Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Quantum Model of Uncertainty for Dynamic Decision Making

Kvantový popis neurčitosti pro dynamické rozhodování

Master's Thesis

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Author's declaration:
I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, January 8, 2024 Aleksej Gaj

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Abstrakt:

Experimenty v oblasti kognitivních věd ukázali, že lidské rozhodování se ukazuje být v rozporu s klasickou teorií rozhodování. Tento rozpor vyvolává řadu paradoxů a nekonzistencí. Bylo ukázáno, že kvantový přístup k rozhodování řeší tento problém, ovšem důvody, proč tomu tak je, zůstávaly neznámy.

Tato práce představuje nový koncept teorie který i) zavádí obecnější formalizaci rozhodovací úlohy ii) krok po kroku ukazuje, že za rozumně odůvodnitelných předpokladů je nalezeno řešení bez potřeby předem zavést pravděpodobnost iii) dokazuje kvantovou povahu neurčitosti a tvrdí, že kvantové modelování je pro rozhodování nezbytné.

Klíčová slova: rozhodování, kvantový model neurčitostí, kvantová pravděpodobnost, plně pravděpodobnostní návrh rozhodovacích strategií, teorie paralelních světů, kvantová teorie rozhodování

Title:

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Abstract: Classical decision theory is in a strong conflict with the observed experimental data coming from cognitive and descriptive decision making research. This conflict yields different paradoxes and inconsistencies. It was shown that quantum-like approach to decision making solves these problems, but the reasons why it does so far stayed unknown.

This thesis presents new framework that i) introduces more general formalisation of decision making task ii) step-by-step shows, that under realistic assumptions a solution is found without prior definition of probability iii) shows quantum nature of uncertainty and claims that quantum models are inevitable for decision making.

Key words: decision making, quantum model of uncertainties, quantum probability, fully probabilistic design of decision strategies, multi-world interpretation, quantum decision theory

To my parents

Contents

Introduction

Disclaimer. The title of this thesis is partially misleading: got inspired by the mathematical apparatus used in quantum mechanics, so we build similar construction for formalization of DM task. The truth is that this work neither directly relates to quantum mechanics nor tries to rebuild/extend it. On the other hand we have physical interpretation of mathematical objects we are dealing with.

Everyone makes decisions on a daily basis. It is the inseparable part of our everyday life. We make decisions based on own (or others) experience, sometimes we rely on logical considerations or use our intuition (whatever it is). Either one-shot decisions (like choosing a meal in a restaurant, a car insurance or a diploma topic) or dynamic decisions (like driving a car or managing a company) are made under uncertainty. Since the main engine of science has always been curiosity and laziness, people want not only to describe the phenomenon of cognition and decision but also to learn how to influence it. This is where decision theory (DT) comes into play.

In classical decision making theory (CDT) [26], [28], rational agents are modelled as "processors" of probabilistic information that updates models in Bayesian way. However general plausibility of the CDT is in strong conflict with significant experimental data: humans make their decisions by violating classical probability laws, for instance the Savage's sure thing principle [28], see also Appendix A. A. Tversky and D. Kahneman demonstrated cases when human behaviour persistently diverges from that predicted by classical probability [18], Ellsberg and Allais formulated paradoxes [1], [13]. One of the reasons is claimed apparent irrationality of humans [7], [39]. Another significant reason that humans are highly sensitive to context, and can be easily disturbed by other observations [18], [7].

Besides people (and their societies) form extremely complex systems having a large number of *practically* unobservable states. Even if we could measure/observe some states, these measurements would be very noisy and highly uncertain.

Judgments (and even weakly related questions) of other humans also may significantly influence judgment of the human in question. For instance a witness of a crime may be significantly influenced by the order in which the pictures are presented to him [35]. So called *order effects* introduce uncertainty into human judgments.

To address the mentioned inconsistency the quantum modelling has been applied [2]. Despite different motivation and different initial points, all of the approaches have confirmed that quantum modelling well describes human decision making and allows to overcome some paradoxes and inconsistencies. However neither of approaches has mathematical justification why quantum models provide better results. Although some of them provide intuitive explanation of different effects.

Representants of contemporary approaches to DM involving quantum way of modelling are: Yukalov, Sornette [39], Khrennikov [4], [5], Busemeyer [7], Pothos [25], Caves [8] and Sozzo [30].

Aim of this thesis is to provide a mathematical justification of quantum modelling in formalisation of decision making tasks involving humans. We would like to begin with a general formulation of decision making task (using closed loop formulation) and via sequence of well-grounded mathematical steps to come to quantum formalisation. Inevitably we will make some assumptions along the way, so we also provide an intuitive explanation that those assumptions are realistic and reasonable for the task we want to solve.

Several approaches can be considered when constructing a mathematically formulated decision theory. Before we get any further we stress that in scope of this work we deal with a prescriptive theory, not descriptive. It means aim of the theory is not (only) to describe some phenomenon (modelling environment/system), but also provide a recommendation on how to choose decisions optimal from the perspective of a given aim.

There exists a great amount of tasks involving decision making (DM) under uncertainty (outer ellipse in Figure 1). Subset of those tasks are tasks which involve human (human can be involved either as the user/agent, for whom the decision should be made or part of system with respect of which the decision should be made. Figure 1 describes the considered main groups of DM tasks. We are searching for prescriptive theory denoted by blue oval, which as much as possible respects the descriptive theory, which analyses how decisions are made when human is involved.

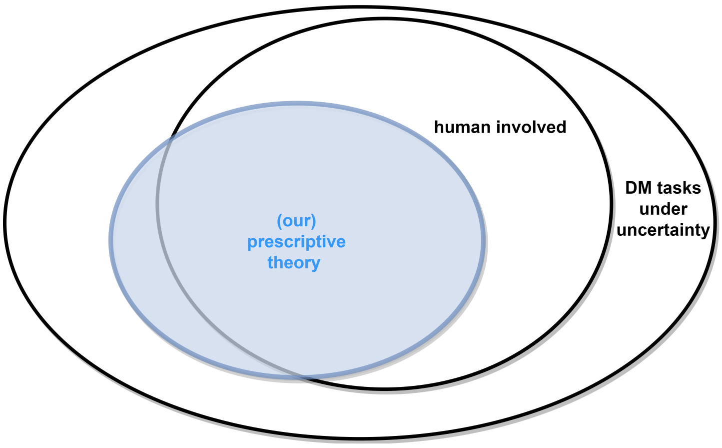

Figure 1: Sets of tasks: outer ellipse represents a set of all DM tasks. Inner ellipse represents such DM tasks where human is involved. Aim of this thesis is propose prescriptive theory which will cover most of inner black ellipse without overlap to the outer ellipse.

Outline. Chapter 1 briefly reminds some mathematical objects and theorems used in this thesis. Also general formulation of decision making task is placed here. Following Chapter 2 reminds (using introduced notation) CPT approach: Markov decision process (MDP) and fully probabilistic design (FPD). Chapter 3 contains a new way to model uncertainties always present in the closed loop model. It is reached by making a sequence of mathematical steps and assumptions. As the main result we get a measure on a space of uncertainties.
At the end of the thesis there are 3 appendix chapters presenting topics related to the core of this work. Note that each of them uses its own notion.

Notations

Shortcut	*Meaning*
DM	decision making
pdf	probability density function
MDP	Markov decision process
FPD	Fully probabilistic design
KLD	Kullback–Leibler divergence
QM	Quantum mechanics
CPT	Classical probability theory (the Kolmogorov's one)
QPT	Quantum probability theory
CDT	Classical decision theory (aka classical DM theory)
QDT	Quantum decision theory
w.r.t.	with respect to
OG	orthogonal
ON	orthonormal

Table 1: Abbreviations used.

Symbol	*Meaning*
$\mathbf{R}$	set of real numbers
$\mathbf{R}^+$	set of real positive numbers
$\mathbf{N}$	set of natural numbers
$\mathbf{X}$, $\mathbf{A}$, $\mathbf{S}$, $\boldsymbol{\Omega}$, ...	sets
A	σ-algebra
$[a, b]$	closed interval from $a \in \mathbf{R}$ to $b \in \mathbf{R}$, $a \leq b$
E	mean value (in CPT)
$\mathrm{D}(f\|g)$	Kullback–Leibler divergence between two probability functions f a g
$\mathbb{A}$	matrix
$\hat{\mathsf{A}}$	operator
$\mathcal{H}$	Hilbert space
$\mathcal{H}^\sharp$	dual space to Hilbert space
$\langle\phi\|\psi\rangle$	bra-ket (according to Dirac notation[11]) of $\phi \in \mathcal{H}^\sharp, \quad \psi \in \mathcal{H}$
$\langle a\|$	bra, covariant vector
$\|a\rangle$	ket, contravariant vector

Table 2: Notations used.

Note. Definitions of the notions are emphasised by *green italic*.

Chapter 1

Preliminaries

Every new beginning comes from some other beginning's end.

– Seneca

This chapter briefly reminds definitions and theorems used in the thesis, using notation introduced in Table 2.

1.1 Hilbert spaces

In this work Hilbert spaces will be one of main mathematical objects to work with. This section briefly reminds what is Hilbert space and how operations on its elements look like.

Definition 1 (Hilbert space)**.** Let's have a vector space V over real ($\mathbf{R}$) or complex ($\mathbf{C}$) field. If V is a complete metric space equipped with scalar product $\langle\cdot|\cdot\rangle : V \times V \to \mathbf{C}$, then it is called *Hilbert space* and noted $\mathcal{H}$.

Remark 1 (Some properties of Hilbert space). Let $\mathcal{H}$ be a Hilbert space. Then:

1. In $\mathcal{H}$ linearity holds: $(\forall\,|\phi,\psi\rangle \in \mathcal{H})\,(\forall a \in \mathbf{R}):\ a\,|\phi\rangle = |a\phi\rangle \in \mathcal{H}$ and $a\,|\phi\rangle + |\psi\rangle \in \mathcal{H}$
2. Every Cauchy sequence $(\psi_n) \subset \mathcal{H}$ is convergent in $\mathcal{H}$.
3. Every tuple of elements of Hilbert space $(\psi_j)_{j\in\mathbf{N}}$, which satisfies:

$$(\forall\,|\phi\rangle \in \mathcal{H})\ (\exists\,\{a_j \text{ in the field}\}_{j\in\mathbf{N}}): \quad |\phi\rangle = \sum_{j\in\mathbf{N}} a_j\,|\psi_j\rangle\,.$$

   is called a basis of the space. If $\forall i,j \in \mathbf{N}, i \neq j : \langle\psi_i|\psi_j\rangle = 0$ then the basis is *orthogonal* (OG).

Remark 2 (Norm in Hilbert space). Let $\mathcal{H}$ be a Hilbert space. Then we define a *norm* on $\mathcal{H}$ $\|\cdot\| : \mathcal{H} \to \mathbf{R}$ as

$$\|\psi\| := \sqrt{\langle\psi|\psi\rangle} \quad \forall\psi \in \mathcal{H}. \tag{1.1}$$

The orthogonal vectors (basis) of unit norm are called *orthonormal* (ON). More details can be found in [22] or [32].

Definition 2 (Subspace of Hilbert space)**.** Let's have a Hilbert space $\mathcal{H}$ over real or complex field. Space $\mathcal{H}_1$ is a *subspace of Hilbert space $\mathcal{H}$* if and only if:

1. $\mathcal{H}_1$ is a subset of $\mathcal{H}$: $\mathcal{H}_1 \subset \mathcal{H}$
2. $\mathcal{H}_1$ is closed under the operation of forming linear combinations:

$$(\forall a_1, a_2 \in \mathbf{R})\,(\forall \psi_1, \psi_2 \in \mathcal{H}_1): \quad a_1 |\psi_1\rangle + a_2 |\psi_2\rangle \in \mathcal{H}_1. \tag{1.2}$$

Notation: $\mathcal{H}_1 \subset\subset \mathcal{H}$.

Remark 3.

- If $\mathcal{H}_1 \subset\subset \mathcal{H}$, subspace $\mathcal{H}_1$ is equipped with the same scalar product as $\mathcal{H}$.
- Every subspace of Hilbert space is also a Hilbert space, [22].

Whenever we describe the evolution of some object, we use a notion of a state.

Definition 3 (State)**.** A *state of an object* is a collection of mathematical quantities such that if we have a prescription for the object's evolution and the current state, we can always determine the next state.

