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Stochastic Tubes in Model Predictive Control with Probabilistic Constraints

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Abstract— **Recent developments in stochastic MPC provided guarantees of closed loop stability and satisfaction of probabilistic and hard constraints. However the required computation can be formidable for anything other than short prediction horizons. This difficulty is removed in the current paper through the use of tubes of fixed cross-section and variable scaling. A model describing the evolution of predicted tube scalings simplifies the computation of stochastic tubes; furthermore this procedure can be performed offline. The resulting MPC scheme has a low online computational load even for long prediction horizons, thus allowing for performance improvements. The approach is illustrated by numerical examples.**

Keywords: **constrained control; stochastic systems; probabilistic constraints.**

I. Introduction

Robust model predictive control (MPC) does not usually take into account the distribution of stochastic model uncertainty. However, this becomes necessary when soft probabilistic constraints are present, and a capability for handling such constraints is particularly important in problems with large unknown disturbances. More generally it can help avoid the trade-off between computation and optimality of robust MPC. This is achieved by replacing the requirement in robust MPC for constraint satisfaction for all uncertainty realizations (which can lead to excessive computation or conservative performance [1]) with a guarantee that constraints are satisfied with specified probabilities.

Problems involving additive disturbances and probabilistic constraints were considered in [2], [3], [4]. Each of these approaches proposed a receding horizon control law, but none considered recursive feasibility or stability of the control strategy in closed loop operation. The same objection applies to a number of reported applications that consider the effects of probabilistic uncertainty, e.g. [5], [6]. To remedy this, recent work [7], [8] proposed a strategy based on minimizing a performance index subject to constraints that restrict the predicted plant state either to nested ellipsoidal sets ([7]) or to nested layered tubes with variable polytopic cross-sections ([8]). By constraining: (i) the probability of transition between tubes, and (ii) the probability of constraint violation within each tube, this strategy ensures satisfaction of soft (also hard) constraints as well as recursive feasibility with respect to these constraints. The constraints were invoked using confidence polytopes for the model uncertainty, and terminal constraints were constructed in order to ensure constraint satisfaction over an infinite horizon.

Tubes have been proposed for different MPC problem formulations: [9], [10], [11], [12] consider the linear case with additive bounded uncertainty, whereas [13] deals with the nonlinear case with no uncertainty, and [14], [15] address nonlinear cases subject to additive uncertainty. The main drawback of [8] is that invoking the transitional and violation probability constraints over a confidence polytope results in large numbers of variables and linear inequalities in the MPC optimization, which can limit application to low-dimensional systems with short horizons even if the number of tube layers is small. For small numbers of tube layers the handling of probabilistic constraints becomes conservative.

The current paper uses layered tubes with fixed cross-sections (for convenience ellipsoids are used, though more general forms are possible), but we allow the scalings and centres of these cross-sections to vary with time. Similar to the deterministic case [12], the evolution of the tubes is described by a simple dynamic system, which implies a significant reduction in the number of optimization variables. However in this context the dynamics governing the tube scalings are stochastic, and the computation of the predicted distributions of tube scalings is facilitated using a process of discretization. The distributions enable bounds to be imposed on the probability of violation of state constraints; these are computed offline and are invoked online by tightening the constraints on the predictions of a nominal model. The offline computation of distributions for the stochastic variables allows many discretized levels to be used, thus allowing for the implicit use of many layered tubes without any concomitant increase in the online computation. Unlike [2], [4] which considered normally distributed disturbances and did not guarantee closed loop feasibility, this paper does not place any restriction on the uncertainty distribution and ensures feasibility recursively.

II. Problem definition

Consider the linear time-invariant model with state $x_k \in \mathbb{R}^{n_x}$, input $u_k \in \mathbb{R}^{n_u}$ and disturbance input $w_k \in \mathbb{R}^{n_w}$,

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k, \tag{1}$$

where w_k is a zero mean random variable (r.v.) and $\{w_k,\, k = 0,1,\ldots\}$ is i.i.d. The system has soft constraints of the form $\Pr(g_i^T x_k \leq h_i) \geq p_i$, $i = 1,\ldots n_c$, for given $g_i \in \mathbb{R}^{n_x}$, $h_i \in \mathbb{R}$ and probabilities $p_i > 0.5$. General linear constraints on states and inputs can be handled using the paper's framework, and hard constraints can be included as

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a special case of soft constraints invoked with probability 1 (w.p. 1). To simplify presentation, the paper's approach is developed for the case of a single soft constraint ($n_c = 1$):

$$\Pr(g^T x_k \le h) \ge p; \quad p > 0.5. \tag{2}$$

The control problem is to minimize, at each time k, the expected quadratic cost

$$J_k = \sum_{i=0}^{\infty} \mathbb{E}_k\big(x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i}\big) \tag{3}$$

subject to $\Pr(g^T x_{k+i} \le h) \ge p$ for all $i > 0$, while ensuring closed loop stability (in a suitable sense) and convergence of x_k to a neighborhood of the origin as $k \to \infty$.

