Eigenfunction approach

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Equations

The equations are given by

$$\frac{\partial u_x}{\partial t} = v_A^2(x,z)\left[\nabla_{||} b_x - \frac{\partial b_{||}}{\partial x}\right],$$

$$\frac{\partial u_\perp}{\partial t} = v_A^2(x,z)\left[\nabla_{||} b_\perp - \nabla_\perp b_{||}\right],$$

$$\frac{\partial b_x}{\partial t} = \nabla_{||} u_x,$$

$$\frac{\partial b_\perp}{\partial t} = \nabla_{||} u_\perp,$$

$$\frac{\partial b_{||}}{\partial t} = -\left[\frac{\partial u_x}{\partial x} + \nabla_\perp u_\perp\right],$$

where

$$\nabla_\perp = \cos\alpha \frac{\partial}{\partial y} - \sin\alpha \frac{\partial}{\partial z},$$

$$\nabla_{||} = \sin\alpha \frac{\partial}{\partial y} + \cos\alpha \frac{\partial}{\partial z}.$$

The field perturbations b_x, $b_\perp$ and $b_{||}$ are dimensionless as they have been normalised by B_0. The background field is given by

$$\boldsymbol{B}_0 = B_0(\sin\alpha\,\hat{\boldsymbol{y}} + \cos\alpha\,\hat{\boldsymbol{z}}).$$

The Alfvén speed is given by $v_A(x,z) = \hat{v}_A(x) v_A(0,z)$, where

$$v_A(0,z) = \begin{cases} v_{A1}, & z \le L_z, \\ v_{A2}, & z > L_z. \end{cases}$$

The system is periodic in z, with period $2L_z$. We assume the variables are of the form

$$f(x,y,z,t) = f'(x,z)\exp\{i[k_\perp(\cos\alpha\, y - \sin\alpha\, z) + \omega t]\}.$$

Note that

$$\nabla_\perp f(x,y,z,t) = \exp\{i[k_\perp(\cos\alpha\, y - \sin\alpha\, z) + \omega t]\}\left(ik_\perp - \sin\alpha\frac{\partial}{\partial z}\right) f',$$

$$\nabla_{||} f(x,y,z,t) = \exp\{i[k_\perp(\cos\alpha\, y - \sin\alpha\, z) + \omega t]\}\cos\alpha\frac{\partial f'}{\partial z}.$$

Therefore, the above equations can be simplified to

$$i\omega u_x' = v_A^2(x,z)\left[\nabla_{||}' b_x' - \frac{\partial b_{||}'}{\partial x}\right],$$

$$i\omega u_\perp' = v_A^2(x,z)\left[\nabla_{||}' b_\perp' - \nabla_\perp' b_{||}'\right],$$

$$i\omega b_x' = \nabla_{||}' u_x',$$

$$i\omega b'_\perp = \nabla'_{||} u'_\perp,$$

$$i\omega b'_{||} = -\left[\frac{\partial u'_x}{\partial x} + \nabla'_\perp u'_\perp\right],$$

where

$$\nabla'_\perp = \left(ik_\perp - \sin\alpha \frac{\partial}{\partial z}\right),$$

$$\nabla'_{||} = \cos\alpha \frac{\partial}{\partial z}.$$

From this point on, to keep the notation clear, we drop the ' notation.

The equations can be rearranged to give

$$-\omega^2 u_x = v_A^2\left[\nabla_{||}^2 u_x - i\omega\frac{\partial b_{||}}{\partial x}\right],$$

$$-\omega^2 u_\perp = v_A^2\left[\nabla_{||}^2 u_\perp - i\omega\nabla_\perp b_{||}\right],$$

hence,

$$\mathcal{L} u_x = i\omega\frac{\partial b_{||}}{\partial x}$$

$$\mathcal{L} u_\perp = i\omega\nabla_\perp b_{||},$$

where

$$\mathcal{L} = \nabla_{||}^2 + \frac{\omega^2}{v_A^2(x,z)},$$

Therefore,

$$\boxed{\frac{\partial u_x}{\partial x} = -\left[i\omega b_{||} + \nabla_\perp u_\perp\right],}$$

$$\boxed{\frac{\partial b_{||}}{\partial x} = -\frac{i}{\omega}\mathcal{L} u_x,}$$

$$\mathcal{L} u_\perp = i\omega\nabla_\perp b_{||}.$$

For $z \neq 0, L_z$,

$$\boxed{\mathcal{L}\nabla_\perp u_\perp = i\omega\nabla_\perp^2 b_{||}.}$$

To solve this, we postulate a solution of the form

$$u_x(x,z) = u_{x0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} u_{xn}^{(1)}(x)\phi_n(z) + u_{xn}^{(2)}(x)\varphi_n(z),$$

$$b_{||}(x,z) = b_{||0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} b_{||n}^{(1)}(x)\phi_n(z) + b_{||n}^{(2)}(x)\varphi_n(z),$$

$$\nabla_\perp u_\perp(x,z) = \Delta_{\perp 0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} \Delta_{\perp n}^{(1)}(x)\phi_n(z) + \Delta_{\perp n}^{(2)}(x)\varphi_n(z),$$

where ϕ_n, φ_n are eigenfunctions, with eigenfrequencies, ω_n, ϖ_n satisfying

$$\left[\nabla_{||}^2 + \frac{\omega_n^2}{v_A^2(0,z)}\right]\phi_n(z) = 0,$$

$$\left[\nabla_{||}^2 + \frac{\varpi_n^2}{v_A^2(0,z)}\right]\varphi_n(z) = 0,$$

as well as the boundary conditions. We will choose ω to be complex, therefore, $\omega \neq \varpi_n$, $\omega \neq \varpi_n$.

Eigenfunctions and eigenfrequencies

The boundary conditions are given by

$$\phi_n(0) = \phi_n(2L_z),\ \varphi_n(0) = \varphi_n(2L_z),\ \left.\frac{\mathrm{d}\phi_n}{\mathrm{d}z}\right|_{z=0} = \left.\frac{\mathrm{d}\phi_n}{\mathrm{d}z}\right|_{z=2L_z},\ \left.\frac{\mathrm{d}\varphi_n}{\mathrm{d}z}\right|_{z=0} = \left.\frac{\mathrm{d}\varphi_n}{\mathrm{d}z}\right|_{z=2L_z}.$$

Also, we require continuity of ϕ, φ, $\mathrm{d}\phi/\mathrm{d}z$, $\mathrm{d}\varphi/\mathrm{d}z$ at $z = L_z$. By symmetry, we assume the eigenfunctions are of the form,

$$\phi_n(z) = \begin{cases} a\cos[k_{z1n}(z - l_z)], & z \le L_z, \\ c\cos[k_{z2n}(z - 3l_z)], & z > L_z, \end{cases}$$

$$\varphi_n(z) = \begin{cases} b\sin[\bar{k}_{z1n}(z - l_z)], & z \le L_z, \\ d\sin[\bar{k}_{z2n}(z - 3l_z)], & z > L_z. \end{cases}$$

Hence

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}z} = \begin{cases} -ak_{z1n}\sin[k_{z1n}(z - l_z)], & z \le L_z, \\ -ck_{z2n}\sin[k_{z2n}(z - 3l_z)], & z > L_z, \end{cases}$$

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}z} = \begin{cases} b\bar{k}_{z1n}\cos[\bar{k}_{z1n}(z - l_z)], & z \le L_z, \\ d\bar{k}_{z2n}\cos[\bar{k}_{z2n}(z - 3l_z)], & z > L_z, \end{cases}$$

where

$$k_{zin} = \frac{\omega_n}{v_{Ai}\cos\alpha} \text{ and } \bar{k}_{zin} = \frac{\varpi_n}{v_{Ai}\cos\alpha}.$$

