Network formation models

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Network Science

Network models

Empirical network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)

Generative models:

- Random graph model (Erdos & Renyi, 1959)
- Preferential attachment model (Barabasi & Albert, 1999)
- "Small world" model (Watts & Strogatz, 1998)

Random graph model

Graph $G\{E, V\}$, nodes $n = |V|$, edges $m = |E|$

Erdos and Renyi, 1959.

Random graph models

- $G_{n,m}$, a randomly selected graph from the set of $C_N^m graphs, N = \frac{n(n-1)}{2}$, with n nodes and m edges
- $G_{n,p}$, each pair out of $N = \frac{n(n-1)}{2}$ pairs of nodes is connected with probability p, m - random number

$$\langle m \rangle = p\frac{n(n-1)}{2}$$

$$\langle k \rangle = \frac{1}{n}\sum_i k_i = \frac{2\langle m \rangle}{n} = p\ (n-1) \approx pn$$

$$\rho = \frac{\langle m \rangle}{n(n-1)/2} = p$$

Random graph model

- Probability that i-th node has a degree $k_i = k$

$$P(k_i = k) = P(k) = C_{n-1}^{k} p^k (1-p)^{n-1-k}$$

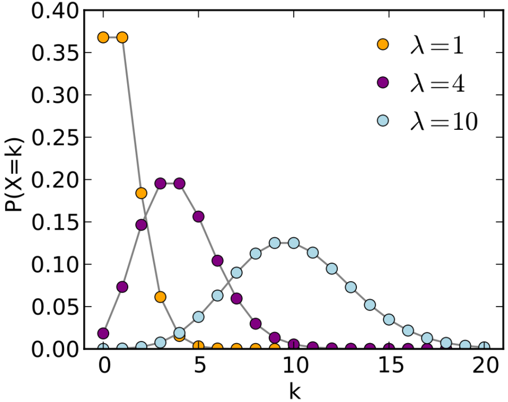
(Bernoulli distribution)
p^k - probability that connects to k nodes (has k-edges)
$(1-p)^{n-k-1}$ - probability that does not connect to any other node
C_{n-1}^{k} - number of ways to select k nodes out of all to connect to

- Limiting case of Bernoulli distribution, when $n \to \infty$ at fixed $\langle k \rangle = pn = \lambda$

$$P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} = \frac{\lambda^k e^{-\lambda}}{k!}$$

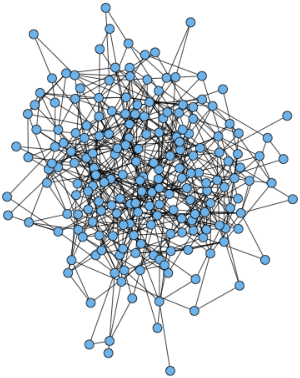
(Poisson distribution)

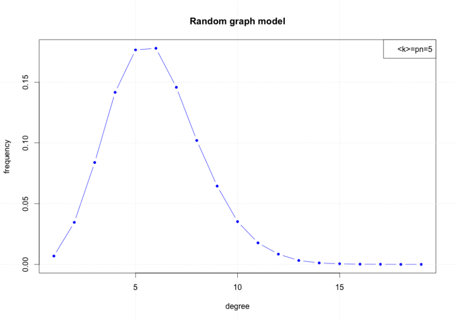
Poisson Distribution



$$P(k_i = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn$$

Random graph



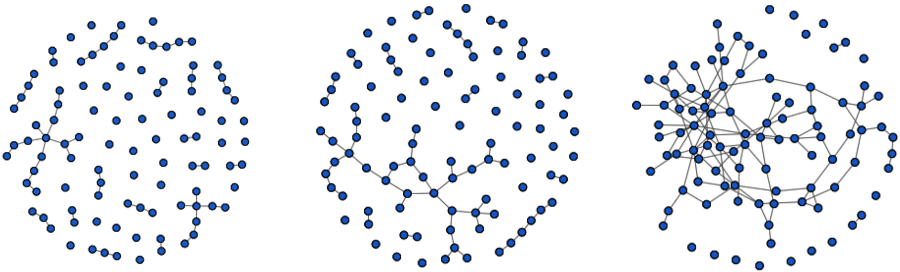


$$\langle k \rangle = pn = 5$$

Random graph model

Consider $G_{n,p}$ as a function of p

- $p = 0$, empty graph - $\langle k \rangle = 0$
- $p = 1$, complete (full) graph - $\langle k \rangle = n - 1$
- n_G -largest connected component, $s = \frac{n_G}{n}$



