Analytic Approximation of the Vacuum Magnetic Field Induced by Infinitely Wide Picture Frame coils

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1 Magnetic field induced by a vertical wire centred at the origin

The magnetic field induced by a vertical wire, centred at the origin, with current, I_0 which points in the $\hat{\mathbf{z}}$ direction and extends from $-z_{min}, z_{max}$ is given by the following Biot-Savart law expression:

$$\begin{aligned}\mathbf{B}_v(R,\phi,z) &= \frac{\mu_0 I_0}{4\pi}\int_{z_{min}}^{z_{max}} \frac{\mathbf{dl'}\times(R\hat{\mathbf{R}}+(z-z')\hat{\mathbf{z}})}{(R^2+(z-z')^2)^{3/2}}\\ &= \frac{\mu_0 I_0}{4\pi}\int_{z_{min}}^{z_{max}} \frac{R dh'\hat{\phi}}{(R^2+(z-z')^2)^{3/2}}.\end{aligned}$$

Let

$$\cos\theta = \frac{R}{\sqrt{R^2+(z-z')^2}},$$

$$\implies \sin\theta = \frac{z-z'}{\sqrt{R^2+(z-z')^2}},$$

$$\implies \tan\theta = \frac{z-z'}{R},$$

$$\implies z' = z - R\tan\theta,$$

$$\implies dz' = -R\sec^2\theta d\theta.$$

Hence

$$\begin{aligned}\mathbf{B}_v(R,\phi,z) &= -\frac{\mu_0 I_0}{4\pi R}\int_{z_{min}}^{z_{max}} \cos\theta d\theta\hat{\phi}\\ &= \frac{\mu_0 I_0}{4\pi R}[-\sin\theta]_{z_{min}}^{z_{max}}\hat{\phi}\\ &= \frac{\mu_0 I_0}{4\pi R}\left[\frac{z_{max}-z}{\sqrt{R^2+(z_{max}-z)^2}} + \frac{z-z_{min}}{\sqrt{R^2+(z-z_{min})^2}}\right]\hat{\phi}.\end{aligned}$$

Let $z_{max} = h$ and $z_{min} = -h$:

$$\boxed{\begin{aligned}\mathbf{B}_v(R,\phi,z) &= -\frac{\mu_0 I_0}{4\pi R}\int_{z_{min}}^{z_{max}} \cos\theta d\theta\hat{\phi}\\ &= \frac{\mu_0 I_0}{4\pi R}[-\sin\theta]_{z_{min}}^{z_{max}}\hat{\phi}\\ &= \frac{\mu_0 I_0}{4\pi R}\left[\frac{h-z}{\sqrt{R^2+(h-z)^2}} + \frac{z+h}{\sqrt{R^2+(z+h)^2}}\right]\hat{\phi}.\end{aligned}}$$

2 Magnetic field induced by a horizontal wire extending from the origin to infinity

Let $\mathbf{B}_{h,c,k}(x, y, z)$ denote the magnetic field induced by a horizontal wire in cartesian coordinates extending from the origin to infinity with current, I_0, in the

$$\hat{\mathbf{R}}_k = \cos(\phi_k)\hat{\mathbf{x}} + \sin(\phi_k)\hat{\mathbf{y}}$$

direction, where

$$\phi_k = k\frac{2\pi}{N},$$

N is the number of picture-frame coils. For $k = 0$, $\mathbf{B}_{h,c,k}(x, y, z)$ points in the $\hat{\mathbf{x}}$ direction and is given by

$$\mathbf{B}_{h,c,0}(x, y, z) = \frac{\mu_0 I_0}{4\pi} \frac{-z\hat{\mathbf{y}} + y\hat{\mathbf{z}}}{y^2 + z^2} \left[1 + \frac{x}{\sqrt{x^2 + y^2 + z^2}}.\right]$$

Let $\mathbf{B}_{h,k}$ denote the same magnetic field as $\mathbf{B}_{h,c,k}$ but in cylindrical coordinates. Then

$$\mathbf{B}_{h,0}(R, \phi, z) = \frac{\mu_0 I_0}{4\pi} \frac{-z\sin\phi\hat{\mathbf{R}} - z\cos\phi\hat{\phi} + R\sin\phi\hat{\mathbf{z}}}{R^2\sin^2\phi + z^2} \left[1 + \frac{R\cos\phi}{\sqrt{R^2 + z^2}}\right].$$

