Analytical Approximations of Vacuum Magnetic Fields in Tokamaks

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1 Magnetic field produced by a vertical wire centred at the origin

The magnetic field produce by a vertical wire, centred at the origin, with current, I_0 which points in the $\hat{\mathbf{z}}$ direction and extends from $-z_{min}, z_{max}$ is given by the following Biot-Savart law expression:

$$\begin{aligned}\mathbf{B}_v(R,\phi,z) &= \frac{\mu_0 I_0}{4\pi}\int_{z_{min}}^{z_{max}} \frac{\mathbf{dh}' \times (R\hat{\mathbf{R}} + (z-z')\hat{\mathbf{z}})}{(R^2+(z-z')^2)^{3/2}} \\ &= \frac{\mu_0 I_0}{4\pi}\int_{z_{min}}^{z_{max}} \frac{R dh' \hat{\phi}}{(R^2+(z-z')^2)^{3/2}}.\end{aligned}$$

Let

$$\begin{aligned}\cos\theta &= \frac{R}{\sqrt{R^2+(z-z')^2}}, \\ \implies \sin\theta &= \frac{z-z'}{\sqrt{R^2+(z-z')^2}}, \\ \implies \tan\theta &= \frac{z-z'}{R}, \\ \implies z' &= z - R\tan\theta, \\ \implies dz' &= -R\sec^2\theta d\theta.\end{aligned}$$

Hence

$$\boxed{\begin{aligned}\mathbf{B}_v(R,\phi,z) &= -\frac{\mu_0 I_0}{4\pi R}\int_{z_{min}}^{z_{max}} \cos\theta d\theta \hat{\phi} \\ &= \frac{\mu_0 I_0}{4\pi R}[-\sin\theta]_{z_{min}}^{z_{max}}\hat{\phi} \\ &= \frac{\mu_0 I_0}{4\pi R}\left[\frac{z_{max}-z}{\sqrt{R^2+(z_{max}-z)^2}} + \frac{z-z_{min}}{\sqrt{R^2+(z-z_{min})^2}}\right]\hat{\phi}.\end{aligned}}$$

2 Vacuum Field from Vertical Wires in Picture-Frame TF Coils

Lets rewrite the above equation in cartesian coordinates:

$$\mathbf{B}_v(x,y,z) = \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max}-z}{\sqrt{x^2+y^2+(z_{max}-z)^2}} + \frac{z-z_{min}}{\sqrt{x^2+y^2+(z-z_{min})^2}}\right]\frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2+y^2}.$$

Hence the magentic field given by the k^{th} vertical wire in the picture-frame TF coils at $x = R_0\cos(\phi_k)$ and $y = R_0\sin(\phi_k)$ is given by

$$\begin{aligned}
&\mathbf{B}_{v,k}(x - R_0\cos(\phi_k), y - R_0\sin(\phi_k), z) = \\
&\frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max} - z}{\sqrt{(x - R_0\cos\phi_k)^2 + (y - R_0\sin\phi_k)^2 + (z_{max} - z)^2}} + \right. \\
&\left.\frac{z - z_{min}}{\sqrt{(x - R_0\cos\phi_k)^2 + (y - R_0\sin\phi_k)^2 + (z - z_{min})^2}}\right] \times \\
&\frac{-(y - R_0\sin\phi_k)\hat{\mathbf{x}} + (x - R_0\cos\phi_k)\hat{\mathbf{y}}}{(x - R_0\cos\phi_k)^2 + (y - R_0\sin\phi_k)^2},
\end{aligned}$$

where

$$\phi_k = 2\pi\frac{k}{N},$$

and N is the number of TF coils. Taking the dot product of the above equation for the magneitc field with $\hat{\mathbf{R}}$ and $\hat{\phi}$ and writing in clyindrical coordinates, we get

$$\begin{aligned}
&B_{R,v,k}(R, \phi, z) \\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max} - z}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z_{max} - z)^2}} + \right. \\
&\left.\frac{z - z_{min}}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z - z_{min})^2}}\right] \times \\
&\frac{-(R\sin\phi - R_0\sin\phi_k)\cos\phi + (R\cos\phi - R_0\cos\phi_k)\sin\phi}{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2} \\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max} - z}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z_{max} - z)^2}} + \right. \\
&\left.\frac{z - z_{min}}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z - z_{min})^2}}\right] \times \\
&\frac{R_0\sin\phi_k\cos\phi - R_0\cos\phi_k\sin\phi}{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2}
\end{aligned}$$

$$\begin{aligned}
&B_{\phi,v,k}(R, \phi, z) \\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max} - z}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z_{max} - z)^2}} + \right. \\
&\left.\frac{z - z_{min}}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z - z_{min})^2}}\right] \times \\
&\frac{(R\sin\phi - R_0\sin\phi_k)\sin\phi + (R\cos\phi - R_0\cos\phi_k)\cos\phi}{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2} \\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max} - z}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z_{max} - z)^2}} + \right. \\
&\left.\frac{z - z_{min}}{\sqrt{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2 + (z - z_{min})^2}}\right] \times \\
&\frac{R - R_0\sin\phi_k\sin\phi - R_0\cos\phi_k\cos\phi}{(R\cos\phi - R_0\cos\phi_k)^2 + (R\sin\phi - R_0\sin\phi_k)^2}
\end{aligned}$$

