

This document is used to assist with the Algebra in Appendix A.2

Note that ∇ has been replaced with '\Delta' to ensure the code works okay.

> restart;

We normalise the velocity coefficients by u_0 and the field components by $(B_0 * u_0 / v_{A+})$.

We start by solving the Matrix equation given by Equation (57).

> eqn1 := $a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + a_{14} \cdot x_4 = y_1$

$$eqn1 := a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + a_{14} x_4 = y_1 \quad (1)$$

> eqn2 := $a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + a_{24} \cdot x_4 = y_2$

$$eqn2 := a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + a_{24} x_4 = y_2 \quad (2)$$

> eqn3 := $a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 + a_{34} \cdot x_4 = y_3$

$$eqn3 := a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + a_{34} x_4 = y_3 \quad (3)$$

> eqn4 := $a_{41} \cdot x_1 + a_{42} \cdot x_2 + a_{43} \cdot x_3 + a_{44} \cdot x_4 = y_4$

$$eqn4 := a_{41} x_1 + a_{42} x_2 + a_{43} x_3 + a_{44} x_4 = y_4 \quad (4)$$

> solns := solve({eqn1, eqn2, eqn3, eqn4}, {x1, x2, x3, x4}) :

> sol1 := rhs(solns[1])

$$sol1 := - \left(a_{12} a_{23} a_{34} y_4 - a_{12} a_{23} a_{44} y_3 - a_{12} a_{24} a_{33} y_4 + a_{12} a_{24} a_{43} y_3 + a_{12} a_{33} a_{44} y_2 - a_{12} a_{34} a_{43} y_2 - a_{13} a_{22} a_{34} y_4 + a_{13} a_{22} a_{44} y_3 + a_{13} a_{24} a_{32} y_4 - a_{13} a_{24} a_{42} y_3 - a_{13} a_{32} a_{44} y_2 + a_{13} a_{34} a_{42} y_2 + a_{14} a_{22} a_{33} y_4 - a_{14} a_{22} a_{43} y_3 - a_{14} a_{23} a_{32} y_4 + a_{14} a_{23} a_{42} y_3 + a_{14} a_{32} a_{43} y_2 - a_{14} a_{33} a_{42} y_2 - a_{22} a_{33} a_{44} y_1 + a_{22} a_{34} a_{43} y_1 + a_{23} a_{32} a_{44} y_1 - a_{23} a_{34} a_{42} y_1 - a_{24} a_{32} a_{43} y_1 + a_{24} a_{33} a_{42} y_1 \right) / \left(a_{11} a_{22} a_{33} a_{44} - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_{44} + a_{11} a_{23} a_{34} a_{42} + a_{11} a_{24} a_{32} a_{43} - a_{11} a_{24} a_{33} a_{42} - a_{12} a_{21} a_{33} a_{44} + a_{12} a_{21} a_{34} a_{43} + a_{12} a_{23} a_{31} a_{44} - a_{12} a_{23} a_{34} a_{41} - a_{12} a_{24} a_{31} a_{43} + a_{12} a_{24} a_{33} a_{41} + a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{42} - a_{13} a_{22} a_{31} a_{44} + a_{13} a_{22} a_{34} a_{41} + a_{13} a_{24} a_{31} a_{42} - a_{13} a_{24} a_{32} a_{41} - a_{14} a_{21} a_{32} a_{43} + a_{14} a_{21} a_{33} a_{42} + a_{14} a_{22} a_{31} a_{43} - a_{14} a_{22} a_{33} a_{41} - a_{14} a_{23} a_{31} a_{42} + a_{14} a_{23} a_{32} a_{41} \right) \quad (5)$$

> sol2 := rhs(solns[2])

$$sol2 := \left(a_{11} a_{23} a_{34} y_4 - a_{11} a_{23} a_{44} y_3 - a_{11} a_{24} a_{33} y_4 + a_{11} a_{24} a_{43} y_3 + a_{11} a_{33} a_{44} y_2 - a_{11} a_{34} a_{43} y_2 - a_{13} a_{21} a_{34} y_4 + a_{13} a_{21} a_{44} y_3 + a_{13} a_{24} a_{31} y_4 - a_{13} a_{24} a_{41} y_3 - a_{13} a_{31} a_{44} y_2 + a_{13} a_{34} a_{41} y_2 + a_{14} a_{21} a_{33} y_4 - a_{14} a_{21} a_{43} y_3 - a_{14} a_{23} a_{31} y_4 + a_{14} a_{23} a_{41} y_3 + a_{14} a_{31} a_{43} y_2 - a_{14} a_{33} a_{41} y_2 - a_{21} a_{33} a_{44} y_1 + a_{21} a_{34} a_{43} y_1 + a_{23} a_{31} a_{44} y_1 - a_{23} a_{34} a_{41} y_1 - a_{24} a_{31} a_{43} y_1 + a_{24} a_{33} a_{41} y_1 \right) / \left(a_{11} a_{22} a_{33} a_{44} - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_{44} + a_{11} a_{23} a_{34} a_{42} + a_{11} a_{24} a_{32} a_{43} - a_{11} a_{24} a_{33} a_{42} - a_{12} a_{21} a_{33} a_{44} + a_{12} a_{21} a_{34} a_{43} + a_{12} a_{23} a_{31} a_{44} - a_{12} a_{23} a_{34} a_{41} - a_{12} a_{24} a_{31} a_{43} + a_{12} a_{24} a_{33} a_{41} + a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{42} - a_{13} a_{22} a_{31} a_{44} + a_{13} a_{22} a_{34} a_{41} \right) \quad (6)$$