3 Fourier series of the magnetic field induced by a horizontal wires extending from the origin to infinity

Let

$$\mathbf{B}_h(R, \phi, z) = \sum_{k=0}^{N-1} \mathbf{B}_{h,k}(R, \phi, z).$$

We now wish to express

$$\mathbf{B}_h(R, \phi, z) = B_{h,R}(R, \phi, z)\hat{\mathbf{R}} + B_{h,\phi}(R, \phi, z)\hat{\phi} + B_{h,z}(R, \phi, z)\hat{\mathbf{z}},$$

as a Fourier series in ϕ. The picture-frame coils are positioned to ensure that $B_{h,R}$ and $B_{h,z}$ are odd functions of ϕ and $B_{h,\phi}$ is an even function of ϕ. Hence

$$B_{h,R}(R, \phi, z) = \sum_{n=N}^{\infty} B_{h,R,n}(R, z)\sin(n\phi),$$

$$B_{h,\phi}(R, \phi, z) = \sum_{n=0}^{\infty} B_{h,\phi,n}(R, z)\cos(n\phi),$$

$$B_{h,z}(R, \phi, z) = \sum_{n=N}^{\infty} B_{h,z,n}(R, z)\sin(n\phi).$$

where $B_{h,R,n}(R, z) = B_{h,\phi,n}(R, z) = B_{h,z,n}(R, z) = 0$ for $n \bmod N \neq 0$. To be clear, $B_{h,R,n}(R, z) \neq B_{h,R,n}(R, \phi, z)$, $B_{h,\phi,n}(R, z) \neq B_{h,\phi,n}(R, \phi, z)$, $B_{h,z,n}(R, z) \neq B_{h,z,n}(R, \phi, z)$. From this point on we will assume $n \bmod N = 0$.

Note that

$$\begin{aligned} B_{h,R,n}(R, z) &= \frac{1}{\pi}\int_0^{2\pi} B_{h,R}(R, \phi, z)\sin(n\phi)d\phi \\ &= \frac{N}{\pi}\int_0^{2\pi} B_{h,R,0}(R, \phi, z)\sin(n\phi)d\phi, \end{aligned}$$

for $n \bmod N = 0$.

$$
\begin{aligned}
B_{h,\phi,0}(R,z) &= \frac{1}{2\pi}\int_0^{2\pi} B_{h,\phi}(R,\phi,z)d\phi \\
&= \frac{N}{2\pi}\int_0^{2\pi} B_{h,\phi,0}(R,\phi,z)d\phi, \\
B_{h,\phi,n}(R,z) &= \frac{1}{\pi}\int_0^{2\pi} B_{h,\phi}(R,\phi,z)\cos(n\phi)d\phi \\
&= \frac{N}{\pi}\int_0^{2\pi} B_{h,\phi,0}(R,\phi,z)\cos(n\phi)d\phi,
\end{aligned}
$$

for $n \bmod N = 0$ and $n \neq 0$.

$$
\begin{aligned}
B_{h,z,n}(R,z) &= \frac{1}{\pi}\int_0^{2\pi} B_{h,z}(R,\phi,z)\sin(n\phi)d\phi \\
&= \frac{N}{\pi}\int_0^{2\pi} B_{h,z,0}(R,\phi,z)\sin(n\phi)d\phi,
\end{aligned}
$$

for $n \bmod N = 0$.

3.1 Fourier coefficients of the radial component of the magnetic field

Our goal in this section is to calculate this integral

$$
\begin{aligned}
B_{R,h,n}(R,z) &= \frac{N}{\pi}\int_0^{2\pi} B_{R,h,0}(R,\phi,z)\sin(n\phi)d\phi \\
&= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\int_0^{2\pi}\frac{z\sin\phi}{R^2\sin^2\phi+z^2}\left(1+\frac{R\cos\phi}{\sqrt{R^2+z^2}}\right)\sin(n\phi)d\phi \\
&= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\int_0^{2\pi}\frac{\Delta\sin\phi}{\sin^2\phi+\Delta^2}\left(1+\frac{\cos\phi}{\sqrt{1+\Delta^2}}\right)\sin(n\phi)d\phi.
\end{aligned}
$$

