol.

- **Isotopes**: Nuclides with the same Z but different N (e.g., ${}^{12}_{6}C$ and ${}^{14}_{6}C$).
- **Isobars**: Nuclides with the same A but different Z (e.g., ${}^{14}_{6}C$ and ${}^{14}_{7}N$).
- **Isotones**: Nuclides with the same N but different Z (e.g., ${}^{14}_{6}C$ and ${}^{15}_{7}N$).

Masses are often given in terms of the **unified atomic mass unit (u)**, defined as one-twelfth of the mass of a neutral carbon-12 atom.

$$u = 931.494\,\mathrm{MeV/c^2}$$

8.2 Shape of the Nucleus

Scattering experiments probe the charge distribution of the nucleus. Scattering assuming point-like, spinless objects, can be derived classically:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2\sin^4(\frac{\theta}{2})} \tag{8.1}$$

If e^- are scattered, spin has to be take into account:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}}[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)] \tag{8.2}$$

The experimental scattering cross section is related to the theoretical point-like cross section (the Mott cross section for spin-1/2 electrons) by a **form factor**, $F(q^2)$, which is the Fourier transform of the spatial charge distribution $f(\vec{x})$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{experimental}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(q^2)|^2$$

From these experiments, the nuclear radius is found to be approximately:

$$\boxed{R = R_0 A^{1/3}} \quad \text{where } R_0 \approx 1.2\,\text{fm}$$

This relation implies that the nuclear volume is proportional to the number of nucleons (A), suggesting that **nuclear density is roughly constant**.

8.3 Nuclear Mass and Binding Energy

The **mass of a nucleus is always less than the sum of the masses of its constituent protons and neutrons**.

$$M(Z, A) < Z \times M_p + N \times M_n \tag{8.3}$$

This **mass deficit** (ΔM) is due to the **binding energy (B)** that holds the nucleons together.

$$\begin{aligned} B(Z, A) = -\Delta M(Z, A) &= -(M(Z, A) - N_p(M_p + m_e) - N_n \times M_n) &(8.4)\\ &= -(M(Z, A) - Z(M_p + m_e) - (A - Z)M_n) &(8.5)\end{aligned}$$

where $M(Z, A)_{\text{atom}}$ is the mass of the neutral atom. A **more tightly bound nucleus** is **more stable** and has a **larger binding energy**. The **binding energy per nucleon (B/A)** is a measure of the relative stability of different nuclei.

8.4 Semi-Empirical Mass Formula (SEMF)

The SEMF provides an **approximate formula for the mass** (and thus binding energy) of a nucleus based on the liquid drop model, with quantum corrections.

$$M(Z, A) = \underbrace{Z(M_p + m_e) + (A - Z)M_n}_{f_0} - a_v A + a_s A^{2/3} + a_C \frac{Z^2}{A^{1/3}} + a_a \frac{(Z - \frac{A}{2})^2}{A} \pm \delta a_p A^{-1/2} \tag{8.6}$$

The terms contributing to the binding energy are:

- **Constituent mass term (f_0):** Adds the masses of all the constituents.
- **Volume Term ($-a_v A$)**: assumptions:
  1. constant density of the nucleons, hence mass is $\propto$ to volume
  2. nucleons bind to each neighbour
  3. binding energy is proportional to the volume of nucleus
  4. consider drop model, hence volume of the nucleus is a perfect sphere with radius R: $V^{1/3} \sim R \sim A^{1/3}$

  Binding energy reduces the mass, hence negative: $f_1(Z, a) = -a_v(A^{1/3})^3 = -a_v A$
- **Surface Term ($+a_s A^{2/3}$)**: Corrects for the fact that nucleons on the surface have fewer neighbours, reducing the overall binding energy. This energy is proportional to the surface area (put together $R \propto A^{1/3}$ and $S \propto R^2$, hence $R^2 \propto A^{2/3}$).

- **Coulomb Term** $(+a_C \frac{Z^2}{A^{1/3}})$: Accounts for the electrostatic repulsion between protons, which reduces binding energy. It is proportional to Z^2 and inversely proportional to the radius.
- **Asymmetry Term** $(+a_A \frac{(A-2Z)^2}{A})$: A quantum mechanical effect favouring nuclei with an equal number of protons and neutrons ($Z = N$). An imbalance increases the total energy and reduces binding. More in section 9.1.1.
- **Pairing Term** $(\pm\delta)$: Nucleons in the same spatial state, but opposite spin directions can combine to give a wavefunction of spin-0 [1]. This bosonic state follows Bose−Einstein statistics and hence the wave-functions overlap very heavily. This means that nucleons are tighter bound if they can be paired.
  * $+\delta$ for odd-odd nuclei (less stable)
  * 0 for odd-even nuclei [2]
  * $-\delta$ for even-even nuclei (more stable)

8.5 Nuclear Stability and Decay

Unstable nuclei decay to reach more energetically favourable states. The number of decays per unit time is the **activity (A)**, and follows:

$$A = -\frac{dN}{dt} = \frac{N}{\tau} \tag{8.7}$$

This leads to the exponential decay law $\boxed{N(t) = N_0 e^{-t/\tau}}$, where τ is the **lifetime** and N the number of nuclei. The **half-life** $(T_{1/2})$ is the time after which half the nuclei have decayed:

$$0.5N_0 = N_0 e^{-t/\tau} \qquad ln0.5 = \frac{T_{1/2}}{\tau} \qquad T_{1/2} = \tau \ln 2 \tag{8.8}$$

Units to measure activity:

- Becquerel 1 Bq = 1 decay/s
- Curie 1 Ci = 3.7×10^{10} decays/s

Exposure measured in Roentgen: 1 R = 2.58 C/Kg
Absorbed dose in rad: 1 rad = 0.01 J/Kg

Biological equivalent dose measured in rem (Roentgen equivalent man) or Sievert Sv, takes the varying biological effects of α, β and γ radiation into account.

8.6 β-Decay

For a fixed mass number A, the SEMF predicts that the mass of isobars follows a parabola as a function of Z (eq. 8.9). The most tightly bound and hence most stable nucleus is at the minimum of this parabola, forming a **valley of stability**. Rearranging Eq. 8.6:

$$M(Z, A) = (\alpha A \pm \lambda A^{-1/2}) - \beta Z + \gamma Z^2 \tag{8.9}$$

where

$$\alpha = M_n - a_v + a_s A^{-1/3} + \frac{a_a}{4}$$
$$\beta = a_a + M_n - M_p - m_e$$

[1] Remember that bosons are classified as particles with spin 0,1,2,3 etc.
[2] SEMF treats this intermediate state as the baseline, and therefore pairing term is defined as zero.

$$\gamma = \frac{a_a}{A} + a_c A^{-1/3}$$
$$\lambda = a_p$$

- For **odd-A** isobars, there is a **single mass parabola** and typically one stable isobar (fig. 8.1).
- For **even-A** isobars, the pairing term creates **two separate parabolas** (one for even-even nuclei, one for odd-odd nuclei, feg. 8.1).
- Nuclei that are not at the minimum of the parabola will undergo β-decay to move towards it (Fig. 8.2).

