Priority Queues

Outline

Priority Queue

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A priority queue (PQ for short) is a data structure that allows us to process keys in order, without storing them in full sorted order all at once

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Fundamental operations: insert and remove the minimum (or maximum)

Priority Queue

Priority Queue

dsa.MinPQ<Key> implements java.lang.Iterable<Key>	
`MinPQ()`	constructs an empty minPQ
`MinPQ(Comparator<Key> c)`	constructs an empty minPQ with the given comparator
`MinPQ(int capacity)`	constructs an empty minPQ with the given capacity
`MinPQ(int capacity, Comparator<Key> c)`	constructs an empty minPQ with the given capacity and comparator
`boolean isEmpty()`	returns `true` if this minPQ is empty, and `false` otherwise
`int size()`	returns the number of keys in this minPQ
`void insert(Key key)`	adds `key` to this minPQ
`Key min()`	returns the smallest key in this minPQ
`Key delMin()`	removes and returns the smallest key in this minPQ
`Iterator<Key> iterator()`	returns an iterator to iterate over the keys in this minPQ in ascending order
`String toString()`	returns a string representation of this minPQ

Priority Queue

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dsa.MaxPQ<Key> implements java.lang.Iterable<Key>	
`MaxPQ()`	constructs an empty maxPQ
`MaxPQ(Comparator<Key> c)`	constructs an empty maxPQ with the given comparator
`MaxPQ(int capacity)`	constructs an empty maxPQ with the given capacity
`MaxPQ(int capacity, Comparator<Key> c)`	constructs an empty maxPQ with the given capacity and comparator
`boolean isEmpty()`	returns `true` if this maxPQ is empty, and `false` otherwise
`int size()`	returns the number of keys in this maxPQ
`void insert(Key key)`	adds `key` to this maxPQ
`Key max()`	returns the largest key in this maxPQ
`Key delMax()`	removes and returns the largest key in this maxPQ
`Iterator<Key> iterator()`	returns an iterator to iterate over the keys in this maxPQ in descending order
`String toString()`	returns a string representation of this maxPQ

Priority Queue

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Program: `TopM.java`

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- Command-line input: m (int)

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```
>_ ~/workspace/dsaj/programs

$ _
```

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- Standard input: sequence of transactions
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```
>_ ~/workspace/dsaj/programs
$ cat ../data/tinyBatch.txt
```

Priority Queue

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>_ ~/workspace/dsaj/programs

```
$ cat ../data/tinyBatch.txt
Turing      6/17/1990   644.08
vonNeumann  3/26/2002  4121.85
Dijkstra    8/22/2007  2678.40
vonNeumann  1/11/1999  4409.74
Dijkstra   11/18/1995   837.42
Hoare       5/10/1993  3229.27
vonNeumann  2/12/1994  4732.35
Hoare       8/18/1992  4381.21
Turing      1/11/2002    66.10
Thompson    2/27/2000  4747.08
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Hoare       8/12/2003  1025.70
vonNeumann 10/13/1993  2520.97
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$ _
```

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$ java TopM 5 < ../data/tinyBatch.txt
```

Program: `TopM.java`

- Command-line input: m (int)
- Standard input: sequence of transactions
- Standard output: top m transactions in decreasing order of amount

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$ _
```

Priority Queue

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TopM.java

```
import dsa.LinkedStack;
import dsa.MinPQ;
import dsa.Transaction;
import stdlib.StdIn;
import stdlib.StdOut;

public class TopM {
    public static void main(String[] args) {
        int m = Integer.parseInt(args[0]);
        MinPQ<Transaction> pq = new MinPQ<Transaction>(m + 1);
        while (StdIn.hasNextLine()) {
            String line = StdIn.readLine();
            Transaction transaction = new Transaction(line);
            pq.insert(transaction);
            if (pq.size() > m) {
                pq.delMin();
            }
        }
        LinkedStack<Transaction> stack = new LinkedStack<Transaction>();
        for (Transaction transaction : pq) {
            stack.push(transaction);
        }
        for (Transaction transaction : stack) {
            StdOut.println(transaction);
        }
    }
}
```

Elementary Implementation

Elementary Implementation

Data Structure	`insert()`	`delMin()` / `delMax()`
Ordered array	n	1
Unordered array	1	n
Ordered linked list	n	1
Unordered linked list	1	n

Heap-based Implementation

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A binary tree is a tree data structure in which each node has at most two children

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The largest key in a max-heap is found at the root

