微分幾何学と位相幾何学に関するノート

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Notes on Differential Geometry

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December 18, 2025

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Preface

Chapter 1

Differential Topology

1 Zermelo-Fraenkel axiom

In different geomeotry, especially manifolds, on proof, instead of contruct by axiom of choice, we only need the basic logical foundation to think and write, that is ZF axiom

Definition 1.1 (ZF Axioms). The axioms of Zermelo–Fraenkel set theory describe the fundamental properties of sets. All variables are understood to range over sets.

1. *Axiom of Extensionality.*
$$\forall A, B \ \big(\forall x \, (x \in A \Leftrightarrow x \in B) \Rightarrow A = B\big).$$
Two sets are equal if and only if they have the same elements.

2. *Axiom of Pairing.*
$$\forall a, b \ \exists c \ \forall x \ \big(x \in c \Leftrightarrow (x = a \lor x = b)\big).$$
For any sets a, b, there exists the set $\{a, b\}$.

3. *Axiom of Union.*
$$\forall A \ \exists U \ \forall x \ \big(x \in U \Leftrightarrow \exists B \in A, \ x \in B\big).$$
For any set A, there exists a set U that contains exactly the elements of elements of A.

4. *Axiom of Power Set.*
$$\forall A \ \exists P \ \forall x \ \big(x \in P \Leftrightarrow x \subseteq A\big).$$
For every set A, there exists the set $\mathcal{P}(A)$ of all subsets of A.

5. *Axiom of Infinity.*
$$\exists I \ \big(\emptyset \in I \land \forall x \in I, \ x \cup \{x\} \in I\big).$$
There exists an infinite set containing $\emptyset$ and closed under the operation $x \mapsto x \cup \{x\}$.

6. *Axiom Schema of Separation (Specification).* For any property $P(x)$,
$$\forall A \ \exists B \ \forall x \ \big(x \in B \Leftrightarrow (x \in A \land P(x))\big).$$
You can form subsets of a given set by filtering with a predicate $P(x)$.

7. *Axiom Schema of Replacement.* If F is a definable function, then the image of any set under F is also a set:
$$\forall A \ \exists B \ \forall y \ \big(y \in B \Leftrightarrow \exists x \in A, \ y = F(x)\big).$$

8. *Axiom of Foundation (Regularity).*

$$\forall A \neq \emptyset \; \exists x \in A, \; (x \cap A = \emptyset).$$

Every nonempty set has an element disjoint from itself; no infinite descending membership chains exist.

Remark 1.2. Adding the *Axiom of Choice (AC)*

$$\forall \{A_i\}_{i \in I} \left(\forall i, \; A_i \neq \emptyset \Rightarrow \exists f : I \to \bigcup A_i, \; f(i) \in A_i \right)$$

yields the theory *ZFC*.

2 Topological Spaces

Definition 1.1. Let X be a set, a *topology on* X is a collection $\mathcal{X} \subseteq \mathcal{P}(X)$ satisfying

1. X and $\varnothing$ are element of $\mathcal{X}$.
2. Union of elements of $\mathcal{X}$ is itself an element of $\mathcal{X}$.
3. Intersection of elements of $\mathcal{X}$ is itself an element of $\mathcal{X}$.

A pair $(X, \mathcal{X})$ is called a *topological space*. Elements of X is called *points* and every set $U \in \mathcal{X}$ is called an *open subset of* X. A *neighborhood* of $p \in X$ is an open subset $U \subseteq X$ containing p.

Definition 1.2 (Closed subsets). Let X be a topological space, a subset $U \subseteq X$ is said to be a *closed subset of* X if $X \backslash U$ is an open subset.

Definition 1.3 (Closure and Interior). Let X be a topological space, *the closure of* A *in* X is the smallest closed subset of X containing A, defined by

$$\overline{A} := \bigcap \{B \subseteq X \mid B \supseteq A \text{ and } B \text{ is closed in } X\}.$$

The interior of A is the largest open subset of X contained by A, defined by

$$\text{Int} A := \bigcup \{C \subseteq X \mid C \subseteq A \text{ and } C \text{ is open in } X\}.$$

The exterior of A is the largest open subset of X outside A, defined by

$$\text{Ext} A := X \backslash \overline{A},$$

and *the boundary of* A is an closed subset of X, defined by

$$\partial A := X \backslash (\text{Int} A \cup \text{Ext} A).$$

Proposition 1.4. Let X be a topological space and let $A \subseteq X$ be any subset.

(1) A point is in $\operatorname{Int} A$ if and only if it has a neighborhood contained in A.

(2) A point is in $\operatorname{Ext} A$ if and only if it has a neighborhood contained in $X \backslash A$.

(3) A point is in ∂A if and only if every neighborhood of it contains both a point of A and a point of $X \backslash A$.

(4) A point is in $\bar{A}$ if and only if every neighborhood of it contains a point of A.

(5) $\bar{A} = A \cup \partial A = \operatorname{Int} A \cup \partial A$.

(6) $\operatorname{Int} A$ and $\operatorname{Ext} A$ are open in X, while $\bar{A}$ and ∂A are closed in X.

(7) The following are equivalent

(a) A is open in X.

(b) $A = \operatorname{Int} A$.

(c) A contains none of its boundary points.

(d) Every point of A has a neighborhood contained in A.

(8) The following are equivalent

(a) A is closed in X.

(b) $A = \overline{A}$.

(c) A contains all of its boundary points.

(d) Every point of $X\backslash A$ has a neighborhood contained in $X\backslash A$.

Proof. (1) This is trivial since for every $a \in \operatorname{Int} A$, one can find an open neighborhood $C \subset A$ which contains a.

(2) By the Morgan law, we can rewrite

$$\operatorname{Ext} A = X\backslash\overline{A} = \bigcup\{B \subset X \mid B \subseteq X\backslash A \text{ and } B \text{ is open in } X\}$$

Let $a \in \operatorname{Ext} A$, one can find a neighborhood $U \subseteq X\backslash A$ which contains a.

(3) Suppose for the sake of condition, that ∂A is nonempty, pick any $a \in \partial A$. Since we have

$$\partial A = (X\backslash\operatorname{Int} A) \cap (X\backslash\operatorname{Ext} A) = (X\backslash\operatorname{Int} A) \cap \overline{A},$$

it follows that any neighborhood U containing a must be contained by the intersection of the following closed subsets. Since $U \cap X\backslash\operatorname{Int} A \neq \varnothing$, which means this is the largest closed set outside A, we can find $u \in C \cap U$ and $C \subseteq X\backslash A$, where C is an closed set. Since $U \cap \overline{A} \neq \varnothing$ and U is open, then one can find $v \in U$ such that $v \in \overline{A}$. We define the set

$$D = \{v \in U \cap \overline{A} \mid v \in A\}.$$

If D is nonempty, one can choose $v \in D$ and we are done. Suppose $D = \varnothing$, then for all $v \in U \cap \overline{A}$, we must have $v \in X\backslash A$. Thus $U \subseteq X\backslash A$, that U is open implies $U \subseteq \operatorname{Ext} A$. Since $\operatorname{Ext} A \cap \overline{A} = \varnothing$ leading to $U \cap \overline{A} = \varnothing$, this contradicts the property that $U \cap \overline{A} \neq \varnothing$.

To establish to reverse implication, we assume the contrary holds, that is, there exists $a \in \partial A$ and its neighborhood U that contain only points in A (or $X\backslash A$). Since U is open, it follows that $U \subseteq \operatorname{Int} A$, and the fact that $\operatorname{Ext} A \cap \operatorname{Int} A = \varnothing$ implies $U \cap \operatorname{Ext} A = \varnothing$, which deduces a contradiction. The case for $U \subseteq X\backslash A$ is clearly similar, and hence we are done.

(4) Suppose the contrary holds, that there exsits $a \in \overline{A}$ and a neighborhood U not containing any point in A. Since U is open, it follows that $U \subseteq \operatorname{Ext} A$ and hence $U \cap \{a\} = \varnothing$, contradiction.

(5) Since

$$\overline{A}\backslash A = (X\backslash\operatorname{Ext} A)\backslash A = X\backslash(\operatorname{Ext} A \cup A) \subseteq X\backslash(\operatorname{Ext} A \cup \operatorname{Int} A) = \partial A \Rightarrow \overline{A} \subseteq \partial A \cup A$$

and

$$A \cup \partial A = A \cup (X\backslash(\operatorname{Ext} A \cup \operatorname{Int} A)) = A \cup [(X\backslash\operatorname{Ext} A) \cap (X\backslash\operatorname{Int} A)] = A \cup (\overline{A} \cap (X\backslash\operatorname{Int} A)) \subseteq A \cup \overline{A} = \overline{A}.$$

Hence $\overline{A} \subseteq A \cup \partial A$.

(6) This is trivial since union of open subsets is open and intersection of closed subsets is closed. □

Definition 1.5. Let X be a topological space and $A \subseteq X$, we say $p \in X$ is a *limit point of* A if for every neighborhood U satisfies $U\backslash\{p\} \cap A \neq \varnothing$. Conversely, a point $p \in A$ is called *isolated point of* A if p has a neighborhood U satisfies $U \cap A = \{p\}$.

Proposition 1.6. A subset of a topological space is closed if and only if it contains all of its limit points.

Proof. ($\Rightarrow$) Let A be a closed subset of a topoglogical space X and $x \in X$ be a limit point of A. Then for any neighborhood U containing p, it follows that $U\backslash\{x\} \cap A \neq \varnothing$. Suppose $x \notin A$, since $X\backslash A$ is open, it follows that $U \cap (X\backslash A)$ is nonempty and open. By the proposition above, we thus have $x \in \partial A \subseteq \overline{A} = A$, contradiction. Thus $x \in A$.

($\Leftarrow$) Since every limit point is in A or ∂A, and by the fact that all of them are in A, we thus have $\partial A \subseteq A$, hence A is closed. $\square$

Definition 1.7. A subset A of a topological space X is said to be *dense in* X if $\overline{A} = X$.

Proposition 1.8. Show that a subset $A \subseteq X$ is dense if and only if every nonempty open subset of X contains a point of A.

Proof. ($\Rightarrow$) Assume the contrary holds, that one can find an open subset $U \subseteq X$ satisfying $A \cap U = \varnothing$. Since $X = A \cup \partial A \Rightarrow X\backslash\partial A \subseteq A$, then

$$(X\backslash\partial A) \cap U = \varnothing$$

Since $U \subseteq X\backslash A$, we thus have $U \subseteq \partial A$. By the above proposition, it follows that $A \cap U \neq \varnothing$, contradiction.

($\Leftarrow$) Suppose A is not dense, in other words $X\backslash\overline{A}$ is an nonempty open subset, consequently, this implies $(X\backslash\overline{A}) \cap A \neq \varnothing$. But since $A \subseteq \overline{A}$, we thus have $X\backslash\overline{A} \subseteq X\backslash A$, or $(X\backslash\overline{A}) \cap A \subset (X\backslash A) \cap A = \varnothing$, which implies a contradiction. Hence we are done. $\square$

3 Convergence and Continuity

Definition 1.1 (Convergence). Let X be a topological space, $(x_n) \subseteq X$ is a sequence of points in X and $x \in X$. We say $\lim\limits_{n\to+\infty} x_n = x$ or $x_n \to x$ if for every neighborhood U of x, there exists $N(U) \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N(U)$.

Proposition 1.2. Let X be a topological space, A is a subset of X and $(x_n) \subset A$. If $x_n \to x \in X$, then $x \in \overline{A}$.

Proof. Let U be a neighborhood of x, since $x_n \to x$, there exists $N(U) \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Thus $U \cap A \neq \varnothing$. The above proposition implies that $x \in A$ or U contains a point in A and a point in $X\backslash A$. In either case, we always have $x \in \overline{A}$. $\square$

Definition 1.3 (Continuity). Let X and Y be a topological spaces, a map $f : X \to Y$ is said to be *continuous* if for every open subset $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in X.

Proposition 1.4. A map between topological spaces is continuous if and only if its preimage of every closed subset is closed.

Proof. Let $f : X \to Y$ be the map between topological spaces satisfying $f^{-1}(U)$ is closed for all closed subsets $U \subseteq Y$. This implies both $Y\backslash U$ and $X\backslash f^{-1}(U)$ is open. Since we have

$$X\backslash f^{-1}(U) = f^{-1}(Y)\backslash f^{-1}(U) = f^{-1}(Y\backslash U) \text{ is open,}$$

and U is abitrary closed subset, f also preserve openess on preimage of the open subsets. Hence f is continuous. $\square$

Proposition 1.5. Let X, Y and Z be topological spaces.

1. Every constant map $f : X \to Y$ is continuous.
2. The indentity map $\mathrm{Id}_X : X \to X$ is continuous.

3. If $f : X \to Y$ and $g : Y \to Z$ are both continuous, then so is their composition $g \circ f : X \to Z$.

Proof. It suffices to prove the third property. Let $U \in Z$ by any open subsets, we need to prove $(g \circ f)^{-1}(U)$ is open, in other words, this can be rewriten as

$$(g \circ f)^{-1}(U) = (f^{-1} \circ g^{-1})(U) = f^{-1}(g^{-1}(U))$$

Since $g^{-1}(U)$ is open, then $f^{-1}(g^{-1}(U))$ is also open. Hence $g \circ f$ is continuous. $\square$

Proposition 1.6. A map $f : X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which the restriction of f is continuous.

Proof. ($\Rightarrow$) If f is continuous and $x \in X$, we simply consider the restriction of $f_U : U \to Y$, where U is any neighborhood of x, and $f_U^{-1}(V) = f^{-1}(V) \cap U$ is open set. Hence f_U is continuous.

($\Leftarrow$) Suppose that f is restrictly continuous on a neighborhood of every point $x \in X$. Let $U \subseteq Y$ be any open subset, it suffices to prove that $f^{-1}(U)$ is open. Let $u \in f^{-1}(U)$, by the hypothesis, one can find a neighborhood $V \subseteq X$ containing u such that $f_V : V \to Y$ is continuous. Thus, the preimage

$$f_V^{-1}(U) = f^{-1}(U) \cap V \text{ is open.}$$

Since $f_V^{-1}(U) \subseteq f^{-1}(U)$, by the above proposition, it follows that $f^{-1}(U)$ is open, hence f is continuous. $\square$

Definition 1.7. A map $f : X \to Y$ is said to be an *open map* if $f(U)$ is open for all open subsets $U \subset X$. Conversely, f is said to be a *closed map* if $f(U)$ is closed for all closed subsets $U \subset X$.

Proposition 1.8. Suppose X and Y are topological spaces, and $f : X \to Y$ is a map.

1. f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
2. f is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.
3. f is continuous if and only if $f^{-1}(\text{Int}B) \subseteq \text{Int}f^{-1}(B)$ for all $B \subseteq X$.

4 Hausdorff Spaces

Definition 1.1. A topological space X is said to be *Hausdorff* if two any distinct points in X can be seperated by disjoint open subsets in X.

Proposition 1.2. Let X be a Hausdorff space.

1. Every finite subsets of X is closed.
2. If a sequence $(x_n) \subseteq X$ converges to a limit $p \in X$, the limit is unique.

Proposition 1.3. Suppose X is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of A, then every neighborhood of p contains infititely many points of A.

5 Bases

Definition 1.1. Let X be a topoglogical space, a basis for the topology X is a collection $\mathcal{B}$ of subsets in X satisfying two conditions:

1. Every element in $\mathcal{B}$ is an open subset of X.
2. Every open subset in X is the union of some collection of elements of $\mathcal{B}$.

6 Connectedness

Definition 1.1. A topological space X is said to be *disconnected* if it can be seperated by two disjoint, nonempty, open subset. If X is not disconnected, then X is said to be *connected*.

Proposition 1.2. A topological space X is connected if and only if the only clopen subsets of X are $\varnothing$ and X itself.

Proof. □

7 Compactness

Definition 1.1. A topological space X is said to be *compact* if every open cover of it attains a finite subcover.

Theorem 1.2. Let $f : X \to Y$ be a continuous map between topological spaces. If X is compact, then $f(X)$ is compact.

8 Topological Manifolds

Definition 1.1 (Topological Space). Let T be a collection of subsets of X such that following axioms hold:

1. $X, \varnothing \in T$.
2. Any union of sets in T is still in T.
3. Any finite intersection of sets in T is still in T.

The pair (X, T) is called *topological space*.

Definition 1.2 (Basis of Topological Space). Let X be a set. A basis $\mathcal{B}$ for a topology on X is a collection of subsets of X satisfying two properties:

1. Covering:
$$\forall x \in X, \exists b \in B : x \in \mathcal{B}$$
2. Intersection condition: For any $B_1, B_2 \in \mathcal{B}$ and any point $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 1.3 (Topological Manifolds). Suppose M is a topological space. We say that M is a *topological manifold of dimension n* or a *topological n-manifold* if it has the following properties:

1. M is a *Hausdorff space*: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
2. M is *second-countable*: there exists a countable basis for the topology of M.
3. M is *locally Euclidean of dimension n*: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Remark 1.4. The third properties means, for each $p \in M$ we can find

1. an open subset $U \subseteq M$ containing p,
2. an open subset $V \subseteq \mathbb{R}^n$, and
3. a homeomorphism $\varphi : U \to V$.

Theorem 1.5. A nonempty topological n-manifold cannot be homeomorphic to an topological m-manifold unless $m = n$.

Definition 1.6. Let M be a topological n-manifold. A *coordinate chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to V$ is a homeomorphism from U to an open subset $V = \varphi(U) \subseteq \mathbb{R}^n$

Definition 1.7. Given a chart (U, φ), we call the set U a *coordinate domain* of each of its point. If $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called *coordinate ball*.

Proposition 1.8. Homeomorphism preserves those following properties of the set: openess, closedness, compactness, precompactness, connectedness, path-connectedness and Hausdorff.

Lemma 1.9. Every topological manifold has a countable basis of precompact coordinate balls.

Proof. We call a topological manifold is *sweet* if it has countable basis satisfies the following required result.

Let M be a topological n-manifold. We first suppose the special case that M could be globally covered by a chart $\varphi : M \to \varphi(M) \subset \mathbb{R}^n$. Let $\mathcal{B}$ be a collection of open balls $B(x, r) \subset \mathbb{R}^n$ where x is a rational point and r is a positive rational number. Since every n-dimensional point x can be placed between two rational balls, we thus can find $B \in \mathcal{B}$ such that $x \in B$. Moreover, given $B_1, B_2 \in \mathcal{B}$ such that $L = B_1 \cap B_2 \neq \varnothing$ and $x_0 \in B_1 \cap B_2$. Since L is open, one can choose an open ball $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. In addition, by definition, since $\mathcal{B}$ is countable and $\varphi(M)$ is open, hence $\varphi(M)$ can be covered by countable subset $\mathcal{B}_1 \subseteq \mathcal{B}$ and if $B \in \mathcal{B}$, it would follow that $B \subset \varphi(M)$.

It suffices to prove $\varphi^{-1}(\mathcal{B}_1) = \{\varphi^{-1}(B) : B \in \mathcal{B}_1\}$ is a basis of precompact of coordinate balls of M. That $\mathcal{B}_1$ is basis of the space $\varphi(M)$ implies that $\varphi^{-1}(B)$ is basis of M. One should check that $\varphi^{-1}(\mathcal{B}_1)$ is countable, since we can write $\mathcal{B}_1$ as a sequence $(B_n)_{n=1}^{\infty} = \mathcal{B}_1$, so does $(\varphi(B_n)^{-1})_{n=1}^{\infty} = \varphi^{-1}(\mathcal{B}_1)$.

Secondly, we need to check that every element of $\varphi^{-1}(\mathcal{B}_1)$ is precompact. Let $B \in \mathcal{B}_1$ be an open ball, since closure(B) is compact, B is compact. Since homeomorphism preverves precompactness, $\varphi^{-1}(B)$ is also precompact. Thus φ^{-1} satisfies every following required properties and. Hence M has φ which is sweet.