The state of an object can be represented via different mathematical quantities. For instance, it can be represented by an integer number (state of simple random walk in 1D), or tuples of values (pressure, volume and temperature of gas), sequences, vectors in vector space (say coordinates of a pendulum) or other mathematical objects. Another example is Markov chain: the current state is the only information (alongside the evolutional equation) we need to determine the next state.

Definition 4 (Tensor product of Hilbert spaces)**.** Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces with scalar products $\langle \bullet|\bullet\rangle_1$ and $\langle \bullet|\bullet\rangle_2$ respectively. Then the *tensor product of two Hilbert spaces* is defined as $\mathcal{H}_3 := \mathcal{H}_1 \otimes \mathcal{H}_2$, [23], with a linear envelope of the elements:

$$(\forall |\alpha_3\rangle \in \mathcal{H}_3)\,(\exists |\alpha_1\rangle \in \mathcal{H}_1, |\alpha_2\rangle \in \mathcal{H}_2) \quad \alpha_3 = \alpha_1 \otimes \alpha_2 \equiv |\alpha_1\, \alpha_2\rangle \tag{1.3}$$

and scalar product $\langle \bullet|\bullet\rangle_3$:

$$\forall \alpha_3, \beta_3 \in \mathcal{H}_3\ \langle \alpha_3|\beta_3\rangle_3 = \langle \alpha_1|\beta_1\rangle_1 + \langle \alpha_2|\beta_2\rangle_2\,. \tag{1.4}$$

Definition 5 (Linear operator on Hilbert space)**.** *Linear operator* is mapping that acts on elements of a space and produces elements of the same space. Symbolically - by operator $\hat{\mathsf{A}}$ on Hilbert space $\mathcal{H}$ we mean : $\hat{\mathsf{A}} : \mathcal{H} \to \mathcal{H}$.

Remark 4. Onwards by operator we always mean *linear* operator.

Definition 6 (Projector)**.** Every (linear) operator $\hat{\mathsf{P}}$ which satisfies $\hat{\mathsf{P}}\hat{\mathsf{P}} = \hat{\mathsf{P}}$ is called a *projector*.

Remark 5 (Bra–ket formalism)*.* P.A.M. Dirac introduced a notational formalism that allows equivalent representation of the state:

- state is represented by an element of Hilbert space: $|\psi\rangle \in \mathcal{H}$, or
- state is represented by a projector: $\hat{\mathsf{P}}_\psi = |\psi\rangle \langle\psi|$.

Details on the formalism can be found in [22].

Let $(u_j)_{j\in\mathbf{N}}$ be a basis in $\mathcal{H}$. Then projection of $|\psi\rangle \in \mathcal{H}$ to the j-th dimension can be found as $|\psi_j\rangle = |u_j\rangle\langle u_j|\psi\rangle$, where the projector is: $\hat{\mathsf{P}}_j = |u_j\rangle\langle u_j|$. The first factor is a vector from the basis (normalized to 1), which determines the direction of projection and the second factor is a scalar product, which determines the coefficient. Indeed,

$$|\psi_j\rangle \equiv \hat{\mathsf{P}}_j|\psi\rangle = \underbrace{|u_j\rangle\langle u_j|}_{\hat{\mathsf{P}}_j}|\psi\rangle = \underbrace{|u_j\rangle}_{\text{direction}}\ \underbrace{\langle u_j|\psi\rangle}_{\text{coef.}} \tag{1.5}$$

Similarly the projection into two-dimensional subspace spanned on (not necessarily orthogonal) unit vectors u_i and u_j reads: $|\psi_{ij}\rangle \equiv \hat{\mathsf{P}}_i\hat{\mathsf{P}}_j|\psi\rangle = |u_i\rangle\langle u_i|u_j\rangle\langle u_j|\psi\rangle$.

Remark 6 (Relation between a projector and a subspace). Every vector of Hilbert space can be represented as a sum of two vectors: a vector that is an element of a subspace of Hilbert space and other OG vector:

$$(\forall\psi\in\mathcal{H})\,(\forall\mathcal{H}_1\subset\subset\mathcal{H})\left(\exists\phi_1\in\mathcal{H}_1,\exists\phi_2\in\mathcal{H}\setminus\mathcal{H}_1,\langle\phi_2|\phi_1\rangle\overset{!}{=}0\right):\quad |\psi\rangle = |\phi_1\rangle + |\phi_2\rangle\,. \tag{1.6}$$

The mapping which assigns $|\psi\rangle \to |\phi_1\rangle$ is the projector on subspace $\mathcal{H}_1$. It can be shown in a following way:

Statement above implies that we expect a projector in a following form:

$$\hat{\mathsf{P}}|\psi\rangle = a|\phi_1\rangle\,, \tag{1.7}$$

where $a\in\mathbf{R}$.

$$\hat{\mathsf{P}}|\psi\rangle \overset{(1.6)}{=} \hat{\mathsf{P}}\left(|\phi_1\rangle + |\phi_2\rangle\right) \overset{!}{=} a|\phi_1\rangle\,. \tag{1.8}$$

That means we search for operator $\hat{\mathsf{P}}:\mathcal{H}\to\mathcal{H}_1$ that has these properties:

$$\begin{aligned}\hat{\mathsf{P}}|\phi_1\rangle &= a|\phi_1\rangle\\ \hat{\mathsf{P}}|\phi_2\rangle &= 0\end{aligned} \tag{1.9}$$

Choice $\hat{\mathsf{P}} := |\phi_1\rangle\langle\phi_1|$ fulfils it:

$$\begin{aligned}|\phi_1\rangle\langle\phi_1|\phi_1\rangle &= |\phi_1\rangle\, a\\ |\phi_1\rangle\langle\phi_1|\phi_2\rangle &= 0.\end{aligned} \tag{1.10}$$

Theorem 1 (Operators in bra–ket formalism)**.** Let $\mathcal{H}$ be a Hilbert space with a countable orthonormal basis $\{u_j\}_{j\in\mathbf{N}}$. Then:

- identity operator can be written as

$$\sum_j |u_j\rangle\langle u_j| = \hat{1} \tag{1.11}$$

  Note the symbol on the right side: operator $\hat{1}$ vs. number 1.

- every operator $\hat{\mathsf{A}}$ on $\mathcal{H}$ can be written as

$$\hat{\mathsf{A}} = \sum_j a_{ij}|u_i\rangle\langle u_j|\,, \tag{1.12}$$

  where $a_{ij} = \langle u_i|\hat{\mathsf{A}}|u_j\rangle \in \mathbf{R}$.

Proof. Straightforward:

- For every $\psi, \phi \in \mathcal{H}$ holds: $\langle\psi|\phi\rangle = \sum_j \langle\psi|u_j\rangle \langle u_j|\phi\rangle = \langle\psi| \sum_j |u_j\rangle \langle u_j|\phi\rangle = \langle\psi| \hat{\mathsf{1}} |\phi\rangle$
- $\hat{\mathsf{A}} = \hat{\mathsf{1}}\hat{\mathsf{A}}\hat{\mathsf{1}} = \sum_i \sum_j |u_i\rangle \langle u_i| \hat{\mathsf{A}} |u_j\rangle \langle u_j| = \sum_i \sum_j |u_i\rangle a_{ij} \langle u_j| = \sum_i \sum_j a_{ij} |u_i\rangle \langle u_j|$

□

Note that with the use of Eq. (1.11) we can write $|\psi\rangle = \sum_j |u_j\rangle \langle u_j|\psi\rangle$ which is basically representation of a vector ψ in a basis $\{u_j\}_{j\in\mathbf{N}}$ in terms of linear algebra.

Definition 7 (Unitary matrix)**.** Matrix $\mathbb{A}$ is called *unitary* if it satisfies:

- $\mathbb{A}$ is square matrix
- $\mathbb{A}$ is complex matrix (it has complex elements)
- $\mathbb{A}$ is invertible: $\exists\mathbb{A}^{-1}$
- its conjugation transpose[1]is also its inverse: $\mathbb{A}^* = \mathbb{A}^{-1}$.

Remark 7. Some sources distinguish between *unitary matrices* (their elements are *complex* numbers) and *orthogonal matrices* (their elements are *real* numbers). Since applying complex conjugation on a real number does not change it, we will keep term unitary matrix as more general one.

Remark 8. Linear operator on a vector space with countable basis (which will be the case in this thesis) can be represented by a matrix.

Theorem 2 (Selected properties of unitary matrices)**.** Let matrix $\mathbb{A}$ be a unitary matrix. Then the following statements hold:

- for any two (complex) vectors $\psi_1, \psi_2 \in \mathcal{H}$ multiplication by $\mathbb{A}$ on $\mathcal{H}$ preserves their inner product, $\langle\mathbb{A}\psi_1|\mathbb{A}\psi_2\rangle = \langle\psi_1|\psi_2\rangle$.
- $\mathbb{A}$ is *normal*, $\mathbb{A}^*\mathbb{A} = \mathbb{A}\mathbb{A}^*$.
- $\mathbb{A}^*\mathbb{A} = \mathbb{A}\mathbb{A}^* = \mathbb{I}$, where $\mathbb{I}$ is identity matrix.
- Eigenspaces[2]of $\mathbb{A}$ are orthogonal.
- Columns of $\mathbb{A}$ form an orthonormal basis (w.r.t. inner product), also $\mathbb{A}^*\mathbb{A} = \mathbb{I}$.
- Rows of $\mathbb{A}$ form an orthonormal basis (w.r.t. inner product), also $\mathbb{A}\mathbb{A}^* = \mathbb{I}$.

Proof. Can be found in [16]. □

[1] *Conjugation transpose* (also Hermitian transpose) is defined by transposing the matrix and applying complex conjugation to each its element.

[2] The set of all eigenvectors of $\mathbb{A}$ corresponding to the same eigenvalue, together with the zero vector, is called an *eigenspace*.

Theorem 3 (Singular value decomposition)**.** Any matrix $\mathbb{A}$ of size $m \times n$ with real or complex values can be decomposed:

$$\mathbb{A} = \mathbb{V} \cdot \mathbb{D} \cdot \mathbb{U}^*, \tag{1.13}$$

where:

- $\mathbb{V}$ is a unitary matrix of size $m \times m$,
- $\mathbb{D}$ is a matrix of size $m \times n$ with non-negative values. Elements on the main diagonal are so called singular values of $\mathbb{A}$, and the remaining elements are zeros,
- $\mathbb{U}$ is a unitary matrix of size $n \times n$.

Proof. Can be found in [17]. □

Definition 8 (Positive semidefinite operator)**.** Let $\hat{\mathsf{A}} : \mathcal{H} \to \mathcal{H}$ be an operator on Hilbert space $\mathcal{H}$. $\hat{\mathsf{A}}$ is *positive semidefinite* (PSD) $\Leftrightarrow \forall \psi \in \mathcal{H} :\ \langle \psi | \, \hat{\mathsf{A}} \, | \psi \rangle \geq 0$.

Remark 9. Note that using Theorem 1 every PSD operator $\hat{\mathsf{A}} : \mathcal{H} \to \mathcal{H}$ can be written as:

$$\hat{\mathsf{A}} = \sum_{j \in \mathbf{j}} w_j \, |\alpha_j\rangle \langle \alpha_j| \,, \tag{1.14}$$

where $w_j \in \mathbf{R}$ are eigenvalues of $\hat{\mathsf{A}}$ and $|\alpha_j\rangle \in \mathcal{H}$ are eigenvectors of $\hat{\mathsf{A}}$. Due to PSD $w_j \geq 0$ and set of all eigenvectors $\{\alpha_j\}_{j \in \mathbf{j}}$ defines a basis in $\mathcal{H}$.

1.2 Quantification of ordering

Let us have a set of objects with a complete transitive ordering defined on it: $(\mathbf{B}, \prec_{\mathbf{B}})$ ($\mathbf{B}$ - set of elements $b_1, b_2, \ldots$). Quantification[3] of the ordering operation $\prec_{\mathbf{B}}$ is required. It is done by finding a real valued mapping $q : \mathbf{B} \to \mathbf{R}$ (where $\mathbf{R}$ is the set of real numbers), such that $b_1 \prec_{\mathbf{B}} b_2 \Leftrightarrow q(b_1) < q(b_2)$ and $b_1 \approx_{\mathbf{B}} b_2 \Leftrightarrow q(b_1) = q(b_2)$.