$$+ a_{13} a_{24} a_{31} a_{42} - a_{13} a_{24} a_{32} a_{41} - a_{14} a_{21} a_{32} a_{43} + a_{14} a_{21} a_{33} a_{42} + a_{14} a_{22} a_{31} a_{43} - a_{14} a_{22} a_{33} a_{41} - a_{14} a_{23} a_{31} a_{42} + a_{14} a_{23} a_{32} a_{41})$$

> sol3 := rhs(solns[3])

$$\begin{aligned} \text{sol3} := & - \left(a_{11} a_{22} a_{34} y_4 - a_{11} a_{22} a_{44} y_3 - a_{11} a_{24} a_{32} y_4 + a_{11} a_{24} a_{42} y_3 + a_{11} a_{32} a_{44} y_2 \right. \\ & - a_{11} a_{34} a_{42} y_2 - a_{12} a_{21} a_{34} y_4 + a_{12} a_{21} a_{44} y_3 + a_{12} a_{24} a_{31} y_4 - a_{12} a_{24} a_{41} y_3 \\ & - a_{12} a_{31} a_{44} y_2 + a_{12} a_{34} a_{41} y_2 + a_{14} a_{21} a_{32} y_4 - a_{14} a_{21} a_{42} y_3 - a_{14} a_{22} a_{31} y_4 \\ & + a_{14} a_{22} a_{41} y_3 + a_{14} a_{31} a_{42} y_2 - a_{14} a_{32} a_{41} y_2 - a_{21} a_{32} a_{44} y_1 + a_{21} a_{34} a_{42} y_1 \\ & + a_{22} a_{31} a_{44} y_1 - a_{22} a_{34} a_{41} y_1 - a_{24} a_{31} a_{42} y_1 + a_{24} a_{32} a_{41} y_1 \Big) / \left(a_{11} a_{22} a_{33} a_{44} \right. \\ & - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_{44} + a_{11} a_{23} a_{34} a_{42} + a_{11} a_{24} a_{32} a_{43} - a_{11} a_{24} a_{33} a_{42} \\ & - a_{12} a_{21} a_{33} a_{44} + a_{12} a_{21} a_{34} a_{43} + a_{12} a_{23} a_{31} a_{44} - a_{12} a_{23} a_{34} a_{41} - a_{12} a_{24} a_{31} a_{43} \\ & + a_{12} a_{24} a_{33} a_{41} + a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{42} - a_{13} a_{22} a_{31} a_{44} + a_{13} a_{22} a_{34} a_{41} \\ & + a_{13} a_{24} a_{31} a_{42} - a_{13} a_{24} a_{32} a_{41} - a_{14} a_{21} a_{32} a_{43} + a_{14} a_{21} a_{33} a_{42} + a_{14} a_{22} a_{31} a_{43} \\ & \left. - a_{14} a_{22} a_{33} a_{41} - a_{14} a_{23} a_{31} a_{42} + a_{14} a_{23} a_{32} a_{41} \right) \end{aligned} \quad (7)$$

