

Eigenfunction approach

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Equations

The equations are given by

$$\begin{aligned}\frac{\partial u_x}{\partial t} &= v_A^2(x, z) \left[\nabla_{||} b_x - \frac{\partial b_{||}}{\partial x} \right], \\ \frac{\partial u_{\perp}}{\partial t} &= v_A^2(x, z) [\nabla_{||} b_{\perp} - \nabla_{\perp} b_{||}], \\ \frac{\partial b_x}{\partial t} &= \nabla_{||} u_x, \\ \frac{\partial b_{\perp}}{\partial t} &= \nabla_{||} u_{\perp}, \\ \frac{\partial b_{||}}{\partial t} &= - \left[\frac{\partial u_x}{\partial x} + \nabla_{\perp} u_{\perp} \right],\end{aligned}$$

where

$$\begin{aligned}\nabla_{\perp} &= \cos \alpha \frac{\partial}{\partial y} - \sin \alpha \frac{\partial}{\partial z}, \\ \nabla_{||} &= \sin \alpha \frac{\partial}{\partial y} + \cos \alpha \frac{\partial}{\partial z}.\end{aligned}$$

The field perturbations b_x , b_{\perp} and $b_{||}$ are dimensionless as they have been normalised by B_0 . The background field is given by

$$\mathbf{B}_0 = B_0 (\sin \alpha \hat{\mathbf{y}} + \cos \alpha \hat{\mathbf{z}}).$$

The Alfvén speed is given by $v_A(x, z) = \hat{v}_A(x) v_A(0, z)$, where

$$v_A(0, z) = \begin{cases} v_{A1}, & z \leq L_z, \\ v_{A2}, & z > L_z. \end{cases}$$

The system is periodic in z , with period $2L_z$. We assume the variables are of the form

$$f(x, y, z, t) = f'(x, z) \exp\{i[k_{\perp}(\cos \alpha y - \sin \alpha z) + \omega t]\}.$$

Note that

$$\begin{aligned}\nabla_{\perp} f(x, y, z, t) &= \exp\{i[k_{\perp}(\cos \alpha y - \sin \alpha z) + \omega t]\} \left(i k_{\perp} - \sin \alpha \frac{\partial}{\partial z} \right) f', \\ \nabla_{||} f(x, y, z, t) &= \exp\{i[k_{\perp}(\cos \alpha y - \sin \alpha z) + \omega t]\} \cos \alpha \frac{\partial f'}{\partial z}.\end{aligned}$$

Therefore, the above equations can be simplified to

$$\begin{aligned}i\omega u'_x &= v_A^2(x, z) \left[\nabla'_{||} b'_x - \frac{\partial b'_{||}}{\partial x} \right], \\ i\omega u'_{\perp} &= v_A^2(x, z) \left[\nabla'_{||} b'_{\perp} - \nabla'_{\perp} b'_{||} \right], \\ i\omega b'_x &= \nabla'_{||} u'_x,\end{aligned}$$

$$i\omega b'_\perp = \nabla'_\parallel u'_\perp,$$

$$i\omega b'_\parallel = -\left[\frac{\partial u'_x}{\partial x} + \nabla'_\perp u'_\perp\right],$$

where

$$\nabla'_\perp = \left(ik_\perp - \sin\alpha \frac{\partial}{\partial z}\right),$$

$$\nabla'_\parallel = \cos\alpha \frac{\partial}{\partial z}.$$

From this point on, to keep the notation clear, we drop the ' notation.

The equations can be rearranged to give

$$-\omega^2 u_x = v_A^2 \left[\nabla_\parallel^2 u_x - i\omega \frac{\partial b_\parallel}{\partial x} \right],$$

$$-\omega^2 u_\perp = v_A^2 \left[\nabla_\parallel^2 u_\perp - i\omega \nabla_\perp b_\parallel \right],$$

hence,

$$\mathcal{L}u_x = i\omega \frac{\partial b_\parallel}{\partial x}$$

$$\mathcal{L}u_\perp = i\omega \nabla_\perp b_\parallel,$$

where

$$\mathcal{L} = \nabla_\parallel^2 + \frac{\omega^2}{v_A^2(x, z)},$$

Therefore,

$$\boxed{\frac{\partial u_x}{\partial x} = -[i\omega b_\parallel + \nabla_\perp u_\perp]},$$

$$\boxed{\frac{\partial b_\parallel}{\partial x} = -\frac{i}{\omega} \mathcal{L}u_x},$$

$$\mathcal{L}u_\perp = i\omega \nabla_\perp b_\parallel.$$

For $z \neq 0, L_z$,

$$\boxed{\mathcal{L} \nabla_\perp u_\perp = i\omega \nabla_\perp^2 b_\parallel}.$$

To solve this, we postulate a solution of the form

$$u_x(x, z) = u_{x0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} u_{xn}^{(1)}(x)\phi_n(z) + u_{xn}^{(2)}(x)\varphi_n(z),$$

$$b_\parallel(x, z) = b_{\parallel 0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} b_{\parallel n}^{(1)}(x)\phi_n(z) + b_{\parallel n}^{(2)}(x)\varphi_n(z),$$

$$\nabla_\perp u_\perp(x, z) = \Delta_{\perp 0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} \Delta_{\perp n}^{(1)}(x)\phi_n(z) + \Delta_{\perp n}^{(2)}(x)\varphi_n(z),$$

where ϕ_n, φ_n are eigenfunctions, with eigenfrequencies, ω_n, ϖ_n satisfying

$$\left[\nabla_\parallel^2 + \frac{\omega_n^2}{v_A^2(0, z)} \right] \phi_n(z) = 0,$$

$$\left[\nabla_\parallel^2 + \frac{\varpi_n^2}{v_A^2(0, z)} \right] \varphi_n(z) = 0,$$

as well as the boundary conditions. We will choose ω to be complex, therefore, $\omega \neq \varpi_n, \omega \neq \varpi_n^*$.

Eigenfunctions and eigenfrequencies

The boundary conditions are given by

$$\phi_n(0) = \phi_n(2L_z), \quad \varphi_n(0) = \varphi_n(2L_z), \quad \left. \frac{d\phi_n}{dz} \right|_{z=0} = \left. \frac{d\phi_n}{dz} \right|_{z=2L_z}, \quad \left. \frac{d\varphi_n}{dz} \right|_{z=0} = \left. \frac{d\varphi_n}{dz} \right|_{z=2L_z}.$$