Definition 9 (Density of a set w.r.t. an ordering on it)**.** Let $(\mathbf{B}, \prec_{\mathbf{B}})$ be a set with a complete transitive ordering. A set $\mathbf{G}$, $\mathbf{G} \subset \mathbf{B}$, is *$\prec_{\mathbf{B}}$-dense* if and only if following holds:

$$(\forall b_1, b_2 \in \mathbf{B})\, (\exists\, b_3 \in \mathbf{G} \subset \mathbf{B}) : \quad b_1 \prec_{\mathbf{B}} b_3 \prec_{\mathbf{B}} b_2. \tag{1.15}$$

Theorem 4 (On quantification of ordering)**.** A continuous loss function, $q : \mathbf{B} \to \mathbf{R}$ that quantifies ordering $\preceq_{\mathbf{B}}$ on $\mathbf{B}$, exists if and only if there exists set $\mathbf{G}$ such that:

1. $\mathbf{G}$ is countable,
2. $\mathbf{G}$ is $\prec_{\mathbf{B}}$-dense.

Proof. Proof can be found in [9]. □

An intuitive interpretation of Theorem 4 states that the quantification of ordering is possible if and only if operation $\prec_{\mathbf{B}}$ on $\mathbf{B}$ is not richer than operation $<$ on real numbers.

[3]In this context - assigning a real number to every element of $\mathbf{B}$ according to the ordering.

1.3 Probability measure on subspaces

Classical probability theory formulated by Andrei Kolmogorov uses σ-algebra as the basic term. Such a σ-algebra is defined via operations of intersection and union on a sample space. A random event is represented as an element of σ-algebra: a set. Therefore probability measure on such random events is a measure defined on sets.

Quantum probability takes unit vectors in Hilbert space as elementary events. It creates events via meet (intersection) and join (linear envelope of union). Thus probability is to be defined on such events. It has a general form given by Gleason's theorem, see below. Table 1.1 compares the models of randomness using the description of subspaces via projectors.

CPT (Kolmogorov)	**QPT (Gleason)**
sample space $\mathbf{\Omega}$	sample space $\mathcal{H}$
set of events: σ-algebra[4] A built on $\mathbf{\Omega}$	**set of subspaces** of $\mathcal{H}$
operation **intersection** $\bigcap$ on A	operation **meet** $\wedge$ on $\mathcal{H}$, which is defined as intersection of subspaces
operation **union** $\bigcup$ on A	operation **join** $\vee$ on $\mathcal{H}$, which is defined as linear span of union of subspaces
de Morgan properties hold	**de Morgan** properties do not hold
commutativity holds: $$A \cap B = B \cap A$$ Order does not matter.	**commutativity:** for $A, B \subset\subset \mathcal{H}$ commutativity does not hold in general: $$\hat{\mathsf{P}}_A \hat{\mathsf{P}}_B \neq \hat{\mathsf{P}}_B \hat{\mathsf{P}}_A$$ That means in assigning probability *order matters.*

Table 1.1: Comparison between CPT and QPT.

A new probability measure is needed since quantum probability theory assigns random events to vectors in Hilbert space. The following theorem (modified version of that in [24]) can be used for it:

Theorem 5 (Gleason's)**.** Let:

- $\mathcal{H}$ be a Hilbert's space such that:
  - $\mathcal{H}$ is separable (within the scope of this thesis is ensured thanks to Theorem 4)
  - $\mathcal{H}$ has dimension $\dim(\mathcal{H}) \geq 3$.
- $\hat{\mathsf{P}}(\mathcal{H})$ is a set of all projectors on $\mathcal{H}$ and for every projector $\hat{\mathsf{P}}$ (representing an event) there exists probability $\mu(\hat{\mathsf{P}})$ of the event such that:

[4] σ-algebra A is defined as a subset of power set of $\mathbf{\Omega}$ that has following properties:

- A is closed under complementation: $(\forall A \in \mathsf{A}) : \mathbf{\Omega} \setminus A \in \mathsf{A}$,
- A is closed under countable unions: $(\forall j, A_j \in \mathsf{A}) : \bigcup_j A_j \in \mathsf{A}$.

Properties mentioned above imply that σ-algebra is also closed under countable intersections, and also that $\varnothing \in \mathsf{A}$, $\mathbf{\Omega} \in \mathsf{A}$.

- $0 \leq \mu(\hat{\mathsf{P}}) \leq 1$
- $\mu(\hat{\mathsf{1}}) = 1$ (where for operator $\hat{\mathsf{1}}$ following holds: $\sum_j |u_j\rangle \langle u_j| = \hat{\mathsf{1}}$).

Then there exists an operator $\hat{\mathsf{T}}_\mu : \mathcal{H} \to \mathcal{H}$ such that the probability can be written as a quadratic form:

$$(\forall \psi \in \mathcal{H}) : \ \mu \left(|\psi\rangle \langle\psi| \right) = \langle\psi| \hat{\mathsf{T}}_\mu |\psi\rangle \, . \tag{1.16}$$

Operator $\hat{\mathsf{T}}_\mu$ is called *density operator* and has the following properties:

1. It is positive semidefinite (PSD), $\hat{\mathsf{T}}_\mu \trianglerighteq 0$
2. Its trace is equal to 1, $\mathrm{tr}\left(\hat{\mathsf{T}}_\mu\right) = 1$.

Proof. Proof can be found in [12]. □

In other words, Theorem 5 states that under some assumptions a probability measure of an element of Hilbert space (for example the state vector) can be represented as a quadratic form. We can find operator $\hat{\mathsf{T}}_\mu$, which in the discrete case is represented by a matrix. Then the right side of (1.16) is a vector multiplied by a matrix and by the same vector, which is a matrix representation of a quadratic form.

1.4 Decision making under uncertainty: examples

Let us consider DM task from the decision-maker's point of view. A decision-maker has some aim concerning the part of the world he[5] is interacting with. His aim can be influencing the world i.e. obtaining the desired behaviour via the interaction. But not always the decision-maker influences the part of the word directly. Let us consider two examples:

1. (DM task: choosing clothes). Imagine a man who has to choose clothes before going outside. His aim is not to get wet or cold during the day (given the unknown weather forecast). The man observes current weather, say, when he looks out of the window. But he is not able to predict weather conditions several hours ahead. The man decides to choose a coat and umbrella, because it's raining and pretty cold. But his decision *does not influence* the weather. i.e. the weather will not change, no matter he chooses a coat or swimsuit. His action will affect only his level of comfort (payoff).
2. (DM task: Air condition problem). Imagine a man in the room equipped with an air conditioner (AC). The man prefers to have temperature in the room at the desired level, say 20C. His decision is to set 20C on AC device. By doing that, he directly *influences* the room temperature (a part of the world).

Remark 10. DM task, its solution and obtained results heavily depend on the formulation of the decision task and involved objects.

To unify formalisation of the different representative examples above, let us model them via *closed loop*, see Figure 1.1, consisting of agent and system. By *system* we call a part of the world

[5]In this text, we refer to the agent as he/him, although based on the agent's nature, pronouns she/her and it/its can be applied too.

the decision-maker is interested in. By *agent* we called decision-maker, which interacts with the system intending to influence it, learn it[6] or both.

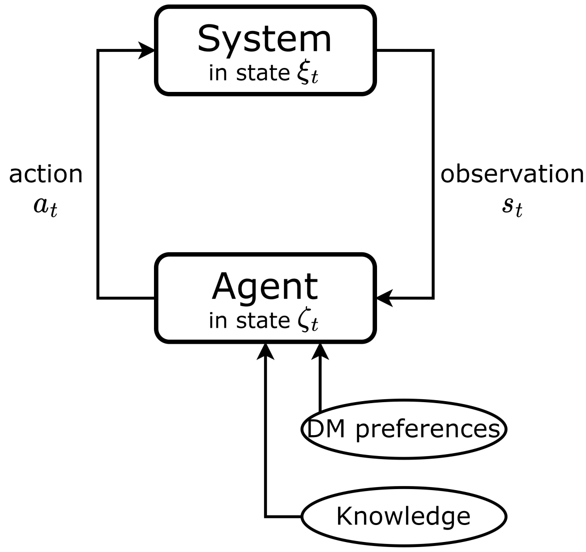

Figure 1.1: Closed-loop formulation of DM task: *agent* chooses actions to influence the *system* (or his knowledge about the system) according to his DM preferences. Also agent receives information from the system via observations and updates his knowledge.

Clearly, any agent's activity influences the whole closed loop. The agent has some *knowledge* (either common, individual or expert-based) and *DM preferences* that express his wishes concerning the closed loop behaviour.

The agent's preferences about system (for example what states or observations he wants the system to be in; or what sequence of states he wants the system to visit) can be expressed via different mathematical objects. One of the possible options is to use *reward (or loss) function*: such function will take state and action chosen by agent and return a reward (payoff in Savage's terminology, see Appendix).

To summarize:

- *Agent* chooses actions $a_t \in \mathbf{a}$ using his knowledge at time t from a given set of actions $\mathbf{a}$ regarding his preferences. At time t agent is has inner state $\zeta_t \in \boldsymbol{\zeta}$.[7] Agent also makes *observations* $s_t \in \mathbf{s}$ on the system.

- *System* is a part of the world related to the agent DM preferences. At time t the system inner (not necessarily observable) state is $\xi_t \in \boldsymbol{\xi}$.

Until now we have not distinguished between static and dynamic DM tasks. *Static DM task* means that there is no time evolution in it. The agent takes only one (possibly multivariate) action (selects either coat of jacket, see DM task: choosing clothes above). One-shot parameter estimation or choosing of the most suitable insurance plan can be an example of such a scenario.

Dynamic DM task means, that the decision making evolves in time: in each time (discrete or continuous) agent makes observations on the system, and chooses optimal action for the

[6]i.e. improve the agent's knowledge about the system.
[7]It may reflect the agent's knowledge about the system.

current time. This optimal action influences the closed-loop. In the next time epoch the agent again makes observation and updates optimal action. Section 2.2 discusses this concept in more details.

Chapter 2

Probabilistic approach to DM under uncertainty

Reality resists imitation through a model.

– E. Schrödinger, *The Present Simulation in Quantum Mechanics*

Since our aim is to describe principles of (human) decision making, we need to formalize them. For a long period of time the classical probability theory (CPT) (pretty successfully) served to it. Recent studies thought suggest an existence of much suitable apparatus for describing human DM behaviour. In this chapter we will briefly summarize, so far classical, probabilistic approach to solution of DM task. It has origin in the classical Savage's formalisation (for details, see Appendix A).

2.1 Decision making as closed loop

The DM formalisation operates on behaviour $b \in \mathbf{b}$, which is a collection of all variables in closed loop in all of the considered time instances.

Assumption 1 (Ordering on behaviours)**.** User's preferences can be expressed/formalised via an ordering $\preceq_{\mathbf{b}}$ on set of behaviours $\mathbf{b}$. It also implies equivalence $\approx_{\mathbf{b}}$ and strong ordering $\prec_{\mathbf{b}}$. The ordering should satisfy the following assumptions:

1. there exist *at least two distinguishable* behaviours[1]: $\exists b_1, b_2 \in \mathbf{b} : b_1 \prec_{\mathbf{b}} b_2$,
2. $\preceq_{\mathbf{b}}$ is a *complete* ordering[2],
3. *transitivity*: for every $b_1, b_2, b_3 \in \mathbf{b}$: if $b_1 \prec_{\mathbf{b}} b_2$ and $b_2 \prec_{\mathbf{b}} b_3$, then $b_1 \prec_{\mathbf{b}} b_3$.

The completeness can be always reached within the considered setup. The transitivity assumption is needed to avoid *money pump*[3].

[1] In the rest of this section subscripts distinguish elements of the set $\mathbf{b}$.

[2] i.e. any pair of behaviours is comparable.

[3] In economic theory, the money pump argument is a thought experiment showing that rational behavior requires transitive preferences.

Definition 10 (Loss function)**.** *Loss function* is a mapping $z : \mathbf{b} \to \mathbf{R}$ that ($\mathbf{R}$ is the set of real numbers):

1. is monotone on $\mathbf{b}$: $b_1 \preceq_{\mathbf{b}} b_2 \;\Leftrightarrow\; z(b_1) \leq z(b_2)$,

2. preserves equivalence on $\mathbf{b}$: $b_1 \approx_{\mathbf{b}} b_2 \;\Leftrightarrow\; z(b_1) = z(b_2)$.

Note that ordering $\preceq_{\mathbf{b}}$ cannot be richer than $\leq$ on a real line – so only countable things are distinguished. This allows us to further deal with at most countable sets. Throughout, we assume that (preferential) ordering $\preceq_{\mathbf{b}}$ can be quantified by loss function $z(\bullet)$.

Definition 11 (Decision rule, strategy)**.** *Decision rule* is a mapping which assigns action a to available knowledge k:

$$S : \mathbf{k} \to \mathbf{a}.$$

Time sequence of decision rules is called *strategy*. In the static DM task a decision rule coincides with the strategy.