Applying boundary and continuity conditions gives

$$\begin{pmatrix} \cos(k_{z1n}l_z) & -\cos(k_{z2n}l_z) \\ k_{z1n}\sin(k_{z1n}l_z) & k_{z2n}\sin(k_{z2n}l_z) \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \sin(k_{z1n}l_z) & \sin(k_{z2n}l_z) \\ k_{z1n}\cos(k_{z1n}l_z) & -k_{z2n}\cos(k_{z2n}l_z) \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, non-trivial solutions exist for

$$k_{z2n}\sin(k_{z2n}l_z)\cos(k_{z1n}l_z) + k_{z1n}\sin(k_{z1n}l_z)\cos(k_{z2n}l_z) = 0,$$

$$\bar{k}_{z2n}\sin(\bar{k}_{z1n}l_z)\cos(\bar{k}_{z2n}l_z) + \bar{k}_{z1n}\sin(\bar{k}_{z2n}l_z)\cos(\bar{k}_{z1n}l_z) = 0,$$

and we use these equations to calculate ω_n, ϖ_n, such that

$$\omega_1 < \omega_2 < \omega_3 ...,\ \varpi_1 < \varpi_2 < \varpi_3 ....$$

Note that if $l_z = n\pi/k_{z1n}$ and $l_z = n\pi/\bar{k}_{z1n}$ the above equations simplify to

$$k_{z2n}\sin\left(n\pi\frac{k_{z2n}}{k_{z1n}}\right)(-1)^n = 0, \quad \bar{k}_{z1n}\sin\left(n\pi\frac{\bar{k}_{z2n}}{\bar{k}_{z1n}}\right)(-1)^n = 0.$$

Hence, for $v_{A1}/v_{A2} = m$, where m is an integer, a subset of the eigenvalues are given by

$$\omega_j = \varpi_j = j\pi\frac{v_{A1}}{l_z}\cos\alpha.$$

Note that if $l_z = (2n+1)\pi/(2k_{z1n})$ and $l_z = (2n+1)\pi/(2\bar{k}_{z1n})$ the above equations simplify to

$$k_{z1n}(-1)^n\cos\left(\frac{1}{2}(2n+1)\pi\frac{k_{z2n}}{k_{z1n}}\right) = 0, \quad \bar{k}_{z2n}(-1)^n\cos\left(\frac{1}{2}(2n+1)\pi\frac{k_{z2n}}{k_{z1n}}\right) = 0.$$

Hence, for $v_{A1}/v_{A2} = m$, where m is an odd integer, a subset of the eigenvalues are given by

$$\omega_j = \varpi_j = j\pi\frac{v_{A1}}{2l_z}\cos\alpha.$$

Hence,

$$\phi_n(z) = A_n \begin{cases} a_n \cos[k_{z1n}(z - l_z)], & z \le L_z, \\ c_n \cos[k_{z2n}(z - 3l_z)], & z > L_z, \end{cases}$$

$$\varphi_n(z) = B_n \begin{cases} b_n \sin[\bar{k}_{z1n}(z - l_z)], & z \le L_z, \\ d_n \sin[\bar{k}_{z2n}(z - 3l_z)], & z > L_z, \end{cases}$$

where

$$a_n = \begin{cases} \cos(k_{z2n}l_z), & \cos(k_{z1n}l_z) \neq 0 \\ k_{z2n}\sin(k_{z2n}l_z), & \cos(k_{z1n}l_z) = 0 \end{cases},$$

$$c_n = \begin{cases} \cos(k_{z1n}l_z), & \cos(k_{z1n}l_z) \neq 0 \\ -k_{z1n}\sin(k_{z1n}l_z), & \cos(k_{z1n}l_z) = 0 \end{cases},$$

$$b_n = \begin{cases} \sin\left(\bar{k}_{z2n}l_z\right), & \sin\left(\bar{k}_{z1n}l_z\right) \neq 0 \\ \bar{k}_{z2n}\cos\left(\bar{k}_{z2n}l_z\right), & \sin\left(\bar{k}_{z1n}l_z\right) = 0 \end{cases},$$

$$d_n = \begin{cases} -\sin\left(\bar{k}_{z1n}l_z\right), & \sin\left(\bar{k}_{z1n}l_z\right) \neq 0 \\ \bar{k}_{z1n}\cos\left(\bar{k}_{z1n}l_z\right), & \sin\left(\bar{k}_{z1n}l_z\right) = 0 \end{cases},$$

We normalise such that $\langle \phi_n \phi_m \rangle = \langle \varphi_n \varphi_m \rangle = \delta_{nm}$, where

$$\langle \phi_n, \phi_m \rangle = \int_0^{2L_z} \frac{\phi_n(z)\phi_m(z)}{v_A^2(0,z)} dz,$$

and δ_{nm} is the Kronecker delta. Note that

$$\begin{aligned}
\frac{\langle \phi_n, \phi_n \rangle}{A_n^2} &= \int_0^{2L_z} \frac{\phi_n^2(z)}{A_n^2 v_A^2(0,z)} dz \\
&= \frac{1}{v_{A1}^2} \int_0^{L_z} a_n^2 \cos^2[k_{z1n}(z - l_z)] dz + \frac{1}{v_{A2}^2} \int_{L_z}^{2L_z} c_n^2 \cos^2[k_{z2n}(z - 3l_z)] dz \\
&= \int_{-l_z}^{l_z} \frac{a_n^2}{v_{A1}^2} \cos^2(k_{z1n}z) + \frac{c_n^2}{v_{A2}^2} \cos^2(k_{z2n}z) dz \\
&= 2\int_0^{l_z} \frac{a_n^2}{v_{A1}^2} \cos^2(k_{z1n}z) + \frac{c_n^2}{v_{A2}^2} \cos^2(k_{z2n}z) dz \\
&= \left\{ \frac{a_n^2}{2v_{A1}^2 k_{z1n}} \Big[2k_{z1n}z + \sin(2k_{z1n}z)\Big] + \frac{c_n^2}{2v_{A2}^2 k_{z2n}} \Big[2k_{z2n}z + \sin(2k_{z2n}z)\Big] \right\}_0^{l_z} \\
&= \frac{a_n^2}{2v_{A1}^2 k_{z1n}} \Big[2k_{z1n}l_z + \sin(2k_{z1n}l_z)\Big] + \frac{c_n^2}{2v_{A2}^2 k_{z2n}} \Big[2k_{z2n}l_z + \sin(2k_{z2n}l_z)\Big],
\end{aligned}$$

$$\begin{aligned}
\frac{\langle \varphi_n, \varphi_n \rangle}{B_n^2} &= \frac{1}{2} \left\{ \frac{b_n^2}{v_{A1}^2 \bar{k}_{z1n}} \Big[2\bar{k}_{z1n}z - \sin\left(2\bar{k}_{z1n}z\right)\Big] + \frac{d_n^2}{v_{A2}^2 \bar{k}_{z2n}} \Big[2\bar{k}_{z2n}z - \sin\left(2\bar{k}_{z2n}z\right)\Big] \right\}_0^{l_z} \\
&= \frac{b_n^2}{2v_{A1}^2 \bar{k}_{z1n}} \Big[2\bar{k}_{z1n}l_z - \sin\left(2\bar{k}_{z1n}l_z\right)\Big] + \frac{d_n^2}{2v_{A2}^2 \bar{k}_{z2n}} \Big[2\bar{k}_{z2n}l_z - \sin\left(2\bar{k}_{z2n}l_z\right)\Big].
\end{aligned}$$

Hence,

$$A_n = \left(\frac{a_n^2}{2v_{A1}^2 k_{z1n}} \Big[2k_{z1n}l_z + \sin(2k_{z1n}l_z)\Big] + \frac{c_n^2}{2v_{A2}^2 k_{z2n}} \Big[2k_{z2n}l_z + \sin(2k_{z2n}l_z)\Big] \right)^{-1/2},$$

$$B_n = \left(\frac{b_n^2}{2v_{A1}^2 \bar{k}_{z1n}} \Big[2\bar{k}_{z1n}l_z - \sin\left(2\bar{k}_{z1n}l_z\right)\Big] + \frac{d_n^2}{2v_{A2}^2 \bar{k}_{z2n}} \Big[2\bar{k}_{z2n}l_z - \sin\left(2\bar{k}_{z2n}l_z\right)\Big] \right)^{-1/2}.$$

