Analytic magnetic field around picture-frame coils

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May 2024

1 Magnetic field around a vertical wire

Consider a vertical wire at the origin with current pointing the in the $\hat{\mathbf{z}}$ direction and extends from $z = z_{min}$ to $z = z_{max}$. Using a formula which is derived in for example:

`https://www.youtube.com/watch?v=BlIEavkDUF8&t=400s`

we know that the magnetic field around this wire is given by:

$$\mathbf{B}_{vwire}(R, \phi, z) = \frac{\mu_0 I}{4\pi} \frac{1}{R} \left[\sin(\theta_1) + \sin(\theta_2)\right] \hat{\phi}.$$

Using

$$\sin(\theta_1) = \frac{z_{max} - z}{\sqrt{R^2 + (z_{max} - z)^2}},$$

$$\sin(\theta_2) = \frac{z - z_{min}}{\sqrt{R^2 + (z - z_{min})^2}},$$

gives

$$\mathbf{B}_{vwire}(R, \phi, z) = \frac{\mu_0 I}{4\pi} \frac{1}{R} \left[\frac{z_{max} - z}{\sqrt{R^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{R^2 + (z - z_{min})^2}}\right] \hat{\phi}.$$

Using

$$R = \sqrt{x^2 + y^2},$$

$$\hat{\phi} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}},$$

we can convert this to Cartesian coordinates to give

$$\mathbf{B}_{vwire}(x, y, z; z_{min}, z_{max}, I) = \frac{\mu_0 I}{4\pi} \frac{1}{x^2 + y^2} \left[\frac{z_{max} - z}{\sqrt{x^2 + y^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{x^2 + y^2 + (z - z_{min})^2}}\right] (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}).$$

Let

$$A_{vwire}(x, y, z; z_{min}, z_{max}, I) = \frac{\mu_0 I}{4\pi} \frac{1}{x^2 + y^2} \left[\frac{z_{max} - z}{\sqrt{x^2 + y^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{x^2 + y^2 + (z - z_{min})^2}}\right].$$

In vector notation $\mathbf{B}_{vwire}$ is given by

$$\boxed{\mathbf{B}_{vwire}(x, y, z; z_{min}, z_{max}, I) = A_{vwire}(x, y, z; z_{min}, z_{max}, I) \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}}$$

2 Magnetic field around a horizontal wire

We can use the formula for the magnetic field around a vertical wire to calculate the magnetic field around a wire that points in the $\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$ direction and extends from $R = R_{inner}$ to R_{outer} at $z = z_0$. We do this by taking a vertical wire which points in the z-direction and is centred at the then apply a 90 degree rotation around the y-axis, and then a ϕ_k rotation around the z-axis.

Using the formula for the rotation matrix about the y and z axes

$$\mathrm{R}_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},$$

$$\mathrm{R}_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Our full rotation matrix is given by

$$\begin{aligned}
\mathrm{R}_k &= \mathrm{R}_z(\phi_k)\mathrm{R}_y(\pi/2) \\
&= \begin{pmatrix} \cos(\phi_k) & -\sin(\phi_k) & 0 \\ \sin(\phi_k) & \cos(\phi_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sin(\phi_k) & \cos(\phi_k) \\ 0 & \cos(\phi_k) & \sin(\phi_k) \\ -1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The inverse rotation matrix is given by

$$\begin{aligned}
\mathrm{R}_k^{-1} &= \mathrm{R}_y(-\pi/2)\mathrm{R}_z(-\phi_k) \\
&= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\phi_k) & \sin(\phi_k) & 0 \\ -\sin(\phi_k) & \cos(\phi_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -1 \\ -\sin(\phi_k) & \cos(\phi_k) & 0 \\ \cos(\phi_k) & \sin(\phi_k) & 1 \end{pmatrix}.
\end{aligned}$$

Let

$$\begin{aligned}
\begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} &= \mathrm{R}_k^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} -z \\ y\cos(\phi_k) - x\sin(\phi_k) \\ x\cos(\phi_k) + y\sin(\phi_k) \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathrm{R}_k \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \\
&= \begin{pmatrix} z_k\cos(\phi_k) - y_k\sin(\phi_k) \\ y_k\cos(\phi_k) + z_k\sin(\phi_k) \\ -x_k \end{pmatrix}.
\end{aligned}$$

Hence, the formula for the magnetic field due to the upper wires in the picture-frame coils is given by

$$\mathbf{B}_{upper,k}(x, y, z) = \mathrm{R}_k\mathbf{B}_{vwire}(x_k + h/2, y_k, z_k; R_{inner}, R_{outer}, I),$$

where h is the height of the coils.

3 Magnetic field in picture-frame coils

The current design has $N_{coil} = 16$ picture-frame TF coils. The k^{th} coil is composed of four wires:

1. A vertical wire at $R = R_{outer}$, $\phi = \phi_k = 2\pi k/N_{coil}$ with current in the negative $\hat{\mathbf{z}}$ direction and extends from $z = -h/2$ to $z = +h/2$, where h is the height of the TF coil.

2. A vertical wire at $R = R_{inner}$, $\phi = \phi_k$ with current in the positive $\hat{\mathbf{z}}$ direction and extends from $z = -h/2$ to $z = +h/2$.

3. A horizontal wire with current which points in the $\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$ direction and extends from $R = R_{inner}$ to R_{outer} at $z = h/2$.

4. A horizontal wire with current that points in the negative $\hat{\mathbf{R}}_k$ direction and extends from $R = R_{inner}$ to R_{outer} at $z = -h/2$.

We can model the magnetic field from the vertical wires with

$$\mathbf{B}_{inner,k}(x, y, z) = \mathbf{B}_{vwire}(x - R_{inner}\cos\phi_k, y - R_{inner}\sin\phi_k, z; -h/2, h/2, I),$$

$$\mathbf{B}_{outer,k}(x, y, z) = \mathbf{B}_{vwire}(x - R_{outer}\cos\phi_k, y - R_{outer}\sin\phi_k, z; -h/2, h/2, -I).$$

We can model the horizontal wires using:

$$\mathbf{B}_{upper,k}(x, y, z) = \mathrm{R}_k\mathbf{B}_{vwire}(x_k + h/2, y_k, z_k; R_{inner}, R_{outer}, I),$$

$$\mathbf{B}_{lower,k}(x, y, z) = \mathrm{R}_k\mathbf{B}_{vwire}(x_k - h/2, y_k, z_k; R_{inner}, R_{outer}, -I).$$

Hence, the full magnetic field from all the TF coils is given by

$$\boxed{\mathbf{B}(x, y, z) = \sum_{k=0}^{N_{coil}-1} \mathbf{B}_{inner,k} + \mathbf{B}_{outer,k} + \mathbf{B}_{lower,k} + \mathbf{B}_{upper,k}.}$$