Slowing down probability density function

Alexander Prokopyszyn

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1 Slowing down PDF

Using the equation in Fig. 5.4.3 of [Wesson, 2004], namely

$$f(E) = \frac{\tau_{se} S}{2E[1 + (E_c/E)^{3/2}]},$$

we can conclude that the PDF of the slowing down distribution is given by

$$f(E; E_c, E_\alpha) = \begin{cases} \frac{A}{E[1+(E_c/E)^{3/2}]}, & E \leq E_\alpha, \\ 0, & E > E_\alpha, \end{cases}$$

where A is a dimensionless normalising constant which ensures

$$\int_0^\infty f(E; E_c, E_\alpha) dE = 1.$$

Hence

$$\begin{aligned} \frac{1}{A} &= \int_0^{E_\alpha} \frac{dE}{E[1 + (E_c/E)^{3/2}]} \\ &= \ln\left[\frac{(E_\alpha^{3/2} + E_c^{3/2})^{2/3}}{E_c}\right], \\ \implies A &= \left\{\ln\left[\frac{(E_\alpha^{3/2} + E_c^{3/2})^{2/3}}{E_c}\right]\right\}^{-1}. \end{aligned}$$

2 Slowing down PDF mean

Now we will calculate the mean,

$$\begin{aligned}
\mu &= \int_0^\infty E f(E; E_c, E_\alpha) dE \\
&= A \int_0^{E_\alpha} \frac{dE}{1 + (E_c/E)^{3/2}} \\
&= A E_c \int_0^{E_\alpha/E_c} \frac{dx}{1 + x^{-3/2}} \\
&= A E_c \left\{ \frac{E_\alpha}{E_c} {}_2F_1 \left[-\frac{2}{3}, 1; \frac{1}{3}; -\left(\frac{E_c}{E_\alpha}\right)^{3/2} \right] - \frac{4\pi}{3\sqrt{3}} \right\}.
\end{aligned}$$

3 Calculate E_c

E_c is the critical energy where the thermonuclear alphas go from colliding mostly with the electrons to colliding mostly with the ions. Equation 5.4.9 of [Wesson, 2004] states that the critical energy is given by

$$E_c = 14.8 \frac{A_\alpha}{A_i^{2/3}} T_e,$$

where A_α and A_i are the atomic masses of the alpha particles and the background ions. T_e is the electron temperature. Substituting $A_\alpha = 4$ gives

$$E_c = 59.2 \frac{T_e}{A_i^{2/3}}.$$

4 Maxwell-Boltzmann PDF

The PDF for the Maxwell-Boltzmann distribution is given by

$$f_{MB}(E) = \sqrt{\frac{4E}{\pi T^3}} \exp\left(-\frac{E}{T}\right).$$

Now we need to choose a T such that

$$\int_0^\infty E f_{MB}(E) dE = \frac{3}{2} T = \mu,$$

$$\implies T = \frac{2}{3}\mu.$$

References

John Wesson. *Tokamaks.* Oxford University Press, Oxford, UK, 3rd edition, 2004.