Leapfrog code

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Normalised equations

The background magnetic field is given by

$$\boldsymbol{B}_0 = (\sin\alpha\hat{\boldsymbol{y}} + \cos\alpha\hat{\boldsymbol{z}}).$$

The background density is given by, $\rho = \rho(x)$. Assume linear waves are imposed on cold, static plasma. Therefore, the velocity is perpendicular to the background field. The velocity is given by

$$\boldsymbol{u}(x,z,y,t) = u_x(x,y,z,t)\hat{\boldsymbol{x}} + u_\perp(x,y,z,t)\hat{\perp},$$

where

$$\hat{\perp} = \hat{\boldsymbol{B}}_0 \times \hat{\boldsymbol{x}} = \cos\alpha\hat{\boldsymbol{y}} - \sin\alpha\hat{\boldsymbol{z}}.$$

The magnetic field perturbation is given by

$$\boldsymbol{b}(x,z,y,t) = b_x(x,z,y,t)\hat{\boldsymbol{x}} + b_\perp(x,z,y,t)\hat{\perp} + b_{||}(x,z,y,t)\hat{\boldsymbol{B}}_0.$$

The momentum equation is given by

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{1}{\rho}(\boldsymbol{B}_0\cdot\boldsymbol{\nabla})\boldsymbol{b} - \frac{1}{\rho}\boldsymbol{\nabla}\left(\boldsymbol{B}_0\cdot\boldsymbol{b}\right),$$

the induction equation is given by

$$\frac{\partial \boldsymbol{b}}{\partial t} = (\boldsymbol{B}_0\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{B}_0(\boldsymbol{\nabla}\cdot\boldsymbol{u}).$$

Hence,

$$\frac{\partial u_x}{\partial t} = \frac{1}{\rho}\left[\left(\sin\alpha\frac{\partial}{\partial y} + \cos\alpha\frac{\partial}{\partial z}\right)b_x - \frac{\partial b_{||}}{\partial x}\right]$$

$$\frac{\partial u_\perp}{\partial t} = \frac{1}{\rho}\left[\left(\sin\alpha\frac{\partial}{\partial y} + \cos\alpha\frac{\partial}{\partial z}\right)b_\perp - \left(\cos\alpha\frac{\partial}{\partial y} - \sin\alpha\frac{\partial}{\partial z}\right)b_{||}\right],$$

$$\frac{\partial b_x}{\partial t} = \left(\sin\alpha\frac{\partial}{\partial y} + \cos\alpha\frac{\partial}{\partial z}\right)u_x,$$

$$\frac{\partial b_\perp}{\partial t} = \left(\sin\alpha\frac{\partial}{\partial y} + \cos\alpha\frac{\partial}{\partial z}\right)u_\perp.$$

$$\frac{\partial b_{||}}{\partial t} = -\left[\frac{\partial u_x}{\partial x} + \left(\cos\alpha\frac{\partial}{\partial y} - \sin\alpha\frac{\partial}{\partial z}\right)u_\perp\right],$$

The grid

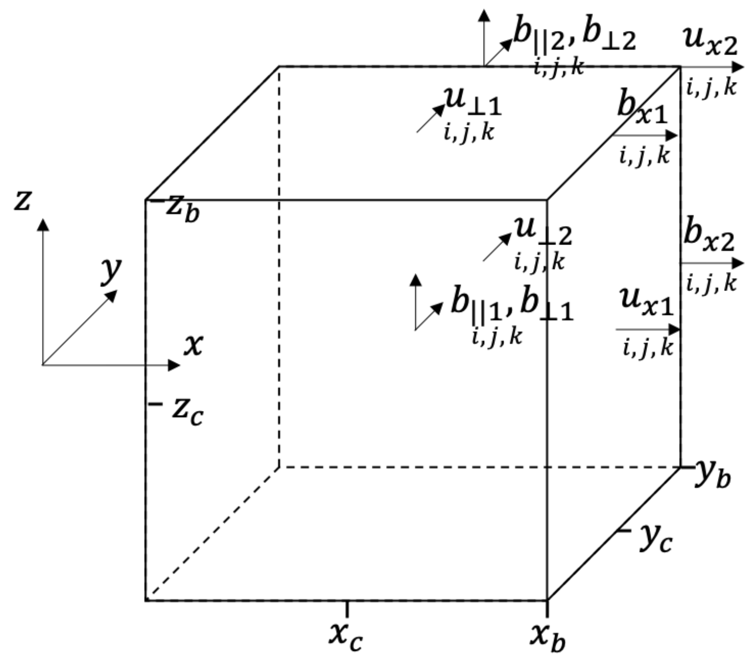

Figure 1: Diagram showing the location where every variable is defined for each cell.