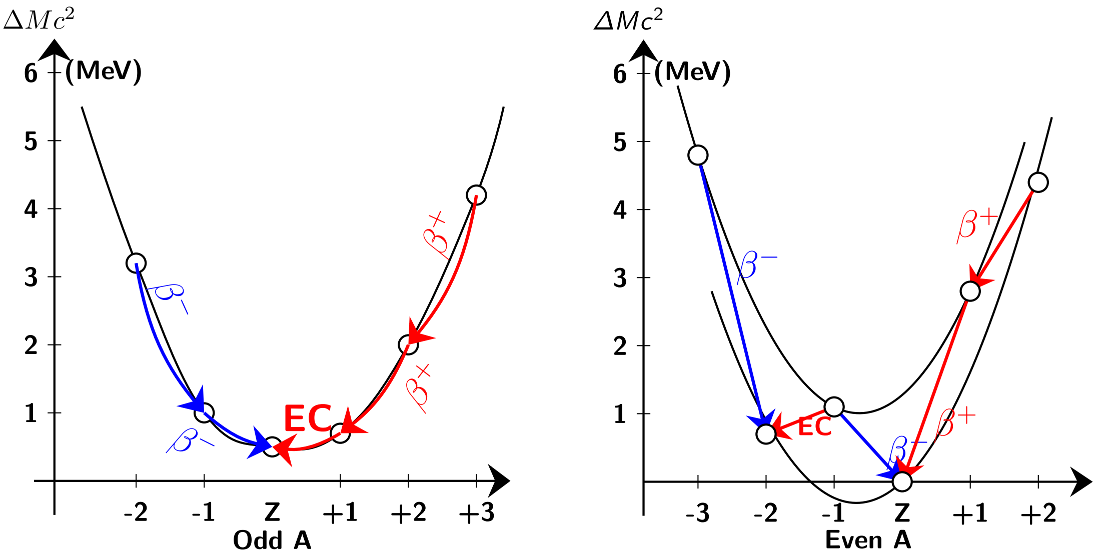


Figure 8.1: Mass parabolas from the Semi-Empirical Mass Formula for isobars with odd A (left) and even A (right), showing the possible β-decay and electron capture (EC) paths towards the most stable nucleus (parabola's minimum).

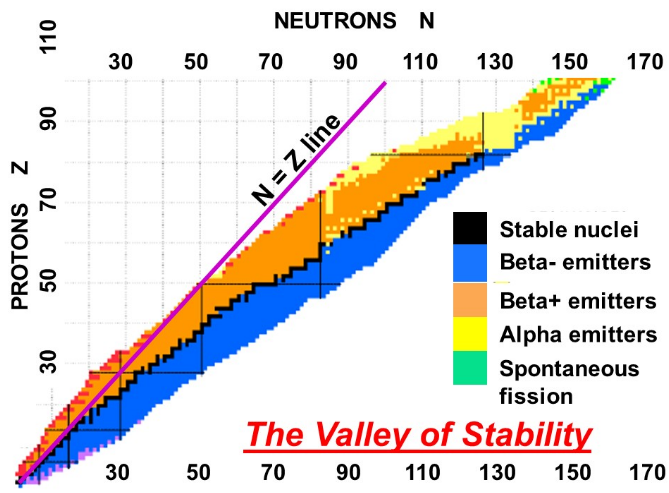


Figure 8.2: Nuclide chart indicating the main decay options of the nuclei. The valley of stability is shown in black. Plot taken from: https://physics.stackexchange.com/questions/783560/how-the-n-z-ratio-affects-the-stability-of-isotopes-and-their-method-of-radioa

Possible stabilization process:

- β^- **Decay**: Occurs in **neutron-rich** nuclei. In this case is is energetically favourable to turn a neutron into a proton.

$$n \to p + e^- + \bar{\nu}_e \qquad\qquad M(Z,A) \to M(Z+1,A) + e^- + \bar{\nu}_e$$

  This is allowed if $M(Z,A) > M(Z+1,A)$.

- β^+ **Decay**: Occurs in **proton-rich** nuclei. In this case is is energetically favourable to turn a proton into a neutron.

$$p \to n + e^+ + \nu_e \qquad\qquad M(Z,A) \to M(Z-1,A) + e^+ + \nu_e$$

  This is allowed if $M(Z,A) > M(Z-1,A) + 2m_e$. The second term appear since we have to take into account the created positron and also the excess electron when going from Z to $Z-1$.

- **Electron Capture (EC)**: An alternative process for **proton-rich** nuclei, where the nucleus captures an atomic electron:

$$p + e^- \to n + \nu_e \qquad\qquad M(Z,A) + e^- \to M(Z-1,A) + \nu_e$$

  This is allowed if $M(Z,A) > M(Z-1,A) + \epsilon$, where ϵ is the atomic binding energy of the captured electron.

8.7 α-Decay

Heavy nuclei ($A > 153$) can decay by emitting an alpha particle (a ${}^4_2$He nucleus), which has a particularly tightly bound configuration ($B \approx 28.3$ MeV). The decay is energetically favourable if the binding energy of the parent nucleus is less than the sum of the binding energies of the daughter and the alpha particle.

$$\text{Condition for } \alpha\text{-decay:} \quad B(Z,A) < B(Z-2,A-4) + B(2,4)$$

Because B/A generally decreases for heavy nuclei, splitting off a tightly bound α particle becomes an effective way to increase the overall binding and release energy.