Now we suppose M be an abitrary n-manifold. By the definition, M is locally Euclidean of dimension n, that is, for every $x \in M$, we can find an open subset $U \subseteq M$ containing x and a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ such that $\varphi(U)$ is open. Thus U restricted by the chart φ, by the argument in the preceding paragraph, U is sweet. In additon, since M is second-countable, M also has countable basis which is sweet by choose countably infinite basis in every set U. Hence we are done. □

Proposition 1.10. Let M be a topological manifold.

1. M is locally path-connected.
2. M is connected if and only if it is path-connected.
3. The components of M are the same as its path components.
4. M has countably many components, each of which is an open subset of M and a connected topological manifold.

Proof. Since each coordinate ball is path-connected, 1. follows from the fact that M has a basis of coordinate balls. □

Definition 1.11. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F : U \to V$ is said to be *smooth* or C^∞ if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a *diffeomorphism*.

Definition 1.12. Let M be a topological n-manifolds. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the *transition map from φ to ψ*. In addition, if either $U \cap V = \varnothing$ or the composite map $\psi \circ \varphi^{-1}$ is a diffeomorphism, two chart (U, φ) and (V, ψ) are said to be *smoothly compatible*, denoted by the relation "$\sim$".

Definition 1.13. Let M be a topological n-manifold and $\mathcal{A}$ be the collection of charts $(U_\alpha, \varphi_\alpha)$ such that $\bigcup U_\alpha = M$, we say $\mathcal{A}$ is an *atlas for* M.

Definition 1.14. A smooth atlas $\mathcal{A}$ on M is *maximal* if it is not properly contained in any larger smooth atlas. If M is a topological manifold, a *smooth structure on* M is a maximal smooth atlas.

Definition 1.15. Let M be topological n-manifold and $\mathcal{A}$ is an atlas for M. If for abitrary $(U, \varphi), (V, \psi) \in \mathcal{A}$, we have $(U, \varphi) \sim (V, \psi)$, then $\mathcal{A}$ is called *a smooth atlas* or *maximal atlas*.

Definition 1.16. A *smooth manifold* is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Proposition 1.17. Let M be a topological manifold.

1. Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called *smooth structure determined by* $\mathcal{A}$.

2. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. 1. Let $\mathcal{A}$ be a smooth atlas for M, and let $[\mathcal{A}]$ be the set of all charts that are smoothly compatible to every chart in $\mathcal{A}$. Since $\mathcal{A} \subseteq [\mathcal{A}]$, it suffices to prove that $[\mathcal{A}]$ is smooth atlas, in other words, let (X, f) and (Y, g) be abitrary two charts in $[\mathcal{A}]$, one should verify that $(X, f) \sim (Y, g)$.

Let $x = f(t) \in f(X \cap Y)$, since $\mathcal{A}$ has domain covering M, one can find $(Z, h)_x \in \mathcal{A}$ such that $t \in Z$ and the hypothesis deduces that $(X, f) \sim (Z, h)_x$ and $(Y, g) \sim (Z, h)_x$. Consider the transition:

$$g \circ f^{-1} : f^{-1}(X \cap Y \cap Z) \to g(X \cap Y \cap Z)$$

It would follow that $g \circ f^{-1} = (g \circ h^{-1}) \circ (h \circ f^{-1})$, where each term of the composition is diffeomorphism, thus $g \sim f$ in a neighborhood of x. Therefore $g \sim f$ on the whole domain $(X \cap Y)$, thus $[\mathcal{A}]$ is smooth atlas. To prove that it is maximal, since $[\mathcal{A}]$ is transitive, if there is a chart (U, φ) which is smoothly compatible with a chart in $[\mathcal{A}]$, one can find $(V, \psi) \in \mathcal{A}$ such that $(U, \varphi) \sim (V, \psi)$, by the definition of $[\mathcal{A}]$, we have $(U, \varphi) \in [\mathcal{A}]$. Moreover, if there is another maximal chart $[\mathcal{B}]$ containing $\mathcal{A}$, also because $[\mathcal{A}]$ is transitive, it would follows that $[\mathcal{B}] \subseteq [\mathcal{A}]$. Thus $[\mathcal{A}]$ is unique maximal atlas.

2. We first prove the foward implication. Suppose $\mathcal{A}$ and $\mathcal{B}$ determine the same atlas $[\mathcal{A}]$. Since every chart in $[\mathcal{A}]$ is compatible with all charts in both atlas $\mathcal{A}$ and $\mathcal{B}$, every atlas in $\mathcal{A}$ is smoothly comptatible with all atlas in $\mathcal{B}$, and vice versa. Thus $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas.

The reverse statement holds since $\mathcal{A} \cup \mathcal{B}$ determine $[\mathcal{A}]$ implies both $\mathcal{A}$ determine $[\mathcal{A}]$ and $\mathcal{B}$ determine $[\mathcal{A}]$. Hence the second proposition is clearly proven. $\square$

Corollary 1.18. The relation $\sim$ is equivalent in maximal atlas.

9 Local Coordinate Representations

Definition 1.1. Let M be a smooth manifold and $[\mathcal{A}]$ be its maximal atlas. Any chart $(U, \varphi) \in [\mathcal{A}]$ is called a *smooth chart*, and the corresponding map φ is called *smooth coordinate map*.

Definition 1.2. We say that a set $B \subseteq M$ is a *regular coordinate ball* if there exists a coordinate ball $B' \supseteq B$ and a smooth coordinate map $\varphi : B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$,

$$\varphi(B) = B(0, r), \quad \varphi(\overline{B}) = \overline{B(0, r)} \quad \text{and} \quad \varphi(B') = B(r', 0).$$

Proposition 1.3. Every smooth manifold has a countable basis of regular coordinate balls.

10 Manifolds with Boundary

Definition 1.1. An n-dimensional topological manifold with boundary is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a relatively open subset of $\mathbb{H}^n$.

11 Problems

Problem 1.1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$ and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

Proof. Consider the continuous map $\pi : X \to M$ be the quotient map satisfying $U \subseteq M$ is open if and only if $\pi^{-1}(U)$ is open in X. We consider two cases:

Case 1: $x_0 \neq 0$, consider the open neighborhood on M of $[x_0]$ pulled back by π^{-1} satisfying

$$I_{[x_0]} = (x_0 - \varepsilon, x_0 + \varepsilon) \times \{-1, 1\},$$

where $\varepsilon > 0$ is abitrary small such that $I_{[x_0]} \cap \{(0, -1), (0, 1)\} = \varnothing$. We define a local chart $\varphi : \pi(I_{[x_0]}) \to (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \mathbb{R}$ satisfying

$$\varphi([t]) = t \text{ for all } [t] \in \pi(I_{[x_0]})$$

It suffices to check that φ is homeomorphism. Since $\varphi([x_1]) = \varphi([x_2])$ implies $\pi(x_1) = \pi(x_2)$ and $[x_1] = [x_2]$, thus φ is injective. φ is also surjective since we can pick any equivalent class $[x]$ for given $x \in \pi(I_{[x_0]})$. The map $\varphi(\pi) : (t, \pm 1) \mapsto t$ implies φ is continuous and the map $\varphi([t, 1])^{-1} = [t, 1] = \pi(i(t))$ is continuous, since the map $i : (x_0 - \varepsilon, x_0 + \varepsilon) \to X$ satisfying $i(t) = (t, 1)$ is continuous. Hence φ is homeomorphism and M is locally Euclidean.

Case 2: $x_0 = 0$, since $(0, 1)$ and $(0, -1)$ are distinct under the following equivalent relation, we just consider the open neighborhood on M for $[(0, 1)]$ (the same construct for $(0, -1)$) satisfying

$$I^+ = (-\varepsilon, +\varepsilon) \times \{1\}$$

where $\varepsilon > 0$ is abitrary. Then we define a local chart $\varphi^+ : \pi(I^+) \to (-\varepsilon, +\varepsilon) \subseteq \mathbb{R}$ such that

$$\varphi^+([t, 1]) = t \text{ for all } [t, 1] \in \varphi(I^+)$$

Notice that φ^+ is well-defined, continuous and bijective since

$$\varphi^+ \circ \pi((t, 1)) = t$$

is continuous and $(\varphi^+)^{-1} = [(t, 1)] = \pi(i(t))$ is also continuous. Thus, φ is homeomorphism and hence M is locally Euclidean.

To prove M is second-countable, it suffices to prove X is countable, we define the set

$$\mathcal{B}_X = \{(a, b) \times \{1\}, (a, b) \times \{-1\} \mid a, b \in \mathbb{Q}\}$$

Since the set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ admits a countable basis for $\mathbb{R}$, therefore $\mathcal{B}_X$ is a countable basis of X. We define the set

$$\mathcal{B}_M = \{\pi(B) \mid B \in \mathcal{B}_X\},$$

Since π is a quotient map, $\{\mathcal{B}_M\}$ is a second-countable basis for M.

Now we aim to prove M is not Hausdorff at two points $(0, 1)$ and $(0, -1)$. Let U and V be abitrary neighborhood of $[(0, 1)]$ and $[(0, -1)]$ in M. Since $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open in Euclidean, one can find an open interval

$$(-\varepsilon_1, +\varepsilon_1) \times \{1\} \subseteq \pi^{-1}(U) \text{ and } (-\varepsilon_1, +\varepsilon_1) \times \{-1\} \subseteq \pi^{-1}(V)$$

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $I = (-\varepsilon_0, +\varepsilon_0)$, we thus have

$$\varnothing \neq \pi(I \times \{1\}) = \pi(I \times \{-1\}) \subseteq \pi(\pi^{-1}(U) \cap \pi^{-1}(V)) \subseteq U \cap V$$

Hence $U \cap V \neq \varnothing$ which implies M is not Hausdorff. $\square$

Problem 1.2. For some $t \in \mathbb{R}$, we denote the set

$$\mathbb{R}_t = \mathbb{R} \times \{t\}$$

Let I be an uncountable set, prove that the set

$$\mathcal{R} = \bigsqcup_{\alpha \in I} \mathbb{R}_\alpha$$

is locally Euclidean and Hausdorff, but not second-countable.

Proof. Let $u \in \mathcal{R}$. Then there exists a unique $\alpha \in I$ such that $u \in \mathbb{R}_\alpha$. Let $\psi_\alpha : \mathbb{R} \to \mathbb{R}_\alpha$ be the homeomorphism satisfying

$$\psi(x) = (x, \alpha),$$

Let $y = \psi^{-1}(u)$ and $U = (y - \varepsilon, y + \varepsilon) \times \{\alpha\}$ be an open neighborhood of u pulled back by ψ. Since ψ is a homeomorphism, it serves as a local chart around u, which implies $\mathcal{R}$ is locally Euclidean.

To prove $\mathcal{R}$ is Hausdorff, let $x = (u, \alpha), y = (v, \beta)$ be distinct points in $\mathcal{R}$, we consider two cases:

Case 1: $\alpha \neq \beta$, let $\varepsilon > 0$ be abitrary, we choose

$$U_\alpha = (u - \varepsilon, u + \varepsilon) \times \{\alpha\},$$
$$V_\beta = (v - \varepsilon, v + \varepsilon) \times \{\beta\}$$

Since U_α and V_β are disjoint, $\mathcal{R}$ is Hausdorff in this case.

Case 2: $\alpha = \beta$. We choose

$$U_\alpha = (u - \varepsilon, u + \varepsilon) \times \{\alpha\},$$
$$V_\beta = (v - \varepsilon, v + \varepsilon) \times \{\beta\},$$

where ε satisfies $0 < \varepsilon < \dfrac{|u - v|}{2}$. Since U_α and V_β are disjoint, every pair of distinct points in $\mathcal{R}$ can be seperated by disjoint open neighborhoods, hence $\mathcal{R}$ is Hausdorff in general.

To prove $\mathcal{R}$ is not second-countable, we consider the following proposition

Proposition 1.1. If a topological space contains uncountably many nonempty disjoint sets, then it is not second-countable.

For every $\alpha \in I$, we denote a corresponding open neighborhood $U_\alpha = (-\varepsilon, +\varepsilon) \times \{\alpha\}$. Since the collection $\{U_\alpha\}_{\alpha \in I}$ consists of uncountably many pairwise disjoint open sets in $\mathcal{R}$, the above proposition implies $\mathcal{R}$ is not second-countable. $\square$

Problem 1.3. Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.

1. Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that U is locally finite.

2. Give an example to show that the converse to (a) may be false.

3. Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

Problem 1.4. Suppose M is a locally Euclidean Hausdorff space. Show that M is secondcountable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

Problem 1.5. Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0, F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

Problem 1.6. Let N denote the north pole $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \ldots, 0, -1)$. Define the stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}$$

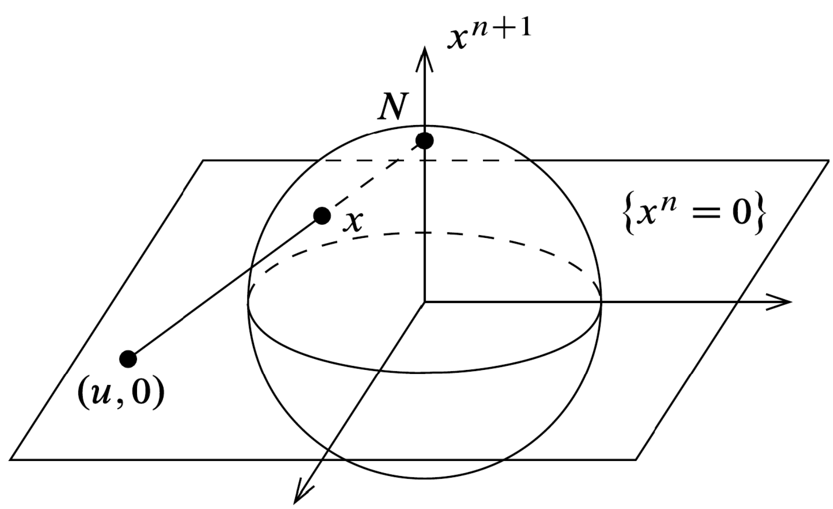
Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

1. For any $x \in \mathbb{S}^n \backslash \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called stereographic projection from the south pole.)

2. Show that σ is bijective, and

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

3. Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$ (The coordinates defined by σ or $\tilde{\sigma}$ are called stereographic coordinates).



Proof. (1) Define the line passing through N and x by

$$L(t) = N + t(x - N) = u(t)$$

Consider the intersection between $L(t)$ and the linear subspace of $\mathbb{R}^{n+1}$ where $x^{n+1} = 0$, we have

$$u(t) = (tx^1, \ldots, tx^n, 1 + t(x^{n+1} - 1)) = (u^1, \ldots, u^n, 0)$$

The $(n+1)-$term implies that $t = \dfrac{1}{1-x^{n+1}}$, thus we have

$$u(t) = \frac{(x^1, \dots, x^n)}{1-x^{n+1}} = \sigma(x)$$

(2) Let $u = (u^1, \dots, u^n) \in \mathbb{R}^n$, it suffices to prove that there exists $(x^1, \dots, x^{n+1}) \in \mathbb{S}^n \backslash \{N\}$ satisfying

$$u^i = \frac{x^i}{1-x^{n+1}} \text{ for all } i = 1, \dots, n.$$

Let $|u|^2 = \sum (u^i)^2$, it follows that

$$\begin{aligned} |u|^2 &= \frac{\sum (x^i)^2}{(1-x^{n+1})^2} = \frac{1-(x^{n+1})^2}{(1-x^{n+1})^2} = \frac{1+x^{n+1}}{1-x^{n+1}} = \frac{2}{1-x^{n+1}} - 1 \\ \Leftrightarrow x^{n+1} &= \frac{|u|^2-1}{|u|^2+1} \end{aligned}$$

Since $-1 < \dfrac{|u|^2-1}{|u|^2+1} < 1$, set $x^{n+1} = \dfrac{|u|^2-1}{|u|^2+1}$ and $x^i = \dfrac{2u^i}{|u|^2+1}$. Thus σ is surjective onto $\mathbb{R}^n$.

To prove that σ is injective, suppose $\sigma(x) = \sigma(y)$, we have

$$\begin{aligned} \frac{x^i}{1-x^{n+1}} = \frac{y^i}{1-y^{n+1}} &\Rightarrow \frac{\sum (x^i)^2}{(1-x^{n+1})^2} = \frac{\sum (y^i)^2}{(1-y^{n+1})^2} \\ &\Leftrightarrow \frac{1+x^{n+1}}{1-x^{n+1}} = \frac{1+y^{n+1}}{1-y^{n+1}} \\ &\Leftrightarrow \frac{2}{1-x^{n+1}} - 1 = \frac{2}{1-y^{n+1}} - 1 \\ &\Leftrightarrow \frac{2}{1-x^{n+1}} = \frac{2}{1-y^{n+1}} \\ &\Leftrightarrow 1-x^{n+1} = 1-y^{n+1}, \end{aligned}$$

which implies $x^{n+1} = y^{n+1}$ and hence $x^i = y^i$ for all $i = 1, \dots, n$. Therefore σ is bijective and its inverse satisfies

$$\sigma^{-1}\left(u^1, \dots, u^n\right) = \frac{\left(2u^1, \dots, 2u^n, |u|^2 - 1\right)}{|u|^2+1}$$

We will verify that $\sigma^{-1}(u) \in \mathbb{S}^n$ for all u. Let $(x^i) = \sigma^{-1}(u)$, since we have

$$\sum (x^i)^2 = \frac{\sum (2u^i)^2 + (|u|^2-1)^2}{(|u|^2+1)^2} = \frac{4|u|^2 + (|u|^2-1)^2}{(|u|^2+1)^2} = \frac{(|u|^2+1)^2}{(|u|^2+1)^2} = 1$$

Thus σ^{-1} maps every point in $\mathbb{R}^n$ into the sphere $\mathbb{S}^n$. (3) By the definition of $\tilde{\sigma}$, it can be written as

$$\tilde{\sigma}(x^1, \dots, x^{n+1}) = \frac{(x^1, \dots, x^n)}{1+x^{n+1}},$$

and the same computation shows that $\tilde{\sigma}$ is bijective. Let $u = (u^i) \in \mathbb{R}^n \backslash \{0\}$, the composition $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is computed by expressing

$$\begin{aligned} \tilde{\sigma} \circ \sigma^{-1}(u) &= \tilde{\sigma}\left(\frac{2u^1}{|u|^2+1}, \dots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right) \\ &= \frac{(u^1, \dots, u^n)}{|u|^2}. \end{aligned}$$

Since $\tilde{\sigma} \circ \sigma^{-1}$ is smooth and a diffeomorphism, σ and $\tilde{\sigma}$ are compatible. Moreover, since the union of their domains covers entire $\mathbb{S}^n$, then they form an atlas which generates a smooth structure on $\mathbb{S}^n$.

$\square$

Problem 1.7. By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An angle function on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Proof. ($\Rightarrow$) Suppose there exists an angle function $\theta : \mathbb{S}^1 \to \mathbb{R}$ which is continuous. We aim to show that θ is discontinuous at $z = 1$. If we write $z = e^{i\pi\phi}$ then it follows that $\theta = \pi\phi + k2\pi$, where k is some integer. Since θ is continuous, k must be fixed. Let

$$a_n = e^{i\pi/n} \text{ and } b_n = e^{i(2\pi - 1/n)}$$

Then we have

$$\lim_{n\to+\infty} \theta(a_n) = \theta(0) = k2\pi \text{ and } \lim_{n\to+\infty} \theta(b_n) = \theta(2\pi) = 2\pi + k2\pi$$

Since $2\pi \neq 0$, θ is not continuous at $z = 1$, hence there is no angle function for the case $U = \mathbb{S}^1$. We define $\alpha(z)$ as a unique function satisfying $z = e^{i\pi\alpha(z)}$ and $\alpha(z) \in [0, 2\pi)$.

If $U \neq \mathbb{S}^1$. In case that $1 \in U$, since there must exsits $z_0 \neq 1$ and $z_0 \notin U$. Since α is continuous, we choose θ satisfying

$$\theta(z) = \alpha(z), (\alpha(z) < \alpha(z_0)) \text{ and}$$
$$\theta(z) = 2\pi - \alpha(z), (\alpha(z) \geq \alpha(z_0))$$

If $1 \notin U$, we choose $\theta(z) = \alpha(z)$ for all $z \in U$.

