Analytical Approximations of Vacuum Magnetic Fields in Tokamaks

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1 Magnetic field produced by a vertical wire centred at the origin

The magnetic field produce by a vertical wire, centred at the origin, with current, I which points in the $\hat{\mathbf{h}}$ direction and extends from $-h_{min}, h_{max}$ is given by the following Biot-Savart law expression:

$$\begin{aligned}\mathbf{B}_v(R,\phi,h) &= \frac{\mu_0 I}{4\pi}\int_{h_{min}}^{h_{max}} \frac{\mathbf{dh'}\times(R\hat{\mathbf{R}}+(h-h')\hat{\mathbf{h}})}{(R^2+(h-h')^2)^{3/2}} \\ &= \frac{\mu_0 I}{4\pi}\int_{h_{min}}^{h_{max}} \frac{R dh'\hat{\phi}}{(R^2+(h-h')^2)^{3/2}}.\end{aligned}$$

Let

$$\cos\theta = \frac{R}{\sqrt{R^2+(h-h')^2}},$$

$$\implies \sin\theta = \frac{h-h'}{\sqrt{R^2+(h-h')^2}},$$

$$\implies \tan\theta = \frac{h-h'}{R},$$

$$\implies h' = h - R\tan\theta,$$

$$\implies dh' = -R\sec^2\theta d\theta.$$

Hence

$$\boxed{\begin{aligned}\mathbf{B}_v(R,\phi,h) &= -\frac{\mu_0 I}{4\pi R}\int_{h_{min}}^{h_{max}} \cos\theta d\theta\hat{\phi} \\ &= \frac{\mu_0 I}{4\pi R}[-\sin\theta]_{h_{min}}^{h_{max}}\hat{\phi} \\ &= \frac{\mu_0 I}{4\pi R}\left[\frac{h_{max}-h}{\sqrt{R^2+(h_{max}-h)^2}}+\frac{h-h_{min}}{\sqrt{R^2+(h-h_{min})^2}}\right]\hat{\phi}.\end{aligned}}$$

2 Vacuum Field from Vertical Wires in Picture-Frame TF Coils

To calculate the vacuum magnetic field induced by the vertical wires in the picture-frame toroidal field coils of a Tokamak, it is convenient to express the field as a complex function, $B_{v,c}$. In this representation, the real and imaginary parts correspond to the x- and y-components of the magnetic field, respectively. Furthermore, the radial and ahimuthal components can be obtained from the real and imaginary parts of $e^{-i\phi}B_{v,c}$, respectively. Let

$$z = x + iy = R\exp(i\phi),$$

$$\bar{z} = x - iy = R\exp(-i\phi).$$

Hence,

$$B_{v,c}(z;h) = i\frac{\mu_0 I}{4\pi}\frac{z}{|z|^2}\left[\frac{h_{max}-h}{\sqrt{|z|^2+(h_{max}-h)^2}} + \frac{h-h_{min}}{\sqrt{|z|^2+(h-h_{min})^2}}\right].$$

Let

$$z_k = R_0 e^{i\phi_k}.$$

Subsituting $z \to z - z_k$ above gives

$$B_{v,c}(z-z_k;h) = i\frac{\mu_0 I}{4\pi}\frac{z-z_k}{|z-z_k|^2}\left[\frac{h_{max}-h}{\sqrt{|z-z_k|^2+(h_{max}-h)^2}} + \frac{h-h_{min}}{\sqrt{|z-z_k|^2+(h-h_{min})^2}}\right].$$

Let

$$\phi_k = 2\pi\frac{k}{N},$$

where N equals the number of toroidal field (TF) coils. Hence, the magnetic field given a complete set of vertical wires in the picture frame TF coils at $R = R_0$ is given by

$$B_{v,\Sigma,c}(z;h,R_0) = i\frac{\mu_0 I}{4\pi}\sum_{k=0}^{N-1}\frac{z-z_k}{|z-z_k|^2}\left[\frac{h_{max}-h}{\sqrt{|z-z_k|^2+(h_{max}-h)^2}} + \frac{h-h_{min}}{\sqrt{|z-z_k|^2+(h-h_{min})^2}}\right].$$

At this stage it convenient to define a new function

$$B_{v,\Sigma,c,p}(z) = B_{v,\Sigma,c}(z)\exp(-i\phi),$$

where the p stands for polar. Note that the real and imaginary components of $B_{v,\Sigma,c}$ give the x and y components of the magnetic field, while the real and imaginary components of $B_{v,\Sigma,c,p}$ give the radial and azimuthal components of the magnetic field.