> sol4 := rhs(solns[4])

$$\begin{aligned} \text{sol4} := & \left(a_{11} a_{22} a_{33} y_4 - a_{11} a_{22} a_{43} y_3 - a_{11} a_{23} a_{32} y_4 + a_{11} a_{23} a_{42} y_3 + a_{11} a_{32} a_{43} y_2 \right. \\ & - a_{11} a_{33} a_{42} y_2 - a_{12} a_{21} a_{33} y_4 + a_{12} a_{21} a_{43} y_3 + a_{12} a_{23} a_{31} y_4 - a_{12} a_{23} a_{41} y_3 \\ & - a_{12} a_{31} a_{43} y_2 + a_{12} a_{33} a_{41} y_2 + a_{13} a_{21} a_{32} y_4 - a_{13} a_{21} a_{42} y_3 - a_{13} a_{22} a_{31} y_4 \\ & + a_{13} a_{22} a_{41} y_3 + a_{13} a_{31} a_{42} y_2 - a_{13} a_{32} a_{41} y_2 - a_{21} a_{32} a_{43} y_1 + a_{21} a_{33} a_{42} y_1 \\ & + a_{22} a_{31} a_{43} y_1 - a_{22} a_{33} a_{41} y_1 - a_{23} a_{31} a_{42} y_1 + a_{23} a_{32} a_{41} y_1 \Big) / \left(a_{11} a_{22} a_{33} a_{44} \right. \\ & - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_{44} + a_{11} a_{23} a_{34} a_{42} + a_{11} a_{24} a_{32} a_{43} - a_{11} a_{24} a_{33} a_{42} \\ & - a_{12} a_{21} a_{33} a_{44} + a_{12} a_{21} a_{34} a_{43} + a_{12} a_{23} a_{31} a_{44} - a_{12} a_{23} a_{34} a_{41} - a_{12} a_{24} a_{31} a_{43} \\ & + a_{12} a_{24} a_{33} a_{41} + a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{42} - a_{13} a_{22} a_{31} a_{44} + a_{13} a_{22} a_{34} a_{41} \\ & + a_{13} a_{24} a_{31} a_{42} - a_{13} a_{24} a_{32} a_{41} - a_{14} a_{21} a_{32} a_{43} + a_{14} a_{21} a_{33} a_{42} + a_{14} a_{22} a_{31} a_{43} \\ & \left. - a_{14} a_{22} a_{33} a_{41} - a_{14} a_{23} a_{31} a_{42} + a_{14} a_{23} a_{32} a_{41} \right) \end{aligned} \quad (8)$$

> a₁₁ := u_{x10} -

$$a_{11} := u_{x10} - \quad (9)$$

> a₁₂ := u_{x40} -

$$a_{12} := u_{x40} - \quad (10)$$

> a₁₃ := -u_{x20} +

$$a_{13} := -u_{x20} + \quad (11)$$

> a₁₄ := -u_{x30} +

$$a_{14} := -u_{x30} + \quad (12)$$

> a₂₁ := k_{z1} - u_{x10} -

	$a_{21} := k_{z1-} u_{x10-}$	(13)
>	$a_{22} := k_{z4-} u_{x40-}$	
	$a_{22} := k_{z4-} u_{x40-}$	(14)
>	$a_{23} := -k_{z2+} u_{x20+}$	
	$a_{23} := -k_{z2+} u_{x20+}$	(15)
>	$a_{24} := -k_{z3+} u_{x30+}$	
	$a_{24} := -k_{z3+} u_{x30+}$	(16)
>	$a_{31} := 1$	
	$a_{31} := 1$	(17)
>	$a_{32} := 1$	
	$a_{32} := 1$	(18)
>	$a_{33} := -1$	
	$a_{33} := -1$	(19)
>	$a_{34} := -1$	
	$a_{34} := -1$	(20)
>	$a_{41} := k_{z1-}$	
	$a_{41} := k_{z1-}$	(21)
>	$a_{42} := k_{z4-}$	
	$a_{42} := k_{z4-}$	(22)
>	$a_{43} := -k_{z2+}$	
	$a_{43} := -k_{z2+}$	(23)
>	$a_{44} := -k_{z3+}$	
	$a_{44} := -k_{z3+}$	(24)
>	$y_1 := u_{x10+}$	
	$y_1 := u_{x10+}$	(25)
>	$y_2 := k_{z1+} u_{x10+}$	
	$y_2 := k_{z1+} u_{x10+}$	(26)
>	$y_3 := 1$	
	$y_3 := 1$	(27)
>	$y_4 := k_{z1+}$	
	$y_4 := k_{z1+}$	(28)

[Note that $u_{x0}[n]$ denotes $\hat{u}_{\{x\}n}$ and $u_{x[n\pm]}$ denotes $u_{\{x\}n\pm}$].