To derive a computationally tractable problem, the n_w-dimensional variable w_k is characterized in terms of a scaling parameter α_k applied to an ellipsoidal set containing w_k,

$$w_k \in \mathcal{E}(W, \alpha_k);\ \mathcal{E}(W, \alpha) = \{w :\ w^T W w \le \alpha\},\ W \succ 0 \tag{4}$$

where $\alpha_k \ge 0$ is a random variable and $\{\alpha_k,\ k = 0, 1, \ldots\}$ is i.i.d. The distribution function F_α, defined by $F_\alpha(a) = \Pr(\alpha_k \le a)$, is assumed to be known, either in closed form or by numerically integrating the density of w, and we make the following assumptions on α.

Assumption 1: F_α is continuously differentiable and α is bounded: $\alpha \in [0, \bar{\alpha}]$.

We make use of the closed loop paradigm and decompose the state of (1) into:

$$x_k = z_k + e_k,\ u_k = K x_k + c_k \tag{5}$$

$$z_{k+1} = \Phi z_k + B_u c_k \tag{6}$$

$$e_{k+1} = \Phi e_k + B_w w_k \tag{7}$$

where $\Phi = A + B_u K$ is assumed to be strictly stable and $\{c_{k+i},\ i = 0, 1, \ldots, N-1\}$ are free variables in a receding horizon optimization at time k. This allows the effect of disturbances on the i-step-ahead predicted state, x_{k+i}, to be considered separately (via e_{k+i}) from the nominal prediction, z_{k+i}, and thus simplifies the handling of constraints.

III. Prediction dynamics for propagation of uncertainty

The uncertainty in the i-step-ahead prediction e_{k+i} can be characterized using (4) and (7) in terms of the scalings β_i of sets containing e_{k+i}, which are taken to be ellipsoidal:

$$e_{k+i} \in \mathcal{E}(V, \beta_i);\ \mathcal{E}(V, \beta) = \{e :\ e^T V e \le \beta\},\ V \succ 0. \tag{8}$$

This section constructs a dynamic system to define the random sequence $\{\beta_i,\ i = 1, 2, \ldots\}$ and proposes a method of approximating numerically the distribution functions of β_i, $i \ge 1$.

Given $e_{k+i} \in \mathcal{E}(V, \beta_i)$ and $w_{k+i} \in \mathcal{E}(W, \alpha)$ we have $e_{k+i+1} \in \mathcal{E}(V, \beta_{i+1})$ if and only if

$$\max_{\substack{e \in \mathcal{E}(V,\beta_i) \\ w \in \mathcal{E}(W,\alpha)}} (\Phi e + B_w w)^T V (\Phi e + B_w w) \le \beta_{i+1}. \tag{9}$$

The problem of minimizing β_{i+1} in (9) is NP-hard, but sufficient conditions for (9) can be stated as follows.

Lemma 1: If β_i , β_{i+1} and V satisfy

$$\beta_{i+1} = \lambda \beta_i + \alpha_i \tag{10}$$

$$V^{-1} - \frac{1}{\lambda} \Phi V^{-1} \Phi^T - B_w W^{-1} B_w^T \succeq 0 \tag{11}$$

for some $\lambda > 0$, then $e_{k+i+1} \in \mathcal{E}(V, \beta_{i+1})$ whenever $e_{k+i} \in \mathcal{E}(V, \beta_i)$ and $w_{k+i} \in \mathcal{E}(W, \alpha_i)$.

Proof: From the S-procedure [16], sufficient conditions for (9) are given by (10) and

$$\begin{bmatrix} \Phi^T \\ B_w^T \end{bmatrix} V \begin{bmatrix} \Phi & B_w \end{bmatrix} \preceq \lambda \begin{bmatrix} I \\ 0 \end{bmatrix} V \begin{bmatrix} I & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ I \end{bmatrix} W \begin{bmatrix} 0 & I \end{bmatrix}$$

for some $\lambda > 0$ and $\mu > 0$. However μ can be removed from this inequality by scaling β_i, β_{i+1} and V, and condition (11) is then obtained from Schur complements. ∎

The distributions of β_i can be determined for all $i > 0$ from the distributions of α and β_0 using (10). In the sequel λ,V in (10) and (11) are taken to be constants independent of α_i,β_i (a procedure for optimizing the values of λ and V is described in section IV). We further assume that the distribution function, F_{β_0}, of β_0 is known and has the following properties.

Assumption 2: F_{β_0} is right-continuous with only a finite number of discontinuities and $\beta_0 \in [0, \bar{\beta}_0]$.

Note that Assumption 2 includes the case of deterministic β_0 (e.g. $\beta_0 = 0$ corresponds to $F_{\beta_0}(X) = 1$ for all $X \ge 0$). The following result shows that F_{β_i} is well-defined if $\lambda \in (0, 1)$.

Theorem 2: If $0 < \lambda < 1$, then:

(i) β_i is bounded for all i: $\beta_i \in [0, \bar{\beta}_i]$, where $\bar{\beta}_{i+1} = \lambda \bar{\beta}_i + \bar{\alpha}$, and $\bar{\beta}_i \le \bar{\beta} = \max\{\bar{\beta}_0, \frac{1}{1-\lambda}\bar{\alpha}\}$.

