# Differential Geometry

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## Contents

# 1 Differential Topology

## 1.1 Topological Spaces

**Definition 1.1.1.** Let $X$ be a set, a *topology on* $X$ is a collection $\mathcal{X} \subseteq \mathcal{P}(X)$ satisfying

1. $X$ and $\varnothing$ are element of $\mathcal{X}$.
2. Union of elements of $\mathcal{X}$ is itself an element of $\mathcal{X}$.
3. Intersection of elements of $\mathcal{X}$ is itself an element of $\mathcal{X}$.

A pair $(X, \mathcal{X})$ is called a *topological space*. Elements of $X$ is called *points* and every set $U \in \mathcal{X}$ is called an *open subset of* $X$. A *neighborhood* of $p \in X$ is an open subset $U \subseteq X$ containing $p$.

**Definition 1.1.2** (Closed subsets). Let $X$ be a topological space, a subset $U \subseteq X$ is said to be a *closed subset of* $X$ if $X \backslash U$ is an open subset.

**Definition 1.1.3** (Closure and Interior). Let $X$ be a topological space, *the closure of* $A$ *in* $X$ is the smallest closed subset of $X$ containing $A$, defined by

$$\overline{A} := \bigcap \{B \subseteq X \mid B \supseteq A \text{ and } B \text{ is closed in } X\}.$$

*The interior of* $A$ is the largest open subset of $X$ contained by $A$, defined by

$$\mathrm{Int} A := \bigcup \{C \subseteq X \mid C \subseteq A \text{ and } C \text{ is open in } X\}.$$

*The exterior of* $A$ is the largest open subset of $X$ outside $A$, defined by

$$\mathrm{Ext} A := X \backslash \overline{A},$$

and *the boundary of* $A$ is an closed subset of $X$, defined by

$$\partial A := X \backslash (\mathrm{Int} A \cup \mathrm{Ext} A).$$

**Proposition 1.1.4.** Let $X$ be a topological space and let $A \subseteq X$ be any subset.

(1) $A$ point is in $\mathrm{Int}\, A$ if and only if it has a neighborhood contained in $A$.

(2) $A$ point is in $\mathrm{Ext}\, A$ if and only if it has a neighborhood contained in $X \backslash A$.

(3) $A$ point is in $\partial A$ if and only if every neighborhood of it contains both a point of $A$ and a point of $X \backslash A$.

(4) A point is in $\bar{A}$ if and only if every neighborhood of it contains a point of $A$.

(5) $\bar{A} = A \cup \partial A = \mathrm{Int}\, A \cup \partial A$.

(6) $\mathrm{Int}\, A$ and $\mathrm{Ext}\, A$ are open in $X$, while $\bar{A}$ and $\partial A$ are closed in $X$.

(7) The following are equivalent
   (a) $A$ is open in $X$.
   (b) $A = \mathrm{Int} A$.
   (c) $A$ contains none of its boundary points.
   (d) Every point of $A$ has a neighborhood contained in $A$.

(8) The following are equivalent
   (a) $A$ is closed in $X$.
   (b) $A = \overline{A}$.
   (c) $A$ contains all of its boundary points.
   (d) Every point of $X\backslash A$ has a neighborhood contained in $X\backslash A$.

*Proof.* (1) This is trivial since for every $a \in \mathrm{Int}A$, one can find an open neighborhood $C \subset A$ which contains $a$.

(2) By the Morgan law, we can rewrite

$$\mathrm{Ext}A = X\backslash\overline{A} = \bigcup\{B \subset X \mid B \subseteq X\backslash A \text{ and } B \text{ is open in } X\}$$

Let $a \in \mathrm{Ext}A$, one can find a neighborhood $U \subseteq X\backslash A$ which contains $a$.

(3) Suppose for the sake of condition, that $\partial A$ is nonempty, pick any $a \in \partial A$. Since we have

$$\partial A = (X\backslash\mathrm{Int}A) \cap (X\backslash\mathrm{Ext}A) = (X\backslash\mathrm{Int}A) \cap \overline{A},$$

it follows that any neighborhood $U$ containing $a$ must be contained by the intersection of the following closed subsets. Since $U \cap X\backslash\mathrm{Int}A \neq \varnothing$, which means this is the largest closed set outside $A$, we can find $u \in C \cap U$ and $C \subseteq X\backslash A$, where $C$ is an closed set. Since $U \cap \overline{A} \neq \varnothing$ and $U$ is open, then one can find $v \in U$ such that $v \in \overline{A}$. We define the set

$$D = \{v \in U \cap \overline{A} \mid v \in A\}.$$

If $D$ is nonempty, one can choose $v \in D$ and we are done. Suppose $D = \varnothing$, then for all $v \in U \cap \overline{A}$, we must have $v \in X\backslash A$. Thus $U \subseteq X\backslash A$, that $U$ is open implies $U \subseteq \mathrm{Ext}A$. Since $\mathrm{Ext}A \cap \overline{A} = \varnothing$ leading to $U \cap \overline{A} = \varnothing$, this contradicts the property that $U \cap \overline{A} \neq \varnothing$.

To establish to reverse implication, we assume the contrary holds, that is, there exists $a \in \partial A$ and its neighborhood $U$ that contain only points in $A$ (or $X\backslash A$). Since $U$ is open, it follows that $U \subseteq \mathrm{Int}A$, and the fact that $\mathrm{Ext}A \cap \mathrm{Int}A = \varnothing$ implies $U \cap \mathrm{Ext}A = \varnothing$, which deduces a contradiction. The case for $U \subseteq X\backslash A$ is clearly similar, and hence we are done.

(4) Suppose the contrary holds, that there exsits $a \in \overline{A}$ and a neighborhood $U$ not containing any point in $A$. Since $U$ is open, it follows that $U \subseteq \mathrm{Ext}A$ and hence $U \cap \{a\} = \varnothing$, contradiction.

(5) Since

$$\overline{A}\backslash A = (X\backslash\mathrm{Ext}A)\backslash A = X\backslash(\mathrm{Ext}A \cup A) \subseteq X\backslash(\mathrm{Ext}A \cup \mathrm{Int}A) = \partial A \Rightarrow \overline{A} \subseteq \partial A \cup A$$

and

$$A \cup \partial A = A \cup (X\backslash(\mathrm{Ext}A \cup \mathrm{Int}A)) = A \cup [(X\backslash\mathrm{Ext}A) \cap (X\backslash\mathrm{Int}A)] = A \cup (\overline{A} \cap (X\backslash\mathrm{Int}A)) \subseteq A \cup \overline{A} = \overline{A}.$$

Hence $\overline{A} \subseteq A \cup \partial A$.

(6) This is trivial since union of open subsets is open and intersection of closed subsets is closed. ☐

**Definition 1.1.5.** Let $X$ be a topological space and $A \subseteq X$, we say $p \in X$ is a *limit point of* $A$ if for every neighborhood $U$ satisfies $U\backslash\{p\} \cap A \neq \varnothing$. Conversely, a point $p \in A$ is called *isolated point of* $A$ if $p$ has a neighborhood $U$ satisfies $U \cap A = \{p\}$.

**Proposition 1.1.6.** A subset of a topological space is closed if and only if it contains all of its limit points.

*Proof.* ($\Rightarrow$) Let $A$ be a closed subset of a topoglogical space $X$ and $x \in X$ be a limit point of $A$. Then for any neighborhood $U$ containing $p$, it follows that $U\backslash\{x\} \cap A \neq \varnothing$. Suppose $x \notin A$, since $X\backslash A$ is open, it follows that $U \cap (X\backslash A)$ is nonempty and open. By the proposition above, we thus have $x \in \partial A \subseteq \overline{A} = A$, contradiction. Thus $x \in A$.

($\Leftarrow$) Since every limit point is in $A$ or $\partial A$, and by the fact that all of them are in $A$, we thus have $\partial A \subseteq A$, hence $A$ is closed. ☐

**Definition 1.1.7.** A subset $A$ of a topological space $X$ is said to be *dense in* $X$ if $\overline{A} = X$.

**Proposition 1.1.8.** Show that a subset $A \subseteq X$ is dense if and only if every nonempty open subset of $X$ contains a point of $A$.

*Proof.* ($\Rightarrow$) Assume the contrary holds, that one can find an open subset $U \subseteq X$ satisfying $A \cap U = \varnothing$. Since $X = A \cup \partial A \Rightarrow X \backslash \partial A \subseteq A$, then

$$(X\backslash \partial A) \cap U = \varnothing$$

Since $U \subseteq X\backslash A$, we thus have $U \subseteq \partial A$. By the above proposition, it follows that $A \cap U \neq \varnothing$, contradiction.

($\Leftarrow$) Suppose $A$ is not dense, in other words $X\backslash \overline{A}$ is an nonempty open subset, consequently, this implies $(X\backslash \overline{A}) \cap A \neq \varnothing$. But since $A \subseteq \overline{A}$, we thus have $X\backslash \overline{A} \subseteq X\backslash A$, or $(X\backslash \overline{A}) \cap A \subset (X\backslash A) \cap A = \varnothing$, which implies a contradiction. Hence we are done. $\square$

### 1.2 Convergence and Continuity

**Definition 1.2.1** (Convergence). Let $X$ be a topological space, $(x_n) \subseteq X$ is a sequence of points in $X$ and $x \in X$. We say $\lim\limits_{n\to+\infty} x_n = x$ or $x_n \to x$ if for every neighborhood $U$ of $x$, there exists $N(U) \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N(U)$.

**Proposition 1.2.2.** Let $X$ be a topological space, $A$ is a subset of $X$ and $(x_n) \subset A$. If $x_n \to x \in X$, then $x \in \overline{A}$.

*Proof.* Let $U$ be a neighborhood of $x$, since $x_n \to x$, there exists $N(U) \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Thus $U \cap A \neq \varnothing$. The above proposition implies that $x \in A$ or $U$ contains a point in $A$ and a point in $X\backslash A$. In either case, we always have $x \in \overline{A}$. $\square$

**Definition 1.2.3** (Continuity). Let $X$ and $Y$ be a topological spaces, a map $f : X \to Y$ is said to be *continuous* if for every open subset $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in $X$.

**Proposition 1.2.4.** A map between topological spaces is continuous if and only if its preimage of every closed subset is closed.

*Proof.* Let $f : X \to Y$ be the map between topological spaces satisfying $f^{-1}(U)$ is closed for all closed subsets $U \subseteq Y$. This implies both $Y\backslash U$ and $X\backslash f^{-1}(U)$ is open. Since we have

$$X\backslash f^{-1}(U) = f^{-1}(Y)\backslash f^{-1}(U) = f^{-1}(Y\backslash U) \text{ is open,}$$

and $U$ is abitrary closed subset, $f$ also preserve openess on preimage of the open subsets. Hence $f$ is continuous. $\square$

**Proposition 1.2.5.** Let $X, Y$ and $Z$ be topological spaces.

1. Every constant map $f : X \to Y$ is continuous.
2. The indentity map $\mathrm{Id}_X : X \to X$ is continuous.
3. If $f : X \to Y$ and $g : Y \to Z$ are both continuous, then so is their composition $g \circ f : X \to Z$.

*Proof.* It suffices to prove the third property. Let $U \in Z$ by any open subsets, we need to prove $(g \circ f)^{-1}(U)$ is open, in other words, this can be rewriten as

$$(g \circ f)^{-1}(U) = (f^{-1} \circ g^{-1})(U) = f^{-1}(g^{-1}(U))$$

Since $g^{-1}(U)$ is open, then $f^{-1}(g^{-1}(U))$ is also open. Hence $g \circ f$ is continuous. $\square$

**Proposition 1.2.6.** A map $f : X \to Y$ between topological spaces is continuous if and only if each point of $X$ has a neighborhood on which the restriction of $f$ is continuous.

*Proof.* ($\Rightarrow$) If $f$ is continuous and $x \in X$, we simply consider the restriction of $f_U : U \to Y$, where $U$ is any neighborhood of $x$, and $f_U^{-1}(V) = f^{-1}(V) \cap U$ is open set. Hence $f_U$ is continuous.

($\Leftarrow$) Suppose that $f$ is restrictly continuous on a neighborhood of every point $x \in X$. Let $U \subseteq Y$ be any open subset, it suffices to prove that $f^{-1}(U)$ is open. Let $u \in f^{-1}(U)$, by the hypothesis, one can find a neighborhood $V \subseteq X$ containing $u$ such that $f_V : V \to Y$ is continuous. Thus, the preimage

$$f_V^{-1}(U) = f^{-1}(U) \cap V \text{ is open.}$$

Since $f_V^{-1}(U) \subseteq f^{-1}(U)$, by the above proposition, it follows that $f^{-1}(U)$ is open, hence $f$ is continuous. $\square$

**Definition 1.2.7.** A map $f : X \to Y$ is said to be an *open map* if $f(U)$ is open for all open subsets $U \subset X$. Conversely, $f$ is said to be a *closed map* if $f(U)$ is closed for all closed subsets $U \subset X$.

**Proposition 1.2.8.** Suppose $X$ and $Y$ are topological spaces, and $f : X \to Y$ is a map.

1. $f$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
2. $f$ is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.
3. $f$ is continuous if and only if $f^{-1}(\mathrm{Int}B) \subseteq \mathrm{Int} f^{-1}(B)$ for all $B \subseteq X$.

### 1.3 Hausdorff Spaces

**Definition 1.3.1.** A topological space $X$ is said to be *Hausdorff* if two any distinct points in $X$ can be seperated by disjoint open subsets in $X$.

**Proposition 1.3.2.** Let $X$ be a Hausdorff space.

1. Every finite subsets of $X$ is closed.
2. If a sequence $(x_n) \subseteq X$ converges to a limit $p \in X$, the limit is unique.

**Proposition 1.3.3.** Suppose $X$ is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of $A$, then every neighborhood of $p$ contains infititely many points of $A$.

### 1.4 Bases

**Definition 1.4.1.** Let $X$ be a topoglogical space, a basis for the topology $X$ is a collection $\mathcal{B}$ of subsets in $X$ satisfying two conditions:

1. Every element in $\mathcal{B}$ is an open subset of $X$.
2. Every open subset in $X$ is the union of some collection of elements of $\mathcal{B}$.

### 1.5 Problems

**Problem 1.1.** Let $X$ be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$ and let $M$ be the quotient of $X$ by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that $M$ is locally Euclidean and second-countable, but not Hausdorff.

*Proof.* Consider the continuous map $\pi : X \to M$ be the quotient map satisfying $U \subseteq M$ is open if and only if $\pi^{-1}(U)$ is open in $X$. We consider two cases:

Case 1: $x_0 \neq 0$, consider the open neighborhood on $M$ of $[x_0]$ pulled back by $\pi^{-1}$ satisfying

$$I_{[x_0]} = (x_0 - \varepsilon, x_0 + \varepsilon) \times \{-1, 1\},$$

where $\varepsilon > 0$ is abitrary small such that $I_{[x_0]} \cap \{(0,-1),(0,1)\} = \varnothing$. We define a local chart $\varphi : \pi(I_{[x_0]}) \to (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \mathbb{R}$ satisfying

$$\varphi([t]) = t \text{ for all } [t] \in \pi(I_{[x_0]})$$

It suffices to check that $\varphi$ is homeomorphism. Since $\varphi([x_1]) = \varphi([x_2])$ implies $\pi(x_1) = \pi(x_2)$ and $[x_1] = [x_2]$, thus $\varphi$ is injective. $\varphi$ is also surjective since we can pick any equivalent class $[x]$ for given $x \in \pi(I_{[x_0]})$. The map $\varphi(\pi) : (t, \pm 1) \mapsto t$ implies $\varphi$ is continuous and the map $\varphi([t,1])^{-1} = [t,1] = \pi(i(t))$ is continuous, since the map $i : (x_0 - \varepsilon, x_0 + \varepsilon) \to X$ satisfying $i(t) = (t,1)$ is continuous. Hence $\varphi$ is homeomorphism and $M$ is locally Euclidean.

Case 2: $x_0 = 0$, since $(0,1)$ and $(0,-1)$ are distinct under the following equivalent relation, we just consider the open neighborhood on $M$ for $[(0,1)]$ (the same construct for $(0,-1)$) satisfying

$$I^+ = (-\varepsilon, +\varepsilon) \times \{1\}$$

where $\varepsilon > 0$ is abitrary. Then we define a local chart $\varphi^+ : \pi(I^+) \to (-\varepsilon, +\varepsilon) \subseteq \mathbb{R}$ such that

$$\varphi^+([t,1]) = t \text{ for all } [t,1] \in \varphi(I^+)$$

Notice that $\varphi^+$ is well-defined, continuous and bijective since

$$\varphi^+ \circ \pi((t,1)) = t$$

is continuous and $(\varphi^+)^{-1} = [(t,1)] = \pi(i(t))$ is also continuous. Thus, $\varphi$ is homeomorphism and hence $M$ is locally Euclidean.

