

Analytic Approximation of the Vacuum Magnetic Field Induced by Infinitely Wide Picture Frame coils

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1 Magnetic field induced by a vertical wire centred at the origin

The magnetic field induced by a vertical wire, centred at the origin, with current, I_0 which points in the $\hat{\mathbf{z}}$ direction and extends from $-z_{min}, z_{max}$ is given by the following Biot-Savart law expression:

$$\begin{aligned}\mathbf{B}_v(R, \phi, z) &= \frac{\mu_0 I_0}{4\pi} \int_{z_{min}}^{z_{max}} \frac{d\mathbf{h}' \times (R\hat{\mathbf{R}} + (z - z')\hat{\mathbf{z}})}{(R^2 + (z - z')^2)^{3/2}} \\ &= \frac{\mu_0 I_0}{4\pi} \int_{z_{min}}^{z_{max}} \frac{R dh' \hat{\phi}}{(R^2 + (z - z')^2)^{3/2}}.\end{aligned}$$

Let

$$\begin{aligned}\cos \theta &= \frac{R}{\sqrt{R^2 + (z - z')^2}}, \\ \implies \sin \theta &= \frac{z - z'}{\sqrt{R^2 + (z - z')^2}}, \\ \implies \tan \theta &= \frac{z - z'}{R}, \\ \implies z' &= z - R \tan \theta, \\ \implies dz' &= -R \sec^2 \theta d\theta.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{B}_v(R, \phi, z) &= -\frac{\mu_0 I_0}{4\pi R} \int_{z_{min}}^{z_{max}} \cos \theta d\theta \hat{\phi} \\ &= \frac{\mu_0 I_0}{4\pi R} [-\sin \theta]_{z_{min}}^{z_{max}} \hat{\phi} \\ &= \frac{\mu_0 I_0}{4\pi R} \left[\frac{z_{max} - z}{\sqrt{R^2 + (z_{max} - z)^2}} + \frac{z - z_{min}}{\sqrt{R^2 + (z - z_{min})^2}} \right] \hat{\phi}.\end{aligned}$$

Let $z_{max} = h$ and $z_{min} = -h$:

$$\begin{aligned}\mathbf{B}_v(R, \phi, z) &= -\frac{\mu_0 I_0}{4\pi R} \int_{z_{min}}^{z_{max}} \cos \theta d\theta \hat{\phi} \\ &= \frac{\mu_0 I_0}{4\pi R} [-\sin \theta]_{z_{min}}^{z_{max}} \hat{\phi} \\ &= \frac{\mu_0 I_0}{4\pi R} \left[\frac{h - z}{\sqrt{R^2 + (h - z)^2}} + \frac{z + h}{\sqrt{R^2 + (z + h)^2}} \right] \hat{\phi}.\end{aligned}$$

2 Magnetic field induced by a horizontal wire extending from the origin to infinity

Let $\mathbf{B}_{h,c,k}(x, y, z)$ denote the magnetic field induced by a horizontal wire in cartesian coordinates extending from the origin to infinity with current, I_0 , in the

$$\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$$

direction, where

$$\phi_k = k \frac{2\pi}{N},$$

N is the number of picture-frame coils. For $k = 0$, $\mathbf{B}_{h,c,k}(x, y, z)$ points in the $\hat{\mathbf{x}}$ direction and is given by

$$\mathbf{B}_{h,c,0}(x, y, z) = \frac{\mu_0 I_0}{4\pi} \frac{-z\hat{\mathbf{y}} + y\hat{\mathbf{z}}}{y^2 + z^2} \left[1 + \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right].$$

Let $\mathbf{B}_{h,k}$ denote the same magnetic field as $\mathbf{B}_{h,c,k}$ but in cylindrical coordinates. Then

$$\mathbf{B}_{h,0}(R, \phi, z) = \frac{\mu_0 I_0}{4\pi} \frac{-z \sin \phi \hat{\mathbf{R}} - z \cos \phi \hat{\phi} + R \sin \phi \hat{\mathbf{z}}}{R^2 \sin^2 \phi + z^2} \left[1 + \frac{R \cos \phi}{\sqrt{R^2 + z^2}} \right].$$

