Approximate Fourier Series of the Magnetic Field Induced by a Single Picture-Frame Coil

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February 19, 2025

In this document we will calculate the fourier series in toroidal angle, ϕ, for a single picture-frame coil. The picture-frame coil is made up of two infinitely long vertical wires which extend from $z \to -\infty$ to $z \to \infty$ and two horizontal wires which extend from $x = 0$ to $x \to \infty$. The current in the vertical wires is I_0. Note that we use infinite wires to simplify the calculations. We will later show numerically that our values agree well with the exact values obtained numerically. We calculate for a single picture-frame coil, however, it's trivial to extend the results to N picture-frame coils evenly space around a Tokamak.

1 Fourier series of vertical wire at $x = R_0$, $y = 0$

The magnetic field induced by an infinite vertical wire at $x = R_0$ and $y = 0$ is given by

$$\begin{aligned}
\mathbf{B}_v(R, \phi, z; R_0) &= \frac{\mu_0 I_0}{2\pi} \frac{-y\hat{\mathbf{x}} + (x - R_0)\hat{\mathbf{y}}}{(x - R_0)^2 + y^2} \\
&= \frac{\mu_0 I_0}{2\pi R} \frac{-R_0 y\hat{\mathbf{R}} + (R^2 - R_0 x)\hat{\phi}}{(x - R_0)^2 + y^2} \\
&= \frac{\mu_0 I_0}{2\pi R} \left\{ \mathrm{Im}\left[\frac{1}{1 - (R_0/R)\exp(-i\phi)} \right] \hat{\mathbf{R}} + \mathrm{Re}\left[\frac{1}{1 - (R_0/R)\exp(-i\phi)} \right] \hat{\phi} \right\} \\
&= -\frac{\mu_0 I_0}{2\pi R} \left\{ \mathrm{Im}\left[\frac{(R/R_0)\exp(i\phi)}{1 - (R/R_0)\exp(i\phi)} \right] \hat{\mathbf{R}} + \mathrm{Re}\left[\frac{(R/R_0)\exp(i\phi)}{1 - (R/R_0)\exp(i\phi)} \right] \hat{\phi} \right\}.
\end{aligned}$$

where

$$\begin{aligned}
R &= x^2 + y^2, \\
x &= R\cos\phi, \\
y &= R\sin\phi, \\
\hat{\mathbf{R}} &= \frac{1}{R}[x\hat{\mathbf{x}} + y\hat{\mathbf{y}}], \\
\hat{\phi} &= \frac{1}{R}[-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}].
\end{aligned}$$

Hence, using the formula for a geometric series,

$$\mathbf{B}_v(R, \phi, z; R_0) = \quad = \frac{\mu_0 I_0}{2\pi R} \sum_{n=0}^{\infty} \left\{ -\left[\left(\frac{R_0}{R} \right)^n \sin(n\phi) \right] \hat{\mathbf{R}} + \left[\left(\frac{R_0}{R} \right)^n \cos(n\phi) \right] \hat{\phi} \right\}.$$

for $R > R_0$ and

$$\mathbf{B}_v(R, \phi, z; R_0) = \quad = -\frac{\mu_0 I_0}{2\pi R} \sum_{n=1}^{\infty} \left\{ \left[\left(\frac{R}{R_0} \right)^n \sin(n\phi) \right] \hat{\mathbf{R}} + \left[\left(\frac{R}{R_0} \right)^n \cos(n\phi) \right] \hat{\phi} \right\},$$

for $R < R_0$.

2 Fourier series of horizontal wire at $y = z = 0$

The magnetic field induced by a vertical which extends from $z = z_{min}$ to $z = z_{max}$ can be calculated using the Biot-Savart law to give

$$\mathbf{B}'_v(R,\phi,z) = \frac{\mu_0 I_0}{4\pi R}\left[\frac{z_{max}-z}{\sqrt{R^2+(z_{max}-z)^2}} + \frac{z-z_{min}}{\sqrt{R^2+(z-z_{min})^2}}\right]\hat{\phi}.$$