(ii) F_{β_i} is continuously differentiable for all $i \ge 1$

(iii) β_i converges in distribution to the random variable $\beta_L = \sum_{k=0}^{\infty} \lambda^k \alpha_k$ as $i \to \infty$.

Proof: Assumption 1 and (10) together imply that $\beta_i \in [0, \bar{\beta}_i]$ with $\bar{\beta}_{i+1} = \lambda \bar{\beta}_i + \bar{\alpha}$. Hence if $\lambda \in (0, 1)$, then $\{\bar{\beta}_i\}$ is monotonic and $\bar{\beta}_i \to \frac{1}{1-\lambda}\bar{\alpha}$ as $i \to \infty$, which proves (i).

For any $i \ge 0$, the distribution of β_{i+1} is given by the convolution integral (see e.g. [17]):

$$F_{\beta_{i+1}}(X) = \frac{1}{\lambda} \int_0^{\bar{\beta}} F_\alpha(X - Y)\ f_{\beta_i}(Y/\lambda)\ \mathrm{d}Y \tag{12}$$

where f_{β_i} denotes the density of β_i. It follows from Assumption 1 and the dominated convergence principle that $F_{\beta_{i+1}}$ is continuously differentiable, as claimed in (ii).

To prove (iii), let β_i' denote the random variable $\beta_i' = \gamma_i + \lambda^i \beta_0$ for $i = 0, 1, \ldots$ where

$$\gamma_{i+1} = \gamma_i + \lambda^i \alpha_i, \quad \gamma_0 = 0. \tag{13}$$

Then $\beta_i' = \sum_{k=0}^{i-1} \lambda^k \alpha_k + \lambda^i \beta_0$, and since (10) gives $\beta_i = \sum_{k=0}^{i-1} \lambda^k \alpha_{i-1-k} + \lambda^i \beta_0$ where $\{\alpha_i\}$ is i.i.d., the distributions of β_i and β_i' are identical for all $i \ge 0$. Furthermore the bounds on α_i and β_0 in Assumptions 1 and 2 imply that for every $\epsilon > 0$ there exists n such that

$$\Pr(|\beta_j' - \beta_i'| < \epsilon) = 1 \ \ \forall\ i > n,\ j > i$$

and it follows that β_i' converges w.p. 1 to a limit as $i \to \infty$ [18]. From the definition of β_L, we therefore have $\beta_i' \to \beta_L$ w.p. 1, and hence β_i converges in distribution to β_L [18]. ■

The simple form of (10) facilitates the approximate computation of the distribution of β_i. Consider for example a numerical integration scheme based on a set of points $\{X_j,\, j = 0, 1, \ldots, \rho\}$ in the interval $[0, \bar{\beta}]$ with

$$0 = X_0 < X_1 < \cdots < X_\rho = \bar{\beta}. \quad (14)$$

Let $\pi_{i,j}$ be an approximation to $F_{\beta_i}(X)$ in the interval $X \in [X_j, X_{j+1})$ for $j = 0, \ldots, \rho-1$, and let $\pi_{i,\rho} = 1$. Then since the convolution in (12) can be written equivalently as

$$F_{\beta_{i+1}}(X) = \lambda \int_0^{\bar{\beta}} F_{\beta_i}(Y)\, f_\alpha(X - \lambda Y)\, \mathrm{d}Y\,, \quad (15)$$

a generic quadrature method enables the vectors $\pi_i = [\pi_{i,0} \ \cdots \ \pi_{i,\rho}]^T$ to be computed over an N-step horizon by setting $\pi_{0,j} = F_{\beta_0}(X_j)$, $j = 0, \ldots, \rho$, and using the recursion

$$\pi_{i+1} = P\pi_i \quad (16)$$

for $i = 0, 1, \ldots, N-1$, where the $(\rho+1) \times (\rho+1)$ elements of the matrix P are determined by the particular numerical integration scheme employed and the density, f_α, of α.

If the data points $\{X_j\}$ satisfy $|X_{j+1} - X_j| \le \delta$ for $j = 0, \ldots, \rho-1$, then the approximation error can be made arbitrarily small if δ is sufficiently small, as we show below. We denote $\hat{F}_{\pi_i,\delta}$ as the piecewise constant function with $\hat{F}_{\pi_i,\delta}(X) = \pi_{i,j}$ for $X \in [X_j, X_{j+1})$, $j = 0, \ldots, \rho-1$.

Lemma 3: For any finite horizon N we have $\hat{F}_{\pi_i,\delta} \to F_{\beta_i}$ as $\delta \to 0$ for $i = 1, \ldots, N$. Also $\hat{F}_{\pi_L,\delta} \to F_{\beta_L}$ as $\delta \to 0$, where π_L is the eigenvector of P associated with an eigenvalue of 1.