Deriving the ODEs

Note that

$$
\begin{aligned}
\mathcal{L}(x,z) &= \nabla_{||}^2 + \frac{\omega_n^2}{v_A^2(0,z)} + \frac{\omega^2/\hat{v}_A^2(x) - \omega_n^2}{v_A^2(0,z)}, \\
\implies \mathcal{L}(x,z)\phi_n(z) &= [\omega^2/\hat{v}_A^2(x) - \omega_n^2]\frac{\phi_n(z)}{v_A^2(0,z)}, \\
\implies \mathcal{L}(x,z)\varphi_n(z) &= [\omega^2/\hat{v}_A^2(x) - \varpi_n^2]\frac{\varphi_n(z)}{v_A^2(0,z)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\mathrm{d}u_{x0}^{(1)}}{\mathrm{d}x}\phi_0(z) + \sum_{n=1}^{\infty}\frac{\mathrm{d}u_{xn}^{(1)}}{\mathrm{d}x}\phi_n(z) + \frac{\mathrm{d}u_{xn}^{(2)}}{\mathrm{d}x}\varphi_n(z) &= -[i\omega b_{||0}^{(1)} + \Delta_{\perp 0}^{(1)}]\phi_0(z) \\
&\quad - \sum_{n=1}^{\infty}\left[i\omega b_{||n}^{(1)}(x) + \Delta_{\perp n}^{(1)}(x)\right]\phi_n(z) \\
&\quad - \sum_{n=1}^{\infty}\left[i\omega b_{||n}^{(2)}(x) + \Delta_{\perp n}^{(2)}(x)\right]\varphi_n(z),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}b_{||0}^{(1)}}{\mathrm{d}x}\phi_0(z) + \sum_{n=1}^{\infty}\frac{\mathrm{d}b_{||n}^{(1)}}{\mathrm{d}x}\phi_n(z) + \frac{\mathrm{d}b_{||n}^{(2)}}{\mathrm{d}x}\varphi_n(z) &= -\frac{i}{\omega}\left[\omega^2/\hat{v}_A^2(x)\right]u_{xn}^{(1)}(x)\frac{\phi_0(z)}{v_A^2(0,z)} \\
&\quad - \frac{i}{\omega}\sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \omega_n^2\right]u_{xn}^{(1)}(x)\frac{\phi_n(z)}{v_A^2(0,z)} \\
&\quad - \frac{i}{\omega}\sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \varpi_n^2\right]u_{xn}^{(2)}(x)\frac{\varphi_n(z)}{v_A^2(0,z)},
\end{aligned}
$$

$$
\begin{aligned}
i\omega\nabla_{\perp}^2\left[b_{||0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty}b_{||n}^{(1)}(x)\phi_n(z) + b_{||n}^{(2)}(x)\varphi_n(z)\right] &= \left[\omega^2/\hat{v}_A^2(x)\right]\Delta_{\perp 0}^{(1)}(x)\frac{\phi_0(z)}{v_A^2(0,z)} \\
&\quad + \sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \omega_n^2\right]\Delta_{\perp n}^{(1)}(x)\frac{\phi_n(z)}{v_A^2(0,z)} \\
&\quad + \sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \varpi_n^2\right]\Delta_{\perp n}^{(2)}(x)\frac{\varphi_n(z)}{v_A^2(0,z)}.
\end{aligned}
$$

Taking the inner product of the above equations with ϕ_m then φ_m and weight function $1/v_A^2(0,z)$ gives

$$
\boxed{\frac{\mathrm{d}u_{xm}^{(1)}}{\mathrm{d}x} = -[i\omega b_{||n}^{(1)} + \Delta_{\perp n}^{(1)}],}
$$

$$
\boxed{\frac{\mathrm{d}u_{xm}^{(2)}}{\mathrm{d}x} = -[i\omega b_{||n}^{(2)} + \Delta_{\perp n}^{(2)}],}
$$

$$
\begin{aligned}
\frac{\mathrm{d}b_{||m}^{(1)}}{\mathrm{d}x} &= -\frac{i}{\omega}\left[\omega^2/\hat{v}_A^2(x)\right]u_{xn}^{(1)}(x)\left\langle\frac{\phi_0(z)}{v_A^2(0,z)}, \phi_m(z)\right\rangle \\
&\quad - \frac{i}{\omega}\sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \omega_n^2\right]u_{xn}^{(1)}(x)\left\langle\frac{\phi_n(z)}{v_A^2(0,z)}, \phi_m(z)\right\rangle \\
&\quad - \frac{i}{\omega}\sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \varpi_n^2\right]u_{xn}^{(2)}(x)\left\langle\frac{\varphi_n(z)}{v_A^2(0,z)}, \phi_m(z)\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}b_{||m}^{(2)}}{\mathrm{d}x} &= -\frac{i}{\omega}\left[\omega^2/\hat{v}_A^2(x)\right]u_{xn}^{(1)}(x)\left\langle \frac{\phi_0(z)}{v_A^2(0,z)}, \varphi_m(z)\right\rangle \\
&- \frac{i}{\omega}\sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \omega_n^2\right]u_{xn}^{(1)}(x)\left\langle \frac{\phi_n(z)}{v_A^2(0,z)}, \varphi_m(z)\right\rangle \\
&- \frac{i}{\omega}\sum_{n=1}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \varpi_n^2\right]u_{xn}^{(2)}(x)\left\langle \frac{\varphi_n(z)}{v_A^2(0,z)}, \varphi_m(z)\right\rangle, \\
\Delta_{\perp m}^{(1)}(x) &= \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2} b_{||0}^{(1)}(x)\left\langle v_A^2(0,z)\nabla_\perp^2\phi_0(z), \phi_m(z)\right\rangle \\
&+ \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2}\sum_{n=1}^{\infty} b_{||n}^{(1)}(x)\left\langle v_A^2(0,z)\nabla_\perp^2\phi_n(z), \phi_m(z)\right\rangle \\
&+ \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2}\sum_{n=1}^{\infty} b_{||n}^{(2)}(x)\left\langle v_A^2(0,z)\nabla_\perp^2\varphi_n(z), \phi_m(z)\right\rangle, \\
\Delta_{\perp m}^{(2)}(x) &= \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2} b_{||0}^{(1)}(x)\left\langle v_A^2(0,z)\nabla_\perp^2\phi_0(z), \varphi_m(z)\right\rangle \\
&+ \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2}\sum_{n=1}^{\infty} b_{||n}^{(1)}(x)\left\langle v_A^2(0,z)\nabla_\perp^2\phi_n(z), \varphi_m(z)\right\rangle \\
&+ \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2}\sum_{n=1}^{\infty} b_{||n}^{(2)}(x)\left\langle v_A^2(0,z)\nabla_\perp^2\varphi_n(z), \varphi_m(z)\right\rangle.
\end{aligned}
$$

Note that

$$
\left\langle \frac{\varphi_n(z)}{v_A^2(0,z)}, \phi_m(z)\right\rangle = \left\langle \frac{\phi_n(z)}{v_A^2(0,z)}, \varphi_m(z)\right\rangle = 0
$$

$$
\nabla_\perp^2 = \sin^2\alpha\frac{\partial^2}{\partial z^2} - 2ik_\perp\sin\alpha\frac{\partial}{\partial z} - k_\perp^2,
$$

$$
\begin{aligned}
\left\langle v_A^2(0,z)\nabla_\perp^2\phi_n(z), \phi_m(z)\right\rangle &= \left\langle v_A^2(0,z)\left(\sin^2\alpha\frac{\partial^2}{\partial z^2} - k_\perp^2\right)\phi_n, \phi_m(z)\right\rangle \\
&= \left\langle -v_A^2(0,z)\left(\tan^2\alpha\frac{\omega_n^2}{v_A^2(0,z)} + k_\perp^2\right)\phi_n, \phi_m(z)\right\rangle \\
&= -\tan^2\alpha\,\omega_n^2\delta_{nm} - k_\perp^2\left\langle v_A^2(0,z)\phi_n(z), \phi_m(z)\right\rangle, \\
\left\langle v_A^2(0,z)\nabla_\perp^2\varphi_n(z), \varphi_m(z)\right\rangle &= -\tan^2\alpha\,\varpi_n^2\delta_{nm} - k_\perp^2\left\langle v_A^2(0,z)\varphi_n(z), \phi_m(z)\right\rangle, \\
\left\langle v_A^2(0,z)\nabla_\perp^2\varphi_n(z), \phi_m(z)\right\rangle &= -2ik_\perp\sin\alpha\left\langle v_A^2(0,z)\frac{\partial\varphi_n(z)}{\partial z}, \phi_m(z)\right\rangle, \\
\left\langle v_A^2(0,z)\nabla_\perp^2\phi_n(z), \varphi_m(z)\right\rangle &= -2ik_\perp\sin\alpha\left\langle v_A^2(0,z)\frac{\partial\phi_n(z)}{\partial z}, \varphi_m(z)\right\rangle.
\end{aligned}
$$