Since θ is continuous and a homeomorphism on from open subset $U \subset \mathbb{S}^1$ onto an open interval $\theta(U)$, the pair (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. $\square$

Problem 1.8. Complex projective n-space, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$\left(x^1 + iy^1, \ldots, x^{n+1} + iy^{n+1}\right) \leftrightarrow \left(x^1, y^1, \ldots, x^{n+1}, y^{n+1}\right)$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.) (Used on pp. 48, 96, 172, 560, 561.)

Proof. Suppose $\mathbb{S}^n$ is an n-dimensional sphere, it suffices to prove the restriction map

$$\pi|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{CP}^n$$

is continuous and surjective. For the sake of condition, we assume to write π instead of $\pi|_{\mathbb{S}^n}$. Given $z \in \mathbb{CP}^n$, by the definiton of projective space, z is an equivalent class satisfying

$$[z^0 : \cdots : z^n] \sim [\lambda z^0 : \cdots : \lambda z^n]$$

for all nonzero complex number λ. To prove π is surjective, let $[z] \in \mathbb{CP}^n$ be abitrary, one can rewrite

$$[z] = [z^0 : \cdots : z^n] \sim \left[\frac{z^0}{|z|} : \cdots : \frac{z^n}{|z|}\right]$$

Since we have

$$\sum \frac{(z^i)^2}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

then $\left(\frac{z^0}{|z|} : \cdots : \frac{z^n}{|z|}\right) \in \mathbb{S}^n$ and hence it follows that $\pi\left(\frac{z^0}{|z|} : \cdots : \frac{z^n}{|z|}\right) = [z]$. Since the natural projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ is already continuous, the its restriction on closed subset $\mathbb{S}^n$ is also continuous. Since

$\mathbb{S}^n$ is closed and bounded in $\mathbb{C}^{n+1}$ by the corresponding identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$, by the Heine-Borrel theorem, $\mathbb{S}^n$ is compact. And since $\pi_{\mathbb{S}^n}(\mathbb{CP}^n) = \mathbb{S}^n$, $\mathbb{CP}^n$ is a compact.

To prove $\mathbb{CP}^n$ is locally Euclidean, for each $i = 0, \dots, n$, let $\tilde{U}_i \subset \mathbb{C}^{n+1}$ be the subset containing all points $x \in \mathbb{C}^{n+1}$ satisfying $x^i \neq 0$ and $\varphi_i : U_i \to \mathbb{C}^n$ be the local chart (where $U_i = \pi(\tilde{U_i})$) satisfying

$$\varphi[z^0 : \cdots : z^n] = \left(\frac{z^0}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i}\right).$$

This map is well-defined since the left-hand side remains if we replace $[z^1 : \cdots : z^n]$ by $\lambda[z^1 : \cdots : z^n]$. To prove φ_i is injective, suppose $\varphi_i[a] = \varphi_i[b]$, then we have

$$\frac{a^j}{a^i} = \frac{b^j}{b^i} \text{ for all } j = 0, \dots, n+1$$

Let $\lambda = \dfrac{a^i}{b^i}$ be fixed, it follows that $a^j = \lambda b^j$ for all $j = 0, \dots, n+1$, hence $[a] = [b]$. To prove φ_i is surjective, if $x \in \mathbb{C}^n$, we choose $[z] \in \mathbb{CP}^n$ satisfying $z^j = x^{j-1}$ for all $j \neq i$ and $z^i = 1$, this implies $\varphi_i[z] = x$. Therefore φ_i is bijective. Moreover, φ is smooth and its inverse given by

$$\varphi_i^{-1}(x^i) = [x^0, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n]$$

is also smooth. Thus φ_i is homeomorphism and $\mathbb{C}^n \cong \mathbb{R}^{2n}$ implies that $\mathbb{CP}^n$ is locally $2n$-dimensional Euclidean. In particular, since φ_i is diffeomorphism, it follows that (U_i, φ_i) is smooth local chart and they are parwise compatible since a composition of two diffeomorphism is again a diffeomorphism. Hence, the atlas $\{(U_i, \varphi_i)\}$ defines a smooth structure on $\mathbb{CP}^n$.

To prove $\mathbb{CP}^n$ is Hausdorff, let $[a], [b]$ be distinct equivalence classes in $\mathbb{CP}^n$, which means $[a] \neq [b]$, pushed forward by φ_i, where i is the non-negative integer satisfying $a^i, b^i \neq 0$. By consider two open disks

$$\begin{aligned}\mathcal{D}_{\varphi_i(a)} &= \{z \in \mathbb{C}^n \mid |z - \varphi_i(a)| < \varepsilon\} \text{ and}\\ \mathcal{D}_{\varphi_i(b)} &= \{z \in \mathbb{C}^n \mid |z - \varphi_i(b)| < \varepsilon\}\end{aligned}$$

where $\varepsilon > 0$ satisfies $\varepsilon < \dfrac{|\varphi(a) - \varphi(b)|}{2}$ implying $\mathcal{D}_{\varphi(a)} \cap \mathcal{D}_{\varphi(b)} = \varnothing$. Since φ_i is homeomorphism, then any distinct points in $\mathbb{CP}^n$ can be seperated by two open disjoint neighborhoods, which is $\mathcal{D}_{\varphi(a)}$ and $\mathcal{D}_{\varphi(b)}$ in this case. Therefore $\mathbb{CP}^n$ is Hausdorff.

The fact that $\mathbb{C}^n$ is second-countable and $\{(U_i, \varphi_i)\}$ is a smooth structure implies that $\mathbb{CP}^n$ is also second-countable. In conclusion, $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, as desired.

□

Problem 1.9. Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \dots, e_k)$ and $(e_{k+1}, \dots, e_n)$, respectively, where e_i is the i th standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k \times k$ identity matrix.

Problem 1.10. Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma : \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

Problem 1.11. Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Chapter 2

Partition of Unity

1 Construction

Lemma 2.1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-1/t}, t > 0, \\ 0, t \leq 0, \end{cases}$$

is smooth.

Lemma 2.2. Given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) = 1$ for $t \leq r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$ and $h(t) = 0$ for $t \geq r_2$.

Proof. Let f be the smooth function of the previous lemma, and set

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$$

Then h satisfies the desired properties. □

Lemma 2.3. Given any positive numbers $r_1 < r_2$, there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H = 1$ on $\overline{B}_{r_1}(0)$, $0 < H < 1$ for all $x \in B_{r_2}(0) \backslash \overline{B}_{r_1}(0)$ and $H = 0$ on $\mathbb{R}^n \backslash B_{r_2}(0)$.

Proof. By setting $H(x) = h(|x|)$ and we are done. □

Definition 2.4. Let $\mathcal{X} = (X_\alpha)_{\alpha\in}$ be an abitrary open cover of M. A *partition of unity subordinate to* $\mathcal{X}$ is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ satisfying the following:

1. $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.

2. $\text{supp}\psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.

3. Every point has a neighborhood that intersects $\text{supp}\psi_\alpha$ for finite values of α.

4. $\displaystyle\sum_{\alpha\in A} \psi_\alpha = 1$ for all $x \in M$.

Theorem 2.5. Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha\in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

Proof. Naturally, if we can find an indexed family support where each of them is a regular coordinate ball, then the construction of smooth function satisfying those following conditions is possible. However, its worth noting that the given open cover $\mathcal{X}$ is not locally finite. Therefore, our idea to find an indexed locally finite refinement of $\mathcal{X}$ and every element also a regular coordinate ball.

The fact that M is smooth manifold implies that there exsists an atlas $\{(U_i, \varphi_i)\}$ where $\{U_i\}$ is a basis for the topology of M and we can define every regular coordinate ball by some charts of this atlas. Since every X_α is itself a smooth manifold, thus it has a basis of regular coordinate balls $\mathcal{B}_\alpha$, and then $\mathcal{B} = \bigcup_{\alpha\in A} \mathcal{B}_\alpha$ defines a basis for the topology on M. Since M is Hausdorff and second-countable, hence it is paracompact, then there exists a subset of $\mathcal{B}$, denoted by $\{B_i\}$, is a locally finite open refinement of $\mathcal{X}$, and hence $\{\overline{B_i}\}$ is also locally finite. Since each B_i is an open subset of some X_α, then there exists a larger coordinate ball $\tilde{B}_i$ of X_α such that $\tilde{B}_i \supset B_i$ and a corresponding local chart $\varphi_i : \tilde{B}_i \to \mathbb{R}^n$ that maps $\varphi(B_i) = B_{r_1}(0)$ and $\varphi(\tilde{B}_i) = B_{r_2}(0)$, where $r_1 < r_2$ are two positive real numbers. Then we can define a smooth function $f_i : M \to \mathbb{R}$ as follows:

$$f_i(x) = \begin{cases} H_i \circ \varphi_i(x) \text{ on } \tilde{B}_i \\ 0 \text{ on } M \backslash \overline{\tilde{B}_i} \end{cases}$$

where H_i is the smooth function defined in the previous lemma for $B_{r_1}(0)$ and $B_{r_2}(0)$.Consequently, it follows that $\text{supp} f_i = \overline{\tilde{B}_i}$ for all i. Since every f_i is non-negative everywhere on M and each point in M is contained by some B_i, then the function $f : M \to \mathbb{R}$ defined by

$$f(x) = \sum_i f_i(x)$$

never vanishing to zero and f is well-defined since the following sum is finite for all point in M. Hence, f is smooth. Let $g_i := \dfrac{f_i}{f}$, we thus have

$$\sum_i g_i(x) = 1 \text{ for all } x \in M.$$

For every $\alpha \in A$, we define ψ_α as the partition sum of g_i satisfying

$$\psi_\alpha = \sum_{i|\tilde{B}_i \subseteq X_\alpha} g_i,$$

This partition of ψ_α satisfies $\sum_{\alpha\in A} \psi_\alpha = 1$ and

$$\text{supp}\psi_\alpha \subseteq \bigcup_{i|\tilde{B}_i \subseteq X_\alpha} \text{supp} g_i \subseteq X_\alpha,$$

and is a smooth function as we can verify. Hence the indexed family $\{\psi_\alpha\}_{\alpha\in A}$ is a smooth partition of unity subordinate to $\mathcal{X}$. $\square$

Definition 2.6. Let M be a topological space, $A \subseteq M$ is a closed subset and $U \subseteq M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for A supported in U* if $0 \le \psi \le 1$ on M, $\psi = 1$ on A and $\text{supp}\psi \subseteq U$.

Theorem 2.7. Let M be a smooth manifold. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.

Proof. Since the collection $\{M\backslash A, U\}$ is an open cover of M, there exists a partition of unity $\{\psi_1, \psi_2\}$ subordinate to M, where $\text{supp}\psi_2 \subseteq U$ and $\psi_2 = 1$ on A since $\psi_1 = 0$ on A. Thus ψ_2 is a bump function for A supported in U. $\square$

Theorem 2.8. Suppose M is a smooth manifold, $A \subseteq M$ is a closed subset, and $f : M \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\text{supp}\tilde{f} \subseteq U$.

Proof. Since every smooth function on closed subset can be extended into another smooth function on a small neighborhood, for each $p \in A$, choose an open neighborhood $W_p \subseteq U$ containing p such that there exists $\tilde{f}_p : W_p \to \mathbb{R}^k$ as a smooth function that agrees with f on $W_p \cap A$. Let $W_0 = M \backslash A$, then the collection $\{W_p\}_{p\in A} \cup \{W_0\}$ induces an open cover on M. Then there exists a smooth partition of unity $\{\psi_p\}_{p\in A} \cup \{\psi_0\}$ subordinate to M. One can define $\tilde{f} : M \to \mathbb{R}^k$ by

$$\tilde{f}(x) = \sum_{p\in A} \psi_p(x)\tilde{f}_p(x) \text{ for all } x \in M$$

Since every product $\psi_p\tilde{f}_p$ is a smooth function and $\tilde{f}$ is well-defined, $\tilde{f}$ is thus a smooth function. Moreover, it's easy to verify that $\mathrm{supp}\tilde{f} \subseteq \bigcup_{p\in A} W_p \subseteq U$ and $\tilde{f}$ agrees with f on A since

$$f\tilde{(x)} = \sum_{p\in A} \psi_p(x)f(x) = \left(\sum_{p\in A} \psi_p(x)\right) f(x) = f(x) \text{ for all } x \in A.$$

Thus, $\tilde{f}$ is indeed an extension of f and $\mathrm{supp}\tilde{f} \subseteq U$. □

Theorem 2.9. Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

Proof. Since every smooth coordinate balls is diffeomorphic to $\mathbb{R}^n$, it suffices to prove there exsits a desired function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = K$, where K is a closed subset of $\mathbb{R}^n$. For every $x \in M\backslash K$, there exists a real number $r > 0$ such that $B_r(x) \subseteq M\backslash K$. Then $M\backslash K$ is the union of countably many such balls $\{B_{r_n}(x_n)\}$. We wish to construct a smooth nonnegative function $f : M \to \mathbb{R}$ such that f vanishes to zero once x reach outside all of those coordinate balls. Let $H : \mathbb{R}^n \to R$ be a smooth bump function that $H = 1$ on $B_q(0)$ and supported in $B_1(0)$, where q is abitrary positive number that $q < 1$.

Since we need f to be nonnegative if x lies in some $B_{r_i}(x_i)$, let $H_i(x) = H\left(\dfrac{x - x_i}{r_i}\right)$ for all i, one can express f as a countably infinite nonnegative linear combination of H_i's. For each positive integer n, let M_i be the bounded constant of the absolute value of h and all of its partial derivation up to order n. We define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(r_n)^n}{2^n M_n} H_n(x)$$

Since every term of this series is nonnegative, bounded by the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, and is continuous, it follows that f is well-defined and continuous. Let $k \in \mathbb{N}$, consider the partial derivation of order k

$$\left\|d^k f(x)\right\| = \left\|\sum_{n=1}^{\infty} \frac{(r_n)^{n-k}}{2^n M_n} d^k H_n(x)\right\| \le \sum_{n=k}^{\infty} \frac{(r_n)^{n-k}}{2^n M_n} \left\|d^k H_n(x)\right\| \le \sum_{n=k}^{\infty} \frac{(r_n)^{n-k}}{2^n M_n} M_n = \sum_{n=k}^{\infty} \frac{(r_n)^{n-k}}{2^n},$$

which is a convergent series, by the criteria of series, $d^k f(x)$ is well-defined for all k, and is continuous. Hence f is smooth.

Let $\{B_\alpha\}_{\alpha\in A}$ be an open cover of M by smooth coordinate balls and K be any closed subset of M. Consider the partition of unity $\{\psi\}_{\alpha\in A}$ subordinate to M. For every $\alpha \in A$, since B_α is diffeomorphic to $\mathbb{R}^n$, then there exsits a smooth nonnegative function $f_\alpha : B_\alpha \to \mathbb{R}$ such that $f_\alpha^{-1}(0) = K \cap B_\alpha$. Let $f(x) = \sum_{\alpha\in A} \psi_\alpha(x) f_\alpha(x)$, it follows that

$$f^{-1}(0) = \bigcup_{\alpha\in A} f_\alpha^{-1}(0) = \bigcup_{\alpha\in A} K \cap B_\alpha = K,$$

as desired. □

2 Problems

Problem 2.1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\left(U \cap f^{-1}(V)\right)$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Problem 2.2. Prove Proposition 2.12 (smoothness of maps into product manifolds).

Problem 2.3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

1. $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the n-th power map $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
2. $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map $\alpha(x) = -x$.
3. $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\left\{(w, z) : |w|^2 + |z|^2 = 1\right\}$ of $\mathbb{C}^2$.

Proof. (1) Since $\mathbb{S}^1 \subset \mathbb{C}$, consider the global coordinate chart $\varphi : \mathbb{S}^1 \to [0, 2\pi)$ satisfying

$$\varphi(z) = \varphi(e^{i\theta}) = \theta \in [0, 2\pi)$$

Thus, the coordinate representation in this case is

$$\tilde{f(x)} = \varphi \circ f \circ \varphi^{-1}(\theta) = n\theta$$

Since φ is smooth and f is a smooth map, it follows that $\tilde{f}$ is also smooth.

(2) Consider the stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{C}^n$ satisfying

$$\sigma\left(z^1, \ldots, z^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}.$$

Then the coordinate representation is computed by

$$\tilde{f}(x) = \sigma \circ f \circ \sigma^{-1}(u) = \sigma\left(\frac{-2u^1}{|u|^2 + 1}, \ldots, \frac{-2u^n}{|u|^2 + 1}, \frac{1 - |u|^2}{|u|^2 + 1}\right) = (-u^1, \ldots, -u^n),$$

which is smooth, the same construction on $\mathbb{S}^n \backslash \{S\}$.

$\square$

Problem 2.4. Show that the inclusion map $\overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary. 2-5. Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

1. Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.
2. Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Problem 2.5. Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{k+1} \backslash \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}, P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \backslash \{0\}$ and $x \in \mathbb{R}^{n+1} \backslash \{0\}$. (Such a function is said to be homogeneous of degree $\boldsymbol{d}$.) Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

Proof. We first prove that $\tilde{P}$ is well-defined. Suppose that $[x] = [y]$, it suffices to prove that $\tilde{P}([x]) = \tilde{P}([y])$ or $[P(x)] = [P(y)]$. Since we have the relation

$$[x^1, \dots, x^{n+1}] = [\lambda x^1, \dots, \lambda x^{n+1}] \text{ for all } \lambda,$$

it follows that

$$\tilde{P}([\lambda x]) = [P(\lambda x)] = [\lambda P(x)] = [P(x)] = P([\lambda x]).$$

Thus $\tilde{P}$ is well-defined. To prove $\tilde{P}$ is smooth, for each $i = 0, \dots, n$, let $\tilde{U}_i \subset \mathbb{R}^{k+1}$ be the subset containing all points $x \in \mathbb{R}^{n+1}$ satisfying $x^i \neq 0$ and $\varphi_i : U_i \to \mathbb{R}^k$ be the local chart (where $U_i = \pi(\tilde{U}_i)$, and π is a natural quotient mapping from $\mathbb{R}^{k+1}$ to $\mathbb{RP}^n$) satisfying

$$\varphi_i[x^1, \dots, x^{k+1}] = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{k+1}}{x^i}\right).$$

As proven above, φ_i is well-defined and smooth, and $\{\varphi_i\}$ defines a smooth structure on $\mathbb{RP}^k$. Let $x \in \mathbb{RP}^n$ and a neighborhood U_i containing x, it suffices to prove the map $\varphi_i \circ \tilde{P} \circ \varphi_i^{-1} : \varphi(U) \to \varphi(U)$ is smooth. Computating sufficiently yields

$$\begin{aligned}\varphi_i \circ \tilde{P} \circ \varphi_i^{-1}(x) &= \varphi_i \circ \tilde{P}([x^1, \dots, x^{i-1}, 1, \dots, x^k]) \\ &= \varphi_i([P(x^1, \dots, x^{i-1}, 1, \dots, x^k)]).\end{aligned}$$

Let $P_i([x]) = P([x^1, \dots, x^{i-1}, 1, \dots, x^k])$, we have

$$\begin{aligned}\varphi_i([P(x^1, \dots, x^{i-1}, 1, \dots, x^k)]) &= \varphi \circ P_i([x]) \\ &= \left(\frac{P_i^1}{P_i^i}, \dots, \frac{P_i^{i-1}}{P_i^i}, \frac{P_i^{i+1}}{P_i^i}, \dots, \frac{P_i^{k+1}}{P_i^i}\right).\end{aligned}$$

Since every component of P is smooth, then the following composition is also smooth. Thus $\tilde{P}$ is smooth, as desired. □

Problem 2.6. Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional. [Hint: show that if $f_1, \dots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]

Proof. We first prove that M contains infinite closed subset. Since M is locally Euclidean, there exists a smooth local chart (U, φ) which maps the open subset $U \subseteq M$ into $\tilde{U} = \varphi(U)$, which is open in $\mathbb{R}^n$. Then we can find an open ball $B(x, q) \subseteq U$, and it contains infinitely disjoint open balls, denoted by the set $\{B(x_\alpha, q_\alpha)\}_{\alpha \in A}$, where A is a countably infinite set. Since φ is a homeomorphism, one can consider the pull back $\{\varphi^{-1}(\overline{B(x_\alpha, q_\alpha)})\}_\alpha$ as a disjoint collection of closed subsets of M.