For $k=0$ we have

$$\begin{aligned}
&B_{R,v,0}(R,\phi,z)\\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max}-z}{\sqrt{R^2+R_0^2-2R_0R\cos\phi+(z_{max}-z)^2}}+\right.\\
&\qquad\left.\frac{z-z_{min}}{\sqrt{R^2+R_0^2-2R_0R\cos\phi+(z-z_{min})^2}}\right]\times\\
&\qquad\frac{-R_0\sin\phi}{R^2+R_0^2-2R_0R\cos\phi}\\
&B_{\phi,v,0}(R,\phi,z)\\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max}-z}{\sqrt{R^2+R_0^2-2R_0R\cos\phi+(z_{max}-z)^2}}+\right.\\
&\qquad\left.\frac{z-z_{min}}{\sqrt{R^2+R_0^2-2R_0R\cos\phi+(z-z_{min})^2}}\right]\times\\
&\qquad\frac{R-R_0\cos\phi}{R^2+R_0^2-2R_0R\cos\phi}.
\end{aligned}$$

Let

$$\epsilon(R_0,z_0)=\frac{2RR_0\cos\phi}{R^2+R_0^2+(z-z_0)^2}.$$

Note that $|\epsilon(R_0,z_0)|\le 1$ by the AM-GM inequality. Hence,

$$\begin{aligned}
&B_{R,v,0}(R,\phi,z)\\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max}-z}{\sqrt{R^2+R_0^2+(z_{max}-z)^2}}\{1+O[\epsilon(R_0,z_{max})]\}+\right.\\
&\qquad\left.\frac{z-z_{min}}{\sqrt{R^2+R_0^2+(z-z_{min})^2}}\{1+O[\epsilon(R_0,z_{min})]\}\right]\times\\
&\qquad\frac{-R_0\sin\phi}{R^2+R_0^2-2R_0R\cos\phi}\\
&B_{\phi,v,0}(R,\phi,z)\\
&= \frac{\mu_0 I_0}{4\pi}\left[\frac{z_{max}-z}{\sqrt{R^2+R_0^2+(z_{max}-z)^2}}\{1+O[\epsilon(R_0,z_{max})]\}+\right.\\
&\qquad\left.\frac{z-z_{min}}{\sqrt{R^2+R_0^2+(z-z_{min})^2}}\{1+O[\epsilon(R_0,z_{min})]\}\right]\times\\
&\qquad\frac{R-R_0\cos\phi}{R^2+R_0^2-2R_0R\cos\phi}.
\end{aligned}$$

Hence, the magnetic field produced by all the vertical wires at $R=R_0$ is given by

$$\mathbf{B}_{v,\Sigma}(R,\phi,z;R_0)=\sum_{k=0}^{N-1}\mathbf{B}_{v,k}(R,\phi,z).$$

Hence,

$$B_{R,v,\Sigma}(R,\phi,z;z_0)=\sum_{k=0}^{N-1}B_{R,v,k}(R,\phi,z),$$

$$B_{\phi,v,\Sigma}(R,\phi,z;z_0)=\sum_{k=0}^{N-1}B_{\phi,v,k}(R,\phi,z).$$

We now wish to express $\mathbf{B}_{v,\Sigma}$ as a fourier series in ϕ. We will assume that $B_{R,v,\Sigma}$ is an odd function of ϕ and $B_{\phi,v,\Sigma}$ is an even function of ϕ. Hence, we can write

$$B_{R,v,\Sigma}(R,\phi,z) = \sum_{n=N}^{\infty} B_{R,v,n}(R,z)\sin(n\phi),$$

$$B_{\phi,v,\Sigma}(R,\phi,z) = \sum_{n=0}^{\infty} B_{\phi,v,n}(R,z)\cos(n\phi),$$

where $B_{R,v,n}(R,z) = B_{\phi,v,n}(R,z) = 0$ for $n \bmod N \neq 0$. To be clear, $B_{R,v,n}(R,z) \neq B_{R,v,n}(R,\phi,z)$ and $B_{\phi,v,n}(R,z) \neq B_{\phi,v,n}(R,\phi,z)$. From this point on we will assume $n \bmod N = 0$.