Hence,

$$
\boxed{\frac{\mathrm{d}b_{||m}^{(1)}}{\mathrm{d}x} - \frac{i}{\omega}\sum_{n=0}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \omega_n^2\right]u_{xn}^{(1)}(x)\left\langle \frac{\phi_n(z)}{v_A^2(0,z)}, \phi_m(z)\right\rangle,}
$$

$$
\boxed{\frac{\mathrm{d}b_{||m}^{(2)}}{\mathrm{d}x} - \frac{i}{\omega}\sum_{n=0}^{\infty}\left[\omega^2/\hat{v}_A^2(x) - \varpi_n^2\right]u_{xn}^{(2)}(x)\left\langle \frac{\varphi_n(z)}{v_A^2(0,z)}, \varphi_m(z)\right\rangle,}
$$

$$
\boxed{
\begin{aligned}
\Delta_{\perp m}^{(1)}(x) = \frac{-i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2}\Bigg\{\tan^2\alpha\,\omega_m^2 b_{||m}^{(1)}(x) + \sum_{n=0}^{\infty}\Bigg[k_\perp^2 b_{||n}^{(1)}(x)\left\langle v_A^2(0,z)\phi_n(z), \phi_m(z)\right\rangle + \\
2ik_\perp\sin\alpha\, b_{||n}^{(2)}(x)\left\langle v_A^2(0,z)\frac{\mathrm{d}\varphi_n}{\mathrm{d}z}, \phi_m(z)\right\rangle\Bigg]\Bigg\},
\end{aligned}
}
$$

$$\boxed{\Delta_{\perp m}^{(2)}(x) = \frac{-i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2}\left\{ \tan^2\alpha\, \varpi_m^2 b_{||m}^{(2)}(x) + \sum_{n=0}^{\infty}\Big[k_\perp^2 b_{||n}^{(2)}(x)\left\langle v_A^2(0,z)\varphi_n(z), \varphi_m(z)\right\rangle + 2ik_\perp \sin\alpha\, b_{||n}^{(1)}(x)\left\langle v_A^2(0,z)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}, \varphi_m(z)\right\rangle\Big]\right\}.}$$

Inner products

Let

$$I_{1nm} = \left\langle \frac{\phi_n(z)}{v_A^2(0,z)}, \phi_m(z)\right\rangle,$$

$$I_{2nm} = \left\langle \frac{\varphi_n(z)}{v_A^2(0,z)}, \varphi_m(z)\right\rangle,$$

$$I_{3nm} = \left\langle v_A^2(0,z)\phi_n(z), \phi_m(z)\right\rangle,$$

$$I_{4nm} = \left\langle v_A^2(0,z)\varphi_n(z), \varphi_m(z)\right\rangle,$$

$$I_{5nm} = \left\langle v_A^2(0,z)\frac{\mathrm{d}\varphi_n}{\mathrm{d}z}, \phi_m(z)\right\rangle,$$

$$I_{6nm} = \left\langle v_A^2(0,z)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}, \varphi_m(z)\right\rangle.$$

For $n \neq m$,

$$\begin{aligned}
\frac{I_{1nm}}{A_nA_m} &= \int_0^{2L_z} \frac{\phi_n(z)\phi_m(z)}{A_nA_m v_A^4(0,z)}dz \\
&= \frac{a_na_m}{v_{A1}^4}\int_0^{L_z}\cos[k_{z1n}(z-l_z)]\cos[k_{z1m}(z-l_z)]dz \\
&+ \frac{c_nc_m}{v_{A2}^4}\int_{L_z}^{2L_z}\cos[k_{z2n}(z-3l_z)]\cos[k_{z2m}(z-3l_z)]dz \\
&= 2\int_0^{l_z}\frac{a_na_m}{v_{A1}^4}\cos(k_{z1n}z)\cos(k_{z1m}z) + \frac{c_nc_m}{v_{A2}^4}\cos(k_{z2n}z)\cos(k_{z2m}z)dz \\
&= 2\frac{a_na_m}{v_{A1}^4}\left[\frac{k_{z1n}\sin(k_{z1n}z)\cos(k_{z1m}z) - k_{z1m}\sin(k_{z1m}z)\cos(k_{z1n}z)}{k_{z1n}^2 - k_{z1m}^2}\right]_0^{l_z} \\
&+ 2\frac{c_nc_m}{v_{A2}^4}\left[\frac{k_{z2n}\sin(k_{z2n}z)\cos(k_{z2m}z) - k_{z2m}\sin(k_{z2m}z)\cos(k_{z2n}z)}{k_{z2n}^2 - k_{z2m}^2}\right]_0^{l_z} \\
&= 2\frac{a_na_m}{v_{A1}^4}\frac{k_{z1n}\sin(k_{z1n}l_z)\cos(k_{z1m}l_z) - k_{z1m}\sin(k_{z1m}l_z)\cos(k_{z1n}l_z)}{k_{z1n}^2 - k_{z1m}^2} \\
&+ 2\frac{c_nc_m}{v_{A2}^4}\frac{k_{z2n}\sin(k_{z2n}l_z)\cos(k_{z2m}l_z) - k_{z2m}\sin(k_{z2m}l_z)\cos(k_{z2n}l_z)}{k_{z2n}^2 - k_{z2m}^2},
\end{aligned}$$

for $n = m$

$$\frac{I_{1nm}}{A_nA_m} = \frac{a_n^2}{2v_{A1}^4 k_{z1n}}\Big[2k_{z1n}l_z + \sin(2k_{z1n}l_z)\Big] + \frac{c_n^2}{2v_{A2}^4 k_{z2n}}\Big[2k_{z2n}l_z + \sin(2k_{z2n}l_z)\Big].$$

For $n \neq m$

$$
\begin{aligned}
\frac{I_{2nm}}{B_n B_m} &= 2\frac{b_n b_m}{v_{A1}^4}\left[\frac{\bar{k}_{z1m}\sin\left(\bar{k}_{z1n}z\right)\cos\left(\bar{k}_{z1m}z\right)-\bar{k}_{z1n}\sin\left(\bar{k}_{z1m}z\right)\cos\left(\bar{k}_{z1n}z\right)}{\bar{k}_{z1n}^2-\bar{k}_{z1m}^2}\right]_0^{l_z} \\
&+ 2\frac{d_n d_m}{v_{A2}^4}\left[\frac{\bar{k}_{z2m}\sin\left(\bar{k}_{z2n}z\right)\cos\left(\bar{k}_{z2m}z\right)-\bar{k}_{z2n}\sin\left(\bar{k}_{z2m}z\right)\cos\left(\bar{k}_{z2n}z\right)}{\bar{k}_{z2n}^2-\bar{k}_{z2m}^2}\right]_0^{l_z} \\
&= 2\frac{b_n b_m}{v_{A1}^4}\frac{\bar{k}_{z1m}\sin\left(\bar{k}_{z1n}l_z\right)\cos\left(\bar{k}_{z1m}l_z\right)-\bar{k}_{z1n}\sin\left(\bar{k}_{z1m}l_z\right)\cos\left(\bar{k}_{z1n}l_z\right)}{\bar{k}_{z1n}^2-\bar{k}_{z1m}^2} \\
&+ 2\frac{d_n d_m}{v_{A2}^4}\frac{\bar{k}_{z2m}\sin\left(\bar{k}_{z2n}l_z\right)\cos\left(\bar{k}_{z2m}l_z\right)-\bar{k}_{z2n}\sin\left(\bar{k}_{z2m}l_z\right)\cos\left(\bar{k}_{z2n}l_z\right)}{\bar{k}_{z2n}^2-\bar{k}_{z2m}^2},
\end{aligned}
$$

for $n = m$

$$
\frac{I_{2nm}}{B_n B_m} = \frac{b_n^2}{2v_{A1}^4\bar{k}_{z1n}}\left[2\bar{k}_{z1n}l_z-\sin\left(2\bar{k}_{z1n}l_z\right)\right]+\frac{d_n^2}{2v_{A2}^4\bar{k}_{z2n}}\left[2\bar{k}_{z2n}l_z-\sin\left(2\bar{k}_{z2n}l_z\right)\right].
$$