We now wish to express $B_{v,\Sigma,c,p}$ as a fourier series given by

$$B_{v,\Sigma,c,p}(z) = \sum_{n=-\infty}^{\infty} B_n(R)e^{in\phi}.$$

Note that

$$\begin{aligned} B_n &= \frac{1}{2\pi}\int_0^{2\pi} B_{v,\Sigma,c,p}(z)e^{-in\phi}d\phi \\ &= \begin{cases} 0 & \text{if } n \bmod N \neq 0, \\ iN\frac{\mu_0 I}{8\pi^2}\int_0^{2\pi}\frac{z-R_0}{|z-R_0|^2}e^{-i\phi}\left[\frac{h_{max}-h}{\sqrt{|z-R_0|^2+(h_{max}-h)^2}} + \frac{h-h_{min}}{\sqrt{|z-R_0|^2+(h-h_{min})^2}}\right]e^{-in\phi}d\phi & \text{if } n \bmod N = 0, \end{cases} \end{aligned}$$

where we have used the fact that $B_{v,\Sigma,c}(R,\phi) = B_{v,\Sigma,c}(R,\phi+\phi_k)$. We now want to calculate this integral which is quite tricky so we will first consider the simpler scenario where $h_{max}, -h_{min} \to \infty$.

2.1 B_n for case where $h_{max}, -h_{min} \to \infty$

$$B_n(R) = iN\frac{\mu_0 I}{4\pi^2}\int_0^{2\pi}\frac{z-R_0}{|z-R_0|^2}e^{-i\phi}e^{-in\phi}d\phi.$$

Let

$$Z = e^{i\phi},$$

$$\implies d\phi = \frac{dZ}{iZ},$$

$$z = RZ.$$

Note that

$$\frac{z-R_0}{|z-R_0|^2}e^{-i\phi} = \frac{1}{\bar{z}-R_0}e^{-i\phi} = \frac{1}{R-R_0e^{i\phi}}.$$

Hence

$$\begin{aligned} B_n(R) &= N\frac{\mu_0 I}{4\pi^2}\oint_{|Z|=1}\frac{1}{(R-R_0Z)Z^{n+1}}dZ \\ &= -N\frac{\mu_0 I}{4\pi^2}\frac{1}{R_0}\oint_{|Z|=1}\frac{1}{(Z-R/R_0)Z^{n+1}}dZ \end{aligned}$$

The $|Z|=1$ disk contains a pole of order n at $Z=0$ if $n\geq 1$ and a pole of order 1 at $Z=R/R_0$ if $R<R_0$.

For $n\leq -1$ and $R<R_0$:

$$\begin{aligned} B_n(R) &= -N\frac{\mu_0 I}{4\pi^2}\frac{2\pi i}{R_0}[Res(Z=R/R_0)] \\ &= -iN\frac{\mu_0 I}{2\pi}\frac{1}{R_0}\left(\frac{R}{R_0}\right)^{-n-1} \\ &= -iN\frac{\mu_0 I}{2\pi}\frac{1}{R}\left(\frac{R}{R_0}\right)^{-n}. \end{aligned}$$

For $n\geq 0$ and $R\leq R_0$:

$$\begin{aligned} B_n(R) &= -N\frac{\mu_0 I}{4\pi^2}\frac{2\pi i}{R_0}[Res(Z=R/R_0)+Res(Z=0)] \\ &= -N\frac{\mu_0 I}{4\pi^2}\frac{2\pi i}{R_0}\left[\left(\frac{R}{R_0}\right)^{-n}-\left(\frac{R}{R_0}\right)^{-n}\right] \\ &= 0 \end{aligned}$$

For $n\leq -1$ and $R>R_0$

$$B_n(R)=0.$$

For $n\geq 0$ and $R>R_0$

$$\begin{aligned} B_n(R) &= -N\frac{\mu_0 I}{4\pi^2}\frac{2\pi i}{R_0}[Res(Z=0)] \\ &= iN\frac{\mu_0 I}{2\pi}\frac{1}{R}\left(\frac{R_0}{R}\right)^{n}. \end{aligned}$$