Note that

$$\begin{aligned} B_{R,v,n}(R,z) &= \frac{1}{\pi}\int_0^{2\pi} B_{R,v,\Sigma}(R,\phi,z)\sin(n\phi)d\phi \\ &= \frac{N}{\pi}\int_0^{2\pi} B_{R,v,0}(R,\phi,z)\sin(n\phi)d\phi, \end{aligned}$$

for $n \bmod N = 0$.

$$\begin{aligned} B_{\phi,v,0}(R,z) &= \frac{1}{2\pi}\int_0^{2\pi} B_{\phi,v,\Sigma}(R,\phi,z)d\phi \\ &= \frac{N}{2\pi}\int_0^{2\pi} B_{\phi,v,0}(R,\phi,z)d\phi, \\ B_{\phi,v,n}(R,z) &= \frac{1}{\pi}\int_0^{2\pi} B_{\phi,v,\Sigma}(R,\phi,z)\cos(n\phi)d\phi \\ &= \frac{N}{\pi}\int_0^{2\pi} B_{\phi,v,0}(R,\phi,z)\cos(n\phi)d\phi, \end{aligned}$$

for $n \bmod N = 0$ and $n \neq 0$.

2.1 Fourier coefficients of the radial component of the magnetic field

Our goal in this section is to calculate this integral

$$B_{R,v,n}(R,z) = \frac{N}{\pi}\int_0^{2\pi} B_{R,v,0}(R,\phi,z)\sin(n\phi)d\phi.$$

Let

$$a_v(R_0,z_0) = \frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{z_0 - z}{\sqrt{R^2 + R_0^2 + (z_0 - z)^2}},$$

$$I_{R,v,n}(R_0,z_0) = a_v(R_0,z_0)\int_0^{2\pi}\frac{R_0\sin\phi}{R^2 + R_0^2 - 2R_0R\cos\phi}\sin(n\phi)d\phi.$$

Note that,

$$B_{R,v,n}(R,z) = I_{R,v,n}(R_0,z_{max})\{1 + O[\epsilon(R_0,z_{max})]\} - I_{R,v,n}(R_0,z_{min})\{1 + O[\epsilon(R_0,z_{max})]\}.$$

To calculate $I_{R,v,n}(R_0,z_0)$ we use a trick where we let $Z = e^{i\phi}$ and substitute

$$\sin\phi = \frac{Z - Z^{-1}}{2i},$$

$$\cos\phi = \frac{Z + Z^{-1}}{2},$$

$$\sin(n\phi) = \frac{Z^n - Z^{-n}}{2i}.$$

Then we take the integral over the unit circle $|Z| = 1$. Note that

$$\frac{dZ}{d\phi} = iZ \implies d\phi = \frac{dZ}{iZ}.$$

Hence,

$$\begin{aligned}
I_{R,v,n}(R_0, z_0) &= a_v \oint_{|Z|=1} \frac{1}{i} \frac{R_0(Z^2-1)}{2Z(R^2+R_0^2) - 2R_0R(Z^2+1)} \frac{Z^n - Z^{-n}}{2i} \frac{dZ}{iZ} \\
&= -\frac{a_v}{4i} \oint_{|Z|=1} \frac{R_0(Z^2-1)Z^{n-1}}{(RZ-R_0)(R-R_0Z)} - \frac{R_0(Z^2-1)Z^{-n-1}}{(RZ-R_0)(R-R_0Z)} dZ \\
&= -\frac{a_v}{4i}(I_1 - I_2).
\end{aligned}$$

We will assume $R \neq R_0$. The integrand of of both I_1 and I_2 have singularities at $Z = R/R_0$ and $Z = R_0/R$. The integrand of I_2 also has a singularity at $Z = 0$. The residues at the simple poles are given by

$$\mathrm{Res}(f, Z_0) = \lim_{Z \to Z_0} (Z - Z_0) f(Z).$$

Note that

$$\begin{aligned}
\lim_{Z \to R/R_0} \frac{(Z - R/R_0)}{R - R_0 Z} &= -\frac{1}{R_0} \\
\lim_{Z \to R_0/R} \frac{(Z - R_0/R)}{RZ - R_0} &= \frac{1}{R}
\end{aligned}$$

Now we will calculate the residues of the I_1 integrand:

$$\begin{aligned}
\mathrm{Res}\left(\frac{R_0(Z^2-1)Z^{n-1}}{(RZ-R_0)(R-R_0Z)}, R/R_0\right) &= -\frac{1}{R_0} \frac{R_0((R/R_0)^2-1)(R/R_0)^{n-1}}{R(R/R_0) - R_0} \\
&= -\frac{1}{R_0} \frac{((R/R_0)^2-1)(R/R_0)^{n-1}}{(R/R_0)^2 - 1} \\
&= -\frac{1}{R_0}(R/R_0)^{n-1} \\
&= -\frac{1}{R}\left(\frac{R}{R_0}\right)^n.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}\left(\frac{R_0(Z^2-1)Z^{n-1}}{(RZ-R_0)(R-R_0Z)}, R_0/R\right) &= \frac{1}{R} \frac{R_0((R_0/R)^2-1)(R_0/R)^{n-1}}{R - R_0(R_0/R)} \\
&= \frac{1}{R} \frac{(R_0/R)((R/R_0)^2-1)(R_0/R)^{n-1}}{1 - (R_0/R)^2} \\
&= -\frac{1}{R}\left(\frac{R_0}{R}\right)^n.
\end{aligned}$$