For $n \neq m$

$$
\begin{aligned}
\frac{I_{3nm}}{A_n A_m} &= \frac{1}{A_n A_m}\int_0^{2L_z}\phi_n(z)\phi_m(z)dz \\
&= 2\int_0^{l_z} a_n a_m\cos(k_{z1n}z)\cos(k_{z1m}z)+c_n c_m\cos(k_{z2n}z)\cos(k_{z2m}z)dz \\
&= 2a_n a_m\frac{k_{z1n}\sin(k_{z1n}l_z)\cos(k_{z1m}l_z)-k_{z1m}\sin(k_{z1m}l_z)\cos(k_{z1n}l_z)}{k_{z1n}^2-k_{z1m}^2} \\
&+ 2c_n c_m\frac{k_{z2n}\sin(k_{z2n}l_z)\cos(k_{z2m}l_z)-k_{z2m}\sin(k_{z2m}l_z)\cos(k_{z2n}l_z)}{k_{z2n}^2-k_{z2m}^2},
\end{aligned}
$$

for $n = m$

$$
\frac{I_{3nm}}{A_n A_m} = \frac{a_n^2}{2k_{z1n}}\left[2k_{z1n}l_z+\sin(2k_{z1n}l_z)\right]+\frac{c_n^2}{2k_{z2n}}\left[2k_{z2n}l_z+\sin(2k_{z2n}l_z)\right].
$$

For $n \neq m$

$$
\begin{aligned}
\frac{I_{4nm}}{B_n B_m} &= 2\int_0^{l_z} b_n b_m\sin\left(\bar{k}_{z1n}z\right)\sin\left(\bar{k}_{z1m}z\right)+d_n d_m\sin\left(\bar{k}_{z2n}z\right)\sin\left(\bar{k}_{z2m}z\right)dz \\
&= 2a_n a_m\frac{k_{z1n}\sin(k_{z1n}l_z)\cos(k_{z1m}l_z)-k_{z1m}\sin(k_{z1m}l_z)\cos(k_{z1n}l_z)}{k_{z1n}^2-k_{z1m}^2} \\
&= 2b_n b_m\frac{\bar{k}_{z1m}\sin\left(\bar{k}_{z1n}l_z\right)\cos\left(\bar{k}_{z1m}l_z\right)-\bar{k}_{z1n}\sin\left(\bar{k}_{z1m}l_z\right)\cos\left(\bar{k}_{z1n}l_z\right)}{\bar{k}_{z1n}^2-\bar{k}_{z1m}^2} \\
&+ 2d_n d_m\frac{\bar{k}_{z2m}\sin\left(\bar{k}_{z2n}l_z\right)\cos\left(\bar{k}_{z2m}l_z\right)-\bar{k}_{z2n}\sin\left(\bar{k}_{z2m}l_z\right)\cos\left(\bar{k}_{z2n}l_z\right)}{\bar{k}_{z2n}^2-\bar{k}_{z2m}^2},
\end{aligned}
$$

for $n = m$

$$
\frac{I_{4nm}}{B_n B_m} = \frac{b_n^2}{2\bar{k}_{z1n}}\left[2\bar{k}_{z1n}l_z-\sin\left(2\bar{k}_{z1n}l_z\right)\right]+\frac{d_n^2}{2\bar{k}_{z2n}}\left[2\bar{k}_{z2n}l_z-\sin\left(2\bar{k}_{z2n}l_z\right)\right].
$$

For $\varpi_n \neq \omega_m$

$$
\begin{aligned}
\frac{I_{5nm}}{B_n A_m} &= \frac{1}{B_n A_m}\int_0^{2L_z} \frac{\mathrm{d}\varphi}{\mathrm{d}z}\phi_m(z)dz \\
&= b_n a_m \bar{k}_{z1n}\int_0^{L_z} \cos[\bar{k}_{z1n}(z-l_z)]\cos[k_{z1m}(z-l_z)]dz \\
&+ d_n c_m \bar{k}_{z2n}\int_{L_z}^{2L_z} \cos[\bar{k}_{z2n}(z-3l_z)]\cos[k_{z2m}(z-3l_z)]dz \\
&= 2\int_0^{l_z} b_n a_m \bar{k}_{z1n}\cos\big(\bar{k}_{z1n}z\big)\cos(k_{z1m}z) + d_n c_m \bar{k}_{z2n}\cos\big(\bar{k}_{z2n}z\big)\cos(k_{z2m}z)dz \\
&= 2b_n a_m \bar{k}_{z1n}\frac{\bar{k}_{z1n}\sin\big(\bar{k}_{z1n}l_z\big)\cos(k_{z1m}l_z) - k_{z1m}\sin(k_{z1m}l_z)\cos\big(\bar{k}_{z1n}l_z\big)}{\bar{k}_{z1n}^2 - k_{z1m}^2} \\
&+ 2d_n c_m \bar{k}_{z2n}\frac{\bar{k}_{z2n}\sin\big(\bar{k}_{z2n}l_z\big)\cos(k_{z2m}l_z) - k_{z2m}\sin(k_{z2m}l_z)\cos\big(\bar{k}_{z2n}l_z\big)}{\bar{k}_{z2n}^2 - k_{z2m}^2},
\end{aligned}
$$

for $\varpi_n = \omega_m$

$$
\frac{I_{5nm}}{B_n A_m} = \frac{b_n a_m}{2}\Big[2k_{z1n}l_z + \sin(2k_{z1n}l_z)\Big] + \frac{d_n c_m}{2}\Big[2k_{z2n}l_z + \sin(2k_{z2n}l_z)\Big].
$$

For $\omega_n \neq \varpi_m$,

$$
\begin{aligned}
\frac{I_{6nm}}{A_n B_m} &= \frac{1}{A_n B_m}\int_0^{2L_z} \frac{\mathrm{d}\phi}{\mathrm{d}z}\varphi_m(z)dz \\
&= -a_n b_m k_{z1n}\int_0^{L_z} \sin[k_{z1n}(z-l_z)]\sin[\bar{k}_{z1m}(z-l_z)]dz \\
&- c_n d_m k_{z2n}\int_{L_z}^{2L_z} \sin[k_{z2n}(z-3l_z)]\sin[\bar{k}_{z2m}(z-3l_z)]dz \\
&= -2\int_0^{l_z} a_n b_m k_{z1n}\sin(k_{z1n}z)\sin\big(\bar{k}_{z1m}z\big) + c_n d_m k_{z2n}\sin(k_{z2n}z)\sin\big(\bar{k}_{z2m}z\big)dz \\
&= -2a_n b_m k_{z1n}\frac{\bar{k}_{z1m}\sin(k_{z1n}l_z)\cos\big(\bar{k}_{z1m}l_z\big) - k_{z1n}\sin\big(\bar{k}_{z1m}l_z\big)\cos(k_{z1n}l_z)}{k_{z1n}^2 - \bar{k}_{z1m}^2} \\
&- 2c_n d_m k_{z2n}\frac{\bar{k}_{z2m}\sin(k_{z2m}l_z)\cos\big(\bar{k}_{z2m}l_z\big) - k_{z2m}\sin\big(\bar{k}_{z2m}l_z\big)\cos(k_{z2m}l_z)}{k_{z2n}^2 - \bar{k}_{z2m}^2},
\end{aligned}
$$

for $\omega_n = \varpi_m$

$$
\frac{I_{6nm}}{A_n B_m} = -\frac{a_n b_m}{2}\Big[2k_{z1n}l_z - \sin(2k_{z1n}l_z)\Big] - \frac{c_n d_m}{2}\Big[2k_{z2n}l_z - \sin(2k_{z2n}l_z)\Big].
$$

Analytic solution

Let $\omega = \omega_r + i\omega_i$, where $\omega_r = \omega_k$ and $\omega_i \ll \omega_r$, $k \in \mathbb{N}$. The singular solution is approximated by

$$
\begin{aligned}
u_x(x,z) &= -\beta_0 \ln(x - ix_i)\nabla_\perp \phi_k(z) \\
&= -\beta_0 \ln(x - ix_i)\left[ik_\perp \phi_k(z) - \sin\alpha \frac{\mathrm{d}\phi_k}{\mathrm{d}z}\right].
\end{aligned}
$$

$$
u_\perp(x,z) = \frac{\beta_0}{x - ix_i}\phi_k(z),
$$

$$
\nabla_\perp b_{||}(x,z) = \frac{\beta_0 \mathcal{L}_1}{i\omega}\phi_k(z),
$$

where we choose the φ_k component equal to zero, $\mathcal{L}_1$ is given by

$$\begin{aligned}\mathcal{L}_1 = &\left.\frac{\partial \mathcal{L}}{\partial x}\right|_{x=0} \\ &= -\frac{2\omega^2}{v_A^3(0,z)}\left.\frac{\partial v_A}{\partial x}\right|_{x=0} \\ &= -\frac{2\omega^2}{a_0 v_A^2(0,z)},\end{aligned}$$

and

$$\begin{aligned}x_i &= \frac{2\omega_i}{\omega_r}\left[\frac{\rho}{\partial\rho/\partial x}\right]_{x=0} \\ &= \frac{2\omega_i}{\omega_r}\left[\frac{1/v_A^2}{-2\ \partial v_A/\partial x\ /v_A^3}\right]_{x=0} \\ &= -\frac{\omega_i}{\omega_r}\left[\frac{v_A}{\partial v_A/\partial x}\right]_{x=0}\end{aligned}$$