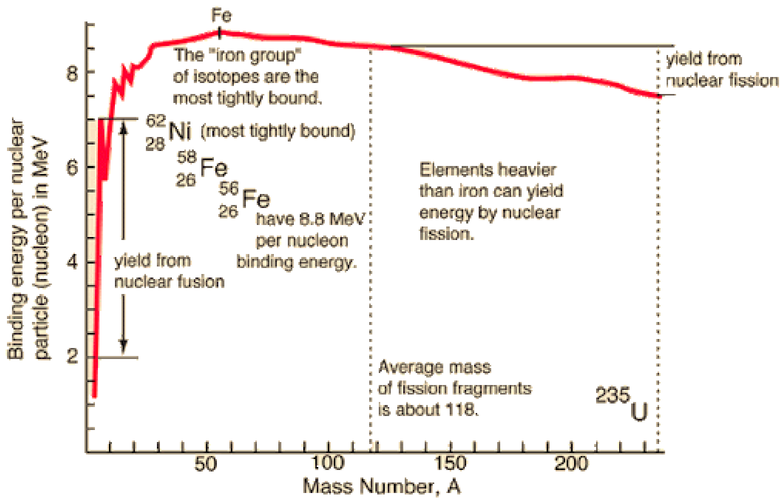


Figure 8.3: Binging energy per nucleon. Plot taken from: http://hyperphysics.phy-astr.gsu.edu/hbase/NucEne/nucbin.html

8.8 γ-Decay

Following an α or β decay, a **daughter nucleus is often left in an excited state**. It **de-excites** to its ground state by **emitting** one or more **high-energy photons**, a process known as γ-decay. This process changes the energy of the nucleus but not its composition (Z and N remain the same).

8.9 Summary

- The mass of a nucleus is less than the sum of its constituent nucleon masses due to the **binding energy**. The stability of a nucleus is best understood by its binding energy per nucleon, B/A.
- The **Semi-Empirical Mass Formula (SEMF)** uses a liquid-drop analogy with quantum corrections (asymmetry, pairing) to model the binding energy and explain the general trend of nuclear stability. Contributions:
  - **Volume:** Larger volume $\Rightarrow$ larger Binding Energy
  - **Surface:** Atoms on the surface are less bound, since they have fewer neighbours
  - **Coulomb:** Electrostatics repulsion between protons
  - **Asymmetric:** Nuclear prefer same proton and neutron number
  - **Pairing:** Nucleons prefer spin paired protons and neutrons. (Remember proton and neutron have different orbital systems)
- The **valley of stability** on the N-Z chart represents the most stable nuclides. Nuclei far from this valley are radioactive.
- Unstable nuclei undergo decay to move towards the valley of stability.
  - Neutron-rich nuclei undergo β^- **decay**
  - Proton-rich nuclei undergo β^+ **decay** or **electron capture**.

  Very heavy nuclei ($A > 153$) undergo α **decay**. γ **decay** occurs when an excited nucleus de-excites.

Chapter 9

Nuclear Structure

This chapter moves beyond the macroscopic properties of nuclei described by the liquid drop model and introduces microscopic models that explain the internal structure and quantum states of nucleons.

9.1 The Fermi Gas Model

The Fermi gas model offers a quantum-statistical explanation for some terms in the SEMF equation, particularly the asymmetry term.

- **Assumption**: Nucleons are treated as non-interacting fermions moving freely within a three-dimensional potential well, analogous to electrons in a metal.
- **Potential Wells**: Protons and neutrons occupy separate potential wells. The proton well is shallower due to **Coulomb repulsion**, which reduces their binding energy compared to neutrons.
- **Fermi Momentum and Energy**: Since nucleons are fermions, they fill the available energy states up to a maximum level, known as the **Fermi energy** (E_F), corresponding to the **Fermi momentum** (p_F). Density of states given by:

$$n(p)dp = dn = \frac{4\pi V}{(2\pi\hbar)^3}p^2 dp \tag{9.1}$$

  Integrating up to fermi momentum gives total number of available states:

$$n = \frac{V(p_F)^3}{6\pi^2\hbar^3} \tag{9.2}$$

  - For a nucleus with volume V, the number of protons (Z) and neutrons (N) are related to their respective Fermi momenta (p_F^p, p_F^n), and taking into account that there are two spin states:

$$Z = \frac{V(p_F^p)^3}{3\pi^2\hbar^3} \qquad N = \frac{V(p_F^n)^3}{3\pi^2\hbar^3} \tag{9.3}$$

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(R_0 A^{\frac{1}{3}})^3 = \frac{4}{3}\pi R_0^3 A \tag{9.4}$$

  - Assume the stability line to be $Z = N = A/2$ than:

$$A/2 = Z = \frac{V(p_F^p)^3}{3\pi^2\hbar^3} = N = \frac{V(p_F^n)^3}{3\pi^2\hbar^3} \tag{9.5}$$

    Substitute 9.4:

$$p_F = p_F^p = p_F^n = \frac{\hbar}{R_0}(\frac{9\pi}{8})^{\frac{1}{3}} \tag{9.6}$$

approximation: $p_F^p = p_F^n$ in reality Fermi momenta for protons and neutrons differ due to the difference in their potentials caused by the Coulomb repulsion felt by protons. The Fermi energy is approximately:

$$E_F = \frac{p_F^2}{2M}$$

9.1.1 Origin of the Asymmetry Term

The Fermi gas model provides a physical origin for the asymmetry term in the SEMF, which **favours nuclei with an equal number of protons and neutrons** ($N = Z$).
Average Kinetic energy per nucleon (denominator is for normalisation):

$$\langle E_{kin}\rangle = \frac{\int_o^{p_f} E_{kin} p^2 dp}{\int_o^{p_f} p^2 dp} = \frac{\int_o^{p_f} \frac{p^2}{2M} p^2 dp}{\int_o^{p_f} p^2 dp} = \frac{3}{5}\frac{p_f^2}{2M} = \frac{3}{10M} p_f^2 \tag{9.7}$$

To get average total kinetic energy of a nucleus, add kinetic energies of all nucleons:

$$E_{kin}(N, Z) = N\langle E_n\rangle + Z\langle E_p\rangle = \frac{3}{10M}(N(p_F^n)^2 + Z(p_F^p)^2) \tag{9.8}$$

Sub. 9.4 into Eq. 9.3 and rearrange for p_F^n, p_F^p:

$$E_{kin}(N, Z) = \frac{3}{10M}(\frac{\hbar}{R_0})^2(\frac{9\pi}{4})^{\frac{2}{3}}(\frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}}) \tag{9.9}$$

For fixed mass number A ($A = N + Z$):

$$E_{kin}(N, Z) = \frac{3}{10M}(\frac{\hbar}{R_0})^2(\frac{9\pi}{4})^{\frac{2}{3}}(\frac{(A - Z)^{\frac{5}{3}} + Z^{\frac{5}{3}}}{A^{\frac{2}{3}}}) \tag{9.10}$$

This formula has a minimum at $Z = N = A/2$. Taylor expand around $A/2$:

$$E_{kin}(Z) = \frac{3}{10M}\left(\frac{\hbar}{R_0}\right)^2\left(\frac{9\pi}{4}\right)^{2/3}\left(\frac{A}{2}\right)^{5/3}\frac{1}{A^{2/3}}\left(1 + \frac{5}{9}\left(\frac{N - Z}{A}\right)^2 + \dots\right) \tag{9.11}$$

Sub. in back $N = A/2$:

$$E_{kin}(N, Z) = \frac{3}{10M}(\frac{\hbar}{R_0})^2(\frac{9\pi}{8})^{\frac{2}{3}}(A + \frac{5}{9}\frac{(N - Z)^2}{A} + \cdots) \tag{9.12}$$

- 1^{st} term gets absorbed in the volume term
- 2^{nd} term reproduces the asymmetry term of the SEMF
- Asymmetric term is **minimised for N = Z**
- **Coulomb repulsion reduces binding energy** for **protons** and dominates for heavier nuclei
- Due to Coulomb term, **heavy nuclei show an excess of neutrons over protons**

9.2 The Shell Model

The shell model describes **nucleons occupying discrete energy levels**, or **shells**, within the nucleus, analogous to the electron shell model in atomic physics. This model successfully explains the existence of **magic numbers**.