Hence,

$$\boxed{B_{v,\Sigma,c,p} = i\frac{\mu_0 NI}{2\pi R}\begin{cases} -(R/R_0)^N e^{-iN\phi}-(R/R_0)^{2N}e^{-2iN\phi}-... & \text{if } R<R_0 \\ 1+(R_0/R)^N e^{iN\phi}+(R_0/R)^{2N}e^{2iN\phi}+... & \text{if } R>R_0. \end{cases}}$$

Note that B_R and B_ϕ are given by the real and imaginary parts of the above expression. Hence

$$\boxed{B_R = \frac{\mu_0 NI}{2\pi R}\begin{cases} -(R/R_0)^N\sin(N\phi)-(R/R_0)^{2N}\sin(2N\phi)-... & \text{if } R<R_0 \\ 1+(R_0/R)^N\sin(N\phi)+(R_0/R)^{2N}\sin(2N\phi)+... & \text{if } R>R_0, \end{cases}}$$

$$\boxed{B_\phi = \frac{\mu_0 NI}{2\pi R}\begin{cases} -(R/R_0)^N\cos(N\phi)-(R/R_0)^{2N}\cos(2N\phi)-... & \text{if } R<R_0 \\ 1+(R_0/R)^N\cos(N\phi)+(R_0/R)^{2N}\cos(2N\phi)+... & \text{if } R>R_0. \end{cases}}$$

2.2 B_n for case where $2RR_0/[(R^2+R_0^2+a^2)]\ll 1$

Consider the function

$$\begin{aligned} f_a(z) &= \frac{a}{\sqrt{|z-R_0|^2+a^2}} \\ &= \frac{a}{\sqrt{R^2+R_0^2+2RR_0\cos\phi+a^2}}. \end{aligned}$$

By the AM-GM inequality

$$\frac{2RR_0}{R^2+R_0^2} \leq 1,$$

with its maximum value at $R = R_0$. Let

$$\epsilon = \frac{2RR_0}{(R^2+R_0^2+a^2)}$$

We can approximate $f_a(z)$ with the following binomial expansion

$$\begin{aligned} f_a(z) &= \frac{a}{\sqrt{R^2+R_0^2+a^2}\sqrt{1+\epsilon\cos\phi}} \\ &= \frac{a}{\sqrt{R^2+R_0^2+a^2}}\left(1-\frac{1}{2}\epsilon\cos\phi+O(\epsilon^2)\right). \end{aligned}$$

Hence, using the equation at the end of Section 2.1 and substituting $a \to h_{max} - h$ and $a \to h - h_{max}$ gives

$$\boxed{B_R(R,\phi,h) = \frac{\mu_0 NI}{4\pi R}\begin{cases} -\left(\frac{R}{R_0}\right)^N\left(\frac{h_{max}-h}{\sqrt{R^2+R_0^2+(h_{max}-h)^2}}+\frac{h-h_{min}}{\sqrt{R^2+R_0^2+(h-h_{min})^2}}+O(\epsilon)\right)\sin(N\phi)+O(\frac{R}{R_0})^{2N} & \text{if } R < R_0 \\ 1+\left(\frac{R}{R_0}\right)^N\left(\frac{h_{max}-h}{\sqrt{R^2+R_0^2+(h_{max}-h)^2}}+\frac{h-h_{min}}{\sqrt{R^2+R_0^2+(h-h_{min})^2}}+O(\epsilon)\right)\sin(N\phi)+O(\frac{R}{R_0})^{2N} & \text{if } R > R_0, \end{cases}}$$

$$\boxed{B_Z(R,\phi,h) = \frac{\mu_0 NI}{4\pi R}\begin{cases} 1+\left(\frac{R}{R_0}\right)^N\left(\frac{h_{max}-h}{\sqrt{R^2+R_0^2+(h_{max}-h)^2}}+\frac{h-h_{min}}{\sqrt{R^2+R_0^2+(h-h_{min})^2}}+O(\epsilon)\right)\cos(N\phi)+O(\frac{R}{R_0})^{2N} & \text{if } R > R_0, \end{cases}}$$