Hence,

$$u_{xn}^{(1)}(x) = -ik_\perp\beta_0 \ln(x-ix_i)\delta_{nk},$$
$$u_{xn}^{(2)}(x) = \beta_0\sin\alpha\ \ln(x-ix_i)\left\langle\frac{\mathrm{d}\phi_k}{\mathrm{d}z},\varphi_n\right\rangle,$$

where for $\omega_k \neq \varpi_n$

$$\begin{aligned}\frac{I_{7kn}}{A_kB_n} &= \left\langle\frac{\mathrm{d}\phi_k}{\mathrm{d}z},\varphi_n\right\rangle \\ &= \frac{1}{A_kB_n}\int_0^{2L_z}\frac{\mathrm{d}\phi_k}{\mathrm{d}z}\frac{\varphi_m}{v_A^2(0,z)}dz \\ &= -a_kb_n\frac{k_{z1k}}{v_{A1}^2}\int_0^{L_z}\sin[k_{z1k}(z-l_z)]\sin[\bar{k}_{z1n}(z-l_z)]dz \\ &- c_kd_n\frac{k_{z2k}}{v_{A2}^2}\int_{L_z}^{2L_z}\sin[k_{z2k}(z-3l_z)]\sin[\bar{k}_{z2n}(z-3l_z)]dz \\ &= -2\int_0^{l_z} a_kb_nk_{z1k}\sin(k_{z1k}z)\sin\big(\bar{k}_{z1n}z\big)+c_kd_nk_{z2k}\sin(k_{z2k}z)\sin\big(\bar{k}_{z2n}z\big)dz \\ &= -2a_kb_n\frac{k_{z1k}}{v_{A1}^2}\frac{\bar{k}_{z1n}\sin(k_{z1k}l_z)\cos\big(\bar{k}_{z1n}l_z\big)-k_{z1k}\sin\big(\bar{k}_{z1n}l_z\big)\cos(k_{z1k}l_z)}{k_{z1k}^2-\bar{k}_{z1n}^2} \\ &- 2c_kd_n\frac{k_{z2k}}{v_{A2}^2}\frac{\bar{k}_{z2n}\sin(k_{z2n}l_z)\cos\big(\bar{k}_{z2n}l_z\big)-k_{z2n}\sin\big(\bar{k}_{z2n}l_z\big)\cos(k_{z2n}l_z)}{k_{z2k}^2-\bar{k}_{z2n}^2},\end{aligned}$$

and for $\omega_k = \varpi_n$

$$\frac{I_{7nm}}{A_nB_m} = -\frac{a_nb_m}{2v_{A1}^2}\Big[2k_{z1n}l_z-\sin(2k_{z1n}l_z)\Big]-\frac{c_nd_m}{2v_{A2}^2}\Big[2k_{z2n}l_z-\sin(2k_{z2n}l_z)\Big].$$

For the case where $\alpha = 0$,

$$b_{||} = 2\frac{\beta_0\omega}{k_\perp a_0}\frac{\phi_k(z)}{v_A^2(0,z)},$$

hence,

$$\begin{aligned}b_{||n}^{(1)} &= 2\frac{\beta_0\omega}{k_\perp a_0}\left\langle\frac{\phi_k(z)}{v_A^2(0,z)},\phi_n\right\rangle \\ &= 2\frac{\beta_0\omega}{k_\perp a_0}I_{1kn}.\end{aligned}$$

For $\alpha \neq 0$

$$\frac{\mathrm{d}b_{||}}{\mathrm{d}z}-i\frac{k_\perp}{\sin\alpha}b_{||} = -\frac{2i\beta_0\omega}{a_0\sin\alpha}\frac{\phi_k}{v_A^2(0,z)}.$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}z}\left[\exp\left(-i\frac{k_\perp z}{\sin\alpha}\right)b_{||}\right] = -\frac{2i\beta_0\omega}{a_0\sin\alpha}\exp\left(-i\frac{k_\perp z}{\sin\alpha}\right)\frac{\phi_k}{v_A^2(0,z)},$$

$$\implies b_{||} = -\frac{2i\beta_0\omega}{a_0\sin\alpha}\exp\left(i\frac{k_\perp z}{\sin\alpha}\right)\left[\int_0^z \exp\left(-i\frac{k_\perp z'}{\sin\alpha}\right)\frac{\phi_k(z')}{v_A^2(0,z')}dz' + C\right]$$

We require

$$b_{||}(0) = -\frac{2i\beta_0\omega}{a_0\sin\alpha}C,$$

to equal

$$b_{||}(2L_z) = -\frac{2i\beta_0\omega}{a_0\sin\alpha}\exp\left(2i\frac{k_\perp L_z}{\sin\alpha}\right)\left[\int_0^{2L_z} \exp\left(-i\frac{k_\perp z'}{\sin\alpha}\right)\frac{\phi_k(z')}{v_A^2(0,z')}dz' + C\right].$$

Solving for C gives

$$\left[\exp\left(-2i\frac{k_\perp L_z}{\sin\alpha}\right) - 1\right]C = \int_0^{2L_z} \exp\left(-i\frac{k_\perp z'}{\sin\alpha}\right)\frac{\phi_k(z')}{v_A^2(0,z')}dz'.$$

Hence, we can solve for C provided $k_\perp L_z/\sin\alpha \neq n\pi$, $n \in \mathbb{Z}$. If $k_\perp L_z/\sin\alpha = n\pi$, then the periodic boundary conditions are automatically satisfied and C is a free parameter. Hence,

$$b_{||n}^{(1)} = \left\langle b_{||}(z), \phi_n(z)\right\rangle,$$

$$b_{||n}^{(2)} = \left\langle b_{||}(z), \varphi_n(z)\right\rangle.$$

Calculate $u_\perp$

Note that

$$\mathcal{L}u_\perp = i\omega\nabla_\perp b_{||}.$$

Assume

$$u_\perp = \sum_{n=0}^{\infty} u_{\perp n}^{(1)}(x)\phi_n(z) + u_{\perp n}^{(2)}(x)\varphi_n(z)$$

Hence,

$$\sum_{n=0}^{\infty} u_{\perp n}^{(1)}[\omega^2/\hat{v}_A^2(x) - \omega_n^2]\frac{\phi_n(z)}{v_A^2(0,z)} + u_{\perp n}^{(1)}[\omega^2/\hat{v}_A^2(x) - \varpi_n^2]\frac{\varphi_n(z)}{v_A^2(0,z)} = i\omega\left[ik_\perp - \sin\alpha\frac{\partial}{\partial z}\right]b_{||},$$

therefore,

$$u_{\perp m}^{(1)} = \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2}\sum_{n=0}^{\infty}\left\{ik_\perp b_{||n}^{(1)}\left\langle v_A^2(0,z)\phi_n, \phi_m\right\rangle - \right.$$
$$\left. \sin\alpha\, b_{||n}^{(2)}\left\langle v_A^2(0,z)\frac{\mathrm{d}\varphi_n}{\mathrm{d}z}, \phi_m\right\rangle\right\},$$

$$u_{\perp m}^{(2)} = \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2}\sum_{n=0}^{\infty}\left\{ik_\perp b_{||n}^{(2)}\left\langle v_A^2(0,z)\varphi_n, \varphi_m\right\rangle - \right.$$
$$\left. \sin\alpha\, b_{||n}^{(1)}\left\langle v_A^2(0,z)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}, \phi_m\right\rangle\right\}.$$

$$\mathcal{L}\nabla_\perp u_\perp = i\omega\nabla_\perp^2 b_{||},$$

$$\begin{aligned}
\nabla_\perp^2 &= -k_\perp^2 - 2ik_\perp\frac{\partial}{\partial z} + \sin^2\alpha\frac{\partial^2}{\partial z^2} \\
&= -k_\perp^2 - 2ik_\perp\frac{\partial}{\partial z} + \tan^2\alpha\nabla_{||}^2, \\
&= -k_\perp^2 - 2ik_\perp\frac{\partial}{\partial z} + \tan^2\alpha\left[\nabla_{||}^2 + \frac{\omega_n^2}{v_A^2(0,z)} - \frac{\omega_n^2}{v_A^2(0,z)}\right], \\
&= -k_\perp^2 - 2ik_\perp\frac{\partial}{\partial z} + \tan^2\alpha\left[\nabla_{||}^2 + \frac{\varpi_n^2}{v_A^2(0,z)} - \frac{\varpi_n^2}{v_A^2(0,z)}\right],
\end{aligned}$$