Hence, the magnetic field induced by a horizontal wire at $y = z = 0$ and extends from $x = 0$ to $x \to \infty$ is given by

$$\begin{aligned}\mathbf{B}_h(x,y,z) &= \frac{\mu_0 I_0}{4\pi}\frac{-z\hat{\mathbf{y}}+y\hat{\mathbf{z}}}{y^2+z^2}\left[1+\frac{x}{\sqrt{x^2+y^2+z^2}}.\right]\\ &= \frac{\mu_0 I_0}{4\pi}\frac{-z\sin\phi\hat{\mathbf{R}} - z\cos\phi\hat{\phi} + R\sin\phi\hat{\mathbf{z}}}{R^2\sin^2\phi+z^2}\left[1+\frac{R\cos\phi}{\sqrt{R^2+z^2}}\right].\end{aligned}$$

We can show that

$$\begin{aligned}B_{R,h}(R,\phi,z) &= -\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\frac{z}{\sqrt{R^2+z^2}}\sum_{n=1}^{\infty}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\sin(n\phi),\\ B_{\phi,h}(R,\phi,z) &= \frac{\mu_0 I_0}{2\pi}\frac{1}{R}\left\{\frac{z}{2\sqrt{R^2+z^2}} - \frac{z}{2|z|} - \frac{z}{|z|}\sum_{n=1}^{\infty}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\cos(n\phi)\right\},\\ B_{z,h}(R,\phi,z) &= \frac{\mu_0 I_0}{2\pi}\frac{1}{R}\frac{R}{\sqrt{R^2+z^2}}\sum_{n=1}^{\infty}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\sin(n\phi).\end{aligned}$$

To prove the above is quite involved, however, by going to the following Desmos link you can see for yourself that the above is almost certainly true: `https://www.desmos.com/calculator/xyl8dvtot4`.

3 Fourier series of a single picture-frame coil

Combing results from the previous section, we can write the magnetic field for a single picture-frame coil as

$$\mathbf{B}(R,\phi,z) = \mathbf{B}_v(R,\phi,z;R_{inner}) - \mathbf{B}_v(R,\phi,z;R_{outer}) + \mathbf{B}_h(R,\phi,z-h) - \mathbf{B}_h(R,\phi,z+h),$$

where R_{inner} and R_{outer} are the inner and outer radii of the picture-frame coil, respectively, and h is the half the height of the picture-frame coil. We can write each component as

$$\boxed{\begin{aligned}B_R(R,\phi,z) = \frac{\mu_0 I_0}{2\pi R}\sum_{n=1}^{\infty}\Bigg[&\left(\frac{R_{outer}}{R}\right)^n - \left(\frac{R}{R_{inner}}\right)^n + \\ &\frac{h-z}{\sqrt{R^2+(z-h)^2}}\left(\frac{\sqrt{R^2+(z-h)^2}-(h-z)}{R}\right)^n + \\ &\frac{z+h}{\sqrt{R^2+(z+h)^2}}\left(\frac{\sqrt{R^2+(z+h)^2}-(z+h)}{R}\right)^n\Bigg]\sin(n\phi),\end{aligned}}$$

$$\boxed{\begin{aligned}B_\phi(R,\phi,z) = \frac{\mu_0 I_0}{2\pi R}\Bigg\{1+\sum_{n=1}^{\infty}\Bigg[&\left(\frac{R}{R_{inner}}\right)^n + \left(\frac{R_{outer}}{R}\right)^n + \\ &\left(\frac{\sqrt{R^2+(z-h)^2}-(h-z)}{R}\right)^n + \\ &\left(\frac{\sqrt{R^2+(z+h)^2}-(z+h)}{R}\right)^n\Bigg]\cos(n\phi)\Bigg\},\end{aligned}}$$

$$\boxed{\begin{aligned}B_z(R,\phi,z) = \frac{\mu_0 I_0}{2\pi R}\sum_{n=1}^{\infty}\Bigg[&\frac{R}{\sqrt{R^2+(z-h)^2}}\left(\frac{\sqrt{R^2+(z-h)^2}-(h-z)}{R}\right)^n - \\ &\frac{R}{\sqrt{R^2+(z+h)^2}}\left(\frac{\sqrt{R^2+(z+h)^2}-(z+h)}{R}\right)^n\Bigg]\sin(n\phi),\end{aligned}}$$

for $R_{inner} < R < R_{outer}$, $-h < z < h$. Note that the $n = 0$ term we set to $\frac{\mu_0 I_0}{2\pi}\hat{\phi}$, despite the forumula from the previous section suggesting otherwise. This is because we know the $n = 0$ term from Ampere's law and the difference is caused by our use of infinite wires instead of finite wires.