In particular, let $U \subseteq M$ be an closed subset, it suffices to construct a smooth function $f \in C^\infty(M)$ which support

$$\operatorname{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

is a subset of U. We consider the following lemma

Lemma 2.1. Suppose M is a smooth manifold with or without boundary, $A \subset M$ is a closed subset, and $f : A \to \mathbb{R}^k$ is a a smooth function. For any open subset U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $f_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$

Proof. Use Partition of unity. □

The above lemma implies that there is a way to contruct a smooth function as required, and we denote the set of nonzero smooth functions $\{f_n\}_{n \in N}$ such that f_i has support is a subset of U_{α_i}. Now we supoose there exists $n \in \mathbb{N}$ and a n-tuple $(a_1, \dots, a_n) \in \mathbb{R}^n$ satisfying

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0 \text{ for all } x \in M$$

Since the support of $f_1, \dots, f_n$ are disjoint. For every $i = 1, \dots, n$ by choosing $x \in U_{\alpha_i}$, it follows that $f_i(x) = 0$ but $f_j(x) \neq 0$ for all $j \neq i$. We thus obtain a homogeneous system of equations

$$\begin{aligned} a_2 f_2 + a_3 f_3 + \cdots + a_n f_n &= 0 \quad \text{for all } x \in M \\ a_1 f_1 + a_3 f_3 + \cdots + a_n f_n &= 0 \quad \text{for all } x \in M \\ &\cdots \\ a_1 f_1 + a_2 f_2 + \cdots + a_{n-1} f_{n-1} &= 0 \quad \text{for all } x \in M \end{aligned} \tag{2.1}$$

which implies $a_i f_i = 0$ for all $i = 1, \dots, n$ and for all $x \in M$. Thus $a_1 = \cdots = a_n$. Therefore $\{f_1, \dots, f_n\}$ is linearly independent in $C^\infty(M)$, but since n is abitrary, $C^\infty(M)$ must be infinite-dimensional, as desired. □

Problem 2.7. Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F\left(x^1, \dots, x^n\right) = \left[x^1, \dots, x^n, 1\right]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G : \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G\left(z^1, \dots, z^n\right) = \left[z^1, \dots, z^n, 1\right]$ (see Problem 1-9).

Proof. Let U be the open subset of $\mathbb{R}^{n+1}$ where $x^{n+1} \neq 0$ and $\tilde{U} = \pi(U)$ is an open subset of $\mathbb{RP}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{RP}^n$ is a natural projection. It suffices to prove the restricted map $F : \mathbb{R}^n \to \tilde{U}$ is a diffeomorphism.

To prove F is injective, suppose $F(x) = F(y)$, then there exists $\lambda \in \mathbb{R}$ such that

$$(x^1, \dots, x^n, 1) = (\lambda y^1, \dots, \lambda y^n, \lambda),$$

which implies $\lambda = 1$ and hence $x = y$. To prove F is surjective, let $[y] \in \tilde{U}$ be abitrary, we have

$$[y^1, \dots, y^{n+1}] = \left[\frac{y^1}{y^{n+1}}, \dots, \frac{y^n}{y^{n+1}}, 1\right] = F\left(\frac{y^1}{y^{n+1}}, \dots, \frac{y^n}{y^{n+1}}\right).$$

Thus, F is a bijection. Since the map $F : \mathbb{R}^n \to \mathbb{RP}^n$ is smooth, then $F : \mathbb{R}^n \to \tilde{U}$ is also smooth. Therefore, F is a diffeomorphism onto $\tilde{U}$. Now we prove $\tilde{U}$ is a dense subset of $\mathbb{RP}^n$. Let $x \in \mathbb{RP}^n \backslash \tilde{U}$, and i be the positive integer such that $x_i \neq 0$, consider the sequence $(x_n) \in \mathbb{R}^{n+1}$ satisfying

$$x_n = \left(\frac{x^1}{x^i}, \dots, \frac{x^n}{x^i}, \frac{1}{n}\right), n \in \mathbb{N}$$

Then we have

$$\lim_{n \to +\infty} \pi([x_n]) = \left[\lim_{n \to +\infty} x_n\right] = \left[\frac{x^1}{x^i}, \dots, \frac{x^n}{x^i}, 0\right] = [x^1, \dots, x^n, 0] = [x]$$

Therefore, for any $[x] \in \mathbb{RP}^n \backslash \tilde{U}$, there exists $(x_n) \in \mathbb{R}^{n+1}$ such that $\pi(y_n) \to [x]$. Hence $\tilde{U}$ is dense in $\mathbb{RP}^n$, as desired. □

Problem 2.8. Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G : \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "G"] \arrow[d, "p"'] & \mathbb{CP}^1 \arrow[d, "\tilde{p}"] \\ \mathbb{C} \arrow[r, "G"] & \mathbb{CP}^1 \end{tikzcd}$$

Proof. It suffices to prove there exsits a unique map $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ satisfying $G \circ p = \tilde{p} \circ G$. Suppose $p(z) = a_n z^n + \cdots + a_1 z + a_0$. By computing sufficiently, we have

$$G \circ p(z) = [a_n z^n + \cdots + a_1 z + a_0, 1] = [p(z), 1].$$

Let $q(z_1, z_2) = \sum_{i=0}^{n} a_i z_1^i z_2^{n-i}$, this implies $q(z_1, z_2) = z_2^n P\left(\frac{z_1}{z_2}\right)$ if $z_2 \neq 0$, then we can construct the map $\tilde{p}$ satisfying

$$\tilde{p}(z_1, z_2) = [q(z_1, z_2), z_2^n].$$

One can verify that $\tilde{p} \circ G(z) = \tilde{p}[z, 1] = [q(z, 1), 1] = [p(z), 1] = G \circ p(z)$. We now prove that $\tilde{p}$ is unique. Assume that there exists $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ satisfying $G \circ p = h \circ G$. If $z_2 \neq 0$, it follows that

$$h[z_1, z_2] = h\left[\frac{z_1}{z_2}, 1\right] = h \circ G\left(\frac{z_1}{z_2}\right) = \left[p\left(\frac{z_1}{z_2}\right), 1\right] = \left[z_2^n p\left(\frac{z_1}{z_2}\right), z_2^n\right] = \tilde{p}[z_1, z_2]$$

for all $z_1 \in \mathbb{C}$. Since h is smooth on $\mathbb{CP}^1$, it must be continuous at $[1, 0]$. Since p is a polynomial, then there exists $K \in \mathbb{N}$ such that $|p(k)| > 0$ for all real numbers $k > K$. Since h is continuous at $[1, 0]$, it follows that

$$h[1, 0] = \lim_{k \to +\infty} h\left[1, \frac{1}{k}\right] = \lim_{k \to +\infty} [p(k), 1] = \lim_{k \to +\infty} \left[1, \frac{1}{p(k)}\right] = [1, 0]$$

Therefore $h[z] = \tilde{p}[z]$ for all $[z] \in \mathbb{CP}^1$, which means $\tilde{p}$ is unique, as desired.

□

Problem 2.9. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.

1. Show that F^* is a linear map.
2. Suppose M and N are smocth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*\left(C^\infty(N)\right) \subseteq C^\infty(M)$.
3. Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

[Remark: this result shows that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subseteq C(M)$. In fact, some authors define a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties; see, e.g., [Nes03].] (Used on p. 75.)

Proof. 1. Since we have

$$F^*(f + g) = (f + g) \circ F = f \circ F + g \circ F,$$

and

$$F^*(\alpha f) = (\alpha f) \circ F = \alpha(f \circ F) = \alpha F^*(f).$$

Hence F^* is linear.

2. Suppose $F : M \to N$ is smooth. Since $F^*(f) = f \circ F$ is smooth in M, we thus have $F^*\left(C^\infty(N)\right) \subseteq C^\infty(M)$. To prove the converse implication, since $\mathrm{Id}_N \in C^\infty(N) \subset C(N)$, we have

$$F^*(\mathrm{Id}_N) = \mathrm{Id}_N \circ F = F \in C^\infty(M),$$

which implies F is smooth on M. □

Problem 2.10. Suppose V is a real vector space of dimension $n \geq 1$. Define the projectivization of V, denoted by $\mathbb{P}(V)$, to be the set of 1 -dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V \backslash \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}\left(\mathbb{R}^n\right) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold. and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E\left[v^1, \ldots, v^n\right] = \left[v^i E_i\right]$ (where brackets denote equivalence classes) is a diffeomorphism. (Used on p. 561.)

Problem 2.11. State and prove an analogue of Problem 2-11 for complex vector spaces.

Problem 2.12. Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

Problem 2.13. Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Proof. The theorem 2.9 implies that there exists two smooth nonnegative real valued f_1, f_2 satisfying $f_1^{-1}(0) = A$ and $f_2^{-1}(0) = B$. Then we set $f(x) = \frac{f_1(x)}{f_1(x)+f_2(x)}$, it follows that $f(x) = 1$ if and only if $f_2(x) = 0$, and $f(x) = 0$ if and only if $f_1 = 0$. Hence we are done.

However, the previous contruction relies mostly on the the fact that the preimage of two closed subsets A and B are simply 0 and 1, but not extend for other general situation. Therefore, we aim to construct a universal method step by step using partition of unity, which can be extended for more complicated cases. First, one can again choose two smooth nonnegative real valued f_1, f_2 such that $f_1^{-1}(0) = A$, $f_2^{-1}(1) = B$, and $0 \le f_1, f_2 \le 1$. Since M is Hausdorff and $M \backslash (A \cup B)$ is open, for every point x outside A and B, then we can choose an open coordinate all B_x for x such that $B_x \cap A = \varnothing$ and $B_x \cap B = \varnothing$, denoted by the collection $\{B_x\}$. Let $B = \{B_\alpha\}_{\alpha \in A} \cup \{B_\beta\}_{\beta \in A}$ be the union of abitrary smooth coordinate balls contained by disjoint open subset $U \supseteq A$ and $V \supseteq B$ for each point in A and B. Then there exsits a partition of unity $\{\psi\}$ subordinate to $B \cup \{B_x\}$. Consider the function

$$f = \sum_{\alpha \in A} \psi_\alpha f_1|_{B_\alpha} + \sum_{\beta \in B} \psi_\beta f_2|_{B_\beta} + \sum_{x \in M \backslash (A \cup B)} \psi_x = f_\alpha + f_\beta + f_x$$

If $x \in A$, then $f_\beta = f_x = 0$, and $f_\alpha = f_1 = 0$.

If $x \in B$, then $f_\alpha = f_x = 0$ and $f_\beta = f_2 = 1$. If $x \in M \backslash (A \cup B)$, then we have

$$f = \sum_{x \in M \backslash (A \cup B)} \psi_x$$

which implies that $0 < f < 1$ since every point is covered by finite coordinate balls.

We consider the case that $x \in U \backslash A$, using the fact that $0 \le f_1 \le$, we thus have

$$f = \sum \psi_\alpha f_1 + \sum_x \psi_x < \sum \psi_\alpha + \sum_x \psi_x < 1$$

and

$$f = \sum \psi_\alpha f_1 + \sum_x \psi_x \ge \sum_x \psi_x > 0$$

Since the case for $x \in V \backslash B$ is analogous, we obtain $f(x) = 1$ if and only if $x \in A$ and $f(x) = 0$ if and only if $x \in B$. Hence f satisfies the desired condition. $\square$

Chapter 3

Tangent Space

1 Derivatives in Multivariable Calculus

Definition 3.1. Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, $x_0 \in E$ be a point, and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We say that f is differentiable at x_0 with derivative L if we have

$$\lim_{x \to x_0, x \in E \setminus \{x_0\}} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0$$

Proposition 3.2. Suppose f is differentiable at x_0 with derivative L_1, and also differentiable at x_0 with derivative L_2. Then $L_1 = L_2$.

Proof. Since f is differentiable at x_0 with derivative L_1 and L_2, we have

$$\begin{aligned}\lim_{x \to x_0} \frac{\|L_1(x - x_0) - L_2(x - x_0)\|}{\|x - x_0\|} &\le \lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L_2(x - x_0)\|}{\|x - x_0\|} + \\ &\quad \lim_{x \to x_0} \frac{\|L_1(x - x_0) - (f(x) - f(x_0))\|}{\|x - x_0\|} \\ &= 0\end{aligned}$$

Let $h = x - x_0$, one obtain that

$$\lim_{h \to 0} \frac{\|L_1(h) - L_2(h)\|}{\|h\|} = 0$$

Given $x \in E$ and t be a scalar such that $t \to 0$, then it follows that $tx \to 0$. Since L_1 and L_2 are linear map, we have

$$\|L_1(x) - L_2(x)\| = \frac{\|L_1(tx) - L2(tx)\|}{\|tx\|} \to 0$$

Thus $L_1 = L_2$, we are done. $\square$

Definition 3.3. Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, let x_0 be an interior point of E, and let v be a vector in $\mathbb{R}^n$. If the limit

$$\lim_{t \to 0^+, x_0 + tv \in E} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exsists, we say that f is *differentiable in the direction v at x_0*, and we denote the above limit by $D_v f(x_0)$:

$$D_v f(x_0) := \lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}$$

Proposition 3.4. If f is differentiable at x_0, then f is also differentiable in the direction v at x_0, and

$$D_v f(x_0) = f'(x_0)v$$

Proof. This is trivial if $v = 0$, assume that $v \neq 0$. Since f is differentiable at a, then there exists a linear map $f'(a) : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$0 = \lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{\|h\|},$$

Replacing h by tv and $t \to 0$ yields

$$0 = \lim_{t \to 0^+} \frac{f(a+tv) - f(a) - tvf'(a)}{t\,\|v\|} = \lim_{t \to 0^-} \frac{f(a+tv) - f(a) - tvf'(a)}{t\,\|v\|}$$

Since $\|v\| \neq 0$, we thus have

$$0 = \lim_{t \to 0} \frac{f(a+tv) - f(a) - tvf'(a)}{t} = D_v|af = f'(a) \cdot v.$$

□

Definition 3.5. Let E be a subset of $\mathbb{R}^n$, let $f : E \to \mathbb{R}^m$ be a function let x_0 be an interior point of E, and let $1 \leq j \leq n$. Then the partial derivative of f respect to the x_j variable at x_0, denoted $\frac{\partial f}{\partial x_j}(x_j)$, is defined by

$$\frac{\partial f}{\partial x_j}(x_j) = \lim_{t \to 0, x_0 + te_j \in E} \frac{f(x_0 + te_j) - f(x_0)}{t} = \frac{d}{dt} f(x_0 + te_j)_{t=0}$$

Theorem 3.6. Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, F be a subset of E, and x_0 be an interior pint of F. If all the partial derivatives $\frac{\partial f}{\partial x_j}$ exist on F and are continuous at x_0, then f is differentiable at x_0. Moreover if $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, the linear transformation $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$f'(x_0)(v) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0)$$

Proof. We first suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that if v is a vector of $\mathbb{R}^n$ and $v = (v_1, v_2, \ldots, v_n)$, then

$$L(v) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0)$$

We attempt to prove that $L = f'(x_0)$, in other words, this is equivalent to

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - L(h)\|}{\|h\|}.$$

Indeed, given $\varepsilon > 0$, it suffices to find $\delta > 0$ such that whenever $h \in B(0, \delta) \backslash \{0\}$, we should have

$$\|f(x_0 + h) - f(x) - L(h)\| < \varepsilon \|h\|.$$

For every $1 \leq j \leq n$, since the partial derivative $\frac{\partial f}{\partial x_j}$ exists, the function $g_j(x) = \frac{\partial f}{\partial x_j}(x)$ is also continuous in at x_0. Thus we can find $\delta_j > 0$ such that whenever $h \in B(0, \delta_i) \backslash \{0\}$, we have $\|g_j(x) - g_j(x_0)\| < \frac{\varepsilon}{mn}$. Let $\delta = \min\{\delta_1, \ldots, \delta_n\}$ and fix $h \in B(0, \delta) \backslash \{0\}$, it is possible to determine the scalars $(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$ such that

$$h = h_1 e_1 + h_2 e_2 + \cdots + h_n e_n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is standard ordered basis of $\mathbb{R}^n$. Indeed, by writing f in components as $f = (f_1, \ldots, f_m)$, it would follow that

$$\frac{\partial f}{\partial x_j}(x_0) = \left(\frac{\partial f_1}{\partial x_j}(x_0), \frac{\partial f_2}{\partial x_j}(x_0), \ldots, \frac{\partial f_n}{\partial x_j}(x_0) \right). \tag{1}$$

Let $1 \le i \le n$, by the Mean Value Theorem, one may locate t_1 between 0 and h_1 such that

$$f_i(x_0 + h_1 e_1) - f_i(x_0) = \frac{\partial f_i}{\partial x_1}(x_0 + t_1 e_1) h_1.$$

In particular, since

$$\left\| \frac{\partial f_i}{\partial x_1}(x_0 + t_1 e_1) - \frac{\partial f_i}{\partial x_1}(x_0) \right\| \le \left\| \frac{\partial f}{\partial x_1}(x_0) - \frac{\partial f}{\partial x_1}(x_0) \right\| < \frac{\varepsilon}{mn}$$

Multiplying both sides by $\|h_1\|$ and replacing the left-side term with (1) obtains that

$$\left\| f_i(x_0 + h_1 e_1) - f_i(x_0) - \frac{\partial f_i}{\partial x_1}(x_0) \right\| < \frac{\varepsilon}{mn} \|h_1\| \le \frac{\varepsilon}{mn} \|h\|$$

whenever $1 \le i \le n$. Adding n above inequalities for specific $f_1, f_2, \dots, f_n$ and using triangle inequality that $\|(y_1, \dots, y_m)\| \le \|y_1\| + \cdots + \|y_m\|$ yields

$$\left\| f(x_0 + h_1 e_1) - f(x_0) - \frac{\partial f}{\partial x_1}(x_0) \right\| < \frac{\varepsilon \|h\|}{n}.$$

Similarly, by developing the analogous process with $x_0 + h_1 e_1$ and $x_0 + h_1 e_1 + h_2 e_2$, we thus have

$$\left\| f(x_0 + h_1 e_1 + h_2 e_2) - f(x_0 + h_1 e_1) - \frac{\partial f}{\partial x_1}(x_0) \right\| < \frac{\varepsilon \|h\|}{n}.$$

whenever $h \in B(0, \delta) \backslash \{0\}$. More generally, for each $0 \le k \le n-1$, one can obtain

$$\|u_k\| = \left\| f\left(x_0 + \sum_{j=1}^{k+1} h_j e_j \right) - f\left(x_0 + \sum_{j=1}^{k} h_j e_j \right) - \frac{\partial f}{\partial x_k}(x_0) \right\| < \frac{\varepsilon \|h\|}{n}.$$

Summing all these terms together implies

$$\begin{aligned}
\|u_1 + u_2 + \cdots + u_n\| &= \left\| f(x_0 + h_1 e_1 + \cdots + h_n e_n) - f(x_0) - \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(x_0) \right\| \\
&= \|f(x_0 + h) - f(x_0) - L(h)\| \\
&\le \|u_1\| + \|u_2\| + \cdots + \|u_n\| \\
&< \varepsilon \|h\|
\end{aligned}$$

Therefore, L is derivative of f at x_0, as desired.