p

Phase transition

Let u – fraction of nodes that do not belong to GCC. The probability that a node does not belong to GCC

$$u = \frac{n - n_G}{n} = P(k=0) + P(k=1) \cdot u + P(k=2) \cdot u^2 + P(k=3) \cdot u^3 ... =$$

$$= \sum_{k=0}^{\infty} P(k) u^k = \sum_{k=0} \frac{\lambda^k e^{-\lambda}}{k!} u^k = e^{-\lambda} e^{\lambda u} = e^{\lambda(u-1)}$$

Let s -fraction of nodes belonging to GCC (size of GCC)

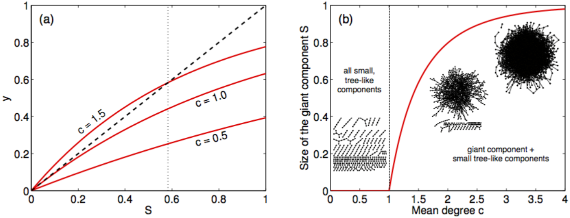
$$s = 1 - u$$

$$1 - s = e^{-\lambda s}$$

when $\lambda \to \infty, \quad s \to 1$
when $\lambda \to 0, \quad s \to 0$
$\lambda = pn = \langle k \rangle$

Phase transition

$$s = 1 - e^{-\lambda s}$$



non-zero solution exists when (at $s = 0$):

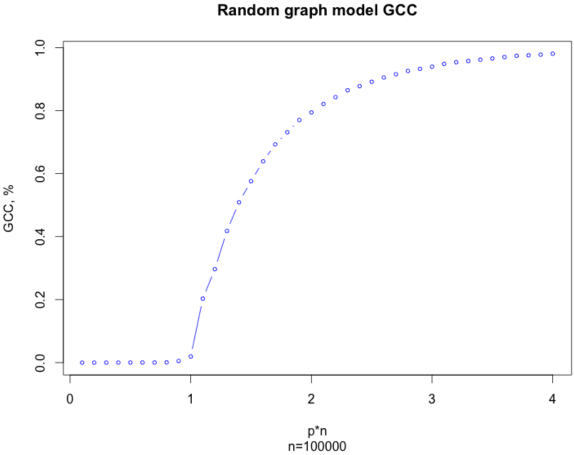
$$\lambda e^{-\lambda s} > 1$$

critical value:

$$\lambda_c = 1$$

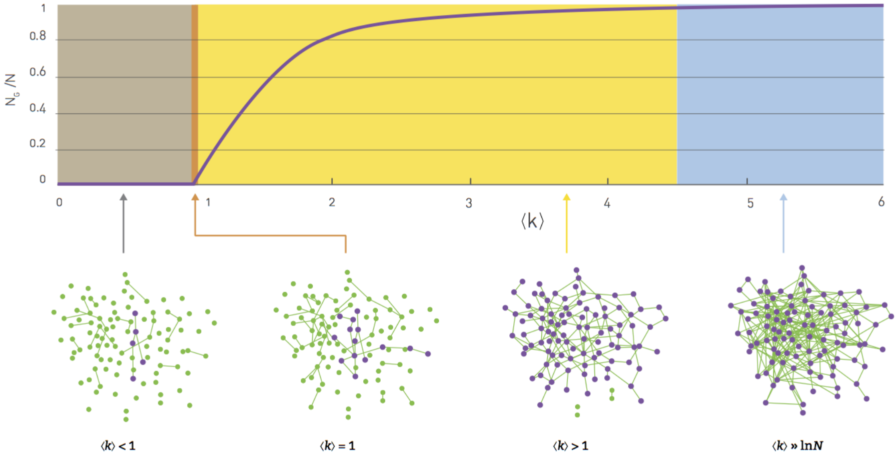
$$\lambda_c = \langle k \rangle = p_c n = 1, \quad p_c = \frac{1}{n}$$