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x10-} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 1-}}{L_{1-} - k_x^2} \\
& \quad \quad \quad u_{x10-} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 1-}}{-k_x^2 + L_{1-}} \tag{29}
\end{aligned}$$

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x10+} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 1+}}{L_{1+} - k_x^2} \\
& \quad \quad \quad u_{x10+} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 1+}}{-k_x^2 + L_{1+}} \tag{30}
\end{aligned}$$

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x20-} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 2-}}{L_{2-} - k_x^2} \\
& \quad \quad \quad u_{x20-} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 2-}}{-k_x^2 + L_{2-}} \tag{31}
\end{aligned}$$

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x20+} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 2+}}{L_{2+} - k_x^2} \\
& \quad \quad \quad u_{x20+} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 2+}}{-k_x^2 + L_{2+}} \tag{32}
\end{aligned}$$

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x30-} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 3-}}{L_{3-} - k_x^2} \\
& \quad \quad \quad u_{x30-} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 3-}}{-k_x^2 + L_{3-}} \tag{33}
\end{aligned}$$

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x30+} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 3+}}{L_{3+} - k_x^2} \\
& \quad \quad \quad u_{x30+} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 3+}}{-k_x^2 + L_{3+}} \tag{34}
\end{aligned}$$

$$\begin{aligned}
& \textcolor{red}{>} \quad u_{x40-} := - \frac{\textcolor{black}{i} \cdot k_x \cdot \Delta_{\perp 4-}}{L_{4-} - k_x^2} \\
& \quad \quad \quad u_{x40-} := \frac{-\textcolor{blue}{I} k_x \Delta_{\perp 4-}}{-k_x^2 + L_{4-}} \tag{35}
\end{aligned}$$

$$\begin{aligned} & \textcolor{red}{>} \quad u_{x40+} := -\frac{i \cdot k_x \cdot \Delta_{\perp 4+}}{L_{4+} - k_x^2} \\ & \quad \quad \quad u_{x40+} := \frac{-i k_x \Delta_{\perp 4+}}{-k_x^2 + L_{4+}} \end{aligned} \quad (36)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{I-} := \Delta_{\parallel I-}^2 + k_{\parallel-}^2 \\ & \quad \quad \quad L_{I-} := k_{\parallel-}^2 + \Delta_{\parallel I-}^2 \end{aligned} \quad (37)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{I+} := \Delta_{\parallel I+}^2 + k_{\parallel+}^2 \\ & \quad \quad \quad L_{I+} := k_{\parallel+}^2 + \Delta_{\parallel I+}^2 \end{aligned} \quad (38)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{2-} := \Delta_{\parallel 2-}^2 + k_{\parallel-}^2 \\ & \quad \quad \quad L_{2-} := k_{\parallel-}^2 + \Delta_{\parallel 2-}^2 \end{aligned} \quad (39)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{2+} := \Delta_{\parallel 2+}^2 + k_{\parallel+}^2 \\ & \quad \quad \quad L_{2+} := k_{\parallel+}^2 + \Delta_{\parallel 2+}^2 \end{aligned} \quad (40)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{3-} := \Delta_{\parallel 3-}^2 + k_{\parallel-}^2 \\ & \quad \quad \quad L_{3-} := k_{\parallel-}^2 + \Delta_{\parallel 3-}^2 \end{aligned} \quad (41)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{3+} := \Delta_{\parallel 3+}^2 + k_{\parallel+}^2 \\ & \quad \quad \quad L_{3+} := k_{\parallel+}^2 + \Delta_{\parallel 3+}^2 \end{aligned} \quad (42)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{4-} := \Delta_{\parallel 4-}^2 + k_{\parallel-}^2 \\ & \quad \quad \quad L_{4-} := k_{\parallel-}^2 + \Delta_{\parallel 4-}^2 \end{aligned} \quad (43)$$

$$\begin{aligned} & \textcolor{red}{>} \quad L_{4+} := \Delta_{\parallel 4+}^2 + k_{\parallel+}^2 \\ & \quad \quad \quad L_{4+} := k_{\parallel+}^2 + \Delta_{\parallel 4+}^2 \end{aligned} \quad (44)$$

$$\begin{aligned} & \textcolor{red}{>} \quad \Delta_{\perp I-} := i \cdot (k_y \cdot \cos(\alpha) - k_{zI-} \cdot \sin(\alpha)) \\ & \quad \quad \quad \Delta_{\perp I-} := i (k_y \cos(\alpha) - k_{zI-} \sin(\alpha)) \end{aligned} \quad (45)$$

$$\begin{aligned} & \textcolor{red}{>} \quad \Delta_{\perp I+} := i \cdot (k_y \cdot \cos(\alpha) - k_{zI+} \cdot \sin(\alpha)) \\ & \quad \quad \quad \Delta_{\perp I+} := i (k_y \cos(\alpha) - k_{zI+} \sin(\alpha)) \end{aligned} \quad (46)$$

$$\begin{aligned} & \textcolor{red}{>} \quad \Delta_{\perp 2-} := i \cdot (k_y \cdot \cos(\alpha) - k_{z2-} \cdot \sin(\alpha)) \\ & \quad \quad \quad \Delta_{\perp 2-} := i (k_y \cos(\alpha) - k_{z2-} \sin(\alpha)) \end{aligned} \quad (47)$$