3 Fourier series of the magnetic field induced by a horizontal wires extending from the origin to infinity

Let

$$\mathbf{B}_h(R, \phi, z) = \sum_{k=0}^{N-1} \mathbf{B}_{h,k}(R, \phi, z).$$

We now wish to express

$$\mathbf{B}_h(R, \phi, z) = B_{h,R}(R, \phi, z)\hat{\mathbf{R}} + B_{h,\phi}(R, \phi, z)\hat{\phi} + B_{h,z}(R, \phi, z)\hat{\mathbf{z}},$$

as a Fourier series in ϕ . The picture-frame coils are positioned to ensure that $B_{h,R}$ and $B_{h,z}$ are odd functions of ϕ and $B_{h,\phi}$ is an even function of ϕ . Hence

$$B_{h,R}(R, \phi, z) = \sum_{n=N}^{\infty} B_{h,R,n}(R, z) \sin(n\phi),$$

$$B_{h,\phi}(R, \phi, z) = \sum_{n=0}^{\infty} B_{h,\phi,n}(R, z) \cos(n\phi),$$

$$B_{h,z}(R, \phi, z) = \sum_{n=N}^{\infty} B_{h,z,n}(R, z) \sin(n\phi).$$

where $B_{h,R,n}(R, z) = B_{h,\phi,n}(R, z) = B_{h,z,n}(R, z) = 0$ for $n \bmod N \neq 0$. To be clear, $B_{h,R,n}(R, z) \neq B_{h,R,n}(R, \phi, z)$, $B_{h,\phi,n}(R, z) \neq B_{h,\phi,n}(R, \phi, z)$, $B_{h,z,n}(R, z) \neq B_{h,z,n}(R, \phi, z)$. From this point on we will assume $n \bmod N = 0$.

Note that

$$\begin{aligned} B_{h,R,n}(R, z) &= \frac{1}{\pi} \int_0^{2\pi} B_{h,R}(R, \phi, z) \sin(n\phi) d\phi \\ &= \frac{N}{\pi} \int_0^{2\pi} B_{h,R,0}(R, \phi, z) \sin(n\phi) d\phi, \end{aligned}$$

for $n \bmod N = 0$.

$$\begin{aligned}
B_{h,\phi,0}(R, z) &= \frac{1}{2\pi} \int_0^{2\pi} B_{h,\phi}(R, \phi, z) d\phi \\
&= \frac{N}{2\pi} \int_0^{2\pi} B_{h,\phi,0}(R, \phi, z) d\phi, \\
B_{h,\phi,n}(R, z) &= \frac{1}{\pi} \int_0^{2\pi} B_{h,\phi}(R, \phi, z) \cos(n\phi) d\phi \\
&= \frac{N}{\pi} \int_0^{2\pi} B_{h,\phi,0}(R, \phi, z) \cos(n\phi) d\phi,
\end{aligned}$$

for $n \bmod N = 0$ and $n \neq 0$.

$$\begin{aligned}
B_{h,z,n}(R, z) &= \frac{1}{\pi} \int_0^{2\pi} B_{h,z}(R, \phi, z) \sin(n\phi) d\phi \\
&= \frac{N}{\pi} \int_0^{2\pi} B_{h,z,0}(R, \phi, z) \sin(n\phi) d\phi,
\end{aligned}$$

for $n \bmod N = 0$.

3.1 Fourier coefficients of the radial component of the magnetic field

Our goal in this section is to calculate this integral

$$\begin{aligned}
B_{R,h,n}(R, z) &= \frac{N}{\pi} \int_0^{2\pi} B_{R,h,0}(R, \phi, z) \sin(n\phi) d\phi \\
&= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \int_0^{2\pi} \frac{z \sin \phi}{R^2 \sin^2 \phi + z^2} \left(1 + \frac{R \cos \phi}{\sqrt{R^2 + z^2}} \right) \sin(n\phi) d\phi \\
&= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \int_0^{2\pi} \frac{\Delta \sin \phi}{\sin^2 \phi + \Delta^2} \left(1 + \frac{\cos \phi}{\sqrt{1 + \Delta^2}} \right) \sin(n\phi) d\phi.
\end{aligned}$$