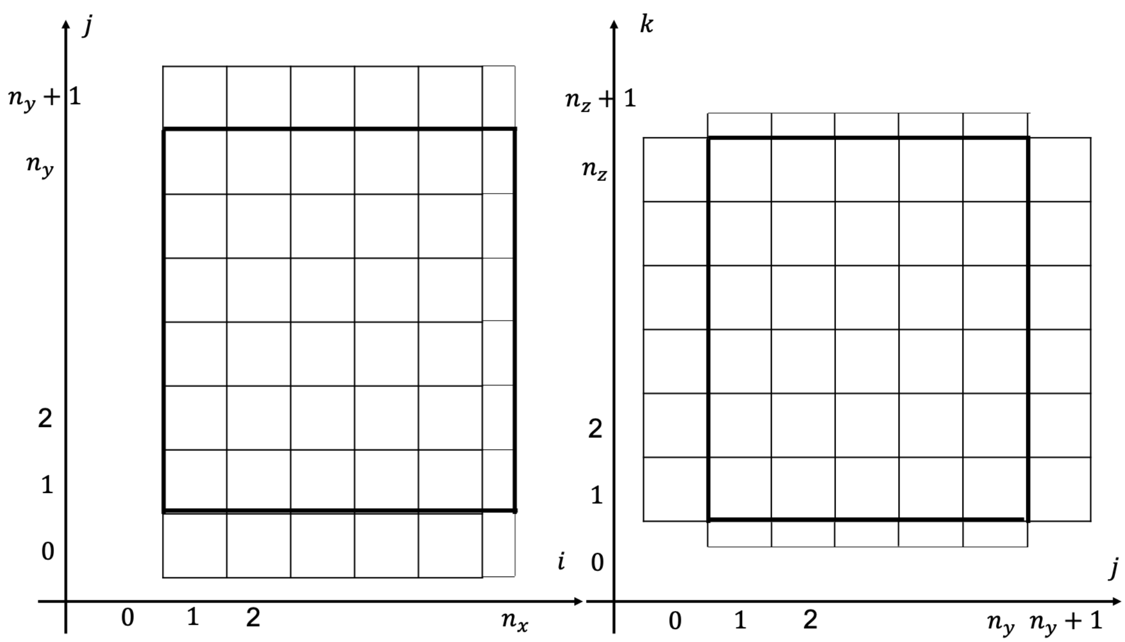

Figure 2: Diagram of the physical and ghost domain. The thick line denotes the boundary between the physical and ghost domain.

We stagger the grid as illustrated in Figure 1. Hence, the MHD equations become:

$$\frac{\partial u_{x1}}{\partial t} = \frac{1}{\rho}\left[\sin\alpha\frac{\partial b_{x2}}{\partial y} + \cos\alpha\frac{\partial b_{x1}}{\partial z} - \frac{\partial b_{||1}}{\partial x}\right]$$