$$\begin{aligned} & \textcolor{red}{>} \quad \Delta_{\perp 2+} := i \cdot (k_y \cdot \cos(\alpha) - k_{z2+} \cdot \sin(\alpha)) \\ & \quad \quad \quad \Delta_{\perp 2+} := i (k_y \cos(\alpha) - k_{z2+} \sin(\alpha)) \end{aligned} \quad (48)$$

$$\Delta_{\perp 2+} := I \left(k_y \cos(\alpha) - k_{z2+} \sin(\alpha) \right) \quad (48)$$

$$\begin{aligned} &> \Delta_{\perp 3-} := i \cdot \left(k_y \cdot \cos(\alpha) - k_{z3-} \cdot \sin(\alpha) \right) \\ &\quad \Delta_{\perp 3-} := I \left(k_y \cos(\alpha) - k_{z3-} \sin(\alpha) \right) \end{aligned} \quad (49)$$

$$\begin{aligned} &> \Delta_{\perp 3+} := i \cdot \left(k_y \cdot \cos(\alpha) - k_{z3+} \cdot \sin(\alpha) \right) \\ &\quad \Delta_{\perp 3+} := I \left(k_y \cos(\alpha) - k_{z3+} \sin(\alpha) \right) \end{aligned} \quad (50)$$

$$\begin{aligned} &> \Delta_{\perp 4-} := i \cdot \left(k_y \cdot \cos(\alpha) - k_{z4-} \cdot \sin(\alpha) \right) \\ &\quad \Delta_{\perp 4-} := I \left(k_y \cos(\alpha) - k_{z4-} \sin(\alpha) \right) \end{aligned} \quad (51)$$

$$\begin{aligned} &> \Delta_{\perp 4+} := i \cdot \left(k_y \cdot \cos(\alpha) - k_{z4+} \cdot \sin(\alpha) \right) \\ &\quad \Delta_{\perp 4+} := I \left(k_y \cos(\alpha) - k_{z4+} \sin(\alpha) \right) \end{aligned} \quad (52)$$

$$\begin{aligned} &> \Delta_{\parallel 1-} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z1-} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 1-} := I \left(k_y \sin(\alpha) + k_{z1-} \cos(\alpha) \right) \end{aligned} \quad (53)$$

$$\begin{aligned} &> \Delta_{\parallel 1+} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z1+} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 1+} := I \left(k_y \sin(\alpha) + k_{z1+} \cos(\alpha) \right) \end{aligned} \quad (54)$$

$$\begin{aligned} &> \Delta_{\parallel 2-} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z2-} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 2-} := I \left(k_y \sin(\alpha) + k_{z2-} \cos(\alpha) \right) \end{aligned} \quad (55)$$

$$\begin{aligned} &> \Delta_{\parallel 2+} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z2+} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 2+} := I \left(k_y \sin(\alpha) + k_{z2+} \cos(\alpha) \right) \end{aligned} \quad (56)$$

$$\begin{aligned} &> \Delta_{\parallel 3-} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z3-} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 3-} := I \left(k_y \sin(\alpha) + k_{z3-} \cos(\alpha) \right) \end{aligned} \quad (57)$$

$$\begin{aligned} &> \Delta_{\parallel 3+} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z3+} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 3+} := I \left(k_y \sin(\alpha) + k_{z3+} \cos(\alpha) \right) \end{aligned} \quad (58)$$

$$\begin{aligned} &> \Delta_{\parallel 4-} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z4-} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 4-} := I \left(k_y \sin(\alpha) + k_{z4-} \cos(\alpha) \right) \end{aligned} \quad (59)$$

$$\begin{aligned} &> \Delta_{\parallel 4+} := i \cdot \left(k_y \cdot \sin(\alpha) + k_{z4+} \cdot \cos(\alpha) \right) \\ &\quad \Delta_{\parallel 4+} := I \left(k_y \sin(\alpha) + k_{z4+} \cos(\alpha) \right) \end{aligned} \quad (60)$$

$$\begin{aligned} &> k_{z1-} := \frac{k_{\parallel-}}{\cos(\alpha)} - k_y \cdot \tan(\alpha) \\ &\quad k_{z1-} := \frac{k_{\parallel-}}{\cos(\alpha)} - k_y \tan(\alpha) \end{aligned} \quad (61)$$