To prove $M$ is second-countable, it suffices to prove $X$ is countable, we define the set

$$\mathcal{B}_X = \{(a,b) \times \{1\}, (a,b) \times \{-1\} \mid a, b \in \mathbb{Q}\}$$

Since the set $\{(a,b) \mid a, b \in \mathbb{Q}\}$ admits a countable basis for $\mathbb{R}$, therefore $\mathcal{B}_X$ is a countable basis of $X$. We define the set

$$\mathcal{B}_M = \{\pi(B) \mid B \in \mathcal{B}_X\},$$

Since $\pi$ is a quotient map, $\{\mathcal{B}_M\}$ is a second-countable basis for $M$.

Now we aim to prove $M$ is not Hausdorff at two points $(0,1)$ and $(0,-1)$. Let $U$ and $V$ be abitrary neighborhood of $[(0,1)]$ and $[(0,-1)]$ in $M$. Since $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open in Euclidean, one can find an open interval

$$(-\varepsilon_1, +\varepsilon_1) \times \{1\} \subseteq \pi^{-1}(U) \text{ and } (-\varepsilon_1, +\varepsilon_1) \times \{-1\} \subseteq \pi^{-1}(V)$$

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $I = (-\varepsilon_0, +\varepsilon_0)$, we thus have

$$\varnothing \neq \pi(I \times \{1\}) = \pi(I \times \{-1\}) \subseteq \pi(\pi^{-1}(U) \cap \pi^{-1}(V)) \subseteq U \cap V$$

Hence $U \cap V \neq \varnothing$ which implies $M$ is not Hausdorff. $\square$

**Problem 1.2.** For some $t \in \mathbb{R}$, we denote the set

$$\mathbb{R}_t = \mathbb{R} \times \{t\}$$

Let $I$ be an uncountable set, prove that the set

$$\mathcal{R} = \bigsqcup_{\alpha \in I} \mathbb{R}_\alpha$$

is locally Euclidean and Hausdorff, but not second-countable.

*Proof.* Let $u \in \mathcal{R}$. Then there exists a unique $\alpha \in I$ such that $u \in \mathbb{R}_\alpha$. Let $\psi_\alpha : \mathbb{R} \to \mathbb{R}_\alpha$ be the homeomorphism satisfying

$$\psi(x) = (x, \alpha),$$

Let $y = \psi^{-1}(u)$ and $U = (y - \varepsilon, y + \varepsilon) \times \{\alpha\}$ be an open neighborhood of $u$ pulled back by $\psi$. Since $\psi$ is a homeomorphism, it serves as a local chart around $u$, which implies $\mathcal{R}$ is locally Euclidean.

To prove $\mathcal{R}$ is Hausdorff, let $x = (u, \alpha), y = (v, \beta)$ be distinct points in $\mathcal{R}$, we consider two cases:

Case 1: $\alpha \neq \beta$, let $\varepsilon > 0$ be abitrary, we choose

$$\begin{aligned} U_\alpha &= (u - \varepsilon, u + \varepsilon) \times \{\alpha\}, \\ V_\beta &= (v - \varepsilon, v + \varepsilon) \times \{\beta\} \end{aligned}$$

Since $U_\alpha$ and $V_\beta$ are disjoint, $\mathcal{R}$ is Hausdorff in this case.

Case 2: $\alpha = \beta$. We choose

$$\begin{aligned} U_\alpha &= (u - \varepsilon, u + \varepsilon) \times \{\alpha\}, \\ V_\beta &= (v - \varepsilon, v + \varepsilon) \times \{\beta\}, \end{aligned}$$

where $\varepsilon$ satisfies $0 < \varepsilon < \dfrac{|u - v|}{2}$. Since $U_\alpha$ and $V_\beta$ are disjoint, every pair of distinct points in $\mathcal{R}$ can be seperated by disjoint open neighborhoods, hence $\mathcal{R}$ is Hausdorff in general.

To prove $\mathcal{R}$ is not second-countable, we consider the following proposition

**Proposition 1.5.1.** If a topological space contains uncountably many nonempty disjoint sets, then it is not second-countable.

For every $\alpha \in I$, we denote a corresponding open neighborhood $U_\alpha = (-\varepsilon, +\varepsilon) \times \{\alpha\}$. Since the collection $\{U_\alpha\}_{\alpha \in I}$ consists of uncountably many pairwise disjoint open sets in $\mathcal{R}$, the above proposition implies $\mathcal{R}$ is not second-countable. $\square$

**Problem 1.3.** Let $M$ be a topological manifold, and let $\mathcal{U}$ be an open cover of $M$.

1. Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $U$ is locally finite.
2. Give an example to show that the converse to (a) may be false.
3. Now assume that the sets in $\mathcal{U}$ are precompact in $M$, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

**Problem 1.4.** Suppose $M$ is a locally Euclidean Hausdorff space. Show that $M$ is secondcountable if and only if it is paracompact and has countably many connected components. [Hint: assuming $M$ is paracompact, show that each component of $M$ has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

**Problem 1.5.** Let $M$ be a nonempty topological manifold of dimension $n \geq 1$. If $M$ has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0, F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

**Problem 1.6.** Let $N$ denote the north pole $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let $S$ denote the south pole $(0, \ldots, 0, -1)$. Define the stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}$$

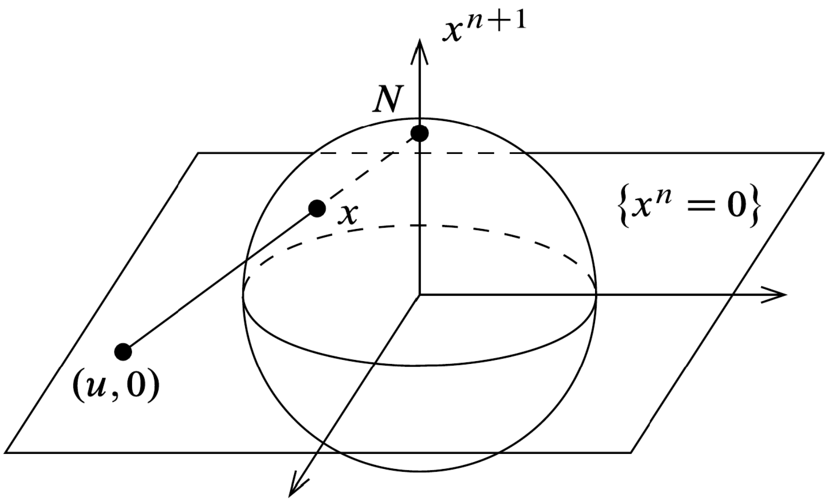
Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

1. For any $x \in \mathbb{S}^n \backslash \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through $N$ and $x$ intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through $S$ and $x$ intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called stereographic projection from the south pole.)

2. Show that $\sigma$ is bijective, and

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

3. Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$ (The coordinates defined by $\sigma$ or $\tilde{\sigma}$ are called stereographic coordinates).




*Proof.* (1) Define the line passing through $N$ and $x$ by

$$L(t) = N + t(x - N) = u(t)$$

Consider the intersection between $L(t)$ and the linear subspace of $\mathbb{R}^{n+1}$ where $x^{n+1} = 0$, we have

$$u(t) = (tx^1, \ldots, tx^n, 1 + t(x^{n+1} - 1)) = (u^1, \ldots, u^n, 0)$$

The $(n+1)-$term implies that $t = \dfrac{1}{1 - x^{n+1}}$, thus we have

$$u(t) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}} = \sigma(x)$$

(2) Let $u = (u^1, \ldots, u^n) \in \mathbb{R}^n$, it suffices to prove that there exists $(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \backslash \{N\}$ satisfying

$$u^i = \frac{x^i}{1 - x^{n+1}} \text{ for all } i = 1, \ldots, n.$$

Let $|u|^2 = \sum (u^i)^2$, it follows that

$$\begin{aligned} |u|^2 &= \frac{\sum (x^i)^2}{(1 - x^{n+1})^2} = \frac{1 - (x^{n+1})^2}{(1 - x^{n+1})^2} = \frac{1 + x^{n+1}}{1 - x^{n+1}} = \frac{2}{1 - x^{n+1}} - 1 \\ \Leftrightarrow x^{n+1} &= \frac{|u|^2 - 1}{|u|^2 + 1} \end{aligned}$$

Since $-1 < \dfrac{|u|^2-1}{|u|^2+1} < 1$, set $x^{n+1} = \dfrac{|u|^2-1}{|u|^2+1}$ and $x^i = \dfrac{2u^i}{|u|^2+1}$. Thus $\sigma$ is surjective onto $\mathbb{R}^n$.

To prove that $\sigma$ is injective, suppose $\sigma(x) = \sigma(y)$, we have

$$\begin{aligned}
\frac{x^i}{1-x^{n+1}} = \frac{y^i}{1-y^{n+1}} \Rightarrow \frac{\sum(x^i)^2}{(1-x^{n+1})^2} &= \frac{\sum(y^i)^2}{(1-y^{n+1})^2} \\
\Leftrightarrow \frac{1+x^{n+1}}{1-x^{n+1}} &= \frac{1+y^{n+1}}{1-y^{n+1}} \\
\Leftrightarrow \frac{2}{1-x^{n+1}} - 1 &= \frac{2}{1-y^{n+1}} - 1 \\
\Leftrightarrow \frac{2}{1-x^{n+1}} &= \frac{2}{1-y^{n+1}} \\
\Leftrightarrow 1-x^{n+1} &= 1-y^{n+1},
\end{aligned}$$

which implies $x^{n+1} = y^{n+1}$ and hence $x^i = y^i$ for all $i = 1, \ldots, n$. Therefore $\sigma$ is bijective and its inverse satisfies

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2-1\right)}{|u|^2+1}$$

We will verify that $\sigma^{-1}(u) \in \mathbb{S}^n$ for all $u$. Let $(x^i) = \sigma^{-1}(u)$, since we have

$$\sum(x^i)^2 = \frac{\sum(2u^i)^2 + (|u|^2-1)^2}{(|u|^2+1)^2} = \frac{4|u|^2 + (|u|^2-1)^2}{(|u|^2+1)^2} = \frac{(|u|^2+1)^2}{(|u|^2+1)^2} = 1$$

Thus $\sigma^{-1}$ maps every point in $\mathbb{R}^n$ into the sphere $\mathbb{S}^n$. (3) By the definition of $\tilde{\sigma}$, it can be written as

$$\tilde{\sigma}(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1+x^{n+1}},$$

and the same computation shows that $\tilde{\sigma}$ is bijective. Let $u = (u^i) \in \mathbb{R}^n \backslash \{0\}$ , the composition $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is computed by expressing

$$\begin{aligned}
\tilde{\sigma} \circ \sigma^{-1}(u) &= \tilde{\sigma}\left(\frac{2u^1}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right) \\
&= \frac{(u^1, \ldots, u^n)}{|u|^2}.
\end{aligned}$$

Since $\tilde{\sigma} \circ \sigma^{-1}$ is smooth and a diffeomorphism, $\sigma$ and $\tilde{\sigma}$ are compatible. Moreover, since the union of their domains covers entire $\mathbb{S}^n$, then they form an atlas which generates a smooth structure on $\mathbb{S}^n$.

$\square$

**Problem 1.7.** By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An angle function on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function $\theta$ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that $(U, \theta)$ is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

*Proof.* ($\Rightarrow$) Suppose there exists an angle function $\theta : \mathbb{S}^1 \to \mathbb{R}$ which is continuous. We aim to show that $\theta$ is discontinuous at $z = 1$. If we write $z = e^{i\pi\phi}$ then it follows that $\theta = \pi\phi + k2\pi$, where $k$ is some integer. Since $\theta$ is continuous, $k$ must be fixed. Let

$$a_n = e^{i\pi/n} \text{ and } b_n = e^{i(2\pi - 1/n)}$$

Then we have

$$\lim_{n\to+\infty} \theta(a_n) = \theta(0) = k2\pi \text{ and } \lim_{n\to+\infty} \theta(b_n) = \theta(2\pi) = 2\pi + k2\pi$$

Since $2\pi \neq 0$, $\theta$ is not continuous at $z = 1$, hence there is no angle function for the case $U = \mathbb{S}^1$. We define $\alpha(z)$ as a unique function satisfying $z = e^{i\pi\alpha(z)}$ and $\alpha(z) \in [0, 2\pi)$.

If $U \neq \mathbb{S}^1$. In case that $1 \in U$, since there must exsits $z_0 \neq 1$ and $z_0 \notin U$. Since $\alpha$ is continuous, we choose $\theta$ satisfying

$$\theta(z) = \alpha(z), (\alpha(z) < \alpha(z_0)) \text{ and}$$
$$\theta(z) = 2\pi - \alpha(z), (\alpha(z) \geq \alpha(z_0))$$

If $1 \notin U$, we choose $\theta(z) = \alpha(z)$ for all $z \in U$.

Since $\theta$ is continuous and a homeomorphism on from open subset $U \subset \mathbb{S}^1$ onto an open interval $\theta(U)$, the pair $(U, \theta)$ is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. □

**Problem 1.8.** Complex projective $n$-space, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$\left(x^1 + iy^1, \ldots, x^{n+1} + iy^{n+1}\right) \leftrightarrow \left(x^1, y^1, \ldots, x^{n+1}, y^{n+1}\right)$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.) (Used on pp. 48, 96, 172, 560, 561.)

*Proof.* Suppose $\mathbb{S}^n$ is an $n$-dimensional sphere, it suffices to prove the restriction map

$$\pi|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{CP}^n$$

is continuous and surjective. For the sake of condition, we assume to write $\pi$ instead of $\pi|_{\mathbb{S}^n}$. Given $z \in \mathbb{CP}^n$, by the definiton of projective space, $z$ is an equivalent class satisfying

$$[z^0 : \cdots : z^n] \sim [\lambda z^0 : \cdots : \lambda z^n]$$

for all nonzero complex number $\lambda$. To prove $\pi$ is surjective, let $[z] \in \mathbb{CP}^n$ be abitrary, one can rewrite

$$[z] = [z^0 : \cdots : z^n] \sim \left[\frac{z^0}{|z|} : \cdots : \frac{z^n}{|z|}\right]$$

Since we have

$$\sum \frac{(z^i)^2}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

then $\left(\frac{z^0}{|z|} : \cdots : \frac{z^n}{|z|}\right) \in \mathbb{S}^n$ and hence it follows that $\pi\left(\frac{z^0}{|z|} : \cdots : \frac{z^n}{|z|}\right) = [z]$. Since the natural projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ is already continuous, the its restriction on closed subset $\mathbb{S}^n$ is also continuous. Since $\mathbb{S}^n$ is closed and bounded in $\mathbb{C}^{n+1}$ by the corresponding identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$, by the Heine-Borrel theorem, $\mathbb{S}^n$ is compact. And since $\pi_{\mathbb{S}^n}(\mathbb{CP}^n) = \mathbb{S}^n$, $\mathbb{CP}^n$ is a compact.

To prove $\mathbb{CP}^n$ is locally Euclidean, for each $i = 0, \ldots, n$, let $\tilde{U_i} \subset \mathbb{C}^{n+1}$ be the subset containing all points $x \in \mathbb{C}^{n+1}$ satisfying $x^i \neq 0$ and $\varphi_i : U_i \to \mathbb{C}^n$ be the local chart (where $U_i = \pi(\tilde{U_i})$) satisfying

$$\varphi[z^0 : \cdots : z^n] = \left(\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^n}{z^i}\right).$$

This map is well-defined since the left-hand side remains if we replace $[z^1 : \cdots : z^n]$ by $\lambda[z^1 : \cdots : z^n]$. To prove $\varphi_i$ is injective, suppose $\varphi_i[a] = \varphi_i[b]$, then we have

$$\frac{a^j}{a^i} = \frac{b^j}{b^i} \text{ for all } j = 0, \ldots, n+1$$

Let $\lambda = \dfrac{a^i}{b^i}$ be fixed, it follows that $a^j = \lambda b^j$ for all $j = 0, \dots, n+1$, hence $[a] = [b]$. To prove $\varphi_i$ is surjective, if $x \in \mathbb{C}^n$, we choose $[z] \in \mathbb{CP}^n$ satisfying $z^j = x^{j-1}$ for all $j \neq i$ and $z^i = 1$, this implies $\varphi_i[z] = x$. Therefore $\varphi_i$ is bijective. Moreover, $\varphi$ is smooth and its inverse given by

$$\varphi_i^{-1}(x^i) = [x^0, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n]$$

is also smooth. Thus $\varphi_i$ is homeomorphism and $\mathbb{C}^n \cong \mathbb{R}^{2n}$ implies that $\mathbb{CP}^n$ is locally $2n$-dimensional Euclidean. In particular, since $\varphi_i$ is diffeomorphism, it follows that $(U_i, \varphi_i)$ is smooth local chart and they are parwise compatible since a composition of two diffeomorphism is again a diffeomorphism. Hence, the atlas $\{(U_i, \varphi_i)\}$ defines a smooth structure on $\mathbb{CP}^n$.

