Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Two-Stage Portfolio Allocation Problems with Affine Recourse

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Abstract— **In this paper we propose an approach based on affine parameterization of the recourse policy for the solution of multi-stage optimization problems that arise in the context of allocation of financial portfolios over multiple periods. Such problems are typically dealt with using the multi-stage stochastic programming paradigm, which has the drawback of being computationally intractable. Here, we show that imposing an affine structure to the recourse policy results in an explicit and exact problem formulation, which is efficiently solvable by means of interior point methods for convex second order cone programs.**

I. INTRODUCTION

The fundamental goal of portfolio theory is to help the investor in allocating money among different securities in an "optimal" way. In the classical Markowitz framework, [11], the selection is guided by a quantitative criterion that considers a tradeoff between the return of an investment and its associated risk. Specifically, in the Markowitz approach, each asset is described by means of its return over a fixed period of time (e.g. one month), and the vector of asset returns is assumed to be random, with known expectation and covariance matrix. An optimal portfolio of assets is hence selected by maximizing the expected return at the end of the period, subject to the constraint that the total risk (as expressed by the portfolio variance, or "volatility") is below some given level. From the computational side, this classical paradigm results in a quadratic programming problem, which may be efficiently solved numerically on a computer.

However, a drawback of this basic approach is that it is tuned to a single period, and it can therefore provide short-sighted strategies of investment, if applied repeatedly over many subsequent periods. To overcome these difficulties, one may formulate from the beginning the problem over a horizon composed of multiple periods ($T = \#$ of periods > 1), with the ultimate goal of maximizing the portfolio return at the final stage T, by rebalancing it at the intermediate stages. The most widely applied and studied multi-period allocation problems are those that consider only two periods. In this paper we also concentrate on this case, but the proposed technique can be extended to the general multi-period case.

In a two-period investment setup, the investor takes some action at the initial stage. Then, he holds the portfolio for the first period of time, and at the end of this period the return of his investment is dictated by a random outcome. A recourse decision can then be made in the second stage, to compensate for any bad effects that might have been experienced as a result of the first-stage decision. The mainstream computational model to solve such kind of recursive decision problems is provided by multi-stage stochastic programming, see e.g. [3]. While stochastic programming may provide a conceptually sound framework for posing multi-stage decision problems, from the computational side it results in numerical optimization problems that are very hard to solve, see e.g. [12]. Various numerical techniques have been proposed in the literature to approximately solve multi-stage stochastic programs, which eventually require the use of a large number of random samples of different possible scenarios, [6]. In the specific context of portfolio allocation, a classical solution method is based on Benders decomposition, and it is proposed in [5]. One key difficulty in two-stage optimization comes from the fact that the second stage decision is actually a decision rule, or "policy," that defines which second-stage action should be taken in response to each random outcome in the first period. These last random outcomes are generally infinite (if the returns over the first period are, as it is commonly assumed, continuous random variables), and also the policy space is infinite dimensional.

In this paper, we propose to consider the second-stage recourse action to be prescribed by a policy with fixed structure. In particular, we shall consider recourse actions that are affine functions of the first period returns, where the coefficients of these functions become the decision variables of our portfolio allocation problem. While with this position we loose some generality, since the policy is now restricted to the affine functions class, we also gain decisive advantages. First, we show that it is possible to express explicitly the expected value and variance of the portfolio at any stage, as a function of the decision variables. Furthermore, the optimization objective and constraints result to be convex in these variables, and therefore the optimal strategy, under the affine recourse hypothesis, can be found *exactly* and *numerically efficiently* by means of standard codes for convex second order cone (SOC) programming. Second, the optimal recourse parameters returned by the algorithm have a simple and insightful interpretation as nominal actions and sensitivities, as further discussed in Remark 1. Finally, the proposed technique can be extended to the general case of problems with an arbitrary number of stages.

The idea of using affine recourse has been inspired by a similar technique recently proposed in [2] in the context of robust optimization (i.e. optimization problems where the data is subject to deterministic unknown-but-bounded

This work has been supported by FIRB funds from the Italian Ministry of University and Research.