Hence,

$$\nabla_\perp^2 \phi_n = \left\{ -k_\perp^2 - 2ik_\perp \frac{\partial}{\partial z} - \tan^2\alpha \frac{\omega_n^2}{v_A^2(0,z)} \right\} \phi_n,$$

$$\nabla_\perp^2 \varphi_n = \left\{ -k_\perp^2 - 2ik_\perp \frac{\partial}{\partial z} - \tan^2\alpha \frac{\varpi_n^2}{v_A^2(0,z)} \right\} \varphi_n.$$

$$\sum_{n=0}^{\infty} \Delta_{\perp n}^{(1)} [\omega^2/\hat{v}_A^2(x) - \omega_n^2] \frac{\phi_n(z)}{v_A^2(0,z)} + \Delta_{\perp n}^{(1)} [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] \frac{\varphi_n(z)}{v_A^2(0,z)} =$$

$$- i\omega \sum_{n=0}^{\infty} b_{||n}^{(1)} \left\{ k_\perp^2 + 2ik_\perp \frac{\partial}{\partial z} + \tan^2\alpha \frac{\omega_n^2}{v_A^2(0,z)} \right\} \phi_n + b_{||n}^{(2)} \left\{ k_\perp^2 + 2ik_\perp \frac{\partial}{\partial z} + \tan^2\alpha \frac{\varpi_n^2}{v_A^2(0,z)} \right\} \varphi_n, ,$$

$$\Delta_{\perp m}^{(1)} = \frac{-i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2} \left\{ \tan^2\alpha\, \omega_n^2 \right.$$

Calculate b'_x and $b'_\perp$

We know that

$$\begin{aligned} b'_x &= \frac{1}{i\omega} \nabla'_{||} u'_x \\ &= \frac{\cos\alpha}{i\omega} \frac{\partial u'_x}{\partial z}, \\ b'_\perp &= \frac{1}{i\omega} \nabla'_{||} u'_\perp \\ &= \frac{\cos\alpha}{i\omega} \frac{\partial u'_\perp}{\partial z}, \end{aligned}$$

hence,

$$b'_x = \frac{\cos\alpha}{i\omega} \sum_{n=0}^{\infty} u_{xn}^{(1)}(x) \frac{\partial \phi_n(z)}{\partial z} + u_{xn}^{(2)}(x) \frac{\partial \varphi_n(z)}{\partial z},$$

$$b'_\perp = \frac{\cos\alpha}{i\omega} \sum_{n=0}^{\infty} u_{\perp n}^{(1)}(x) \frac{\partial \phi_n(z)}{\partial z} + u_{\perp n}^{(2)}(x) \frac{\partial \varphi_n(z)}{\partial z},$$

Arrays

Let

$$\boldsymbol{u}_x^{(1)} = \left(u_{x0}^{(1)}, u_{x1}^{(1)}, ..., u_{xN}^{(1)} \right)$$

$$\boldsymbol{u}_x^{(2)} = \left(u_{x0}^{(2)}, u_{x1}^{(2)}, ..., u_{xN}^{(2)} \right)$$

$$\boldsymbol{b}_{||}^{(1)} = \left(b_{||0}^{(1)}, b_{||1}^{(1)}, ..., b_{||N}^{(1)} \right)$$

$$\boldsymbol{b}_{||}^{(2)} = \left(b_{||0}^{(2)}, b_{||1}^{(2)}, ..., b_{||N}^{(2)} \right)$$

$$\boldsymbol{\Delta}_\perp^{(1)} = \left(\Delta_{\perp 0}^{(1)}, \Delta_{\perp 1}^{(1)}, ..., \Delta_{\perp N}^{(1)} \right)$$

$$\boldsymbol{\Delta}_\perp^{(2)} = \left(\Delta_{\perp 0}^{(2)}, \Delta_{\perp 1}^{(2)}, ..., \Delta_{\perp N}^{(2)} \right)$$

$$\boldsymbol{k}_{z1} = \left(k_{z1,0}, k_{z1,1}, ..., k_{z1,N} \right)$$

$$\boldsymbol{k}_{z2} = \left(k_{z2,0}, k_{z2,1}, ..., k_{z2,N} \right)$$

$$\bar{\boldsymbol{k}}_{z1} = \left(\bar{k}_{z1,0}, \bar{k}_{z1,1}, ..., \bar{k}_{z1,N} \right)$$

$$\bar{\boldsymbol{k}}_{z2} = \left(\bar{k}_{z2,0}, \bar{k}_{z2,1}, ..., \bar{k}_{z2,N}\right)$$

$$\boldsymbol{U}_x^{(1)} = \left(\boldsymbol{u}_x^{(1)T}, \boldsymbol{u}_x^{(1)T}, ..., \boldsymbol{u}_x^{(1)T}\right)$$

$$\boldsymbol{U}_x^{(2)} = \left(\boldsymbol{u}_x^{(2)T}, \boldsymbol{u}_x^{(2)T}, ..., \boldsymbol{u}_x^{(2)T}\right)$$

$$\boldsymbol{B}_{||}^{(1)} = \left(\boldsymbol{B}_{||0}^{(1T)}, \boldsymbol{B}_{||1}^{(1T)}, ..., \boldsymbol{B}_{||N}^{(1T)}\right)$$

$$\boldsymbol{B}_{||}^{(2)} = \left(\boldsymbol{B}_{||0}^{(2T)}, \boldsymbol{B}_{||1}^{(2T)}, ..., \boldsymbol{B}_{||N}^{(2T)}\right)$$

$$\boldsymbol{K}_{z1} = \left(\boldsymbol{k}_{z1}^T, \boldsymbol{k}_{z1}^T, ..., \boldsymbol{k}_{z1}^T\right)$$

$$\boldsymbol{K}_{z2} = \left(\boldsymbol{k}_{z2}^T, \boldsymbol{k}_{z2}^T, ..., \boldsymbol{k}_{z2}^T\right)$$

$$\bar{\boldsymbol{K}}_{z1} = \left(\bar{\boldsymbol{k}}_{z1}^T, \bar{\boldsymbol{k}}_{z1}^T, ..., \bar{\boldsymbol{k}}_{z1}^T\right)$$

$$\bar{\boldsymbol{K}}_{z1} = \left(\bar{\boldsymbol{k}}_{z1}^T, \bar{\boldsymbol{k}}_{z1}^T, ..., \bar{\boldsymbol{k}}_{z1}^T\right)$$

Total energy

The total energy density is given by

$$\begin{aligned}
e_{tot} &= \frac{1}{2}\left\{\rho\left[\mathrm{Re}(u_x)^2 + \mathrm{Re}(u_\perp)^2\right] + \frac{1}{\mu}\left[\mathrm{Re}(b_x)^2 + \mathrm{Re}(b_\perp)^2 + \mathrm{Re}(b_{||})^2\right]\right\} \\
&= \frac{B_0^2}{2\mu}\left\{\frac{1}{v_A^2}\left[\mathrm{Re}(u_x)^2 + \mathrm{Re}(u_\perp)^2\right] + \mathrm{Re}(\hat{b}_x)^2 + \mathrm{Re}(\hat{b}_\perp)^2 + \mathrm{Re}(\hat{b}_{||})^2\right\}
\end{aligned}$$

Hence,

$$\frac{\partial e_{tot}}{\partial t} = \frac{B_0^2}{\mu}\left\{\frac{1}{v_A^2}[\mathrm{Re}(u_x)\,\mathrm{Re}(i\omega u_x) + \mathrm{Re}(u_\perp)\,\mathrm{Re}(i\omega u_\perp)] + \mathrm{Re}(\hat{b}_x)\,\mathrm{Re}(i\omega\hat{b}_x) + \mathrm{Re}(\hat{b}_\perp)\,\mathrm{Re}(i\hat{b}_\perp) + \mathrm{Re}(\hat{b}_{||})\,\mathrm{Re}(i\omega\hat{b}_{||})\right\}.$$