Now we will calculate the residues of the I_2 integrand:

$$\begin{aligned}
\mathrm{Res}\left(\frac{R_0(Z^2-1)Z^{-n-1}}{(RZ-R_0)(R-R_0Z)}, R/R_0\right) &= -\frac{1}{R_0} \frac{R_0((R/R_0)^2-1)(R/R_0)^{-n-1}}{R(R/R_0) - R_0} \\
&= -\frac{1}{R_0} \frac{((R/R_0)^2-1)(R/R_0)^{-n-1}}{(R/R_0)^2 - 1} \\
&= -\frac{1}{R_0}(R/R_0)^{-n-1} \\
&= -\frac{1}{R}\left(\frac{R_0}{R}\right)^n.
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Res}\left(\frac{R_0(Z^2-1)Z^{-n-1}}{(RZ-R_0)(R-R_0Z)}, R_0/R\right) &= \frac{1}{R}\frac{R_0((R_0/R)^2-1)(R_0/R)^{-n-1}}{R-R_0(R_0/R)} \\
&= \frac{1}{R}\frac{R_0/R((R/R_0)^2-1)(R/R_0)^{-n-1}}{1-(R/R_0)^2} \\
&= -\frac{1}{R}\left(\frac{R}{R_0}\right)^n.
\end{aligned}
$$

Now, to calculate the residue at the order $n+1$ pole at $Z=0$, we use Wolfram Alpha. See the link here. To get

$$
\operatorname{Res}\left(\frac{R_0(Z^2-1)Z^{-n-1}}{(RZ-R_0)(R-R_0Z)}, 0\right) = \frac{1}{R}\left(\frac{R}{R_0}\right)^n + \frac{1}{R}\left(\frac{R_0}{R}\right)^n
$$

Hence, by the residue theorem,

$$
\boxed{I_{R,v,n}(R_0, z_0) = \pi a_v(R_0, z_0)\frac{1}{R}\left(\frac{R}{R_0}\right)^n}
$$

for $R < R_0$, $n \bmod N = 0$ and

$$
\boxed{I_{R,v,n}(R_0, z_0) = \pi a_v(R_0, z_0)\frac{1}{R}\left(\frac{R_0}{R}\right)^n}
$$

for $R > R_0$, $n \bmod N = 0$.

$$
B_{R,v,n}(R, z) = I_{R,v,n}(R_0, z_{max})\{1 + O[\epsilon(R_0, z_{max})]\} - I_{R,v,n}(R_0, z_{min})\{1 + O[\epsilon(R_0, z_{max})]\}.
$$

$$
a_v(R_0, z_0) = \frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{z_0 - z}{\sqrt{R^2 + R_0^2 + (z_0 - z)^2}},
$$

Hence the radial magnetic field from the all the veritical fields at $R = R_0$ is given by

$$
\boxed{
\begin{aligned}
B_{R,v}(R, \phi, z) =& \frac{\mu_0 I_0}{4\pi}\left(\frac{z_{max} - z}{\sqrt{R^2 + R_0^2 + (z_{max} - z)^2}} + O[\epsilon(R_0, z_{max})]\right. \\
&\left. + \frac{z - z_{min}}{\sqrt{R^2 + R_0^2 + (z_{min} - z)^2}} + O[\epsilon(R_0, z_{min})]\right) \\
&\sum_{n=N}^{\infty}\frac{1}{R}\left(\frac{R}{R_0}\right)^n \sin(n\phi),
\end{aligned}
}
$$

for $R < R_0$, $n \bmod N = 0$ and

$$
\boxed{
\begin{aligned}
B_{R,v}(R, \phi, z) =& \frac{\mu_0 I_0}{4\pi}\left(\frac{z_{max} - z}{\sqrt{R^2 + R_0^2 + (z_{max} - z)^2}} + O[\epsilon(R_0, z_{max})]\right. \\
&\left. + \frac{z - z_{min}}{\sqrt{R^2 + R_0^2 + (z_{min} - z)^2}} + O[\epsilon(R_0, z_{min})]\right) \\
&\sum_{n=N}^{\infty}\frac{1}{R}\left(\frac{R}{R_0}\right)^n \sin(n\phi),
\end{aligned}
}
$$

for $R > R_0$, $n \bmod N = 0$

3 Vacuum Field from Horizontal Wires in Picture-Frame TF Coils

To calculate the magnetic field induced by a horizontal wire which points in the $\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$ direction and extends from R_{inner} to R_{outer} at $h = h_0$ we rotate our expression from the end of Section 1.