Numerical simulations



$$\langle k \rangle = pn$$

evolution of random network



from A-L. Barabasi, 2016

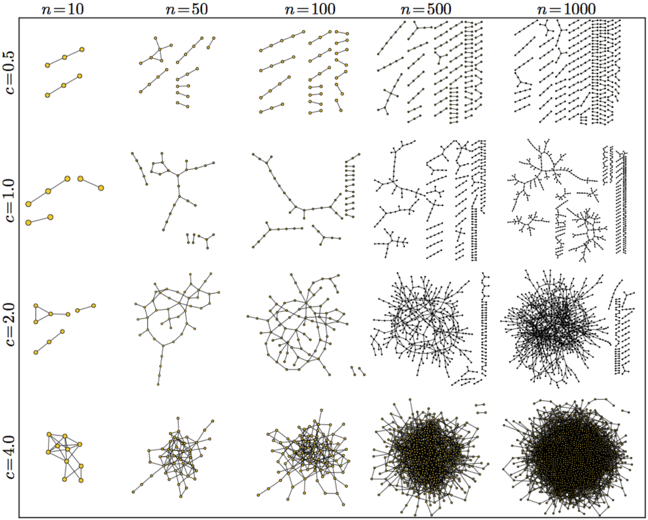
Phase transition

Graph $G(n, p)$, for $n \to \infty$, critical value $p_c = 1/n$

- Subcritical regime: $p < p_c$, $\langle k \rangle < 1$ there is no components with more than $O(\ln n)$ nodes, largest component is a tree
- Critical point: $p = p_c$, $\langle k \rangle = 1$ the largest component has $O(n^{2/3})$ nodes
- Supercritical regime: $p > p_c$, $\langle k \rangle > 1$ gigantic component has all $O((p - p_c)n)$ nodes
- Connected regime: $p >> \ln n/n$, $\langle k \rangle > \ln n$ gigantic component has all $O(n)$ nodes

Critical value: $\langle k \rangle = p_c n = 1$- on average one neighbor for a node

Numerical simulation

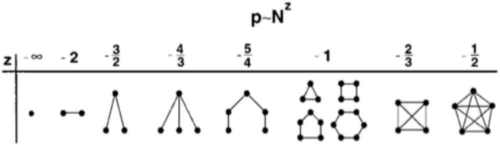


Clauset, 2014

Threshold probabilities

Graph $G(n, p)$

Threshold probabilities when different subgraphs of k-nodes and l-edges appear in a random graph $p_s \sim n^{-k/l}$



When $p > p_s$:

- $p_s \sim n^{-k/(k-1)}$, having a tree with k nodes
- $p_s \sim n^{-1}$, having a cycle with k nodes
- $p_s \sim n^{-2/(k-1)}$, complete subgraph with k nodes

Barabasi, 2002

Clustering coefficient

- Clustering coefficient (probability that two neighbors link to each other):

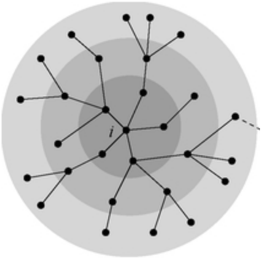
$$C_i(k) = \frac{\#\text{of links between NN}}{\#\text{max number of links NN}} = \frac{pk(k-1)/2}{k(k-1)/2} = p$$

$$C = p = \frac{\langle k \rangle}{n}$$

- when $n \to \infty, \quad C \to 0$

Graph diameter

- $G(n, p)$ is locally tree-like (GCC) (no loops; low clustering coefficient)



- on average, the number of nodes d steps away from a node

$$n = 1 + \langle k \rangle + \langle k \rangle^2 + ... \langle k \rangle^D = \frac{\langle k \rangle^{D+1} - 1}{\langle k \rangle - 1} \approx \langle k \rangle^D$$

- in GCC, around p_c , $\langle k \rangle^D \sim n$,

$$D \sim \frac{\ln n}{\ln \langle k \rangle}$$

Random graph model

- Node degree distribution function - Poisson:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn = \langle k \rangle$$