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uncertainties), where the authors impose on some of the variables an affine dependence on the data, so that they can "tune themselves to varying data." Also, the affine recourse is reminiscent of the classical linear feedback laws used for control of dynamical systems.

II. Preliminaries

A. Basic definitions

Let $p_i(k)$ denote the price of a certain security "i" at time k, where k denotes the time instant at the end of the k-th period of time of fixed length Δ (Δ may be for instance one day, a month, or a year). The (simple) *return* of an investment in security i over the period of time $[k-1, k]$ is defined as

$$r_i(k) \doteq \frac{p_i(k) - p_i(k-1)}{p_i(k-1)} = \frac{p_i(k)}{p_i(k-1)} - 1.$$

The one-period *gain* (or factor return) of an investment in security i is $g_i(k) \doteq p_i(k)/p_i(k-1) = r_i(k) + 1$.

If $p(k)$ denotes a collection of prices of n assets: $p(k) = [p_1(k)\ p_2(k)\ \cdots\ p_n(k)]^T$, the return and gain vectors are defined accordingly. The asset gains $g(k)$ are assumed to be random quantities that follow a given stochastic process.

B. Portfolio dynamics

Let $x_i(k)$, $i = 1, \ldots, n$ denote the Euro value of an investor's holdings in asset i, at a certain time k. Vector $x(k) \doteq [x_1(k)\ \cdots\ x_n(k)]^T$ represents the investor portfolio at time k. The total wealth at time k is $w(k) = 1^T x(k)$, where 1 represents a vector of ones of suitable dimension.

Let $x(0)$ be the portfolio composition at some initial time $k = 0$. At $k = 0$, we have the opportunity of conducting transactions on the market and therefore adjusting the portfolio by increasing or decreasing the amount invested in each asset. We denote by $u(0) \doteq [u_1(0)\ \cdots\ u_n(0)]^T$ the Euro amount transacted in each asset, with $u_i > 0$ for buying, and $u_i < 0$ for selling. Just after transactions, the adjusted portfolio is $x^+(0) = x(0) + u(0)$. Clearly, constraints exist on the portfolio holdings and on the adjustment vector $u(0)$. These constraints are dictated both by market regulations and by the investor (exposure to risk, etc.). We shall discuss in more detail portfolio constraints in Section III-A.

Suppose now that the portfolio is held unchanged for one fixed period of time Δ. At the end of this first period, the portfolio composition is

$$x(1) = G(1)x^+(0) = G(1)x(0) + G(1)u(0),$$

where $G(1) = \mathrm{diag}(g_1(1), \ldots, g_n(1))$ is a diagonal matrix of the asset gains over the period from time 0 to time Δ. At the end of this first time period, we perform again an adjustment of the portfolio: $x^+(1) = x(1) + u(1)$, and then hold the updated portfolio for another period of duration Δ. At time 2Δ the portfolio composition is hence

$$x(2) = G(2)x^+(1) = G(2)x(1) + G(2)u(1).$$

Proceeding in this way for $k = 0, 1, 2, \ldots$, we determine the iterative dynamic equations of the portfolio composition at the end of period $(k+1)\Delta$

$$x(k+1) = G(k+1)x(k) + G(k+1)u(k) \qquad (1)$$

as well as the equations for portfolio composition just after the $(k+1)$-th transaction

$$x^+(k) = x(k) + u(k).$$

Notice that, since the asset gains over a given period are random quantities, recursion (1) does not generate a deterministic sequence of portfolios, but it rather defines a stochastic process.

III. Multi-stage portfolio allocation

Given the initial portfolio composition $x(0)$ and a final horizon T, our objective is to determine an optimal sequence of portfolio adjustments $u(0), u(1), \ldots, u(T-1)$, in order to maximize a utility function at the end of period $k = T$. Specifically, we shall consider a classical Markowitz problem of maximizing the expectation of the portfolio total gain $J(T) \doteq E\{w(T)/w(0)\}$, under a constraint on exposure to risk, expressed in the form of a bound on the variance of $w(T)$. Notice that, since $w(0)$ is given and constant, maximizing $J(T)$ is equivalent to minimizing $-E\{w(T)\}$, which is henceforth used as the optimization objective. Other constraints are also typically included in the problem, as discussed in the next section.