Also, we require continuity of ϕ , φ , $d\phi/dz$, $d\varphi/dz$ at $z = L_z$. By symmetry, we assume the eigenfunctions are of the form,

$$\begin{aligned} \phi_n(z) &= \begin{cases} a \cos[k_{z1n}(z - l_z)], & z \leq L_z, \\ c \cos[k_{z2n}(z - 3l_z)], & z > L_z, \end{cases} \\ \varphi_n(z) &= \begin{cases} b \sin[\bar{k}_{z1n}(z - l_z)], & z \leq L_z, \\ d \sin[\bar{k}_{z2n}(z - 3l_z)], & z > L_z. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \frac{d\phi_n}{dz} &= \begin{cases} -ak_{z1n} \sin[k_{z1n}(z - l_z)], & z \leq L_z, \\ -ck_{z2n} \sin[k_{z2n}(z - 3l_z)], & z > L_z, \end{cases} \\ \frac{d\varphi_n}{dz} &= \begin{cases} b\bar{k}_{z1n} \cos[\bar{k}_{z1n}(z - l_z)], & z \leq L_z, \\ d\bar{k}_{z2n} \cos[\bar{k}_{z2n}(z - 3l_z)], & z > L_z, \end{cases} \end{aligned}$$

where

$$k_{zin} = \frac{\omega_n}{v_{Ai} \cos \alpha} \quad \text{and} \quad \bar{k}_{zin} = \frac{\varpi_n}{v_{Ai} \cos \alpha}.$$

Applying boundary and continuity conditions gives

$$\begin{aligned} \begin{pmatrix} \cos(k_{z1n}l_z) & -\cos(k_{z2n}l_z) \\ k_{z1n} \sin(k_{z1n}l_z) & k_{z2n} \sin(k_{z2n}l_z) \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} \sin(k_{z1n}l_z) & \sin(k_{z2n}l_z) \\ k_{z1n} \cos(k_{z1n}l_z) & -k_{z2n} \cos(k_{z2n}l_z) \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence, non-trivial solutions exist for

$$k_{z2n} \sin(k_{z2n}l_z) \cos(k_{z1n}l_z) + k_{z1n} \sin(k_{z1n}l_z) \cos(k_{z2n}l_z) = 0,$$

$$\bar{k}_{z2n} \sin(\bar{k}_{z2n}l_z) \cos(\bar{k}_{z1n}l_z) + \bar{k}_{z1n} \sin(\bar{k}_{z1n}l_z) \cos(\bar{k}_{z2n}l_z) = 0,$$

and we use these equations to calculate ω_n , ϖ_n , such that

$$\omega_1 < \omega_2 < \omega_3 \dots, \quad \varpi_1 < \varpi_2 < \varpi_3 \dots$$

Note that if $l_z = n\pi/k_{z1n}$ and $l_z = n\pi/\bar{k}_{z1n}$ the above equations simplify to

$$k_{z2n} \sin\left(n\pi \frac{k_{z2n}}{k_{z1n}}\right)(-1)^n = 0, \quad \bar{k}_{z1n} \sin\left(n\pi \frac{\bar{k}_{z2n}}{\bar{k}_{z1n}}\right)(-1)^n = 0.$$

Hence, for $v_{A1}/v_{A2} = m$, where m is an integer, a subset of the eigenvalues are given by

$$\omega_j = \varpi_j = j\pi \frac{v_{A1}}{l_z} \cos \alpha.$$

Note that if $l_z = (2n+1)\pi/(2k_{z1n})$ and $l_z = (2n+1)\pi/(2\bar{k}_{z1n})$ the above equations simplify to

$$k_{z1n}(-1)^n \cos\left(\frac{1}{2}(2n+1)\pi \frac{k_{z2n}}{k_{z1n}}\right) = 0, \quad \bar{k}_{z2n}(-1)^n \cos\left(\frac{1}{2}(2n+1)\pi \frac{\bar{k}_{z2n}}{\bar{k}_{z1n}}\right) = 0.$$

Hence, for $v_{A1}/v_{A2} = m$, where m is an odd integer, a subset of the eigenvalues are given by

$$\omega_j = \varpi_j = j\pi \frac{v_{A1}}{2l_z} \cos \alpha.$$

Hence,

$$\begin{aligned}\phi_n(z) &= A_n \begin{cases} a_n \cos[k_{z1n}(z - l_z)], & z \leq L_z, \\ c_n \cos[k_{z2n}(z - 3l_z)], & z > L_z, \end{cases} \\ \varphi_n(z) &= B_n \begin{cases} b_n \sin[\bar{k}_{z1n}(z - l_z)], & z \leq L_z, \\ d_n \sin[\bar{k}_{z2n}(z - 3l_z)], & z > L_z, \end{cases}\end{aligned}$$

where

$$\begin{aligned}a_n &= \begin{cases} \cos(k_{z2n}l_z), & \cos(k_{z1n}l_z) \neq 0 \\ k_{z2n} \sin(k_{z2n}l_z), & \cos(k_{z1n}l_z) = 0 \end{cases}, \\ c_n &= \begin{cases} \cos(k_{z1n}l_z), & \cos(k_{z1n}l_z) \neq 0 \\ -k_{z1n} \sin(k_{z1n}l_z), & \cos(k_{z1n}l_z) = 0 \end{cases}, \\ b_n &= \begin{cases} \sin(\bar{k}_{z2n}l_z), & \sin(\bar{k}_{z1n}l_z) \neq 0 \\ \bar{k}_{z2n} \cos(\bar{k}_{z2n}l_z), & \sin(\bar{k}_{z1n}l_z) = 0 \end{cases}, \\ d_n &= \begin{cases} -\sin(\bar{k}_{z1n}l_z), & \sin(\bar{k}_{z1n}l_z) \neq 0 \\ \bar{k}_{z1n} \cos(\bar{k}_{z1n}l_z), & \sin(\bar{k}_{z1n}l_z) = 0 \end{cases},\end{aligned}$$

We normalise such that $\langle \phi_n \phi_m \rangle = \langle \varphi_n \varphi_m \rangle = \delta_{nm}$, where

$$\langle \phi_n, \phi_m \rangle = \int_0^{2L_z} \frac{\phi_n(z) \phi_m(z)}{v_A^2(0, z)} dz,$$