Proof: By Assumptions 1-2 and theorem 2, the integrand in (15) is piecewise continuous for $i = 0$ and continuous for $i > 0$. It can therefore be shown that, as $\delta \to 0$

$$\max_{X \in [0, \bar{\beta}]} |\hat{F}_{\pi_i,\delta}(X) - F_{\beta_i}(X)| = O\big(m\delta + i\kappa\delta^\nu\big) \quad (17)$$

for $i = 1, \ldots, N$, where m denotes the number of discontinuities in F_{β_0} and κ, $\nu \ge 1$ are constants dictated by the numerical integration scheme employed. This implies $\hat{F}_{\pi_i,\delta} \to F_{\beta_i}$ as $\delta \to 0$ since m and N are finite. Combining the error bounds used to derive (17) with $\pi_{i+1} = P\pi_i$, we have, as $\delta \to 0$

$$\hat{F}_{\pi_{i+1},\delta}(X) = \lambda \int_0^{\bar{\beta}} \hat{F}_{\pi_i,\delta}(Y)\, f_\alpha(X - \lambda Y)\, \mathrm{d}Y + O(\kappa\delta^\nu),$$

and since π_L satisfies $\pi_L = P\pi_L$, it follows that, as $\delta \to 0$

$$\hat{F}_{\pi_L,\delta}(X) = \lambda \int_0^{\bar{\beta}} \hat{F}_{\pi_L,\delta}(Y)\, f_\alpha(X - \lambda Y)\, \mathrm{d}Y + O(\kappa\delta^\nu). \quad (18)$$

But by theorem 2, F_{β_L} is the unique solution of

$$F_{\beta_L}(X) = \lambda \int_0^{\bar{\beta}} F_{\beta_L}(Y)\, f_\alpha(X - \lambda Y)\, \mathrm{d}Y.$$

Therefore (18) implies $\hat{F}_{\pi_L,\delta} \to F_{\beta_L}$ as $\delta \to 0$. ■

Lemma 3 shows that the eigenvector π_L provides an approximation to the steady state distribution of (10) despite the linear growth of the error bound in (17) with N. In the sequel we assume that δ is chosen to be sufficiently small so that the approximation errors associated with π_i for $i = 0, \ldots, N$ and π_L may be neglected.

Remark 4: The matrix $T^{-1}PT$, where T is a lower-triangular matrix of 1's, is the transition matrix associated with a Markov chain. The (j, k)th element of $T^{-1}PT$ gives the probability of $X_{j-1} \le \beta_{i+1} < X_j$ given $X_{k-1} \le \beta_i < X_k$. The elements of $T^{-1}PT$ are therefore non-negative and each column sums to 1, so P necessarily has an eigenvalue equal to 1 (e.g. [19]).

Remark 5: The definitions of P and π_0 ensure that each π_i generated by (16) belongs to the set $\mathbb{S} = \{\pi \in \mathbb{R}^{\rho+1} : 0 \le \pi_0 \le \cdots \le \pi_{\rho+1} = 1\}$.

We next show how the sequence $\{\pi_i\}$ can be used to construct ellipsoidal sets that contain the predicted state of (7) with a specified level of confidence. For a given sequence $\{X_j\}$ satisfying (14), and for any $\pi \in \mathbb{S}$ and $p \in [0, 1]$, let $\mathrm{ind}(\pi, p)$ and $b(\pi, p)$ denote the functions that extract respectively the index and the value of X_j corresponding to a confidence level p

$$\mathrm{ind}(\pi, p) = \min\{j : \pi_j \ge p\}, \quad (19a)$$

$$b(\pi, p) = X_j,\ \ j = \mathrm{ind}(\pi, p). \quad (19b)$$

Theorem 6: The i-step-ahead prediction e_{k+i} satisfies

$$e_{k+i} \in \mathcal{E}\big(V, b(\pi_i, p)\big) \text{ w.p. } p. \quad (20)$$

Proof: If π_0 is determined using the known distribution for β_0, then by construction $\beta_i \le b(\pi_i, p)$ holds w.p. p, so (20) follows from the condition $e_{k+i} \in \mathcal{E}(V, \beta_i)$. ■

IV. THE MPC STRATEGY

To avoid an infinite dimensional optimization problem, we adopt the usual dual prediction mode MPC paradigm [20]. Accordingly, at time k, mode 1 consists of the first N steps of the prediction horizon over which the control moves c_{k+i} in (6-7) are free, and mode 2 comprises the remainder of the horizon, with $c_{k+i} = 0$ for all $i \ge N$. Because of the additive uncertainty appearing in (1), the quadratic cost (3) is unbounded. Hence the modified cost of [8] is used:

$$J_k = \sum_{i=0}^{\infty} \big(\mathbb{E}_k(L_{k+i}) - L_{ss}\big), \quad L_{ss} = \lim_{i\to\infty} \mathbb{E}_k(L_{k+i}) \quad (21)$$

where $L_{k+i} = x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i}$. This cost is shown in [8] to be a quadratic function of the vector $\mathbf{c}_k$ of free variables $\{c_{k+i},\, i = 0, \ldots, N-1\}$ at time k:

$$J_k = \begin{bmatrix} x_k^T & \mathbf{c}_k^T & 1 \end{bmatrix} \Omega \begin{bmatrix} x_k^T & \mathbf{c}_k^T & 1 \end{bmatrix}^T \quad (22)$$

for a suitably defined positive definite matrix Ω. Constraints are invoked explicitly in mode 1 and implicitly, through the use of a terminal set, S, in mode 2.