A. Portfolio constraints

a) Budget constraints: Cash may be part of the portfolio assets. Each time the portfolio is adjusted, money value is transferred from one asset to another, but the net value remains unchanged, except for possible loss due transaction costs. This fact is expressed by the *budget constraints*

$$1^T u(k) + c(u(k)) = 0, \quad k = 0, 1, \ldots$$

where $c(u(k))$ denotes all costs associated to the transactions. Since the focus of this paper is on a new technique for multi-stage problems, we shall assume for simplicity in the sequel that no transaction costs are present. Under this assumption, the budget constraints become $1^T u(k) = 0$, $k = 0, 1, \ldots$

b) Volatility constraint: A bound on the maximum allowed volatility of the final wealth is imposed by constraining the variance of the final wealth $w(T)$ to be smaller than a given value $\sigma^2_{\max}$: $\mathrm{var}\{w(T)\} \leq \sigma^2_{\max}$. Similar constraints can also be imposed on any intermediate stage.

c) Portfolio composition constraints: At each period k when the portfolio is rebalanced, we can impose constraints ensuring that, with high probability $1 - \delta$, $\delta \in [0, 1]$, the portfolio content $x_i^+(k)$ in each single security i is between given minimum and maximum levels,

$$\mathrm{Prob}\{\underline{b}_i(k) \leq x_i^+(k)\} \geq 1 - \delta, \quad i = 1, \ldots, n; k = 0, 1, \ldots$$
$$\mathrm{Prob}\{x_i^+(k) \leq \bar{b}_i(k)\} \geq 1 - \delta, \quad i = 1, \ldots, n; k = 0, 1, \ldots$$

where $\underline{b}_i(k), \bar{b}_i(k)$ are the given lower and upper bound on portfolio holding in security i at time k, after rebalancing. For instance, if no shortselling is allowed, we impose these constraints by taking $\underline{b}_i(k) = 0$, $\bar{b}_i(k) = \infty$, i.e.

$$\mathrm{Prob}\{x_i^+(k) \geq 0\} \geq 1 - \delta, \ i = 1, \ldots, n; \ k = 0, 1, \ldots \quad (2)$$

To enforce this probabilistic constraint, one can exploit the Chebychev inequality, which guarantees that (2) is satisfied with probability at least $1 - \delta$, irrespective of the actual probability distribution of $x_i^+(k)$, if

$$E\{x_i^+(k)\} \geq \nu\sqrt{\mathrm{var}\{x_i^+(k)\}}, \quad (3)$$

where $\nu = \sqrt{1/\delta}$, see for instance [4].

B. The feedback action strategy

A key and well-known observation is now that in tackling the described optimization problem, two strategies are available: an "open loop" strategy and a "closed loop" one.

In the open loop strategy, the whole action sequence $u(0), u(1), \ldots, u(T-1)$ is determined at the decision time $k = 0$. This setup fails to exploit the sequential nature of the decision problem at hand. Indeed, we remark that at time $k = 0$ the whole future sequence of asset gains is uncertain. However, only action $u(0)$ is actually applied to the portfolio at $k = 0$. At the subsequent time $k = 1$, the *actual outcome* of asset returns over the first period is revealed to the decision maker, and therefore his next action $u(1)$ should take into account this knowledge. In other words, at the decision time $k = 1$, only the future asset gains relative to period 2 are still uncertain. In general, at the decision stage k, the past gains over periods $1, \ldots, k$ have been observed, and hence are exactly known, while the future gains of periods $k+1, \ldots, T$ are still uncertain. Thus, uncertainty is reduced as the decision stage moves forward.

In this dynamic optimization setting, one seeks for an optimal action as a function of the information state of the system, i.e. $u(k)$ is given by a policy, or rule, π_k that associates an action to a given state of knowledge (closed loop strategy). In the problem at hand, the information state is the observed sequence $(x(1), \ldots, x(k))$, so that $u(k) = \pi_k(x(1), \ldots, x(k))$. We here show that we can equivalently take $(g(1), \ldots, g(k))$ as information state.