For reference, the rotation matrix to rotate a vector an angle θ about the y-axis is given by

$$\mathrm{R}_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},$$

and about the z-axis is given by

$$\mathrm{R}_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Our full rotation matrix is given by

$$\begin{aligned} \mathrm{R}_k &= \mathrm{R}_z(\phi_k)\mathrm{R}_y(\pi/2) \\ &= \begin{pmatrix} \cos(\phi_k) & -\sin(\phi_k) & 0 \\ \sin(\phi_k) & \cos(\phi_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\sin(\phi_k) & \cos(\phi_k) \\ 0 & \cos(\phi_k) & \sin(\phi_k) \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The inverse rotation matrix is given by

$$\begin{aligned} \mathrm{R}_k^{-1} &= \mathrm{R}_y(-\pi/2)\mathrm{R}_z(-\phi_k) \\ &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\phi_k) & \sin(\phi_k) & 0 \\ -\sin(\phi_k) & \cos(\phi_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1 \\ -\sin(\phi_k) & \cos(\phi_k) & 0 \\ \cos(\phi_k) & \sin(\phi_k) & 1 \end{pmatrix}. \end{aligned}$$

Note that

$$\mathbf{B}_v(R, \phi, z; z_{min}, z_{max}) = \frac{\mu_0 I_0}{4\pi R}\left[\frac{z_{max} - z}{\sqrt{R^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{R^2 + (z - z_{min})^2}}\right]\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix}.$$

$$\implies \mathbf{B}_v(x, y, z; z_{min}, z_{max}) = \frac{\mu_0 I_0}{4\pi}\frac{1}{x^2 + y^2}\left[\frac{z_{max} - z}{\sqrt{x^2 + y^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{x^2 + y^2 + (z - z_{min})^2}}\right]\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

Let

$$\begin{aligned} \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} &= \mathrm{R}_k^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} -z \\ y\cos(\phi_k) - x\sin(\phi_k) \\ x\cos(\phi_k) + y\sin(\phi_k) \end{pmatrix}. \end{aligned}$$

Hence

$$\implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathrm{R}_k \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}$$
$$= \begin{pmatrix} z_k \cos(\phi_k) - y_k \sin(\phi_k) \\ y_k \cos(\phi_k) + z_k \sin(\phi_k) \\ -x_k \end{pmatrix}.$$

The magnetic field generated by the kth horizontal wire at $z = z_0$ is given

$$\begin{aligned}
\mathbf{B}_{h,k}(x, y, z) &= \mathrm{R}_k \mathbf{B}_v(x_k + z_0, y_k, z_k; R_{inner}, R_{outer}) \\
&= \frac{\mu_0 I_0}{4\pi} \frac{1}{(x_k + z_0)^2 + y_k^2} \left[\frac{R_{outer} - z_k}{\sqrt{(x_k + z_0)^2 + y_k^2 + (R_{outer} - z_k)^2}} \right. \\
&\qquad \left. + \frac{z_k - R_{inner}}{\sqrt{(x_k + z_0)^2 + y_k^2 + (z_k - R_{inner})^2}} \right] R_k \begin{pmatrix} -y_k \\ x_k + z_0 \\ 0 \end{pmatrix} \\
&= \frac{\mu_0 I_0}{4\pi} \frac{1}{y_k^2 + (z - z_0)^2} \left[\frac{R_{outer} - z_k}{\sqrt{(R_{outer} - z_k)^2 + y_k^2 + (z - z_0)^2}} \right. \\
&\qquad \left. + \frac{z_k - R_{inner}}{\sqrt{(z_k - R_{inner})^2 + y_k^2 + (z - z_0)^2}} \right] \begin{pmatrix} -(x_k + z_0)\sin(\phi_k) \\ (x_k + z_0)\cos(\phi_k) \\ y_k \end{pmatrix}.
\end{aligned}$$

For $k = 0$, we have

$$\begin{aligned}
\mathbf{B}_{h,0}(x, y, z) = \frac{\mu_0 I_0}{4\pi} \frac{1}{y^2 + (z_0 - z)^2} &\left[\frac{R_{outer} - x}{\sqrt{(R_{outer} - x)^2 + y^2 + (z - z_0)^2}} \right. \\
&\left. + \frac{x - R_{inner}}{\sqrt{(x - R_{inner})^2 + y^2 + (z - z_0)^2}} \right] \begin{pmatrix} 0 \\ z_0 - z \\ y \end{pmatrix}.
\end{aligned}$$

In cylindrical coordinates and taking the dot product with $\hat{\mathbf{R}}$, $\hat{\phi}$ and $\hat{\mathbf{z}}$ we get

$$\begin{aligned}
B_{R,h,0}(R, \phi, z) = \frac{\mu_0 I_0}{4\pi} \frac{\sin\phi(z_0 - z)}{R^2 \sin^2\phi + (z_0 - z)^2} &\left[\frac{R_{outer} - R\cos(\phi)}{\sqrt{R^2 + R_{outer}^2 - 2RR_{outer}\cos\phi + (z - z_0)^2}} \right. \\
&\left. + \frac{R\cos(\phi) - R_{inner}}{\sqrt{R^2 + R_{inner}^2 - 2RR_{inner}\cos\phi + (z - z_0)^2}} \right] \\
B_{\phi,h,0}(R, \phi, z) = \frac{\mu_0 I_0}{4\pi} \frac{\cos\phi(z_0 - z)}{R^2 \sin^2\phi + (z_0 - z)^2} &\left[\frac{R_{outer} - R\cos(\phi)}{\sqrt{R^2 + R_{outer}^2 - 2RR_{outer}\cos\phi + (z - z_0)^2}} \right. \\
&\left. + \frac{R\cos(\phi) - R_{inner}}{\sqrt{R^2 + R_{inner}^2 - 2RR_{inner}\cos\phi + (z - z_0)^2}} \right] \\
B_{z,h,0}(R, \phi, z) = \frac{\mu_0 I_0}{4\pi} \frac{R\sin\phi}{R^2 \sin^2\phi + (z_0 - z)^2} &\left[\frac{R_{outer} - R\cos(\phi)}{\sqrt{R^2 + R_{outer}^2 - 2RR_{outer}\cos\phi + (z - z_0)^2}} \right. \\
&\left. + \frac{R\cos(\phi) - R_{inner}}{\sqrt{R^2 + R_{inner}^2 - 2RR_{inner}\cos\phi + (z - z_0)^2}} \right].
\end{aligned}$$