$$> k_{z1+} := \frac{k_{\parallel+}}{\cos(\alpha)} - k_y \cdot \tan(\alpha)$$

$$k_{z1+} := \frac{k_{||+}}{\cos(\alpha)} - k_y \tan(\alpha) \quad (62)$$

$$> k_{z2-} := -\frac{k_{||-}}{\cos(\alpha)} - k_y \cdot \tan(\alpha)$$

$$k_{z2-} := -\frac{k_{||-}}{\cos(\alpha)} - k_y \tan(\alpha) \quad (63)$$

$$> k_{z2+} := -\frac{k_{||+}}{\cos(\alpha)} - k_y \cdot \tan(\alpha)$$

$$k_{z2+} := -\frac{k_{||+}}{\cos(\alpha)} - k_y \tan(\alpha) \quad (64)$$

$$> k_{z3-} := i \cdot \sqrt{k_x^2 + k_y^2 - k_{||-}^2}$$

$$k_{z3-} := I \sqrt{k_x^2 + k_y^2 - k_{||-}^2} \quad (65)$$

$$> k_{z3+} := i \cdot \sqrt{k_x^2 + k_y^2 - k_{||+}^2}$$

$$k_{z3+} := I \sqrt{k_x^2 + k_y^2 - k_{||+}^2} \quad (66)$$

$$> k_{z4-} := -i \cdot \sqrt{k_x^2 + k_y^2 - k_{||-}^2}$$

$$k_{z4-} := -I \sqrt{k_x^2 + k_y^2 - k_{||-}^2} \quad (67)$$

$$> k_{z4+} := -i \cdot \sqrt{k_x^2 + k_y^2 - k_{||+}^2}$$

$$k_{z4+} := -I \sqrt{k_x^2 + k_y^2 - k_{||+}^2} \quad (68)$$

>

$$> k_x := \frac{k_{||+}}{\epsilon_+}$$

$$k_x := \frac{k_{||+}}{\epsilon_{+,+}} \quad (69)$$

$$> k_y := a_y \cdot k_{||+}$$

$$k_y := a_y k_{||+} \quad (70)$$

$$> k_{||-} := \frac{k_x}{\epsilon_-}$$

$$k_{||-} := \frac{k_{||+}}{\epsilon_{+,+}, \epsilon_{-,+}} \quad (71)$$

Need to help maple to take limit as epsilon_- and epsilon_+ go to zero by taking terms that go to infinity out of the square root.

$$\begin{aligned}
> k_{z3-} &:= \frac{i \cdot \sqrt{\epsilon_-^2 \cdot k_{\parallel+}^2 + \epsilon_-^2 \cdot \epsilon_+^2 \cdot k_y^2 - k_{\parallel+}^2}}{\epsilon_- \cdot \epsilon_+} \\
&\quad k_{z3-} := \frac{I \sqrt{a_y^2 \epsilon_{-,}^2 \epsilon_{+,}^2 k_{\parallel+}^2 + \epsilon_{-,}^2 k_{\parallel+}^2 - k_{\parallel+}^2}}{\epsilon_{-,} \epsilon_{+,}}
\end{aligned} \tag{72}$$

$$\begin{aligned}
> k_{z3+} &:= \frac{i \cdot \sqrt{k_{\parallel+}^2 + \epsilon_+^2 \cdot k_y^2 - \epsilon_+^2 \cdot k_{\parallel+}^2}}{\epsilon_+} \\
&\quad k_{z3+} := \frac{I \sqrt{a_y^2 \epsilon_{+,}^2 k_{\parallel+}^2 - \epsilon_{+,}^2 k_{\parallel+}^2 + k_{\parallel+}^2}}{\epsilon_{+,}}
\end{aligned} \tag{73}$$

$$\begin{aligned}
> k_{z4-} &:= - \frac{i \cdot \sqrt{\epsilon_-^2 \cdot k_{\parallel+}^2 + \epsilon_-^2 \cdot \epsilon_+^2 \cdot k_y^2 - k_{\parallel+}^2}}{\epsilon_- \cdot \epsilon_+} \\
&\quad k_{z4-} := \frac{-I \sqrt{a_y^2 \epsilon_{-,}^2 \epsilon_{+,}^2 k_{\parallel+}^2 + \epsilon_{-,}^2 k_{\parallel+}^2 - k_{\parallel+}^2}}{\epsilon_{-,} \epsilon_{+,}}
\end{aligned} \tag{74}$$

$$\begin{aligned}
> k_{z4+} &:= - \frac{i \cdot \sqrt{k_{\parallel+}^2 + \epsilon_+^2 \cdot k_y^2 - \epsilon_+^2 \cdot k_{\parallel+}^2}}{\epsilon_+} \\
&\quad k_{z4+} := \frac{-I \sqrt{a_y^2 \epsilon_{+,}^2 k_{\parallel+}^2 - \epsilon_{+,}^2 k_{\parallel+}^2 + k_{\parallel+}^2}}{\epsilon_{+,}}
\end{aligned} \tag{75}$$