2.2 Probabilistic formalization of DM task: MDP and FPD.

We consider DM task in *closed loop* formulation, see Figure 1.1). The agent's aim[4] related to the system is any of the following: i) to influence the system; ii) to learn it, or both. The solved DM task is to chose action (or sequence of actions) that influence the system itself or improve agent's knowledge about the system.

Formally at time $t \in \mathbf{T}$, $\quad \mathbf{T} := \{0, 1, \dots T-1\}$ the system generates state $s_t \in \mathbf{S}$ observed by the agent[5]. The agent (based on sequence of observed states, his aim and prior knowledge) generates action $a_t \in \mathbf{A}$, which influences the system state at the next time $s_{t+1} \in \mathbf{S}$. To dynamically select appropriate action, the agent needs to have optimal *decision rule* (aka decision function) which maps knowledge to actions. Time sequence of decision rules forms *DM strategy*.

To describe the "agent-system" interaction, a notion of closed-loop behaviour is used.

Definition 12 (Closed-loop behaviour)**.** The *closed–loop behaviour* at time t is a sequence of pairs a_t, s_t:

$$b_t := \left(\underbrace{a_T, s_T, \;\dots,\; s_{t+1}}_{g_{t+1}}, a_t, \underbrace{s_t, \;\dots,\; a_1, s_1}_{k_t} \right) = (g_{t+1}, a_t, k_t) \tag{2.1}$$

where a_t and and s_t are action and system state at time $t \in \mathbf{T}$, respectively. Note that b is a sequence, so generally it can be interpreted as a vector.

In (2.1):

- $k_t \in \mathbf{k}$ represents a *knowledge* possessed by the agent. For example: data (sequence of previous observations and actions), structural knowledge, any kind of prior knowledge, etc. Initial action a_0 and state s_0 are supposed to be known and implicitly included in knowledge k_0. $\mathbf{k}$ – set of knowledges[6]

[4]Often represented by agent DM preferences.
[5]This is a simplification as system states are quite often observed indirectly and partially
[6]These unusual plural forms (knowledges, behaviours, ignorances, ...) are necessary in a context of this thesis to distinguish between a (sub)set and its element.

- $g_t \in \mathbf{g}$ is so–called *ignorance* containing anything that the agent considers but cannot use for the choice of action (may contain: future observations, unobserved parts of state, ...), $\mathbf{g}$ – set of ignorances

Definition 13 (Behaviour of the closed loop)**.** Tuple

$$b_t := (g_t, a_t, k_t) \tag{2.2}$$

represents *behaviour of the closed loop* and $\mathbf{b}$ is a set of all possible behaviours of the closed loop.

Interpretation of b_t in (2.2) is: a_t is an action chosen by the agent based on knowledge k_t under ignorance g_t. In a static DM task (when single $a \in \mathbf{a}$ is chosen), time subscript is dropped. Generally, with k_0, a_0, s_0 in the, often implicit, condition:

$$b_t = (g_t, a_t, k_t) = (s_t,\ a_{t-1}, s_{t-1},\ \ldots,\ a_\tau, s_\tau,\ \ldots,\ a_1, s_1) \tag{2.3}$$

is a complete description of the closed loop in time t: all states observed and actions chosen before current time t belong to knowledge k_t, while all future observations and actions are part of ignorance g_t. With every new time step ignorance is reduced (next action was chosen and observed state received) and knowledge k_t is larger (or at least not smaller).

Let us model closed-loop behaviour, Definition 12, probabilistically and factorise the model $p(b_T)$ using the chain rule

$$\begin{aligned} p(b_T) &= p\left(s_T, a_{T-1}, s_{T-1},\ \ldots, a_1, s_1\right) \\ &= \prod_{t\in\mathbf{T}} p\left(s_{t+1}, a_t | \underbrace{a_{t-1}, \ldots, a_0, s_t, \ldots, s_0}_{k_t,\ \text{see (2.1)}}\right) \\ &= \prod_{t\in\mathbf{T}} p\left(s_{t+1}|a_t, k_t\right) p\left(a_t|k_t\right) \end{aligned} \tag{2.4}$$

Definition 14 (DM rule and DM strategy)**.** *DM rule* is conditional pdf that represents probability of choosing action a_t when state s_t is observed:

$$p\left(a_t|s_t, s_{t-1}, \ldots, s_0, a_{t-1}, \ldots, a_0\right).$$

DM strategy (or simply *strategy*) is a time sequence of DM rules:

$$S = S^T = \left(p\left(a_t|s_t, s_{t-1}, \ldots, s_0, a_{t-1}, \ldots, a_0\right)\right)_{t\in\mathbf{T}} = \left(p\left(a_t|k_t\right)\right)_{t\in\mathbf{T}}.$$

Strategy is also called *policy*.

We already have got an interpretation for two objects on the right-hand side of (2.4):

$$p(b_T) = \prod_{t\in\mathbf{T}} \underbrace{p\left(s_{t+1}|a_t, k_t\right)}_{\text{model of the system}} \underbrace{p\left(a_t|k_t\right)}_{\text{DM rule}}. \tag{2.5}$$

Solving DM task means to design an algorithm for selecting a DM strategy that is optimal regarding the agent's aim. There exist many different formulations of this problem and *Markov decision process* (MDP), [26], is one of them. MDP uses so–called Markov assumption that replaces conditions in (2.5) only by the latest values, which significantly simplifies the model.

Under some assumptions the MDP solution is an optimal strategy that maximises expected reward (cf. with payoff in Appendix A):

$$S^{opt} = \arg\max_{S \in \mathbf{S}} \mathrm{E}_S \left[\sum_{t \in \mathbf{T}} R\left(s_{t+1}, a_t, s_t\right) \right], \tag{2.6}$$

where E_S is a strategy-dependent expectation and $R : \mathbf{S} \times \mathbf{A} \times \mathbf{S} \to \mathbf{R}$ is a *reward function* assigning every triple (s_{t+1}, a_t, s_t) a real value, and $\mathbf{S}$ is a set of all possible strategies. If a *loss function* $l : \mathbf{S} \times \mathbf{A} \times \mathbf{S} \to \mathbf{R}$, $l = -R$ is used instead of reward function, then in (2.6) minimisation replaces maximisation. Note that Chapter 3 will use loss function for quantification.

Another formulation of DM problem generalising MDP is so-called Fully Probabilistic Design, [20], solution of which is given by Theorem 6, see below.

Definition 15 (Fully probabilistic design (FPD))**.** Let's have a Markov model of the system $p\left(s_{t+1}|a_t, s_t\right)$ and probability function ${}^{I}p(b_T)$ that describes preferable (from agent's perspective) behaviour of the closed loop[7]. A strategy which ensures minimal divergence between probability describing behaviours of closed loop b_T and its ideal counterpart is called *FPD-optimal strategy*. In other words:

$$S^{opt} \in \arg\min_{S \in \mathbf{S}} \mathrm{D}\left(p_S(b_T) \| {}^{I}p(b_T) \right), \tag{2.7}$$

where $\mathrm{D}(\cdot\|\cdot)$ is Kullback–Leibler divergence, for more see [10].

Remark 11. Axiomatic justification of this definition is provided in [21].

Theorem 6 (Solution of FPD)**.** Let $\mathbf{A}$, $\mathbf{S}$ to be discrete sets and $a_t \in \mathbf{A}$, $s_t \in \mathbf{S}$ for $\forall t \in \mathbf{T}$. Let's have model of system $p\left(s_{t+1}|a_t, s_t\right)$ and ideal pdf of closed loop behaviour factorized similarly as Eq. (2.5).

Then optimal DM rule for time $t \in \mathbf{T}$ is to be found in a way:

$$p^{opt}(a_t|s_t) = {}^{I}p\left(a_t|s_t\right) \frac{\exp\left(-d(a_t, s_t)\right)}{h(s_t)}, \tag{2.8}$$

where

$$d(a_t, s_t) = \sum_{s_{t+1} \in \mathbf{S}} p\left(s_{t+1}|a_t, s_t\right) \ln \frac{p\left(s_{t+1}|a_t, s_t\right)}{{}^{I}p\left(s_{t+1}|a_t, s_t\right) h(s_{t+1})}, \tag{2.9}$$

$$h(s_t) = \sum_{a_t \in \mathbf{A}} {}^{I}p\left(a_t|s_t\right) \exp\left(-d(a_t, s_t)\right) \tag{2.10}$$

We are searching for optimal decision rule for every time epoch $t \in \mathbf{T}$. We are going though time backwards, using so called backward induction. The computation starts at $t = T$ with:

$$h(s_T) = 1 \quad \forall s_T \in \mathbf{S}. \tag{2.11}$$

The optimal strategy is a sequence of the optimal DM rules:

$$S^{opt} = \left(p^{opt}(a_t|s_t) \right)_{t \in \mathbf{T}}. \tag{2.12}$$

Proof. Proof can be found in [29]. □

[7]This pdf is often called the ideal pdf.

Chapter 3

A new way of formalisation of DM under uncertainty

> *Once we have granted that any physical theory is essentially only a model for the world of experience, we must renounce all hope of finding anything like "the correct theory".*
>
> – H. Everett III., *The Theory of Universal Wave Function*

In this chapter we formalise DM in the closed loop from the scratch and arrive to the need for the quantum probability as a tool for the optimal decision making. This avoids the classical approach [28] relying on pre-existing notion of probability.

3.1 Let the magic begin (static case)

Let us first consider a static case, i. e. when the action is selected only once. Keeping the introduced notation (see Section 1.4) we obtain that t = fixed, g = fixed, k = fixed. The time index t is omitted in this section.

Assumption 2 (DM under uncertainty)**.** Every closed–loop behaviour b can be expressed as a function Φ of two arguments: strategy S, which is explicitly chosen by the agent, and uncertainty[1] U, ($U \in \mathbf{U}$, $\mathrm{card}\,(\mathbf{U}) \geq 2$) that is completely independent of the agent and is not accessible to the agent:

$$b = \Phi\,(S, U)\,. \tag{3.1}$$

In Assumption 2 U represents overall uncertainty existing in the closed DM loop, i. e.

- anything that prevents from accurately determining b for the chosen strategy, S;
- anything that the agent cannot influence, no matter which strategy he chooses.

Applying loss function $z(\bullet)$ on both sides of (3.1), we get:

$$z(b) = z\,(\Phi\,(S, U))\,. \tag{3.2}$$

[1] We will not specify at this moment what is the mathematical nature of it.

According to Theorem 4, the assumed existence of a loss function implies that distinguishable behaviours, strategies, and uncertainties are densely populated with countable subsets. Thus, we can handle these countable subsets symbolically. With some notation abuse, we denote both the strategy (i. e. a sequence of mappings) and the integer pointer to it by the same symbol S. We use the same multiply interpretable notation to uncertainty, i. e. U denotes the uncertainty itself and a pointer to it. The meaning of specific notations will be apparent from the context.

Now let us consider S to be an index (or pointer) representing a strategy and U be an index (or pointer) representing an uncertainty. We can introduce matrix $\mathbb{L}$ that contains values $z\left(\Phi\left(S,U\right)\right)$ for every index values S,U, i.e.

$$\mathbb{L}_{S,U} \equiv \left[\mathbb{L}\right]_{S,U} := z\left(\Phi\left(S,U\right)\right), \tag{3.3}$$

where $\operatorname{card}\left(\mathbf{S}\right) \leq \operatorname{card}\left(\mathbf{N}\right), S \in \mathbf{S},\ \operatorname{card}\left(\mathbf{U}\right) \leq \operatorname{card}\left(\mathbf{N}\right), U \in \mathbf{U}$. From the definition of loss function $z(\bullet)$ it follows that elements of matrix $\mathbb{L}$ are real numbers.

Assumption 3 (No hidden feedback)**.** *U is* not *dependent on strategy S and evolves irrespectively of S.*

This assumption means that no hidden feedback is present in the closed loop.

From here onwards, we accept Assumptions 1, 2, 3. Matrix $\mathbb{L}$ contains loss values for every "scenario", i. e. for every combination of S and U.