To see this, note that substituting $u(k) = \pi_k(x(1), \ldots, x(k))$ in (1) and solving onward up to time j, $x(j)$ is obtained as a function of $(g(1), \ldots, g(j))$ (plus $x(0)$, which is however a given deterministic quantity). Thus, the information in $(x(1), \ldots, x(k))$ is no larger than the information in $(g(1), \ldots, g(k))$. Viceversa, from (1) we see that $g(j)$ can be determined from $x(j)$, $x(j-1)$ and $u(j-1) = \pi_{j-1}(x(1), \ldots, x(j-1))$, so that the information in $(g(1), \ldots, g(k))$ is no larger than the information in $(x(1), \ldots, x(k))$. Thus, for any policy of the form $u(k) = \pi_k(x(1), \ldots, x(k))$, $(x(1), \ldots, x(k))$ and $(g(1), \ldots, g(k))$ are equivalent information states.

Let us ask now the following question: is it true that the set of policies of the form $u(k) = \pi_k(x(1), \ldots, x(k))$ is equivalent to the set of policies of the form $u(k) = \pi_k(g(1), \ldots, g(k))$? The answer is indeed positive. In fact, if we take a policy $u(k) = \pi_k(x(1), \ldots, x(k))$, from the argument developed in the previous paragraph, the information state $(x(1), \ldots, x(k))$ can be replaced by the information state $(g(1), \ldots, g(k))$ and, therefore, $u(k) = \pi_k(x(1), \ldots, x(k))$ can be rewritten as $u(k) = \pi_k(g(1), \ldots, g(k))$ (for some different π_k). Conversely, suppose we apply $u(k) = \pi_k(g(1), \ldots, g(k))$. If, by inductive assumption, $(g(1), \ldots, g(k))$ is equivalent to $(x(1), \ldots, x(k))$ up to a given time, say j, then for $k \leq j$ action $u(k)$ can be rewritten as $u(k) = \pi_k(x(1), \ldots, x(k))$. But then, applying the usual argument of the previous pragraph for $k \leq j+1$, we see that $(x(1), \ldots, x(j+1))$ is equivalent to $(g(1), \ldots, g(j+1))$ so substaining induction to show that $u(j+1) = \pi_{j+1}(g(1), \ldots, g(j+1))$ can be rewritten as $u(j+1) = \pi_{j+1}(x(1), \ldots, x(j+1))$. Repeating on subsequent steps, we obtain that $u(k) = \pi_k(g(1), \ldots, g(k))$ can be rewritten as $u(k) = \pi_k(x(1), \ldots, x(k))$ for any k. In the sequel, we make reference to control policy in the form $u(k) = \pi_k(g(1), \ldots, g(k))$.

Notice that in this setting our optimization problem is a functional one, since we need to search over the infinite-dimensional space of policies π_k. An exact solution to this problem is numerically unfeasible. Techniques such as (constrained) dynamic programming, or stochastic optimization try to solve approximations of the problem, and invariably result in NP-hard problem formulations, see, e.g., [3], [12].

We here propose a solution approach based on parameterizations of the policies. Suppose that policy π_k is restricted to belong to a family Π_k of functions of $g(1), \ldots, g(k)$ of given structure, that are *affinely parameterized* by a collection of parameters θ_k. Then, it can be immediately verified that also the portfolio composition $x(k)$ is an affine function of the decision parameters θ_k. Moreover, the expectation of $x(k)$ will also be affine in the θ_k's, and the covariance matrix of $x(k)$ will be a convex quadratic function of the θ_k's. These desirable properties hold in general, for any stochastic model of the market (i.e. for any stochastic description of the $G(k)$'s), and any given functional dependence of π_k on $g(1), \ldots, g(k-1)$, provided that the parameterization of the policy family Π_k is affine in θ_k. We now formally state this assumption.