The height of a complete binary tree of size n is $\lfloor \lg n \rfloor$

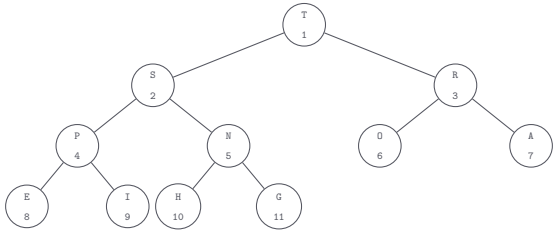
Heap-based Implementation

Heap-based Implementation

Representation

a[]

0	1	2	3	4	5	6	7	8	9	10	11
-	T	S	R	P	N	O	A	E	I	H	G



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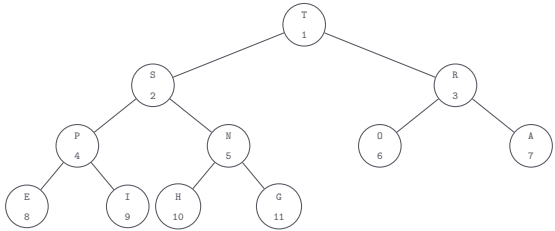

The left and right children of node i are nodes $2i$ and $2i + 1$ respectively

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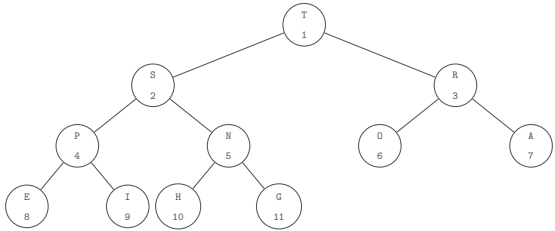

The left and right children of node i are nodes $2i$ and $2i + 1$ respectively

The parent of node i is $\left\lfloor \frac{i}{2} \right\rfloor$

Heap-based Implementation

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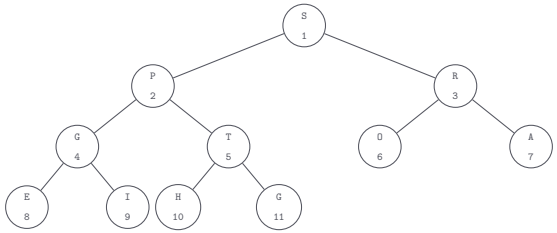
Bottom-up reheapify (swim)



Heap-based Implementation

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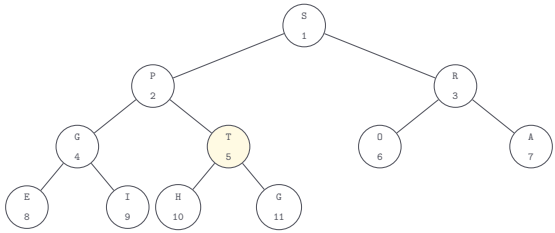

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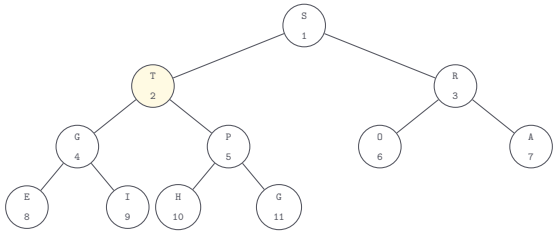

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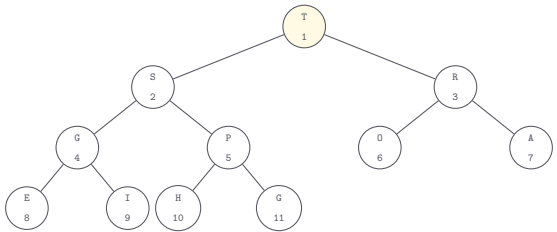

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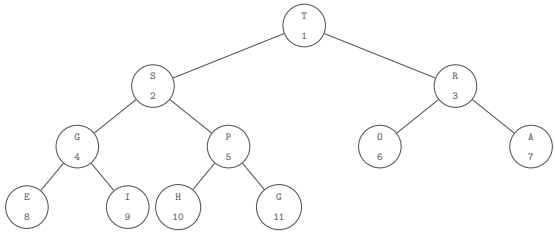

Heap-based Implementation

Heap-based Implementation

```
private void swim(int i) {
    while (i > 1 && less(i / 2, i)) {
        exchange(i, i / 2);
        i /= 2;
    }
}
```

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private void swim(int i) {
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Insert: add the new key at the end of the array, increment the size of the heap, and swim up to restore the heap order