To prove $\mathbb{CP}^n$ is Hausdorff, let $[a], [b]$ be distinct equivalence classes in $\mathbb{CP}^n$, which means $[a] \neq [b]$, pushed forward by $\varphi_i$, where $i$ is the non-negative integer satisfying $a^i, b^i \neq 0$. By consider two open disks

$$\begin{aligned} \mathcal{D}_{\varphi_i(a)} &= \{z \in \mathbb{C}^n \mid |z - \varphi_i(a)| < \varepsilon\} \text{ and} \\ \mathcal{D}_{\varphi_i(b)} &= \{z \in \mathbb{C}^n \mid |z - \varphi_i(b)| < \varepsilon\} \end{aligned}$$

where $\varepsilon > 0$ satisfies $\varepsilon < \dfrac{|\varphi(a) - \varphi(b)|}{2}$ implying $\mathcal{D}_{\varphi(a)} \cap \mathcal{D}_{\varphi(b)} = \varnothing$. Since $\varphi_i$ is homeomorphism, then any distinct points in $\mathbb{CP}^n$ can be seperated by two open disjoint neighborhoods, which is $\mathcal{D}_{\varphi(a)}$ and $\mathcal{D}_{\varphi(b)}$ in this case. Therefore $\mathbb{CP}^n$ is Hausdorff.

The fact that $\mathbb{C}^n$ is second-countable and $\{(U_i, \varphi_i)\}$ is a smooth structure implies that $\mathbb{CP}^n$ is also second-countable. In conclusion, $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, as desired.

$\square$

**Problem 1.9.** Let $k$ and $n$ be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \dots, e_k)$ and $(e_{k+1}, \dots, e_n)$, respectively, where $e_i$ is the $i$ th standard basis vector for $\mathbb{R}^n$. For any $k$-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with $Q$, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix $B$ such that $S$ is spanned by the columns of the matrix $\binom{I_k}{B}$, where $I_k$ denotes the $k \times k$ identity matrix.

**Problem 1.10.** Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that $M$ is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma : \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and $\pi$ is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

**Problem 1.11.** Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

## 2 Partition of Unity

### 2.1 Construction

**Lemma 2.1.1.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-1/t}, t > 0, \\ 0, t \leq 0, \end{cases}$$

is smooth.

**Lemma 2.1.2.** Given any real numbers $r_1$ and $r_2$ such that $r_1 < r_2$, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) = 1$ for $t \leq r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$ and $h(t) = 0$ for $t \geq r_2$.

*Proof.* Let $f$ be the smooth function of the previous lemma, and set

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$$

Then $h$ satisfies the desired properties. □

**Lemma 2.1.3.** Given any positive numbers $r_1 < r_2$, there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H = 1$ on $\overline{B}_{r_1}(0)$, $0 < H < 1$ for all $x \in B_{r_2}(0) \backslash \overline{B}_{r_1}(0)$ and $H = 0$ on $\mathbb{R}^n \backslash B_{r_2}(0)$.

*Proof.* By setting $H(x) = h(|x|)$ and we are done. □

**Definition 2.1.4.** Let $\mathcal{X} = (X_\alpha)_{\alpha\in}$ be an abitrary open cover of $M$. A *partition of unity subordinate to* $\mathcal{X}$ is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ satisfying the following:

1. $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.
2. $\mathrm{supp}\psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.
3. Every point has a neighborhood that intersects $\mathrm{supp}\psi_\alpha$ for finite values of $\alpha$.
4. $\displaystyle\sum_{\alpha\in A} \psi_\alpha = 1$ for all $x \in M$.

**Theorem 2.1.5.** Suppose $M$ is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha\in A}$ is any indexed open cover of $M$. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

*Proof.* Naturally, if we can find an indexed family support where each of them is a regular coordinate ball, then the construction of smooth function satisfying those following conditions is possible. However, its worth noting that the given open cover $\mathcal{X}$ is not locally finite. Therefore, our idea to find an indexed locally finite refinement of $\mathcal{X}$ and every element also a regular coordinate ball.

The fact that $M$ is smooth manifold implies that there exsists an atlas $\{(U_i, \varphi_i)\}$ where $\{U_i\}$ is a basis for the topology of $M$ and we can define every regular coordinate ball by some charts of this atlas. Since every $X_\alpha$ is itself a smooth manifold, thus it has a basis of regular coordinate balls $\mathcal{B}_\alpha$, and then $\mathcal{B} = \bigcup_{\alpha\in A} \mathcal{B}_\alpha$ defines a basis for the topology on $M$. Since $M$ is Hausdorff and second-countable, hence it is paracompact, then there exists a subset of $\mathcal{B}$, denoted by $\{B_i\}$, is a locally finite open refinement of $\mathcal{X}$, and hence $\{\overline{B_i}\}$ is also locally finite. Since each $B_i$ is an open subset of some $X_\alpha$, then there exists a larger coordinate ball $\tilde{B}_i$ of $X_\alpha$ such that $\tilde{B}_i \supset B_i$ and a corresponding local chart $\varphi_i : \tilde{B}_i \to \mathbb{R}^n$ that maps $\varphi(B_i) = B_{r_1}(0)$ and $\varphi(\tilde{B}_i) = B_{r_2}(0)$, where $r_1 < r_2$ are two positive real numbers. Then we can define a smooth function $f_i : M \to \mathbb{R}$ as follows:

$$f_i(x) = \begin{cases} H_i \circ \varphi_i(x) \text{ on } \tilde{B}_i \\ 0 \text{ on } M \backslash \overline{\tilde{B}_i} \end{cases}$$

where $H_i$ is the smooth function defined in the previous lemma for $B_{r_1}(0)$ and $B_{r_2}(0)$.Consequently, it follows that $\mathrm{supp} f_i = \overline{\tilde{B}_i}$ for all $i$. Since every $f_i$ is non-negative everywhere on $M$ and each point in $M$ is contained by some $B_i$, then the function $f : M \to \mathbb{R}$ defined by

$$f(x) = \sum_i f_i(x)$$

never vanishing to zero and $f$ is well-defined since the following sum is finite for all point in $M$. Hence, $f$ is smooth. Let $g_i := \dfrac{f_i}{f}$, we thus have

$$\sum_i g_i(x) = 1 \text{ for all } x \in M.$$

For every $\alpha \in A$, we define $\psi_\alpha$ as the partition sum of $g_i$ satisfying

$$\psi_\alpha = \sum_{i|\tilde{B}_i \subseteq X_\alpha} g_i,$$

This partition of $\psi_\alpha$ satisfies $\sum_{\alpha\in A} \psi_\alpha = 1$ and

$$\text{supp}\psi_\alpha \subseteq \bigcup_{i|\tilde{B}_i \subseteq X_\alpha} \text{supp} g_i \subseteq X_\alpha,$$

and is a smooth function as we can verify. Hence the indexed family $\{\psi_\alpha\}_{\alpha\in A}$ is a smooth partition of unity subordinate to $\mathcal{X}$. ☐

**Definition 2.1.6.** Let $M$ be a topological space, $A \subseteq M$ is a closed subset and $U \subseteq M$ is an open subset containing $A$, a continuous function $\psi : M \to \mathbb{R}$ is called a *bump function for $A$ supported in $U$* if $0 \le \psi \le 1$ on $M$, $\psi = 1$ on $A$ and $\text{supp}\psi \subseteq U$.

**Theorem 2.1.7.** Let $M$ be a smooth manifold. For any closed subset $A \subseteq M$ and any open subset $U$ containing $A$, there exists a smooth bump function for $A$ supported in $U$.

*Proof.* Since the collection $\{M\backslash A, U\}$ is an open cover of $M$, there exists a partition of unity $\{\psi_1, \psi_2\}$ subordinate to $M$, where $\text{supp}\psi_2 \subseteq U$ and $\psi_2 = 1$ on $A$ since $\psi_1 = 0$ on $A$. Thus $\psi_2$ is a bump function for $A$ supported in $U$. ☐

**Theorem 2.1.8.** Suppose $M$ is a smooth manifold, $A \subseteq M$ is a closed subset, and $f : M \to \mathbb{R}^k$ is a smooth function. For any open subset $U$ containing $A$, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\text{supp}\tilde{f} \subseteq U$.

*Proof.* Since every smooth function on closed subset can be extended into another smooth function on a small neighborhood, for each $p \in A$, choose an open neighborhood $W_p \subseteq U$ containing $p$ such that there exists $\tilde{f}_p : W_p \to \mathbb{R}^k$ as a smooth function that agrees with $f$ on $W_p \cap A$. Let $W_0 = M\backslash A$, then the collection $\{W_p\}_{p\in A} \cup \{W_0\}$ induces an open cover on $M$. Then there exists a smooth partition of unity $\{\psi_p\}_{p\in A} \cup \{\psi_0\}$ subordinate to $M$. One can define $\tilde{f} : M \to \mathbb{R}^k$ by

$$\tilde{f}(x) = \sum_{p\in A} \psi_p(x)\tilde{f}_p(x) \text{ for all } x \in M$$

Since every product $\psi_p\tilde{f}_p$ is a smooth function and $\tilde{f}$ is well-defined, $\tilde{f}$ is thus a smooth function. Moreover, it's easy to verify that $\text{supp}\tilde{f} \subseteq \bigcup_{p\in A} W_p \subseteq U$ and $\tilde{f}$ agrees with $f$ on $A$ since

$$f\tilde{(x)} = \sum_{p\in A} \psi_p(x)f(x) = \left(\sum_{p\in A} \psi_p(x)\right) f(x) = f(x) \text{ for all } x \in A.$$

Thus, $\tilde{f}$ is indeed an extension of $f$ and $\text{supp}\tilde{f} \subseteq U$. ☐

**Theorem 2.1.9.** Let $M$ be a smooth manifold. If $K$ is any closed subset of $M$, there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

*Proof.* Since every smooth coordinate balls is diffeomorphic to $\mathbb{R}^n$, it suffices to prove there exsits a desired function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = K$, where $K$ is a closed subset of $\mathbb{R}^n$. For every $x \in M\backslash K$, there exists a real number $r > 0$ such that $B_r(x) \subseteq M\backslash K$. Then $M\backslash K$ is the union of countably many such balls $\{B_{r_n}(x_n)\}$. We wish to construct a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f$ vanishes to zero once $x$ reach outside all of those coordinate balls. Let $H : \mathbb{R}^n \to R$ be a smooth bump function that $H = 1$ on $B_q(0)$ and supported in $B_1(0)$, where $q$ is abitrary positive number that $q < 1$.

Since we need $f$ to be nonnegative if $x$ lies in some $B_{r_i}(x_i)$, let $H_i(x) = H\left(\dfrac{x - x_i}{r_i}\right)$ for all $i$, one can express $f$ as a countably infinite nonnegative linear combination of $H_i$'s. For each positive integer $n$, let $M_i$ be the bounded constant of the absolute value of $h$ and all of its partial derivation up to order $n$. We define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{(r_n)^n}{2^n M_n} H_n(x)$$

Since every term of this series is nonnegative, bounded by the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, and is continuous, it follows that $f$ is well-defined and continuous. Let $k \in \mathbb{N}$, consider the partial derivation of order $k$

$$\left\|d^k f(x)\right\| = \left\|\sum_{n=1}^{\infty} \frac{(r_n)^{n-k}}{2^n M_n} d^k H_n(x)\right\| \leq \sum_{n=k}^{\infty} \frac{(r_n)^{n-k}}{2^n M_n} \left\|d^k H_n(x)\right\| \leq \sum_{n=k}^{\infty} \frac{(r_n)^{n-k}}{2^n M_n} M_n = \sum_{n=k}^{\infty} \frac{(r_n)^{n-k}}{2^n},$$

which is a convergent series, by the criteria of series, $d^k f(x)$ is well-defined for all $k$, and is continuous. Hence $f$ is smooth.

Let $\{B_\alpha\}_{\alpha \in A}$ be an open cover of $M$ by smooth coordinate balls and $K$ be any closed subset of $M$. Consider the partition of unity $\{\psi\}_{\alpha \in A}$ subordinate to $M$. For every $\alpha \in A$, since $B_\alpha$ is diffeomorphic to $\mathbb{R}^n$, then there exsits a smooth nonnegative function $f_\alpha : B_\alpha \to \mathbb{R}$ such that $f_\alpha^{-1}(0) = K \cap B_\alpha$. Let $f(x) = \sum_{\alpha \in A} \psi_\alpha(x) f_\alpha(x)$, it follows that

$$f^{-1}(0) = \bigcup_{\alpha \in A} f_\alpha^{-1}(0) = \bigcup_{\alpha \in A} K \cap B_\alpha = K,$$

as desired. $\square$

### 2.2 Problems

**Problem 2.1.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts $(U, \varphi)$ containing $x$ and $(V, \psi)$ containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\left(U \cap f^{-1}(V)\right)$ to $\psi(V)$, but $f$ is not smooth in the sense we have defined in this chapter.

**Problem 2.2.** Prove Proposition 2.12 (smoothness of maps into product manifolds).

**Problem 2.3.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

1. $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the $n$-th power map $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
2. $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map $\alpha(x) = -x$.
3. $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\left\{(w, z) : |w|^2 + |z|^2 = 1\right\}$ of $\mathbb{C}^2$.

*Proof.* (1) Since $\mathbb{S}^1 \subset \mathbb{C}$, consider the global coordinate chart $\varphi : \mathbb{S}^1 \to [0, 2\pi)$ satisfying

$$\varphi(z) = \varphi(e^{i\theta}) = \theta \in [0, 2\pi)$$

Thus, the coordinate representation in this case is

$$\tilde{f(x)} = \varphi \circ f \circ \varphi^{-1}(\theta) = n\theta$$

Since $\varphi$ is smooth and $f$ is a smooth map, it follows that $\tilde{f}$ is also smooth.

(2) Consider the stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{C}^n$ satisfying

$$\sigma\left(z^1, \ldots, z^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}.$$

Then the coordinate representation is computed by

$$\tilde{f}(x) = \sigma \circ f \circ \sigma^{-1}(u) = \sigma\left(\frac{-2u^1}{|u|^2+1}, \ldots, \frac{-2u^n}{|u|^2+1}, \frac{1-|u|^2}{|u|^2+1}\right) = (-u^1, \ldots, -u^n),$$

which is smooth, the same construction on $\mathbb{S}^n \backslash \{S\}$.

□

**Problem 2.4.** Show that the inclusion map $\overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary. 2-5. Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

1. Show that $f$ is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.
2. Show that $f$ is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever $n$ is not an integral multiple of 3.

**Problem 2.5.** Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{k+1} \backslash \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}, P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \backslash \{0\}$ and $x \in \mathbb{R}^{n+1} \backslash \{0\}$. (Such a function is said to be homogeneous of degree $\boldsymbol{d}$.) Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is well defined and smooth.