$\square$

Definition 3.7. If $f : E \to \mathbb{R}$ is a real-valued function, and we define the gradient $df(x_0)$ of f at x_0 to be the n-dimensional row vector

$$df(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right)$$

2 Tangent Vectors of Euclidean Space

Definition 3.1. A derivation at $p \in \mathbb{R}^n$ is a linear map $\omega : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz product rule:

$$\omega(fg) = f\omega(g) + g\omega(f) \text{ for all smooth maps } g, f \in C^\infty(M)$$

We denote the set $T_p\mathbb{R}^n$ to be the set containing all kind of following derivation, that is

$$T_p\mathbb{R}^n := \{\omega : C^\infty(M) \to \mathbb{R} \mid \omega \text{ linear and Leibniz}\}.$$

Proposition 3.2. Let $a \in \mathbb{R}^n$, for each geometric tangent vector $v_a \in \mathbb{R}$, the map $d \cdot v_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at a. Moreover, the operation $v_a \mapsto d(a) \cdot v_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.

Proof. The fact $d \cdot v_a$ is a consequence of the linearity and Lebniz implies that it is a derivation. It suffices to prove that $\mathcal{L} : v_a \mapsto d \cdot v_a$ is an isomorphism.

For injectivity, suppose $\mathcal{L}(v_a) = \mathcal{L}(w_a)$ for some geometric tangent vector v_a, w_a, it folows that

$$\begin{aligned} 0 = \mathcal{L}(v_a - w_a)(x^j) &= d \cdot (v_a - w_a)(x^j) = \left(\frac{\partial}{\partial x^1}(x^j), \dots, \frac{\partial}{\partial x^n}(x^j) \right)(v_a - w_a) \\ &= \frac{\partial}{\partial x_i}(x^j)(v^i - w^i) = \delta_{ij}(v^i - w^i) = v^j - w^j, \end{aligned}$$

for all $j = 1, \dots, n$. Thus $\mathcal{L}$ is injective. To prove surjectivity, let $w \in T_a\mathbb{R}^n$ be abitrary and let v the the tangent vector of $\mathbb{R}^n_a$ such that $w(x^i) = v^i$, it suffices to prove that $w = d \cdot v_a$. By the Taylor's theorem, one can write

$$f(x) = f(a) + \left[\frac{\partial f}{\partial x^i}(a)(x^i - a^i) \right] + \left[(x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i x^j}(a + t(x-a))dt \right]$$

Differentiating both sides by w yields

$$w(f(x)) = w(f(a)) + \frac{\partial f}{\partial x^i}(a)(w(x^i) - a^i) = \frac{\partial f}{\partial x^i}(a)v^i = d \cdot v_a(a)$$

Hence, $\mathbb{R}^n_a$ is isomorphic to $T_a\mathbb{R}^n$ by the following operation. □

Definition 3.3 (Tangent Vectors on Manifolds). Let M be a smooth manifold, a derivation at $p \in M$ is a linear map $\omega : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz product rule:

$$\omega(fg) = f\omega(g) + g\omega(f) \text{ for all smooth maps } g, f \in C^\infty(M)$$

We denote the set $T_p(M)$ to be the set containing all kind of following derivation, that is

$$T_pM := \{\omega : C^\infty(M) \to \mathbb{R} \mid \omega \text{ linear and Leibniz}\}.$$

Definition 3.4 (The Differential of a Smooth Map). Let $F : M \to N$ be the smooth map between smooth manifolds, for each $p \in M$, we define the map

$$dF_p : T_pM \to T_{F(p)}N,$$

called the *differential of F at p* that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Proposition 3.5. The operation dF_p defined above is a derivation.

Proposition 3.6. Let M be a smooth manifold, $p \in M$ and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.

Proposition 3.7. If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.

Definition 3.8 (Second definition of Tangent Space). Let M be a smooth manifold and $p \in M$. We say every the smooth function $\zeta : (-\varepsilon, +\varepsilon) \to M$ such that $\zeta(0) = p$ is *a p-path.* Two p-path α and β is said to satisfies the $\sim$ equivalent relation if

$$\frac{d}{dt}(f(\alpha(t)))\bigg|_{t=0} = \frac{d}{dt}(f(\beta(t)))\bigg|_{t=0}$$

for all smooth map $f \in C^\infty(M)$. Then the tangent space at p is defined as:

$$T_p(M) := \{[\zeta'(0)] \mid \text{Smooth curve } \zeta : (-\varepsilon, +\varepsilon) \to M, \zeta(0) = p\}$$

Proposition 3.9. The tangent spaces at p in first and second definition are naturally isomorphic.

Proposition 3.10. If M is a smooth manifolds with dimension n and let $(x_1, x_2, \dots, x_n)$ be a smooth local chart around $p \in M$, then the set

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \frac{\partial}{\partial x_2}\bigg|_p, \dots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$$

forms a basis for T_pM.

For convenience, if $v \in T_pM$ we write $v = (dx_1, dx_2, \dots, dx_n)$.

Definition 3.11 (Tangent Bundle). Let M be a smooth manifold, we define the *tangent bundle of* M to be the disjoint union of the tangent spaces at all points on M

$$TM = \coprod_{p \in M} T_pM.$$

Proposition 3.12. For any n-dimensional smooth manifold M, the tangent bundle TM has a natural topology and a smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth.

Proof. We first prove that π is smooth. Consider the local coordinate chart (U, φ) for M which has the form $(x^1, \dots, x^n)$. Notice that $\pi^{-1}(U)$ is an open set in TM and

$$\pi^{-1}(U) = TU = \coprod_{p \in U} T_pM,$$

which motivates us to consider the local chart $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ satisfying

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}(p)\right) = (x^i(p), v^i) \text{ and } \tilde{\varphi}^{-1}(x^i, v^i) = v^i \frac{\partial}{\partial x^i}(\varphi^{-1}(x))$$

for all $v = v^i \dfrac{\partial}{\partial x^i} \in T_pM$ and $p \in U$. Let (V, ψ) and $(\pi^{-1}(V), \tilde{\psi})$ be the similarly defined smooth local chart on M and TM, respectively. Since φ and ψ is a homeomorphism, they are smoothly compatible. Then it suffices to verify the compatibility of the transition map $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \tilde{\varphi}(\pi^{-1}(U) \cap \pi^{-1}(V)) \to \tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V))$, which can be computed sufficiently

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^i(p), v^i) = \tilde{\psi}\left(v^i \frac{\partial}{\partial x^i}(p)\right) = (\tilde{x}^i(p), \tilde{v}^i)$$

which is clearly a smooth function. Since M is second-countable, one can choose a countable cover $\{U_i\}$ for M and a smooth structure $\{(U_i, \varphi_i)\}$ defined above. Hence we obtain a countable cover $\{\pi U_i\}$ for TM which implies that TM is locally Euclidean and second-countable. The Hausdorff property is trival since two points in the same fiber can be seperated by the same chart and in different fiber can be mapped through π onto M, which is Hausdorff. Hence TM is a manifold, and the corresponding smooth structure $\{(\pi^{-1}(U_i), \tilde{\varphi}_i)\}$ implies that TM is smooth manifold. Finally, to see that π is smooth, with respect to the charts (U, φ) and $(\pi^{-1}(U), \tilde{\varphi})$, the coordinate representation is $\pi(x, v) = x$. Hence we are done. $\square$

3 Problems

Problem 3.1. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. Since the converse implication is trival, we only prove the forward one. Let $(x^1, \ldots, x^n)$ be a local coordinate chart for a neighborhood U of p. Let $v \in T_pM$, then it can be expressed as a linear combination

$$v = v^i \frac{\partial}{\partial x^i}(p)$$

Differentiating both sides by dF_p yields

$$0 = dF_p(v)(x^j) = v^i \frac{\partial(x^j \circ F)}{\partial x^i}(p) = v^i \frac{\partial F^j}{\partial x^i}(p)$$

Since this is true for all $v \in T_pM$, choosing $v = \partial_j$ implies that $\dfrac{\partial F^j}{\partial x^j}(p)$ for all j. As a consequence, each component F_j is constant, then F is also constant. ☐

Problem 3.2. Let $M_1, \ldots, M_k$ be smooth manifolds and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_i)_p(v))$$

is an isomorphism.

Proof. We first verify that α is well-defined. For every $j = 1, \ldots, k$, since the differential

$$d(\pi_j)_p : T_p(M_1 \times \cdots \times M_k) \to T_{\pi_j(p)}M_j = T_{p_j}M_j$$

sends the product tangent vector v to every seperate tangent space $T_{p_j}M_j$. This ensures that the image of α always lies on the direct sum of following tangent spaces. We now prove that α is injective. Suppose $\alpha(v) = \alpha(w)$, since α is linear due to the linearity of the differentials, we thus have $\alpha(v) - \alpha(w) = \alpha(v - w) = 0$ implies that $d(\pi_i)_p(v - w) = 0$ for all $i = 1, \ldots, k$. Let $k_i = \dim M_i$ for all i and consider the local coordinate $(x_1^1, \ldots, x_1^{k_1}, x_2^1, \ldots, x_k^{k_k}) = (x^1, \ldots, x^{k_1 + \cdots + k_k})$. Let $j = 1, \ldots, k$ and $t = 1, \ldots, k_j$ be fixed and i varies from 1 to $k_1 + \cdots + k_k$, expressing the vector $u = v - w$ in coordinate $u = u^i \frac{\partial}{\partial x^i}(p)$ yields

$$0 = d(\pi_j)_p(u)(x^t) = u^i \frac{\partial(x^t \circ \pi_j)}{\partial x^i}(p) = u^i \frac{\partial x_j^t}{\partial x^i}(p) = u^i \frac{\partial x^{k_{t-1}+j}}{\partial x^i}(p) = u^{k_{t-1}+j}$$

Since t and j was abitrary, then it follows that $u^i = v^i - u^i = 0$, hence α is injective. Let $w = (w_1, \ldots, w_k) \in T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$, and let

$$v = w_i^j \frac{\partial}{\partial x^{k_{i-1}+j}}.$$

The computation above implies that $\alpha(v) = w$. Thus α is bijective. ☐

Problem 3.3. Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Problem 3.4. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Proof. We begin to construct by viewing $\mathbb{S}^1$ in $\mathbb{C}$. Let (U, θ) and (V, ψ) be the angle local chart such that $\theta : \mathbb{S}^1 \backslash \{1\} \to (0, 2\pi)$ and $\psi : \mathbb{S}^1 \backslash \{-1\} \to (-\pi, \pi)$. We consider the map $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ satisfying

$$F\left(x, v\partial_\theta|_x\right) = (x, v) \text{ on } U \text{ and } F\left(x, v\partial_\psi|_x\right) = (x, v) \text{ on } V.$$

Since we have $\theta = \psi - \pi$ and $\dfrac{d\psi}{d\theta} = 1$ then $\partial_\theta|_x = \partial_\psi|_x$ on the restriction to F on $U \cap V$ since , it follows that F is well-defined. To check that F is bijection, define the candidate function $G : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$ satisfying

$$G(x, v) = \begin{cases} \left(x, v\partial_\theta|_x\right), & x \in U, \\ \left(x, v\partial_\psi|_x\right), & x \in V \end{cases}$$

Since G is well-defined and $F \circ G(x,v) = G \circ F(x,v) = (x,v)$, thus F is bijective.

In addition, F is smooth on U and V since the coordinate expression $(x,v) \mapsto (x,v)$ is smooth on U and V, and agrees smoothly on $U \cap V$. Computing the transition $\theta \circ \psi^{-1} : \psi(U \cap V) \to \theta(U \cap V)$, we obtain

$$\theta \circ \psi^{-1}(\psi(x)) = \theta(x)$$

which is smooth, the gluing lemma implies that F is smooth globally. As G is smooth by the analogous argument, hence F is diffeomorphism $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$. □

Problem 3.5. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x,y) : \max(|x|,|y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no diffeomorphism with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF\left(\gamma'(t)\right)$ on the coordinate functions x and y.] (Used on p. 123.)

Proof. Suppose that there is a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. Following to the hint, let $\gamma : [0, 2\pi) \to \mathbb{S}^1$ be the smooth curve satisfying $\gamma(t) = (\cos(t), \sin(t))$. Since γ is a homeomorphism and has smooth inverse, which is

$$\gamma^{-1}(x,y) = \gamma^{-1}(x, \pm\sqrt{1-x^2}) = \arctan\left(\frac{\pm\sqrt{1-x^2}}{x}\right)$$

Then γ is also a diffeomorphism, thus the composition $F \circ \gamma : [0, 2\pi) \to K$ is also a diffeomorphism. Since $F \circ \gamma$ is onto, then there exists t_0 such that $F \circ \gamma(t_0) = (1,1)$. Let I be an open interval containing $(F \circ \gamma)^{-1}(1,1)$, then one can split I such that $F \circ \gamma(I_x) \subseteq (-1,1) \times \{1\}$ and $F \circ \gamma(I_y) \subseteq \{1\} \times \subseteq (-1,1)$. Consequently, we have

$$F \circ \gamma(t) = \begin{cases} (x \circ F \circ \gamma(t), 1) & \text{if } t \in I_x \\ (1, y \circ F \circ \gamma(t)) & \text{if } t \in I_y \end{cases} \tag{3.1}$$

Since the velocity of $F \circ \gamma$ at t_0 is

$$\left(\frac{d(x \circ F \circ \gamma)}{dt}(t_0), 0\right) \text{ and } \left(0, \frac{d(y \circ F \circ \gamma)}{dt}(t_0)\right)$$

by (3.1), respectively, then two expressions for $(F \circ \gamma)(t_0)$ must concide, which means $(F \circ \gamma)'(t_0) = 0$, it follows that $dF(\gamma'(t_0)) = (F \circ \gamma)'(t_0) = 0$, which leads to contradiction since $dF_{\gamma'(t_0)}$ is globally homeomorphism and $\gamma'(t_0) \neq 0$, hence no such diffeomorphism F can exists. □

Problem 3.6. Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = \left(z^1, z^2\right) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.

Proof. Let $z^1 = x^1 + iy^1$and $z^2 = x^2 + iy^2$, then we can write γ_z under the usual identification as

$$\begin{aligned}
\gamma_z(t) = (e^itz^1, e^itz^2) &= \big((\cos(t) + i\sin(t))(x^1 + iy^1), (\cos(t) + i\sin(t))(x^2 + iy^2)\big) \\
&= (x^1\cos(t) - y^1\sin(t) + i(x^1\sin(t) + y^1\cos(t)), x^2\cos(t) - y^2\sin(t) + i(x^2\sin(t) + y^2\cos(t))) \\
&\Leftrightarrow (x^1\cos(t) - y^1\sin(t), x^1\sin(t) + y^1\cos(t), x^2\cos(t) - y^2\sin(t), x^2\sin(t) + y^2\cos(t))
\end{aligned} \tag{3.2}$$

Let $\alpha(t) = y^1\sin(t) + x^1\cos(t)$ and $\beta(t) = y^2\sin(t) + x^2\cos(t)$, one can rewrite

$$\gamma_z(t) = (\alpha'(t), \alpha(t), \beta'(t), \beta(t)) \tag{3.3}$$

Since the components α and β are smooth, it follows that $\gamma_z(t)$ is smooth. Consider the local chart (U_1^+, φ_1^+) for $\mathbb{S}^3$, where $U_1^+ = \{(x^1, \ldots, x^4) \in \mathbb{S}^3 \mid x^1 > 0\}$ and φ_1^+ satisfies

$$\varphi_1^+(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4)$$

Then the coordinate representation respect to this chart is $\varphi_1^+ \circ \gamma : \gamma_z^{-1}(U_1^+) \to \mathbb{R}^3$ and $\varphi_1^+ \circ \gamma_z(t) = (\alpha(t), \beta'(t), \beta(t))$. Then the velocity of $\varphi_1^+ \circ \gamma_z$ is

$$(\varphi_1^+ \circ \gamma_z)'(t) = (\alpha'(t), \beta''(t), \beta'(t)) = (\alpha'(t), -\beta(t), \beta'(t))$$

Since $\alpha(t)' > 0$, then its velocity is nonzero for all $t \in \gamma_z^{-1}(U_1^+)$. An analogous computation for the chart $(U_1^-, \varphi_1^-), (U_2^\pm, \varphi_2^\pm), \ldots, (U_4^\pm, \varphi_4^\pm)$, since the domain of the collection $\{(U_i^\pm, \varphi_i^\pm)\}$ covers $\mathbb{S}^3$, it follows that the velocity of the smooth curve γ_z is never zero. $\square$

Problem 3.7. Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_p M \to T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. (Used on p. 72.)

Chapter 4

Embeddings

1 Immersion and Submersion

Definition 4.1. Let M^m and N^n be smooth manifolds, and $F : M \to N$ is a smooth map. For abitrary point $p \in M$, we define the rank of F at p to be the rank of the differential dF_p. If $\text{rank}(dF_p) = r$ for all $p \in M$, we say F *has the constant rank* r.

Lemma 4.2. Let $F : M^m \to N^n$ be a smooth map. For abitrary $p \in M$, dF_p is injective if and only if $\text{rank}(dF_p) = n$ and is surjective if and only if $\text{rank}(dF_p) = m$. Consequently, dF_p is bijective if and only if $\text{rank}(dF_p) = m = n$.

Definition 4.3. Let $F : M^m \to N^n$ be the smooth map. Then F is called a smooth immersion if its differential is injective at each point, and is called a smooth submersion if its is surjective at each point.

Proposition 4.4. Injectivity and surjectivity of the differential implies local immersion and submersion, respectively.

Theorem 4.5 (Inverse Function Theorem on Manifolds). Let $F : M^m \to N^n$ be a smooth map. If $p \in M$ is point such that dF_p is invertable, then there are connected neighborhoods U_p and $V_{F(p)}$ such that $F|_{U_p} : U_p \to V_{F(p)}$ is a diffeomorphism.

2 Constant Rank Theorem

Theorem 4.1 (Constant Rank Theorem). Let M^m and N^n be smooth manifolds and $F : M \to N$ be a smooth map with constant rank r. For each $p \in M$, there exists smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\psi \circ F \circ \varphi^{-1}(x^1, \dots, x^r, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0) \tag{4.1}$$

Proof. Since the theorem is local, one can view M and N as open subsets U and V in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. For the sake of condition, we can reorder coordinate such that the $r \times r$ block of DF_p has nonzero determinant. Consider the indentification

$$\begin{aligned}(x, y) &\Leftrightarrow (x^1, \dots, x^r, y^1, \dots, y^{m-r}),\\ (u, v) &\Leftrightarrow (u^1, \dots, u^r, v^1, \dots, v^{n-r}).\end{aligned}$$

Then one can view $F(x, y) = (F_1(x, y), F_2,(x, y))$ for some smooth map $F_1 : U \to \mathbb{R}^r$ and $F_2 : U \to \mathbb{R}^{n-r}$. Let $p = (x_p, y_p)$ and $\varphi : U \to \mathbb{R}^m$ satisfying $\varphi(x, y) = (F_1(x, y), y)$, we have

$$D\varphi(x_p, y_p) = \begin{bmatrix} DF_1(x, y) & \dfrac{\partial F_1}{\partial y}(x, y) \\ 0 & \delta^i_j \end{bmatrix}$$

The above assumption implies that $DF_1(x,y)$ and δ_i^j are invertable, then $D\varphi(x_p,y_p)$ is invertable. Apply the inverse function theorem for the map φ and $p \in U$, there exists connected neighborhoods U_p such that $\varphi|_{U_p} : U_p \to \varphi(U_p)$ is a diffeomorphism. Then we can consider the local chart (U_p, φ) which is a restricted diffeomorphism. On the following restriction, we can let $A, B : \varphi(U_p) \to U_p$ be the smooth function satisfying $\varphi^{-1}(x,y) = (A(x,y), B(x,y))$, then we obtain

$$(x,y) = \varphi(A(x,y), B(x,y)) = (F_1(A(x,y), B(x,y)), B(x,y)).$$

It follows that $B(x,y) = y$ and $F_1(A(x,y), y) = x$. Hence $F \circ \varphi^{-1}$ is calculated by

$$F \circ \varphi^{-1}(x,y) = F(A(x,y), y) = (F_1(A(x,y), y), F_2(A(x,y), y)) = (x, C(x,y)),$$

where $C : \varphi(U_p) \to \mathbb{R}^{n-r}$ defined by $C(x,y) = F_2(A(x,y), y)$. Moreover, since the Jacobian of $F \circ \varphi^{-1}$ at abitrary point $(x,y) \in \varphi(U_p)$ is

$$D(F \circ \varphi^{-1})(x,y) = \begin{bmatrix} \delta_j^i & 0 \\ \dfrac{\partial C^i}{\partial x^j}(x,y) & \dfrac{\partial C^i}{\partial y^j}(x,y) \end{bmatrix},$$

where its rank remains at r and the block δ_j^i has the rank r. Then the block $\dfrac{\partial C^i}{\partial y^j}(x,y)$ must vanish to zero. Therefore $C(x,y)$ is independent of y hence we can set $S(x) = C(x,y) = C(x, y_p)$ for all y and then we obtain

$$F \circ \varphi^{-1}(x,y) = (x, S(x)) \tag{4.2}$$

Let $V_p \subseteq V$ be an open subset satisfying $V_p = \{(u,v) \in V \mid (u, \varphi^2(p)) \in \varphi(U_p)\}$. Then V_p is a neighborhood of $\varphi(p)$ and the definition of $F \circ \varphi^{-1}$ in 4.2 implies that $F \circ \varphi^{-1}(\varphi(U_p)) = F(U_p) \subseteq V_p$. Then we can consider the local chart (V_p, ψ) such that $\psi : V_p \to \mathbb{R}^n$ satisfying $\psi(u,v) = (u, v - S(u))$. Notice that (V_p, ψ) is a smooth chart, following from 4.2, we thus have

$$\psi \circ F \circ \varphi^{-1}(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0).$$

Since (φ, U_p) and (ψ, V_p) satisfy the following conditions, the proof is done. □

Corollary 4.2. Let $F : M^m \to N^n$ be a smooth map between smooth manifolds and suppose M is connected. Then the followings are equivalent:

1. For each $p \in M$, there exists smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.

2. F has constant rank.

Proof. First we suppose F has linear coordinate representation in a neighborhoods of each point. Since every linear map has constant rank, it follows that F has constant rank on the following neighborhoods and the fact that M is connected implies that the rank of F is constant on every point of M.

