Analytic Test of RT-GSFit

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1 Deriving Bessel's Eqn from the Grad-Shafranov Eqn

The Grad-Shafranov equation is given by

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = -\mu_o R^2 p'(\psi) - F(\psi)F'(\psi).$$

We will assume a plasma beta, $\beta = 0$. We are interested in solving the Grad-Shafranov equation inside the LCFS. We will take the large aspect-ratio limit and assume $R \to \infty$. Hence, we can simplify the Grad-Shafranov to

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{\partial^2 \psi}{\partial z^2} = -F(\psi)F'(\psi).$$

We will assume poloidal symmetry, and so we assume $\psi = \psi(\rho)$. Let

$$\rho = \sqrt{(R - R_o)^2 + (z - z_o)^2}.$$

$$R - R_o = \rho\cos(\theta),$$
$$z - z_o = \rho\sin(\theta),$$
$$\hat{\rho} = \cos(\theta)\hat{\mathbf{R}} + \sin(\theta)\hat{\mathbf{z}},$$
$$\hat{\theta} = -\sin(\theta)\hat{\mathbf{R}} + \cos(\theta)\hat{\mathbf{z}}.$$

Note that

$$\begin{aligned}
\frac{\partial \psi}{\partial R} &= \frac{\partial \rho}{\partial R}\frac{d\psi}{d\rho} \\
&= \frac{(R - R_o)}{\sqrt{(R - R_o)^2 + (z - z_o)^2}}\frac{d\psi}{d\rho} \\
&= \cos(\theta)\frac{d\psi}{d\rho},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial z} &= \frac{\partial \rho}{\partial z}\frac{d\psi}{d\rho} \\
&= \frac{(z - z_o)}{\sqrt{(R - R_o)^2 + (z - z_o)^2}}\frac{d\psi}{d\rho} \\
&= \sin(\theta)\frac{d\psi}{d\rho},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial R}\hat{\mathbf{R}} + \frac{\partial \psi}{\partial z}\hat{\mathbf{z}} &= \frac{d\psi}{d\rho}(\cos(\theta)\hat{\mathbf{R}} + \sin(\theta)\hat{\mathbf{z}}) \\
&= \frac{d\psi}{d\rho}\hat{\rho},
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{R}} \cdot \frac{\partial \hat{\rho}}{\partial R} &= \frac{\partial}{\partial R}\left(\frac{R - R_o}{\sqrt{(R - R_o^2) + (z - z_o^2)}}\right) \\
&= \frac{\sin^2\theta}{\rho}
\end{aligned}$$

$$
\begin{aligned}
\hat{\mathbf{z}} \cdot \frac{\partial \hat{\rho}}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{z - R_o}{\sqrt{(R - R_o^2) + (z - z_o^2)}}\right) \\
&= \frac{\cos^2\theta}{\rho}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \psi}{\partial R^2} + \frac{\partial^2 \psi}{\partial z^2} &= \left(\hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial \psi}{\partial R}\hat{\mathbf{R}} + \frac{\partial \psi}{\partial z}\hat{\mathbf{z}}\right) \\
&= \left(\hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{d\psi}{d\rho}\hat{\rho}\right) \\
&= \cos(\theta)\frac{\partial}{\partial R}\left(\frac{d\psi}{d\rho}\right) + \frac{d\psi}{d\rho}\hat{\mathbf{R}} \cdot \frac{\partial \hat{\rho}}{\partial R} + \sin(\theta)\frac{\partial}{\partial z}\left(\frac{d\psi}{d\rho}\right) + \frac{d\psi}{d\rho}\hat{\mathbf{z}} \cdot \frac{\partial \hat{\rho}}{\partial z} \\
&= \frac{d^2\psi}{d\rho^2} + \frac{1}{\rho}\frac{d\psi}{d\rho}.
\end{aligned}
$$

Hence, we can rewrite the Grad-Shafranov equation as

$$
\frac{d^2\psi}{d\rho^2} + \frac{1}{\rho}\frac{d\psi}{d\rho} = -F(\psi)F'(\psi).
$$