Note that

$$
\int_0^{2\pi}\frac{\sin(\phi)\sin(n\phi)}{\sin^2\phi+\Delta^2}d\phi = 0,
$$

for n odd.

$$
\begin{aligned}
\int_0^{2\pi}\frac{\sin(\phi)\cos(\phi)\sin(n\phi)}{\sin^2\phi+\Delta^2}d\phi &= \frac{1}{2}\int_0^{2\pi}\frac{\sin(2\phi)\sin(n\phi)}{\sin^2\phi+\Delta^2}d\phi \\
&= \int_0^{2\pi}\frac{\sin(2\phi)\sin(n\phi)}{1+2\Delta^2-\cos(2\phi)}d\phi.
\end{aligned}
$$

Let

$$
\begin{aligned}
\Delta^2 &= \frac{(1-a)^2}{4a}, \\
a &= \left(\sqrt{1+\Delta^2}-|\Delta|\right)^2.
\end{aligned}
$$

Hence,

$$
\int_0^{2\pi}\frac{\sin(\phi)\cos(\phi)\sin(n\phi)}{\sin^2\phi+\Delta^2}d\phi = 2\int_0^{2\pi}\frac{a\sin(2\phi)}{1+a^2-2a\cos(2\phi)}\sin(n\phi)d\phi.
$$

Note that

$$
\begin{aligned}
\frac{1}{1-a\exp(2i\phi)} &= \frac{1-a\cos(2\phi)}{1+a^2-2a\cos(2\phi)} + i\frac{a\sin(2\phi)}{1+a^2-2a\cos(2\phi)} \\
&= \sum_{k=0}^{\infty} a^k\exp(2ik\phi).
\end{aligned}
$$

Taking the imaginary part gives

$$\frac{a\sin(2\phi)}{1+a^2-2a\cos(2\phi)} = \sum_{k=1} a^k \sin(2k\phi).$$

Hence,

$$\begin{aligned}\int_0^{2\pi} \frac{\sin(\phi)\cos(\phi)\sin(n\phi)}{\sin^2\phi+\Delta^2}d\phi &= 2\pi a^{n/2}\\ &= 2\pi\Big(\sqrt{1+\Delta^2}-|\Delta|\Big)^n\end{aligned}$$

Therefore,

$$\begin{aligned}B_{R,h,n}(R,z) &= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\int_0^{2\pi}\frac{\Delta\sin\phi}{\sin^2\phi+\Delta^2}\left(1+\frac{\cos\phi}{\sqrt{1+\Delta^2}}\right)\sin(n\phi)d\phi\\ &= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\frac{\Delta}{\sqrt{1+\Delta^2}}2\pi\Big(\sqrt{1+\Delta^2}-|\Delta|\Big)^n\\ &= -N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\frac{z}{\sqrt{R^2+z^2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\end{aligned}$$

3.2 Fourier coefficients of the azimuthal component of the magnetic field

Our goal in this section is to calculate these integrals

$$\begin{aligned}B_{h,\phi,0}(R,z) &= \frac{1}{2\pi}\int_0^{2\pi} B_{h,\phi}(R,\phi,z)d\phi\\ &= -\frac{N}{2\pi}\frac{\mu_0 I_0}{4\pi}\int_0^{2\pi}\frac{z\cos\phi}{R^2\sin^2\phi+z^2}\left(1+\frac{R\cos\phi}{\sqrt{R^2+z^2}}\right)d\phi\\ &= -\frac{N}{2\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\int_0^{2\pi}\frac{\Delta\cos\phi}{\sin^2\phi+\Delta^2}\left(1+\frac{\cos\phi}{\sqrt{1+\Delta^2}}\right)d\phi,\end{aligned}$$

$$\begin{aligned}B_{h,\phi,n}(R,z) &= \frac{1}{\pi}\int_0^{2\pi} B_{h,\phi}(R,\phi,z)\cos(n\phi)d\phi\\ &= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\int_0^{2\pi}\frac{z\cos\phi}{R^2\sin^2\phi+z^2}\left(1+\frac{R\cos\phi}{\sqrt{R^2+z^2}}\right)\cos(n\phi)d\phi\\ &= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\int_0^{2\pi}\frac{\Delta\cos\phi}{\sin^2\phi+\Delta^2}\left(1+\frac{\cos\phi}{\sqrt{1+\Delta^2}}\right)\cos(n\phi)d\phi,\end{aligned}$$

for $n \bmod N = 0$ and $n \neq 0$, where

$$\Delta = z/R.$$

Note that,

$$\int_0^{2\pi}\frac{\cos\phi}{\sin^2\phi+\Delta^2}d\phi = 0,$$

$$\int_0^{2\pi}\frac{\cos^2\phi}{\sin^2\phi+\Delta^2}d\phi = 2\pi\left(\frac{\sqrt{1+\Delta^2}}{|\Delta|}-1\right).$$