A. Constraint handling in mode 1

The distribution of β_0 is dictated by the information available on the plant state at the beginning of the prediction horizon. Assuming that x_k is known at time k, we set $z_k = x_k$, $e_k = 0$ and $\beta_0 = 0$, so that $F_{\beta_0}(X) = 1$ for all $X \geq 0$. Then, using $v_{\cdot|k}$ to denote the corresponding predictions of variable v at time k, we have from (7),

$$e_{k|k} = 0,\ e_{k+i|k} = \Phi^{i-1}B_w w_k + \cdots + B_w w_{k+i-1},$$

for $i = 1, 2, \ldots$. The scalings of sets defining the stochastic tube containing $e_{k+i|k}$ can therefore be computed offline, and theorem 6 gives $e_{k+i|k} \in \mathcal{E}\big(V, b(\pi_{i|0}, p)\big)$ w.p. p, where

$$\pi_{i|0} = P^i \pi_{0|0},\ i = 1, 2, \ldots, \quad \pi_{0|0} = [1\ \cdots\ 1]^T.$$

The probabilistic constraint (2) can be enforced as follows.

Lemma 7: The constraint $\Pr(g^T x_{k+i|k} \leq h) \geq p$ for $p > 0.5$ is ensured by the condition

$$g^T z_{k+i|k} \leq h - \big[b(\pi_{i|0}, q)\big]^{1/2}\sqrt{g^T V^{-1} g}; \quad q = 2p-1 \quad (23)$$

where $z_{k+i+1|k} = \Phi z_{k+i|k} + B_u c_{k+i|k}$ with $z_{k|k} = x_k$.

Proof: Assume that the distribution of $g^T e_{k+i|k}$ is symmetric about 0 (note that there is no loss of generality in this assumption because of the symmetric bound employed in (8)). Then $g^T z_{k+i|k} + g^T e_{k+i|k} \leq h$ holds w.p. $p > 0.5$ iff $g^T z_{k+i|k} + |g^T e_{k+i|k}| \leq h$ holds w.p. $q = 2p - 1$. The sufficient condition (23) follows since $e_{k+i|k} \in \mathcal{E}\big(V, b(\pi_{i|0}, p)\big)$ and $\max_{e \in \mathcal{E}(V,b)} |g^T e| = b^{1/2}\sqrt{g^T V^{-1} g}$. ■

Although (23), when invoked for $i = 1, \ldots, N$, ensures that at time k the predicted sequence $\{x_{k+i|k}\}$ satisfies (2), it does not ensure recursive feasibility, namely that there exists a predicted sequence at time $k + 1$ satisfying (2). This is because constraints of the form $\Pr(g^T x_{k+i|k} \leq h) \geq p$ do not guarantee that $g^T x_{k+i|k+1} \leq h$ holds with probability p. However the predicted state is contained at all times within the robust tube associated with upper bounds on β_i, i.e.

$$e_{k+i|k} \in \mathcal{E}\big(V, b(\pi_{i|0}, 1)\big),\ b(\pi_{i|0}, 1) = \frac{1 - \lambda^i}{1 - \lambda}\bar{\alpha},\ i = 1, 2, \ldots \quad (24)$$

with probability 1. The stochastic tubes that define constraints such as (23) at times $k + 1, k + 2, \ldots$ are therefore based on initial distributions for β_0 which can be inferred from the robust tube at time k, and this enables the construction of constraints to ensure recursive feasibility.

Let $\pi_{i|j}$ denote the discrete approximation to the distribution of β_i that is obtained if β_j $(j < i)$ is equal to the upper bound $b(\pi_{j|0}, 1)$, so that $F_{\beta_j}(X) = 0$ for $X \in [0, b(\pi_{j|0}, 1))$ and $F_{\beta_j}(X) = 1$ for $X \geq b(\pi_{j|0}, 1)$. Then applying (16) over the interval $j, \ldots, i - 1$ gives

$$\pi_{i|j} = P^{i-j}\pi_{j|j}, \quad \pi_{j|j} = u(\mathrm{ind}(\pi_{j|0}, 1) \quad (25)$$

where $u(j) = [0\ \cdots\ 0\ 1 \cdots 1]^T$ denotes the jth column of the lower-triangular matrix of 1s.

Lemma 8: For any $i > j$, the value of $e_{k+i|k+j}$ predicted at time k (given x_k) satisfies

$$e_{k+i|k+j} \in \mathcal{E}\big(V, b(\pi_{i|j}, p)\big) \text{ w.p. } p. \quad (26)$$

Proof: From $e_{k+j+1} = \Phi e_{k+j} + B_w w_{k+j}$ and (24), the prediction at time k of $e_{k+j+1|k+j}$ satisfies

$$e_{k+j+1|k+j} \in \Phi\mathcal{E}\big(V, b(\pi_{j|0}, 1)\big) \oplus \mathcal{E}\big(V, b(\pi_{1|0}, p)\big) \text{ w.p. } p$$

where $\oplus$ denotes Minkowski addition. But, by lemma 1 and (25) we have

$$\Phi\mathcal{E}\big(V, b(\pi_{j|0}, 1)\big) \oplus \mathcal{E}\big(V, b(\pi_{1|0}, p)\big) \subseteq \mathcal{E}\big(V, b(\pi_{j+1|j}, p)\big),$$

which establishes (26) for $i = j + 1$. Similarly

$$e_{k+j+2|k+j} \in \Phi\mathcal{E}\big(V, b(\pi_{j+1|j}, p)\big) \oplus \mathcal{E}\big(V, b(\pi_{1|0}, p)\big) \text{ w.p. } p$$

so $e_{k+j+2|k+j} \in \mathcal{E}\big(V, b(\pi_{j+2|j}, p)\big)$ w.p. p, and (26) follows by induction for all $i > j$. ■