RTGSFit assumes FF' is of the form

$$
FF' = a(1 - \psi_N),
$$

where a is a constant and

$$
\psi_N = \frac{\psi_o - \psi}{\psi_o - \psi_b},
$$

where ψ_b is ψ at the last closed flux surface and ψ_o is the value of ψ at the o-point. Let

$$
\psi(\rho) = \psi'(\rho) + \psi_b,
$$

$$
a' = \frac{a}{(\psi_o - \psi_b)},
$$

hence by substituting our expression for ψ in terms of ψ' and a in terms of a' we get

$$
\frac{d^2\psi'}{d\rho^2} + \frac{1}{\rho}\frac{d\psi'}{d\rho} = -a'\psi'.
$$

Multiplying through by ρ^2 and rearranging we arrive at an equation that is nearly Bessel's equation of order 0

$$
\rho^2\frac{d^2\psi'}{d\rho^2} + \rho\frac{d\psi'}{d\rho} + a'\rho^2\psi' = 0.
$$

Let

$$
s = \frac{a'}{|a'|}\sqrt{|a'|}\rho,
$$

$$
\implies \rho = \frac{a'}{|a'|}\frac{1}{\sqrt{|a'|}}s
$$

$$
\implies \frac{\partial}{\partial \rho} = \frac{a'}{|a'|}\sqrt{|a'|}\frac{\partial}{\partial s},
$$

$$
\implies \frac{\partial^2}{\partial \rho^2} = a'\frac{\partial^2}{\partial s^2},
$$

$$
\implies \rho^2 = \frac{s^2}{a'}
$$

Hence, the Grad-Shafranov equation becomes

$$
s^2\frac{\partial^2\psi'}{\partial s^2} + s\frac{\partial\psi'}{\partial s} + s^2\psi' = 0,
$$

which is exactly of the same form as Bessel's equation of order 0.

2 Solving Bessels' equation

We know that ψ' is of the form

$$\psi' = C_1 J_0(s) + C_2 Y_0(s),$$

where J_0 and Y_0 are Bessel function of order zero of the first and second kind respectively. We know that ψ is finite at $R = R_o$, $z = z_o$, hence, $C_2 = 0$. Past the last closed flux surface, there is no plasma current. Hence, the full solution for ψ' is

$$\psi'(\rho) = C_1 J_0(\sqrt{a'}\rho).$$

3 Tidying up solution

We know that

$$\psi = C_1 J_0(\sqrt{a'}\rho) + \psi_b,$$

which we can rewrite as

$$\psi = C_1 J_0\left(a^* \frac{\rho}{\rho_b}\right) + \psi_b,$$

where

$$a^* = \sqrt{\frac{a\rho_b^2}{\psi_o - \psi_b}}.$$

We know that $\psi(\rho_b) = \psi_b$, hence,

$$C_1 J_0(a^*) = 0.$$

We will assume that we have chosen a such that a^* is a zero of J_0. We know that $\psi(0) = \psi_o$, hence

$$\psi = (\psi_o - \psi_b) J_0\left(a^* \frac{\rho}{\rho_b}\right) + \psi_b,$$

4 Solution for $\rho > \rho_b$

For $\rho > \rho_b$ there is no plasma current and the Grad-Shafranov equation is given by

$$\rho^2 \frac{d^2\psi}{d\rho^2} + \rho \frac{d\psi}{d\rho} = 0$$

The equation has the solution

$$\psi = C_1 \ln\left(\frac{\rho}{\rho_b}\right) + \psi_b,$$

and satisfies continuity at $\rho = \rho_b$. We also require continuity of the derivative of ψ, note that

$$\frac{d}{d\rho}\left[(\psi_o - \psi_b) J_0\left(a^* \frac{\rho}{\rho_b}\right)\right] = -\frac{(\psi_o - \psi_b)a^*}{\rho_b} J_1\left(a^* \frac{\rho}{\rho_b}\right),$$

$$\frac{d}{d\rho} C_1 \ln\left(\frac{\rho}{\rho_b}\right) = \frac{C_1}{\rho},$$

hence,

$$\frac{C_1}{\rho_b} = -\frac{(\psi_o - \psi_b)a^*}{\rho_b} J_1(a^*),$$

$$\implies C_1 = -(\psi_o - \psi_b)a^* J_1(a^*),$$

Hence, for $\rho \geq \rho_b$

$$\psi = -(\psi_o - \psi_b)a^* J_1(a^*) \ln\left(\frac{\rho}{\rho_b}\right) + \psi_b.$$