Note that

$$\int_0^{2\pi} \frac{\sin(\phi) \sin(n\phi)}{\sin^2 \phi + \Delta^2} d\phi = 0,$$

for n odd.

$$\begin{aligned}
\int_0^{2\pi} \frac{\sin(\phi) \cos(\phi) \sin(n\phi)}{\sin^2 \phi + \Delta^2} d\phi &= \frac{1}{2} \int_0^{2\pi} \frac{\sin(2\phi) \sin(n\phi)}{\sin^2 \phi + \Delta^2} d\phi \\
&= \int_0^{2\pi} \frac{\sin(2\phi) \sin(n\phi)}{1 + 2\Delta^2 - \cos(2\phi)} d\phi.
\end{aligned}$$

Let

$$\begin{aligned}
\Delta^2 &= \frac{(1-a)^2}{4a}, \\
a &= \left(\sqrt{1 + \Delta^2} - |\Delta| \right)^2.
\end{aligned}$$

Hence,

$$\int_0^{2\pi} \frac{\sin(\phi) \cos(\phi) \sin(n\phi)}{\sin^2 \phi + \Delta^2} d\phi = 2 \int_0^{2\pi} \frac{a \sin(2\phi)}{1 + a^2 - 2a \cos(2\phi)} \sin(n\phi) d\phi.$$

Note that

$$\begin{aligned}
\frac{1}{1 - a \exp(2i\phi)} &= \frac{1 - a \cos(2\phi)}{1 + a^2 - 2a \cos(2\phi)} + i \frac{a \sin(2\phi)}{1 + a^2 - 2a \cos(2\phi)} \\
&= \sum_{k=0}^{\infty} a^k \exp(2ik\phi).
\end{aligned}$$

Taking the imaginary part gives

$$\frac{a \sin(2\phi)}{1 + a^2 - 2a \cos(2\phi)} = \sum_{k=1}^{\infty} a^k \sin(2k\phi).$$

Hence,

$$\begin{aligned} \int_0^{2\pi} \frac{\sin(\phi) \cos(\phi) \sin(n\phi)}{\sin^2 \phi + \Delta^2} d\phi &= 2\pi a^{n/2} \\ &= 2\pi \left(\sqrt{1 + \Delta^2} - |\Delta| \right)^n \end{aligned}$$

Therefore,

$$\begin{aligned} B_{R,h,n}(R, z) &= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \int_0^{2\pi} \frac{\Delta \sin \phi}{\sin^2 \phi + \Delta^2} \left(1 + \frac{\cos \phi}{\sqrt{1 + \Delta^2}} \right) \sin(n\phi) d\phi \\ &= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \frac{\Delta}{\sqrt{1 + \Delta^2}} 2\pi \left(\sqrt{1 + \Delta^2} - |\Delta| \right)^n \\ &= -N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \frac{z}{\sqrt{R^2 + z^2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \end{aligned}$$

3.2 Fourier coefficients of the azimuthal component of the magnetic field

Our goal in this section is to calculate these integrals

$$\begin{aligned} B_{h,\phi,0}(R, z) &= \frac{1}{2\pi} \int_0^{2\pi} B_{h,\phi}(R, \phi, z) d\phi \\ &= -\frac{N}{2\pi} \frac{\mu_0 I_0}{4\pi} \int_0^{2\pi} \frac{z \cos \phi}{R^2 \sin^2 \phi + z^2} \left(1 + \frac{R \cos \phi}{\sqrt{R^2 + z^2}} \right) d\phi \\ &= -\frac{N}{2\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \int_0^{2\pi} \frac{\Delta \cos \phi}{\sin^2 \phi + \Delta^2} \left(1 + \frac{\cos \phi}{\sqrt{1 + \Delta^2}} \right) d\phi, \\ B_{h,\phi,n}(R, z) &= \frac{1}{\pi} \int_0^{2\pi} B_{h,\phi}(R, \phi, z) \cos(n\phi) d\phi \\ &= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \int_0^{2\pi} \frac{z \cos \phi}{R^2 \sin^2 \phi + z^2} \left(1 + \frac{R \cos \phi}{\sqrt{R^2 + z^2}} \right) \cos(n\phi) d\phi \\ &= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \int_0^{2\pi} \frac{\Delta \cos \phi}{\sin^2 \phi + \Delta^2} \left(1 + \frac{\cos \phi}{\sqrt{1 + \Delta^2}} \right) \cos(n\phi) d\phi, \end{aligned}$$