Using the expression for $\epsilon(R_0, z_0)$ introduced in the previous section, namely,

$$\epsilon(R_0, z_0) = \frac{2RR_0\cos\phi}{R^2 + R_0^2 + (z - z_0)^2},$$

we can simplify the above to give

$$B_{R,h,0}(R,\phi,z) = \frac{\mu_0 I_0}{4\pi}\frac{\sin\phi(z_0-z)}{R^2\sin^2\phi+(z_0-z)^2}\left[\frac{R_{outer}-R\cos(\phi)}{\sqrt{R^2+R_{outer}^2+(z-z_0)^2}}\{1+O[\epsilon(R_{outer},z_0)]\}\right.$$
$$\left. + \frac{R\cos(\phi)-R_{inner}}{\sqrt{R^2+R_{inner}^2+(z-z_0)^2}}\{1+O[\epsilon(R_0,z_{max})]\}\right]$$

$$B_{\phi,h,0}(R,\phi,z) = \frac{\mu_0 I_0}{4\pi}\frac{\cos\phi(z_0-z)}{R^2\sin^2\phi+(z_0-z)^2}\left[\frac{R_{outer}-R\cos(\phi)}{\sqrt{R^2+R_{outer}^2+(z-z_0)^2}}\{1+O[\epsilon(R_{outer},z_0)]\}\right.$$
$$\left. + \frac{R\cos(\phi)-R_{inner}}{\sqrt{R^2+R_{inner}^2+(z-z_0)^2}}\{1+O[\epsilon(R_{inner},z_0)]\}\right]$$

The magnetic field produced by all the horizontal wires at $z=z_0$ is given by

$$\mathbf{B}_{h,\Sigma}(R,\phi,z;z_0) = \sum_{k=0}^{N-1}\mathbf{B}_{h,k}(R,\phi,z).$$

Hence,

$$B_{R,h,\Sigma}(R,\phi,z;z_0) = \sum_{k=0}^{N-1} B_{R,h,k}(R,\phi,z),$$

$$B_{\phi,h,\Sigma}(R,\phi,z;z_0) = \sum_{k=0}^{N-1} B_{\phi,h,k}(R,\phi,z),$$

$$B_{z,h,\Sigma}(R,\phi,z;z_0) = \sum_{k=0}^{N-1} B_{z,h,k}(R,\phi,z).$$

We now wish to express $\mathbf{B}_{h,\Sigma}$ as a fourier series in ϕ. We will assume that $B_{R,h,\Sigma}$, $B_{z,h,\Sigma}$ are odd functions of ϕ and $B_{\phi,h,\Sigma}$ is an even function of ϕ. Hence, we can write

$$B_{R,h,\Sigma}(R,\phi,z) = \sum_{n=N}^{\infty} B_{R,h,n}(R,z)\sin(n\phi),$$

$$B_{\phi,h,\Sigma}(R,\phi,z) = \sum_{n=0}^{\infty} B_{\phi,h,n}(R,z)\cos(n\phi),$$

$$B_{z,h,\Sigma}(R,\phi,z) = \sum_{n=N}^{\infty} B_{z,h,n}(R,z)\sin(n\phi),$$

where $B_{R,h,n}(R,z) = B_{\phi,h,n}(R,z) = B_{z,h,n}(R,z) = 0$ for $n \bmod N \neq 0$. To be clear, $B_{R,h,n}(R,z) \neq B_{R,h,n}(R,\phi,z)$, $B_{\phi,h,n}(R,z) \neq B_{\phi,h,n}(R,\phi,z)$, $B_{z,h,n}(R,z) \neq B_{z,h,n}(R,\phi,z)$. From this point on we will assume $n \bmod N = 0$.