*Proof.* We first prove that $\tilde{P}$ is well-defined. Suppose that $[x] = [y]$, it suffices to prove that $\tilde{P}([x]) = \tilde{P}([y])$ or $[P(x)] = [P(y)]$. Since we have the relation

$$[x^1, \ldots, x^{n+1}] = [\lambda x^1, \ldots, \lambda x^{n+1}] \text{ for all } \lambda,$$

it follows that

$$\tilde{P}([\lambda x]) = [P(\lambda x)] = [\lambda P(x)] = [P(x)] = P([\lambda x]).$$

Thus $\tilde{P}$ is well-defined. To prove $\tilde{P}$ is smooth, for each $i = 0, \ldots, n$, let $\tilde{U}_i \subset \mathbb{R}^{k+1}$ be the subset containing all points $x \in \mathbb{R}^{n+1}$ satisfying $x^i \neq 0$ and $\varphi_i : U_i \to \mathbb{R}^k$ be the local chart (where $U_i = \pi(\tilde{U}_i)$, and $\pi$ is a natural quotient mapping from $\mathbb{R}^{k+1}$ to $\mathbb{RP}^n$) satisfying

$$\varphi_i[x^1, \ldots, x^{k+1}] = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{k+1}}{x^i}\right).$$

As proven above, $\varphi_i$ is well-defined and smooth, and $\{\varphi_i\}$ defines a smooth structure on $\mathbb{RP}^k$. Let $x \in \mathbb{RP}^n$ and a neighborhood $U_i$ containing $x$, it suffices to prove the map $\varphi_i \circ \tilde{P} \circ \varphi_i^{-1} : \varphi(U) \to \varphi(U)$ is smooth. Computating sufficiently yields

$$\begin{aligned} \varphi_i \circ \tilde{P} \circ \varphi_i^{-1}(x) &= \varphi_i \circ \tilde{P}([x^1, \ldots, x^{i-1}, 1, \ldots, x^k]) \\ &= \varphi_i([P(x^1, \ldots, x^{i-1}, 1, \ldots, x^k)]). \end{aligned}$$

Let $P_i([x]) = P([x^1, \ldots, x^{i-1}, 1, \ldots, x^k])$, we have

$$\begin{aligned} \varphi_i([P(x^1, \ldots, x^{i-1}, 1, \ldots, x^k)]) &= \varphi \circ P_i([x]) \\ &= \left(\frac{P_i^1}{P_i^i}, \ldots, \frac{P_i^{i-1}}{P_i^i}, \frac{P_i^{i+1}}{P_i^i}, \ldots, \frac{P_i^{k+1}}{P_i^i}\right). \end{aligned}$$

Since every component of $P$ is smooth, then the following composition is also smooth. Thus $\tilde{P}$ is smooth, as desired. □

**Problem 2.6.** Let $M$ be a nonempty smooth $n$-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional. [Hint: show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]

*Proof.* We first prove that $M$ contains infinite closed subset. Since $M$ is locally Euclidean, there exists a smooth local chart $(U, \varphi)$ which maps the open subset $U \subseteq M$ into $\tilde{U} = \varphi(U)$, which is open in $\mathbb{R}^n$. Then we can find an open ball $B(x, q) \subseteq U$, and it contains infinitely disjoint open balls, denoted by the set $\overline{\{B(x_\alpha, q_\alpha)\}_{\alpha \in A}}$, where $A$ is a countably infinite set. Since $\varphi$ is a homeomorphism, one can consider the pull back $\{\varphi^{-1}(\overline{B(x_\alpha, q_\alpha)})\}_\alpha$ as a disjoint collection of closed subsets of $M$.

In particular, let $U \subseteq M$ be an closed subset, it suffices to construct a smooth function $f \in C^\infty(M)$ which support

$$\operatorname{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

is a subset of $U$. We consider the following lemma

**Lemma 2.2.1.** Suppose $M$ is a smooth manifold with or without boundary, $A \subset M$ is a closed subset, and $f : A \to \mathbb{R}^k$ is a a smooth function. For any open subset $U$ containing $A$, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $f_A = f$ and $\operatorname{supp}\tilde{f} \subseteq U$

*Proof.* Use Partition of unity. □

The above lemma implies that there is a way to contruct a smooth function as required, and we denote the set of nonzero smooth functions $\{f_n\}_{n \in N}$ such that $f_i$ has support is a subset of $U_{\alpha_i}$. Now we supoose there exists $n \in \mathbb{N}$ and a $n$-tuple $(a_1, \ldots, a_n) \in \mathbb{R}^n$ satisfying

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0 \text{ for all } x \in M$$

Since the support of $f_1, \ldots, f_n$ are disjoint. For every $i = 1, \ldots, n$ by choosing $x \in U_{\alpha_i}$, it follows that $f_i(x) = 0$ but $f_j(x) \neq 0$ for all $j \neq i$. We thus obtain a homogeneous system of equations

$$\begin{aligned} a_2 f_2 + a_3 f_3 + \cdots + a_n f_n &= 0 \quad \text{for all } x \in M \\ a_1 f_1 + a_3 f_3 + \cdots + a_n f_n &= 0 \quad \text{for all } x \in M \\ &\cdots \\ a_1 f_1 + a_2 f_2 + \cdots + a_{n-1} f_{n-1} &= 0 \quad \text{for all } x \in M \end{aligned} \tag{1}$$

which implies $a_i f_i = 0$ for all $i = 1, \ldots, n$ and for all $x \in M$. Thus $a_1 = \cdots = a_n$. Therefore $\{f_1, \ldots, f_n\}$ is linearly independent in $C^\infty(M)$, but since $n$ is abitrary, $C^\infty(M)$ must be infinite-dimensional, as desired. □

**Problem 2.7.** Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F\left(x^1, \ldots, x^n\right) = \left[x^1, \ldots, x^n, 1\right]$. Show that $F$ is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G : \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G\left(z^1, \ldots, z^n\right) = \left[z^1, \ldots, z^n, 1\right]$ (see Problem 1-9).

*Proof.* Let $U$ be the open subset of $\mathbb{R}^{n+1}$ where $x^{n+1} \neq 0$ and $\tilde{U} = \pi(U)$ is an open subset of $\mathbb{RP}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{RP}^n$ is a natural projection. It suffices to prove the restricted map $F : \mathbb{R}^n \to \tilde{U}$ is a diffeomorphism.

To prove $F$ is injective, suppose $F(x) = F(y)$, then there exists $\lambda \in \mathbb{R}$ such that

$$(x^1, \ldots, x^n, 1) = (\lambda y^1, \ldots, \lambda y^n, \lambda),$$

which implies $\lambda = 1$ and hence $x = y$. To prove $F$ is surjective, let $[y] \in \tilde{U}$ be abitrary, we have

$$[y^1, \ldots, y^{n+1}] = \left[\frac{y^1}{y^{n+1}}, \ldots, \frac{y^n}{y^{n+1}}, 1\right] = F\left(\frac{y^1}{y^{n+1}}, \ldots, \frac{y^n}{y^{n+1}}\right).$$

Thus, $F$ is a bijection. Since the map $F : \mathbb{R}^n \to \mathbb{RP}^n$ is smooth, then $F : \mathbb{R}^n \to \tilde{U}$ is also smooth. Therefore, $F$ is a diffeomorphism onto $\tilde{U}$. Now we prove $\tilde{U}$ is a dense subset of $\mathbb{RP}^n$. Let $x \in \mathbb{RP}^n \backslash \tilde{U}$, and $i$ be the positive integer such that $x_i \neq 0$, consider the sequence $(x_n) \in \mathbb{R}^{n+1}$ satisfying

$$x_n = \left(\frac{x^1}{x^i}, \ldots, \frac{x^n}{x^i}, \frac{1}{n}\right), n \in \mathbb{N}$$

Then we have

$$\lim_{n \to +\infty} \pi([x_n]) = \left[\lim_{n \to +\infty} x_n\right] = \left[\frac{x^1}{x^i}, \ldots, \frac{x^n}{x^i}, 0\right] = [x^1, \ldots, x^n, 0] = [x]$$

Therefore, for any $[x] \in \mathbb{RP}^n \backslash \tilde{U}$, there exists $(x_n) \in \mathbb{R}^{n+1}$ such that $\pi(y_n) \to [x]$. Hence $\tilde{U}$ is dense in $\mathbb{RP}^n$, as desired. □

**Problem 2.8.** Given a polynomial $p$ in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G : \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \end{array}$$

*Proof.* It suffices to prove there exsits a unique map $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ satisfying $G \circ p = \tilde{p} \circ G$. Suppose $p(z) = a_n z^n + \cdots + a_1 z + a_0$. By computing sufficiently, we have

$$G \circ p(z) = [a_n z^n + \cdots + a_1 z + a_0, 1] = [p(z), 1].$$

Let $q(z_1, z_2) = \sum_{i=0}^{n} a_i z_1^i z_2^{n-i}$, this implies $q(z_1, z_2) = z_2^n P\left(\frac{z_1}{z_2}\right)$ if $z_2 \neq 0$, then we can construct the map $\tilde{p}$ satisfying

$$\tilde{p}(z_1, z_2) = [q(z_1, z_2), z_2^n].$$

One can verify that $\tilde{p} \circ G(z) = \tilde{p}[z, 1] = [q(z, 1), 1] = [p(z), 1] = G \circ p(z)$. We now prove that $\tilde{p}$ is unique. Assume that there exists $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ satisfying $G \circ p = h \circ G$. If $z_2 \neq 0$, it follows that

$$h[z_1, z_2] = h\left[\frac{z_1}{z_2}, 1\right] = h \circ G\left(\frac{z_1}{z_2}\right) = \left[p\left(\frac{z_1}{z_2}\right), 1\right] = \left[z_2^n p\left(\frac{z_1}{z_2}\right), z_2^n\right] = \tilde{p}[z_1, z_2]$$

for all $z_1 \in \mathbb{C}$. Since $h$ is smooth on $\mathbb{CP}^1$, it must be continuous at $[1, 0]$. Since $p$ is a polynomial, then there exists $K \in \mathbb{N}$ such that $|p(k)| > 0$ for all real numbers $k > K$. Since $h$ is continuous at $[1, 0]$, it follows that

$$h[1, 0] = \lim_{k \to +\infty} h\left[1, \frac{1}{k}\right] = \lim_{k \to +\infty} [p(k), 1] = \lim_{k \to +\infty} \left[1, \frac{1}{p(k)}\right] = [1, 0]$$

Therefore $h[z] = \tilde{p}[z]$ for all $[z] \in \mathbb{CP}^1$, which means $\tilde{p}$ is unique, as desired.

□

**Problem 2.9.** For any topological space $M$, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.

1. Show that $F^*$ is a linear map.

2. Suppose $M$ and $N$ are smocth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

3. Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if $F^*$ restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

[Remark: this result shows that in a certain sense, the entire smooth structure of $M$ is encoded in the subset $C^{\infty}(M) \subseteq C(M)$. In fact, some authors define a smooth structure on a topological manifold $M$ to be a subalgebra of $C(M)$ with certain properties; see, e.g., [Nes03].] (Used on p. 75.)

*Proof.* 1. Since we have

$$F^*(f+g) = (f+g) \circ F = f \circ F + g \circ F,$$

and

$$F^*(\alpha f) = (\alpha f) \circ F = \alpha(f \circ F) = \alpha F^*(f).$$

Hence $F^*$ is linear.

2. Suppose $F : M \to N$ is smooth. Since $F^*(f) = f \circ F$ is smooth in $M$, we thus have $F^*\left(C^{\infty}(N)\right) \subseteq C^{\infty}(M)$. To prove the converse implication, since $\mathrm{Id}_N \in C^{\infty}(N) \subset C(N)$, we have

$$F^*(\mathrm{Id}_N) = \mathrm{Id}_N \circ F = F \in C^{\infty}(M),$$

which implies $F$ is smooth on $M$. □

**Problem 2.10.** Suppose $V$ is a real vector space of dimension $n \geq 1$. Define the projectivization of $V$, denoted by $\mathbb{P}(V)$, to be the set of 1 -dimensional linear subspaces of $V$, with the quotient topology induced by the map $\pi : V \backslash \{0\} \to \mathbb{P}(V)$ that sends $x$ to its span. (Thus $\mathbb{P}\left(\mathbb{R}^n\right) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold. and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ for $V$, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E\left[v^1, \ldots, v^n\right] = \left[v^i E_i\right]$ (where brackets denote equivalence classes) is a diffeomorphism. (Used on p. 561.)

**Problem 2.11.** State and prove an analogue of Problem 2-11 for complex vector spaces.

**Problem 2.12.** Suppose $M$ is a topological space with the property that for every indexed open cover $\mathcal{X}$ of $M$, there exists a partition of unity subordinate to $\mathcal{X}$. Show that $M$ is paracompact.

**Problem 2.13.** Suppose $A$ and $B$ are disjoint closed subsets of a smooth manifold $M$. Show that there exists $f \in C^{\infty}(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

*Proof.* The theorem 2.1.9 implies that there exists two smooth nonnegative real valued $f_1, f_2$ satisfying $f_1^{-1}(0) = A$ and $f_2^{-1}(0) = B$. Then we set $f(x) = \frac{f_1(x)}{f_1(x)+f_2(x)}$, it follows that $f(x) = 1$ if and only if $f_2(x) = 0$, and $f(x) = 0$ if and only if $f_1 = 0$. Hence we are done.

However, the previous contruction relies mostly on the the fact that the preimage of two closed subsets $A$ and $B$ are simply 0 and 1, but not extend for other general situation. Therefore, we aim to construct a universal method step by step using partition of unity, which can be extended for more complicated cases. First, one can again choose two smooth nonnegative real valued $f_1, f_2$ such that $f_1^{-1}(0) = A$, $f_2^{-1}(1) = B$, and $0 \leq f_1, f_2 \leq 1$. Since $M$ is Hausdorff and $M \backslash (A \cup B)$ is open, for every point $x$ outside $A$ and $B$, then we can choose an open coordinate all $B_x$ for $x$ such that $B_x \cap A = \varnothing$ and $B_x \cap B = \varnothing$, denoted by the collection $\{B_x\}$. Let $B = \{B_\alpha\}_{\alpha \in A} \cup \{B_\beta\}_{\beta \in A}$ be the union of abitrary smooth coordinate balls contained by disjoint open subset $U \supseteq A$ and $V \supseteq B$ for each point in $A$ and $B$. Then there exsits a partition of unity $\{\psi\}$ subordinate to $B \cup \{B_x\}$. Consider the function

$$f = \sum_{\alpha \in A} \psi_\alpha f_1|_{B_\alpha} + \sum_{\beta \in B} \psi_\beta f_2|_{B_\beta} + \sum_{x \in M \backslash (A \cup B)} \psi_x = f_\alpha + f_\beta + f_x$$

If $x \in A$, then $f_\beta = f_x = 0$, and $f_\alpha = f_1 = 0$.

If $x \in B$, then $f_\alpha = f_x = 0$ and $f_\beta = f_2 = 1$. If $x \in M \backslash (A \cup B)$, then we have

$$f = \sum_{x \in M \backslash (A \cup B)} \psi_x$$

which implies that $0 < f < 1$ since every point is covered by finite coordinate balls.

We consider the case that $x \in U \backslash A$, using the fact that $0 \leq f_1 \leq$, we thus have

$$f = \sum \psi_\alpha f_1 + \sum_x \psi_x < \sum \psi_\alpha + \sum_x \psi_x < 1$$

and

$$f = \sum \psi_\alpha f_1 + \sum_x \psi_x \geq \sum_x \psi_x > 0$$

Since the case for $x \in V \backslash B$ is analogous, we obtain $f(x) = 1$ if and only if $x \in A$ and $f(x) = 0$ if and only if $x \in B$. Hence $f$ satisfies the desired condition. □

# 3 Tangent Space

## 3.1 Derivatives in Multivariable Calculus

**Definition 3.1.1.** Let $E$ be a subset of $\mathbb{R}^n, f : E \to \mathbb{R}^m$ be a function, $x_0 \in E$ be a point, and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We say that $f$ is differentiable at $x_0$ with derivative $L$ if we have

$$\lim_{x \to x_0, x \in E \backslash \{x_0\}} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0$$

**Proposition 3.1.2.** Suppose $f$ is differentiable at $x_0$ with derivative $L_1$, and also differentiable at $x_0$ with derivative $L_2$. Then $L_1 = L_2$.

