TTK4250 Sensor Fusion Assignment 2

Hand in: *Wednesday 9. September 16.00* on Blackboard.

Tasks are to be solved on paper if you are not told otherwise, and you are supposed to show how you got to a particular answer. It is, however, encouraged to use Python, MATLAB, Maple, etc. to verify your answers. Rottmann's mathematical formula collection is allowed at both the exercises and the exam.

Task 1: *Transformation of Gaussian random variables*

Let $x \in \mathbb{R}^n$ be $\mathcal{N}(\mu, \Sigma)$. Find the distribution and see if you recognize it:

Hint: they are all given in the book.

(a) $z = \Sigma^{-\frac{1}{2}}(x - \mu)$, where $\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^T = \Sigma$

Hint: If you are using theorem 2.4.1, you might need $\det(A^{\frac{1}{2}}) = \det(A)^{\frac{1}{2}}$, $(A^{-1})^T = (A^T)^{-1}$, and $\det(A^T) = \det(A)$ whenever A has full rank.

(b) Use transformation of random variables to find $y_i = z_i^2$, where z_i is the i'th variable in the vector z.

(c) $y = (x - \mu)^T \Sigma^{-1}(x - \mu) = z^T z = \sum z_i^2 = \sum y_i$.

Hint: The MGF of y_i is given in the book through example 2.8 and 2.10. Example 2.6 might also be handy.

Task 2: *Sensor fusion*

In this task we want to find out if a boat is above the line $y = x + 2$. In order to do this we will fuse measurements from two sensors with our prior belief: A drone-mounted camera, and a maritime surveillance radar. You have some prior knowledge of the state of the boat. You get 1 measurement from each sensor that are processed so that you know them to be (approximately) Gaussian [1] conditioned on the position.

To be more specific, let us denote the state by x and our prior Gaussian by $\mathcal{N}(x; \bar{x}, P)$. The measurement from the camera is given by $z^c = H^c x + v^c$ and the measurement from the radar by $z^r = H^r x + v^r$, where v^j, $j \in \{c, r\}$ denotes the measurement noise and is distributed according to $\mathcal{N}(0, R^c)$ and $\mathcal{N}(0, R^r)$, respectively.

The boat is assumed to move around according to the model $x^+ = Fx + w$, where w is $\mathcal{N}(0, Q)$. Note the similarity to the measurement models.

Only insert the numbers when asked to. The needed values are given by

$$\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \qquad P = 25 I_2, \qquad H^c = H^r = I_2,$$
$$R^c = \begin{bmatrix} 79 & 36 \\ 36 & 36 \end{bmatrix}, \qquad R^r = \begin{bmatrix} 28 & 4 \\ 4 & 22 \end{bmatrix}, \qquad z_c = \begin{bmatrix} 2 & 14 \end{bmatrix}^T, \qquad z_r = \begin{bmatrix} -4 & 6 \end{bmatrix}^T$$

(a) What is $p(z^c | x)$.

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[1] In reality, a camera measures a bearing from a point while a radar measures in polar coordinates. However, with some knowledge of which plane/distance something is operating in, we can extract an approximate 3d cartesian measurement and approximate it as Gaussian (more on that later in the course). At a certain distance a Gaussian in polar coordinates with small enough covariance can safely be approximated by a Gaussian in cartesian coordinates.

(b) Show that the joint $p(x, z^c)$ can be written as a Gaussian distribution.

Hint: conditional probability and the proof of theorem 3.3.1.

(c) Find the marginal $p(z^c)$ and the conditional $p(x|z^c)$, using the above and either theorems from the book or calculations.

(d) Can what was found above be reused to find the marginal $p(x^+)$ and/or $p(x|z^r)$? If so, state them.

(e) What is the MMSE and MAP estimate of x given z^c? You do not need to do calculations to find the answer, but briefly state what you would do if you had to.

(f) Using Python. Use what you have found to condition x on z for each sensor to find the conditional mean and covariance, and plot and comment. Ie., insert the values to find the parameters of $p(x|z^c)$ and $p(x|z^r)$.

Hint: See the attached Python script for details

(g) Using Python. Perform the update of the other sensor (for both) to get $p(x|z^r, z^c)$ and investigate. Are the distributions the same? does it matter which order we condition?

(h) You now want to know the probability that the boat is above the line, $\Pr(x_2 - x_1 > 5)$. Find this probability using the appropriate linear transform and the CDF.

Hint: scipy.stats.norm have the function cdf and sf which can be imported as `from scipy.stats import norm` and then used as `norm.cdf(value, mean, std)`.

Task 3: *Working with the canonical form*

In Section 3.3 the fundamental product identity was stuedied using a moment-based parametrization. Clearly, it must also be possible to establish an equivalent result using the canonical representation. In this exercise we shall therefore consider the product

$$\mathcal{N}^{-1}(\mathbf{x}\,;\,\mathbf{a}, \mathbf{B})\mathcal{N}^{-1}(\mathbf{y}\,;\,\mathbf{C}\mathbf{x}, \mathbf{D}). \tag{6}$$

(a) Show that (6) is identical to the Gaussian

$$\mathcal{N}^{-1}\left(\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\,;\,\begin{bmatrix}\mathbf{a}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C} & -\mathbf{C}^\mathsf{T}\\ -\mathbf{C} & \mathbf{D}\end{bmatrix}\right). \tag{7}$$

Hint: Taking the logarithm of the form (3.17) in the book with (3.20) inserted give a relatively simple way to the goal, after the terms constant in $\mathbf{x}$ and $\mathbf{y}$ are subtracted. Also

$$\mathbf{a}^\mathsf{T}\mathbf{A}\mathbf{a} + \mathbf{b}^\mathsf{T}\mathbf{B}\mathbf{b} + 2\mathbf{a}^\mathsf{T}\mathbf{C}\mathbf{b} = \mathbf{a}^\mathsf{T}\mathbf{A}\mathbf{a} + \mathbf{b}^\mathsf{T}\mathbf{B}\mathbf{b} + \mathbf{a}^\mathsf{T}\mathbf{C}\mathbf{b} + \mathbf{b}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{a} = \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}^\mathsf{T}\begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{C}^\mathsf{T} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix},$$

is handy, and valid for any vectors $\mathbf{a}$ and $\mathbf{b}$ and matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of interest (variable names are not related to the task).

(b) Show that the marginal distribution of $\mathbf{y}$, from the joint density (7), is

$$\mathcal{N}^{-1}\left(\mathbf{y}\,;\,\mathbf{C}^\mathsf{T}(\mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{a}, \mathbf{D} - \mathbf{C}(\mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{C}^\mathsf{T}\right). \tag{8}$$

Hint: Theorem 3.4.1

(c) Show that the conditional distribution of $\mathbf{x}$ given $\mathbf{y}$ is

$$\mathcal{N}^{-1}\left(\mathbf{x}\,;\,\mathbf{a} + \mathbf{C}^\mathsf{T}\mathbf{y}, \mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{D}^{-1}\mathbf{C}\right). \tag{9}$$

Hint: Theorem 3.4.1

(d) Let us now return to the original formulation of the product identity in Theorem 3.3.1. Use the result from c) to show that

$$\hat{\mathbf{P}}^{-1} = \bar{\mathbf{P}}^{-1} + \mathbf{H}^\mathsf{T}\mathbf{R}^{-1}\mathbf{H}. \tag{10}$$

Hint: Match variables in (3.21) with (3.10) in the book.