Note that

$$\begin{aligned} B_{R,h,n}(R,z) &= \frac{1}{\pi}\int_0^{2\pi} B_{R,h,\Sigma}(R,\phi,z)\sin(n\phi)d\phi \\ &= \frac{N}{\pi}\int_0^{2\pi} B_{R,h,0}(R,\phi,z)\sin(n\phi)d\phi, \end{aligned}$$

for $n \bmod N = 0$.

$$\begin{aligned} B_{\phi,h,0}(R,z) &= \frac{1}{2\pi}\int_0^{2\pi} B_{\phi,h,\Sigma}(R,\phi,z)d\phi \\ &= \frac{N}{2\pi}\int_0^{2\pi} B_{\phi,h,0}(R,\phi,z)d\phi, \end{aligned}$$

$$B_{\phi,h,n}(R,z) = \frac{1}{\pi}\int_0^{2\pi} B_{\phi,h,\Sigma}(R,\phi,z)\cos(n\phi)d\phi$$
$$= \frac{N}{\pi}\int_0^{2\pi} B_{\phi,h,0}(R,\phi,z)\cos(n\phi)d\phi,$$

for $n \bmod N = 0$ and $n \neq 0$.

$$B_{z,h,n}(R,z) = \frac{1}{\pi}\int_0^{2\pi} B_{z,h,\Sigma}(R,\phi,z)\sin(n\phi)d\phi$$
$$= \frac{N}{\pi}\int_0^{2\pi} B_{z,h,0}(R,\phi,z)\sin(n\phi)d\phi,$$

for $n \bmod N = 0$.

3.1 Fourier coefficients of the radial component of the magnetic field

Our goal in this section is to calculate this integral

$$B_{R,h,n}(R,z) = \frac{N}{\pi}\int_0^{2\pi} B_{R,h,0}(R,\phi,z)\sin(n\phi)d\phi.$$

Let

$$a_h(R_0,z_0) = \frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{\sqrt{R^2+R_0^2+(z_0-z)^2}},$$
$$I_{R,h,n}(R_0,z_0) = a_h(R_0,z_0)\int_0^{2\pi}\frac{\sin\phi(z_0-z)(R_0-R\cos\phi)}{R^2\sin^2\phi+(z_0-z)^2}\sin(n\phi)d\phi.$$

Note that if nw is even we can simplify this integral to give

$$I_{R,h,n}(R_0,z_0) = a_h(R_0,z_0)\int_0^{2\pi}\frac{\sin\phi(z_0-z)(R_0-R\cos\phi)}{R^2\sin^2\phi+(z_0-z)^2}\sin(n\phi)d\phi.$$

Note that, for n even

$$I_{R,h,n}(R_0,z_0) = -a_h(R_0,z_0)\int_0^{2\pi}\frac{\sin\phi(z_0-z)R\cos\phi}{R^2\sin^2\phi+(z_0-z)^2}\sin(n\phi)d\phi$$
$$= -\frac{1}{2}a_h(R_0,z_0)\int_0^{2\pi}\frac{R(z_0-z)\sin 2\phi}{R^2\sin^2\phi+(z_0-z)^2}\sin(n\phi)d\phi$$
$$= -\frac{1}{2}a_h(R_0,z_0)\int_0^{2\pi}\frac{R(z_0-z)\sin 2\phi}{R^2+(z_0-z)^2-R^2\cos^2\phi}\sin(n\phi)d\phi$$
$$= -\frac{1}{2}a_h(R_0,z_0)\frac{R(z_0-z)}{R^2+(z_0-z)^2}\int_0^{2\pi}\frac{\sin 2\phi}{1-\delta^2(R_0,z_0)\cos^2\phi}\sin(n\phi)d\phi,$$

where

$$\delta^2(R_0,z0) = \frac{R^2}{R^2+(z_0-z)^2}.$$

For proof see this Wolfram alpha link.

$$B_{R,h,n}(R,z) = I_{R,h,n}(R_{max},z_0)\{1+O[\epsilon(R_0,z_{max})]\} - I_{R,h,n}(R_{min},z_0)\{1+O[\epsilon(R_0,z_{max})]\}.$$

To calculate $I_{R,v,n}(R_0,z_0)$ we use a trick where we let $Z = e^{i\phi}$ and substitute

$$\sin(n\phi) = \frac{Z^n - Z^{-n}}{2i}.$$

Then we take the integral over the unit circle $|Z| = 1$. Note that

$$\frac{dZ}{d\phi} = iZ \implies d\phi = \frac{dZ}{iZ}.$$

Hence,

$$I_{R,h,n}(R_0, z_0) = -\frac{1}{2} a_h(R_0, z_0) \frac{R(z_0 - z)}{R^2 + (z_0 - z)^2} \int_0^{2\pi} \frac{\left(\frac{Z^2 - Z^{-2}}{2i}\right)}{1 - \delta^2(R_0, z_0)\left(\frac{Z + Z^{-1}}{2}\right)^2} \left(\frac{Z^n - Z^{-n}}{2i}\right) \frac{dZ}{iZ}.$$

For n even this is given by

$$\begin{aligned}
I_{R,h,n}(R_0, z_0) &= -\frac{1}{2} a_h(R_0, z_0) \frac{R(z_0 - z)}{R^2 + (z_0 - z)^2} 2\pi i \left[-\frac{i}{2}\left(\frac{\delta}{2}\right)^{n-2} + O(\delta^n)\right] \\
&= -\frac{\pi}{2} a_h(R_0, z_0) \frac{z_0 - z}{R} \left[4\left(\frac{\delta}{2}\right)^n + O(\delta^{n+2})\right] \\
&= -2\pi a_h(R_0, z_0) \frac{z_0 - z}{R} \left[\left(\frac{\delta}{2}\right)^n + O(\delta^{n+2})\right] \\
&= -N \frac{\mu_0 I_0}{2\pi} \frac{1}{\sqrt{R^2 + R_0^2 + (z_0 - z)^2}} \frac{z_0 - z}{R} \left[\left(\frac{\delta}{2}\right)^n + O(\delta^{n+2})\right]
\end{aligned}$$

For proof see case where $n = 6$: link1, link2, link3, link4.