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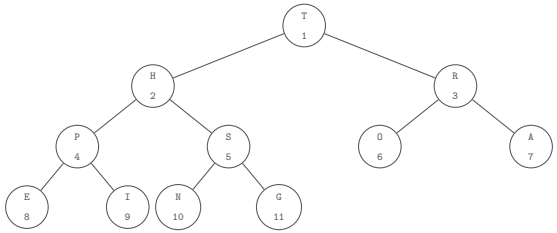
$T(n) = \log n$

Heap-based Implementation

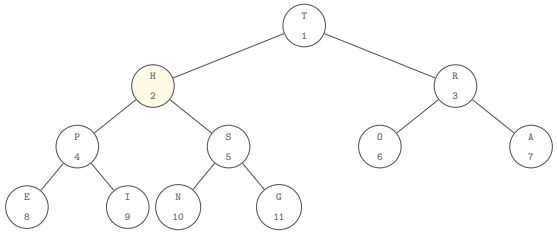
Heap-based Implementation

Top-down reheapify (sink)



Heap-based Implementation

Top-down reheapify (sink)



Heap-based Implementation

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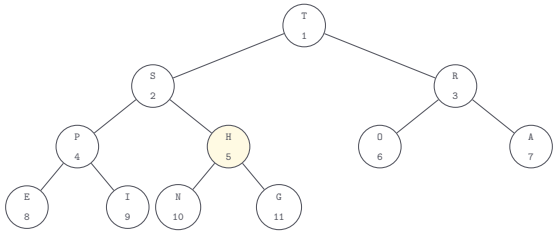

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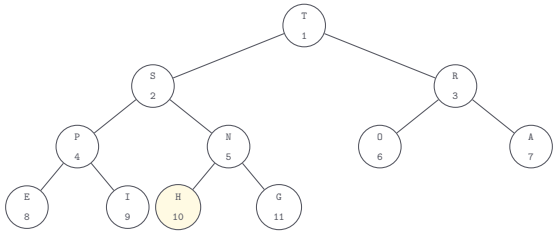

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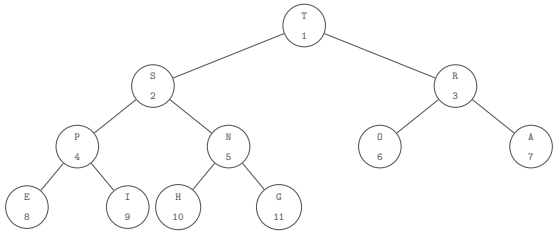

Heap-based Implementation

Heap-based Implementation

```
private void sink(int i) {
    while (2 * i <= n) {
        int j = 2 * i;
        if (j < n && less(j, j + 1)) {
            j++;
        }
        if (!less(i, j)) {
            break;
        }
        exchange(i, j);
        i = j;
    }
}
```

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Remove the maximum: take the largest key off the top, put the key from the end of the heap at the top, decrement the size of the heap, and sink down to restore the heap order

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$T(n) = \log n$

Heap-based Implementation

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MaxPQ.java

```
package dsa;

import java.util.Comparator;
import java.util.Iterator;
import java.util.NoSuchElementException;

import stdlib.StdIn;
import stdlib.StdOut;

public class MaxPQ<Key> implements Iterable<Key> {
    private Key[] pq;
    private int n;
    private Comparator<Key> c;

    public MaxPQ() {
        this(1);
    }

    public MaxPQ(int capacity) {
        pq = (Key[]) new Object[capacity + 1];
        n = 0;
    }

    public MaxPQ(Comparator<Key> c) {
        this(1, c);
    }

    public MaxPQ(int capacity, Comparator<Key> c) {
        pq = (Key[]) new Object[capacity + 1];
        n = 0;
        this.c = c;
    }

    public boolean isEmpty() {
        return n == 0;
```

Heap-based Implementation

MaxPQ.java

```
    }

    public int size() {
        return n;
    }

    public void insert(Key key) {
        if (n == pq.length - 1) {
            resize(2 * pq.length);
        }
        pq[++n] = key;
        swim(n);
    }

    public Key max() {
        if (isEmpty()) {
            throw new NoSuchElementException("Priority queue is empty");
        }
        return pq[1];
    }

    public Key delMax() {
        if (isEmpty()) {
            throw new NoSuchElementException("Priority queue is empty");
        }
        Key max = pq[1];
        exchange(1, n--);
        sink(1);
        pq[n + 1] = null;
        if (n > 0 && n == (pq.length - 1) / 4) {
            resize(pq.length / 2);
        }
        return max;
    }
```