and δ_{nm} is the Kronecker delta. Note that

$$\begin{aligned}\frac{\langle \phi_n, \phi_n \rangle}{A_n^2} &= \int_0^{2L_z} \frac{\phi_n^2(z)}{A_n^2 v_A^2(0, z)} dz \\ &= \frac{1}{v_{A1}^2} \int_0^{L_z} a_n^2 \cos^2[k_{z1n}(z - l_z)] dz + \frac{1}{v_{A2}^2} \int_{L_z}^{2L_z} c_n^2 \cos^2[k_{z2n}(z - 3l_z)] dz \\ &= \int_{-l_z}^{l_z} \frac{a_n^2}{v_{A1}^2} \cos^2(k_{z1n}z) + \frac{c_n^2}{v_{A2}^2} \cos^2(k_{z2n}z) dz \\ &= 2 \int_0^{l_z} \frac{a_n^2}{v_{A1}^2} \cos^2(k_{z1n}z) + \frac{c_n^2}{v_{A2}^2} \cos^2(k_{z2n}z) dz \\ &= \left\{ \frac{a_n^2}{2v_{A1}^2 k_{z1n}} [2k_{z1n}z + \sin(2k_{z1n}z)] + \frac{c_n^2}{2v_{A2}^2 k_{z2n}} [2k_{z2n}z + \sin(2k_{z2n}z)] \right\}_0^{l_z} \\ &= \frac{a_n^2}{2v_{A1}^2 k_{z1n}} [2k_{z1n}l_z + \sin(2k_{z1n}l_z)] + \frac{c_n^2}{2v_{A2}^2 k_{z2n}} [2k_{z2n}l_z + \sin(2k_{z2n}l_z)], \\ \frac{\langle \varphi_n, \varphi_n \rangle}{B_n^2} &= \frac{1}{2} \left\{ \frac{b_n^2}{v_{A1}^2 \bar{k}_{z1n}} [2\bar{k}_{z1n}z - \sin(2\bar{k}_{z1n}z)] + \frac{d_n^2}{v_{A2}^2 \bar{k}_{z2n}} [2\bar{k}_{z2n}z - \sin(2\bar{k}_{z2n}z)] \right\}_0^{l_z} \\ &= \frac{b_n^2}{2v_{A1}^2 \bar{k}_{z1n}} [2\bar{k}_{z1n}l_z - \sin(2\bar{k}_{z1n}l_z)] + \frac{d_n^2}{2v_{A2}^2 \bar{k}_{z2n}} [2\bar{k}_{z2n}l_z - \sin(2\bar{k}_{z2n}l_z)].\end{aligned}$$

Hence,

$$\begin{aligned}A_n &= \left(\frac{a_n^2}{2v_{A1}^2 k_{z1n}} [2k_{z1n}l_z + \sin(2k_{z1n}l_z)] + \frac{c_n^2}{2v_{A2}^2 k_{z2n}} [2k_{z2n}l_z + \sin(2k_{z2n}l_z)] \right)^{-1/2}, \\ B_n &= \left(\frac{b_n^2}{2v_{A1}^2 \bar{k}_{z1n}} [2\bar{k}_{z1n}l_z - \sin(2\bar{k}_{z1n}l_z)] + \frac{d_n^2}{2v_{A2}^2 \bar{k}_{z2n}} [2\bar{k}_{z2n}l_z - \sin(2\bar{k}_{z2n}l_z)] \right)^{-1/2}.\end{aligned}$$

Deriving the ODEs

Note that

$$\begin{aligned}\mathcal{L}(x, z) &= \nabla_{||}^2 + \frac{\omega_n^2}{v_A^2(0, z)} + \frac{\omega^2/\hat{v}_A^2(x) - \omega_n^2}{v_A^2(0, z)}, \\ \implies \mathcal{L}(x, z)\phi_n(z) &= [\omega^2/\hat{v}_A^2(x) - \omega_n^2] \frac{\phi_n(z)}{v_A^2(0, z)}, \\ \implies \mathcal{L}(x, z)\varphi_n(z) &= [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] \frac{\varphi_n(z)}{v_A^2(0, z)}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{du_{x0}^{(1)}}{dx}\phi_0(z) + \sum_{n=1}^{\infty} \frac{du_{xn}^{(1)}}{dx}\phi_n(z) + \frac{du_{xn}^{(2)}}{dx}\varphi_n(z) &= -[i\omega b_{||0}^{(1)} + \Delta_{\perp 0}^{(1)}]\phi_0(z) \\ &\quad - \sum_{n=1}^{\infty} [i\omega b_{||n}^{(1)}(x) + \Delta_{\perp n}^{(1)}(x)]\phi_n(z) \\ &\quad - \sum_{n=1}^{\infty} [i\omega b_{||n}^{(2)}(x) + \Delta_{\perp n}^{(2)}(x)]\varphi_n(z), \\ \frac{db_{||0}^{(1)}}{dx}\phi_0(z) + \sum_{n=1}^{\infty} \frac{db_{||n}^{(1)}}{dx}\phi_n(z) + \frac{db_{||n}^{(2)}}{dx}\varphi_n(z) &= -\frac{i}{\omega} [\omega^2/\hat{v}_A^2(x)] u_{xn}^{(1)}(x) \frac{\phi_0(z)}{v_A^2(0, z)} \\ &\quad - \frac{i}{\omega} \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \omega_n^2] u_{xn}^{(1)}(x) \frac{\phi_n(z)}{v_A^2(0, z)} \\ &\quad - \frac{i}{\omega} \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] u_{xn}^{(2)}(x) \frac{\varphi_n(z)}{v_A^2(0, z)}, \\ i\omega \nabla_{\perp}^2 \left[b_{||0}^{(1)}(x)\phi_0(z) + \sum_{n=1}^{\infty} b_{||n}^{(1)}(x)\phi_n(z) + b_{||n}^{(2)}(x)\varphi_n(z) \right] &= [\omega^2/\hat{v}_A^2(x)] \Delta_{\perp 0}^{(1)}(x) \frac{\phi_0(z)}{v_A^2(0, z)} \\ &\quad + \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \omega_n^2] \Delta_{\perp n}^{(1)}(x) \frac{\phi_n(z)}{v_A^2(0, z)} \\ &\quad + \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] \Delta_{\perp n}^{(2)}(x) \frac{\varphi_n(z)}{v_A^2(0, z)}.\end{aligned}$$

Taking the inner product of the above equations with ϕ_m then φ_m and weight function $1/v_A^2(0, z)$ gives

$$\begin{aligned}\boxed{\frac{du_{xm}^{(1)}}{dx} = -[i\omega b_{||n}^{(1)} + \Delta_{\perp n}^{(1)}]}, \\ \boxed{\frac{du_{xm}^{(2)}}{dx} = -[i\omega b_{||n}^{(2)} + \Delta_{\perp n}^{(2)}]}, \\ \frac{db_{||m}^{(1)}}{dx} = -\frac{i}{\omega} [\omega^2/\hat{v}_A^2(x)] u_{xm}^{(1)}(x) \left\langle \frac{\phi_0(z)}{v_A^2(0, z)}, \phi_m(z) \right\rangle \\ - \frac{i}{\omega} \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \omega_n^2] u_{xn}^{(1)}(x) \left\langle \frac{\phi_n(z)}{v_A^2(0, z)}, \phi_m(z) \right\rangle \\ - \frac{i}{\omega} \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] u_{xn}^{(2)}(x) \left\langle \frac{\varphi_n(z)}{v_A^2(0, z)}, \phi_m(z) \right\rangle,\end{aligned}$$