- Average path length:

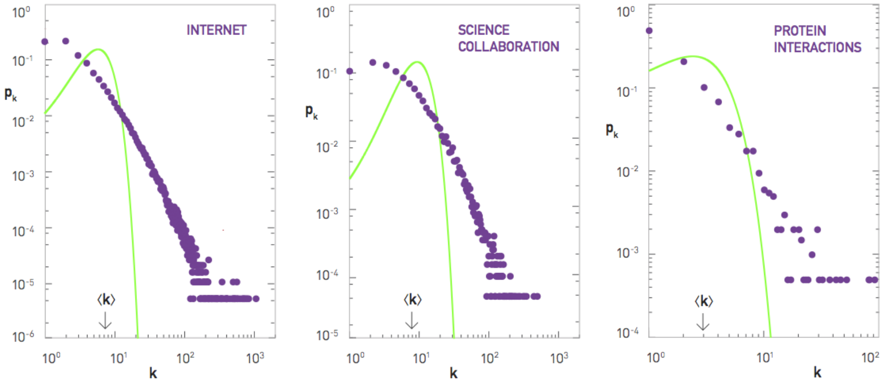
$$\langle L \rangle = \frac{\ln n}{\ln \langle k \rangle}$$

- Clustering coefficient:

$$C = \frac{\langle k \rangle}{n}$$

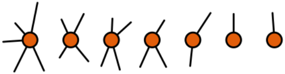
Real networks

Degree distribution in real networks



Configuration model

- Random graph with n nodes with a given degree sequence: $D = \{k_1, k_2, k_3..k_n\}$ and $m = 1/2 \sum_i k_i$ edges.
- Construct by randomly matching two stubs and connecting them by an edge.

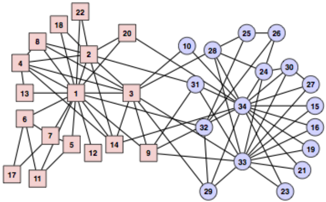


- Can contain self loops and multiple edges
- Probability that two nodes i and j are connected

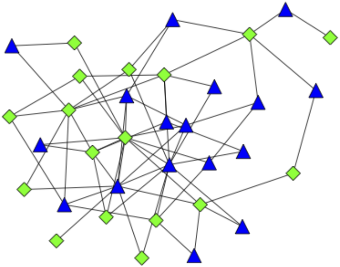
$$p_{ij} = \frac{k_i k_j}{2m - 1}$$

- Will be a simple graph for special "graphical degree sequence"

Can be used as a "null model" for comparative network analysis



karate club



configuration model

Clauset, 2014

Motivation of Evolutional Random Graph Model

Most of the networks we study are evolving over time, they expand by adding new nodes:

- Citation networks
- Collaboration networks
- Web
- Social networks

Preferential attachment model

Barabasi and Albert, 1999

Dynamic growth model

Start at $t = 0$ with n_0 nodes and some edges $m_0 \geq n_0$

1. **Growth**

   At each time step add a new node with m edges $(m \leq n_0)$, connecting to m nodes already in netwrok $k_i(i) = m$

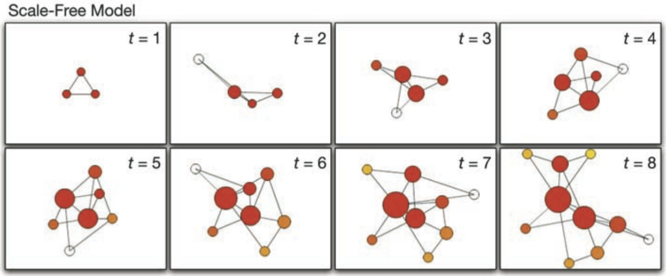
2. **Preferential attachment**

   The probability of linking to existing node i is proportional to the node degree k_i

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i}$$

after t timesteps: $t + n_0$ nodes, $mt + m_0$ edges

Preferential attachment model



Barabasi, 1999

Preferential attachment

Continues approximation: continues time, real variable node degree
$\langle k_i(t) \rangle$- expected value over multiple realizations
Time-dependent degree of a single node:

$$k_i(t + \delta t) = k_i(t) + m\Pi(k_i)\delta t$$

$$\frac{dk_i(t)}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_i k_i} = \frac{mk_i}{2mt}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t}$$