$$
\begin{aligned}
\frac{\partial u_{x2}}{\partial t} &= \frac{1}{\rho}\left[\sin\alpha\frac{\partial b_{x1}}{\partial y} + \cos\alpha\frac{\partial b_{x2}}{\partial z} - \frac{\partial b_{||2}}{\partial x}\right] \\
\frac{\partial u_{\perp 1}}{\partial t} &= \frac{1}{\rho}\left[\sin\alpha\frac{\partial b_{\perp 2}}{\partial y} + \cos\alpha\frac{\partial b_{\perp 1}}{\partial z} - \cos\alpha\frac{\partial b_{||2}}{\partial y} + \sin\alpha\frac{\partial b_{||1}}{\partial z}\right], \\
\frac{\partial u_{\perp 2}}{\partial t} &= \frac{1}{\rho}\left[\sin\alpha\frac{\partial b_{\perp 1}}{\partial y} + \cos\alpha\frac{\partial b_{\perp 2}}{\partial z} - \cos\alpha\frac{\partial b_{||1}}{\partial y} + \sin\alpha\frac{\partial b_{||2}}{\partial z}\right], \\
\frac{\partial b_{x1}}{\partial t} &= \sin\alpha\frac{\partial u_{x2}}{\partial y} + \cos\alpha\frac{\partial u_{x1}}{\partial z}, \\
\frac{\partial b_{x2}}{\partial t} &= \sin\alpha\frac{\partial u_{x1}}{\partial y} + \cos\alpha\frac{\partial u_{x2}}{\partial z}, \\
\frac{\partial b_{\perp 1}}{\partial t} &= \sin\alpha\frac{\partial u_{\perp 2}}{\partial y} + \cos\alpha\frac{\partial u_{\perp 1}}{\partial z}, \\
\frac{\partial b_{\perp 2}}{\partial t} &= \sin\alpha\frac{\partial u_{\perp 1}}{\partial y} + \cos\alpha\frac{\partial u_{\perp 2}}{\partial z}, \\
\frac{\partial b_{||1}}{\partial t} &= -\left[\frac{\partial u_{x1}}{\partial x} + \cos\alpha\frac{\partial u_{\perp 2}}{\partial y} - \sin\alpha\frac{\partial u_{\perp 1}}{\partial z}\right], \\
\frac{\partial b_{||2}}{\partial t} &= -\left[\frac{\partial u_{x2}}{\partial x} + \cos\alpha\frac{\partial u_{\perp 1}}{\partial y} - \sin\alpha\frac{\partial u_{\perp 2}}{\partial z}\right].
\end{aligned}
$$

Where:

- u_{x1} is defined at (x_b, y_c, z_c).
- u_{x2} is defined at (x_b, y_b, z_b).
- $u_{\perp 1}$ is defined at (x_c, y_c, z_b).
- $u_{\perp 2}$ is defined at (x_c, y_b, z_c).
- b_{x1} is defined at (x_b, y_c, z_b).
- b_{x2} is defined at (x_b, y_b, z_c).
- $b_{\perp 1}$ and $b_{||1}$ are defined at (x_c, y_c, z_c).
- $b_{\perp 2}$ and $b_{||2}$ are defined at (x_c, y_b, z_b).

Figure 2 illustrates the number of cells used and how they are arranged. The physical domain is defined by

- `xc(1:nx)`,
- `xb(0:nx-1)`,
- `yc(1:ny)`,
- `yb(1:ny)`,
- `zc(1:nz)`,
- `zb(0:nz)`,

the full computational domain, including ghost cells is given by

- `xc(1:nx)`,
- `xb(0:nx-1)`,

- `yc(0:ny+1)`,
- `yb(0:ny+1)`,
- `zc(0:nz+1)`,
- `zb(0:nz)`,

where b_x, $b_{||}$ and $b_\perp$ do not have ghost cells in the z-direction because the velocity is set to zero at the boundaries.

Boundary conditions

The physical domain is given by

$$0 \leq x \leq L_x,$$
$$0 \leq y \leq L_y$$
$$0 \leq z \leq L_z$$

The boundary conditions are:

- At $x = 0$ we impose
  $$u_x = b_x = 0,$$
  this should ensure that
  $$\frac{\partial u_\perp}{\partial x} = \frac{\partial b_\perp}{\partial x} = \frac{\partial b_{||}}{\partial x} = 0,$$
  on the boundary.
- At $x = L_x$ we impose
  $$u_\perp = b_\perp = b_{||} = 0$$
  this should ensure that
  $$\frac{\partial u_x}{\partial x} = \frac{\partial b_x}{\partial x} = 0,$$
  on the boundary.
- At $y = 0$ and $y = L_y$ periodic boundaries are imposed.
- At $z = 0$ and $z = L_z$ we impose
  $$u_x = u_\perp = 0.$$
  We also impose that the ghost flow must be the negative mirror image of the domain flow. No boundary conditions are imposed on the magnetic field directly.