Heap-based Implementation

MaxPQ.java

```
    public Iterator<Key> iterator() {
        return new HeapIterator();
    }

    public String toString() {
        StringBuilder sb = new StringBuilder();
        for (Key key : this) {
            sb.append(key);
            sb.append(", ");
        }
        return n > 0 ? "[" + sb.substring(0, sb.length() - 2) + "]" : "[]";
    }

    private void resize(int capacity) {
        Key[] temp = (Key[]) new Object[capacity];
        for (int i = 1; i <= n; i++) {
            temp[i] = pq[i];
        }
        pq = temp;
    }

    private void swim(int i) {
        while (i > 1 && less(i / 2, i)) {
            exchange(i, i / 2);
            i /= 2;
        }
    }

    private void sink(int i) {
        while (2 * i <= n) {
            int j = 2 * i;
            if (j < n && less(j, j + 1)) {
                j++;
            }
            if (!less(i, j)) {
```

Heap-based Implementation

MaxPQ.java

```
                break;
            }
            exchange(i, j);
            i = j;
        }
    }

    private boolean less(int i, int j) {
        if (c == null) {
            return ((Comparable) pq[i]).compareTo(pq[j]) < 0;
        }
        return c.compare(pq[i], pq[j]) < 0;
    }

    private void exchange(int i, int j) {
        Key swap = pq[i];
        pq[i] = pq[j];
        pq[j] = swap;
    }

    private class HeapIterator implements Iterator<Key> {
        private MaxPQ<Key> copy;

        public HeapIterator() {
            copy = (c == null) ? new MaxPQ<Key>(n) : new MaxPQ<Key>(n, c);
            for (int i = 1; i <= n; i++) {
                copy.insert(pq[i]);
            }
        }

        public boolean hasNext() {
            return !copy.isEmpty();
        }

        public Key next() {
```

Heap-based Implementation

MaxPQ.java

```
            if (!hasNext()) {
                throw new NoSuchElementException("Iterator is empty");
            }
            return copy.delMax();
        }
    }

    public static void main(String[] args) {
        MaxPQ<String> pq = new MaxPQ<String>();
        while (!StdIn.isEmpty()) {
            String item = StdIn.readString();
            if (!item.equals("-")) {
                pq.insert(item);
            } else if (!pq.isEmpty()) {
                StdOut.print(pq.delMax() + " ");
            }
        }
        StdOut.println();
        StdOut.println(pq.size() + " keys left in the pq");
        StdOut.println(pq);
    }
}
```

Indexed Priority Queue

Indexed Priority Queue

dsa.IndexMinPQ<Key extends java.lang.Comparable<Key>> implements java.lang.Iterable<Key>	
`IndexMinPQ(int maxN)`	constructs an empty indexMinPQ with indices from the interval `[0, maxN)`
`boolean isEmpty()`	returns `true` if this indexMinPQ is empty, and `false` otherwise
`int size()`	returns the number of keys in this indexMinPQ
`void insert(int i, Key key)`	associates `key` with index `i` in this indexMinPQ
`void change(int i, Key key)`	changes the key associated with index `i` to `key` in this indexMinPQ
`boolean contains(int i)`	returns `true` if `i` is an index in this indexMinPQ, and `false` otherwise
`int minIndex()`	returns the index associated with the smallest key in this indexMinPQ
`Key minKey()`	returns the smallest key in this indexMinPQ
`Key keyOf(int i)`	returns the key associated with index `i` in this indexMinPQ
`int delMin()`	removes the smallest key from this indexMinPQ and returns its associated index
`void delete(int i)`	removes the key associated with index `i` in this indexMinPQ
`Iterator<Integer> iterator()`	returns an iterator to iterate over the indices in this indexMinPQ in ascending order of the associated keys
`String toString()`	returns a string representation of this indexMinPQ

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`Key minKey()`	returns the smallest key in this indexMinPQ
`Key keyOf(int i)`	returns the key associated with index `i` in this indexMinPQ
`int delMin()`	removes the smallest key from this indexMinPQ and returns its associated index
`void delete(int i)`	removes the key associated with index `i` in this indexMinPQ
`Iterator<Integer> iterator()`	returns an iterator to iterate over the indices in this indexMinPQ in ascending order of the associated keys
`String toString()`	returns a string representation of this indexMinPQ

Indexed Priority Queue

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Program: `Multiway.java`

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- Command-line input: names of files with sorted strings

Indexed Priority Queue

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Indexed Priority Queue

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```
>_ ~/workspace/dsaj/programs
$ _
```

Indexed Priority Queue

Program: `Multiway.java`

- Command-line input: names of files with sorted strings
- Standard output: strings from all the files in sorted order

```
>_ ~/workspace/dsaj/programs

$ cat ../data/m1.txt
```

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```
>_ ~/workspace/dsaj/programs

$ cat ../data/m1.txt
A B C F G I I Z
$ _
```