5 Total current

For $\rho > \rho_b$, the poloidal field is given by

$$
\begin{aligned}
B_\theta &= \frac{1}{R}\frac{d\psi}{d\rho}, \\
&= -\frac{(\psi_o - \psi_b)a^* J_1(a^*)}{(R_o + \rho\cos\theta)\rho}.
\end{aligned}
$$

Using Ampere's law, namely,

$$
\begin{aligned}
2\pi\rho B_\theta &= \mu_0 I_{\text{total}}, \\
\oint_{\rho=\rho_0} B_\theta \rho_0 d\theta &= \frac{(\psi_o - \psi_b)a^* J_1(a^*)}{R_o}\frac{2\pi}{\sqrt{1-\frac{\rho_0^2}{R_o^2}}} \\
&= 2\pi\frac{(\psi_o - \psi_b)a^* J_1(a^*)}{R_o} \quad (\text{for } \rho_0/R_o \to 0) \\
&= \mu_0 I_{\text{total}}.
\end{aligned}
$$

Taking the leading-order term, we know that

$$
\begin{aligned}
I_{total} &= \frac{2\pi}{R_o\mu_0}(\psi_o - \psi_b)a^* J_1(a^*), \\
\implies \psi_o - \psi_b &= \frac{\mu_0 I_{\text{total}} R_o}{2\pi a^* J_1(a^*)}
\end{aligned}
$$

6 Calculating expressions for ψ_o, ψ_b

We need to ensure that ψ is zero at $(R, z) = (0, 0)$. Where we assume this is outside the region $\rho < \rho_b$. Let

$$
d = \sqrt{R_o^2 - z_o^2}.
$$

We impose $\psi(d) = 0$. Hence

$$
\begin{aligned}
\psi_b &= \frac{\mu_0 I_{\text{total}} R_o}{2\pi}\ln\left(\frac{d}{\rho_b}\right), \\
\implies \psi_o &= \frac{\mu_0 I_{\text{total}} R_o}{2\pi}\left[\ln\left(\frac{d}{\rho_b}\right) + \frac{1}{a^* J_1(a^*)}\right]
\end{aligned}
$$

7 Full solution

The full solution for ψ is given by

$$
\psi(\rho) = \begin{cases} \Delta_\psi J_0\left(a^*\frac{\rho}{\rho_b}\right) + \psi_b, & \rho \le \rho_b, \\ -\Delta_\psi a^* J_1(a^*)\ln\left(\frac{\rho}{\rho_b}\right) + \psi_b, & \rho \ge \rho_b, \end{cases}
$$

where

$$
\begin{aligned}
\Delta_\psi &= \psi_o - \psi_b \\
\psi_o &= \frac{\mu_0}{2\pi} I_{\text{total}} R_o\left[\ln\left(\frac{d}{\rho_b}\right) + \frac{1}{a^* J_1(a^*)}\right], \\
\psi_b &= \frac{\mu_0}{2\pi} I_{\text{total}} R_o \ln\left(\frac{d}{\rho_b}\right), \\
\Delta_\psi &= \frac{\mu_0 I_{\text{total}} R_o}{2\pi a^* J_1(a^*)}, \\
a &= \frac{(a^*)^2(\psi_o - \psi_b)}{\rho_b^2} \\
&= \frac{\mu_0 a^* I_{\text{total}} R_o}{2\pi J_1(a^*)\rho_b^2},
\end{aligned}
$$

and a^* is the coordinate of a positive zero of J_0. Hence, the solution is not unique but if we impose that ψ needs to decrease monotonically away from ψ_o then this forces a^* to be the first positive zero of J_0.

8 Magnetic field

The magnetic field is given by

$$\begin{aligned}
B_R &= B_\theta \vec{\hat{\theta}} \cdot \vec{\hat{R}} \\
&= -\frac{1}{R}\sin(\theta)\frac{d\psi}{d\rho} \\
&= -\frac{1}{R}\frac{z - z_o}{\rho}\frac{d\psi}{d\rho} \\
B_z &= B_\theta \vec{\hat{\theta}} \cdot \vec{\hat{z}} \\
&= \frac{1}{R}\cos(\theta)\frac{d\psi}{d\rho} \\
&= \frac{1}{R}\frac{R - R_o}{\rho}\frac{d\psi}{d\rho}
\end{aligned}$$