*Proof.* Since $f$ is differentiable at $x_0$ with derivative $L_1$ and $L_2$, we have

$$\begin{aligned}\lim_{x \to x_0} \frac{\|L_1(x - x_0) - L_2(x - x_0)\|}{\|x - x_0\|} \leq & \lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L_2(x - x_0)\|}{\|x - x_0\|} + \\ & \lim_{x \to x_0} \frac{\|L_1(x - x_0) - (f(x) - f(x_0))\|}{\|x - x_0\|} \\ & = 0\end{aligned}$$

Let $h = x - x_0$, one obtain that

$$\lim_{h \to 0} \frac{\|L_1(h) - L_2(h)\|}{\|h\|} = 0$$

Given $x \in E$ and $t$ be a scalar such that $t \to 0$, then it follows that $tx \to 0$. Since $L_1$ and $L_2$ are linear map, we have

$$\|L_1(x) - L_2(x)\| = \frac{\|L_1(tx) - L2(tx)\|}{\|tx\|} \to 0$$

Thus $L_1 = L_2$, we are done. □

**Definition 3.1.3.** Let $E$ be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, let $x_0$ be an interior point of $E$, and let $v$ be a vector in $\mathbb{R}^n$. If the limit

$$\lim_{t \to 0^+, x_0 + tv \in E} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exsists, we say that $f$ is *differentiable in the direction* $v$ *at* $x_0$, and we denote the above limit by $D_v f(x_0)$:

$$D_v f(x_0) := \lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}$$

**Proposition 3.1.4.** If $f$ is differentiable at $x_0$, then $f$ is also differentiable in the direction $v$ at $x_0$, and

$$D_v f(x_0) = f'(x_0)v$$

*Proof.* This is trivial if $v = 0$, assume that $v \neq 0$. Since $f$ is differentiable at $a$, then there exists a linear map $f'(a) : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$0 = \lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{\|h\|},$$

Replacing $h$ by $tv$ and $t \to 0$ yields

$$0 = \lim_{t \to 0^+} \frac{f(a+tv) - f(a) - tvf'(a)}{t\,\|v\|} = \lim_{t \to 0^-} \frac{f(a+tv) - f(a) - tvf'(a)}{t\,\|v\|}$$

Since $\|v\| \neq 0$, we thus have

$$0 = \lim_{t \to 0} \frac{f(a+tv) - f(a) - tvf'(a)}{t} = D_v|af = f'(a) \cdot v.$$

□

**Definition 3.1.5.** Let $E$ be a subset of $\mathbb{R}^n$, let $f : E \to \mathbb{R}^m$ be a function let $x_0$ be an interior point of $E$, and let $1 \leq j \leq n$. Then the partial derivative of $f$ respect to the $x_j$ variable at $x_0$, denoted $\frac{\partial f}{\partial x_j}(x_j)$, is defined by

$$\frac{\partial f}{\partial x_j}(x_j) = \lim_{t \to 0, x_0 + te_j \in E} \frac{f(x_0 + te_j) - f(x_0)}{t} = \frac{d}{dt} f(x_0 + te_j)_{t=0}$$

**Theorem 3.1.6.** Let $E$ be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, $F$ be a subset of $E$, and $x_0$ be an interior pint of $F$. If all the partial derivatives $\frac{\partial f}{\partial x_j}$ exist on $F$ and are continuous at $x_0$, then $f$ is differentiable at $x_0$. Moreover if $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, the linear transformation $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$f'(x_0)(v) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0)$$

*Proof.* We first suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that if $v$ is a vector of $\mathbb{R}^n$ and $v = (v_1, v_2, \ldots, v_n)$, then

$$L(v) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0)$$

We attempt to prove that $L = f'(x_0)$, in other words, this is equivalent to

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - L(h)\|}{\|h\|}.$$

Indeed, given $\varepsilon > 0$, it suffices to find $\delta > 0$ such that whenever $h \in B(0, \delta) \backslash \{0\}$, we should have

$$\|f(x_0 + h) - f(x) - L(h)\| < \varepsilon \|h\|.$$

For every $1 \leq j \leq n$, since the partial derivative $\frac{\partial f}{\partial x_j}$ exists, the function $g_j(x) = \frac{\partial f}{\partial x_j}(x)$ is also continuous in at $x_0$. Thus we can find $\delta_j > 0$ such that whenever $h \in B(0, \delta_i) \backslash \{0\}$, we have $\|g_j(x) - g_j(x_0)\| < \frac{\varepsilon}{mn}$. Let $\delta = \min\{\delta_1, \ldots, \delta_n\}$ and fix $h \in B(0, \delta) \backslash \{0\}$, it is possible to determine the scalars $(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$ such that

$$h = h_1 e_1 + h_2 e_2 + \cdots + h_n e_n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is standard ordered basis of $\mathbb{R}^n$. Indeed, by writing $f$ in components as $f = (f_1, \ldots, f_m)$, it would follow that

$$\frac{\partial f}{\partial x_j}(x_0) = \left( \frac{\partial f_1}{\partial x_j}(x_0), \frac{\partial f_2}{\partial x_j}(x_0), \ldots, \frac{\partial f_n}{\partial x_j}(x_0) \right). \tag{1}$$

Let $1 \leq i \leq n$, by the Mean Value Theorem, one may locate $t_1$ between 0 and $h_1$ such that

$$f_i(x_0 + h_1 e_1) - f_i(x_0) = \frac{\partial f_i}{\partial x_1}(x_0 + t_1 e_1) h_1.$$

In particular, since

$$\left\|\frac{\partial f_i}{\partial x_1}(x_0+t_1e_1)-\frac{\partial f_i}{\partial x_1}(x_0)\right\| \le \left\|\frac{\partial f}{\partial x_1}(x_0)-\frac{\partial f}{\partial x_1}(x_0)\right\| < \frac{\varepsilon}{mn}$$

Multiplying both sides by $\|h_1\|$ and replacing the left-side term with (1) obtains that

$$\left\|f_i(x_0+h_1e_1)-f_i(x_0)-\frac{\partial f_i}{\partial x_1}(x_0)\right\| < \frac{\varepsilon}{mn}\|h_1\| \le \frac{\varepsilon}{mn}\|h\|$$

whenever $1 \le i \le n$. Adding $n$ above inequalities for specific $f_1, f_2, \ldots, f_n$ and using triangle inequality that $\|(y_1,\ldots,y_m)\| \le \|y_1\|+\cdots+\|y_m\|$ yields

$$\left\|f(x_0+h_1e_1)-f(x_0)-\frac{\partial f}{\partial x_1}(x_0)\right\| < \frac{\varepsilon\|h\|}{n}.$$

Similarly, by developing the analogous process with $x_0+h_1e_1$ and $x_0+h_1e_1+h_2e_2$, we thus have

$$\left\|f(x_0+h_1e_1+h_2e_2)-f(x_0+h_1e_1)-\frac{\partial f}{\partial x_1}(x_0)\right\| < \frac{\varepsilon\|h\|}{n}.$$

whenever $h \in B(0,\delta)\backslash\{0\}$. More generally, for each $0 \le k \le n-1$, one can obtain

$$\|u_k\| = \left\|f\left(x_0+\sum_{j=1}^{k+1}h_je_j\right)-f\left(x_0+\sum_{j=1}^{k}h_je_j\right)-\frac{\partial f}{\partial x_k}(x_0)\right\| < \frac{\varepsilon\|h\|}{n}.$$

Summing all these terms together implies

$$\begin{aligned}\|u_1+u_2+\cdots+u_n\| &= \left\|f(x_0+h_1e_1+\cdots+h_ne_n)-f(x_0)-\sum_{j=1}^{n}h_j\frac{\partial f}{\partial x_j}(x_0)\right\| \\ &= \|f(x_0+h)-f(x_0)-L(h)\| \\ &\le \|u_1\|+\|u_2\|+\cdots+\|u_n\| \\ &< \varepsilon\|h\|\end{aligned}$$

Therefore, $L$ is derivative of $f$ at $x_0$, as desired.

□

**Definition 3.1.7.** If $f : E \to \mathbb{R}$ is a real-valued function, and we define the gradient $df(x_0)$ of $f$ at $x_0$ to be the $n$-dimensional row vector

$$df(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0)\right)$$

### 3.2 Tangent Vectors of Euclidean Space

**Definition 3.2.1.** A derivation at $p \in \mathbb{R}^n$ is a linear map $\omega : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz product rule:

$$\omega(fg) = f\omega(g) + g\omega(f) \text{ for all smooth maps } g, f \in C^\infty(M)$$

We denote the set $T_p\mathbb{R}^n$ to be the set containing all kind of following derivation, that is

$$T_p\mathbb{R}^n := \{\omega : C^\infty(M) \to \mathbb{R} \mid \omega \text{ linear and Leibniz}\}.$$

**Proposition 3.2.2.** Let $a \in \mathbb{R}^n$, for each geometric tangent vector $v_a \in \mathbb{R}$, the map $d \cdot v_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at $a$. Moreover, the operation $v_a \mapsto d(a) \cdot v_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.

*Proof.* The fact $d \cdot v_a$ is a consequence of the linearity and Lebniz implies that it is a derivation. It suffices to prove that $\mathcal{L} : v_a \mapsto d \cdot v_a$ is an isomorphism.

For injectivity, suppose $\mathcal{L}(v_a) = \mathcal{L}(w_a)$ for some geometric tangent vector $v_a, w_a$, it folows that

$$\begin{aligned} 0 = \mathcal{L}(v_a - w_a)(x^j) = d \cdot (v_a - w_a)(x^j) &= \left( \frac{\partial}{\partial x^1}(x^j), \dots, \frac{\partial}{\partial x^n}(x^j) \right) (v_a - w_a) \\ = \frac{\partial}{\partial x_i}(x^j)(v^i - w^i) = \delta_{ij}(v^i - w^i) = v^j - w^j, \end{aligned}$$

for all $j = 1, \dots, n$. Thus $\mathcal{L}$ is injective. To prove surjectivity, let $w \in T_a\mathbb{R}^n$ be abitrary and let $v$ the the tangent vector of $\mathbb{R}^n_a$ such that $w(x^i) = v^i$, it suffices to prove that $w = d \cdot v_a$. By the Taylor's theorem, one can write

$$f(x) = f(a) + \left[ \frac{\partial f}{\partial x^i}(a)(x^i - a^i) \right] + \left[ (x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i x^j}(a + t(x-a)) dt \right]$$

Differentiating both sides by $w$ yields

$$w(f(x)) = w(f(a)) + \frac{\partial f}{\partial x^i}(a)(w(x^i) - a^i) = \frac{\partial f}{\partial x^i}(a) v^i = d \cdot v_a(a)$$

Hence, $\mathbb{R}^n_a$ is isomorphic to $T_a\mathbb{R}^n$ by the following operation. □

**Definition 3.2.3** (Tangent Vectors on Manifolds). Let $M$ be a smooth manifold, a derivation at $p \in M$ is a linear map $\omega : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz product rule:

$$\omega(fg) = f\omega(g) + g\omega(f) \text{ for all smooth maps } g, f \in C^\infty(M)$$

We denote the set $T_p(M)$ to be the set containing all kind of following derivation, that is

$$T_pM := \{\omega : C^\infty(M) \to \mathbb{R} \mid \omega \text{ linear and Leibniz}\}.$$

**Definition 3.2.4** (The Differential of a Smooth Map). Let $F : M \to N$ be the smooth map between smooth manifolds, for each $p \in M$, we define the map

$$dF_p : T_pM \to T_{F(p)}N,$$

called the *differential of* $F$ *at* $p$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

**Proposition 3.2.5.** The operation $dF_p$ defined above is a derivation.

**Proposition 3.2.6.** Let $M$ be a smooth manifold, $p \in M$ and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of $p$, then $vf = vg$.

**Proposition 3.2.7.** If $M$ is an $n$-dimensional smooth manifold, then for each $p \in M$, the tangent space $T_pM$ is an $n$-dimensional vector space.

**Definition 3.2.8** (Second definition of Tangent Space). Let $M$ be a smooth manifold and $p \in M$. We say every the smooth function $\zeta : (-\varepsilon, +\varepsilon) \to M$ such that $\zeta(0) = p$ is *a p-path*. Two $p$-path $\alpha$ and $\beta$ is said to satisfies the $\sim$ equivalent relation if

$$\frac{d}{dt}(f(\alpha(t)))\bigg|_{t=0} = \frac{d}{dt}(f(\beta(t)))\bigg|_{t=0}$$

for all smooth map $f \in C^\infty(M)$. Then the tangent space at $p$ is defined as:

$$T_p(M) := \{[\zeta'(0)] \mid \text{Smooth curve } \zeta : (-\varepsilon, +\varepsilon) \to M, \zeta(0) = p\}$$

**Proposition 3.2.9.** The tangent spaces at $p$ in first and second definition are naturally isomorphic.

**Proposition 3.2.10.** If $M$ is a smooth manifolds with dimension $n$ and let $(x_1, x_2, \ldots, x_n)$ be a smooth local chart around $p \in M$, then the set

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \frac{\partial}{\partial x_2}\bigg|_p, \ldots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$$

forms a basis for $T_pM$.

For convenience, if $v \in T_pM$ we write $v = (dx_1, dx_2, \ldots, dx_n)$.

**Definition 3.2.11** (Tangent Bundle). Let $M$ be a smooth manifold, we define the *tangent bundle of* $M$ to be the disjoint union of the tangent spaces at all points on $M$

$$TM = \coprod_{p \in M} T_pM.$$

**Proposition 3.2.12.** For any $n$-dimensional smooth manifold $M$, the tangent bundle $TM$ has a natural topology and a smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth.

*Proof.* We first prove that $\pi$ is smooth. Consider the local coordinate chart $(U, \varphi)$ for $M$ which has the form $(x^1, \ldots, x^n)$. Notice that $\pi^{-1}(U)$ is an open set in $TM$ and

$$\pi^{-1}(U) = TU = \coprod_{p \in U} T_pM,$$

which motivates us to consider the local chart $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ satisfying

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}(p)\right) = (x^i(p), v^i) \text{ and } \tilde{\varphi}^{-1}(x^i, v^i) = v^i \frac{\partial}{\partial x^i}(\varphi^{-1}(x))$$

for all $v = v^i \dfrac{\partial}{\partial x^i} \in T_pM$ and $p \in U$. Let $(V, \psi)$ and $(\pi^{-1}(V), \tilde{\psi})$ be the similarly defined smooth local chart on $M$ and $TM$, respectively. Since $\varphi$ and $\psi$ is a homeomorphism, they are smoothly compatible. Then it suffices to verify the compatibility of the transition map $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \tilde{\varphi}(\pi^{-1}(U) \cap \pi^{-1}(V)) \to \tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V))$, which can be computed sufficiently

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^i(p), v^i) = \tilde{\psi}\left(v^i \frac{\partial}{\partial x^i}(p)\right) = (\tilde{x}^i(p), \tilde{v}^i)$$

which is clearly a smooth function. Since $M$ is second-countable, one can choose a countable cover $\{U_i\}$ for $M$ and a smooth structure $\{(U_i, \varphi_i)\}$ defined above. Hence we obtain a countable cover $\{\pi U_i\}$ for $TM$ which implies that $TM$ is locally Euclidean and second-countable. The Hausdorff property is trival since two points in the same fiber can be seperated by the same chart and in different fiber can be mapped through $\pi$ onto $M$, which is Hausdorff. Hence $TM$ is a manifold, and the corresponding smooth structure $\{(\pi^{-1}(U_i), \tilde{\varphi}_i)\}$ implies that $TM$ is smooth manifold. Finally, to see that $\pi$ is smooth, with respect to the charts $(U, \varphi)$ and $(\pi^{-1}(U), \tilde{\varphi})$, the coordinate representation is $\pi(x, v) = x$. Hence we are done. □

### 3.3 Problems

**Problem 3.1.** Suppose $M$ and $N$ are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if $F$ is constant on each component of $M$.

*Proof.* Since the converse implication is trival, we only prove the forward one. Let $(x^1, \dots, x^n)$ be a local coordinate chart for a neighborhood $U$ of $p$. Let $v \in T_pM$, then it can be expressed as a linear combination

$$v = v^i \frac{\partial}{\partial x^i}(p)$$

Differentiating both sides by $dF_p$ yields

$$0 = dF_p(v)(x^j) = v^i \frac{\partial(x^j \circ F)}{\partial x^i}(p) = v^i \frac{\partial F^j}{\partial x^i}(p)$$

Since this is true for all $v \in T_pM$, choosing $v = \partial_j$ implies that $\dfrac{\partial F^j}{\partial x^j}(p)$ for all $j$. As a consequence, each component $F_j$ is constant, then $F$ is also constant. □

**Problem 3.2.** Let $M_1, \dots, M_k$ be smooth manifolds and for each $j$, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the $M_j$ factor. For any point $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_i)_p(v))$$

is an isomorphism.