9.2.1 Evidence for the Shell Model: Magic Numbers

Certain numbers of protons or neutrons result in exceptionally stable nuclei. These are the magic numbers.

- **Magic Numbers**: **2, 8, 20, 28, 50, 82, 126**. Nuclei with a magic number of protons or neutrons are called **magic nuclei**.
- Nuclei where **both N** and **Z are magic** are called **doubly magic nuclei** (e.g., ${}^{4}_{2}\text{He}$, ${}^{16}_{8}\text{O}$, ${}^{208}_{82}\text{Pb}$).
- **Evidence**:
  - Higher binding energy compared to neighbours (i.e., large deviations from the SEMF prediction).
  - Higher energy required for the first excited state.
  - Smaller neutron capture cross-sections (Very stable nucleons prefer not to increase their atomic mass).

9.2.2 Nuclear Potential and Spin-Orbit Coupling

Early attempts using simple potentials (e.g., square well, harmonic oscillator) failed to reproduce all the magic numbers. The key was the inclusion of spin-orbit coupling.

- **Woods-Saxon Potential**: A more realistic potential that approximates the nuclear charge distribution:
$$V_{\text{WS}}(r) = \frac{-V_0}{1 + \exp\left(\frac{r-R}{a}\right)}$$
However, this potential alone is still not sufficient.
- **Spin-Orbit Coupling**: The crucial addition to the potential is a strong coupling between the orbital angular momentum ($\vec{L}$) and the intrinsic spin ($\vec{S}$) of each nucleon.
$$V(r) = V_{\text{WS}}(r) + V_{ls}(r)\frac{\langle \vec{L}\cdot\vec{S}\rangle}{\hbar^2}$$
This interaction splits the energy levels with orbital angular momentum $l > 0$ into two sublevels based on the total angular momentum **J**:
$$J^2 = (\vec{S} + \vec{L})^2 = S^2 + 2\vec{L}\cdot\vec{S} + L^2 \tag{9.13}$$
$$J^2 - S^2 - L^2 = 2\vec{L}\cdot\vec{S} \tag{9.14}$$
Use eigenvalues:
$$[j(j+1) - s(s+1) - l(l+1)]\,\hbar^2 = 2\langle \vec{L}\cdot\vec{S}\rangle \tag{9.15}$$
$$\frac{\langle \vec{L}\cdot\vec{S}\rangle}{\hbar^2} = \frac{j(j+1) - s(s+1) - l(l+1)}{2} = \begin{cases} l/2 & \text{for } j = l + 1/2 \\ -(l+1)/2 & \text{for } j = l - 1/2 \end{cases} \tag{9.16}$$
  - The energy splitting is significant and increases with l. The state with **higher j** is pushed to a **lower energy**.
  - This splitting **correctly reproduces** all the observed **magic numbers**. Each level with total angular momentum J can hold **2J+1 nucleons**. Example: $1d_{\frac{5}{2}}$ can accommodate 6 nucleons.

9.2.3 Predictions of the Shell Model

The shell model, incorporating the pairing hypothesis, allows for predictions of ground-state properties.

- **The Pairing Hypothesis**: Nucleons have a strong tendency to form pairs with opposite spin projections, such that the total angular momentum of the pair is zero ($\vec{J}_{\text{pair}} = 0$). This is energetically favourable.
- **Nuclear Spin (s_N)** = the total angular momentum of the nucleus in its ground state.
  - **Even-Even Nuclei**: All nucleons are paired. The total angular momentum is always $s_N = 0$.
  - **Odd-A Nuclei** (Even-Odd or Odd-Even): The total spin is determined entirely by the **J of the last, unpaired nucleon**.
  - **Odd-Odd Nuclei**: The spin is determined by coupling the total angular momenta of the unpaired proton (J_p) and neutron (J_n). The total spin lies in the range $|J_p - J_n| \leq s_N \leq J_p + J_n$.
- **Parity (P)**: A quantum number related to the spatial inversion symmetry of the wavefunction.
  - The parity of a single nucleon state is given by $\boxed{P = (-1)^l}$, where l is its orbital angular momentum quantum number.
  - Parity is a **multiplicative** quantum number, $P = \prod P_i$. For a paired system, $P = (-1)^l \times (-1)^l = +1$.
  - **Even-Even Nuclei**: All nucleons are paired, so the total parity is always **even (+)**.
  - **Odd-A Nuclei**: The parity is determined by the **l of the last, unpaired nucleon**.
  - **Odd-Odd Nuclei**: The total parity is the product of the parities of the two unpaired nucleons: $P = P_p \times P_n = (-1)^{l_p} \times (-1)^{l_n}$.
- **Magnetic moment:**
  - Magnetic moment of a nucleus is given by:

$$\vec{\mu} = g_j \frac{\langle \vec{j} \rangle}{\hbar} \mu_N = \frac{\mu_N}{\hbar} \sum_{i=0}^{A} \{ g_l \vec{l_i} + g_s \vec{s_i} \} \tag{9.17}$$

    where $\mu_N = \frac{e\hbar}{2m_p}$ is the nuclear magneton, g_j Landre g-factor, $\vec{s_i}$ spins, $\vec{l_i}$ orbital angular momenta of the contributing nucleons.
  - Predictions of the **shell model for nuclear magnetic moments are not very reliable**.
- **Excited States**: These arise when a nucleon is promoted from its ground-state level to a higher, unoccupied level.
  - Predicting the correct order of these excited states proves to be difficult.