Hence,

$$\begin{aligned}B_{h,\phi,0}(R,z) &= -\frac{N}{2\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\frac{\Delta}{\sqrt{1+\Delta^2}}2\pi\left(\frac{\sqrt{1+\Delta^2}}{|\Delta|}-1\right)\\ &= N\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\left(\frac{z}{\sqrt{R^2+z^2}}-\frac{z}{|z|}\right).\end{aligned}$$

Note that

$$\int_0^{2\pi} \frac{\cos\phi}{\sin^2\phi + \Delta^2}\cos(n\phi)d\phi = 0,$$

for n even.

$$\int_0^{2\pi} \frac{\cos^2\phi}{\sin^2\phi + \Delta^2}\cos(n\phi)d\phi = \frac{2\pi}{|\Delta|}\sqrt{1+\Delta^2}\Big(\sqrt{1+\Delta^2} - |\Delta|\Big)^n,$$

for $n \geq 2$ and even. For proof, see this stack exchange link. Hence,

$$\begin{aligned}
B_{h,\phi,n}(R,z) &= -\frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\frac{1}{R}\frac{\Delta}{\sqrt{1+\Delta^2}}\frac{2\pi}{|\Delta|}\sqrt{1+\Delta^2}\Big(\sqrt{1+\Delta^2} - |\Delta|\Big)^n \\
&= -\frac{z}{|z|}N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\left(\sqrt{1+\Big(\frac{z}{R}\Big)^2} - \Big|\frac{z}{R}\Big|\right)^n \\
&= -\frac{z}{|z|}N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\left(\frac{\sqrt{R^2+z^2} - |z|}{R}\right)^n.
\end{aligned}$$

3.3 Fourier coefficients of the vertical component of the magnetic field

Our goal in this section is to calculate this integral

$$\begin{aligned}
B_{h,z,n}(R,z) &= \frac{N}{\pi}\int_0^{2\pi} B_{h,z,0}(R,\phi,z)\sin(n\phi)d\phi \\
&= \frac{N}{\pi}\frac{\mu_0 I_0}{4\pi}\int_0^{2\pi}\frac{R\sin\phi}{R^2\sin^2\phi + z^2}\left(1 + \frac{R\cos\phi}{\sqrt{R^2+z^2}}\right)\sin(n\phi)d\phi \\
&= -\frac{R}{z}B_{h,R,n}(R,z)
\end{aligned}$$

for $n \bmod N = 0$. Since

$$B_{R,h,n}(R,z) = -N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\frac{z}{\sqrt{R^2+z^2}}\left(\frac{\sqrt{R^2+z^2} - |z|}{R}\right)^n$$

we know that

$$B_{z,h,n}(R,z) = N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\frac{R}{\sqrt{R^2+z^2}}\left(\frac{\sqrt{R^2+z^2} - |z|}{R}\right)^n$$

4 Full magnetic field

The full magnetic field is given by

$$\mathbf{B}(R,\phi,z) = N\mathbf{B}_v(R,\phi,z) + \mathbf{B}_h(R,\phi,z-h) - \mathbf{B}_h(R,\phi,z+h).$$