Let us apply the singular value decomposition (SVD) (see Theorem 3) to matrix $\mathbb{L}$ (3.3):

$$\mathbb{L} \overset{\text{SVD}}{=} \mathbb{V}\cdot\mathbb{D}\cdot\mathbb{U}^{*} = \underbrace{\mathbb{V}\cdot\mathbb{D}}_{\mathbb{S}}\cdot\mathbb{U}^{*} = \mathbb{S}\cdot\mathbb{U}^{*}. \tag{3.4}$$

In (3.4) matrix $\mathbb{S} := \mathbb{V}\cdot\mathbb{D}$ depends only on strategy index S while matrix $\mathbb{U}$ depends only on uncertainty index U. It can be seen when $\mathbb{L}_{S,N}$ is rewritten in the following way:

$$\mathbb{L}_{S,U} = \sum_{j}\mathbb{S}_{S,j}\mathbb{U}^{*}_{j,U}, \tag{3.5}$$

where j is a summation index. SVD serves to separate the dependencies of $\mathbb{L}$ on S and U and this guarantees that important Assumption3 is structurally met.

The S-th row of matrix $\mathbb{L}$ is a strategy-dependent linear combination of rows of $\mathbb{U}^{*}$ that describes uncertainty. Clearly uncertainties enter (3.4) in a linear way and have Hilbert's space structure. Since $\mathbb{U}$ is a unitary matrix, columns of $\mathbb{U}$ are orthonormal vectors that form a basis of Hilbert space on which $\mathbb{S}$ operates and "projects" them on losses (S-dependent $\mathbb{D}$ makes this projection). Multiplication of uncertainties (rows of $\mathbb{U}^{*}$) by zero elements of $\mathbb{S}$[2] remove their influence on losses $\mathbb{L}$, (3.5).

We insert the identity matrix, $\mathbb{I} = \operatorname{diag}\left(1,\ldots,1\right)$, expressed via any unitary matrix $\mathbb{P}$ into (3.4):

$$\mathbb{L} = \mathbb{S}\cdot\mathbb{U}^{*} = \mathbb{S}\cdot\underbrace{\mathbb{P}\cdot\mathbb{P}^{*}}_{=\mathbb{I}}\cdot\mathbb{U}^{*}. \tag{3.6}$$

This does not change matrix $\mathbb{L}$ and keeps $\mathbb{P}^{*}\cdot\mathbb{U}^{*}$ unitary. Since matrix $\mathbb{P}$ is unitary, it represents a rotation. This fact has a very attractive interpretation. This allows us to conclude that $\mathbb{P}^{*}\cdot\mathbb{U}^{*}$ is a rotation of the Hilbert space that describes uncertainty. It maps a line to a line,

[2] corresponding to zero rows of $\mathbb{D}$.

a plane to a plane, ..., a subspace of a given dimension to a subspace of the same dimension. Similarly, $\mathbb{S} \cdot \mathbb{P}$ is a rotation of a "projection" influenced by the agent's strategies. Neither of the rotations affects the resulting matrix, $\mathbb{L}$.

The effect of the uncertainties represented by those rows of $\mathbb{U}^*$ that are multiplied by non-zero elements of $\mathbb{D}$ is indistinguishable from the point of view of the values of $\mathbb{L}$.

Thus subspaces spanned on those uncertainties matter. Probabilities (arising in quantitative ordering of strategies, see Section 3.2) have to be assigned to them. Random event is a subspace of Hilbert's space. This is key difference from Kolmogorov's modelling of randomness.

3.2 Quantification of ordering on set of strategies

In context of DM task we are solving, we are given: set of all possible behaviours $\mathbf{b}$ and ordering $\preceq_{\mathbf{b}}$ which determines DM aim that needs to be reached (which behaviour is more preferable). Optimization on set $\mathbf{b}$ is not possible, because behaviour is influenced both by choice of strategy (made by agent) and choice of uncertainty (agent has no influence on it).

Aim of DM is to choose a strategy $S \in \mathbf{S}$ such that respects given preferences on behaviours (they were given via ordering $\preceq_{\mathbf{b}}$). To make the choice we need to be able to decide which $S \in \mathbf{S}$ is "better"/more preferred. Mathematically speaking we need to build $\preceq_{\mathbf{S}}$ that will respect $\preceq_{\mathbf{b}}$. Note that we have not introduced any kind of probability yet. Its necessity arises here.

Our next step is *to build an ordering on set of available strategies:* $\preceq_{\mathbf{S}}$ (respecting ordering on set of behaviours $\preceq_{\mathbf{b}}$). It can be done using an extended version of Riesz representation of a functional. There appears a measure on random events, i.e. probability on subspaces – which is exactly formalism used in quantum mechanics. Here it applies also to macro-world of DM.

First let us inspect a set of functions whose only argument is uncertainty U. For the fixed strategy, S, the considered function maps uncertainty on values of the loss, i.e.

$$l_S(U) := \mathbb{L}_{S,U}. \tag{3.7}$$

The set of such functions looks like:

$$\Lambda := \{l_S : \mathbf{U} \to \mathbf{R} \mid \exists S \in \mathbf{S} : \; l_S(U) = \mathbb{L}_{S,U}\}. \tag{3.8}$$

The functions in (3.8) are assumed to be bounded. It is unrestrictive as they quantify ordering $\preceq_{\mathbf{B}}$.

Definition 16 (Correspondence between preferences on strategy and preferences on loss function)**.** Let $S_1, S_2 \in \mathbf{S}$. We define *ordering* $\preceq_{\mathbf{\Lambda}}$ on $\mathbf{\Lambda}$

$$S_1 \preceq_{\mathbf{S}} S_2 \overset{!}{\iff} l_{S_1} \preceq_{\mathbf{\Lambda}} l_{S_2}. \tag{3.9}$$

Strategy S_1 is more preferable than S_2 if S_1 leads to a "better" loss function l_{S_1} that l_{S_2}. Definition 16 implies that once we quantify ordering on $\mathbf{\Lambda}$, we can use (3.9) to quantify ordering on $\mathbf{S}$. We shall use Rao's theorem [27], which generally represents a functional $\Upsilon : \Lambda \to \mathbf{R}$, under several widely acceptable conditions:

1. Υ should be *somewhat continuous*[3]: small changes of S and U should cause only small changes of value $\Upsilon\left(l_S(U)\right)$,

[3]For exact definition see [27], Theorem 5, Chapter 9.

2. the specific linearity should hold $(\forall S_1, S_2 \in \mathbf{S})\,(\forall U \in \mathbf{U})$:

$$l_{S_1}(U) \cdot l_{S_2}(U) = 0 \implies \Upsilon\left(l_{S_1}(U) + l_{S_2}(U)\right) \stackrel{!}{=} \Upsilon\left(l_{S_1}(U)\right) + \Upsilon\left(l_{S_2}(U)\right).$$

According to [27] (Chapter 9, Theorem 5) such a functional is represented in the following way:

$$\Upsilon\left(l_S\right) := \int_{U \in \mathbf{U}} \mathcal{K}(l_S(U), U)\, \mathrm{d}p(U), \tag{3.10}$$

where $\mathcal{K}$ is a kernel for which the integral is well defined, $\mathcal{K}(0, U) = 0$ and p is Kolmogorov's probabilistic measure.

The whole quantification scheme is thus as follows:

$$S_1 \preceq_{\mathbf{S}} S_2 \overset{\text{Definition 16}}{\iff} l_{S_1} \preceq_{\mathbf{\Lambda}} l_{S_2} \overset{\text{quantification via }\Upsilon}{\iff} \Upsilon\left(l_{S_1}\right) \leq \Upsilon\left(l_{S_2}\right) \tag{3.11}$$

where the last inequality compares two real numbers. Let us simplify 3.10 for the discrete countable set $\mathbf{U}$ under consideration. For simplicity we assume it to be finite

$$\mathbf{U} = \{U_1, U_2, \ldots, U_n\} \equiv \{U_j\}_{j \in \mathbf{j}}, \tag{3.12}$$

where $j \in \mathbf{j} \equiv \{1, \ldots, n\}$, $n = \mathrm{card}(\mathbf{U})$. Now (3.10) can be rewritten as:

$$\Upsilon\left(l_S\right) := \sum_{j \in \mathbf{j}} \mathcal{K}(l_S(U_j), U_j) p(U_j). \tag{3.13}$$

The used theorem ([27], Chapter 9, Theorem 5) models uncertainty by assigning probability to elementary random events U_j and then to a σ-algebra on $\mathbf{U}$. Thus it holds $(\forall j \in \mathbf{j})\, p(U_j) \geq 0$ and $\sum_{j \in \mathbf{j}} p(U_j) = 1$. We can however regard the atomic random events (U_j) as elements in Hilbert space. For each atomic[4] random event U_j we can define a unique unit vector $|\eta_j\rangle$ in $\mathcal{H}$ (which implies $\dim(\mathcal{H}) = \mathrm{card}(\mathbf{U})$):

$$(\forall U_j \in \mathbf{U}) \left(\exists^1 |\eta_j\rangle \in \mathcal{H}\right) : \ |\eta_j\rangle_k := \delta_{j,k}, \ j, k \in \mathbf{j} \tag{3.14}$$

where $\delta_{j,k}$ is Kronecker's delta symbol[5]. In other words we define $|\eta_j\rangle$ as a vector of length equal to $\mathrm{card}(\mathbf{U})$ which has 1 on j-th position and zeros everywhere else. A set of such $|\eta_j\rangle$ forms a basis of $\mathcal{H}$.

Now we are ready to use Theorem 5, Section 1.3 and express any measure μ on $\mathcal{H}$

$$(\forall j \in \mathbf{j}) : \ \mu(\eta_j) = \langle\eta_j| \hat{\mathsf{T}}_\mu |\eta_j\rangle . \tag{3.15}$$

How does operator $\hat{\mathsf{T}}_\mu$ look like to get a quantum equivalent of (3.13)? Because of our choice of vectors $|\eta_j\rangle$, only the diagonal elements in the matrix representing $\hat{\mathsf{T}}_\mu$ will influence $\mu(\eta_j)$, so it is natural to choose the matrix representing $\hat{\mathsf{T}}_\mu$ in the following form:

$$\hat{\mathsf{T}}_\mu := \mathrm{diag}\left(p(U_1), p(U_2), \ldots, p(U_n)\right) \overset{\text{Remark 8}}{=} \mathbb{T}_\mu. \tag{3.16}$$

[4]Event E is called an *atom* if $\mu(E) > 0$ and $E_1 \subset E$ implies $\mu\left(E_1\right) = 0$.

[5]$\delta_{j,k} = 1$ when $j = k$ and $\delta_{j,k} = 0$ otherwise.

Then

$$
\begin{aligned}
\mu(\eta_j) &= \langle \eta_j | \, \hat{\Upsilon}_\mu \, | \eta_j \rangle \\
&= (0, 0, \ldots, 1, \ldots, 0) \begin{pmatrix} p(U_1) & 0 & \ldots & 0 \\ 0 & p(U_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & p(U_n) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \\
&= p(U_j), \qquad \forall j \in \mathbf{j}.
\end{aligned}
\tag{3.17}
$$

We have shown that starting from a very general formulation of the DM problem and using Hilbert space to represent the uncertainties in a closed loop, our solution for a *static case* (one-shot DM problem) is consistent with the classical probability theory (the Kolmogorov probability). Functional (3.13) can be written using Kolmogorov's probability or via quantum probability:

$$
\Upsilon\,(l_S) := \sum_{j \in \mathbf{j}} \mathcal{K}(l_S(U_j), U_j)\, p(U_j) \overset{(3.17)}{=} \sum_{j \in \mathbf{j}} \mathcal{K}(l_S(U_j), U_j)\, \mu(\eta_j). \tag{3.18}
$$

The quantified $\preceq_{\mathbf{S}}$, (3.11), implies that optimal strategy $S \in \mathbf{S}$ can be found by minimising $\Upsilon\,(l_S)$ over $\mathbf{S}$:

$$
\begin{aligned}
S^{(\mathrm{opt})} &= \arg\min_{S \in \mathbf{S}} \Upsilon\,(l_S) \\
&= \arg\min_{S \in \mathbf{S}} \sum_{j \in \mathbf{j}} \mathcal{K}(l_S(U_j), U_j)\, p(U_j).
\end{aligned}
\tag{3.19}
$$

Remark 12. The above equations are written under the assumption $\mathrm{card}(\mathbf{U}) = n < +\infty$, but they can be rewritten for the case where $\mathrm{card}(\mathbf{U}) = +\infty$ without any loss.