9.2.4 Example: Spin, parity and exited states

Spins and parity predictions (see fig. 9.1):

- For $^{40}_{18}$Ar, all nucleons are paired
  - spin of the nucleus is 0
  - even parity
- For $^{17}_{7}$N, unpaired proton occupies $1p_{1/2}$

  - spin is $J = 1/2$
  - $l = p = 1$ in spectroscopic notation, parity $P = (-1)^1$ is odd
- For ${}^{16}_{7}\mathrm{N}$, unpaired proton occupies $1p_{1/2}$, unpaired neutron occupies $1d_{2/5}$,
  - spin will be in the range from $|1/2 - 5/2| = 2$ to $1/2 + 2/5 = 3$
  - $l_{proton} = 1$, $l_{neutron} = 2$, $P = P_p \times P_n = (-1)^1 \times (-1)^2 = -1$

Possible shell configurations for the excited states of ${}^{17}_{7}\mathrm{N}$ are represented in fig. 9.2.

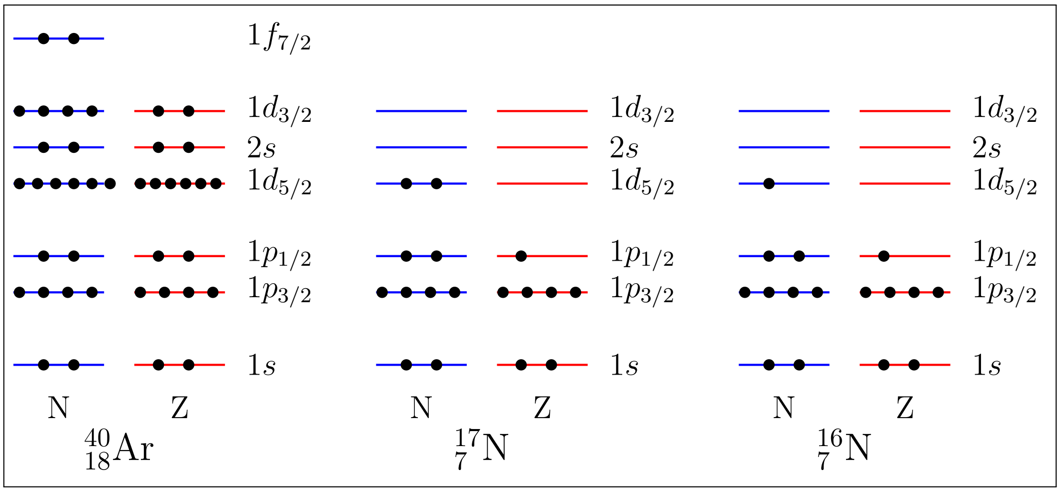


Figure 9.1: Shell configurations for the nucleus of ${}^{40}_{18}\mathrm{Ar}$, ${}^{17}_{7}\mathrm{N}$, ${}^{16}_{7}\mathrm{N}$

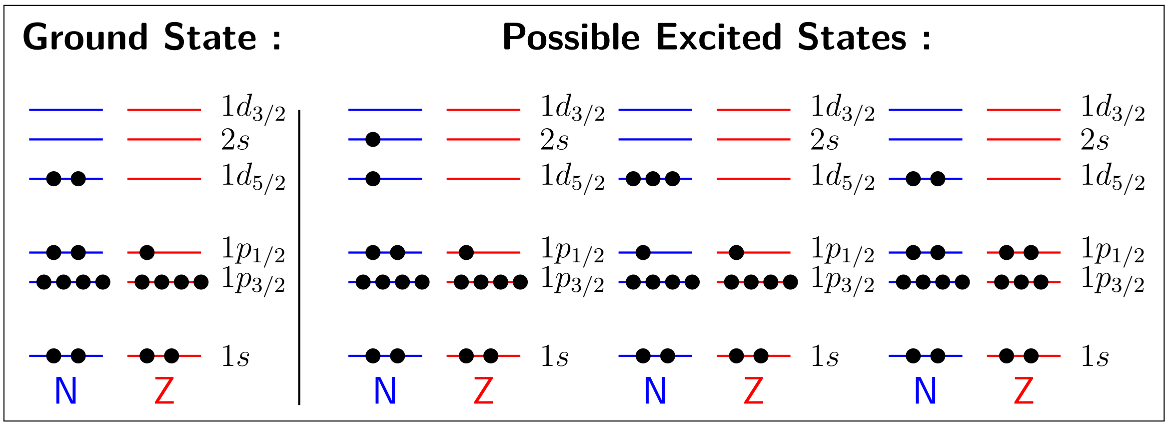


Figure 9.2: Shell configurations for the excited states of ${}^{17}_{7}\mathrm{N}$.

9.3 The Collective Model

The collective model provides a better description by **incorporating features** of both the **shell model** and the **liquid drop model**.

- It treats the nucleus as having a core of paired nucleons (as in the shell model) and valence nucleons that can induce a collective, non-spherical motion of the core.
- These collective motions can be either **vibrations** (surface oscillations) or **rotations** of a permanently deformed shape.

9.4 Theory of Nuclear β-Decay

- **Fermi's Golden Rule**: Fermis second Golden rule gives the transition rate ω as:

$$\omega = \frac{2\pi}{\hbar}|M_{fi}|^2 n(E) \tag{9.18}$$

where M_{fi} is the matrix element going from initial state i to final state f, $n(E)$ is the density of states. After some lengthy calculation (look at page 107 of the original notes) you get the differential decay rate as a function of the electron's momentum (p_e):

$$\frac{d\omega}{dp_e} = \frac{G_F^2}{2\pi^3\hbar^7 c^3} p_e^2 (E_0 - E_e)^2 \tag{9.19}$$

- **Screening Factor** $F(Z, p_e)$: This factor, accounts for the Coulomb interaction between the emitted electron/positron and the daughter nucleus:

$$F(Z, p_e) \sim \frac{2\pi\eta}{1 - e^{-2\pi\eta}} \qquad \text{where } \eta = \pm\frac{Ze^2}{4\pi\epsilon_0 \hbar c \beta_e} \tag{9.20}$$

  - In Fig. 9.3 positrons are repelled by the positive charges of the protons, while electrons are attracted.
  - β^- spectrum is shifted towards lower momenta and the β^+ spectrum towards higher momenta.
  - The effect increases with the charge and hence the number of protons Z in the nucleus.

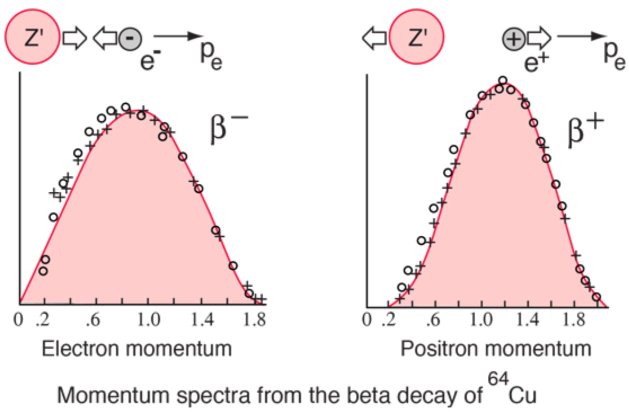


Figure 9.3: Screening factor $F(Z, p_e)$ for β^- and β^+. Plot taken from: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/beta2.html

9.4.1 The Kurie Plot

A **Kurie plot** is a linearized representation of the β-decay spectrum, used to determine the endpoint energy and investigate the neutrino mass.