$$\begin{aligned}
\frac{db_{||m}^{(2)}}{dx} &= -\frac{i}{\omega} [\omega^2/\hat{v}_A^2(x)] u_{xn}^{(1)}(x) \left\langle \frac{\phi_0(z)}{v_A^2(0, z)}, \varphi_m(z) \right\rangle \\
&\quad - \frac{i}{\omega} \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \omega_n^2] u_{xn}^{(1)}(x) \left\langle \frac{\phi_n(z)}{v_A^2(0, z)}, \varphi_m(z) \right\rangle \\
&\quad - \frac{i}{\omega} \sum_{n=1}^{\infty} [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] u_{xn}^{(2)}(x) \left\langle \frac{\varphi_n(z)}{v_A^2(0, z)}, \varphi_m(z) \right\rangle, \\
\Delta_{\perp m}^{(1)}(x) &= \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2} b_{||0}^{(1)}(x) \langle v_A^2(0, z) \nabla_{\perp}^2 \phi_0(z), \phi_m(z) \rangle \\
&\quad + \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2} \sum_{n=1}^{\infty} b_{||n}^{(1)}(x) \langle v_A^2(0, z) \nabla_{\perp}^2 \phi_n(z), \phi_m(z) \rangle \\
&\quad + \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2} \sum_{n=1}^{\infty} b_{||n}^{(2)}(x) \langle v_A^2(0, z) \nabla_{\perp}^2 \varphi_n(z), \phi_m(z) \rangle, \\
\Delta_{\perp m}^{(2)}(x) &= \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2} b_{||0}^{(1)}(x) \langle v_A^2(0, z) \nabla_{\perp}^2 \phi_0(z), \varphi_m(z) \rangle \\
&\quad + \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2} \sum_{n=1}^{\infty} b_{||n}^{(1)}(x) \langle v_A^2(0, z) \nabla_{\perp}^2 \phi_n(z), \varphi_m(z) \rangle \\
&\quad + \frac{i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2} \sum_{n=1}^{\infty} b_{||n}^{(2)}(x) \langle v_A^2(0, z) \nabla_{\perp}^2 \varphi_n(z), \varphi_m(z) \rangle.
\end{aligned}$$

Note that

$$\begin{aligned}
\left\langle \frac{\varphi_n(z)}{v_A^2(0, z)}, \phi_m(z) \right\rangle &= \left\langle \frac{\phi_n(z)}{v_A^2(0, z)}, \varphi_m(z) \right\rangle = 0 \\
\nabla_{\perp}^2 &= \sin^2 \alpha \frac{\partial^2}{\partial z^2} - 2ik_{\perp} \sin \alpha \frac{\partial}{\partial z} - k_{\perp}^2, \\
\langle v_A^2(0, z) \nabla_{\perp}^2 \phi_n(z), \phi_m(z) \rangle &= \left\langle v_A^2(0, z) \left(\sin^2 \alpha \frac{\partial^2}{\partial z^2} - k_{\perp}^2 \right) \phi_n, \phi_m(z) \right\rangle \\
&= \left\langle -v_A^2(0, z) \left(\tan^2 \alpha \frac{\omega_n^2}{v_A^2(0, z)} + k_{\perp}^2 \right) \phi_n, \phi_m(z) \right\rangle \\
&= -\tan^2 \alpha \omega_n^2 \delta_{nm} - k_{\perp}^2 \langle v_A^2(0, z) \phi_n(z), \phi_m(z) \rangle, \\
\langle v_A^2(0, z) \nabla_{\perp}^2 \varphi_n(z), \varphi_m(z) \rangle &= -\tan^2 \alpha \varpi_n^2 \delta_{nm} - k_{\perp}^2 \langle v_A^2(0, z) \varphi_n(z), \phi_m(z) \rangle, \\
\langle v_A^2(0, z) \nabla_{\perp}^2 \varphi_n(z), \phi_m(z) \rangle &= -2ik_{\perp} \sin \alpha \left\langle v_A^2(0, z) \frac{\partial \varphi_n(z)}{\partial z}, \phi_m(z) \right\rangle, \\
\langle v_A^2(0, z) \nabla_{\perp}^2 \phi_n(z), \varphi_m(z) \rangle &= -2ik_{\perp} \sin \alpha \left\langle v_A^2(0, z) \frac{\partial \phi_n(z)}{\partial z}, \varphi_m(z) \right\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\boxed{\frac{db_{||m}^{(1)}}{dx} - \frac{i}{\omega} \sum_{n=0}^{\infty} [\omega^2/\hat{v}_A^2(x) - \omega_n^2] u_{xn}^{(1)}(x) \left\langle \frac{\phi_n(z)}{v_A^2(0, z)}, \phi_m(z) \right\rangle,} \\
&\boxed{\frac{db_{||m}^{(2)}}{dx} - \frac{i}{\omega} \sum_{n=0}^{\infty} [\omega^2/\hat{v}_A^2(x) - \varpi_n^2] u_{xn}^{(2)}(x) \left\langle \frac{\varphi_n(z)}{v_A^2(0, z)}, \varphi_m(z) \right\rangle,} \\
&\boxed{\Delta_{\perp m}^{(1)}(x) = \frac{-i\omega}{\omega^2/\hat{v}_A^2(x) - \omega_m^2} \left\{ \tan^2 \alpha \omega_m^2 b_{||m}^{(1)}(x) + \sum_{n=0}^{\infty} \left[k_{\perp}^2 b_{||n}^{(1)}(x) \langle v_A^2(0, z) \phi_n(z), \phi_m(z) \rangle + \right. \right.} \\
&\quad \left. \left. 2ik_{\perp} \sin \alpha b_{||n}^{(2)}(x) \left\langle v_A^2(0, z) \frac{d\varphi_n}{dz}, \phi_m(z) \right\rangle \right] \right\},}
\end{aligned}$$

$$\Delta_{\perp m}^{(2)}(x) = \frac{-i\omega}{\omega^2/\hat{v}_A^2(x) - \varpi_m^2} \left\{ \tan^2 \alpha \varpi_m^2 b_{||m}^{(2)}(x) + \sum_{n=0}^{\infty} \left[k_{\perp}^2 b_{||n}^{(2)}(x) \langle v_A^2(0, z) \varphi_n(z), \varphi_m(z) \rangle + 2ik_{\perp} \sin \alpha b_{||n}^{(1)}(x) \left\langle v_A^2(0, z) \frac{d\varphi_n}{dz}, \varphi_m(z) \right\rangle \right] \right\}.$$

Inner products

Let

$$\begin{aligned} I_{1nm} &= \left\langle \frac{\phi_n(z)}{v_A^2(0, z)}, \phi_m(z) \right\rangle, \\ I_{2nm} &= \left\langle \frac{\varphi_n(z)}{v_A^2(0, z)}, \varphi_m(z) \right\rangle, \\ I_{3nm} &= \langle v_A^2(0, z) \phi_n(z), \phi_m(z) \rangle, \\ I_{4nm} &= \langle v_A^2(0, z) \varphi_n(z), \varphi_m(z) \rangle, \\ I_{5nm} &= \left\langle v_A^2(0, z) \frac{d\varphi_n}{dz}, \phi_m(z) \right\rangle, \\ I_{6nm} &= \left\langle v_A^2(0, z) \frac{d\phi_n}{dz}, \varphi_m(z) \right\rangle. \end{aligned}$$