ux leading order terms

$$\begin{aligned}
> u_{x1-} &:= u_{x10-} \cdot \text{sol1} : \\
> u_{x4-} &:= u_{x40-} \cdot \text{sol2} : \\
> u_{x1+} &:= u_{x10+} : \\
> u_{x2+} &:= u_{x20+} \cdot \text{sol3} : \\
> u_{x3+} &:= u_{x30+} \cdot \text{sol4} : \\
> \text{simplify}(\text{mtaylor}(u_{x1-}, [\epsilon_-, \epsilon_+], 3)) \\
&\quad -2 I \sin(\alpha) \epsilon_{-,} \epsilon_{+,} \text{csgn}(k_{\parallel+})
\end{aligned} \tag{76}$$

$$\begin{aligned}
> \text{simplify}(\text{mtaylor}(u_{x4-}, [\epsilon_-, \epsilon_+], 4)) \\
&\quad - \frac{2 \epsilon_{-,}^2 \epsilon_{+,} \cos(\alpha)}{\sin(\alpha)}
\end{aligned} \tag{77}$$

$$> \text{simplify}(\text{mtaylor}(u_{x1+}, [\epsilon_-, \epsilon_+], 2))$$

$$\frac{\epsilon_{+,} \left(-a_y + \sin(\alpha) \right)}{\cos(\alpha)} \quad (78)$$

> simplify(mtaylor(u_{x2+}, [ϵ_{-} , ϵ_{+}], 2))

$$\frac{\epsilon_{+,} \left(a_y + \sin(\alpha) \right)}{\cos(\alpha)} \quad (79)$$

> simplify(mtaylor(u_{x3+}, [ϵ_{-} , ϵ_{+}], 2))

$$- \frac{2 \sin(\alpha) \epsilon_{+,}}{\cos(\alpha)} \quad (80)$$

u_perp leading order terms

> simplify(mtaylor(sol1, [ϵ_{-} , ϵ_{+}], 4))

$$-2 \text{I} \epsilon_{-,}^2 \epsilon_{+,} \cos(\alpha) \text{csgn}(k_{||+}) \quad (81)$$

> simplify(mtaylor(sol2, [ϵ_{-} , ϵ_{+}], 3))

$$\frac{2 \text{I} \cos(\alpha) \epsilon_{-,} \epsilon_{+,} k_{||+}}{\sqrt{-k_{||+}^2}} \quad (82)$$

> simplify(mtaylor(sol3, [ϵ_{-} , ϵ_{+}], 1))

$$-1 \quad (83)$$

> simplify(mtaylor(sol4, [ϵ_{-} , ϵ_{+}], 2))

$$\frac{2 \text{I} \epsilon_{+,} \text{csgn}(k_{||+}) \sin(\alpha)^2}{\cos(\alpha)} \quad (84)$$

bx leading order terms

$$b_{x1-} := \frac{\Delta_{||1-} \cdot u_{x1-}}{\text{i} \cdot k_{||+}} :$$

$$b_{x4-} := \frac{\Delta_{||4-} \cdot u_{x4-}}{\text{i} \cdot k_{||+}} :$$

$$b_{x1+} := \frac{\Delta_{||1+} \cdot u_{x1+}}{\text{i} \cdot k_{||+}} :$$

$$b_{x2+} := \frac{\Delta_{||2+} \cdot u_{x2+}}{\text{i} \cdot k_{||+}} :$$

$$b_{x3+} := \frac{\Delta_{||3+} \cdot u_{x3+}}{\text{i} \cdot k_{||+}} :$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{x1-}, [\epsilon_-, \epsilon_+], 1)) \\ &\quad -2 I \sin(\alpha) \text{csgn}(k_{\parallel+}) \end{aligned} \quad (85)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{x4-}, [\epsilon_-, \epsilon_+], 2)) \\ &\quad \frac{-2 I k_{\parallel+} \epsilon_{-} \cos(\alpha)^2}{\sqrt{-k_{\parallel+}^2} \sin(\alpha)} \end{aligned} \quad (86)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{x1+}, [\epsilon_-, \epsilon_+], 2)) \\ &\quad \frac{\epsilon_{+} (-a_y + \sin(\alpha))}{\cos(\alpha)} \end{aligned} \quad (87)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{x2+}, [\epsilon_-, \epsilon_+], 2)) \\ &\quad - \frac{\epsilon_{+} (a_y + \sin(\alpha))}{\cos(\alpha)} \end{aligned} \quad (88)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{x3+}, [\epsilon_-, \epsilon_+], 1)) \\ &\quad -2 I \sin(\alpha) \text{csgn}(k_{\parallel+}) \end{aligned} \quad (89)$$

