

# Analytic magnetic field around picture-frame coils

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## 1 Magnetic field around a vertical wire

Consider a vertical wire at the origin with current pointing the in the  $\hat{\mathbf{z}}$  direction and extends from  $z = z_{min}$  to  $z = z_{max}$ . Using a formula which is derived in for example:

<https://www.youtube.com/watch?v=BlIEavkDUF8&t=400s>

we know that the magnetic field around this wire is given by:

$$\mathbf{B}_{wire}(R, \phi, z) = \frac{\mu_0 I}{4\pi} \frac{1}{R} [\sin(\theta_1) + \sin(\theta_2)] \hat{\phi}.$$

Using

$$\sin(\theta_1) = \frac{z_{max} - z}{\sqrt{R^2 + (z_{max} - z)^2}},$$

$$\sin(\theta_2) = \frac{z - z_{min}}{\sqrt{R^2 + (z - z_{min})^2}},$$

gives

$$\mathbf{B}_{wire}(R, \phi, z) = \frac{\mu_0 I}{4\pi} \frac{1}{R} \left[ \frac{z_{max} - z}{\sqrt{R^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{R^2 + (z - z_{min})^2}} \right] \hat{\phi}.$$

Using

$$R = \sqrt{x^2 + y^2},$$

$$\hat{\phi} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}},$$

we can convert this to Cartesian coordinates to give

$$\mathbf{B}_{wire}(x, y, z; z_{min}, z_{max}, I) = \frac{\mu_0 I}{4\pi} \frac{1}{x^2 + y^2} \left[ \frac{z_{max} - z}{\sqrt{x^2 + y^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{x^2 + y^2 + (z - z_{min})^2}} \right] (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}).$$

Let

$$A_{wire}(x, y, z; z_{min}, z_{max}, I) = \frac{\mu_0 I}{4\pi} \frac{1}{x^2 + y^2} \left[ \frac{z_{max} - z}{\sqrt{x^2 + y^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{x^2 + y^2 + (z - z_{min})^2}} \right].$$

In vector notation  $\mathbf{B}_{wire}$  is given by

$$\boxed{\mathbf{B}_{wire}(x, y, z; z_{min}, z_{max}, I) = A_{wire}(x, y, z; z_{min}, z_{max}, I) \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}}$$

## 2 Magnetic field around a horizontal wire

We can use the formula for the magnetic field around a vertical wire to calculate the magnetic field around a wire that points in the  $\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$  direction and extends from  $R = R_{inner}$  to  $R_{outer}$  at  $z = z_0$ . We do this by taking a vertical wire which points in the  $z$ -direction and is centred at the then apply a 90 degree rotation around the  $y$ -axis, and then a  $\phi_k$  rotation around the  $z$ -axis.

Using the formula for the rotation matrix about the  $y$  and  $z$  axes

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},$$

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Our full rotation matrix is given by

$$\begin{aligned} \mathbf{R}_k &= \mathbf{R}_z(\phi_k)\mathbf{R}_y(\pi/2) \\ &= \begin{pmatrix} \cos(\phi_k) & -\sin(\phi_k) & 0 \\ \sin(\phi_k) & \cos(\phi_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\sin(\phi_k) & \cos(\phi_k) \\ 0 & \cos(\phi_k) & \sin(\phi_k) \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The inverse rotation matrix is given by

$$\begin{aligned} \mathbf{R}_k^{-1} &= \mathbf{R}_y(-\pi/2)\mathbf{R}_z(-\phi_k) \\ &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\phi_k) & \sin(\phi_k) & 0 \\ -\sin(\phi_k) & \cos(\phi_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1 \\ -\sin(\phi_k) & \cos(\phi_k) & 0 \\ \cos(\phi_k) & \sin(\phi_k) & 1 \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned} \begin{pmatrix} x'_k \\ y'_k \\ z'_k \end{pmatrix} &= \mathbf{R}_k^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} -z \\ y \cos(\phi_k) - x \sin(\phi_k) \\ x \cos(\phi_k) + y \sin(\phi_k) \end{pmatrix}. \\ \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \mathbf{R}_k \begin{pmatrix} x'_k \\ y'_k \\ z'_k \end{pmatrix} \\ &= \begin{pmatrix} z'_k \cos(\phi_k) - y'_k \sin(\phi_k) \\ y'_k \cos(\phi_k) + z'_k \sin(\phi_k) \\ -x'_k \end{pmatrix}. \end{aligned}$$

Hence, the formula for the magnetic field due to the upper wires in the picture-frame coils is given by

$$\mathbf{B}_{upper,k}(x, y, z) = \mathbf{R}_k \mathbf{B}_{vwire}(x'_k + h'/2, y'_k, z'_k; R_{inner}, R_{outer}, I),$$

where  $h$  is the height of the coils.

### 3 Magnetic field in picture-frame coils

The current design has  $N_{coil} = 16$  picture-frame TF coils. The  $k^{\text{th}}$  coil is composed of four wires:

1. A vertical wire at  $R = R_{outer}$ ,  $\phi = \phi_k = 2\pi k/N_{coil}$  with current in the negative  $\hat{\mathbf{z}}$  direction and extends from  $z = -h/2$  to  $z = +h/2$ , where  $h$  is the height of the TF coil.
2. A vertical wire at  $R = R_{inner}$ ,  $\phi = \phi_k$  with current in the positive  $\hat{\mathbf{z}}$  direction and extends from  $z = -h/2$  to  $z = +h/2$ .
3. A horizontal wire with current which points in the  $\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$  direction and extends from  $R = R_{inner}$  to  $R_{outer}$  at  $z = h/2$ .
4. A horizontal wire with current that points in the negative  $\hat{\mathbf{R}}_k$  direction and extends from  $R = R_{inner}$  to  $R_{outer}$  at  $z = -h/2$ .

We can model the magnetic field from the vertical wires with

$$\mathbf{B}_{inner,k}(x, y, z) = \mathbf{B}_{wire}(x - R_{inner} \cos \phi_k, y - R_{inner} \sin \phi_k, z; -h/2, h/2, I),$$

$$\mathbf{B}_{outer,k}(x, y, z) = \mathbf{B}_{wire}(x - R_{outer} \cos \phi_k, y - R_{outer} \sin \phi_k, z; -h/2, h/2, -I).$$

We can model the horizontal wires using:

$$\mathbf{B}_{upper,k}(x, y, z) = R_k \mathbf{B}_{wire}(x'_k + h'/2, y'_k, z'_k; R_{inner}, R_{outer}, I),$$

$$\mathbf{B}_{lower,k}(x, y, z) = R_k \mathbf{B}_{wire}(x'_k - h'/2, y'_k, z'_k; R_{inner}, R_{outer}, -I).$$

Hence, the full magnetic field from all the TF coils is given by

$$\mathbf{B}(x, y, z) = \sum_{k=0}^{N_{coil}-1} \mathbf{B}_{inner,k} + \mathbf{B}_{outer,k} + \mathbf{B}_{lower,k} + \mathbf{B}_{upper,k}.$$