For $n \neq m$,

$$\begin{aligned} \frac{I_{1nm}}{A_n A_m} &= \int_0^{2L_z} \frac{\phi_n(z) \phi_m(z)}{A_n A_m v_A^4(0, z)} dz \\ &= \frac{a_n a_m}{v_{A1}^4} \int_0^{L_z} \cos[k_{z1n}(z - l_z)] \cos[k_{z1m}(z - l_z)] dz \\ &\quad + \frac{c_n c_m}{v_{A2}^4} \int_{L_z}^{2L_z} \cos[k_{z2n}(z - 3l_z)] \cos[k_{z2m}(z - 3l_z)] dz \\ &= 2 \int_0^{l_z} \frac{a_n a_m}{v_{A1}^4} \cos(k_{z1n}z) \cos(k_{z1m}z) + \frac{c_n c_m}{v_{A2}^4} \cos(k_{z2n}z) \cos(k_{z2m}z) dz \\ &= 2 \frac{a_n a_m}{v_{A1}^4} \left[\frac{k_{z1n} \sin(k_{z1n}z) \cos(k_{z1m}z) - k_{z1m} \sin(k_{z1m}z) \cos(k_{z1n}z)}{k_{z1n}^2 - k_{z1m}^2} \right]_0^{l_z} \\ &\quad + 2 \frac{c_n c_m}{v_{A2}^4} \left[\frac{k_{z2n} \sin(k_{z2n}z) \cos(k_{z2m}z) - k_{z2m} \sin(k_{z2m}z) \cos(k_{z2n}z)}{k_{z2n}^2 - k_{z2m}^2} \right]_0^{l_z} \\ &= 2 \frac{a_n a_m}{v_{A1}^4} \frac{k_{z1n} \sin(k_{z1n}l_z) \cos(k_{z1m}l_z) - k_{z1m} \sin(k_{z1m}l_z) \cos(k_{z1n}l_z)}{k_{z1n}^2 - k_{z1m}^2} \\ &\quad + 2 \frac{c_n c_m}{v_{A2}^4} \frac{k_{z2n} \sin(k_{z2n}l_z) \cos(k_{z2m}l_z) - k_{z2m} \sin(k_{z2m}l_z) \cos(k_{z2n}l_z)}{k_{z2n}^2 - k_{z2m}^2}, \end{aligned}$$

for $n = m$

$$\frac{I_{1nm}}{A_n A_m} = \frac{a_n^2}{2v_{A1}^4 k_{z1n}} [2k_{z1n}l_z + \sin(2k_{z1n}l_z)] + \frac{c_n^2}{2v_{A2}^4 k_{z2n}} [2k_{z2n}l_z + \sin(2k_{z2n}l_z)].$$

For $n \neq m$

$$\begin{aligned}
\frac{I_{2nm}}{B_n B_m} &= 2 \frac{b_n b_m}{v_{A1}^4} \left[\frac{\bar{k}_{z1m} \sin(\bar{k}_{z1n} z) \cos(\bar{k}_{z1m} z) - \bar{k}_{z1n} \sin(\bar{k}_{z1m} z) \cos(\bar{k}_{z1n} z)}{\bar{k}_{z1n}^2 - \bar{k}_{z1m}^2} \right]_0^{l_z} \\
&+ 2 \frac{d_n d_m}{v_{A2}^4} \left[\frac{\bar{k}_{z2m} \sin(\bar{k}_{z2n} z) \cos(\bar{k}_{z2m} z) - \bar{k}_{z2n} \sin(\bar{k}_{z2m} z) \cos(\bar{k}_{z2n} z)}{\bar{k}_{z2n}^2 - \bar{k}_{z2m}^2} \right]_0^{l_z} \\
&= 2 \frac{b_n b_m}{v_{A1}^4} \frac{\bar{k}_{z1m} \sin(\bar{k}_{z1n} l_z) \cos(\bar{k}_{z1m} l_z) - \bar{k}_{z1n} \sin(\bar{k}_{z1m} l_z) \cos(\bar{k}_{z1n} l_z)}{\bar{k}_{z1n}^2 - \bar{k}_{z1m}^2} \\
&+ 2 \frac{d_n d_m}{v_{A2}^4} \frac{\bar{k}_{z2m} \sin(\bar{k}_{z2n} l_z) \cos(\bar{k}_{z2m} l_z) - \bar{k}_{z2n} \sin(\bar{k}_{z2m} l_z) \cos(\bar{k}_{z2n} l_z)}{\bar{k}_{z2n}^2 - \bar{k}_{z2m}^2},
\end{aligned}$$

for $n = m$

$$\frac{I_{2nm}}{B_n B_m} = \frac{b_n^2}{2v_{A1}^4 \bar{k}_{z1n}} \left[2\bar{k}_{z1n} l_z - \sin(2\bar{k}_{z1n} l_z) \right] + \frac{d_n^2}{2v_{A2}^4 \bar{k}_{z2n}} \left[2\bar{k}_{z2n} l_z - \sin(2\bar{k}_{z2n} l_z) \right].$$

For $n \neq m$

$$\begin{aligned}
\frac{I_{3nm}}{A_n A_m} &= \frac{1}{A_n A_m} \int_0^{2L_z} \phi_n(z) \phi_m(z) dz \\
&= 2 \int_0^{l_z} a_n a_m \cos(k_{z1n} z) \cos(k_{z1m} z) + c_n c_m \cos(k_{z2n} z) \cos(k_{z2m} z) dz \\
&= 2a_n a_m \frac{k_{z1n} \sin(k_{z1n} l_z) \cos(k_{z1m} l_z) - k_{z1m} \sin(k_{z1m} l_z) \cos(k_{z1n} l_z)}{k_{z1n}^2 - k_{z1m}^2} \\
&+ 2c_n c_m \frac{k_{z2n} \sin(k_{z2n} l_z) \cos(k_{z2m} l_z) - k_{z2m} \sin(k_{z2m} l_z) \cos(k_{z2n} l_z)}{k_{z2n}^2 - k_{z2m}^2},
\end{aligned}$$

for $n = m$

$$\frac{I_{3nm}}{A_n A_m} = \frac{a_n^2}{2k_{z1n}} \left[2k_{z1n} l_z + \sin(2k_{z1n} l_z) \right] + \frac{c_n^2}{2k_{z2n}} \left[2k_{z2n} l_z + \sin(2k_{z2n} l_z) \right].$$