*Proof.* We first verify that $\alpha$ is well-defined. For every $j = 1, \dots, k$, since the differential

$$d(\pi_j)_p : T_p(M_1 \times \cdots \times M_k) \to T_{\pi_j(p)}M_j = T_{p_j}M_j$$

sends the product tangent vector $v$ to every seperate tangent space $T_{p_j}M_j$. This ensures that the image of $\alpha$ always lies on the direct sum of following tangent spaces. We now prove that $\alpha$ is injective. Suppose $\alpha(v) = \alpha(w)$, since $\alpha$ is linear due to the linearity of the differentials, we thus have $\alpha(v) - \alpha(w) = \alpha(v - w) = 0$ implies that $d(\pi_i)_p(v - w) = 0$ for all $i = 1, \dots, k$. Let $k_i = \dim M_i$ for all $i$ and consider the local coordinate $(x_1^1, \dots, x_1^{k_1}, x_2^1, \dots, x_k^{k_k}) = (x^1, \dots, x^{k_1 + \cdots + k_k})$. Let $j = 1, \dots, k$ and $t = 1, \dots, k_j$ be fixed and $i$ varies from 1 to $k_1 + \cdots + k_k$, expressing the vector $u = v - w$ in coordinate $u = u^i \frac{\partial}{\partial x^i}(p)$ yields

$$0 = d(\pi_j)_p(u)(x^t) = u^i \frac{\partial(x^t \circ \pi_j)}{\partial x^i}(p) = u^i \frac{\partial x_j^t}{\partial x^i}(p) = u^i \frac{\partial x^{k_{t-1}+j}}{\partial x^i}(p) = u^{k_{t-1}+j}$$

Since $t$ and $j$ was abitrary, then it follows that $u^i = v^i - u^i = 0$, hence $\alpha$ is injective. Let $w = (w_1, \dots, w_k) \in T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$, and let

$$v = w_i^j \frac{\partial}{\partial x^{k_{i-1}+j}}.$$

The computation above implies that $\alpha(v) = w$. Thus $\alpha$ is bijective. □

**Problem 3.3.** Prove that if $M$ and $N$ are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

**Problem 3.4.** Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

*Proof.* We begin to construct by viewing $\mathbb{S}^1$ in $\mathbb{C}$. Let $(U, \theta)$ and $(V, \psi)$ be the angle local chart such that $\theta : \mathbb{S}^1 \backslash \{1\} \to (0, 2\pi)$ and $\psi : \mathbb{S}^1 \backslash \{-1\} \to (-\pi, \pi)$. We consider the map $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ satisfying

$$F\left(x, v\partial_\theta|_x\right) = (x, v) \text{ on } U \text{ and } F\left(x, v\partial_\psi|_x\right) = (x, v) \text{ on } V.$$

Since we have $\theta = \psi - \pi$ and $\dfrac{d\psi}{d\theta} = 1$ then $\partial_\theta|_x = \partial_\psi|_x$ on the restriction to $F$ on $U \cap V$ since , it follows that $F$ is well-defined. To check that $F$ is bijection, define the candidate function $G : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$ satisfying

$$G(x, v) = \begin{cases} \left(x, v\partial_\theta|_x\right), & x \in U, \\ \left(x, v\partial_\psi|_x\right), & x \in V \end{cases}$$

Since $G$ is well-defined and $F \circ G(x, v) = G \circ F(x, v) = (x, v)$, thus $F$ is bijective.

In addition, $F$ is smooth on $U$ and $V$ since the coordinate expression $(x, v) \mapsto (x, v)$ is smooth on $U$ and $V$, and agrees smoothly on $U \cap V$. Computing the transition $\theta \circ \psi^{-1} : \psi(U \cap V) \to \theta(U \cap V)$, we obtain

$$\theta \circ \psi^{-1}(\psi(x)) = \theta(x)$$

which is smooth, the gluing lemma implies that $F$ is smooth globally. As $G$ is smooth by the analogous argument, hence $F$ is diffeomorphism $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$. □

**Problem 3.5.** Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no diffeomorphism with the same property. [Hint: let $\gamma$ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF\left(\gamma'(t)\right)$ on the coordinate functions $x$ and $y$.] (Used on p. 123.)

*Proof.* Suppose that there is a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. Following to the hint, let $\gamma : [0, 2\pi) \to \mathbb{S}^1$ be the smooth curve satisfying $\gamma(t) = (\cos(t), \sin(t))$. Since $\gamma$ is a homeomorphism and has smooth inverse, which is

$$\gamma^{-1}(x, y) = \gamma^{-1}(x, \pm\sqrt{1 - x^2}) = \arctan\left(\frac{\pm\sqrt{1 - x^2}}{x}\right)$$

Then $\gamma$ is also a diffeomorphism, thus the composition $F \circ \gamma : [0, 2\pi) \to K$ is also a diffeomorphism. Since $F \circ \gamma$ is onto, then there exists $t_0$ such that $F \circ \gamma(t_0) = (1, 1)$. Let $I$ be an open interval containing $(F \circ \gamma)^{-1}(1, 1)$, then one can split $I$ such that $F \circ \gamma(I_x) \subseteq (-1, 1) \times \{1\}$ and $F \circ \gamma(I_y) \subseteq \{1\} \times \subseteq (-1, 1)$. Consequently, we have

$$F \circ \gamma(t) = \begin{cases} (x \circ F \circ \gamma(t), 1) & \text{if } t \in I_x \\ (1, y \circ F \circ \gamma(t)) & \text{if } t \in I_y \end{cases} \tag{2}$$

Since the velocity of $F \circ \gamma$ at $t_0$ is

$$\left(\frac{d(x \circ F \circ \gamma)}{dt}(t_0), 0\right) \text{ and } \left(0, \frac{d(y \circ F \circ \gamma)}{dt}(t_0)\right)$$

by (2), respectively, then two expressions for $(F \circ \gamma)(t_0)$ must concide, which means $(F \circ \gamma)'(t_0) = 0$, it follows that $dF(\gamma'(t_0)) = (F \circ \gamma)'(t_0) = 0$, which leads to contradiction since $dF_{\gamma'(t_0)}$ is globally homeomorphism and $\gamma'(t_0) \neq 0$, hence no such diffeomorphism $F$ can exists. □

**Problem 3.6.** Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = \left(z^1, z^2\right) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (e^{it} z^1, e^{it} z^2)$. Show that $\gamma_z$ is a smooth curve whose velocity is never zero.

*Proof.* Let $z^1 = x^1 + iy^1$and $z^2 = x^2 + iy^2$, then we can write $\gamma_z$ under the usual identification as

$$\begin{aligned} \gamma_z(t) &= (e^i t z^1, e^i t z^2) = \big((\cos(t) + i\sin(t))(x^1 + iy^1), (\cos(t) + i\sin(t))(x^2 + iy^2)\big) \\ &= (x^1\cos(t) - y^1\sin(t) + i(x^1\sin(t) + y^1\cos(t)), x^2\cos(t) - y^2\sin(t) + i(x^2\sin(t) + y^2\cos(t))) \\ &\Leftrightarrow (x^1\cos(t) - y^1\sin(t), x^1\sin(t) + y^1\cos(t), x^2\cos(t) - y^2\sin(t), x^2\sin(t) + y^2\cos(t)) \end{aligned} \tag{3}$$

Let $\alpha(t) = y^1\sin(t) + x^1\cos(t)$ and $\beta(t) = y^2\sin(t) + x^2\cos(t)$, one can rewrite

$$\gamma_z(t) = (\alpha'(t), \alpha(t), \beta'(t), \beta(t)) \tag{4}$$

Since the components $\alpha$ and $\beta$ are smooth, it follows that $\gamma_z(t)$ is smooth. Consider the local chart $(U_1^+, \varphi_1^+)$ for $\mathbb{S}^3$, where $U_1^+ = \{(x^1, \dots, x^4) \in \mathbb{S}^3 \mid x^1 > 0\}$ and $\varphi_1^+$ satisfies

$$\varphi_1^+(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4)$$

Then the coordinate representation respect to this chart is $\varphi_1^+ \circ \gamma : \gamma_z^{-1}(U_1^+) \to \mathbb{R}^3$ and $\varphi_1^+ \circ \gamma_z(t) = (\alpha(t), \beta'(t), \beta(t))$. Then the velocity of $\varphi_1^+ \circ \gamma_z$ is

$$(\varphi_1^+ \circ \gamma_z)'(t) = (\alpha'(t), \beta''(t), \beta'(t)) = (\alpha'(t), -\beta(t), \beta'(t))$$

Since $\alpha(t)' > 0$, then its velocity is nonzero for all $t \in \gamma_z^{-1}(U_1^+)$. An analogous computation for the chart $(U_1^-, \varphi_1^-), (U_2^\pm, \varphi_2^\pm), \ldots, (U_4^\pm, \varphi_4^\pm)$, since the domain of the collection $\{(U_i^\pm, \varphi_i^\pm)\}$ covers $\mathbb{S}^3$, it follows that the velocity of the smooth curve $\gamma_z$ is never zero. □

**Problem 3.7.** Let $M$ be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of smooth curves starting at $p$ under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function $f$ defined in a neighborhood of $p$. Show that the map $\Psi : \mathcal{V}_p M \to T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. (Used on p. 72.)

# 4 Embeddings

## 4.1 Immersion and Submersion

**Definition 4.1.1.** Let $M^m$ and $N^n$ be smooth manifolds, and $F : M \to N$ is a smooth map. For abitrary point $p \in M$, we define the rank of $F$ at $p$ to be the rank of the differential $dF_p$. If $\operatorname{rank}(dF_p) = r$ for all $p \in M$, we say $F$ *has the constant rank* $r$.

**Lemma 4.1.2.** Let $F : M^m \to N^n$ be a smooth map. For abitrary $p \in M$, $dF_p$ is injective if and only if $\operatorname{rank}(dF_p) = n$ and is surjective if and only if $\operatorname{rank}(dF_p) = m$. Consequently, $dF_p$ is bijective if and only if $\operatorname{rank}(dF_p) = m = n$.

**Definition 4.1.3.** Let $F : M^m \to N^n$ be the smooth map. Then $F$ is called a smooth immersion if its differential is injective at each point, and is called a smooth submersion if its is surjective at each point.

**Proposition 4.1.4.** Injectivity and surjectivity of the differential implies local immersion and submersion, respectively.

**Theorem 4.1.5** (Inverse Function Theorem on Manifolds). Let $F : M^m \to N^n$ be a smooth map. If $p \in M$ is point such that $dF_p$ is invertable, then there are connected neighborhoods $U_p$ and $V_{F(p)}$ such that $F|_{U_p} : U_p \to V_{F(p)}$ is a diffeomorphism.

## 4.2 Constant Rank Theorem

**Theorem 4.2.1** (Constant Rank Theorem). Let $M^m$ and $N^n$ be smooth manifolds and $F : M \to N$ be a smooth map with constant rank $r$. For each $p \in M$, there exists smooth charts $(U, \varphi)$ for $M$ containing $p$ and $(V, \psi)$ for $N$ containing $F(p)$ such that $F(U) \subseteq V$, in which $F$ has a coordinate representation of the form

$$\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^r, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0) \tag{5}$$

*Proof.* Since the theorem is local, one can view $M$ and $N$ as open subsets $U$ and $V$ in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. For the sake of condition, we can reorder coordinate such that the $r \times r$ block of $DF_p$ has nonzero determinant. Consider the indentification

$$\begin{aligned}(x, y) &\Leftrightarrow (x^1, \ldots, x^r, y^1, \ldots, y^{m-r}),\\ (u, v) &\Leftrightarrow (u^1, \ldots, u^r, v^1, \ldots, v^{n-r}).\end{aligned}$$

Then one can view $F(x, y) = (F_1(x, y), F_2, (x, y))$ for some smooth map $F_1 : U \to \mathbb{R}^r$ and $F_2 : U \to \mathbb{R}^{n-r}$. Let $p = (x_p, y_p)$ and $\varphi : U \to \mathbb{R}^m$ satisfying $\varphi(x, y) = (F_1(x, y), y)$, we have

$$D\varphi(x_p, y_p) = \begin{bmatrix} DF_1(x, y) & \dfrac{\partial F_1}{\partial y}(x, y) \\ 0 & \delta^i_j \end{bmatrix}$$

The above assumption implies that $DF_1(x,y)$ and $\delta_i^j$ are invertable, then $D\varphi(x_p,y_p)$ is invertable. Apply the inverse function theorem for the map $\varphi$ and $p \in U$, there exists connected neighborhoods $U_p$ such that $\varphi|_{U_p} : U_p \to \varphi(U_p)$ is a diffeomorphism. Then we can consider the local chart $(U_p, \varphi)$ which is a restricted diffeomorphism. On the following restriction, we can let $A, B : \varphi(U_p) \to U_p$ be the smooth function satisfying $\varphi^{-1}(x,y) = (A(x,y), B(x,y))$, then we obtain

$$(x,y) = \varphi(A(x,y), B(x,y)) = (F_1(A(x,y), B(x,y)), B(x,y)).$$

It follows that $B(x,y) = y$ and $F_1(A(x,y), y) = x$. Hence $F \circ \varphi^{-1}$ is calculated by

$$F \circ \varphi^{-1}(x,y) = F(A(x,y), y) = (F_1(A(x,y), y), F_2(A(x,y), y)) = (x, C(x,y)),$$

where $C : \varphi(U_p) \to \mathbb{R}^{n-r}$ defined by $C(x,y) = F_2(A(x,y), y)$. Moreover, since the Jacobian of $F \circ \varphi^{-1}$ at abitrary point $(x,y) \in \varphi(U_p)$ is

$$D(F \circ \varphi^{-1})(x,y) = \begin{bmatrix} \delta_j^i & 0 \\ \dfrac{\partial C^i}{\partial x^j}(x,y) & \dfrac{\partial C^i}{\partial y^j}(x,y) \end{bmatrix},$$

where its rank remains at $r$ and the block $\delta_j^i$ has the rank $r$. Then the block $\dfrac{\partial C^i}{\partial y^j}(x,y)$ must vanish to zero. Therefore $C(x,y)$ is independent of $y$ hence we can set $S(x) = C(x,y) = C(x,y_p)$ for all $y$ and then we obtain

$$F \circ \varphi^{-1}(x,y) = (x, S(x)) \tag{6}$$

Let $V_p \subseteq V$ be an open subset satisfying $V_p = \{(u,v) \in V \mid (u, \varphi^2(p)) \in \varphi(U_p)\}$. Then $V_p$ is a neighborhood of $\varphi(p)$ and the definition of $F \circ \varphi^{-1}$ in 6 implies that $F \circ \varphi^{-1}(\varphi(U_p)) = F(U_p) \subseteq V_p$. Then we can consider the local chart $(V_p, \psi)$ such that $\psi : V_p \to \mathbb{R}^n$ satisfying $\psi(u,v) = (u, v - S(u))$. Notice that $(V_p, \psi)$ is a smooth chart, following from 6, we thus have

$$\psi \circ F \circ \varphi^{-1}(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0).$$

Since $(\varphi, U_p)$ and $(\psi, V_p)$ satisfy the following conditions, the proof is done. □

**Corollary 4.2.2.** Let $F : M^m \to N^n$ be a smooth map between smooth manifolds and suppose $M$ is connected. Then the followings are equivalent:

1. For each $p \in M$, there exists smooth charts containing $p$ and $F(p)$ in which the coordinate representation of $F$ is linear.

2. $F$ has constant rank.

*Proof.* First we suppose $F$ has linear coordinate representation in a neighborhoods of each point. Since every linear map has constant rank, it follows that $F$ has constant rank on the following neighborhoods and the fact that $M$ is connected implies that the rank of $F$ is constant on every point of $M$.

Conversely, if $F$ has constant rank, it follows from the previous theorem that there exists following smooth charts such that $F$ has the representation of the form 5, which is linear. □

**Theorem 4.2.3.** Let $F : M^m \to N^n$ be a smooth map of constant rank between smooth manifolds.

1. If $F$ is surjective, then it is a smooth submersion.

2. If $F$ is injective, then it is a smooth immersion.

3. If $F$ is bijective, then it is a diffeomorphism.

*Proof.* (1) Suppose $F$ is surjective and has constant rank $r < n$. Let $p \in M$, by the Constant Rank Theorem, there exists smooth local charts $(U_p, \varphi_p)$ and $(V_{F(p)}, \psi_{F(p)})$ corresponding to $p$ and $F(p)$ such that $F(U_p) \subseteq V_{F(p)}$ in which $F$ has a coordinate representation of the form

$$\tilde{F}_p(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots). \tag{7}$$

For the sake of condition, we can shrink $U$ into an $r$-dimensional regular coordinate ball and $F(\overline{U_p}) \subseteq V$. Then $F(\overline{U_p})$ is a compact subset of the set

$$\{y \in V \mid y^{r+1} = \cdots = y^n = 0\}.$$

Since such a closed $r$-dimensional cannot contain any $n$-dimensional regular coordinate bals in $N$, $F(\overline{U_p})$ does not include any open subset of $N$. Hence it is nowhere dense in $N$.