Conversely, if F has constant rank, it follows from the previous theorem that there exists following smooth charts such that F has the representation of the form 4.1, which is linear. □

Theorem 4.3. Let $F : M^m \to N^n$ be a smooth map of constant rank between smooth manifolds.

1. If F is surjective, then it is a smooth submersion.

2. If F is injective, then it is a smooth immersion.

3. If F is bijective, then it is a diffeomorphism.

Proof. (1) Suppose F is surjective and has constant rank $r < n$. Let $p \in M$, by the Constant Rank Theorem, there exists smooth local charts (U_p, φ_p) and $(V_{F(p)}, \psi_{F(p)})$ corresponding to p and $F(p)$ such that $F(U_p) \subseteq V_{F(p)}$ in which F has a coordinate representation of the form

$$\tilde{F}_p(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots). \tag{4.3}$$

For the sake of condition, we can shrink U into an r-dimensional regular coordinate ball and $F(\overline{U_p}) \subseteq V$. Then $F(\overline{U_p})$ is a compact subset of the set

$$\{y \in V \mid y^{r+1} = \cdots = y^n = 0\}.$$

Since such a closed r-dimensional cannot contain any n-dimensional regular coordinate bals in N, $F(\overline{U_p})$ does not include any open subset of N. Hence it is nowhere dense in N.

Since charts $\{(U_p, \varphi_p)\}_{p \in M}$ is an open cover of M, then we can choose a subcover $\{(U_p, \varphi_p)\}_{p \in A}$ on M. Since $F(M)$ is covered by the corresponding subcover $\{F(\overline{U_p})\}_{p \in A}$, which is countable and every element is nowhere dense, by the Baire category theorem, $\overline{F(M)} = \varnothing$, which contradicts the fact that F is surjective, hence it must be a smooth submersion.

(2) Suppose F is injective and $r < m$, follows from (7), we have

$$\tilde{F}_p(x^1, \dots, x^m) = \tilde{F}_p(x^1, \dots, x^r, 0, \dots)$$

which implies that U is r-dimensional, contradiction. Hence F must be a smooth immersion.

(3) Follows from (1) and (2), F is both smooth submersion and immersion, it thus is local diffeomorphism. The fact that F is bijective implies that it is diffeomorphism from M onto N. $\square$

3 Embeddings

Definition 4.1. Let M^m and N^n be smooth manifolds. A smooth embedding of M into N is a smooth immersion $F : M \to N$ that yields a homeomorphism onto its image $F(M) \subseteq N$.

Proposition 4.2. Let $F : M^m \to N^n$ be an injective smooth immersion. If any of the following holds, then F is a smooth embedding.

1. F is an open or closed map.
2. F is a proper map.
3. M is compact.
4. M has empty boundary and $\dim M = \dim N$.

Proof. If F is open or closed map, then the inverse map $F^{-1} : F(M) \to M$ is continuous and thus F is a smooth embedding. If F is proper, then it is closed, and if M is compact, F is also closed. Finally assume M has empty boundary and $\dim M = \dim N$. Since F is both smooth immersion and submersion, then dF_p is nonsingular everywhere, and the problem 4.2 implies that $F(M) \subseteq \mathrm{Int}(N)$. Then $F : M \to \mathrm{Int}N$ is locally diffeomorphism, so it is open map. Hence $F : M \to N$ is an embedding. $\square$

Theorem 4.3 (Local Embedding Theorem). Suppose M^m and N^n are smooth manifolds, and $F : M \to N$ is a smooth map. Then F is a submersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a smooth embedding.

4 Submersions

Definition 4.1. Let $\pi : M \to N$ be a continuous map, a *section of* π is a continuous right inver for π, which is a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$. Analogously, a *local section of* π is a continuous map $\sigma : U \to M$ defined by some open subset $U \subseteq N$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

Theorem 4.2. Let M^m and N^n be smooth manifolds, and $\pi : M \to N$ be a smooth map. Then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

Theorem 4.3. Let M and N be smooth manifolds, and suppose $\pi : M \to N$ is a smooth submersion. Then π is an open map, and if it is surjective it is a quotient map.

Theorem 4.4 (**Characteristic Property Theorem**). Let $\pi : M^m \to N^n$ be a surjective smooth submersion. For any smooth manifold P with or without boundary, a map $F : N \to P$ is smooth if and only if $F \circ \pi$ is smooth.

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "F\circ\pi"'] & N \arrow[d, "F"] \\ & P \end{tikzcd}$$

5 Smooth Covering Maps

Definition 4.1. Let $\pi : M \to N$ is called a *smooth covering map* if

1. π is smooth and surjective.
2. For abitrary $q \in N$, there exists an open neighborhoods V containing q such that
$$\pi^{-1}(V) = \bigsqcup_{\alpha} U_\alpha,$$
where $U_\alpha \subseteq M$ are open, disjoint and connected.
3. For each α, we have $\pi_{U_\alpha} : U_\alpha \to V$ is a diffeomorphism.

Proposition 4.2. Let E and M be nonempty connected smooth manifolds. If $\pi : E \to M$ is a proper local diffeomorphism, then π is a smooth covering map.

Lemma 4.3. Let $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ be the map satisfying
$$\varepsilon^n(x^j) = \left(e^{2\pi i x^j}\right).$$
We claim that ε^n is a smooth covering map.

Proof. Observe that ε^n is smooth and surjective, it suffices to prove that ε^n is local diffeomorphism. For any $x \in \mathbb{R}^n$, consider the open cube
$$U_x = x + \left(-\frac{1}{2}, \frac{1}{2}\right)^n$$
Consider the restriction map $\varepsilon^n|_{U_x} : U_x \to \varepsilon^n(U_x)$. If $a, b \in U_x$ and $\varepsilon^n(a) = \varepsilon^n(b)$, it follows that $a - b \in \mathbb{Z}^n$ since $|a^i - b^i| < 1$ for all $i = 1, \ldots, n$. Thus $\varepsilon^n|_{U_x}$ is injective and has smooth inverse, thus it is diffeomorphism, following that ε^n is local diffeomorphism. Now we decay the preimage of $V = \varepsilon^n(U)$ into union of disjoint connected subsets, which is
$$\left(\varepsilon^n\right)^{-1}(V) = \bigsqcup_{k \in \mathbb{Z}^n} (U_x + k)$$
And since each map : $\varepsilon^n : U_x + k \to \mathbb{T}^n$ is already a diffeomorphism, hence ε^n is indeed a smooth covering map. $\square$

6 Problems

Problem 4.1. Use the inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that the Inverse Function Theorem does not extend to the case which M is a manifold with boundary.

Proof. Suppose the theorem holds for manifolds with boundary. Let $p = (x^1, \dots, x^{n-1}, 0) \in \partial\mathbb{H}^n$, then there exists open neighborhoods U_0 and V_0 centered at p and $F(p)$, respectively such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism. This leads to contradiction since U_0 and V_0 have dimension n and $n-1$, respectively. □

Problem 4.2. Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F : M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \operatorname{Int} N$. (Used on pp. 80, 87.)

Proof. Suppose the contrary holds that $F(p) \in \partial N$. Let (x^i) for $i = 1, \dots, m$ and $y^j(x^i)$ and $j = 1, \dots, n$ be the coordinate chart containing p and the boundary chart containing $F(p)$. Since $F(p) \in \partial N$, it follows that $y^n(x^i)(x) = 0$ for all $x \in V$, we thus have

$$\left.\frac{\partial(y^n)}{\partial x^i}\right|_{\varphi(p)} = 0$$

for all $i = 1, 2, \dots, m$. Since we have

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_{F(p)}$$

Then the last row of DF_p vanishes to zero, which means dF_p is singular, contradiction. □

Problem 4.3. Formulate and prove a version of the rank theorem for a map of constant rank whose domain is a smooth manifold with boundary. [Hint: after extending F arbitrarily as we did in the proof of Theorem 4.15, follow through the proof of the rank theorem until the point at which the constant-rank hypothesis is used, and then explain how to modify the extended map so that it has constant rank.]

Problem 4.4. We denote $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-dimensional torus. Let $\alpha \in \mathbb{R}$ be an irrational number and $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve satisfying:

$$\gamma(t) = (e^{2\pi\alpha t}, e^{2\pi\alpha i t})$$

Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

Proof. We aim to prove the following lemma:

Lemma 4.1 (Kronecker's Density Theorem). Let $\alpha \in \mathbb{R}$ be an irrational number, then the set

$$S = \{\{n\alpha\} \mid n \in \mathbb{Z}\}$$

is dense in $[0, 1)$.

Proof of the lemma. Observe that every element of S is dinstict since $\{n_1\alpha\} = \{n_2\alpha\}$ implies that

$$n_1\alpha - \lfloor n_1\alpha \rfloor = n_2\alpha - \lfloor n_2\alpha \rfloor \Leftrightarrow (n_1 - n_2)\alpha = \lfloor n_1\alpha \rfloor - \lfloor n_2\alpha \rfloor.$$

Then it follows that α is rational, which leads to contradiction. Let $m \in N$, divide $(0, 1)$ into the following interval:

$$\left[0, \frac{1}{m}\right), \left[\frac{1}{m}, \frac{2}{m}\right), \dots, \left[\frac{m-1}{m}, \frac{m}{m}\right)$$

The Dirichlet principle implies that there exists $k \le m$ and $n_1, n_2 \in \mathbb{Z}$ such that

$$\{n_1\alpha\}, \{n_2\alpha\} \in \left[\frac{k-1}{m}, \frac{k}{m}\right).$$

For the sake of condition, we can assume that $n_1 < n_2$. Since we have the following expression

$$\begin{aligned} \{(n_2 - n_1)\alpha\} &= (n_2 - n_1)\alpha - \lfloor (n_2 - n_1)\alpha \rfloor \\ &= \lfloor n_2\alpha \rfloor + \{n_2\alpha\} - \lfloor n_1\alpha \rfloor - \{n_1\alpha\} - \lfloor (n_2 - n_1)\alpha \rfloor \end{aligned} \tag{4.4}$$

If $\lfloor n_2\alpha \rfloor = \lfloor n_1\alpha \rfloor + \lfloor (n_2 - n_1)\alpha \rfloor$, we have

$$\begin{aligned} \{(n_2 - n_1)\alpha\} &= \lfloor n_2\alpha \rfloor + \{n_2\alpha\} - \lfloor n_1\alpha \rfloor - \{n_1\alpha\} - \lfloor (n_2 - n_1)\alpha \rfloor \\ &= \{n_2\alpha\} - \{n_1\alpha\} < \frac{k}{m} - \frac{k-1}{m} = \frac{1}{m} \end{aligned} \tag{4.5}$$

Let $q = \{(n_2 - n_1)\alpha\} < \frac{1}{m}$, for each $p \leq m$, we need to find $N \in \mathbb{N}$ and such that

$$\frac{p-1}{m} < Nq < \frac{p}{m} \tag{4.6}$$

As we calculate the distance between two side of the inequality

$$\frac{1}{q}\left(\frac{p}{m} - \frac{p-1}{m}\right) = \frac{1}{qm} > 1$$

Then it is possible to choose an integer N satisfying 4.6. And since $q < \frac{1}{m}$, we thus have

$$\{N(n_2 - n_1)\alpha\} = qN \in \left[\frac{p-1}{m}, \frac{p}{m}\right).$$

If $\lfloor n_2\alpha \rfloor = \lfloor n_1\alpha \rfloor + \lfloor (n_2 - n_1)\alpha \rfloor + 1$, we also have

$$\begin{aligned} \{(n_2 - n_1)\alpha\} &= \lfloor n_2\alpha \rfloor + \{n_2\alpha\} - \lfloor n_1\alpha \rfloor - \{n_1\alpha\} - \lfloor (n_2 - n_1)\alpha \rfloor \\ &= \{n_2\alpha\} - \{n_1\alpha\} + 1 \geq \frac{k-1}{m} - \frac{k}{m} + 1 = \frac{m-1}{m} \end{aligned}$$

Let $q = \{(n_2 - n_1)\alpha\} > \frac{m-1}{m}$, we wish to scale q by by integer such that $qN = \{N(n_2 - n_1)\alpha\} < \frac{1}{m}$. Indeed, this motivates us to prove the following approximation lemma:

Lemma 4.2 (Dirichlet's Approximation Theorem). Given $\alpha \in \mathbb{R}$ and any positive integer N. Then there exists $m, n \in \mathbb{N}$ such that

$$|n\alpha - m| < \frac{1}{N}$$

Proof of the Dirichlet's lemma. Let $f(x) = \{x\}$. Apply the Dirichlet's principle for $N + 1$ numbers

$$\{f(i\alpha) : i = 0, \ldots, N\}$$

lying in

$$\left[0, \frac{1}{N}\right), \ldots, \left[\frac{N-1}{N}, 1\right)$$

Then there exists $f(i\alpha)$ and $f(j\alpha)$ lies in one of the subintervals above. This implies that

$$|f(j\alpha) - f(i\alpha)| < \frac{1}{N}$$

Setting $n = j - i$ and $m = \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor$ and we are done. ☐

Back to the main lemma, since there exists $m, n \in \mathbb{N}$ such that

$$|nq - m| < \frac{1}{m}.$$

Then we can choose $N = n$ and then our wishing statement is proven. Since $\{N(n_2 - n_1)\alpha\} \in \left(0, \frac{1}{m}\right]$, we establish the same proof in 4.5. In general, for any $m \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $k \leq m$, the open interval

$$I_m^k = \left(\frac{k-1}{m}, \frac{k}{m}\right)$$

contains at least an element in S. And since the collection $\{I_m^k\}$ defines an open cover of $[0, 1)$. Hence every open subset of $[0, 1)$ contains at least an element in S. S thus is dense in $[0, 1)$. $\square$

We would rather use the below corollary of the above lemma.

Corollary 4.3. Let $\alpha \in \mathbb{R}$ be an irrational number, then the set

$$S = \{m + n\alpha \mid m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$

Back to the problem, let $\alpha(x, y) = (e^{2\pi ix}, e^{2\pi iy}) \in \mathbb{T}^2$ be abitrary. Using the notation $\exp(x)$ for $e^{2\pi ix}$, we consider the following substraction

$$\|\gamma(t) - \alpha(x, y)\| = \left\|e^{2\pi it} - e^{2\pi ix}, e^{2\pi i\alpha t} - e^{2\pi iy}\right\| \tag{4.7}$$

Let $m, n \in \mathbb{N}$ be abitrary and substitute $t = x + n$. Since $2p\pi \equiv 0 \pmod{2\pi}$ for all $p \in \mathbb{N}$, we obtain

$$\begin{aligned}\|\gamma(t) - \alpha(x, y)\| &= \|\exp(x + n) - \exp(x), \exp(\alpha x + \alpha n + m) - \exp(y)\| \\ &= \|0, \exp(\alpha x + \alpha n + m) - \exp(y)\|\end{aligned} \tag{4.8}$$

Since we have

$$|e^{it_1} - e^{it_2}| \leq |t_1 - t_2|$$

for all $t_1, t_2 \in \mathbb{R}$, using this inequality for 4.8 yields

$$\begin{aligned}\|\gamma(t) - \alpha(x, y)\| &= |\exp(\alpha x + \alpha n + m) - \exp(y)| \\ &\leq |\alpha x + \alpha n + m - y| \\ &= |(\alpha n + m) - (y - \alpha x)|\end{aligned} \tag{4.9}$$

Since $y - \alpha x \in \mathbb{R}$ and the set

$$S = \{\alpha n + m \mid n, m \in \mathbb{N}\}$$

is dense in $\mathbb{R}$, then one can choose a sequence $(a_n)_{n\in\mathbb{N}} \subseteq S$ which has the form $a_n = x_n\alpha + y_n$ and

$$\lim_{n\to+\infty} a_n = y - \alpha x.$$

For each k, set $n = x_k$ and $m = y_k$. From 4.9, we obtain

$$\|\gamma(t) - \alpha(x, y)\| \leq |a_n - (y - \alpha x)| \to 0$$

Therefore, for all $x, y \in \mathbb{R}$, since we can already choose $t_n \in \mathbb{R}$ such that $\|\gamma(t) - \alpha(x, y)\| \to 0$, then $\gamma(\mathbb{R})$ must be dense in $\mathbb{T}^2$. $\square$

Problem 4.5. Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space, as defined in Problem 1-9. (Used on pp. 172, 560.)

1. Show that the quotient map $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.

2. Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

Proof. Since π is already surjective, it suffices to prove that it is smooth submersion. For each $i = 1, \dots, n+1$, let $U_i \subset N$ be the set of equivalent classes satisfying

$$U_i = \{[z^1, \dots, z^{n+1}] \mid (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \backslash \{0\} \text{ and } z^i \neq 0\}$$

Consider the map $\sigma_i : U_i \to M$ satisfying

$$\sigma_i[z^1, \dots, z^{n+1}] = \left(\frac{z^1}{z^i}, \dots, \frac{z^{n+1}}{z^i} \right).$$

In fact, we already proved that σ_i is well-defined and a homeomorphism. Moreover, σ_i is a local section of π since we have

$$\pi \circ \sigma_i[z^1, \dots, z^{n+1}] = \left[\frac{z^1}{z^i}, \dots, \frac{z^{n+1}}{z^i} \right] = [z^1, \dots, z^{n+1}] = \mathrm{Id}_{\sigma_i(N)}$$

As the domain of $\{(U_i, \sigma_i)\}$ admits an open cover of $\mathbb{C}^{n+1} \backslash \{0\}$, then every point of $\mathbb{C}^{n+1} \backslash \{0\}$ is in the image of some smooth local section of π. Therefore π is a surjective smooth submersion. $\square$

Problem 4.6. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Proof. Assume the contrary holds that there exists a smooth submersion $F : M \to \mathbb{R}^k$ for some $k > 0$. Since F is continuous, then $F(M)$ is compact, hence $F(M)$ is bounded and closed by the Heine-Borel theorem. Applying Constant Rank theorem, there exists local coordinate charts such that one can write F as the following coordinate representation

$$\tilde{F}(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

which is an open map, then F is also an open map. Hence $F(M)$ is open, which contradicts the fact that there is no open compact subset in $\mathbb{R}^k$. $\square$

Problem 4.7. Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F : \tilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\tilde{N}$. [Remark: this shows that the property described in Theorem 4.29 is "characteristic" in the same sense as that in which Theorem A.27(a) is characteristic of the quotient topology.]