For $n \neq m$

$$\begin{aligned}
\frac{I_{4nm}}{B_n B_m} &= 2 \int_0^{l_z} b_n b_m \sin(\bar{k}_{z1n} z) \sin(\bar{k}_{z1m} z) + d_n d_m \sin(\bar{k}_{z2n} z) \sin(\bar{k}_{z2m} z) dz \\
&= 2a_n a_m \frac{k_{z1n} \sin(k_{z1n} l_z) \cos(k_{z1m} l_z) - k_{z1m} \sin(k_{z1m} l_z) \cos(k_{z1n} l_z)}{k_{z1n}^2 - k_{z1m}^2} \\
&= 2b_n b_m \frac{\bar{k}_{z1m} \sin(\bar{k}_{z1n} l_z) \cos(\bar{k}_{z1m} l_z) - \bar{k}_{z1n} \sin(\bar{k}_{z1m} l_z) \cos(\bar{k}_{z1n} l_z)}{\bar{k}_{z1n}^2 - \bar{k}_{z1m}^2} \\
&+ 2d_n d_m \frac{\bar{k}_{z2m} \sin(\bar{k}_{z2n} l_z) \cos(\bar{k}_{z2m} l_z) - \bar{k}_{z2n} \sin(\bar{k}_{z2m} l_z) \cos(\bar{k}_{z2n} l_z)}{\bar{k}_{z2n}^2 - \bar{k}_{z2m}^2},
\end{aligned}$$

for $n = m$

$$\frac{I_{4nm}}{B_n B_m} = \frac{b_n^2}{2\bar{k}_{z1n}} \left[2\bar{k}_{z1n} l_z - \sin(2\bar{k}_{z1n} l_z) \right] + \frac{d_n^2}{2\bar{k}_{z2n}} \left[2\bar{k}_{z2n} l_z - \sin(2\bar{k}_{z2n} l_z) \right].$$

For $\varpi_n \neq \omega_m$

$$\begin{aligned}
\frac{I_{5nm}}{B_n A_m} &= \frac{1}{B_n A_m} \int_0^{2L_z} \frac{d\varphi}{dz} \phi_m(z) dz \\
&= b_n a_m \bar{k}_{z1n} \int_0^{L_z} \cos[\bar{k}_{z1n}(z - l_z)] \cos[k_{z1m}(z - l_z)] dz \\
&\quad + d_n c_m \bar{k}_{z2n} \int_{L_z}^{2L_z} \cos[\bar{k}_{z2n}(z - 3l_z)] \cos[k_{z2m}(z - 3l_z)] dz \\
&= 2 \int_0^{l_z} b_n a_m \bar{k}_{z1n} \cos(\bar{k}_{z1n} z) \cos(k_{z1m} z) + d_n c_m \bar{k}_{z2n} \cos(\bar{k}_{z2n} z) \cos(k_{z2m} z) dz \\
&= 2b_n a_m \bar{k}_{z1n} \frac{\bar{k}_{z1n} \sin(\bar{k}_{z1n} l_z) \cos(k_{z1m} l_z) - k_{z1m} \sin(k_{z1m} l_z) \cos(\bar{k}_{z1n} l_z)}{\bar{k}_{z1n}^2 - k_{z1m}^2} \\
&\quad + 2d_n c_m \bar{k}_{z2n} \frac{\bar{k}_{z2n} \sin(\bar{k}_{z2n} l_z) \cos(k_{z2m} l_z) - k_{z2m} \sin(k_{z2m} l_z) \cos(\bar{k}_{z2n} l_z)}{\bar{k}_{z2n}^2 - k_{z2m}^2},
\end{aligned}$$

for $\varpi_n = \omega_m$

$$\frac{I_{5nm}}{B_n A_m} = \frac{b_n a_m}{2} [2k_{z1n} l_z + \sin(2k_{z1n} l_z)] + \frac{d_n c_m}{2} [2k_{z2n} l_z + \sin(2k_{z2n} l_z)].$$

For $\omega_n \neq \varpi_m$,

$$\begin{aligned}
\frac{I_{6nm}}{A_n B_m} &= \frac{1}{A_n B_m} \int_0^{2L_z} \frac{d\phi}{dz} \varphi_m(z) dz \\
&= -a_n b_m k_{z1n} \int_0^{L_z} \sin[k_{z1n}(z - l_z)] \sin[\bar{k}_{z1m}(z - l_z)] dz \\
&\quad - c_n d_m k_{z2n} \int_{L_z}^{2L_z} \sin[k_{z2n}(z - 3l_z)] \sin[\bar{k}_{z2m}(z - 3l_z)] dz \\
&= -2 \int_0^{l_z} a_n b_m k_{z1n} \sin(k_{z1n} z) \sin(\bar{k}_{z1m} z) + c_n d_m k_{z2n} \sin(k_{z2n} z) \sin(\bar{k}_{z2m} z) dz \\
&= -2a_n b_m k_{z1n} \frac{\bar{k}_{z1m} \sin(k_{z1n} l_z) \cos(\bar{k}_{z1m} l_z) - k_{z1n} \sin(\bar{k}_{z1m} l_z) \cos(k_{z1n} l_z)}{k_{z1n}^2 - \bar{k}_{z1m}^2} \\
&\quad - 2c_n d_m k_{z2n} \frac{\bar{k}_{z2m} \sin(k_{z2n} l_z) \cos(\bar{k}_{z2m} l_z) - k_{z2n} \sin(\bar{k}_{z2m} l_z) \cos(k_{z2n} l_z)}{k_{z2n}^2 - \bar{k}_{z2m}^2},
\end{aligned}$$

for $\omega_n = \varpi_m$

$$\frac{I_{6nm}}{A_n B_m} = -\frac{a_n b_m}{2} [2k_{z1n} l_z - \sin(2k_{z1n} l_z)] - \frac{c_n d_m}{2} [2k_{z2n} l_z - \sin(2k_{z2n} l_z)].$$

Analytic solution

Let $\omega = \omega_r + i\omega_i$, where $\omega_r = \omega_k$ and $\omega_i \ll \omega_r$, $k \in \mathbb{N}$. The singular solution is approximated by

$$\begin{aligned}
u_x(x, z) &= -\beta_0 \ln(x - ix_i) \nabla_{\perp} \phi_k(z) \\
&= -\beta_0 \ln(x - ix_i) \left[ik_{\perp} \phi_k(z) - \sin \alpha \frac{d\phi_k}{dz} \right].
\end{aligned}$$

$$u_{\perp}(x, z) = \frac{\beta_0}{x - ix_i} \phi_k(z),$$

$$\nabla_{\perp} b_{||}(x, z) = \frac{\beta_0 \mathcal{L}_1}{i\omega} \phi_k(z),$$

where we choose the φ_k component equal to zero, \mathcal{L}_1 is given by

$$\begin{aligned}\mathcal{L}_1 &= \left. \frac{\partial \mathcal{L}}{\partial x} \right|_{x=0} \\ &= -\frac{2\omega^2}{v_A^3(0, z)} \left. \frac{\partial v_A}{\partial x} \right|_{x=0} \\ &= -\frac{2\omega^2}{a_0 v_A^2(0, z)},\end{aligned}$$

and

$$\begin{aligned}x_i &= \frac{2\omega_i}{\omega_r} \left[\frac{\rho}{\partial \rho / \partial x} \right]_{x=0} \\ &= \frac{2\omega_i}{\omega_r} \left[\frac{1/v_A^2}{-2 \partial v_A / \partial x / v_A^3} \right]_{x=0} \\ &= -\frac{\omega_i}{\omega_r} \left[\frac{v_A}{\partial v_A / \partial x} \right]_{x=0}\end{aligned}$$