Hence,

$$\boxed{\begin{aligned}B_R(R,\phi,z) = N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\sum_{n=N}^{\infty}\Bigg[&\frac{h-z}{\sqrt{R^2+(z-h)^2}}\left(\frac{\sqrt{R^2+(z-h)^2} - |z-h|}{R}\right)^n + \\
&\frac{z+h}{\sqrt{R^2+(z+h)^2}}\left(\frac{\sqrt{R^2+(z+h)^2} - |z+h|}{R}\right)^n\Bigg]\sin(n\phi),\end{aligned}}$$

$$\boxed{\begin{aligned}B_\phi(R,\phi,z) = N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\Bigg\{1 + \sum_{n=N}^{\infty}\Bigg[&\left(\frac{\sqrt{R^2+(z-h)^2} - |z-h|}{R}\right)^n + \\
&\left(\frac{\sqrt{R^2+(z+h)^2} - |z+h|}{R}\right)^n\Bigg]\Bigg\}\cos(n\phi),\end{aligned}}$$

$$\boxed{\begin{aligned}B_z(R,\phi,z) = N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\sum_{n=N}^{\infty}\Bigg[&\frac{R}{\sqrt{R^2+(z-h)^2}}\left(\frac{\sqrt{R^2+(z-h)^2}-|z-h|}{R}\right)^n - \\ &\frac{R}{\sqrt{R^2+(z+h)^2}}\left(\frac{\sqrt{R^2+(z+h)^2}-|z+h|}{R}\right)^n\Bigg]\sin(n\phi),\end{aligned}}$$

for $n \bmod N = 0$ and $-h < z < h$.

5 Check divergence is zero

$$\nabla\cdot\mathbf{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}.$$

$$\begin{aligned}
\frac{1}{R}\frac{\partial}{\partial R}(RB_{R,h,n}(R,z)) &= \frac{1}{R}\frac{\partial}{\partial R}\left\{-N\frac{\mu_0 I_0}{2\pi}\frac{z}{\sqrt{R^2+z^2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\right\} \\
&= -N\frac{\mu_0 I_0}{2\pi}\frac{z}{R}\frac{\partial}{\partial R}\left\{\frac{1}{\sqrt{R^2+z^2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\right\} \\
&= -N\frac{\mu_0 I_0}{2\pi}\frac{z}{R}\Bigg\{-\frac{R}{(R^2+z^2)^{3/2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n + \\
&\qquad n\frac{|z|\sqrt{R^2+z^2}-z^2}{R^2(R^2+z^2)}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^{n-1}\Bigg\} \\
&= -N\frac{\mu_0 I_0}{2\pi}\frac{z}{R}\Bigg\{-\frac{R}{(R^2+z^2)^{3/2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n + \\
&\qquad n\frac{|z|}{R(R^2+z^2)}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\Bigg\} \\
&= -N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\Bigg\{-\frac{zR}{(R^2+z^2)^{3/2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n + \\
&\qquad n\frac{|z|}{z}\frac{z^2}{R(R^2+z^2)}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial B_{z,h,n}(R,z)}{\partial z} &= N\frac{\mu_0 I_0}{2\pi}\frac{\partial}{\partial z}\left\{\frac{1}{\sqrt{R^2+z^2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n\right\} \\
&= N\frac{\mu_0 I_0}{2\pi}\left\{-\frac{z}{(R^2+z^2)^{3/2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n + \right. \\
&\qquad\qquad \left. n\frac{z|z|-z\sqrt{R^2+z^2}}{R|z|(R^2+z^2)}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^{n-1}\right\} \\
&= N\frac{\mu_0 I_0}{2\pi}\left\{-\frac{z}{(R^2+z^2)^{3/2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n - \right. \\
&\qquad\qquad \left. n\frac{z}{|z|(R^2+z^2)}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^{n}\right\} \\
&= N\frac{\mu_0 I_0}{2\pi}\frac{1}{R}\left\{-\frac{zR}{(R^2+z^2)^{3/2}}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n - \right. \\
&\qquad\qquad \left. n\frac{z}{|z|}\frac{R^2}{R(R^2+z^2)}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^{n}\right\}.
\end{aligned}
$$

Hence,

$$
\frac{1}{R}\frac{\partial}{\partial R}(RB_{R,h,n}) + \frac{\partial B_{z,h,n}}{\partial z} = -n\frac{z}{|z|}N\frac{\mu_0 I_0}{2\pi}\frac{1}{R^2}\left(\frac{\sqrt{R^2+z^2}-|z|}{R}\right)^n.
$$

$$
\implies \frac{1}{R}\frac{\partial}{\partial R}(RB_{R,h,n}) - \frac{n}{R}B_{\phi,h,n} + \frac{\partial B_{z,h,n}}{\partial z} = 0,
$$

which confirms that the divergence of the magnetic field is zero.