Hence, the radial component of the magnetic field induced by the horizontal wires at $z = z_0$ is given by

$$\boxed{\begin{aligned}
B_{R,h,n}(R, \phi, z) = N \frac{\mu_0 I_0}{2\pi} \frac{z_0 - z}{R} \Bigg\{ &\frac{1}{\sqrt{R^2 + R_{min}^2 + (z_0 - z)^2}} + O[\epsilon(R_{max}, z_0)] \\
&- \frac{1}{\sqrt{R^2 + R_{max}^2 + (z_0 - z)^2}} + O[\epsilon(R_{max}, z_0)] \Bigg\} \times \\
\sum_{n=N}^{\infty} \Bigg\{ &\left(\frac{R}{2\sqrt{R^2 + (z_0 - z)^2}}\right)^n + O\left[\left(\frac{R}{2\sqrt{R^2 + (z_0 - z)^2}}\right)^{n+2}\right] \Bigg\} \sin(n\phi)
\end{aligned}}$$

3.2 Fourier coefficients of the toroidal component of the magnetic field

Our goal in this section is to calculate these integrals

$$B_{\phi,h,0}(R, z) = \frac{N}{2\pi} \int_0^{2\pi} B_{\phi,h,0}(R, \phi, z) d\phi,$$

and

$$B_{\phi,h,n}(R, z) = \frac{N}{\pi} \int_0^{2\pi} B_{\phi,h,0}(R, \phi, z) \cos(n\phi) d\phi,$$

for $n \bmod N = 0$ and $n \neq 0$. Let

$$a_h(R_0, z_0) = \frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{\sqrt{R^2 + R_0^2 + (z_0 - z)^2}},$$

$$I_{\phi,h,0}(R_0, z_0) = \frac{a_h(R_0, z_0)}{2} \int_0^{2\pi} \frac{\cos\phi(z_0 - z)(R_0 - R\cos\phi)}{R^2 \sin^2\phi + (z_0 - z)^2} d\phi.$$

$$I_{\phi,h,n}(R_0, z_0) = a_h(R_0, z_0) \int_0^{2\pi} \frac{\cos\phi(z_0 - z)(R_0 - R\cos\phi)}{R^2 \sin^2\phi + (z_0 - z)^2} \cos(n\phi) d\phi,$$

for $n \neq 0$. Note that,

$$B_{\phi,v,n}(R,z) = I_{\phi,h,n}(R_{outer}, z_0)\{1 + O[\epsilon(R_{outer}, z_0)]\} - I_{\phi,h,n}(R_{inner}, z_0)\{1 + O[\epsilon(R_{inner}, z_{max})]\}.$$

To calculate $I_{\phi,v,n}(R_0, z_0)$ we use a trick where we let $Z = e^{i\phi}$ and substitute

$$\sin\phi = \frac{Z - Z^{-1}}{2i},$$

$$\cos\phi = \frac{Z + Z^{-1}}{2},$$

$$\cos(n\phi) = \frac{Z^n + Z^{-n}}{2}.$$

Then we take the integral over the unit circle $|Z| = 1$. Note that

$$\frac{dZ}{d\phi} = iZ \implies d\phi = \frac{dZ}{iZ}.$$

Hence,

$$\begin{aligned} I_{R,v,0}(R_0, z_0) &= \frac{a_h}{2}\int_0^{2\pi} \frac{\left(\frac{Z+Z^{-1}}{2}\right)(z_0 - z)(R_0 - R\left(\frac{Z+Z^{-1}}{2}\right))}{R^2\left(\frac{Z-Z^{-1}}{2i}\right)^2 + (z_0 - z)^2}\frac{dZ}{iZ} \\ &= \frac{a_h}{2i}\int_0^{2\pi} \frac{(Z^2+1)(z_0 - z)[2R_0 Z - R(Z^2+1)]}{4Z^2(z_0-z)^2 - R^2(Z^2-1)^2}\frac{dZ}{Z} \end{aligned}$$

The integrand of $I_{R,v,0}(R_0, z_0)$ has simple poles at

$$Z = \pm\sqrt{\frac{R + (z_0 - z)}{R}},$$

however, these lie outside the unit circle and so we can ignore them. The integrand of $I_{R,v,0}(R_0, z_0)$ also has simple poles at $Z = 0$ and

$$Z = \pm\sqrt{\frac{R - (z_0 - z)}{R}},$$