A simple way to ensure the recurrence of feasibility is to require the probabilistic constraints (2) to hold for all future predictions. Lemma 8 shows that the worst case predictions of (26) for $j = 0, 1, \ldots$ can be handled by selecting at each prediction time i the largest scaling:

$$\bar{b}_i(q) = \max\{b(\pi_{i|0}, q), \ldots, b(\pi_{i|i-1}, q)\} \quad (27)$$

and by replacing $b(\pi_{i|0}, q)$ with $\bar{b}_i(q)$ in (23):

$$g^T z_{k+i|k} \leq h - \big[\bar{b}_i(q)\big]^{1/2}\sqrt{g^T V^{-1} g}, \quad q = 2p - 1. \quad (28)$$

The sets $\mathcal{E}(V, \bar{b}_i(p))$, for $i = 1, 2, \ldots$, define the cross-sections of a recurrently feasible stochastic tube. The preceding argument is summarized as follows.

Theorem 9: If at $k = 0$ there exists $\mathbf{c}_0$ satisfying (28) for all $i > 0$, then there exists $\mathbf{c}_k$ satisfying (28) for all $k, i > 0$, and hence the probabilistic constraint (2) is feasible for all $k > 0$.

Proof: Immediate from lemma 7 and lemma 8. ■

Remark 10: Note that the computation of recurrently feasible stochastic tubes does not depend on the information available at time k, and hence can be performed offline. Consequently the online computation does not depend on the dimension of π, so this can be chosen sufficiently large that the approximation errors discussed in lemma 3 are negligible.

B. Constraint handling in mode 2

The constraint handling framework of mode 1 can also be used to define a terminal constraint, $z_{k+N|k} \in S$, which ensures that (28) is satisfied for all $i > N$. Since (28) constitutes a set of linear constraints on the deterministic predicted trajectory $\{z_{k+i|k}\}$, the infinite horizon of mode 2 can be accounted for using techniques similar to those deployed in [21] for the computation of maximal invariant sets. Before describing the construction of S, we first derive some fundamental properties of the constraints (28).

Lemma 11: The scaling of the tube cross-section defined in (27) satisfies $\bar{b}_i(q) \leq \bar{b}_{i+1}(q)$ for all i and all $q \in [0, 1]$, and converges as $i \to \infty$ to a limit which is bounded by

$$\lim_{i\to\infty} \bar{b}_i(q) \leq \bar{b}_\infty, \quad \bar{b}_\infty = \frac{\bar{\alpha}}{1 - \lambda}. \quad (29)$$

Proof: The monotonic increase of the scaling $b(\pi_{i|0}, 1)$ of the robust tube with i in (24) and the definition of $\pi_{j|j}$ in terms of the robust tube in (25) implies (by lemma 1) that

$b(P^{i-j}\pi_{j|j},q) \le b(P^{i-j}\pi_{j+1|j+1},q)$, and hence $b(\pi_{i|j},q) \le b(\pi_{i+1|j+1},q)$ for any $i > j$. This implies $\bar{b}_i(q) \le \bar{b}_{i+1}(q)$ for all i, and since $\bar{b}_i(q) \le b(\pi_{i|0},1)$ for all i, $\bar{b}_i(q)$ must converge as $i \to \infty$ to a limit no greater than the asymptotic value of the robust tube scaling, $\bar{b}_\infty$. ∎

Lemma 12: Let S_∞ be the maximal set with the property that (28) holds for all $i > N$ whenever $z_{k+N|k} \in S_\infty$, then sufficient conditions for S_∞ to be non-empty and compact are:

$$h \ge \bar{b}_\infty^{1/2}\sqrt{g^T V^{-1} g}\,, \tag{30}$$

and observability of (Φ, g^T), respectively.

Proof: Satisfaction of constraints (28) over the infinite horizon of mode 2 requires that

$$g^T\Phi^i z \le h - \big[\bar{b}_{N+i}(q)\big]^{1/2}\sqrt{g^T V^{-1} g}, \quad i = 1,2,\ldots \tag{31}$$

where for convenience the initial state of mode 2 is labelled z. By lemma 11, $\bar{b}_i(q)$ increases monotonically with i, and Φ is strictly stable by assumption, so (30) is obtained by taking the limit of (31) as $i \to \infty$ and using the upper bound in (29). Furthermore S_∞ is a subset of $\{z : g^T\Phi^i z \le h,\ i = 1,2,\ldots\}$, which is necessarily compact if (Φ, g^T) is observable. ∎