- The plot is constructed by plotting the function $H(E_e)$ against the electron's kinetic energy E_e:

$$H(E_e) = \sqrt{\frac{d\omega/dp_e}{p_e^2 F(Z, p_e)}} \sim (E_0 - E_e)$$

- **Massless Neutrino**: If the neutrino is massless, the Kurie plot is a **straight line** that intersects the energy axis at the endpoint energy E_0.

- **Massive Neutrino**: If the neutrino has a non-zero mass ($m_\nu > 0$), the relation is modified near the endpoint:

$$H(E_e) \sim \sqrt{(E_0 - E_e)\sqrt{(E_0 - E_e)^2 - m_\nu^2 c^4}}$$

  This causes the Kurie plot to **curve downwards** (Fig. 9.4) and intersect the energy axis vertically at $E_0 - m_\nu c^2$. Measuring this deviation is a **key method for determining the absolute neutrino mass scale**.

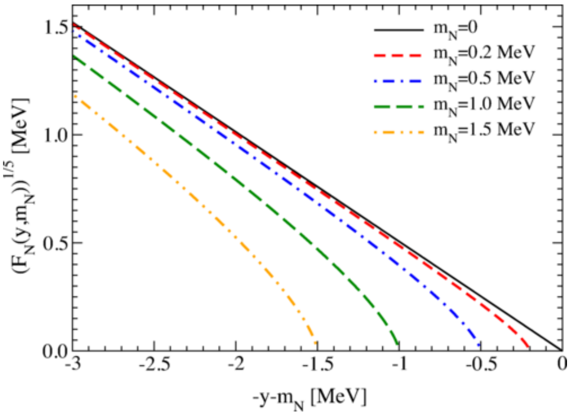


Figure 9.4: : Kurie plot for different non-zero neutrino masses.. Plot taken from: https://journals.aps.org/prd/abstract/10.1103/PhysRevD.103.055019

9.5 Summary

- The **Fermi Gas Model** explains the origin of the **asymmetry term** in the SEMF as a consequence of the kinetic energy cost of having an unequal number of protons and neutrons.
  - Nucleons are moving freely in a 3D potential well
  - Protons and neutrons occupy separate potential wells
- The **Shell Model**, with including a strong **spin-orbit coupling** term, explains the existence of **magic numbers** and allows for the prediction of ground-state spins and parities for many nuclei.
- The shape of the β-decay spectrum, when linearized in a **Kurie plot**, is highly sensitive to the mass of the neutrino. A deviation from a straight line at the endpoint is direct evidence for a non-zero neutrino mass.

Chapter 10

Fission and Fusion

10.1 Key principles

- The mass of a nucleus is always less than the sum of the masses of its individual constituent nucleons (protons and neutrons). This difference is the **mass deficit**, which corresponds to the **binding energy** of the nucleus.
- The **binding energy per nucleon** varies with the mass number (A), peaking for iron (^{56}Fe).

10.2 Fission

10.2.1 Basics

- Energy is released when a very heavy nucleus splits into two or more lighter nuclei. The daughter nuclei have a higher total binding energy (and thus a lower total mass) than the parent nucleus, releasing the energy difference.
- Example is the fission of Uranium-235:

$$^{235}_{92}\mathrm{U} \quad \rightarrow \quad ^{92}_{37}\mathrm{Rb} +^{140}_{55}\mathrm{Cs} + 3n + X\ \mathrm{MeV} \tag{10.1}$$

- The energy released can be estimated using the **Semi-Empirical Mass Formula (SEMF)**. Ignore the very small pairing term:
  1. Volume term: $-\,a_v(A_U - A_{Rb} - A_{Cs}) = -46.7$ MeV
  2. Surface term: $+\,a_s(A_U^{2/3} - A_{Rb}^{2/3} - A_{Cs}^{2/3}) = -160$ MeV
  3. Coulomb term: $+\,a_c\left(\frac{Z_U^2}{A_U^{1/3}} - \frac{Z_{Rb}^2}{A_{Rb}^{1/3}} - \frac{Z_{Cs}^2}{A_{Cs}^{1/3}}\right) = +339$ MeV
  4. Asymmetry term: $+\,a_a\left(\frac{(Z_U - A_U/2)^2}{A_U} - \frac{(Z_{Rb} - A_{Rb}/2)^2}{A_{Rb}} - \frac{(Z_{Cs} - A_{Cs}/2)^2}{A_{Cs}}\right) = +26.1$ MeV
- Summing all the above will give the energy realised, after including the energy from subsequent decays of the fission products, energy released per fission event is $\approx$ **200 MeV**.
- Products of fission undergo their own decay chains, often emitting **delayed neutrons**, this process crucial for reactor control.

10.2.2 Spontaneous and induced fission

Spontaneous fission

- Occurs when a nucleus **fissions without any external trigger**. This is energetically favourable if the reduction in Coulomb energy outweighs the increase in surface energy upon deformation. Why?
  - Imagine the fission starting with the deformation of a spherical nucleus into a elongated shape, leading to the eventual splitting into two nuclei.
  - As the nucleus stretches into a elongated shape, the surface term will increase and the Coulomb term will decrease (assuming the volume remains the same).
  - If the change in Coulomb term is larger than the change in the surface term then the deformed shape will be energetically favourable and the nucleus is unstable, meaning that spontaneous fission can occur.
- This condition is met for very heavy nuclei ($Z \geq 116$ and $A \geq 270$), where:

$$\frac{Z^2}{A} \geq \frac{2a_s}{a_c} \approx 49 \tag{10.2}$$

- For nuclei lighter than this, there is an energy barrier known as the **activation energy** or **fission barrier** that must be overcome.
  - Spontaneous fission can occur in this case via a quantum mechanical tunnelling through the barrier, however the probability is extremely small.