b_perp leading order terms

$$> b_{\perp 1-} := \frac{\Delta_{\parallel 1-} \cdot \text{sol1}}{i \cdot k_{\parallel+}} :$$

$$> b_{\perp 4-} := \frac{\Delta_{\parallel 4-} \cdot \text{sol2}}{i \cdot k_{\parallel+}} :$$

$$> b_{\perp 1+} := \frac{\Delta_{\parallel 1+} \cdot 1}{i \cdot k_{\parallel+}} :$$

$$> b_{\perp 2+} := \frac{\Delta_{\parallel 2+} \cdot \text{sol3}}{i \cdot k_{\parallel+}} :$$

$$> b_{\perp 3+} := \frac{\Delta_{\parallel 3+} \cdot \text{sol4}}{i \cdot k_{\parallel+}} :$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{\perp 1-}, [\epsilon_-, \epsilon_+], 2)) \\ &\quad -2 I \epsilon_{-} \cos(\alpha) \text{csgn}(k_{\parallel+}) \end{aligned} \quad (90)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{\perp 4-}, [\epsilon_-, \epsilon_+], 1)) \\ &\quad 2 \cos(\alpha)^2 \end{aligned} \quad (91)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{\perp 1+}, [\epsilon_-, \epsilon_+], 5)) \\ &\quad 1 \end{aligned} \quad (92)$$

$$\begin{aligned} &> \text{simplify}(\text{mtaylor}(b_{\perp 2+}, [\epsilon_-, \epsilon_+], 1)) \\ &\quad 1 \end{aligned} \quad (93)$$

$$\begin{aligned} & \text{> simplify}\left(\text{mtaylor}\left(b_{\perp 3+}, \left[\epsilon_{-}, \epsilon_{+}\right], 1\right)\right) \\ & \qquad \qquad \qquad -2 \sin(\alpha)^2 \end{aligned} \tag{94}$$

b_par leading order terms

$$\begin{aligned} & \text{> } b_{\parallel 1-} := -\frac{i \cdot k_x \cdot u_{x1-} + \Delta_{\perp 1-} \cdot \text{sol1}}{i \cdot k_{\parallel+}} : \\ & \text{> } b_{\parallel 4-} := -\frac{i \cdot k_x \cdot u_{x4-} + \Delta_{\perp 4-} \cdot \text{sol2}}{i \cdot k_{\parallel+}} : \\ & \text{> } b_{\parallel 1+} := -\frac{i \cdot k_x \cdot u_{x1+} + \Delta_{\perp 1+}}{i \cdot k_{\parallel+}} : \\ & \text{> } b_{\parallel 2+} := -\frac{i \cdot k_x \cdot u_{x2+} + \Delta_{\perp 2+} \cdot \text{sol3}}{i \cdot k_{\parallel+}} : \\ & \text{> } b_{\parallel 3+} := -\frac{i \cdot k_x \cdot u_{x3+} + \Delta_{\perp 3+} \cdot \text{sol4}}{i \cdot k_{\parallel+}} : \\ & \text{> simplify}\left(\text{mtaylor}\left(b_{\parallel 1-}, \left[\epsilon_{-}, \epsilon_{+}\right], 2\right)\right) \\ & \qquad \qquad \qquad 0 \end{aligned} \tag{95}$$

$$\begin{aligned} & \text{> simplify}\left(\text{mtaylor}\left(b_{\parallel 4-}, \left[\epsilon_{-}, \epsilon_{+}\right], 1\right)\right) \\ & \qquad \qquad \qquad 2 \sin(\alpha) \cos(\alpha) \end{aligned} \tag{96}$$

$$\begin{aligned} & \text{> simplify}\left(\text{mtaylor}\left(b_{\parallel 1+}, \left[\epsilon_{-}, \epsilon_{+}\right], 1\right)\right) \\ & \qquad \qquad \qquad 0 \end{aligned} \tag{97}$$

$$\begin{aligned} & \text{> simplify}\left(\text{mtaylor}\left(b_{\parallel 2+}, \left[\epsilon_{-}, \epsilon_{+}\right], 1\right)\right) \\ & \qquad \qquad \qquad 0 \end{aligned} \tag{98}$$

$$\begin{aligned} & \text{> simplify}\left(\text{mtaylor}\left(b_{\parallel 3+}, \left[\epsilon_{-}, \epsilon_{+}\right], 1\right)\right) \\ & \qquad \qquad \qquad 2 \sin(\alpha) \cos(\alpha) \end{aligned} \tag{99}$$