Assumption 1: Each policy π_k is assumed to be a function of given structure of $g(1), \ldots, g(k)$, affinely parameterized by a collection θ_k of free parameters. $\star$

For example, the π_k's can be taken to be polynomials of some fixed order in $g(1), \ldots, g(k)$, with coefficients described by θ_k. A special case is given by polynomials families of degree one, chosen as

$$\pi_k(g(1), \ldots, g(k)) = \theta_0(k) + \sum_{i=1}^{k} \Theta_i(k)(g(i) - \bar{g}(i)) \quad (4)$$

where $\theta_k = (\theta_0(k), \Theta_1(k), \ldots, \Theta_k(k))$, with $\theta_0(k) \in \mathbb{R}^n$, $\Theta_i(k) \in \mathbb{R}^{n,n}$, constitute the collection of parameters, and $\bar{g}(k) \doteq E\{g(k)\}$. This particular choice leads to a simple statistical set-up and is made in the specific two stage problem discussed in Section IV.

C. Stochastic market model

We shall make in the sequel the following standard assumptions on the stochastic behavior of the asset gains. These hypotheses are not necessary within our framework, but their introduction simplifies calculations. Generalizations are possible.

Assumption 2:

1) $g_i(k_1)$ is statistically independent of $g_j(k_2)$, for all i, j and for all $k_2 \neq k_1$. In other words, the returns over different periods are assumed to be independent.
2) The first two moments of the gain vectors $g(k)$, $k = 1, 2, \ldots$, are known.

$\star$

These assumptions are compatible with the classical "efficient market hypothesis" (EMH), see for example [10]. We define

$$\begin{aligned} \bar{g}(k) &\doteq E\{g(k)\}, \quad k = 1, 2, \ldots \\ M(k) &\doteq E\{g(k)g^T(k)\}, \quad k = 1, 2, \ldots \end{aligned}$$

and the covariance matrices

$$\Sigma(k) \doteq \text{var}\{g(k)\} = M(k) - \bar{g}(k)\bar{g}^T(k), \quad k = 1, 2, \ldots$$

In order to convey our ideas in a clear way, in the rest of this paper we concentrate on the simplest form of multi-stage optimization problem, that is on a problem with two stages. Whilst this setup is simple enough not to let the notation obscure the main line of discourse, it contains the main ingredients and difficulties of dynamic optimization under uncertainty.

IV. A two-stage portfolio problem

Let the time horizon be of $T = 2$ periods (Figure 1). Our objective is to determine an optimal sequence of portfolio adjustments $u(0), u(1)$ so to maximize $E\{w(T)\}$, under a bound on risk and imposing that shortselling is avoided, with high probability (and thus imposing constraints of type (3) at each stage).

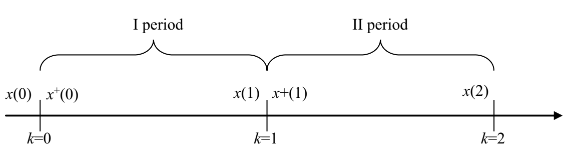

Fig. 1. Time axis and periods for two-stage portfolio optimization.

The open loop optimization strategy would require the solution of the optimization problem

$$\min_{u(0),u(1)} \; -E\{1^T x(2)\} \qquad \text{subject to:} \tag{5}$$

$$\text{var}\{1^T x(2)\} \leq \sigma^2_{\max} \tag{6}$$

$$1^T u(k) = 0, \qquad k = 0, 1 \tag{7}$$

$$E\{x_i^+(k)\} \geq \nu\sqrt{\text{var}\{x_i^+(k)\}}, \quad i = 1, \ldots, n; \; k = 0, 1 \tag{8}$$

where the objective (5) is (minus) the expectation of the final total wealth $w(T)$, (6) is a bound on the volatility of $w(T)$, (7) are the budget constraints, and (8) enforce no shortselling, with high probability.

However, we already discussed the fact that a closed loop strategy is much preferable in this dynamic context. Hence, in the following section we develop the explicit optimization model for the two-stage problem in closed loop, assuming that the recourse policies are affinely parameterized.