Note that back substitution

$$
U = \Phi^{-1}\,(S, b) \tag{3.20}
$$

gives the functional

$$
\Upsilon\,(l_S) = \sum_{b \in \mathbf{b}} \mathcal{K}(l_S(\Phi^{-1}\,(S, b)), \Phi^{-1}\,(S, b))\, p(\Phi^{-1}\,(S, b)). \tag{3.21}
$$

For convenience let us denote:

$$
p(\Phi^{-1}\,(S, b)) =: \mu_S\,(b) \;\text{ and }\; \mathcal{K}_{\mathcal{S}}(z(b), b) = \mathcal{K}(\underbrace{l_S(\Phi^{-1}\,(S, b))}_{z(b),\ \text{see (3.7)}}, \Phi^{-1})\, \mu_S\,(b) \tag{3.22}
$$

then (3.21) reads

$$
\Upsilon\,(l_S) = \sum_{b \in \mathbf{b}} \mathcal{K}_S\,(z(b), b)\, p_S\,(b) = \sum_{b \in \mathbf{b}} \mathcal{K}_S\,(z(b), b)\, \mu_S\,(b)\,. \tag{3.23}
$$

There Kolmogorov's probability is related to quantum probability via a diagonal S-dependent density operator $\hat{\Upsilon}_S$.

3.3 Dynamic case

In dynamic case the agent chooses a sequence of actions: $a_1, a_2, \ldots, a_t, a_{t+1}, \ldots$. In order to see what may happen in infinitesimal time increase we distinguish discrete time of acting (t) and continuous time of evolution of uncertainties (τ).[6]

[6]Later on we shall call them *acting time step* and *evolutional time step* respectively.

Uncertainty U evolves independently on strategy (due to Assumption 3) and we accept the following time formalism:

Assumption 4 (Two time scales formalism)**.** After the agent chooses action a_t, uncertainty present in the closed loop evolves in (continuous) time $\tau \in [t, t+1)$, see Figure 3.1.

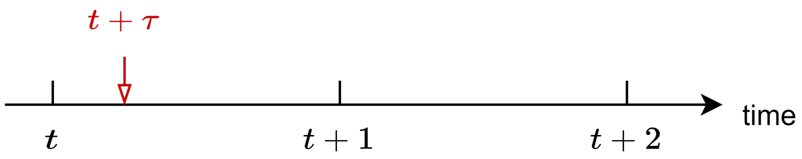

Figure 3.1: Time scales

Note that no actions could been chosen between those two times (and the same holds for observations). With the adopted scaling $\tau \leq 1$ always holds and when $\tau = 1$, then $t + \tau = t + 1$ and τ is set to zero: $\tau = 0$. In dynamic case the matrices in (3.6) become time dependent.

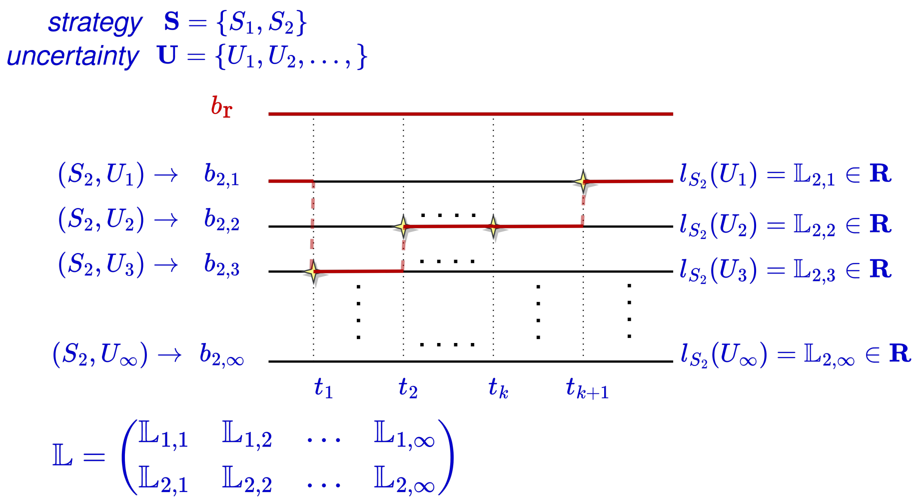

Figure 3.2: Let's imagine a simplified example. Each pair (S_i, U_j) leads to behaviour of closed loop $b_{i,j}$, which is quantified by value $l_{S_i}(U_j)$. Let's say that we choose always only strategy S_2. Then black lines represent time evolution for cases when for any time t U_1 holds, U_2 holds, etc. But uncertainty somehow changes between discrete time steps - so actual (real) evolution switches between those possible evolutions. If red line represents real behaviour b_r of closed loop, we can imagine that when another uncertainty actualises, it switches the evolution to another one (from one "black line" to another).

Between times t and $t + \tau$ uncertainty generally changes from:

$$\mathbb{L}_t = \mathbb{S}_t \cdot \mathbb{U}_t^*, \tag{3.24}$$

to

$$\mathbb{L}_{t+\tau} = \mathbb{S}_{t+\tau} \cdot \mathbb{U}_{t+\tau}^* \tag{3.25}$$

and the agent has no influence on $\mathbb{U}$. Note that the matrix of losses on the left side of (3.24) and of (3.25) is the same. Time index just emphasises different decompositions. We express this change by introducing matrix $\mathbb{M}$, which may depend on both time t and time τ:

$$\mathbb{U}_{t+\tau} = \mathbb{M}_{t,\tau} \cdot \mathbb{U}_t. \tag{3.26}$$

Matrices $\mathbb{U}_t$ and $\mathbb{U}_{t+\tau}$ are unitary, so matrix $\mathbb{M}_{t,\tau}$ has to be unitary too. Now we shall express $\mathbb{M}_{t,\tau}$ as:

$$\mathbb{M}_{t,\tau} = \exp\left(i\tau\mathbb{H}_t\right), \tag{3.27}$$

where i is imaginary unit and $\mathbb{H}_t$ is some positive semi-definite symmetric matrix that generally can change with time t. It can be verified that any unitary matrix can be expressed in this way.

By inserting (3.27) into (3.26), we get:

$$\mathbb{U}_{t+\tau} = \exp\left(i\tau\mathbb{H}_t\right) \cdot \mathbb{U}_t. \tag{3.28}$$

This looks very similar to the ansatz from which the Schrödinger equation is derived. This means we are able to derive it too:

$$\mathbb{U}_{t+\tau} = \exp\left(i\tau\mathbb{H}_t\right) \cdot \mathbb{U}_t \quad \left| \frac{\partial}{\partial\tau} \right. \tag{3.29}$$

$$\frac{\partial}{\partial\tau}\mathbb{U}_{t+\tau} = i\mathbb{H}_t \underbrace{\exp\left(i\tau\mathbb{H}_t\right) \cdot \mathbb{U}_t}_{\overset{(3.28)}{=}\mathbb{U}_{t+\tau}} + \exp\left(i\tau\mathbb{H}_t\right) \cdot \underbrace{\frac{\partial}{\partial\tau}\left(\mathbb{U}_t\right)}_{=0} \tag{3.30}$$

and we recognize the *Schrödinger equation*:

$$\frac{\partial}{\partial\tau}\mathbb{U}_{t+\tau} = i\mathbb{H}_t \cdot \mathbb{U}_{t+\tau}, \tag{3.31}$$

where $\mathbb{H}_t$ can be interpreted as energy influencing the uncertainty in the closed loop at time t.

Technical steps above lead us to the important conclusion: between two actions (say, a_t and a_{t+1}) uncertainty in the closed loop evolves in time τ via the Schrödinger equation.

Brief summary of what has been done:

- Ordering of behaviours $\preceq_{\mathbf{b}}$ was quantified via loss function $z : \mathbf{b} \to \mathbf{R}$, and as a consequence only countable sets of behaviours and uncertainties are relevant for DM.
- Uncertainty U independent of the strategy, S, was introduced.
- Proposed SVD of matrix of losses $\mathbb{L}(S, U) = \mathbb{L}_{S,U}$ ensures that uncertainty is stored in matrix $\mathbb{U}^*$ which is not influenced by chosen strategy S. Hilbert's structure of uncertainties determine subspaces as random events.
- We have built an ordering on set of available strategies: $\preceq_{\mathbf{S}}$ (respecting ordering on set of behaviours $\preceq_{\mathbf{b}}$). It has been done using an extended version of Riesz representation of functional Υ, (3.21). There appears a measure on random events.
- In the *static case* they can have a standard Kolmogorov's structure, which can be embedded into a set of events represented by subspaces of Hilbert space: into formalism of quantum mechanics.

- In the *dynamic case* the diagonal embedding is lost and quantum description is inevitable. This follow from the following statement. Even if the probabilistic measure on uncertainty is given by the diagonal density operator, $\hat{\mathsf{T}}_{\mu,t} = \mathbb{T}_{\mu,t}$, (3.16), the inevitable rotation of uncertainties, (3.26), rotates $\mathbb{T}_{\mu,t}$ to:

$$\mathbb{T}_{\mu,t+\tau} = \mathbb{M}_{t,\tau}\, \mathbb{T}_{\mu,t}\, \mathbb{M}^{*}_{t,\tau} \tag{3.32}$$

  and the original embedding of quantum description to Kolmogorov's description is lost. The *non-diagonal* density operator $\hat{\mathsf{T}}_{\mu,t+\tau} = \mathbb{T}_{\mu,t+\tau}$ has to be used in the considered macro-world DM.

3.4 Quantum DM

The density operator $\hat{\mathsf{T}}_{\mu}$, (3.32), is generally represented by a matrix with an infinite number of rows and columns. For practical use, it must be generated in a way similar to partially observable Markov decision processes (POMDP). This means that the Hilbert space on b is constructed as the tensor product of Hilbert spaces on behaviour components, $\{s_t, a_t\}_{t\in\mathbf{t}}$. Then the quantum analogy of POMDP arises. The solution for a specific state evolution can be found in [3]. We conjecture that extension to FPD version is possible in a way similar to the classical extension [19], which strongly relies on the used non-standard Riesz representation of functional.

Conclusion

Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning.

– Winston Churchill

The work proposes novel formalisation and solution of DM task. The formalisation builds upon agent-system interaction and uses the quantum description of uncertainty. Many previous works have confirmed the usefulness of quantum models in decision-making. However, to the author's best knowledge, none of the earlier works has answered the question: "Why are the quantum models so successful in describing human-like DM and existing paradoxes?"

Main contribution of the proposed work is that it answers this question and step-by-step shows that under realistic assumptions, quantum models become inevitable for describing DM under uncertainty. Indeed, uncertainty significantly influences the system and agent. Since the nature of uncertainty is generally unknown, deterministic modelling can hardly help. The starting point did not rely on the concept of probability, so the necessity of probabilistic description rose after a detailed analysis of the role of uncertainty and its modelling. The proposed solution has the following features:

1. *static* DM: The solution coincides with solution obtained by the classical decision theory.
2. *dynamic* DM: It is shown that uncertainty inherent to DM task evolves according to Schrödinger equation and requires quantum modelling - both in the micro- and macro-world. Notably, this conclusion is fully consistent with the results of Busemeyer and Pothos, [7].
3. *Mathematics*: being a general mathematical model the obtained solution emphasises that quantum modelling can be of help in any application where uncertainty is present (for instance economy, biology, social sciences, physics, ...).

Surprisingly, the formalisation developed shares some common features with two very different domains. When compared with Savage's subjective utility theory, key differences are:

DM theory: *Savage's approach* operates with three orderings: order in states, order in payoffs, and order in decision rules. We operate with order in behaviours and order in strategies. Our operating with strategies is more general than with decision rules (strategy is a sequence of decision rules in time). Moreover we do not rely on pre-existing probabilities.

Physics: There is a deep relation to physics. In the dynamic case, loss functions for every realisation of uncertainty are defined, but at each time step, another uncertainty can be realised. It means reality is “switching” in discrete times between different trajectories (recall Figure 3.2). In other words, several possible trajectories exist, and a concrete value of loss function is assigned to each of them, but only one trajectory is realised. It can be interpreted as parallel realities that exist, and the agent is aware of them but does not know in which one he will find himself. This interpretation is close to the formulation of *many-worlds interpretation* introduced by Hugh Everett (see Appendix C).

This work is just a beginning and calls for a further research. Two immediate non-trivial research tasks are:

- Detailed and thorough elaboration of quantum version of the partially observable Markov decision process (q-POMDP);
- Elaboration of quantum version of fully probabilistic design, which is enabled by the outlined quantum version of functional Υ, see Section 3.2.

Appendix A

Approach to decision making through subjective probability

Disclaimer. Each Appendix has own notations.

Theory of subjective (personal) probability was developed by Leonard J. Savage in 1954 [28]. His idea was to use a numerical representation for a preference ordering of decision functions.