Induced fission

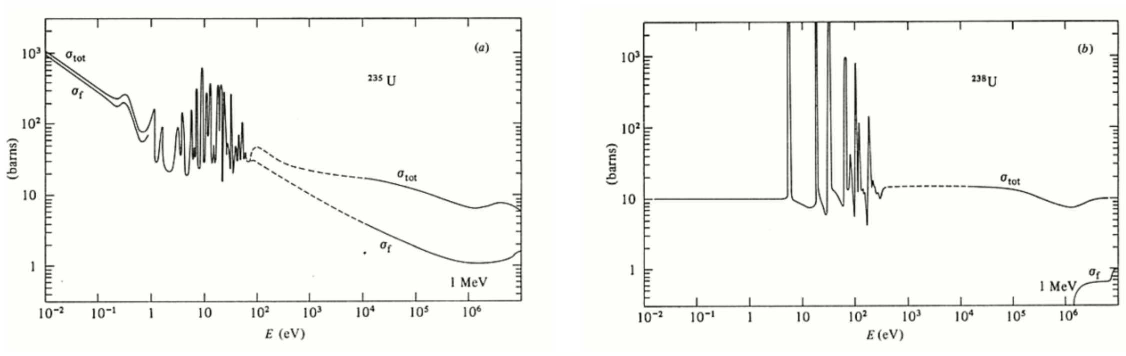
- Occurs when **energy is supplied to the nucleus to overcome the activation barrier**. This is typically achieved by the **capture of a neutron**.
- The binding energy released by capturing a neutron can be sufficient to exceed the activation energy. This depends on the pairing term of the SEMF.
  - **Fissile materials:** The capture of a neutron by an even-odd nucleus, like ^{235}U, forms a more tightly bound even-even nucleus. Binding energy of the last neutron is 6.5 Mev, activation energy of fission in ^{235}U is 5 MeV. Therefore the fission can occur via **slow neutrons** (KE can be zero).
  - **Non-fissile materials:** The capture of a neutron by an even-even nucleus, like ^{238}U, forms a less tightly bound even-odd nucleus. Binding energy of the last neutron is 4.8 Mev, activation energy of fission in ^{239}U is 6 MeV. Therefore ^{238}U, require **fast neutrons** with significant kinetic energy to induce fission (1.2 MeV).

10.2.3 Nucleus-neutron cross sections

- The probability of fission is described by the **fission cross section**. See Fig. 10.1.
- For fissile ^{235}U, the fission cross section (σ_f) is very large for low-energy (thermal) neutrons and decreases with increasing neutron energy.
- For non-fissile ^{238}U, the fission cross section is zero below a threshold energy of about 1.2 MeV. It also exhibits large **resonant capture** cross sections in the 1 eV to 1 keV range, where it is likely to absorb a neutron without fissioning.

10.2.4 Fission chain reactions

- A **chain reaction** is possible because fission events release neutrons that can then induce subsequent fissions.



(a) (b)

Figure 10.1: Nucleus neutron cross sections versus the neutron kinetic energy for (a) uranium-235 and (b) uranium-238.. Plot taken from: NO IDEA

- Probability that a given neutron induces fission is q and each fission reaction produces an average of n neutrons.
- Each neutron will lead to $(nq - 1)$ additional neutrons in a time tp, where tp is the average time before absorption of the neutron occurs, -1 because you remove neutron that created the initial fission.
- If there are $N(t)$ neutrons at time t, Than at time $t + \delta t$:

$$N(t + \delta t) = N(t) + N(t)\frac{(nq - 1)\delta t}{t_p}$$

- Rearrange, notice that the LHS is the derivative dN/dt:

$$\lim_{\delta t \to 0} \frac{N(t + \delta t) - N(t)}{\delta t} = \frac{N(t)(nq - 1)}{t_p}$$

$$\frac{dN}{dt} = \frac{N(t)(nq - 1)}{t_p}$$

- Integrate using variable separation

$$\int_{N(0)}^{N(t)} \frac{dN}{N(t)} = \int_0^t \frac{(nq - 1)}{t_p} dt$$

$$\ln[N(t)] - \ln[N(0)] = \frac{(nq - 1)t}{t_p}$$

- The neutron population evolves as:

$$N(t) = N(0)e^{(nq-1)t/t_p} \tag{10.3}$$

- There are three possible scenarios for the chain reaction:
  - **Subcritical** $(nq < 1)$: The neutron population decreases exponentially, and the reaction dies out.
  - **Critical** $(nq = 1)$: The neutron population remains constant, leading to a sustained, controlled energy release, as required for a **nuclear power reactor**.
  - **Supercritical** $(nq > 1)$: The neutron population increases exponentially, leading to a rapid, uncontrolled release of energy, as in a **nuclear fission bomb**.

10.2.5 Nuclear fission bombs

- A nuclear bomb requires achieving a **supercritical** condition ($nq > 1$).
- Consider ^{235}U, which has on average $n \approx 2.5$ and $t_p \approx 10^{-8}s$, hence we require $q < 0.4$.
- A key concept is the **critical mass**, which is the minimum mass of fissile material needed to ensure that enough neutrons induce further fission before escaping the material.
  - If the sphere of uranium is small enough that neutrons are likely to reach the edge of the volume before t_f, the reaction will die.
  - Consider non-relativistic neutron with energy 2 Mev (average neutron coming from fission), using $T = 1/2mv^2$, $v = 2 \times 10^7 ms^{-1}$
  - Distance travelled in time t_p is 20 cm.
  - However neutron will not travel in a straight line because an average of 6 collisions is expected before inducing fission, and if we assume that the direction of the neutron changes randomly after each collision, the average distance travelled is actually 7 cm.
  - Not all these neutrons will induce fission, as some will escape the material and some will be captured in nuclei without inducing fission.
  - This means that a larger sphere is required and it turns out that a radius of 9 cm, corresponding to a critical mass of 50 kg, is required to ensure a supercritical reaction.

10.2.6 Nuclear fission power reactors

- Power reactors are designed to maintain a precisely **critical** state ($nq = 1$).
- Natural uranium is only 0.7% ^{235}U, the rest is ^{238}U.
- Neutron is much more likely to interact with a nucleus of ^{238}U. A 2 MeV neutron has little chance of inducing fission in natural uranium. It is more likely to scatter inelastically with a ^{238}U nucleus (as seen in Fig. 10.1), losing energy in the process. To sustain a chain reaction, reactors must:
  - Use **enriched uranium** with a higher concentration of ^{235}U (typically 2-3%).
  - Use a **moderator** (like heavy water or graphite) to slow down the fast neutrons from fission. These slow neutrons are more likely to cause fission in ^{235}U and avoid resonant capture by ^{238}U.
- The reaction is managed using **control rods** made of neutron-absorbing materials (e.g., cadmium) that can be inserted or withdrawn to fine-tune the neutron population and keep the reaction critical.
- Reactor safety and control depend critically on **delayed neutrons**.
  - These neutrons, come from the fission of decay products that have undergone a series of β decays, emitted seconds after the initial fission.
  - Taking account of delayed neutrons we need to replace nq with $(n_{prompt} + n_{delayed})$.
  - Reactors are operated to be subcritical with respect to prompt neutrons alone.