A. Affine recourse policy

In our specific problem with two periods, we have a so-called "here-and-now" variable, which is the first portfolio rebalancing vector $u(0)$, and a "wait-and-see" variable, which is the second portfolio adjustment $u(1)$. We parameterize the policy π_1 by imposing an affine functional dependence on $u(1)$, according to the structure (4)

$$u(1) = \theta + \Theta(g(1) - \bar{g}(1)) \tag{9}$$

where $\theta_1 = (\theta, \Theta)$, with $\theta \in \mathbb{R}^n$, $\Theta \in \mathbb{R}^{n,n}$ is the collection of parameters. The new optimization variables are $u(0)$ and the "recourse" parameters $\theta \in \mathbb{R}^n$, $\Theta \in \mathbb{R}^{n,n}$, whose purpose is to correct $u(1)$ in function of the discrepancy between the expectation on the first period gain $\bar{g}(1)$, and its actual outcome $g(1)$.

B. The controlled portfolio

A key advantage in imposing the specific dependence (9) for the wait-and-see variable is that the expected value and variance of the portfolio can be simply expressed. These expressions are derived next. First notice that we have

$$\begin{aligned} x^+(0) &= x(0) + u(0) \\ x(1) &= G(1)(x(0) + u(0)) \\ x^+(1) &= G(1)(x(0) + u(0)) + \theta + \Theta(g(1) - \bar{g}(1)) \\ x(2) &= G(2)G(1)(x(0) + u(0)) + G(2)\theta + \\ & \quad +G(2)\Theta(g(1) - \bar{g}(1)). \end{aligned}$$

To simplify the notation, let

$$\begin{aligned} \xi &\doteq x(0) + u(0) \\ g &\doteq [g_1(1)g_1(2) \;\; g_2(1)g_2(2) \;\; \cdots \;\; g_n(1)g_n(2)]^T \\ h &\doteq \text{vec}(\Theta) \\ G &\doteq \text{diag}(g), \end{aligned}$$

where $\text{vec}(\cdot)$ is the column vectorization operator, and define also $\bar{g} \doteq E\{g\}$, $\tilde{g}(1) \doteq g(1) - \bar{g}(1)$, $\tilde{g}(2) \doteq g(2) - \bar{g}(2)$, $\tilde{g} \doteq g - \bar{g}$, $\bar{G}(1) \doteq \text{diag}(\bar{g}(1))$, $\bar{G}(2) \doteq \text{diag}(\bar{g}(2))$, $\bar{G} \doteq \text{diag}(\bar{g})$. Then, we rewrite

$$\begin{aligned} x^+(0) &= \xi \\ x(1) &= G(1)\xi \\ x^+(1) &= G(1)\xi + \theta + \Theta\tilde{g}(1) \\ x(2) &= G\xi + G(2)\theta + G(2)\Theta\tilde{g}(1). \end{aligned}$$

The total wealth at final time $T = 2$ is

$$w(2) = 1^T x(2) = g^T\xi + g^T(2)\theta + g^T(2)\Theta\tilde{g}(1).$$

1) Expectation and variance of $w(2)$: For the expectation, since $g(1), g(2)$ are independent, we simply have

$$\bar{w} \doteq E\{w(2)\} = \bar{g}^T\xi + \bar{g}^T(2)\theta$$

and hence

$$\begin{aligned} \tilde{w} \doteq w(2) - E\{w(2)\} &= \\ &= \tilde{g}^T\xi + \tilde{g}^T(2)\theta + g^T(2)\Theta\tilde{g}(1) \\ &= [\xi^T \; \theta^T \; h^T] \begin{bmatrix} \tilde{g} \\ \tilde{g}(2) \\ \tilde{g}(1) \otimes g(2) \end{bmatrix} \end{aligned}$$

where $\otimes$ denotes the Kronecker product: if $X \in \mathbb{R}^{n,m}$, $X \otimes Y$ is a block matrix with n row-blocks and m column-blocks, with the (i,j)-th block being $X_{i,j}Y$. Also, in the sequel $\odot$ denotes the Hadamard product, i.e. the entry-wise product of two matrices. The variance of $w(2)$ is therefore given by