Lemma 11 implies that $\bar{b}_{N+i}(q)$ in (31) can be replaced by $\bar{b}_\infty$ for all $i \ge \hat{N}$, for some finite $\hat{N}$, in order to define the terminal set S. The corresponding conditions of (31) are conservative, but the implied constraint tightening can be made insignificant with sufficiently large $\hat{N}$, as shown next. In this setting, the terminal set is defined by $S = S_{\hat{N}}$, where

$$\begin{aligned} S_{\hat{N}} = \{z : g^T\Phi^i z \le h - \big[\bar{b}_{N+i}(q)\big]^{1/2}\sqrt{g^T V^{-1} g},\ i = 1,\ldots,\hat{N} \\ g^T\Phi^{\hat{N}+j} z \le h - \bar{b}_\infty^{1/2}\sqrt{g^T V^{-1} g}, \qquad j = 1,2,\ldots\} \end{aligned} \tag{32}$$

Lemma 13: Under the assumptions of lemma 12 we have $S_{\hat{N}} \to S_\infty$ as $\hat{N} \to \infty$.

Proof: Lemma 11 and the definition of $S_{\hat{N}}$ in (32) imply that $S_{\hat{N}} \subseteq S_{\hat{N}+1} \subseteq S_\infty$ for all $\hat{N} \ge 0$ and that $S_{\hat{N}}$ is compact for all $\hat{N} \ge 0$. Since S_∞ is compact, $S_{\hat{N}}$ therefore converges to a limit, which by construction is equal to S_∞. ∎

Although (32) involves an infinite number of inequalities, $S_{\hat{N}}$ has the form of a maximal (output admissible) invariant set. The procedure of [21] can therefore be used to determine an equivalent representation of $S_{\hat{N}}$ in terms of a finite number of inequalities, the existence of which is ensured by the assumptions of lemma 12. This involves solving a sequence of linear programs to find the smallest integer n^* such that $g^T\Phi^{n^*+1} z \le h - \bar{b}_\infty^{1/2}\sqrt{g^T V^{-1} g}$ for all z satisfying $g^T\Phi^j z \le h - \bar{b}_\infty^{1/2}\sqrt{g^T V^{-1} g}$ for $j = 1,\ldots,n^*$, then $S_{\hat{N}}$ in (32) is given by

$$\begin{aligned} S_{\hat{N}} = \{z : g^T\Phi^i z \le h - \big[\bar{b}_{N+i}(q)\big]^{1/2}\sqrt{g^T V^{-1} g},\ i = 1,\ldots,\hat{N} \\ g^T\Phi^{\hat{N}+j} z \le h - \bar{b}_\infty^{1/2}\sqrt{g^T V^{-1} g}, \qquad j = 1,\ldots,n^*\} \end{aligned}$$

Remark 14: The scalings of the tube cross-sections in (32) are computed on the basis of (10); this is done for convenience given the simple form of (10). However tighter tubes would be obtained if (9) were used in place of (10) to generate the robust bounds $b(\pi_{i|0},1)$ of (24) and the probabilistic bounds $\bar{b}_i(q)$ of (27). The implied computation is practicable even though (9) involves an implicit optimization because of the discretization of the range for β in (14).

C. Stochastic MPC algorithm

Given the definition of the cost (22), constraints (28), and terminal set (32), the MPC algorithm is as follows.

Algorithm 1 (Stochastic MPC): Offline: compute the scalings $\bar{b}_i(q)$, $i = 1,\ldots,N+\hat{N}$, and the value of n^* allowing $S = S_{\hat{N}}$ to be expressed in terms of a finite number of inequalities.
Online: at each time-step $k = 0,1,\ldots$

1. solve the quadratic programming optimization

$$\begin{aligned} \mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} J_k \text{ subject to (28), } i = 1,\ldots,N-1 \\ \text{and } z_{k+N|k} \in S \end{aligned} \tag{33}$$

2. implement $u_k = Kx_k + c_k^*$.

Theorem 15: Algorithm 1 is feasible for all $k \ge 1$ if feasible at $k = 0$. The closed loop system satisfies probabilistic constraints (2) and satisfies the quadratic stability criterion:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \mathbb{E}_0(L_k) \le L_{ss}. \tag{34}$$

Proof: Recurrence of feasibility follows from theorem 9, which implies that the tail: $\mathbf{c}_{k+1|k} = \{c_{k+1},\ldots,c_{k+N-1},0\}$ of the minimizing sequence: $\mathbf{c}_k^* = \{c_k,\ldots,c_{k+N-1}\}$ of (33) at time k is feasible for (33) at time $k+1$. The feasibility of the tail also implies that the optimal cost satisfies the bound: $J_k^* - \mathbb{E}_k(J_{k+1}^*) \ge L_k - L_{ss}$, which implies (34). ∎

The offline computation of Algorithm 1 requires knowledge of the tube parameters V and λ. A possible choice is the pair V,λ that minimizes the effect on the constraints in (28) and (32) of the upper bound $\bar{b}_\infty$ on the tube scaling $\bar{b}_i(q)$. This must be performed subject to (11), which suggests defining V,λ as the solutions of the following optimization.

$$\begin{aligned} &\arg\min_{V^{-1},\lambda}\ \frac{1}{(1-\lambda)}\ g^T V^{-1} g \\ &\quad\text{subject to } V^{-1} - \frac{1}{\lambda}\Phi V^{-1}\Phi^T - B_w W^{-1} B_w^T \succeq 0 \end{aligned} \tag{35}$$

For a fixed value of $\lambda > 0$, this is a semidefinite program (SDP) in the variable V^{-1}. Given that λ is univariate and restricted to the interval $(0,1)$, (35) can be solved by solving successive SDPs for V^{-1} with alternate iterations of a univariate optimization (such as bisection) for λ.