10.3 Fusion

10.3.1 Basics

- **Fusion** is the process of combining two light nuclei to form a single, heavier nucleus, releasing energy because the final product has a larger binding energy per nucleon.
- Energy realised in fusion is less than in fission.

10.3.2 Coulomb barrier

- For fusion to occur, nuclei must overcome the electrostatic **Coulomb repulsion** between their positive charges. This requires extremely high kinetic energies, corresponding to temperatures of hundreds of millions of Kelvin.
- The energy required to overcome the barrier is given by the **Coulomb potential**:

$$V_C = \frac{1}{4\pi\epsilon_0}\frac{ZZ'e^2}{R+R'} \tag{10.4}$$

  where Z and Z' are the two atomic numbers and R and R' are their effective radii.
- Example:
  - For simplicity $A = A' = 2Z = 2Z'$
  - Then $V_C \approx 0.15A^{5/3}$ MeV, for $A = 8$, $V_C \approx 4.8$ MeV
  - Using $E = k_B T$, $T \approx 5.6^{10}$K for particles with energy 4.8 MeV
- Fusion can occur at temperatures lower than classically required due to two effects:
  - **Quantum tunnelling**, which allows nuclei to penetrate the barrier even if they do not have enough energy to go over it.
    * Probability for tunnelling is e^{-G}, where G is Gamow factor: $G = \sqrt{E_G/E}$ where E_G increases as the barrier increases and E is the particles energy.
  - The **Maxwellian distribution** of energies in the plasma, which ensures that some nuclei in the high-energy tail of the distribution have sufficient energy to fuse.
- Combining the two effects gives a peak at a certain energy within which fusion can take place. See fig. 10.2

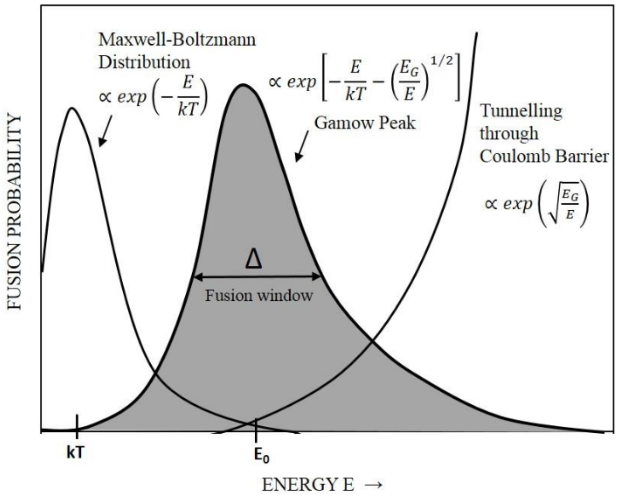


Figure 10.2: Tunnelling and the Maxwellian distribution of energy. Plot made by yee West - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=150007179

10.3.3 Stellar fusion

- The Sun's energy is produced by nuclear fusion, primarily through the **proton-proton chain (PPI)**:

  1. Two protons fuse to form deuterium:

  $${}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \quad \rightarrow \quad {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e + 0.42\ \mathrm{MeV}$$

  2. Deuterium fuses with more protons to produce Helium:

  $${}^{1}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \quad \rightarrow \quad {}^{3}_{2}\mathrm{He} + \gamma + 5.49\ \mathrm{MeV}$$

  3. Two Helium nuclei fuse to form α particle:

  $${}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \quad \rightarrow \quad {}^{4}_{2}\mathrm{He} + {}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} + 12.86\ \mathrm{MeV}$$

  The overall reaction is:

  $$4\,{}^{1}_{1}\mathrm{H} \quad \rightarrow \quad {}^{4}_{2}\mathrm{He} + 2e^{+} + 2\nu_e + 2\gamma + 24.68\ \mathrm{MeV}$$

  - Energy release due to the difference in binding energy between the initial and final states is: $2(0.42 + 5.49) + 12.86 = 24.68\ MeV$
  - Positrons will annihilate with electrons, 2 e^{+} and 2 e^{-}, extra 2.04 MeV per PPI chain.
  - Each neutrino will carry off on average 0.26 MeV, this will be lost in space.

- The first step of the chain involves the weak interaction, making it extremely slow and thus setting the long timescale for the Sun's lifetime.
- Other processes involving the fusion of heavier nuclei also occur producing additional energy.

10.3.4 Fusion reactors

- The proton-proton chain is too slow to be used as power source.
- Possible candidate for fusion are:

1. Deuterium-deuterium

$$^2_1\mathrm{H} +^2_1\mathrm{H} \;\rightarrow\; ^3_2\mathrm{He} + n + 3.27\text{ MeV} \qquad \textbf{or} \qquad ^2_1\mathrm{H} +^2_1\mathrm{H} \;\rightarrow\; ^3_1\mathrm{H} + p + 4.03\text{ MeV}$$

2. Deuterium-tritium

$$^2_1\mathrm{H} +^3_1\mathrm{H} \;\rightarrow\; ^4_2\mathrm{He} + n + 17.62\text{ MeV}$$

- The **most promising reaction** is **deuterium-tritium (D-T) fusion** due to its **high energy release** and **larger cross section** at lower temperatures.
- The primary challenge is achieving and containing a plasma at the required high temperatures (For D-T fusion, fur nuclei with energy 20 MeV, $T = \frac{E}{k_B} = 2 \times 10^8$ K).
- Two main methods for plasma containment are:
  - **Magnetic confinement**: Using strong magnetic fields to hold the charged plasma in a toroidal (doughnut) shape (e.g., a tokamak).
  - **Inertial confinement**: Using high-power lasers to rapidly compress and heat a small fuel pellet.
- The **Lawson criterion** defines the conditions (in terms of temperature, density, and confinement time) needed for a fusion reactor to achieve net energy gain.

10.4 Summary

- **Fission** releases energy by splitting heavy nuclei (like ^{235}U), while **fusion** releases energy by combining light nuclei. Both processes move towards the peak of the binding energy per nucleon curve.
- **Fissile** materials (^{235}U, ^{239}Pu) can undergo fission with slow neutrons, whereas **non-fissile** materials (^{238}U) require fast neutrons.
- A **chain reaction** occurs when neutrons from one fission induce more fissions. The state can be subcritical ($nq < 1$), critical ($nq = 1$), or supercritical ($nq > 1$).
- Nuclear reactors operate in a **critical** state, using **moderators** to slow neutrons and **control rods** to absorb them.
- Fusion requires overcoming the immense **Coulomb barrier** between nuclei, which necessitates extremely high temperatures.
- **Quantum tunneling** and the **Maxwellian energy distribution** allow fusion to occur at temperatures lower than classically predicted.