Since charts $\{(U_p, \varphi_p)\}_{p\in M}$ is an open cover of $M$, then we can choose a subcover $\{(U_p, \varphi_p)\}_{p\in A}$ on $M$. Since $F(M)$ is covered by the corresponding subcover $\{F(\overline{U_p})\}_{p\in A}$, which is countable and every element is nowhere dense, by the Baire category theorem, $\overline{F(M)} = \varnothing$, which contradicts the fact that $F$ is surjective, hence it must be a smooth submersion.

(2) Suppose $F$ is injective and $r < m$, follows from (7), we have

$$\tilde{F}_p(x^1, \dots, x^m) = \tilde{F}_p(x^1, \dots, x^r, 0, \dots)$$

which implies that $U$ is $r$-dimensional, contradiction. Hence $F$ must be a smooth immersion.

(3) Follows from (1) and (2), $F$ is both smooth submersion and immersion, it thus is local diffeomorphism. The fact that $F$ is bijective implies that it is diffeomorphism from $M$ onto $N$. ☐

## 5 Cotangent Space

**Definition 5.0.1.** Suppose $M$ is a smooth manifolds and $T_pM$ is a tangent space of $M$ at $p \in M$, then *cotangent space* $T_p^*M$ is defined as

$$T_p^*M = Hom(T_pM; \mathbb{R}) = \{\omega \mid \omega : T_pM \to \mathbb{R} \text{ linear}\}.$$

Its element $\omega_p \in T_p^*M$ is called *covector*, one can be written as:

$$\omega_p(v) = a_1 dx_1 + a_2 dx_2 + \cdots + a_n dx_n.$$

## 6 Wedge Product

**Definition 6.0.1** (Alternating Tensors). A covariant $(0, k)$-tensor $\alpha$ is said to be *alternating* if

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

The subspace of all alternating covariant $(0, k)$-tensor is denoted by $\Lambda(V^*) \subseteq T^k(V^*)$.

**Proposition 6.0.2.** Let $\alpha$ be a covariant $(0, k)$-tensor on $V$. The followings are equivalent:

1. $\alpha$ is alternanting.
2. $\alpha(v_1, \dots, v_k) = 0$ if and only if the $k$-tuple $(v_1, \dots, v_k)$ is linearly dependent.
3. $\alpha(v_1, \dots, w, \dots, w, \dots, v_k) = 0$.

**Definition 6.0.3.** We define the alternating projection $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ as follows:

$$\mathrm{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \alpha(v_{\sigma_1} \dots, v_{\sigma_k})$$

**Definition 6.0.4.** Let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, define the *Wedge product* to be the following $(k+l)$-covector:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta).$$

**Definition 6.0.5** (Elementary Alternating Tensors). Let $V$ be an $n$-dimensional vector space and $\{\varepsilon_1, \dots, \varepsilon_n\}$ be a basis of $V^*$. We define a covariant $(0,k)$-tensor $\varepsilon^I = \varepsilon^{i_1,\dots,i_k}$ by

$$\varepsilon^I(v_1,\dots,v_k) = \det\left[\overrightarrow{\varepsilon^I(v_1)}, \dots, \overrightarrow{\varepsilon^I(v_k)}\right] = \det\begin{bmatrix} \varepsilon_{i_1}(v_1) & \cdots & \varepsilon_{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon_{i_k}(v_1) & \cdots & \varepsilon_{i_k}(v_k) \end{bmatrix} = \det\begin{bmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{bmatrix}$$

As it is a $k$-covector, $\varepsilon^I$ is called *elementary $k$-vector.*

**Proposition 6.0.6.** Let $(E_i)$ be a basis for $V$, $(\varepsilon_i)$ be the dual basis for $V^*$, and let $\varepsilon^I$ be an elementary $k$-covector be dual to $(E_i)$. Since $\varepsilon^I$ is alternating, then these followings hold.

1. If $I$ has a repeated index, then $\varepsilon^I = 0$.
2. If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = \mathrm{sgn}(\sigma)\varepsilon^J$.
3. $\varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J$.

**Definition 6.0.7.** A multi-index $I = (i_1, \dots, i_k)$ is said to be *increasing* if $i_1 < i_2 < \cdots < i_k$.

**Theorem 6.0.8.** Let $V$ be an $n$-dimensional vector space. If $(\varepsilon^i)$ is any basis for $V^*$, then for each positive integer $k \le n$, the collection of $k$-covectors

$$\mathcal{F} = \{\varepsilon^I \mid I \text{ is increasing of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Consequently,

$$\dim \Lambda^k(V^*) = \binom{n}{k}.$$

*Proof.* Let $\omega \in \Lambda^k(V)$, and $(E_i)$ be the basis of $V$ dual to $V^*$ since $\omega$ is alternating, for all abitrary and increasing multi-index representation of $S_n$, say $(j_1, \dots, j_k)$ and $(i_1, \dots, i_k)$, respectively, one can rewrite

$$\begin{aligned} \omega &= \omega_I \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_k} = \omega(E_{j_1}, \dots, E_{j_k})\varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_k} \\ &= \omega(E_{i_1}, \dots, E_{i_k}) \left( \sum_{\sigma \in (i_1,\dots,i_k)} \mathrm{sgn}(\sigma)\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \right) \\ &= \omega(E_{i_1}, \dots, E_{i_k})\varepsilon^{(i_1,\dots,i_k)}. \end{aligned}$$

Therefore $\mathcal{F}$ generates the $\Lambda^k(V^*)$. To prove $\mathcal{F}$ is linearly independent, suppose $\omega = 0$ and applying both sides for each $(E_{j_1}, \dots, E_{j_k})$ yields

$$0 = \omega(E_{i_1}, \dots, E_{i_k}) \left[\varepsilon^{(i_1,\dots,i_k)}(E_{j_1}, \dots, E_{j_k})\right] = \omega(E_{j_1}, \dots, E_{j_k})\delta^J_J = \omega(E_{j_1}, \dots, E_{j_k}).$$

Thus, $\mathcal{F}$ is basis for $\Lambda^k(V^*)$, and the rule of counting implies $\dim \Lambda^k(V^*) = \binom{n}{k}$, as desired. $\square$

**Theorem 6.0.9.** Suppose $V$ is an $n$-dimensional vector space and $\omega \in \Lambda^n(V^*)$. If $T : V \to V$ is any linear map and $v_1, \dots, v_n$ are abitrary vectors in $V$, then

$$\omega(Tv_1, \dots, Tv_n) = (\det T)\omega(v_1, \dots v_n).$$

*Proof.* Let $(\varepsilon_i)$ be a dual basis for $V^*$ and $(E_i)$ be a basis for $V$ dual to $(\varepsilon_i)$. Since $\mathcal{F}$ forms a basis for $\Lambda^n(V^*)$, one can express $\omega$ as

$$\omega(Tv_1,\dots,Tv_n)=\omega(E_{i_1},\dots,E_{i_k})\varepsilon^{(i_1,\dots,i_k)}(Tv_1,\dots,Tv_n)=\omega(E_{i_1},\dots,E_{i_k})\det\begin{bmatrix}(Tv_1)^{i_1} & \cdots & (Tv_k)^{i_1}\\ \vdots & \ddots & \vdots\\ (Tv_1)^{i_k} & \cdots & (Tv_k)^{i_k}\end{bmatrix}$$
$$=\omega(E_{i_1},\dots,E_{i_k})\det\begin{bmatrix}\sum_{j=1}^n T_{i_1,j}v_1^j & \cdots & \sum_{j=1}^n T_{i_1,j}v_k^j\\ \vdots & \ddots & \vdots\\ \sum_{j=1}^n T_{i_k,j}v_1^j & \cdots & \sum_{j=1}^n T_{i_k,j}v_k^j\end{bmatrix}=\omega(E_{i_1},\dots,E_{i_k})\det\left(T\cdot\begin{bmatrix}v_1^{i_1} & \cdots & v_k^{i_1}\\ \vdots & \ddots & \vdots\\ v_1^{i^k} & \cdots & v_k^{i^k}\end{bmatrix}\right)$$
$$=(\det T)\omega(E_{i_1},\dots,E_{i_k})\varepsilon^{(i_1,\dots,i_k)}(v_1,\dots,v_n)=(\det T)\omega(v_1,\dots,v_n).$$

Hence, we are done. □

**Lemma 6.0.10.** Suppose $\alpha\in\Lambda^m(V^*),\beta\in\Lambda^n(V^*),\omega\in\Lambda^k(V^*)$, then we have $\alpha\wedge\beta\wedge\omega:=(\alpha\wedge\beta)\wedge\omega=\alpha\wedge(\beta\wedge\omega)$.

*Proof.* We have

$$\begin{aligned}(\alpha\wedge\beta)\wedge\omega&=\frac{1}{(m+n)!k!}\sum_{\eta\in S_{m+n+k}}\operatorname{sgn}(\eta)\left(\frac{1}{m!n!}\sum_{\sigma\in S_{m+n}}\operatorname{sgn}(\sigma)\alpha\otimes\beta\right)\otimes\omega(v^\sigma)\\
&=\frac{1}{m!n!k!(m+n)!}\sum_{\eta\in S_{m+n+k}}\sum_{\sigma\in S_{m+n}}\operatorname{sgn}(\eta)\operatorname{sgn}(\sigma)\alpha\otimes\beta\otimes\omega(v^\sigma)\\
&=\frac{1}{m!n!k!}\sum_{\eta\in S_{m+n+k}}\operatorname{sgn}(\eta)\alpha\otimes\beta\otimes\omega(v^\sigma)\\
&=\frac{1}{m!n!k!(n+k)!}\sum_{\eta\in S_{m+n+k}}\sum_{\phi\in S_{n+k}}\operatorname{sgn}(\eta)\operatorname{sgn}(\phi)\alpha\otimes\beta\otimes\omega(v^\sigma)\\
&=\alpha\wedge(\beta\wedge\omega).\end{aligned}$$

Hence, we are done. □

**Lemma 6.0.11.** Let $(\varepsilon^i)$ be any basis for $V^*$ and $I=(i_1,\dots,i_k)$ be any multi-index, then we have

$$\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}=\varepsilon^I$$

*Proof.* Let $(v_i)$ be abitrary $k$-vector tuple in $V$ represented by the basis $(E_i)$ of $V$ dual to $(\varepsilon^i)$. We will show the following holds by induction. For $k=1$, this is trivial since $\varepsilon^I=\varepsilon^{i_1}$. For $n=2$, we have

$$\varepsilon^{i_1}\wedge\varepsilon^{i_2}(v_1,v_2)=\varepsilon^{i_1}\otimes\varepsilon^{i_2}(v_1,v_2)-\varepsilon^{i_1}\otimes\varepsilon^{i_2}(v_2,v_1)=\det\begin{bmatrix}v_1^1 & v_2^1\\ v_1^2 & v_2^2\end{bmatrix}=\varepsilon^{(i_1,i_2)}(v_1,v_2).$$

Suppose the following hypothesis holds for $k \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \wedge \varepsilon^{i_{k+1}}(v_1, \ldots, v_{k+1}) = (\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}) \wedge \varepsilon^{i_{k+1}}(v_1, \ldots, v_{k+1}) = \varepsilon^{(i_1,\ldots,i_k)} \wedge \varepsilon^{i_{k+1}}(v_1, \ldots, v_{k+1}) \\
&= \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma) \varepsilon^{(i_1,\ldots,i_k)} \otimes \varepsilon^{i_{k+1}}(v_1, \ldots, v_{k+1}) = \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma) \varepsilon^{(i_1,\ldots,i_k)} \otimes \varepsilon^{i_{k+1}}(v_{\sigma_1}, \ldots, v_{\sigma_{k+1}}) \\
&= \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma) \varepsilon^{(i_1,\ldots,i_k)}(v_{\sigma_1}, \ldots, v_{\sigma_k}) \cdot \varepsilon^{i_{k+1}}(v_{\sigma_{k+1}}) = \sum_{i=1}^{k} (-1)^{k+1+i} \varepsilon^{(i_1,\ldots,i_k)}(v_1, \ldots, v_{i-1}, v_{i+1}) \varepsilon^{i_{k+1}}(v_i) \\
&= \det \begin{bmatrix} v_1^{i_1} & \ldots & v_{k+1}^{i_{k+1}} \\ \vdots & \ddots & \vdots \\ v_1^{i_{k+1}} & \ldots & v_{k+1}^{i_{k+1}} \end{bmatrix} = \varepsilon^{(i_1,\ldots,i_{k+1})}
\end{aligned}$$

Hence the following is proven, as desired.

□

## 7 Differential Forms on Manifolds

**Definition 7.0.1** ($m$-forms). Suppose $M$ is a smooth manifold with dimension $n$ and $p \in M$, an $m$**-form at** $p$ is a covector $\omega_p$, that is

$$\omega_p : (T_pM)^m \to \mathbb{R} \text{ linear, or } \omega \in \Lambda^m(T_p^*M)$$

Alternatively, a $m$-form at $p$ takes $m$ tangent vectors and returns a real number linearly. We denote the vector space of smooth $k$-forms by

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M).$$

Consider a local smooth chart $(\varphi, U)$ for some neighborhood of $p$, for $m = 1$, as $\omega_p$ is linear, one can rewrite the output of $\omega_p$ by the following formula:

$$\omega_p(v) = a_1 dx^1 + a_2 dx^2 + \cdots + a_n dx^n = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \cdot v$$

For $m = 2$, let $u = u^i \dfrac{\partial}{\partial x^i}$ and $v = v^i \dfrac{\partial}{\partial x^i}$ (Einstein's summation convention), since $\omega_p$ is multilinear and alternating, it would follow that

$$\begin{aligned}
\omega_p(u, v) = \omega_p\left(u^i \frac{\partial}{\partial x^i}, v^i \frac{\partial}{\partial x^i}\right) &= \sum_{i \neq j} u^i v^j \omega_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \sum_{i<j} (u^i v^j - u^j v^i) \omega_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)
\end{aligned}$$

Let $dx^i \wedge dx^j(u, v) := u^i v^j - u^j v^i = \det \begin{pmatrix} u^i & v^i \\ u^j & v^j \end{pmatrix}$, and $\omega_{ij} := \omega_p\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right)$, one can reduce the following equality to

$$\omega_p(u, v) = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j(u, v)$$

or equivalently, i would like to use the reduced notation based on Einstein's summation convention,

$$\omega_p = \omega_N dx^i \wedge dx^j,$$

for each $N \in [n]^2 := \{(i, j) \mid i, j \in [n]\}$. By our definition of $dx^i \wedge dx^j$, since

$$\det[u^{i,j}, v^{i,j}] = dx^i \wedge dx^j(u, v)$$

is multilinear, one can be extended to the generalized operation, which is

$$\boxed{dx^{i_1} \wedge \cdots \wedge dx^{i_n}(v_1, \ldots, v_m) := \det[v_1^{i_1,\ldots,i_n}, \ldots, v_n^{i_1,\ldots,i_m}] = \det[v_i^N]}. \tag{*}$$

This is called the **Wedge product**. Moreover, it is worth noting that this operation holds only for the basis of tangent space, we want it also works for any two forms. But beforehand, we state the general formula for any higher $m$-forms:

**Theorem 7.0.2.** Any $m$-form of dimension $n$ can be expressed uniquely as

$$\omega_p = \omega_N dx^{i_1} \wedge \cdots \wedge dx^{i_m}$$

where $N \in [n]^m := \{(i_1, \ldots, i_m) \mid i_1, \ldots, i_m \in [n]\}$. Generally, since $(T_p^* M)^m$ is a vector space, it thus have basis, which is

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_m} \mid i_1 < i_2 < \cdots < i_m \in [n]\}$$

Consequently, the dimension of $(T_p^* M)^m$ is $\binom{n}{m}$.