Proof. For the sake of condition, let $\pi_1 : M \to N$ be a surjective smooth submersion and $\pi_2 : M \to \tilde{N}$ be the surjective map that agrees with π_1 on any points in M. Since the map $\mathrm{Id}_N : N \to \tilde{N}$ is a bijection, it thus have inverse. Since we have

$$\mathrm{Id}_N^{-1} \circ \pi_2 = \pi_1$$

which is a smooth map, then the above hypothesis implies that Id_N^{-1} is smooth. Let $i : \tilde{N} \to \tilde{N}$ be the inclusion, since $\tilde{N}$ is a smooth manifold, then i is smooth. Hence it follows that $i \circ \pi_2 = \pi_2$ is smooth. Analogously, the map

$$\mathrm{Id}_N \circ \pi_1 = \pi_2$$

is smooth. Apply the theorem 4.29, we obtain that Id_N is smooth. Therefore, Id_N is a diffeomorphism. $\square$

Problem 4.8. This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Proof. For every $t \in \mathbb{R}$, just pick $\pi(t, 1) = t$, then π is surjective. Since the maps $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(x, y) = x$ and $g(x, y) = y$ is smooth, then π is also smooth. Let $F : \mathbb{R} \to P$ the any map. The fact F is smooth implies that the composition $F \circ \pi : \mathbb{R}^2 \to P$ is also smooth. Now we assume that $F \circ \pi$ is smooth, then the resitricted map $F \circ \pi : \mathbb{R} \times \{1\} \to \mathbb{R}$ is smooth, hence F is smooth. Let $p = (0, 0) \in \mathbb{R}^2$ and $v_p \in T_p\mathbb{R}^2$, suppose $d\pi_p(v_p) = w_{\pi(p)} \neq 0$, we have

$$D\pi_p(v_p) = \begin{bmatrix} y & x \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = w_{\pi(p)}$$

Then it follows that $w_{\pi(p)} = 0$, thus π is not a smooth submersion. □

Problem 4.9. Let M be a connected smooth manifold, and let $\pi : E \to M$ be a topological covering map. Complete the proof of Proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map. [Hint: use the existence of smooth local sections.]

Problem 4.10. Show that the map $q : \mathbb{S}^n \to \mathbb{RP}^n$ defined by

$$q(x^1, \dots, x^n) = [x^1, \dots, x^n]$$

is a smooth covering map.

Proof. We already proved that q is smooth and surjective. For each $x \in \mathbb{S}^n$, consider the neighborhood U_x containing x where U_x is abitrarily small such that $U_x \cap (-U_x) = \varnothing$ The fact that $q(a) = q(b)$ implies that $a = \pm b$, since $U_x \cap (-U_x) = \varnothing$. Then it follows that $a = b$ and q is injective. Then the restricted map $q : U_x \to q(U_x)$ is diffeomorphism since it is also smooth and has smooth inverse. Let $V_x = q(U_x)$, since we have

$$q^{-1}(V_x) = U_x \sqcup U_{-x}$$

and each of component of $q^{-1}(V_x)$ is mapped diffeomorpically into corresponding U_x and U_{-x}, hence q is a smooth covering map. □

Problem 4.11. Show that a topological covering map is proper if and only if its fibers are finite, and therefore the converse of Proposition 4.46 is false.

Problem 4.12. Using the covering map $\varepsilon^2 : \mathbb{R}^2 \to \mathbb{T}^2$ (see Example 4.35), show that the immersion $X : \mathbb{R}^2 \to \mathbb{R}^3$ defined in Example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\tilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$, and then show that $\tilde{X}$ is a smooth embedding whose image is the given surface of revolution.

Problem 4.13. Define a map $F : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x, y, z) = \big(x^2 - y^2, xy, xz, yz\big)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

Proof. Let $U = \{[x, y, z] \mid (x, y, z) \in \mathbb{S}^2\}$, then we have $U = \mathbb{RP}^4$. Consider the map $\tilde{F} : U \to \mathbb{R}^4$ be the map satisfying $F = \tilde{F} \circ \pi$. Then we have

$$\tilde{F}[x, y, z] = (x^2 - y^2, xy, xz, yz),$$

Where $x^2 + y^2 + z^2 = 1$. Since $[x, y, z] = [\lambda x, \lambda y, \lambda z]$ implies that $\lambda = \pm 1$, then it follows that

$$\tilde{F}[-x, -y, -z] = (x^2 - y^2, xy, xz, yz) = \tilde{F}[x, y, z].$$

Thus $\tilde{F}$ is well-defined. Since F is smooth and π is a smooth covering, then the above proposition follows that $\tilde{F}$ is smooth. It is clear to show that $\tilde{F}$ is bijective onto $\tilde{F}(\mathbb{R}^4)$. Let $p = (x, y, z)$ and $v_p = v^i \dfrac{\partial}{x^i}\Big|_p \in T_p(\mathbb{S}^2)$ be abitrary. Suppose we have $d\tilde{F}_{[p]}([v_p]) = 0$, it follows that

$$dF_p(v_p) = d\tilde{F}_{[p]} \circ d\pi_p(v_p) = d\tilde{F}_{[p]}([v_p]) = 0$$

Computing the Jacobian DF_p we obtain

$$DF_p(v_p) = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

Since x, y, z cannot vanish simultaneously, then the only solution for $dF_p(v) = 0$ is $v = 0$, thus $\ker(d\tilde{F}_p) = \{0\}$. Therefore, $\tilde{F}$ is an immersion and since $\mathbb{RP}^2$ is compact, $\tilde{F}$ must be a smooth embedding. □

Chapter 5

Level Sets Theorem

1 Embedded Submanifolds

Definition 5.1. Let M^m be a smooth manifold, an *embedded submanifold of* M is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion $S \to M$ is a smooth embedding.

Definition 5.2. If S is an embedded submanifold of M, the difference $\dim M - \dim S$ is called the *codimension of S in M*. An *embedding hypersurface* is an embedded submanifold of codimension 1.

Proposition 5.3. Suppose M^m is a smooth manifold. Then embedded submanifolds of codimension 0 in M are exactly open submanifolds.

Proposition 5.4. Let M and N be smooth manifolds, and $F : M \to N$ be a smooth embedding. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.

Proof. If we let S inherit subspace topology from M, then S is a topological manifold. Since F is a homeomorphism onto its image, for any chart (U, φ) in N, then we can choose the corresponding chart $(F(U), \varphi \circ F^{-1})$ for S. Since every different choosed chart in S is compatible, it follows that S attains a smooth structure. Then F is local diffeomorphism, the Gluing lemma follows that F is diffeomorphism onto its image, hence S has a unique smooth structure sent from N. Let $i : S \hookrightarrow M$ be the inclusion. Since $i = F \circ F^{-1}$ which is a composition of two smooth embedding, then i is also a smooth embedding. Hence S is a embedded submanifold of M. ☐

Proposition 5.5. Suppose M and N are smooth manifolds. For each $p \in N$, the slice $M \times \{p\}$ is an embedded submanifold of $M \times N$ diffeomorphic to M.

Proof. Since $M \times \{p\}$ is an image of the smooth embedding $x \mapsto (x, p)$, it follows from the above proposition that $M \times \{p\}$ is an embedded submanifold. ☐

Proposition 5.6. Suppose M is a smooth m-manifold without boundary. N is a smooth n manifold, $U \subseteq M$ is open, and $f : U \to N$ is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote the graph of f

$$\Gamma(f) = \{(x, y) \in M \times N \mid x \in U, y = f(x)\}.$$

Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$.

Proof. Let $\gamma : U \to M \times N$ be the map satisfying $\gamma(x) = (x, f(x))$. Since the projection $\pi_M : M \times N \to M$ mapping $(x, y) \mapsto x$ which is a surjective submersion. Then composition $\pi_M \circ \gamma = i : U \hookrightarrow U$ is a smooth map. Hence γ is a smooth map by the Characteristic Property Theorem and is injective.

In addition, the differential $d(\pi_M \circ \gamma)_p : T_pM \to T_pM$ satisfying

$$d(\pi_M \circ \gamma)_p = d(\pi_M)_{\gamma(p)} \circ d(\gamma)_p = \mathrm{Id}_{T_pM}$$

Thus $d(\gamma)_p$ is injective and it follows that γ is a smooth immersion. Hence γ is a smooth embedding, then its image $\Gamma(f)$ is an embdedded m-dimensional submanifold of $M \times N$. $\square$

2 Slice Charts

Definition 5.1. Let $U \subseteq \mathbb{R}^n$ be an open subset. A *k-slice of U* is any subset S which has the form

$$S = \{(x^1, \dots, x^n) \in U \mid (x^{k+1}, \dots, x^n) = \text{constant}\}.$$

Moreover, let M^m be a smooth manifold and (U, φ) be a smooth chart on M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S is a k-slice of U.

3 Level Sets

Definition 5.1. Let $\Phi : M^m \to N^n$ be any map and $c \in N$, we call the set $\Phi^{-1}(c)$ a *level set of* Φ.

4 Cotangent Space

Definition 5.1. Suppose M is a smooth manifolds and T_pM is a tangent space of M at $p \in M$, then *cotangent space* T_p^*M is defined as

$$T_p^*M = \mathrm{Hom}(T_pM; \mathbb{R}) = \{\omega \mid \omega : T_pM \to \mathbb{R} \text{ linear}\}.$$

Its element $\omega_p \in T_p^*M$ is called *covector*, one can be written as:

$$\omega_p(v) = a_1 dx_1 + a_2 dx_2 + \cdots + a_n dx_n.$$

Chapter 6

Wedge Product

Definition 6.1 (Alternating Tensors). A covariant $(0,k)$-tensor α is said to be *alternating* if

$$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k) = -\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k).$$

The subspace of all alternating covariant $(0,k)$-tensor is denoted by $\Lambda(V^*) \subseteq T^k(V^*)$.

Proposition 6.2. Let α be a covariant $(0,k)$-tensor on V. The followings are equivalent:

1. α is alternanting.
2. $\alpha(v_1,\dots,v_k) = 0$ if and only if the k-tuple $(v_1,\dots,v_k)$ is linearly dependent.
3. $\alpha(v_1,\dots,w,\dots,w,\dots,v_k) = 0$.

Definition 6.3. We define the alternating projection $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ as follows:

$$\mathrm{Alt}(\alpha) = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\alpha(v_{\sigma_1}\dots,v_{\sigma_k})$$

Definition 6.4. Let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, define the *Wedge product* to be the following $(k+l)$-covector:

$$\omega\wedge\eta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega\otimes\eta).$$

Definition 6.5 (Elementary Alternating Tensors). Let V be an n-dimensional vector space and $\{\varepsilon_1,\dots,\varepsilon_n\}$ be a basis of V^*. We define a covariant $(0,k)$-tensor $\varepsilon^I = \varepsilon^{i_1,\dots,i_k}$ by

$$\varepsilon^I(v_1,\dots,v_k) = \det\begin{bmatrix}\overrightarrow{\varepsilon^I(v_1)},\dots,\overrightarrow{\varepsilon^I(v_k)}\end{bmatrix} = \det\begin{bmatrix}\varepsilon_{i_1}(v_1) & \cdots & \varepsilon_{i_1}(v_k)\\ \vdots & \ddots & \vdots\\ \varepsilon_{i_k}(v_1) & \cdots & \varepsilon_{i_k}(v_k)\end{bmatrix} = \det\begin{bmatrix}v_1^{i_1} & \cdots & v_k^{i_1}\\ \vdots & \ddots & \vdots\\ v_1^{i_k} & \cdots & v_k^{i_k}\end{bmatrix}$$

As it is a k-covector, ε^I is called *elementary k-vector.*

Proposition 6.6. Let (E_i) be a basis for V, (ε_i) be the dual basis for V^*, and let ε^I be an elementary k-covector be dual to (E_i). Since ε^I is alternating, then these followings hold.

1. If I has a repeated index, then $\varepsilon^I = 0$.
2. If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = \mathrm{sgn}(\sigma)\varepsilon^J$.
3. $\varepsilon^I(E_{j_1},\dots,E_{j_k}) = \delta^I_J$.

Definition 6.7. A multi-index $I = (i_1, \dots, i_k)$ is said to be *increasing* if $i_1 < i_2 < \cdots < i_k$.

Theorem 6.8. Let V be an n-dimensional vector space. If (ε^i) is any basis for V^*, then for each positive integer $k \le n$, the collection of k-covectors

$$\mathcal{F} = \{\varepsilon^I \mid I \text{ is increasing of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Consequently,

$$\dim \Lambda^k(V^*) = \binom{n}{k}.$$

Proof. Let $\omega \in \Lambda^k(V)$, and (E_i) be the basis of V dual to V^* since ω is alternating, for all abitrary and increasing multi-index representation of S_n, say $(j_1, \dots, j_k)$ and $(i_1, \dots, i_k)$, respectively, one can rewrite

$$\begin{aligned}\omega &= \omega_I \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_k} = \omega(E_{j_1}, \dots, E_{j_k})\varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_k} \\ &= \omega(E_{i_1}, \dots, E_{i_k}) \left(\sum_{\sigma \in (i_1, \dots, i_k)} \operatorname{sgn}(\sigma) \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \right) \\ &= \omega(E_{i_1}, \dots, E_{i_k}) \varepsilon^{(i_1, \dots, i_k)}.\end{aligned}$$

Therefore $\mathcal{F}$ generates the $\Lambda^k(V^*)$. To prove $\mathcal{F}$ is linearly independent, suppose $\omega = 0$ and applying both sides for each $(E_{j_1}, \dots, E_{j_k})$ yields

$$0 = \omega(E_{i_1}, \dots, E_{i_k}) \left[\varepsilon^{(i_1, \dots, i_k)}(E_{j_1}, \dots, E_{j_k}) \right] = \omega(E_{j_1}, \dots, E_{j_k}) \delta^J_J = \omega(E_{j_1}, \dots, E_{j_k}).$$

Thus, $\mathcal{F}$ is basis for $\Lambda^k(V^*)$, and the rule of counting implies $\dim \Lambda^k(V^*) = \binom{n}{k}$, as desired. □

Theorem 6.9. Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V^*)$. If $T : V \to V$ is any linear map and $v_1, \dots, v_n$ are abitrary vectors in V, then

$$\omega(Tv_1, \dots, Tv_n) = (\det T)\omega(v_1, \dots v_n).$$

Proof. Let (ε_i) be a dual basis for V^* and (E_i) be a basis for V dual to (ε_i). Since $\mathcal{F}$ forms a basis for $\Lambda^n(V^*)$, one can express ω as

$$\omega(Tv_1, \dots, Tv_n) = \omega(E_{i_1}, \dots, E_{i_k}) \varepsilon^{(i_1, \dots, i_k)}(Tv_1, \dots, Tv_n) = \omega(E_{i_1}, \dots, E_{i_k}) \det \begin{bmatrix} (Tv_1)^{i_1} & \cdots & (Tv_k)^{i_1} \\ \vdots & \ddots & \vdots \\ (Tv_1)^{i_k} & \cdots & (Tv_k)^{i_k} \end{bmatrix}$$

$$= \omega(E_{i_1}, \dots, E_{i_k}) \det \begin{bmatrix} \displaystyle\sum_{j=1}^n T_{i_1,j} v_1^j & \cdots & \displaystyle\sum_{j=1}^n T_{i_1,j} v_k^j \\ \vdots & \ddots & \vdots \\ \displaystyle\sum_{j=1}^n T_{i_k,j} v_1^j & \cdots & \displaystyle\sum_{j=1}^n T_{i_k,j} v_k^j \end{bmatrix} = \omega(E_{i_1}, \dots, E_{i_k}) \det \left(T \cdot \begin{bmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i^k} & \cdots & v_k^{i^k} \end{bmatrix} \right)$$

$$= (\det T) \omega(E_{i_1}, \dots, E_{i_k}) \varepsilon^{(i_1, \dots, i_k)}(v_1, \dots, v_n) = (\det T)\omega(v_1, \dots, v_n).$$

Hence, we are done. □

Lemma 6.10. Suppose $\alpha \in \Lambda^m(V^*), \beta \in \Lambda^n(V^*), \omega \in \Lambda^k(V^*)$, then we have $\alpha \wedge \beta \wedge \omega := (\alpha \wedge \beta) \wedge \omega = \alpha \wedge (\beta \wedge \omega)$.

Proof. We have

$$\begin{aligned}(\alpha\wedge\beta)\wedge\omega &= \frac{1}{(m+n)!k!}\sum_{\eta\in S_{m+n+k}}\operatorname{sgn}(\eta)\left(\frac{1}{m!n!}\sum_{\sigma\in S_{m+n}}\operatorname{sgn}(\sigma)\alpha\otimes\beta\right)\otimes\omega(v^\sigma)\\ &= \frac{1}{m!n!k!(m+n)!}\sum_{\eta\in S_{m+n+k}}\sum_{\sigma\in S_{m+n}}\operatorname{sgn}(\eta)\operatorname{sgn}(\sigma)\alpha\otimes\beta\otimes\omega(v^\sigma)\\ &= \frac{1}{m!n!k!}\sum_{\eta\in S_{m+n+k}}\operatorname{sgn}(\eta)\alpha\otimes\beta\otimes\omega(v^\sigma)\\ &= \frac{1}{m!n!k!(n+k)!}\sum_{\eta\in S_{m+n+k}}\sum_{\phi\in S_{n+k}}\operatorname{sgn}(\eta)\operatorname{sgn}(\phi)\alpha\otimes\beta\otimes\omega(v^\sigma)\\ &= \alpha\wedge(\beta\wedge\omega).\end{aligned}$$

Hence, we are done. □

Lemma 6.11. Let (ε^i) be any basis for V^* and $I=(i_1,\ldots,i_k)$ be any multi-index, then we have

$$\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}=\varepsilon^I$$

Proof. Let (v_i) be abitrary k-vector tuple in V represented by the basis (E_i) of V dual to (ε^i). We will show the following holds by induction. For $k=1$, this is trivial since $\varepsilon^I=\varepsilon^{i_1}$. For $n=2$, we have

$$\varepsilon^{i_1}\wedge\varepsilon^{i_2}(v_1,v_2)=\varepsilon^{i_1}\otimes\varepsilon^{i_2}(v_1,v_2)-\varepsilon^{i_1}\otimes\varepsilon^{i_2}(v_2,v_1)=\det\begin{bmatrix}v_1^1 & v_2^1\\ v_1^2 & v_2^2\end{bmatrix}=\varepsilon^{(i_1,i_2)}(v_1,v_2).$$

Suppose the following hypothesis holds for $k\in\mathbb{N}$, it follows that

$$\begin{aligned}&\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}\wedge\varepsilon^{i_{k+1}}(v_1,\ldots,v_{k+1})=(\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k})\wedge\varepsilon^{i_{k+1}}(v_1,\ldots,v_{k+1})=\varepsilon^{(i_1,\ldots,i_k)}\wedge\varepsilon^{i_{k+1}}(v_1,\ldots,v_{k+1})\\ &=\frac{1}{k!}\sum_{\sigma\in S_{k+1}}\operatorname{sgn}(\sigma)\varepsilon^{(i_1,\ldots,i_k)}\otimes\varepsilon^{i_{k+1}}(v_1,\ldots,v_{k+1})=\frac{1}{k!}\sum_{\sigma\in S_{k+1}}\operatorname{sgn}(\sigma)\varepsilon^{(i_1,\ldots,i_k)}\otimes\varepsilon^{i_{k+1}}(v_{\sigma_1},\ldots,v_{\sigma_{k+1}})\\ &=\frac{1}{k!}\sum_{\sigma\in S_{k+1}}\operatorname{sgn}(\sigma)\varepsilon^{(i_1,\ldots,i_k)}(v_{\sigma_1},\ldots,v_{\sigma_k})\cdot\varepsilon^{i_{k+1}}(v_{\sigma_{k+1}})=\sum_{i=1}^{k}(-1)^{k+1+i}\varepsilon^{(i_1,\ldots,i_k)}(v_1,\ldots,v_{i-1},v_{i+1})\varepsilon^{i_{k+1}}(v_i)\\ &=\det\begin{bmatrix}v_1^{i_1} & \ldots & v_{k+1}^{i_{k+1}}\\ \vdots & \ddots & \vdots\\ v_1^{i_{k+1}} & \ldots & v_{k+1}^{i_{k+1}}\end{bmatrix}=\varepsilon^{(i_1,\ldots,i_{k+1})}\end{aligned}$$

Hence the following is proven, as desired.