V. Illustrative examples

Example 1. The system model and constraints are defined

$$\begin{aligned} &A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix},\ B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_w = I,\ W = I \\ &\Pr(g^T x_k \le h) \ge p,\ g = [1\ 0.2]^T,\ h = 1.2,\ p = 0.8 \end{aligned} \tag{36}$$

The disturbance w_k has a truncated normal distribution, with zero mean, variance $\mathbb{E}(w_k w_k^T) = W^{-1}/144$ and bounds

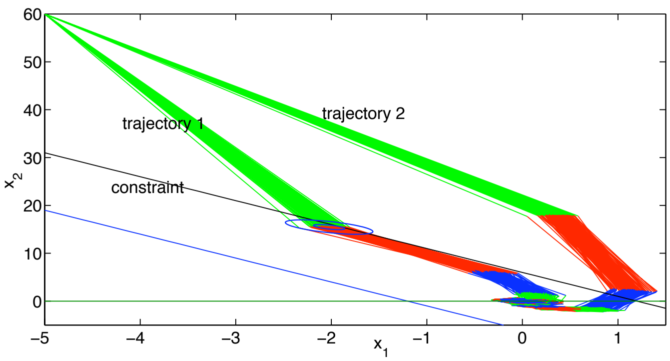

Fig. 1. Closed loop response of (36) under Algorithm 1 (trajectory 1) and unconstrained LQG control (trajectory 2). Ellipsoidal tube cross-sections corresponding to $q = 1$ and $q = 0.6$ predicted at time $k = 0$ are shown.

$-0.224 \le (w_k)_{1,2} \le 0.224$. Thus α has a modified χ-squared distribution with $\bar{\alpha} = 0.1$. The cost weights are $Q = I$, $R = 1$, and K in (5) is chosen as the LQG optimal feedback for the unconstrained case: $K = [1.041\ 1.047]$.

The offline optimization of V, λ in (35) gives

$$\lambda = 0.422, \quad V = \begin{bmatrix} 0.786 & 0.150 \\ 0.150 & 0.072 \end{bmatrix}$$

The robust tube has asymptotic scaling $\bar{b}_\infty = 0.173$, and the interval $[0, 0.173]$ is divided into 1000 equal subintervals to define $\{X_j\}$ and hence compute P and $\bar{b}_i(q)$. Horizons $N = 6$ and $\hat{N} = 7$, give $n^* = 0$, and the scalings of the recurrently feasible stochastic tube are

$$\{\bar{b}_i(q)\} = \{0.013,\ 0.055,\ 0.073,\ 0.080,\ 0.083,\ 0.085,\ 0.085,\ 0.086,\ 0.086, \ldots,\ 0.086\}.$$

Figure 1 shows two trajectories with the same initial condition resulting from 200 realizations of the disturbance sequence: trajectory 1 is the stochastic MPC law of Algorithm 1; trajectory 2 is unconstrained LQG optimal control law. The ellipsoids on trajectory 1 show the tube cross-sections with scalings $\bar{b}_1(q)$ and $b(\pi_{1|0}, 1)$ corresponding to the recurrently feasible stochastic tube with $q = 0.6$ (inner ellipse) and the robust tube (outer ellipse). Algorithm 1 departs from the unconstrained optimal in order to meet constraints with probability greater than 0.8, while allowing some constraint violations (9.5% at $k = 1$). As expected, LQG control achieves a lower value of average cost:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big(\mathbb{E}_0(L_k) - L_{ss}\big) = \begin{cases} 7659 & \text{(LQG)} \\ 7745 & \text{(Algorithm 1)} \end{cases}$$

but achieves this at the expense of violating the probabilistic constraint of the problem.

Example 2. The system model and constraints are defined

$$A = \begin{bmatrix} 1.8 & 0.5 \\ 1.5 & 1.2 \end{bmatrix},\ B_u = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ B_w = I,\ W = I$$
$$\Pr(g^T x_k \le h) \ge p,\ g = [1.1\ 1]^T,\ h = 2.2,\ p = 0.6 \tag{37}$$

The disturbance and cost weights are as defined in example 1. In this case the average closed loop costs are similar: 481.9 for LQG, 485.3 for Algorithm 1, but the rate of constraint violation at $k = 1$, evaluated for 200 disturbance realizations, is 22% for Algorithm 1 and 100% for LQG control.

VI. Conclusions

The proposed MPC strategy handles probabilistic constraints with an online computational load similar to nominal MPC. This is achieved by fixing the cross-sectional shapes of tubes containing predicted states, but allowing their centres and scalings to vary with time. The probability distributions of predicted tube scalings are computed offline, and the online optimization is performed on the disturbance-free model with tightened constraints. The MPC law has guarantees of recursive feasibility and mean-square convergence.

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