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```
>_ ~/workspace/dsaj/programs

$ cat ../data/m1.txt
A B C F G I I Z
$ cat ../data/m2.txt
```

Indexed Priority Queue

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- Standard output: strings from all the files in sorted order

```
>_ ~/workspace/dsaj/programs

$ cat ../data/m1.txt
A B C F G I I Z
$ cat ../data/m2.txt
B D H P Q Q
$ _
```

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- Command-line input: names of files with sorted strings
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```
>_ ~/workspace/dsaj/programs

$ cat ../data/m1.txt
A B C F G I I Z
$ cat ../data/m2.txt
B D H P Q Q
$ cat ../data/m3.txt
```

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```
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$ cat ../data/m1.txt
A B C F G I I Z
$ cat ../data/m2.txt
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$ cat ../data/m3.txt
A B E F J N
$ _
```

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```
>_ ~/workspace/dsaj/programs

$ cat ../data/m1.txt
A B C F G I I Z
$ cat ../data/m2.txt
B D H P Q Q
$ cat ../data/m3.txt
A B E F J N
$ java Multiway ../data/m1.txt ../data/m2.txt ../data/m3.txt
```

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$ cat ../data/m1.txt
A B C F G I I Z
$ cat ../data/m2.txt
B D H P Q Q
$ cat ../data/m3.txt
A B E F J N
$ java Multiway ../data/m1.txt ../data/m2.txt ../data/m3.txt
A A B B B C D E F F G H I I J N P Q Q Z
$ _
```

Indexed Priority Queue

Indexed Priority Queue

Multiway.java

```
import dsa.IndexMinPQ;
import stdlib.In;
import stdlib.StdOut;

public class Multiway {
    public static void main(String[] args) {
        int n = args.length;
        In[] streams = new In[n];
        for (int i = 0; i < n; i++) {
            streams[i] = new In(args[i]);
        }
        merge(streams);
    }

    private static void merge(In[] streams) {
        int n = streams.length;
        IndexMinPQ<String> pq = new IndexMinPQ<String>(n);
        for (int i = 0; i < n; i++) {
            if (!streams[i].isEmpty()) {
                pq.insert(i, streams[i].readString());
            }
        }
        while (!pq.isEmpty()) {
            StdOut.print(pq.minKey() + " ");
            int i = pq.delMin();
            if (!streams[i].isEmpty()) {
                pq.insert(i, streams[i].readString());
            }
        }
        StdOut.println();
    }
}
```

Heap Sort

Heap Sort

Heapify `a[]` containing n items to a max-heap, and repeatedly exchange the maximum key with the key at index n and reheapify the first $n - 1$ keys

Heap Sort

Heapify `a[]` containing n items to a max-heap, and repeatedly exchange the maximum key with the key at index n and reheapify the first $n-1$ keys

$T(n) = n \log n$

Heap Sort

Heap Sort

Heap.java

```
public class Heap {
    public static void sort(Comparable[] a) {
        int n = a.length;
        for (int i = n / 2; i >= 1; i--) {
            sink(a, i, n);
        }
        while (n > 1) {
            exchange(a, 1, n--);
            sink(a, 1, n);
        }
    }

    public static void sort(Object[] a, Comparator c) {
        int n = a.length;
        for (int i = n / 2; i >= 1; i--) {
            sink(a, i, n, c);
        }
        while (n > 1) {
            exchange(a, 1, n--);
            sink(a, 1, n, c);
        }
    }

    private static void sink(Comparable[] a, int i, int n) {
        while (2 * i <= n) {
            int j = 2 * i;
            if (j < n && less(a, j, j + 1)) {
                j++;
            }
            if (!less(a, i, j)) {
                break;
            }
            exchange(a, i, j);
            i = j;
        }
```

Heap Sort

Heap.java

```
    }

    private static void sink(Object[] a, int i, int n, Comparator c) {
        while (2 * i <= n) {
            int j = 2 * i;
            if (j < n && less(a, j, j + 1, c)) {
                j++;
            }
            if (!less(a, i, j, c)) {
                break;
            }
            exchange(a, i, j);
            i = j;
        }
    }

    private static boolean less(Comparable[] a, int i, int j) {
        return a[i - 1].compareTo(a[j - 1]) < 0;
    }

    private static boolean less(Object[] a, int i, int j, Comparator c) {
        return c.compare(a[i - 1], a[j - 1]) < 0;
    }

    private static void exchange(Object[] a, int i, int j) {
        Object swap = a[i - 1];
        a[i - 1] = a[j - 1];
        a[j - 1] = swap;
    }
}
```