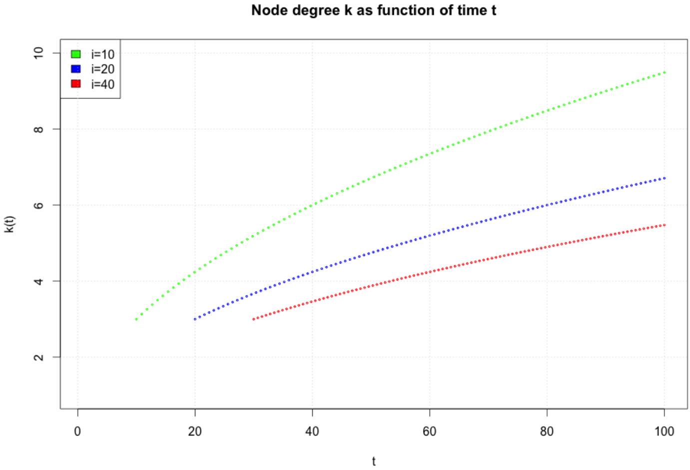
initial conditions: $k_i(t = i) = m$

$$\int_m^{k_i(t)} \frac{dk_i}{k_i} = \int_i^t \frac{dt}{2t}$$

Solution:

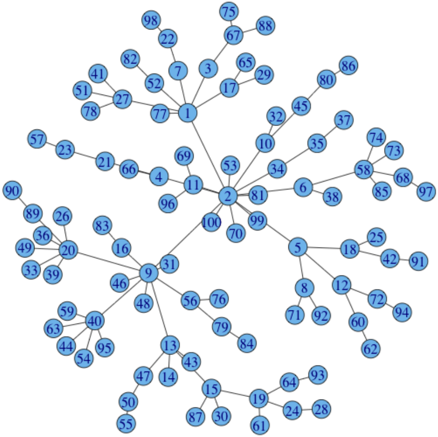
$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Preferential attachement



$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Preferential attachement



Preferential attachment

Time evolution of a node degree

$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Nodes with $k_i(t) \leq k$:

$$m\left(\frac{t}{i}\right)^{1/2} \leq k$$

$$i \geq \frac{m^2}{k^2} t$$

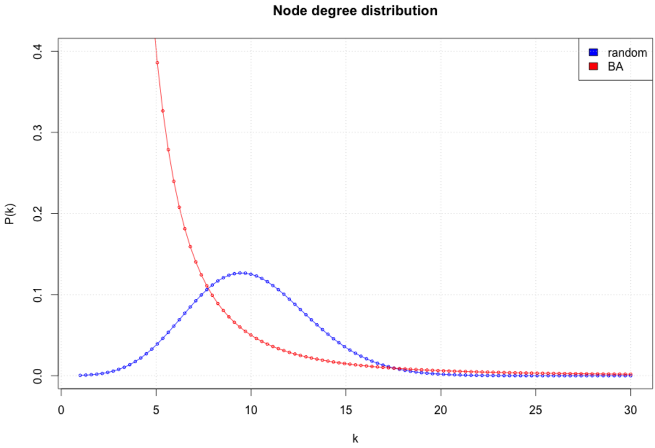
Probability of randomly selected node to have $k' \leq k$ (fraction of nodes with $k' \leq k$)

$$F(k) = P(k' \leq k) = \frac{n_0 + t - m^2 t / k^2}{n_0 + t} \approx 1 - \frac{m^2}{k^2}$$

Distribution function:

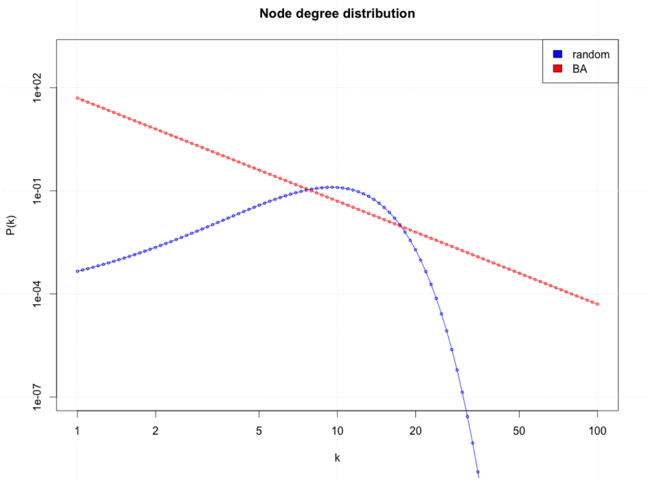
$$P(k) = \frac{d}{dk} F(k) = \frac{2m^2}{k^3}$$

Preferential attachment vs random graph



$$BA : P(k) = \frac{2m^2}{k^3}, \qquad ER : P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}, \quad \langle k \rangle = pn$$

Preferential attachment vs random graph



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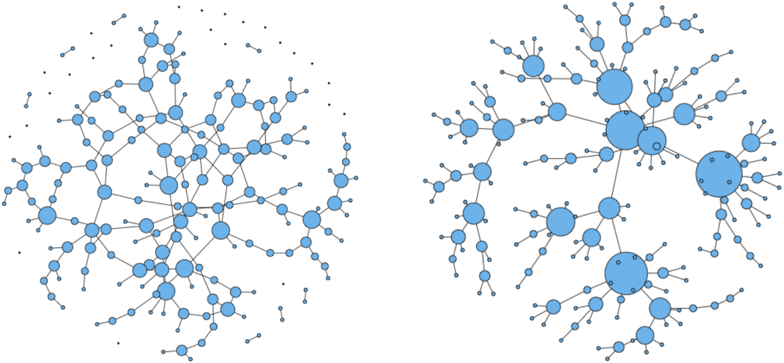
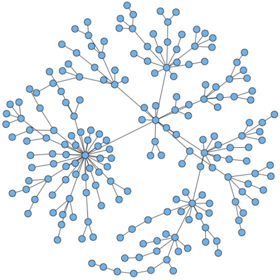
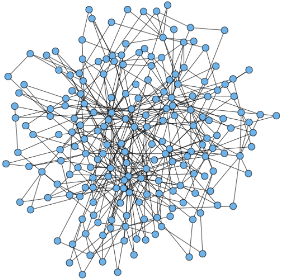
Preferential attachment vs random graph



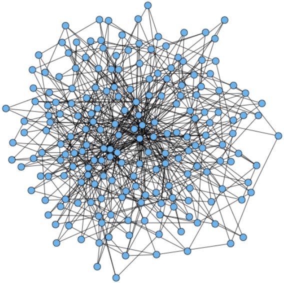
Preferential attachment model



$m = 1$



$m = 2$



$m = 3$

Growing random graph

1. **Growth**
   At each time step add a new node with m edges ($m \leq n_0$), connecting to m nodes already in network $k_i(i) = m$
2. **~~Preferential attachment~~ Uniformly at random**
   The probability of linking to existing node i is

$$\Pi(k_i) = \frac{1}{n_0 + t - 1}$$

Node degree growth:

$$k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Node degree distribution function:

$$P(k) = \frac{e}{m} \exp\left(-\frac{k}{m}\right)$$

Preferential attachment

- Power law distribution function:

$$P(k) = \frac{2m^2}{k^3}$$

- Average path length (analytical result) :

$$\langle L \rangle \sim \log(N)/\log(\log(N))$$

- Clustering coefficient (numerical result):

$$C \sim N^{-0.75}$$

Many more models

Some other models that produce scale-free networks:

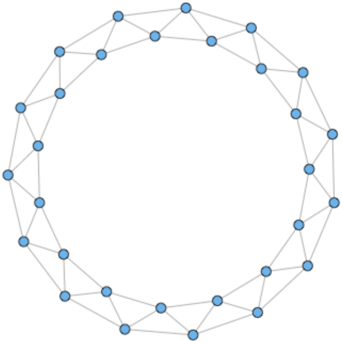
- Non-linear preferential attachment
- Link selection model
- Copying model
- Cost-optimization model
- ...