$$\mathrm{var}\{w(2)\} = E\{\tilde{w}^2\} = \left[\begin{array}{ccc} \xi^T & \theta^T & h^T \end{array}\right] \Gamma \left[\begin{array}{ccc} \xi^T & \theta^T & h^T \end{array}\right]^T$$

being

$$\Gamma \doteq E\left\{ \left[\begin{array}{ccc} \tilde{g}\tilde{g}^T & \tilde{g}\tilde{g}^T(2) & \tilde{g}(\tilde{g}^T(1) \otimes g^T(2)) \\ * & \tilde{g}(2)\tilde{g}^T(2) & \tilde{g}(2)(\tilde{g}^T(1) \otimes g^T(2)) \\ * & * & (\tilde{g}(1) \otimes g(2))(\tilde{g}^T(1) \otimes g^T(2)) \end{array}\right] \right\}$$

where $*$ denotes elements whose value is inferred from symmetry. The evaluation of the expectations for the six blocks in the above matrix requires lengthy manipulations, that are omitted for brevity. The result is the following:

$$\Gamma \doteq \left[\begin{array}{ccc} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ * & \Gamma_{22} & 0 \\ * & * & \Gamma_{33} \end{array}\right]$$

with

$$\begin{aligned} \Gamma_{11} &= \Sigma(1) \odot \Sigma(2) + \Sigma(1) \odot [\bar{g}(2)\bar{g}^T(2)] + \\ &\quad + \Sigma(2) \odot [\bar{g}(1)\bar{g}^T(1)] \\ \Gamma_{22} &= \Sigma(2) \\ \Gamma_{33} &= \Sigma(1) \otimes M(2) \\ \Gamma_{12} &= \bar{G}(1)\Sigma(2) \\ \Gamma_{13} &= \left[\begin{array}{c} \Sigma_1(1) \otimes M_1(2) \\ \Sigma_2(1) \otimes M_2(2) \\ \cdots \\ \Sigma_n(1) \otimes M_n(2) \end{array}\right] \end{aligned}$$

where $\Sigma_i(1)$ denotes the i-th row of $\Sigma(1)$, and $M_i(2)$ denotes the i-th row of the matrix of second order moments

$$M(2) = E\{g(2)g^T(2)\} = \Sigma(2) + \bar{g}(2)\bar{g}^T(2).$$

The key remark at this point is that the expectation and variance of the end-of-period wealth can be computed from the available information. It then follows that the volatility constraint (6) can be explicitly imposed as a convex quadratic constraint on the decision variables ξ, θ, h:

$$\left[\begin{array}{ccc} \xi^T & \theta^T & h^T \end{array}\right] \Gamma \left[\begin{array}{ccc} \xi^T & \theta^T & h^T \end{array}\right]^T \leq \sigma^2_{\max}$$

(notice that this constraint is convex, since Γ is positive semidefinite).

2) Expectation and variance of $x_i^+(1)$: We proceed in a similar way for the computation of the expectation and variance of $x_i^+(1)$, which are needed in (8). For the expectation, we have

$$\overline{x}^+(1) \doteq E\{x^+(1)\} = \bar{G}(1)\xi + \theta$$

and therefore $\overline{x}_i^+(1) \doteq E\{x_i^+(1)\} = \bar{g}_i(1)\xi_i + \theta_i$. For the variance, it can be easily checked that

$$\mathrm{var}\{x_i^+(1)\} = (\xi_i e_i^T + \Theta_i)\Sigma(1)(\xi_i e_i^T + \Theta_i)^T$$

where $e_i \in \mathbb{R}^n$ has all zero entries, except for the i-th entry, which is equal to one, and Θ_i denotes the i-th row of matrix Θ. It follows that constraint (8) can be explicitly imposed as

$$\nu \|\Sigma^{1/2}(1)(\xi_i e_i^T + \Theta_i)^T\| \leq \bar{g}_i(1)\xi_i + \theta_i$$

where $\Sigma^{1/2}(1)$ denotes the symmetric matrix-square-root of $\Sigma(1)$. This latter is a convex second order cone constraint (SOC) on the decision variables, see [1] for further details on second order cone programming.