Hence,

$$\begin{aligned}u_{xn}^{(1)}(x) &= -ik_{\perp} \beta_0 \ln(x - ix_i) \delta_{nk}, \\ u_{xn}^{(2)}(x) &= \beta_0 \sin \alpha \ln(x - ix_i) \left\langle \frac{d\phi_k}{dz}, \varphi_n \right\rangle,\end{aligned}$$

where for $\omega_k \neq \varpi_n$

$$\begin{aligned}\frac{I_{7kn}}{A_k B_n} &= \left\langle \frac{d\phi_k}{dz}, \varphi_n \right\rangle \\ &= \frac{1}{A_k B_n} \int_0^{2L_z} \frac{d\phi_k}{dz} \frac{\varphi_m}{v_A^2(0, z)} dz \\ &= -a_k b_n \frac{k_{z1k}}{v_{A1}^2} \int_0^{L_z} \sin[k_{z1k}(z - l_z)] \sin[\bar{k}_{z1n}(z - l_z)] dz \\ &\quad - c_k d_n \frac{k_{z2k}}{v_{A2}^2} \int_{L_z}^{2L_z} \sin[k_{z2k}(z - 3l_z)] \sin[\bar{k}_{z2n}(z - 3l_z)] dz \\ &= -2 \int_0^{l_z} a_k b_n k_{z1k} \sin(k_{z1k} z) \sin(\bar{k}_{z1n} z) + c_k d_n k_{z2k} \sin(k_{z2k} z) \sin(\bar{k}_{z2n} z) dz \\ &= -2a_k b_n \frac{k_{z1k}}{v_{A1}^2} \frac{\bar{k}_{z1n} \sin(k_{z1k} l_z) \cos(\bar{k}_{z1n} l_z) - k_{z1k} \sin(\bar{k}_{z1n} l_z) \cos(k_{z1k} l_z)}{k_{z1k}^2 - \bar{k}_{z1n}^2} \\ &\quad - 2c_k d_n \frac{k_{z2k}}{v_{A2}^2} \frac{\bar{k}_{z2n} \sin(k_{z2n} l_z) \cos(\bar{k}_{z2n} l_z) - k_{z2n} \sin(\bar{k}_{z2n} l_z) \cos(k_{z2n} l_z)}{k_{z2k}^2 - \bar{k}_{z2n}^2},\end{aligned}$$

and for $\omega_k = \varpi_n$

$$\frac{I_{7nm}}{A_n B_m} = -\frac{a_n b_m}{2v_{A1}^2} [2k_{z1n} l_z - \sin(2k_{z1n} l_z)] - \frac{c_n d_m}{2v_{A2}^2} [2k_{z2n} l_z - \sin(2k_{z2n} l_z)].$$

For the case where $\alpha = 0$,

$$b_{||} = 2 \frac{\beta_0 \omega}{k_{\perp} a_0} \frac{\phi_k(z)}{v_A^2(0, z)},$$

hence,

$$\begin{aligned}b_{||n}^{(1)} &= 2 \frac{\beta_0 \omega}{k_{\perp} a_0} \left\langle \frac{\phi_k(z)}{v_A^2(0, z)}, \phi_n \right\rangle \\ &= 2 \frac{\beta_0 \omega}{k_{\perp} a_0} I_{1kn}.\end{aligned}$$

For $\alpha \neq 0$

$$\frac{db_{||}}{dz} - i \frac{k_{\perp}}{\sin \alpha} b_{||} = -\frac{2i\beta_0 \omega}{a_0 \sin \alpha} \frac{\phi_k}{v_A^2(0, z)}.$$

$$\begin{aligned} &\Rightarrow \frac{d}{dz} \left[\exp \left(-i \frac{k_{\perp} z}{\sin \alpha} \right) b_{\parallel} \right] = -\frac{2i\beta_0\omega}{a_0 \sin \alpha} \exp \left(-i \frac{k_{\perp} z}{\sin \alpha} \right) \frac{\phi_k}{v_A^2(0, z)}, \\ &\Rightarrow b_{\parallel} = -\frac{2i\beta_0\omega}{a_0 \sin \alpha} \exp \left(i \frac{k_{\perp} z}{\sin \alpha} \right) \left[\int_0^z \exp \left(-i \frac{k_{\perp} z'}{\sin \alpha} \right) \frac{\phi_k(z')}{v_A^2(0, z')} dz' + C \right] \end{aligned}$$

We require

$$b_{\parallel}(0) = -\frac{2i\beta_0\omega}{a_0 \sin \alpha} C,$$

to equal

$$b_{\parallel}(2L_z) = -\frac{2i\beta_0\omega}{a_0 \sin \alpha} \exp \left(2i \frac{k_{\perp} L_z}{\sin \alpha} \right) \left[\int_0^{2L_z} \exp \left(-i \frac{k_{\perp} z'}{\sin \alpha} \right) \frac{\phi_k(z')}{v_A^2(0, z')} dz' + C \right].$$

Solving for C gives

$$\left[\exp \left(-2i \frac{k_{\perp} L_z}{\sin \alpha} \right) - 1 \right] C = \int_0^{2L_z} \exp \left(-i \frac{k_{\perp} z'}{\sin \alpha} \right) \frac{\phi_k(z')}{v_A^2(0, z')} dz'.$$

Hence, we can solve for C provided $k_{\perp} L_z / \sin \alpha \neq n\pi$, $n \in \mathbb{Z}$. If $k_{\perp} L_z / \sin \alpha = n\pi$, then the periodic boundary conditions are automatically satisfied and C is a free parameter. Hence,

$$b_{\parallel n}^{(1)} = \langle b_{\parallel}(z), \phi_n(z) \rangle,$$

$$b_{\parallel n}^{(2)} = \langle b_{\parallel}(z), \varphi_n(z) \rangle.$$

Calculate u_{\perp}

Note that

$$\mathcal{L}u_{\perp} = i\omega \nabla_{\perp} b_{\parallel}.$$

Assume

$$u_{\perp} = \sum_{n=0}^{\infty} u_{\perp n}^{(1)}(x) \phi_n(z) + u_{\perp n}^{(2)}(x) \varphi_n(z)$$