Historical note

- Polya urn model, George Polya, 1923
- Yule process, Udny Yule, 1925
- Distribution of wealth, Herbert Simon,1955
- Evolution of citation networks, cumulative advantage, Derek de Solla Price, 1976
- Preferential attachment network model, Barabasi and Albert, 1999

Small world

Motivation: keep high clustering, get small diameter



Clustering coefficient $C = 1/2$
Graph diameter $d = 8$

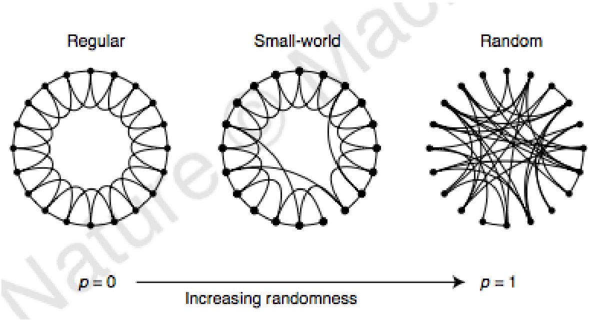
Small world

Watts and Strogatz, 1998

Single parameter model, interpolation between regular lattice and random graph

- start with regular lattice with n nodes, k edges per vertex (node degree), $k << n$
- randomly connect with other nodes with probability p, forms $pnk/2$ "long distance" connections from total of $nk/2$ edges
- $p = 0$ regular lattice, $p = 1$ random graph

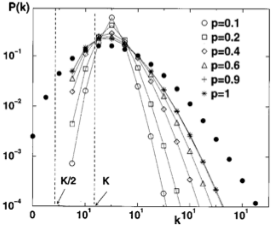
Small world

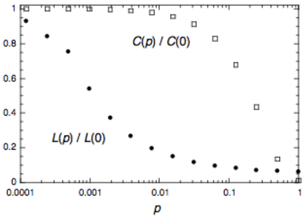


Watts, 1998

Small world model

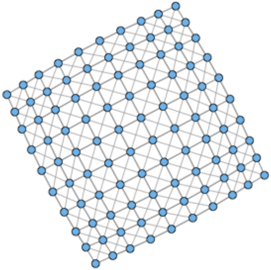
- Node degree distribution:
  Poisson like
- Ave. path length $\langle L(p) \rangle$:
  $p \to 0$, ring lattice, $\langle L(0) \rangle = 2n/k$
  $p \to 1$, random graph, $\langle L(1) \rangle = \log(n)/\log(k)$
- Clustering coefficient $C(p)$:
  $p \to 0$, ring lattice, $C(0) = 3/4 = const$
  $p \to 1$, random graph, $C(1) = k/n$

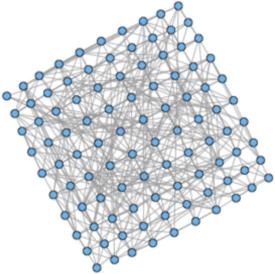




Watts, 1998

Small world model





20% rewiring:

ave. path length = 3.58 $\rightarrow$ ave. path length = 2.32

clust. coeff = 0.49 $\rightarrow$ clust. coeff = 0.19

Model comparison

	Random	BA model	WS model	Empirical networks
P(k)	$\frac{\lambda^k e^{-\lambda}}{k!}$	k^{-3}	poisson like	power law
C	$\langle k \rangle / N$	$N^{-0.75}$	const	large
$\langle L \rangle$	$\frac{\log(N)}{\log(\langle k \rangle)}$	$\frac{\log(N)}{\log\log(N)}$	$\log(N)$	small

References

- On random graphs I, P. Erdos and A. Renyi, Publicationes Mathematicae 6, 290–297 (1959).
- On the evolution of random graphs, P. Erdos and A. Renyi, Publicaton of the Mathematical Institute of the Hungarian Academy of Sciences, 17-61 (1960)
- Emergence of Scaling in Random Networks, A.L. Barabasi and R. Albert, Science 286, 509-512, 1999
- Collective dynamics of small-world networks. Duncan J. Watts and Steven H. Strogatz. Nature 393 (6684): 440-442, 1998