Now we construct a Wedge product operation between any forms. Consider vectors

$$\{v_1, \ldots, v_{m+n}\} \in (T_p M)^{m+n},$$

and $m$-form $\omega_1$ and $n$-form $\omega_2$. By writiting $\omega_1$ and $\omega_2$ in basis of $(*)$ and since Wedge is multilinear to forms, it follows that

$$\begin{aligned}
&\omega_1 \wedge \omega_2(v_1, \ldots, v_{m+n}) = (\sum \omega_{1M} dx^{i_1} \wedge \cdots \wedge dx^{i_m}) \wedge (\sum \omega_{2N} dx^{j_1} \wedge \cdots \wedge dx^{j_n}) \\
&= \sum \omega_{1M}\omega_{2N} dx^{i_1} \wedge \cdots \wedge dx^{i_m} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\
&= \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \operatorname{sgn}(\sigma) \omega_{\sigma_1 \ldots \sigma_m} dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_m} \omega_{\sigma_{m+1} \ldots \sigma_{m+n}} dx^{\sigma_{m+1}} \wedge \cdots \wedge dx^{\sigma_{m+n}} \\
&= \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \operatorname{sgn}(\sigma) \omega_{\sigma_1 \ldots \sigma_m}(v_{\sigma_1}, \ldots, v_{\sigma_m}) \omega_{\sigma m+1 \ldots \sigma m+n}(v_{\sigma_{m+1}}, \ldots, v_{\sigma_{m+n}})
\end{aligned}$$

**Proposition 7.0.3.** Given two any $m$-form $x$, $n$-form $y$ and $p$-form $z$ , then we have

1. $x \wedge x = 0$,
2. $x \wedge y = (-1)^{nm} y \wedge x$,
3. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
4. If $m = n$ then $(x + y) \wedge z = x \wedge z + y \wedge z$.
5. ($m$ - form)$\wedge$($n$ - form) = $(m + n)$ - form.

**Definition 7.0.4.** Let $F : M \to N$ be a smooth map between smooth manifolds $M$ and $N$, $\omega$ is a differential form on $N$ . The *pullback* $F^*\omega$ is a differential form on $M$, which is defined as

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))$$

**Proposition 7.0.5.** Let $F : M \to N$ be smooth map.

1. $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.
2. $F^*(\omega \wedge \eta) = (F^*(\omega)) \wedge (F^*(\eta))$.

3. In any smooth chart,

$$F^*\left(\sum \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = \sum (\omega_I \circ F) d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

*Proof.* Let $(v_i)$ and $(w_i)$ be abitrary $k$-vector tuple of $T_pM$, $\alpha$ be abitrary real number. We have

$$\begin{aligned}(F^*\omega)_p(v_1+\alpha w_1,\ldots,v_k+\alpha w_k) &= \omega_{F(p)}(dF_p(v_1+w_1),\ldots dF_p(v_k+w_k))\\ &= \omega_{F(p)}(dF_p(v_1)+dF_p(\alpha w_1),\ldots dF_p(v_k)+dF_p(\alpha w_k))\\ &= \omega_{F(p)}(dF_p(v_1),\ldots,dF_p(v_k)) + \alpha\omega_{F(p)}(dF_p(w_1),\ldots,dF_p(w_k))\\ &= (F^*\omega)_p(v_1,\ldots,v_k) + \alpha(F^*\omega)_p(w_1,\ldots,w_k).\end{aligned}$$

Thus $F^*$ is linear over $\mathbb{R}$. To prove the second property, suppose $\omega$ is $k$-form and $\eta$ is $l$-form on $N$, one can be expressed so that

$$\begin{aligned}F^*(\omega\wedge\eta) = \omega_{F(p)} \wedge \eta_{F(p)}(dF_p) &= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}} \operatorname{sgn}(\sigma)\omega_{F(p)} \otimes \eta_{F(p)}(d(v_{\sigma_1}),\ldots,d(v_{\sigma k+l}))\\ &= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}} (F^*\omega)_p \otimes (F^*\eta)(d(v_{\sigma_1}),\ldots,d(v_{\sigma k+l}))\\ &= \frac{(k+l)!}{k!l!}\operatorname{Alt}(F^*(\omega)_p \otimes F^*(\eta)_p)\\ &= F^*(\omega)_p \wedge F^*(\eta)_p.\end{aligned}$$

Since $F^*$ is proven to be linear and pull back of a real-valued function is just a composition, one can obtain

$$\begin{aligned}F^*\left(\sum \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) &= \sum F^*\left(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\\ &= \sum (\omega_I \circ F) F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k})\\ &= \sum (\omega_I \circ F) dx^{i_1}(dF_p^1) \wedge \cdots \wedge dx^{i_1}(dF_p^k)\\ &= \sum (\omega_I \circ F) d(x^{i_1} \circ F) \wedge \ldots d(x^{i_k} \circ F)\end{aligned}$$

Hence, we are done. □

**Theorem 7.0.6.** Let $F : M \to N$ be smooth map between $n$-dimensional manifolds, $(x^i)$ and $(y^i)$ be smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and $u$ be a continuous real-valued function on $V$, then the following holds on $U \cap F^{-1}(V)$:

$$F^*(u dy^1 \wedge \ldots dy^n) = (u \circ F)(\det J_F) dx^1 \wedge \cdots \wedge dx^n.$$

*Proof.* Since $F^*$ is linear over $\mathbb{R}$, it follows that

$$F^*(u dy^1 \wedge \ldots dy^n) = (u \circ F) d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) = (u \circ F) dF^1 \wedge \cdots \wedge dF^n$$

For abitrary $1 \le j \le n$, by writings $dF^j$ in basis of $(x^i)$

$$dF^j = \frac{\partial F^j}{\partial x^i} dx^i.$$

Since $a \wedge a = 0$ for any form, this implies

$$
\begin{aligned}
dF^1 \wedge \cdots \wedge dF^n &= \sum_{\sigma \in S_n} \left( \frac{\partial F^1}{\partial x^{\sigma_1}} \cdots \frac{\partial F^n}{\partial x^{\sigma_n}} \right) dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_n} \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left( \frac{\partial F^1}{\partial x^{\sigma_1}} \cdots \frac{\partial F^n}{\partial x^{\sigma_n}} \right) dx^1 \wedge \cdots \wedge dx^n \\
&= \det \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{x^1} & \cdots & \frac{\partial F_n}{x^n} \end{bmatrix} dx^1 \wedge \cdots \wedge dx^n \\
&= J_F dx^1 \wedge \cdots \wedge dx^n
\end{aligned}
$$

.

Hence we are done. □

# 8 Measure Theory

**Definition 8.0.1** ($\sigma$-algebra). Let $X$ be a set, a collection $\mathcal{X} \in \mathcal{P}(X)$ is called a $\sigma$-algebra if it is

1. The set $X$ is itself in $\mathcal{X}$.
2. *Closed under unions:* Let $(E_i) \subseteq \mathcal{X}$ be countable subset of $\mathcal{X}$, then

$$\bigcup_{n=1}^{\infty} E_i \in X.$$

3. *Closed under complements:* If $A \in \mathcal{X}$, then

$$A^c := X \backslash A \in \mathcal{X}$$

**Proposition 8.0.2.** If $\mathcal{X}$ is a $\sigma$-algebra and $(X_i) \subseteq \mathcal{X}$ be countable subset of $\mathcal{X}$, then we have

$$\bigcap_{n=1}^{\infty} X_n \in \mathcal{X}$$

*Proof.* By the Morgan law, we have

$$\bigcap_{n=1}^{\infty} E_n = X \backslash \bigcap_{n=1}^{\infty} (X \backslash X_n) = \left( \bigcup_{n=1}^{\infty} E_n^c \right)^c.$$

Using those above condition yields

$$
\begin{aligned}
X_n^c \in \mathcal{X} \Rightarrow E_n^c \in \mathcal{X} \text{ for all n}, & \qquad \text{(condition 3)}, \\
\Rightarrow \bigcup_{n=1}^{\infty} E_n^c \in \mathcal{X} & \qquad \text{(condition 2)}, \\
\Rightarrow \left( \bigcup_{n=1}^{\infty} X_n^c \right)^c \in \mathcal{X} & \qquad \text{(condition 3)}, \\
\Rightarrow \bigcap_{n=1}^{\infty} E_n \in X &
\end{aligned}
$$

Hence, we are done. □

**Definition 8.0.3** (Measure). Let $X$ be a set with $\sigma$-algebra $\mathcal{M}$. A *measure* on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfying

1. $\mu(\varnothing) = 0$,
2. If $(E_n)$ is a countably disjoint subset of $\mathcal{M}$, then
$$\mu\left(\bigcup_{n=1}^{\infty} E_i\right) = \sum_{n=1}^{\infty} \mu(E_i).$$

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space* and the pair $(X, \mathcal{M})$ is called a *measurable space*.

**Theorem 8.0.4.** Let $(X, \mathcal{M}, \mu)$ be a measure space, then the followings hold.

1. *Monotonicity:* If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. *Subadditivity:* If $(E_n)$ is a countably disjoint subset of $\mathcal{M}$, then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$
3. *Continuity from below:* If $(E_n) \subset \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to+\infty} \mu(E_n)$$
4. *Continuity from above:* If $(E_n) \subset \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$, then
$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to+\infty} \mu(E_n)$$

*Proof.* 1. Since $F\backslash E$ and $E$ is disjoint, we have $\mu(F) = \mu((F\backslash E) \cup E) = \mu(F\backslash E) + \mu(E) \geq \mu(E)$.

2. Let $X_n = E_n \cup E_{n+1} \cup \dots$, it follows that
$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left((E_1\backslash X_1) \cup \bigcup_{n=2}^{\infty} E_n\right) = \mu(E_1\backslash X_1) + \mu\left(\bigcup_{n=2}^{\infty} E_n\right) \\
&\leq \mu(E_1) + \mu\left(\bigcup_{n=2}^{\infty} E_n\right) \\
&\leq \mu(E_1) + \mu(E_2)\mu\left(\bigcup_{n=3}^{\infty} E_n\right) \\
&\leq \cdots \\
&\leq \sum_{n=1}^{\infty} \mu(E_n).
\end{aligned}$$

3. Let $n \in \mathbb{N}$, we have
$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu(E_n)$$

Hence,
$$\lim_{n\to+\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \lim_{n\to+\infty} \mu(E_n)$$

4. Similar to the third property, since

$$\mu\left(\bigcap_{i=1}^{n} E_i\right) = \mu(E_n)$$

This implies

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) \lim_{n\to+\infty} \mu(E_n)$$

□

**Definition 8.0.5.** Let $(X, \mathcal{M}, \mu)$ be a measure space, define the set

$$\ker(\mu) := \{E \in \mathcal{M} \mid \mu(E) = 0\}.$$

A measure whose domain includes all subset of an element $E \in \ker(\mu)$ is called *complete*.

**Theorem 8.0.6.** Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\overline{\mathcal{M}}$ be the collection satisfying

$$\overline{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \in \ker(\mu)\}.$$

Then $\overline{\mathcal{M}}$ is a $\sigma$-algebra, and there is a unique extension $\overline{\mu}$ to $\mu$ to a comple measure on $\overline{\mathcal{M}}$.

## 9 Orientations

**Definition 9.0.1.** Two ordered basis $(E_1, \dots, E_n)$ and $(\tilde{E_1}, \dots, \tilde{E_n})$ for $V$ are called *consistently oriented* if the transition matrix $[B]_{E\to\tilde{E}}$ defined by

$$[E] = B[\tilde{E}]$$

has positive determeniant.

**Definition 9.0.2.** Let $V$ be a vector space, We denote the set

$$\mathcal{O}(V) := \mathcal{B}(V)/\mathcal{R}(V),$$

as the set of *all possible orientation of* $V$, where $\mathcal{R}(V) := \left\{\left([E_i], [\tilde{E_i}]\right) \in V^2 \mid \det[E_i] \cdot \det[\tilde{E_i}] > 0\right\}$, and each element of $\mathcal{O}(V)$ (which is a equivalent class) is said to be the *orientation for* $V$.

**Definition 9.0.3.** A vector space with a choice of orientation $(V, [E_i])$ (where $[E_i] \in \mathcal{O}(V)$) is called an *oriented vector space*. Any ordered basis $(E_i)$ that is in the given orientation is said to be *positively oriented*. Any basis that is not in the given orientation is said to be *negatively oriented*.

**Remark 9.0.4.** To prove a set of ordered basis determines an orientation for a vector space $V$, it suffices to prove every element of the following set belongs to only one equivalent class of $\mathcal{O}(V)$.

**Theorem 9.0.5.** Let $V$ be a vector space of dimension $n \geq 1$. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation of $V$ as the set of ordered bases $(E_i)$ such that $\omega(E_i) > 0$. Two nonzero $n$-covectors $\omega$ and $\eta$ determines the same orientation if and only if $\omega \cdot \eta > 0$.

*Proof.* Let $(E_i)$ and $(\tilde{E_i})$ be the ordered basis of $V$, we denote the set

$$\mathcal{O}_\omega = \left\{\left([E_i], [\tilde{E_i}]\right) \mid \omega(E_i) > 0 \text{ and } \omega(\tilde{E_i}) > 0\right\}.$$

Since those two sets are linearly independent, one can find a linear map $\mathcal{B} : V \to V$ such that $[E_i] = B[\tilde{E_i}]$. By the above proposition, it follows that

$$\omega(E_i) = \det(\mathcal{B})\omega(\tilde{E_i})$$

which implies $\det(\mathcal{B}) > 0$ or $\det[E_i] \cdot \det[\tilde{E_i}] > 0$ if and only if $\omega(E_i) > 0$ and $\omega(\tilde{E_i}) > 0$. Thus $\mathcal{O}_\omega$ is an equivalent class of $\mathcal{O}(V)$, which implies it is an orientation of $V$. In addition, that $\omega$ and $\eta$ determines the same orientation implies that $\omega(E_i) \cdot \eta(E_i) > 0$ since $\eta(E_i) > 0$, and vice versa, as desired. □

**Definition 9.0.6.** If $V$ is an oriented $n$-dimensional vector space and $\omega$ is an $n$-covector determines the orientation of $V$ that satifying the above theorem, $\omega$ is called a *positively oriented* $n$-covector.

**Definition 9.0.7.** Let $M$ be a smooth manifold, a *vector field on* $M$ is a map

$$\begin{aligned} X : M &\to TM \\ p &\mapsto X(p) \in T_p(M) \end{aligned}$$

such that $X$ is a smooth section of the tangent bundle $\pi \circ X = Id_M$, where $\pi : TM \to M$ is a projection.

**Definition 9.0.8.** Let $(E_1, \ldots, E_n)$ be a collection of vector fields determined on the open set $U \subseteq M$. If for every point $p \in M$, the set $(E_i(p))$ form a basis of the tangent space $T_pM$, the collection $(E_i)$ is said to be *a local frame*.

**Definition 9.0.9.** Let $M$ be a smooth $n$-manifold, $(E_i)$ be a local frame for $TM$ and let $\varphi$ be the map satisfying

$$\sigma : M \to \{-1, +1\}$$

We say that $(E_i)$ is *positively oriented* if $(E_i)$ is a positively oriented basis for $(T_pM, \sigma(p))$. The function $\sigma$ is called *pointwise orientation*.

**Definition 9.0.10.** A pointwise orientation $\sigma$ is said to be continuous if for every point $p \in M$, there exists an open neighborhood $U$ of $p$ and the local frame $(E_i)$ defined $U$ such that $(E_i(q))$ is positively oriented respect to restricted $\sigma_U$ for all $q \in U$. $\sigma$ is called an *orientation on* $M$.

**Definition 9.0.11.** An *oriented manifold* is an ordered pair $(M, \mathcal{O})$, where $M$ is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for $M$.

## 10 Integration on Manifolds

**Definition 10.0.1** (Measure Zero set). An *open rectangle* is the set of the form

$$C_a^b := (a^1, b^1) \times \cdots \times (a^n, b^n),$$

where $a^i < b^i$. The *volume of* $C_a^b$ is denoted by:

$$\mathrm{Vol}(C_a^b) := (b^1 - a^1) \ldots (b^n - a^n).$$

A subset $X \subset \mathbb{R}^n$ is said to have *measure zero* if for every $\varepsilon > 0$, there exists a countable cover of open rectangles $\{C_i\}$ such that

$$\sum_i \mathrm{Vol}(C_i) < \varepsilon.$$

A *domain of integration* in $\mathbb{R}^n$ is the bounded subset whose boundary has measure zero.

**Definition 10.0.2.** Let $D \subset \mathbb{R}^n$ be a domain of integration and $\omega = f dx^1 \wedge \cdots \wedge dx^n$ be a differential n-form on $\overline{D}$, where $f : \overline{D} \to \mathbb{R}$ is some continuous function. We define the integral of $\omega$ over $D$ as

$$\int_D \omega = \int_D f dx^1 \ldots dx^n = \int_D f dV.$$

More generally, let $U$ be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $D$ be any domain of integration containing the compact set $\mathrm{supp}(\omega)$, then

$$\int_U \omega = \int_D \omega.$$