The approach is very general and abstract. Hence, for the purpose of this work it will be simplified. Let system be a part of the world. The system is in definite state at each time moment.[1]. In the Table A.1 are mentioned main objects to work with.

Symbol	*Meaning*
$\mathbf{S}$	set of states $s \in \mathbf{S}$ of the world
$\prec$	(comparative probability) ordering between subsets of $\mathbf{S}$
$\mathbf{C}$	set of abstract consequences $c \in \mathbf{C}$ or payoffs
$\prec_{\mathbf{C}}$	ordering between elements of $\mathbf{C}$
$\mathbf{D} = \{d(\bullet) : \mathbf{S} \to \mathbf{C}\}$	set of functions (mappings) from $\mathbf{S}$ to $\mathbf{C}$ (decision rules)
$\prec_{\mathbf{D}}$	ordering between elements of $\mathbf{D}$

Table A.1: Notions of the subjective probability theory

In the considered formulation $\mathbf{S}$ is a set of states of the system and $\mathbf{D}$ is a set of decision rules. Payoff $c \in \mathbf{C}$ is a consequence of a decision rule $d(s)$ when $s \in \mathbf{S}$ is the *true* state of the world. The decision problem is to find an optimal decision rule.

We define $\prec_{\mathbf{C}}$ and $\prec$ in terms of $\prec_{\mathbf{D}}$ as follows. For any given $c \in \mathbf{C}$ we introduce $d_c(\bullet) \in \mathbf{D}$ as

$$\forall s \in \mathbf{S} : d_c(s) = c. \tag{A.1}$$

For each $c_1, c_2 \in \mathbf{C}$ we define

$$c_1 \prec_{\mathbf{C}} c_2 \Leftrightarrow d_{c_1}(\bullet) \prec_{\mathbf{D}} d_{c_2}(\bullet).$$

Remark 13. Using bullet ($\bullet$) as argument of a function just emphasizes that the object is a function by itself, not a value of the function in a certain point.

[1]There exist many different approaches how to start explaining basic terms of DM. In [28] L. Savage expects intuitive understanding of what decision and decision maker is.

Subsets $A, B, \ldots$ of $\mathbf{S}$ represent *events*. Let's denote

$$d^A_{c_1,c_2}(s) = \begin{cases} c_1 & \text{if } s \in A \\ c_2 & \text{if } s \in \mathbf{S}\backslash A. \end{cases} \tag{A.2}$$

Now we are able to define comparative probability ordering between events $A, B \subset \mathbf{S}$:

$$A \prec B \Leftrightarrow (\forall c_1, c_2 \in \mathbf{C},\ c_1 \prec_{\mathbf{C}} c_2) : d^A_{c_1,c_2}(\bullet) \prec_{\mathbf{D}} d^B_{c_1,c_2}(\bullet) \tag{A.3}$$

Those above definitions of ordering operations ($\prec$, $\prec_{\mathbf{D}}$) are reasonable because better decision can be obtained either

- by improving the payoff[2] for a fixed events, or
- by increasing chances to get a better payoff.

At first, let us illustrate the terms represented by elements of those sets.

Example. Let's imagine we are playing a simplified dice game with one die. We bet one dollar on one of the numbers between 1 and 6. After that, the die is rolled, and if the number we bet on is rolled, we win the dollar. Otherwise we lose.

In this example, our world is represented by the dice. Possible states of the world are

$$\mathbf{S} = \{\ \text{“The die rolls 1.”, “The die rolls 2.”, ... , “The die rolls 6.”}\ \}.$$

Set of our payoffs (consequences) are

$$\mathbf{C} = \{\ \text{“We win 1 dollar.”, “We lose 1 dollar.”}\}.$$

Elements of $\mathbf{D}$ are functions that maps any given state of the thrown die to our payoff (in terms of dollars)[3]. Our decision (which number we want to bet on) is implicitly a part of these functions. For instance, if we bet on number 6, and die rolls number 3:

$$d(\text{“The die rolls a 3.”}) = \text{“We lose 1 dollar.”}.$$

Now we have three sets, each set with its own ordering operation.

L. J. Savage formulated seven axioms for personalistic rational DM. Before we recall the axioms, one more definition needs to be specified:

Definition 17 (Comparison of "conditional" decision rules). For event $A \subset \mathbf{S}$ and for each decision rules $f, g \in \mathbf{D}$ we define $f(\bullet) \prec_{\mathbf{D}} g(\bullet)$ *given* A as follows:

$$f(\bullet) \prec_{\mathbf{D}} g(\bullet) \text{ given } A \Leftrightarrow \begin{pmatrix} (\forall f' \in \mathbf{D})\,(\forall s \in A):\ f'(s) = f(s) \\ (\forall g' \in \mathbf{D})\,(\forall s \in A):\ g'(s) = g(s) \\ (\forall f' \in \mathbf{D})\,(\forall s \in \mathbf{S}\backslash A):\ f'(s) = g'(s) \end{pmatrix} (f'(\bullet) \prec_{\mathbf{D}} g'(\bullet)) \tag{A.4}$$

The comparing operator defined above has meaning similar to conditional probability (in Kolmogorov sense). It compares two decision rules but only for states in a given event A.

Now we are able to state the axioms ([15]):

[2]By payoff we understand any abstract consequences related to the state/event/decision.

[3]Here elements of $\mathbf{D}$ are called *decision rules* or simply *decisions*. In [28] those functions are called *acts*.

Axiom 1. $(\exists c_1, c_2 \in \mathbf{C}) : \ c_1 \prec_{\mathbf{C}} c_2$
(assuring non-triviality of the DM task)

Axiom 2. $\prec_{\mathbf{D}}$ is complete order (i.e. transitive, reflexive, binary relation)
(any pair of decision functions is comparable)

Axiom 3. $(\forall d_1(\bullet), d_2(\bullet) \in \mathbf{D})\,(\forall A \subset \mathbf{S})$ holds either $d_1(\bullet) \prec_{\mathbf{D}} d_2(\bullet)$ given A or $d_2(\bullet) \prec_{\mathbf{D}} d_1(\bullet)$ given A
(all "conditional" decisions are comparable)

Axiom 4. If $A \neq \emptyset$, then equivalency holds: $(\ (\forall s \in A)\ (d_{c_1}(s) \prec_{\mathbf{D}} d_{c_2}(s))\ ) \Leftrightarrow (c_1 \prec_{\mathbf{C}} c_2)$
*(so-called **sure-thing principle**: if person would not prefer decision rule $d_1(\bullet)$ to $d_2(\bullet)$, either event $A \subset \mathbf{S}$ obtained or not, then he does not prefer $d_1(\bullet)$ to $d_2(\bullet)$ at all)*

Axiom 5. $(\forall A, B \subset \mathbf{S}) : \ A \prec B$ or $B \prec A$

Axiom 6. Let $\mathbf{P}$ be a finite disjoint cover of set $\mathbf{S}$:

$$\mathbf{P} = \{\mathbf{S_j}\}, \quad \cup_j \mathbf{S_j} = \mathbf{P}.$$

For $(\forall f, g \in \mathbf{D},\ f(\bullet) \prec_{\mathbf{D}} g(\bullet))$ $(\forall c \in \mathbf{C})$ $(\exists \mathbf{P})$ $(\forall \mathbf{S_j} \in \mathbf{P})$ let's define $f_j, g_j \in \mathbf{D}$ as follows:

$$f_j(s) = \begin{cases} f(s) & \text{if } s \notin \mathbf{S_j} \\ c & \text{if } s \in \mathbf{S_j} \end{cases}$$

$$g_j(s) = \begin{cases} g(s) & \text{if } s \notin \mathbf{S_j} \\ c & \text{if } s \in \mathbf{S_j} \end{cases}$$

Then $f_j(\bullet) \prec_{\mathbf{D}} g(\bullet)$ and $g_j(\bullet) \prec_{\mathbf{D}} f(\bullet)$.

Axiom 7.

$$(\forall A \subset \mathbf{S})\,(\forall s \in A) : f(\bullet) \prec_{\mathbf{D}} d_{g(s)}(\bullet) \text{ given } A \Rightarrow f(\bullet) \prec_{\mathbf{D}} g(\bullet) \text{ given } A \tag{A.5}$$

and

$$(\forall A \subset \mathbf{S})\,(\forall s \in A) : d_{g(s)}(\bullet) \prec_{\mathbf{D}} f(\bullet) \text{ given } A \Rightarrow g(\bullet) \prec_{\mathbf{D}} f(\bullet) \text{ given } A \tag{A.6}$$

(This axiom is needed only in situations where set of consequences $\mathbf{C}$ is not finite.)

Accepting those axioms, we implicitly believe that states $s \in \mathbf{S}$ and consequences $c \in \mathbf{C}$ are unrelated entities, i.e. choice of consequences does not influence states or probability, with which those states occur. Also choices of states and events does not influence the desirability of consequences (i.e. our preferences do not change no matter in which state the world is).

Theorem 7 (Savage's subjective probability)*.* Under Axioms 1-7 (page ii) there exists a uniquely defined, finitely additive set function[4] $P(\bullet) : \mathbf{S} \to [0,1]$ agreeing with ordering operation $\prec$ on $\mathbf{S}$ and having following property:

$$(\forall A \subseteq \mathbf{S})\,(\forall \rho \in [0,1])\,(\exists B \subseteq A) : \ \ P(B) = \rho P(A) \tag{A.7}$$

Furthermore there exists a bounded (utility) function $u(\bullet) : \mathbf{C} \to \mathbf{R}$ unique up to linear transformation, such that

$$f(\bullet) \prec_{\mathbf{D}} g(\bullet) \Leftrightarrow \mathrm{E}\left[u(f(s))\right] \leq \mathrm{E}\left[u(g(s))\right], \tag{A.8}$$

where $\mathrm{E}[\bullet]$ is expected value with respect to the function P (Eq. (A.7)).

Proof. Proof can be found in [28].

This theorem says nothing about the *interpretation* of the probability P. As long as $\prec_{\mathbf{D}}$ is subjectively defined operation (it depends on us, which decision rule should be considered as more preferable), P is also subjective in origin (Savage's emphasis on *personalistic* DM). The purpose of P in this approach was not to make empirically correct statements about occurrence of the events, but the purpose was to represent $\prec_{\mathbf{D}}$ in "probability manner", as stated in Theorem 7.

Appendix B

Quantum decision theory (Yukalov & Sornette)

Application of the quantum theory to psychological and cognitive phenomena operating on composite events and non-commuting observables produced a variety of quantum models [7] and theories, see survey [2]. A special attention deserves quantum decision theory (QDT) [36] based on generalisation of the quantum theory of measurement [34]. The key assumption behind that there is a strong correspondence between measurements and decisions and composite measurements are equivalent to composite decisions. QDT is formulated as a self-consistent mathematical theory and allows to explain all reported paradoxes. The detailed description of the theory can be found in [36]. It is quite advanced but definitely not finished and commonly accepted. This makes us to give up unification in notation and to use the notations of [36]. They are related to the previous presentations and to numerous attempts to get QDT. Below we summarise the main relevant features/aspects of QDT.

- QDT operates on events that can be: an event in decision theory or probability theory, or the result of a measurement in the quantum theory of measurements.
- Decision-maker is an open system described by a statistical operator not by wave function.
- Memory of decision maker is nothing but delayed interactions.
- Observable quantities are represented by self-adjoint operators, A. Measuring an eigenvalue A_n of the operator is interpreted as the occurrence of event A_n. The corresponding eigenvector $|n\rangle$ [1] is an event (decision) mode.
- Operator $\hat{P}_n \equiv |n\rangle\langle n|$ [2] is an event operator. The collection $\{\hat{P}_n\}$ is a projector-valued measure.
- The space of decision modes is given by the Hilbert space $\mathcal{H}_A = \mathsf{span}\{|n\rangle\}$.
- Decision-maker state is characterised by a statistical operator $\hat{\rho}$, thus pair $\{\mathcal{H}, \hat{\rho}\}$ is a decision ensemble.

[1] microstate in physical interpretation
[2] measurement projector in physical interpretation

- Probability of an event A_n[3], is given by $p(A_n) = \mathsf{Tr}_A\hat{\rho}\hat{P}_n \equiv \langle\hat{P}_n\rangle$ with Tr_A denotes trace of operator A over space $\mathcal{H}_A$.

- QDT interprets the measurement of two observables (occurrence of two events) as a composite event called *prospect* and defines joint and conditional probability of events.