for $n \bmod N = 0$ and $n \neq 0$, where

$$\Delta = z/R.$$

Note that,

$$\begin{aligned} \int_0^{2\pi} \frac{\cos \phi}{\sin^2 \phi + \Delta^2} d\phi &= 0, \\ \int_0^{2\pi} \frac{\cos^2 \phi}{\sin^2 \phi + \Delta^2} d\phi &= 2\pi \left(\frac{\sqrt{1 + \Delta^2}}{|\Delta|} - 1 \right). \end{aligned}$$

Hence,

$$\begin{aligned} B_{h,\phi,0}(R, z) &= -\frac{N}{2\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \frac{\Delta}{\sqrt{1 + \Delta^2}} 2\pi \left(\frac{\sqrt{1 + \Delta^2}}{|\Delta|} - 1 \right) \\ &= N \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \left(\frac{z}{\sqrt{R^2 + z^2}} - \frac{z}{|z|} \right). \end{aligned}$$

Note that

$$\int_0^{2\pi} \frac{\cos \phi}{\sin^2 \phi + \Delta^2} \cos(n\phi) d\phi = 0,$$

for n even.

$$\int_0^{2\pi} \frac{\cos^2 \phi}{\sin^2 \phi + \Delta^2} \cos(n\phi) d\phi = \frac{2\pi}{|\Delta|} \sqrt{1 + \Delta^2} \left(\sqrt{1 + \Delta^2} - |\Delta| \right)^n,$$

for $n \geq 2$ and even. For proof, see this stack exchange link. Hence,

$$\begin{aligned} B_{h,\phi,n}(R, z) &= -\frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \frac{1}{R} \frac{\Delta}{\sqrt{1 + \Delta^2}} \frac{2\pi}{|\Delta|} \sqrt{1 + \Delta^2} \left(\sqrt{1 + \Delta^2} - |\Delta| \right)^n \\ &= -\frac{z}{|z|} N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \left(\sqrt{1 + \left(\frac{z}{R} \right)^2} - \left| \frac{z}{R} \right| \right)^n \\ &= -\frac{z}{|z|} N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n. \end{aligned}$$

3.3 Fourier coefficients of the vertical component of the magnetic field

Our goal in this section is to calculate this integral

$$\begin{aligned} B_{h,z,n}(R, z) &= \frac{N}{\pi} \int_0^{2\pi} B_{h,z,0}(R, \phi, z) \sin(n\phi) d\phi \\ &= \frac{N}{\pi} \frac{\mu_0 I_0}{4\pi} \int_0^{2\pi} \frac{R \sin \phi}{R^2 \sin^2 \phi + z^2} \left(1 + \frac{R \cos \phi}{\sqrt{R^2 + z^2}} \right) \sin(n\phi) d\phi \\ &= -\frac{R}{z} B_{h,R,n}(R, z) \end{aligned}$$

for $n \bmod N = 0$. Since

$$B_{R,h,n}(R, z) = -N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \frac{z}{\sqrt{R^2 + z^2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n$$

we know that

$$B_{z,h,n}(R, z) = N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \frac{R}{\sqrt{R^2 + z^2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n$$

4 Full magnetic field

The full magnetic field is given by

$$\mathbf{B}(R, \phi, z) = N \mathbf{B}_v(R, \phi, z) + \mathbf{B}_h(R, \phi, z - h) - \mathbf{B}_h(R, \phi, z + h).$$