Note that

$$\begin{aligned}
u_x &= u_x' \exp(-\omega_i t)\exp\{i[k_\perp(\cos\alpha\, y - \sin\alpha\, z) + \omega_r t]\}, \\
\mathrm{Re}(u_x)\,\mathrm{Re}(i\omega u_x) &= -\,\mathrm{Re}(u_x)\,\mathrm{Im}(\omega u_x) \\
&= -\left(\frac{u_x + u_x^*}{2}\right)\left(\frac{\omega u_x - \omega^* u_x^*}{2i}\right) \\
&= -\frac{1}{4i}\left[\omega u_x^2 - \omega^*(u_x^*)^2 + u_x' u_x'^* \exp(-2\omega_i t)(\omega - \omega^*)\right]
\end{aligned}$$

Hence, the average in y is given by

$$\langle \mathrm{Re}(u_x)\,\mathrm{Re}(i\omega u_x)\rangle = -\frac{1}{2}|u_x'|^2\,\mathrm{Im}(\omega)$$

$$\left\langle \frac{\partial e_{tot}}{\partial t}\right\rangle = -\omega_i \frac{B_0^2}{2\mu}\exp(-2\omega_i t)\left\{\frac{1}{v_A^2}\left[|u_x'|^2 + |u_\perp'|^2\right] + \left|\hat{b}_x'\right|^2 + \left|\hat{b}_\perp'\right|^2 + \left|\hat{b}_{||}'\right|^2\right\}.$$

The Poynting flux is given by

$$\begin{aligned}
\boldsymbol{S} &= \frac{\boldsymbol{E}\times\boldsymbol{b}}{\mu} \\
&= \frac{1}{\mu}\{[\boldsymbol{B}_0\cdot\mathrm{Re}(\boldsymbol{b})]\,\mathrm{Re}(\boldsymbol{u}) - [\mathrm{Re}(\boldsymbol{u})\cdot\mathrm{Re}(\boldsymbol{b})]\boldsymbol{B}_0\} \\
&= \frac{B_0^2}{\mu}\Big[\mathrm{Re}(\hat{b}_{||})\,\mathrm{Re}(\boldsymbol{u}) - \Big[\mathrm{Re}(\boldsymbol{u})\cdot\mathrm{Re}(\hat{\boldsymbol{b}})\Big]\,(\sin\alpha\,\hat{\boldsymbol{y}} + \cos\alpha\,\hat{\boldsymbol{z}})\Big].
\end{aligned}$$

Hence, the Poynting flux in the x-direction is given by

$$
\begin{aligned}
S_x &= \boldsymbol{S}\cdot\hat{\boldsymbol{x}} \\
&= \frac{B_0^2}{\mu}\,\mathrm{Re}(\hat{b}_{||})\,\mathrm{Re}(u_x) \\
&= \frac{B_0^2}{4\mu}\Big[\hat{b}_{||}u_x + \hat{b}_{||}^{*}u_x^{*} + \left(\hat{b}_{||}'u_x'^{*} + \hat{b}_{||}'^{*}u_x'\right)\exp(-2\omega_i t)\Big].
\end{aligned}
$$

Hence, the average in y is given by

$$\langle S_x\rangle = \frac{B_0^2}{2\mu}\,\mathrm{Re}(\hat{b}_{||}'u_x'^{*})\exp(-2\omega_i t).$$

From the energy equation,

$$\frac{\partial e_{tot}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0,$$

we know that

$$\int_{z=z_{min}}^{z_{max}}\int_{x=x_{min}}^{x_{max}}\left\langle\frac{\partial e_{tot}}{\partial t}\right\rangle dx + [\langle S_x\rangle]_{x_{min}}^{x_{max}} dz = 0.$$

Analytic solution

$$u_\perp'(x,z) \approx \frac{u_0 x_{i,m}}{x - x_{res,m}}\phi_m(z),$$

$$u_x'(x,z) \approx -u_0 x_{i,m}\ln(x - x_{res,m})\nabla_\perp'\phi_m(z),$$

For $\sin\alpha = 0$,

$$\hat{b}_{||}'(z) \approx \frac{2u_0\omega x_{i,m}}{a(0)v_A^2(0,z)k_\perp}\phi_m(z),$$

For $\sin\alpha \neq 0$,

$$\frac{\mathrm{d}\hat{b}_{||}'}{\mathrm{d}z} - \frac{ik_\perp}{\sin\alpha}\hat{b}_{||}'(z) \approx \frac{-2iu_0\omega x_{i,m}\phi_m(z)}{a(0)v_A^2(0,z)\sin\alpha},$$

this can be solved using the integrating factor method

$$\hat{b}_{||}'(z) \approx -\frac{2i\beta_0\omega}{a_0\sin\alpha}\exp\left(i\frac{k_\perp z}{\sin\alpha}\right)\left[\int_{-L_z}^{z}\exp\left(-i\frac{k_\perp z'}{\sin\alpha}\right)\frac{\phi_m(z')}{v_A^2(0,z')}dz' + C\right].$$

Applying the periodic boundary conditions we can solve for C to give

$$\sin\left(\frac{k_\perp L_z}{\sin\alpha}\right)C = -\frac{1}{2i}\exp\left(i\frac{k_\perp L_z}{\sin\alpha}\right)\int_{-L_z}^{L_z}\exp\left(-i\frac{k_\perp z'}{\sin\alpha}\right)\frac{\phi_m(z')}{v_A^2(0,z')}dz'.$$

Therefore, C is unique if and only if $k_\perp L_z/\sin\alpha \neq n\pi$, where $n\in\mathbb{Z}$.

$$\hat{b}_\perp'(x,z) \approx \frac{\cos\alpha}{i\omega}\frac{u_0 x_{i,m}}{x - x_{res,m}}\frac{\mathrm{d}\phi_m}{\mathrm{d}z},$$

$$\hat{b}_x'(x,z) \approx -\frac{\cos\alpha}{i\omega}u_0 x_{i,m}\ln(x - x_{res,m})\nabla_\perp'\frac{\mathrm{d}\phi_m}{\mathrm{d}z},$$

Eigenfrequencies for the case where $v_{A+}/v_{A-} = 3$

The eigenfrequencies are given by

$$k_{zn+}\sin(k_{zn+}l_z)\cos(k_{zn-}l_z) + k_{zn-}\sin(k_{zn-}l_z)\cos(k_{zn+}l_z) = 0,$$

$$\bar{k}_{zn+}\sin(\bar{k}_{zn-}l_z)\cos(\bar{k}_{zn+}l_z) + \bar{k}_{zn-}\sin(\bar{k}_{zn+}l_z)\cos(\bar{k}_{zn-}l_z) = 0.$$

Note that

$$k_{zn+} = \frac{\omega_n}{v_{A+}\cos\alpha},$$

$$k_{zn-} = \frac{\omega_n}{v_{A+}\cos\alpha}\frac{v_{A+}}{v_{A-}},$$

$$\bar{k}_{zn+} = \frac{\varpi_n}{v_{A+}\cos\alpha},$$

$$\bar{k}_{zn-} = \frac{\varpi_n}{v_{A+}\cos\alpha}\frac{v_{A+}}{v_{A-}},$$

let

$$x = \frac{\omega_n l_z}{v_{A+}\cos\alpha},$$

$$y = \frac{\varpi_n l_z}{v_{A+}\cos\alpha},$$

hence,

$$\sin(x)\cos(3x) + 3\sin(3x)\cos(x) = 0,$$

$$\sin(3y)\cos(y) + 3\sin(y)\cos(3x) = 0.$$

The solutions are given by

$$x = \begin{cases} n\pi/2, \\ n\pi - \tan^{-1}(\sqrt{5/3}), \\ n\pi + \tan^{-1}(\sqrt{5/3}), \end{cases}$$

$$y = \begin{cases} n\pi/2, \\ n\pi - \tan^{-1}(\sqrt{3/5}), \\ n\pi + \tan^{-1}(\sqrt{3/5}), \end{cases}$$

for $n \in \mathbb{N}$. Hence,

$$\omega_n = \frac{v_{A+}\cos\alpha}{l_z}\begin{cases} n\pi/2, \\ n\pi - \tan^{-1}(\sqrt{5/3}), \\ n\pi + \tan^{-1}(\sqrt{5/3}), \end{cases}$$

$$\varpi_n = \frac{v_{A+}\cos\alpha}{l_z}\begin{cases} n\pi/2, \\ n\pi - \tan^{-1}(\sqrt{3/5}), \\ n\pi + \tan^{-1}(\sqrt{3/5}). \end{cases}$$