- Probability of uncertain composite event π_n in quantum form is described by $p(\pi_n) = \mathsf{Tr}_{AB}\hat{\rho}_{AB}\hat{P}_n$ where A and B are two events with event operators $\hat{A}_n \equiv |A\rangle\langle A|$ and $\hat{P}_B \equiv |B\rangle\langle B|$ respectively. Note $\sum_n p(\pi_n) = 1$ and $0 \leq p(\pi_n) \leq 1$.

- Probability of prospect (composite event) can be written as a sum $p(\pi_n) = f(\pi_n) + q(\pi_n)$, where $f(\pi_n)$ describes the *utility factor* and $q(\pi_n)$ is an *attractor factor*[4]. Once the second term goes zero, quantum probability reduces to the classic one: $q(\pi_n) \to 0$ implies $p(\pi_n) \to f(\pi_n)$. This allows to interpret decisions under uncertainty in such way that π_1 is *more useful* than π_2 if $f(\pi_1) > f(\pi_2)$; π_1 is *more attractive* than π_2 if $q(\pi_1) > q(\pi_2)$; π_1 is *more preferable* than π_2 if $p(\pi_1) > p(\pi_2)$.

- Prospect $\pi^* = \mathsf{argmax}_j p(\pi_n)$ is called optimal. In QDT the concept of an optimal decision is replaced by a probabilistic decision, when the prospect π that makes $p(\pi_n)$ maximal, is the one which corresponds best to the given strategic state of mind of the decision maker.

- The dependence of probability of prospect on the additional information QDT expresses as $p(\pi_n, \mu) = f(\pi_n) + q(\pi_n, \mu)$, where μ information measure, [37]. Notably that the utility factor does not dependent on additional information while attractor (subjective by its nature) factor does. Work [38] shows that the attraction factor decreases with the received additional information, which explains effect of preference reversal.

[3] equivalent to the probability of measuring eigenvalue of A_n

[4] Later also *informational factor* was introduced:

$$p(\pi_n) = \underbrace{f(\pi_n)}_{\substack{\text{utility}\\ \text{factor}}} + \underbrace{q(\pi_n)}_{\substack{\text{attraction}\\ \text{factor}}} + \underbrace{h(\pi_n)}_{\substack{\text{informational}\\ \text{factor}}}$$

Appendix C

Everett's approach: one wave function to rule them all

"One Ring to rule them all, One Ring to find them, One Ring to bring them all and in the darkness bind them."

– J. R. R. Tolkien, *The Ring's inscription, translated*

Conventional quantum theory states that a physical system is completely described by a state function $\psi \in \mathcal{H}$. It specifies the probabilities of results of various observations which are made on the system by external observers[1]. There are two different ways in which state function can change:

Process 1. Discontinuous change of the state caused by observation of quantity with eigenstates $\phi_1, \phi_2, \ldots$. The state will change to the state ϕ_j with probability $|\langle\psi|\phi_j\rangle|^2$.

Process 2. Continuous (and deterministic) change of the state with time according to a wave equation

$$\frac{\partial\psi}{\partial t} = \hat{\mathsf{A}}\psi,$$

where $\hat{\mathsf{A}}$ is linear operator.

This formulation corresponds with experience. No experimental evidence is known to contradict it.

But not all situations/setups fit into that formulation. Let's consider a composite system, consisting of physical system and observer, who observes that physical system. Can the time change of the composite system be described via Process 2?

If yes, then it appears no discontinuous change of state (Process 1) takes place. If no, we need to admit that such composite systems cannot be described by the same quantum–mechanical description used for other physical systems.

[1]Such observer can be either human being, or some measuring apparatus, or even an algorithm. In this work we will refer to an observer as "he" in all generality.

C.1 Everett's interpretation of quantum mechanics

The many–worlds interpretation of quantum mechanics is close to the boundary between a "formulation" and "interpretation": its founder, Hugh Everett III., called it "the relative state formulation", while its successor Bryce DeWitt continued to develop it under the name "the many–worlds interpretation"[31].

In this theory, Everett criticises usual concept of collapse of the wave function. The question changes from "What happens to the world (system)?" to "What happens in particular story line with the system?" Let's recall extremely famous example with Schrödinger cat, which is so-called to be "dead and alive at the same time". Bohr interpretation poses the question "What happens?" and the answer is: when observer opens the box, the probability to observe a dead cat is equal to 0.5, and the probability to observe alive (and very angry) cat is also 0.5. In Everett's interpretation, that is not the right question to pose. There is one story line, where observer opens the box and sees a dead cat (for sure), and another story line, where observer sees alive cat after he opens the box. The probabilities for observer to find himself in one of the story lines is still 0.5. So Everett question is "What happens in particular story line?"[2] The wave function representing state never collapses, it keeps splitting.

C.1.1 Setup

Everett's approach [14] profit from wave mechanics. So his starting point is a postulate *that a wave function that obeys linear wave equation* (everywhere and at any time) *is a complete mathematical model for an isolated physical system.* Furthermore, every system that can be a subject for external observation can be considered as a part of (larger) isolated system. Simply speaking: any even non–isolated system, whose states can someone observe, can be a part of some bigger isolated system.

To put theory into correspondence with experience, it is necessary to formulate abstract models of observers (observer models), such that:

- Observer models should be treatable as physical systems.
- Observer models should consider following structure: there exists an isolated system, observer is a part of it. Observer is interacting with other subsystems in that system, see Figure C.1.
- It should be able to deduce the changes in observer as a consequence of these interactions.

In his work, Everett introduces the concept of *relativity of states*, which can be reformulated as follows: Let's have an isolated system (composite system). There are several other systems inside of it, we shall call them subsystems. A subsystem cannot be said to be in any well–defined state, independently from the rest of the system. For every chosen state of the subsystem there will exists exactly one *relative state* of the rest of the system. Usually this relative state depends on the state of chosen subsystem. In other words, all states of subsystems are correlated (not independent). Such correlation appears whenever subsystems interact. In this formulation, all measurements/observations are considered as interactions between subsystems.

So deductions about the observer state are done relatively to the object system.

[2]The philosophy behind "right" and "wrong" questions can be illustrated even simpler. Question "How far is Paris?" does not make sense, while question "How far is Paris from Prague?" does.

C.1.2 The concept of relative state

Now, let's investigate the consequences of applying wave mechanics formalism to the composite systems (a system, containing several subsystems). Let's assume we have a system S, associated with Hilbert space $\mathcal{H}$. S is composed of two subsystems: S_1 and S_2 (being associated with $\mathcal{H}_1$ and $\mathcal{H}_2$). Then, according to the formalism of composite systems $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ (tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$). Let $\{\xi_i^{S_1}\}$ be an orthonormal (ON) set of states for subsystem S_1, and $\{\eta_j^{S_2}\}$ be an ON set of states for subsystem S_2[3].

Then state ψ^S of the system Scan be found as a superposition of states of subsystems:

$$\psi^S = \sum_i \sum_j a_{ij}\, \xi_i^{S_1}\, \eta_j^{S_2} \tag{C.1}$$

Even if system S is in state ψ^S, we cannot say its subsystems are in definite states independently on one another (except the special case when all but one numbers a_{ij} are equal 0). However it is possible to assign for any state of one subsystem a corresponding *relative* state of another subsystem.

That brings us to the concept of *relative state.* Let's choose a state ξ_k as the state for S_1. Composite system is in the state ψ^S, given by Equation (C.1). Then the corresponding relative state will be:

$$\psi\left(S_2,\ \mathrm{rel}\,\xi_k, S_1\right) = N_k \sum_j a_{kj} \eta_j^{s_2} \tag{C.2}$$

Left side of Equation (C.2) reads "state of S_2 when S_1 is in state ξ_k".

Visually definition of relative state reminds conditional probability in CPT formulation, and there really is a connection: in conventional formulation, the relative state $\psi\left(S_2,\ \mathrm{rel}\,\phi, S_1\right)$ of S_2 for a state ϕ^{S_1} of system S_1 gives conditional probability distribution of the results of all measurements done on system S_2, given that S_1 has been measured and found to be in state ϕ^{S_1}. In other words it means that ϕ^{S_1} is an eigenfunction of the measurement that corresponds to the observed eigenvalue in S_1.

C.1.3 Observer as a subsystem

According to this approach, for now on, the observers will be treated as physical system. Aim of further steps will be to make deductions about the consequences of this assumption. Let's explore the present properties of such observer in the light of his past experience.

If we are saying that an observer O has observed event A, it means that the state of O has changed from it former state to a new state, which depends on event A. Such observer has his own memory (recording device). In order to make a conclusion about anything O observed in the past it is sufficient to analyse his current contents of the memory.

Here it should be mentioned, that Everett in his work assumes that observer could be considered as a machine, equipped with sensors (recording data) and recording device (storing data). Such a machine is able to perform a sequence of observations (measurements), and even will be capable to decide upon its future experiments on the basis of past results of observations. For such machine it is possible to use statement "machine is aware of A" if event A has been observed and the result of this observation is stored in machine's memory.

When describing observer O in terms of composite systems (see Figure C.1), we assign a state function ψ^O to it. When observer has observed events $A, B, \ldots, C$[4], we denote this via

[3] We leave out Dirac's bra-ket notation as Everett does.

[4] In order of occurrence, i.e. time dependent order.

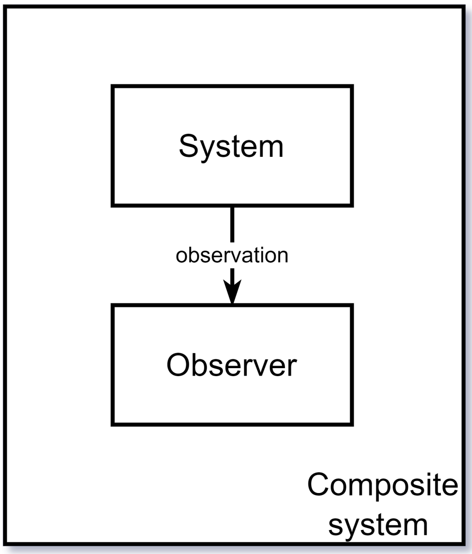

Figure C.1: Everett concept of an observer and a system as a part of composite system

lower index, representing contents of observer's memory:

$$\psi^O_{[A,B,\ldots,C]}.$$

Interaction between observer and system are treated within the framework of Process 2 (see page vii).

Let's define what a "good" observation is.

Definition 18 (Good observation). We have system S and observer O. Observer is in state ψ^O. A *good observation of quantity A* consists of an interaction, which transforms state of the composite system $\psi^{S+O} = \phi_j\, \psi^O_{[\ldots]}$ into a newer state $\psi^{S+O'} = \phi_j\, \psi^O_{[\ldots,\alpha_j]}$ in a specified period of time, where α_j are eigenvalues of quantity A and ϕ_j are eigenfunctions of of quantity A.

Because of an observation, state of the system S does not change, but the state of an observer does change according to new record in his memory (the change of observer's state describes that observer is "aware" of which eigenfunction corresponds to observed eigenstate). As a result, the state of composite system $(S + O)$ also changes.

In case that system S has not been observed in its eigenstate, we can write current general state ψ as a superposition of eigenstates. Then final total state will have the form:

$$\psi^{S+O'} = \sum_j a_j \phi_j\, \psi^O_{[\ldots,\alpha_j]},$$

where $\psi^S = \sum_j a_j \phi_j$. Coefficients a_j are given by $a_j = \langle \phi_j | \psi^S \rangle$.

Based on this steps, Everett defines two rules that describe how the total state of composite system changes after observer makes an observation (in other words, after observer interacts with subsystem).

C.2 Summary

Everett's interpretation tries to describe a composite system consisted of a physical subsystem (object system) and an observer observing the object subsystem. The proposed concept can be applied to composite systems containing several subsystems and several observers. The state of a subsystem will be observed in the same way by every observer. However each observer will be in a different state after observing the same subsystem. One can interpret this as an existence of “parallel universes”: each observer has his own new state.

Important consequences of this interpretation are, [6]

> “... a state of a composite system leads to a *joint* distributions over subsystem quantities which are generally not independent. Conditional distributions and expectations for subsystems are obtained from *relative states*, and subsystem marginal distributions and expectations are given by *density matrices*. There does not, in general, exist anything like a single state for one subsystem of a composite system. That is, subsystems do not possess states independent of the states of the remainder of the system, so that the subsystem states re generally *correlated*. One can arbitrarily choose a state for one subsystem, and to be led to the *relative state* for the other subsystems. Thus we are faced with a fundamental *relativity of states*, which is implied by the formalism of composite systems. It is meaningless to ask the absolute value of a subsystem - one can only ask the state relative to a given state of the remained of the system.”

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