Hence,

$$\sum_{n=0}^{\infty} u_{\perp n}^{(1)} [\omega^2 / \hat{v}_A^2(x) - \omega_n^2] \frac{\phi_n(z)}{v_A^2(0, z)} + u_{\perp n}^{(2)} [\omega^2 / \hat{v}_A^2(x) - \varpi_n^2] \frac{\varphi_n(z)}{v_A^2(0, z)} = i\omega \left[ik_{\perp} - \sin \alpha \frac{\partial}{\partial z} \right] b_{\parallel},$$

therefore,

$$\begin{aligned} u_{\perp m}^{(1)} &= \frac{i\omega}{\omega^2 / \hat{v}_A^2(x) - \omega_m^2} \sum_{n=0}^{\infty} \left\{ ik_{\perp} b_{\parallel n}^{(1)} \langle v_A^2(0, z) \phi_n, \phi_m \rangle - \right. \\ &\quad \left. \sin \alpha b_{\parallel n}^{(2)} \langle v_A^2(0, z) \frac{d\varphi_n}{dz}, \phi_m \rangle \right\}, \\ u_{\perp m}^{(2)} &= \frac{i\omega}{\omega^2 / \hat{v}_A^2(x) - \varpi_m^2} \sum_{n=0}^{\infty} \left\{ ik_{\perp} b_{\parallel n}^{(2)} \langle v_A^2(0, z) \varphi_n, \varphi_m \rangle - \right. \\ &\quad \left. \sin \alpha b_{\parallel n}^{(1)} \langle v_A^2(0, z) \frac{d\phi_n}{dz}, \varphi_m \rangle \right\}. \end{aligned}$$

$$\mathcal{L} \nabla_{\perp} u_{\perp} = i\omega \nabla_{\perp}^2 b_{\parallel},$$

$$\begin{aligned} \nabla_{\perp}^2 &= -k_{\perp}^2 - 2ik_{\perp} \frac{\partial}{\partial z} + \sin^2 \alpha \frac{\partial^2}{\partial z^2} \\ &= -k_{\perp}^2 - 2ik_{\perp} \frac{\partial}{\partial z} + \tan^2 \alpha \nabla_{\parallel}^2, \\ &= -k_{\perp}^2 - 2ik_{\perp} \frac{\partial}{\partial z} + \tan^2 \alpha \left[\nabla_{\parallel}^2 + \frac{\omega_n^2}{v_A^2(0, z)} - \frac{\omega_n^2}{v_A^2(0, z)} \right], \\ &= -k_{\perp}^2 - 2ik_{\perp} \frac{\partial}{\partial z} + \tan^2 \alpha \left[\nabla_{\parallel}^2 + \frac{\varpi_n^2}{v_A^2(0, z)} - \frac{\varpi_n^2}{v_A^2(0, z)} \right], \end{aligned}$$

Hence,

$$\begin{aligned}
\nabla_{\perp}^2 \phi_n &= \left\{ -k_{\perp}^2 - 2ik_{\perp} \frac{\partial}{\partial z} - \tan^2 \alpha \frac{\omega_n^2}{v_A^2(0, z)} \right\} \phi_n, \\
\nabla_{\perp}^2 \varphi_n &= \left\{ -k_{\perp}^2 - 2ik_{\perp} \frac{\partial}{\partial z} - \tan^2 \alpha \frac{\varpi_n^2}{v_A^2(0, z)} \right\} \varphi_n. \\
\sum_{n=0}^{\infty} \Delta_{\perp n}^{(1)} [\omega^2 / \hat{v}_A^2(x) - \omega_n^2] \frac{\phi_n(z)}{v_A^2(0, z)} + \Delta_{\perp n}^{(1)} [\omega^2 / \hat{v}_A^2(x) - \varpi_n^2] \frac{\varphi_n(z)}{v_A^2(0, z)} = \\
- i\omega \sum_{n=0}^{\infty} b_{||n}^{(1)} \left\{ k_{\perp}^2 + 2ik_{\perp} \frac{\partial}{\partial z} + \tan^2 \alpha \frac{\omega_n^2}{v_A^2(0, z)} \right\} \phi_n + b_{||n}^{(2)} \left\{ k_{\perp}^2 + 2ik_{\perp} \frac{\partial}{\partial z} + \tan^2 \alpha \frac{\varpi_n^2}{v_A^2(0, z)} \right\} \varphi_n, \\
\Delta_{\perp m}^{(1)} &= \frac{-i\omega}{\omega^2 / \hat{v}_A^2(x) - \omega_m^2} \left\{ \tan^2 \alpha \omega_n^2 \right.
\end{aligned}$$

Arrays

Let

$$\begin{aligned}
\mathbf{u}_x^{(1)} &= (u_{x0}^{(1)}, u_{x1}^{(1)}, \dots, u_{xN}^{(1)}) \\
\mathbf{u}_x^{(2)} &= (u_{x0}^{(2)}, u_{x1}^{(2)}, \dots, u_{xN}^{(2)}) \\
\mathbf{b}_{||}^{(1)} &= (b_{||0}^{(1)}, b_{||1}^{(1)}, \dots, b_{||N}^{(1)}) \\
\mathbf{b}_{||}^{(2)} &= (b_{||0}^{(2)}, b_{||1}^{(2)}, \dots, b_{||N}^{(2)}) \\
\Delta_{\perp}^{(1)} &= (\Delta_{\perp 0}^{(1)}, \Delta_{\perp 1}^{(1)}, \dots, \Delta_{\perp N}^{(1)}) \\
\Delta_{\perp}^{(2)} &= (\Delta_{\perp 0}^{(2)}, \Delta_{\perp 1}^{(2)}, \dots, \Delta_{\perp N}^{(2)}) \\
\mathbf{k}_{z1} &= (k_{z1,0}, k_{z1,1}, \dots, k_{z1,N}) \\
\mathbf{k}_{z2} &= (k_{z2,0}, k_{z2,1}, \dots, k_{z2,N}) \\
\bar{\mathbf{k}}_{z1} &= (\bar{k}_{z1,0}, \bar{k}_{z1,1}, \dots, \bar{k}_{z1,N}) \\
\bar{\mathbf{k}}_{z2} &= (\bar{k}_{z2,0}, \bar{k}_{z2,1}, \dots, \bar{k}_{z2,N}) \\
\mathbf{U}_x^{(1)} &= (\mathbf{u}_x^{(1)T}, \mathbf{u}_x^{(1)T}, \dots, \mathbf{u}_x^{(1)T}) \\
\mathbf{U}_x^{(2)} &= (\mathbf{u}_x^{(2)T}, \mathbf{u}_x^{(2)T}, \dots, \mathbf{u}_x^{(2)T}) \\
\mathbf{B}_{||}^{(1)} &= (\mathbf{B}_{||0}^{(1T)}, \mathbf{B}_{||1}^{(1T)}, \dots, \mathbf{B}_{||N}^{(1T)}) \\
\mathbf{B}_{||}^{(2)} &= (\mathbf{B}_{||0}^{(2T)}, \mathbf{B}_{||1}^{(2T)}, \dots, \mathbf{B}_{||N}^{(2T)}) \\
\mathbf{K}_{z1} &= (\mathbf{k}_{z1}^T, \mathbf{k}_{z1}^T, \dots, \mathbf{k}_{z1}^T) \\
\mathbf{K}_{z2} &= (\mathbf{k}_{z2}^T, \mathbf{k}_{z2}^T, \dots, \mathbf{k}_{z2}^T) \\
\bar{\mathbf{K}}_{z1} &= (\bar{\mathbf{k}}_{z1}^T, \bar{\mathbf{k}}_{z1}^T, \dots, \bar{\mathbf{k}}_{z1}^T) \\
\bar{\mathbf{K}}_{z1} &= (\bar{\mathbf{k}}_{z1}^T, \bar{\mathbf{k}}_{z1}^T, \dots, \bar{\mathbf{k}}_{z1}^T)
\end{aligned}$$

Total energy