Hence,

$$B_R(R, \phi, z) = N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \sum_{n=N}^{\infty} \left[\frac{h - z}{\sqrt{R^2 + (z - h)^2}} \left(\frac{\sqrt{R^2 + (z - h)^2} - |z - h|}{R} \right)^n + \frac{z + h}{\sqrt{R^2 + (z + h)^2}} \left(\frac{\sqrt{R^2 + (z + h)^2} - |z + h|}{R} \right)^n \right] \sin(n\phi),$$

$$B_\phi(R, \phi, z) = N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \left\{ 1 + \sum_{n=N}^{\infty} \left[\left(\frac{\sqrt{R^2 + (z - h)^2} - |z - h|}{R} \right)^n + \left(\frac{\sqrt{R^2 + (z + h)^2} - |z + h|}{R} \right)^n \right] \right\} \cos(n\phi),$$

$$B_z(R, \phi, z) = N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \sum_{n=N}^{\infty} \left[\frac{R}{\sqrt{R^2 + (z-h)^2}} \left(\frac{\sqrt{R^2 + (z-h)^2} - |z-h|}{R} \right)^n - \frac{R}{\sqrt{R^2 + (z+h)^2}} \left(\frac{\sqrt{R^2 + (z+h)^2} - |z+h|}{R} \right)^n \right] \sin(n\phi),$$

for $n \bmod N = 0$ and $-h < z < h$.

5 Check divergence is zero

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \frac{1}{R} \frac{\partial}{\partial R} (R B_R) + \frac{1}{R} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}. \\ \frac{1}{R} \frac{\partial}{\partial R} (R B_{R,h,n}(R, z)) &= \frac{1}{R} \frac{\partial}{\partial R} \left\{ -N \frac{\mu_0 I_0}{2\pi} \frac{z}{\sqrt{R^2 + z^2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\} \\ &= -N \frac{\mu_0 I_0}{2\pi} \frac{z}{R} \frac{\partial}{\partial R} \left\{ \frac{1}{\sqrt{R^2 + z^2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\} \\ &= -N \frac{\mu_0 I_0}{2\pi} \frac{z}{R} \left\{ -\frac{R}{(R^2 + z^2)^{3/2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n + \right. \\ &\quad \left. n \frac{|z| \sqrt{R^2 + z^2} - z^2}{R^2 (R^2 + z^2)} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^{n-1} \right\} \\ &= -N \frac{\mu_0 I_0}{2\pi} \frac{z}{R} \left\{ -\frac{R}{(R^2 + z^2)^{3/2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n + \right. \\ &\quad \left. n \frac{|z|}{R (R^2 + z^2)} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\} \\ &= -N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \left\{ -\frac{zR}{(R^2 + z^2)^{3/2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n + \right. \\ &\quad \left. n \frac{|z|}{z} \frac{z^2}{R (R^2 + z^2)} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\} \end{aligned}$$

$$\begin{aligned}
\frac{\partial B_{z,h,n}(R,z)}{\partial z} &= N \frac{\mu_0 I_0}{2\pi} \frac{\partial}{\partial z} \left\{ \frac{1}{\sqrt{R^2 + z^2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\} \\
&= N \frac{\mu_0 I_0}{2\pi} \left\{ -\frac{z}{(R^2 + z^2)^{3/2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n + \right. \\
&\quad \left. n \frac{z|z| - z\sqrt{R^2 + z^2}}{R|z|(R^2 + z^2)} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^{n-1} \right\} \\
&= N \frac{\mu_0 I_0}{2\pi} \left\{ -\frac{z}{(R^2 + z^2)^{3/2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n - \right. \\
&\quad \left. n \frac{z}{|z|(R^2 + z^2)} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\} \\
&= N \frac{\mu_0 I_0}{2\pi} \frac{1}{R} \left\{ -\frac{zR}{(R^2 + z^2)^{3/2}} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n - \right. \\
&\quad \left. n \frac{z}{|z|} \frac{R^2}{R(R^2 + z^2)} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{R} \frac{\partial}{\partial R} (RB_{R,h,n}) + \frac{\partial B_{z,h,n}}{\partial z} &= -n \frac{z}{|z|} N \frac{\mu_0 I_0}{2\pi} \frac{1}{R^2} \left(\frac{\sqrt{R^2 + z^2} - |z|}{R} \right)^n. \\
\implies \frac{1}{R} \frac{\partial}{\partial R} (RB_{R,h,n}) - \frac{n}{R} B_{\phi,h,n} + \frac{\partial B_{z,h,n}}{\partial z} &= 0,
\end{aligned}$$

which confirms that the divergence of the magnetic field is zero.