□

Chapter 7

Differential Forms on Manifolds

Definition 7.1 (m-forms). Suppose M is a smooth manifold with dimension n and $p \in M$, an **m-form at p** is a covector ω_p, that is

$$\omega_p : (T_pM)^m \to \mathbb{R} \text{ linear, or } \omega \in \Lambda^m(T_p^*M)$$

Alternatively, a m-form at p takes m tangent vectors and returns a real number linearly. We denote the vector space of smooth k-forms by

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M).$$

Consider a local smooth chart (φ, U) for some neighborhood of p, for $m = 1$, as ω_p is linear, one can rewrite the output of ω_p by the following formula:

$$\omega_p(v) = a_1 dx^1 + a_2 dx^2 + \cdots + a_n dx^n = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \cdot v$$

For $m = 2$, let $u = u^i \dfrac{\partial}{\partial x^i}$ and $v = v^i \dfrac{\partial}{\partial x^i}$ (Einstein's summation convention), since ω_p is multilinear and alternating, it would follow that

$$\begin{aligned} \omega_p(u, v) = \omega_p \left(u^i \frac{\partial}{\partial x^i}, v^i \frac{\partial}{\partial x^i} \right) &= \sum_{i \neq j} u^i v^j \omega_p \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i<j} (u^i v^j - u^j v^i) \omega_p \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \end{aligned}$$

Let $dx^i \wedge dx^j(u, v) := u^i v^j - u^j v^i = \det \begin{pmatrix} u^i & v^i \\ u^j & v^j \end{pmatrix}$, and $\omega_{ij} := \omega_p \left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j} \right)$, one can reduce the following equality to

$$\omega_p(u, v) = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j(u, v)$$

or equivalently, i would like to use the reduced notation based on Einstein's summation convention,

$$\omega_p = \omega_N dx^i \wedge dx^j,$$

for each $N \in [n]^2 := \{(i, j) \mid i, j \in [n]\}$. By our definition of $dx^i \wedge dx^j$, since

$$\det[u^{i,j}, v^{i,j}] = dx^i \wedge dx^j(u, v)$$

is multilinear, one can be extended to the generalized operation, which is

$$\boxed{dx^{i_1} \wedge \cdots \wedge dx^{i_n}(v_1, \dots, v_m) := \det[v_1^{i_1, \dots, i_n}, \dots, v_n^{i_1, \dots, i_m}] = \det[v_i^N]}. \tag{*}$$

This is called the **Wedge product**. Moreover, it is worth noting that this operation holds only for the basis of tangent space, we want it also works for any two forms. But beforehand, we state the general formula for any higher m-forms:

Theorem 7.2. Any m-form of dimension n can be expressed uniquely as

$$\omega_p = \omega_N dx^{i_1} \wedge \cdots \wedge dx^{i_m}$$

where $N \in [n]^m := \{(i_1, \ldots, i_m) \mid i_1, \ldots, i_m \in [n]\}$. Generally, since $(T_p^* M)^m$ is a vector space, it thus have basis, which is

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_m} \mid i_1 < i_2 < \cdots < i_m \in [n]\}$$

Consequently, the dimension of $(T_p^* M)^m$ is $\binom{n}{m}$.

Now we construct a Wedge product operation between any forms. Consider vectors

$$\{v_1, \ldots, v_{m+n}\} \in (T_p M)^{m+n},$$

and m-form ω_1 and n-form ω_2. By writiting ω_1 and ω_2 in basis of $(*)$ and since Wedge is multilinear to forms, it follows that

$$\begin{aligned}
&\omega_1 \wedge \omega_2(v_1, \ldots, v_{m+n}) = (\sum \omega_{1M} dx^{i_1} \wedge \cdots \wedge dx^{i_m}) \wedge (\sum \omega_{2N} dx^{j_1} \wedge \cdots \wedge dx^{j_n}) \\
&= \sum \omega_{1M}\omega_{2N} dx^{i_1} \wedge \cdots \wedge dx^{i_m} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\
&= \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \operatorname{sgn}(\sigma) \omega_{\sigma_1 \ldots \sigma_m} dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_m} \omega_{\sigma_{m+1} \ldots \sigma_{m+n}} dx^{\sigma_{m+1}} \wedge \cdots \wedge dx^{\sigma_{m+n}} \\
&= \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \operatorname{sgn}(\sigma) \omega_{\sigma_1 \ldots \sigma_m}(v_{\sigma_1}, \ldots, v_{\sigma_m}) \omega_{\sigma m+1 \ldots \sigma m+n}(v_{\sigma_{m+1}}, \ldots, v_{\sigma_{m+n}})
\end{aligned}$$

Proposition 7.3. Given two any m-form x, n-form y and p-form z , then we have

1. $x \wedge x = 0$,
2. $x \wedge y = (-1)^{nm} y \wedge x$,
3. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
4. If $m = n$ then $(x + y) \wedge z = x \wedge z + y \wedge z$.
5. (m - form)$\wedge$(n - form) = $(m + n)$ - form.

Definition 7.4. Let $F : M \to N$ be a smooth map between smooth manifolds M and N, ω is a differential form on N . The *pullback* $F^*\omega$ is a differential form on M, which is defined as

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))$$

Proposition 7.5. Let $F : M \to N$ be smooth map.

1. $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.
2. $F^*(\omega \wedge \eta) = (F^*(\omega)) \wedge (F^*(\eta))$.
3. In any smooth chart,

$$F^* \left(\sum \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) = \sum (\omega_I \circ F) d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

Proof. Let (v_i) and (w_i) be abitrary k-vector tuple of T_pM, α be abitrary real number. We have

$$\begin{aligned}(F^*\omega)_p(v_1+\alpha w_1,\dots,v_k+\alpha w_k) &= \omega_{F(p)}(dF_p(v_1+w_1),\dots dF_p(v_k+w_k))\\ &= \omega_{F(p)}(dF_p(v_1)+dF_p(\alpha w_1),\dots dF_p(v_k)+dF_p(\alpha w_k))\\ &= \omega_{F(p)}(dF_p(v_1),\dots,dF_p(v_k))+\alpha\omega_{F(p)}(dF_p(w_1),\dots,dF_p(w_k))\\ &= (F^*\omega)_p(v_1,\dots,v_k)+\alpha(F^*\omega)_p(w_1,\dots,w_k).\end{aligned}$$

Thus F^* is linear over $\mathbb{R}$. To prove the second property, suppose ω is k-form and η is l-form on N, one can be expressed so that

$$\begin{aligned}F^*(\omega\wedge\eta) = \omega_{F(p)}\wedge\eta_{F(p)}(dF_p) &= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\omega_{F(p)}\otimes\eta_{F(p)}(d(v_{\sigma_1}),\dots,d(v_{\sigma k+l}))\\ &= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(F^*\omega)_p\otimes(F^*\eta)(d(v_{\sigma_1}),\dots,d(v_{\sigma k+l}))\\ &= \frac{(k+l)!}{k!l!}\operatorname{Alt}(F^*(\omega)_p\otimes F^*(\eta)_p)\\ &= F^*(\omega)_p\wedge F^*(\eta)_p.\end{aligned}$$

Since F^* is proven to be linear and pull back of a real-valued function is just a composition, one can obtain

$$\begin{aligned}F^*\left(\sum\omega_I dx^{i_1}\wedge\dots\wedge dx^{i_k}\right) &= \sum F^*\left(\omega_I dx^{i_1}\wedge\dots\wedge dx^{i_k}\right)\\ &= \sum(\omega_I\circ F)F^*(dx^{i_1})\wedge\dots\wedge F^*(dx^{i_k})\\ &= \sum(\omega_I\circ F)dx^{i_1}(dF_p^1)\wedge\dots\wedge dx^{i_1}(dF_p^k)\\ &= \sum(\omega_I\circ F)d(x^{i_1}\circ F)\wedge\dots d(x^{i_k}\circ F)\end{aligned}$$

Hence, we are done. □

Theorem 7.6. Let $F: M\to N$ be smooth map between n-dimensional manifolds, (x^i) and (y^i) be smooth coordinates on open subsets $U\subseteq M$ and $V\subseteq N$, respectively, and u be a continuous real-valued function on V, then the following holds on $U\cap F^{-1}(V)$:

$$F^*(u dy^1\wedge\dots dy^n) = (u\circ F)(\det J_F)dx^1\wedge\dots\wedge dx^n.$$

Proof. Since F^* is linear over $\mathbb{R}$, it follows that

$$F^*(u dy^1\wedge\dots dy^n) = (u\circ F)d(y^1\circ F)\wedge\dots\wedge d(y^n\circ F) = (u\circ F)dF^1\wedge\dots\wedge dF^n$$

For abitrary $1\le j\le n$, by writings dF^j in basis of (x^i)

$$dF^j = \frac{\partial F^j}{\partial x^i}dx^i.$$

Since $a\wedge a = 0$ for any form, this implies

$$\begin{aligned}dF^1\wedge\dots\wedge dF^n &= \sum_{\sigma\in S_n}\left(\frac{\partial F^1}{\partial x^{\sigma_1}}\cdots\frac{\partial F^n}{\partial x^{\sigma_n}}\right)dx^{\sigma_1}\wedge\dots\wedge dx^{\sigma_n}\\ &= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\left(\frac{\partial F^1}{\partial x^{\sigma_1}}\cdots\frac{\partial F^n}{\partial x^{\sigma_n}}\right)dx^1\wedge\dots\wedge dx^n\\ &= \det\begin{bmatrix}\frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial F_n}{x^1} & \cdots & \frac{\partial F_n}{x^n}\end{bmatrix}dx^1\wedge\dots\wedge dx^n\\ &= J_F dx^1\wedge\dots\wedge dx^n\end{aligned}$$

.

Hence we are done. $\square$

Chapter 8

Measure Theory

Definition 8.1 (σ-algebra). Let X be a set, a collection $\mathcal{X} \in \mathcal{P}(X)$ is called a σ-algebra if it is

1. The set X is itself in $\mathcal{X}$.
2. *Closed under unions:* Let $(E_i) \subseteq \mathcal{X}$ be countable subset of $\mathcal{X}$, then
$$\bigcup_{n=1}^{\infty} E_i \in X.$$
3. *Closed under complements:* If $A \in \mathcal{X}$, then
$$A^c := X \backslash A \in \mathcal{X}$$

Proposition 8.2. If $\mathcal{X}$ is a σ-algebra and $(X_i) \subseteq \mathcal{X}$ be countable subset of $\mathcal{X}$, then we have
$$\bigcap_{n=1}^{\infty} X_n \in \mathcal{X}$$

Proof. By the Morgan law, we have
$$\bigcap_{n=1}^{\infty} E_n = X \backslash \bigcap_{n=1}^{\infty} (X \backslash X_n) = \left(\bigcup_{n=1}^{\infty} E_n^c \right)^c.$$

Using those above condition yields
$$\begin{aligned} X_n^c \in \mathcal{X} \Rightarrow E_n^c \in \mathcal{X} \text{ for all n}, & \qquad \text{(condition 3)}, \\ \Rightarrow \bigcup_{n=1}^{\infty} E_n^c \in \mathcal{X} & \qquad \text{(condition 2)}, \\ \Rightarrow \left(\bigcup_{n=1}^{\infty} X_n^c \right)^c \in \mathcal{X} & \qquad \text{(condition 3)}, \\ \Rightarrow \bigcap_{n=1}^{\infty} E_n \in X & \end{aligned}$$

Hence, we are done. □

Definition 8.3 (Measure). Let X be a set with σ-algebra $\mathcal{M}$. A *measure* on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfying

1. $\mu(\varnothing) = 0$,
2. If (E_n) is a countably disjoint subset of $\mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_i\right) = \sum_{n=1}^{\infty} \mu(E_i).$$

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space* and the pair $(X, \mathcal{M})$ is called a *measurable space.*

Theorem 8.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, then the followings hold.

1. *Monotonicity:* If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. *Subadditivity:* If (E_n) is a countably disjoint subset of $\mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

3. *Continuity from below:* If $(E_n) \subset \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to+\infty} \mu(E_n)$$

4. *Continuity from above:* If $(E_n) \subset \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to+\infty} \mu(E_n)$$

Proof. 1. Since $F \backslash E$ and E is disjoint, we have $\mu(F) = \mu((F\backslash E) \cup E) = \mu(F\backslash E) + \mu(E) \geq \mu(E)$.
2. Let $X_n = E_n \cup E_{n+1} \cup \dots$, it follows that

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left((E_1\backslash X_1) \cup \bigcup_{n=2}^{\infty} E_n\right) = \mu(E_1\backslash X_1) + \mu\left(\bigcup_{n=2}^{\infty} E_n\right)\\ &\leq \mu(E_1) + \mu\left(\bigcup_{n=2}^{\infty} E_n\right)\\ &\leq \mu(E_1) + \mu(E_2)\mu\left(\bigcup_{n=3}^{\infty} E_n\right)\\ &\leq \cdots\\ &\leq \sum_{n=1}^{\infty} \mu(E_n).\end{aligned}$$

3. Let $n \in \mathbb{N}$, we have

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu(E_n)$$

Hence,

$$\lim_{n\to+\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \lim_{n\to+\infty} \mu(E_n)$$

4. Similar to the third property, since
$$\mu\left(\bigcap_{i=1}^{n} E_i\right) = \mu(E_n)$$
This implies
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) \lim_{n\to+\infty} \mu(E_n)$$
$\square$

Definition 8.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, define the set
$$\ker(\mu) := \{E \in \mathcal{M} \mid \mu(E) = 0\}.$$
A measure whose domain includes all subset of an element $E \in \ker(\mu)$ is called *complete*.

Theorem 8.6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\overline{\mathcal{M}}$ be the collection satisfying
$$\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \in \ker(\mu)\}.$$
Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ to μ to a comple measure on $\overline{\mathcal{M}}$.

Chapter 9

Orientations

Definition 9.1. Two ordered basis $(E_1, \ldots, E_n)$ and $(\tilde{E}_1, \ldots, \tilde{E}_n)$ for V are called *consistently oriented* if the transition matrix $[B]_{E \to \tilde{E}}$ defined by

$$[E] = B[\tilde{E}]$$

has positive determeniant.

Definition 9.2. Let V be a vector space, We denote the set

$$\mathcal{O}(V) := \mathcal{B}(V)/\mathcal{R}(V),$$

as the set of *all possible orientation of* V, where $\mathcal{R}(V) := \left\{ \left([E_i], [\tilde{E}_i]\right) \in V^2 \mid \det[E_i] \cdot \det[\tilde{E}_i] > 0 \right\}$, and each element of $\mathcal{O}(V)$ (which is a equivalent class) is said to be the *orientation for* V.

Definition 9.3. A vector space with a choice of orientation $(V, [E_i])$ (where $[E_i] \in \mathcal{O}(V)$) is called an *oriented vector space*. Any ordered basis (E_i) that is in the given orientation is said to be *positively oriented*. Any basis that is not in the given orientation is said to be *negatively oriented*.

Remark 9.4. To prove a set of ordered basis determines an orientation for a vector space V, it suffices to prove every element of the following set belongs to only one equivalent class of $\mathcal{O}(V)$.

Theorem 9.5. Let V be a vector space of dimension $n \geq 1$. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation of V as the set of ordered bases (E_i) such that $\omega(E_i) > 0$. Two nonzero n-covectors ω and η determines the same orientation if and only if $\omega \cdot \eta > 0$.

Proof. Let (E_i) and $(\tilde{E}_i)$ be the ordered basis of V, we denote the set

$$\mathcal{O}_\omega = \left\{ \left([E_i], [\tilde{E}_i]\right) \mid \omega(E_i) > 0 \text{ and } \omega(\tilde{E}_i) > 0 \right\}.$$

Since those two sets are linearly independent, one can find a linear map $\mathcal{B} : V \to V$ such that $[E_i] = B[\tilde{E}_i]$. By the above proposition, it follows that

$$\omega(E_i) = \det(\mathcal{B})\omega(\tilde{E}_i)$$

which implies $\det(\mathcal{B}) > 0$ or $\det[E_i] \cdot \det[\tilde{E}_i] > 0$ if and only if $\omega(E_i) > 0$ and $\omega(\tilde{E}_i) > 0$. Thus $\mathcal{O}_\omega$ is an equivalent class of $\mathcal{O}(V)$, which implies it is an orientation of V. In addition, that ω and η determines the same orientation implies that $\omega(E_i) \cdot \eta(E_i) > 0$ since $\eta(E_i) > 0$, and vice versa, as desired. ☐

Definition 9.6. If V is an oriented n-dimensional vector space and ω is an n-covector determines the orientation of V that satifying the above theorem, ω is called a *positively oriented* n-covector.

Definition 9.7. Let M be a smooth manifold, a *vector field on M* is a map

$$\begin{aligned} X : M &\to TM \\ p &\mapsto X(p) \in T_p(M) \end{aligned}$$

such that X is a smooth section of the tangent bundle $\pi \circ X = Id_M$, where $\pi : TM \to M$ is a projection.

Definition 9.8. Let $(E_1, \ldots, E_n)$ be a collection of vector fields determined on the open set $U \subseteq M$. If for every point $p \in M$, the set $(E_i(p))$ form a basis of the tangent space T_pM, the collection (E_i) is said to be *a local frame.*

Definition 9.9. Let M be a smooth n-manifold, (E_i) be a local frame for TM and let φ be the map satisfying

$$\sigma : M \to \{-1, +1\}$$

We say that (E_i) is *positively oriented* if (E_i) is a positively oriented basis for $(T_pM, \sigma(p))$. The function σ is called *pointwise orientation.*

Definition 9.10. A pointwise orientation σ is said to be continuous if for every point $p \in M$, there exists an open neighborhood U of p and the local frame (E_i) defined U such that $(E_i(q))$ is positively oriented respect to restricted σ_U for all $q \in U$. σ is called an *orientation on M*.

Definition 9.11. An *oriented manifold* is an ordered pair $(M, \mathcal{O})$, where M is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for M.

Chapter 10

Integration on Manifolds

Definition 10.1 (Measure Zero set). An *open rectangle* is the set of the form

$$C_a^b := (a^1, b^1) \times \cdots \times (a^n, b^n),$$

where $a^i < b^i$. The *volume of* C_a^b is denoted by:

$$\mathrm{Vol}(C_a^b) := (b^1 - a^1) \dots (b^n - a^n).$$

A subset $X \subset \mathbb{R}^n$ is said to have *measure zero* if for every $\varepsilon > 0$, there exists a countable cover of open rectangles $\{C_i\}$ such that

$$\sum_i \mathrm{Vol}(C_i) < \varepsilon.$$

A *domain of integration* in $\mathbb{R}^n$ is the bounded subset whose boundary has measure zero.

Definition 10.2. Let $D \subset \mathbb{R}^n$ be a domain of integration and $\omega = f dx^1 \wedge \cdots \wedge dx^n$ be a differential n-form on $\overline{D}$, where $f : \overline{D} \to \mathbb{R}$ is some continuous function. We define the integral of ω over D as

$$\int_D \omega = \int_D f dx^1 \dots dx^n = \int_D f dV.$$

More generally, let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and D be any domain of integration containing the compact set $\mathrm{supp}(\omega)$, then

$$\int_U \omega = \int_D \omega.$$