We finally remark that the no-shortselling constraint (8) at $k = 0$ is simply enforced by the entry-wise linear inequality $x^+(0) \geq 0$, i.e. $\xi \geq 0$. Also, the budget constraint (7) for $k = 0$ writes $1^T u(0) = 0$, which is enforced by the linear equality $1^T\xi = 1^T x(0)$, whereas, for $k = 1$, the budget constraint can be enforced by imposing $1^T\theta = 0, \quad 1^T\Theta = 0$.

C. The explicit two-period program

Collecting our previous results, we can write the two-stage portfolio optimization problem in the explicit form of a convex optimization program on the variables $\xi, \theta \in \mathbb{R}^n$ and $\Theta \in \mathbb{R}^{n,n}$ (or, equivalently, $h = \mathrm{vec}(\Theta) \in \mathbb{R}^{n^2}$):

$$\begin{array}{l} \min_{\xi,\theta,\Theta} \ -\bar{g}^T\xi - \bar{g}^T(2)\theta \quad \text{subject to:} \\ \left[\begin{array}{ccc} \xi^T & \theta^T & h^T \end{array}\right] \Gamma \left[\begin{array}{ccc} \xi^T & \theta^T & h^T \end{array}\right]^T \leq \sigma^2_{\max} \\ 1^T\xi = 1^T x(0) \\ 1^T\theta = 0 \\ 1^T\Theta = 0 \\ \xi \geq 0 \\ \nu\|\Sigma^{1/2}(1)(\xi_i e_i^T + \Theta_i)^T\| \leq \bar{g}_i(1)\xi_i + \theta_i, \quad i = 1, \ldots, n. \end{array}$$

D. Numerical example

To exemplify our approach, we consider next a portfolio composed of six blue chips from the Milan stock exchange (tickers: AUTS.MI, CPTA.MI, ENI.MI, GASI.MI, PG.MI, SPMI.MI), plus cash. Cash is considered to be a riskless asset, having unit gain over each period. The mean return and covariance matrix of the risky assets have instead been estimated via standard methods, using historical data for closing prices, from 03-May-2004 to 17-Jan-2005, with an exponential forgetting factor of .99. We here assume that the system is stationary, i.e. that its statistics do not change from period to period. The expected gain over each period (assuming a period of 20 trading days) results to be

$$\begin{aligned} &\bar{g}(1) = \bar{g}(2) = \\ &\qquad [1.0535 \ \ 1.0473 \ \ 1.0139 \ \ 1.0183 \ \ 1.0170 \ \ 1.0268 \ \ 1]^T \end{aligned}$$

and the covariance is

$$\Sigma(1) = \Sigma(2) = \left[\begin{array}{ccccccc} 1.3058 & 0.4628 & 0.3996 & 0.2589 & 0.5024 & 0.1886 & 0 \\ 0.4628 & 4.1217 & 0.6221 & 0.7037 & 1.2662 & 0.1857 & 0 \\ 0.3996 & 0.6221 & 1.9690 & 0.4737 & 0.5141 & 1.4340 & 0 \\ 0.2589 & 0.7037 & 0.4737 & 0.8004 & 0.5493 & 0.2300 & 0 \\ 0.5024 & 1.2662 & 0.5141 & 0.5493 & 10.6348 & 0.0551 & 0 \\ 0.1886 & 0.1857 & 1.4340 & 0.2300 & 0.0551 & 3.7108 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \times 10^{-3}.$$

We considered an initial portfolio holding of a unit in cash, i.e. $x(0) = [0 \ 0 \ \cdots \ 0 \ 1]^T$, and we fixed $\nu = 3.16$, which ensures satisfaction of the no-shortselling constraints at the second stage, with probability at least 0.9, regardless of the data distribution. We conducted the numerical experiment with the objective of evaluating the improvement brought by the recourse action (9), with respect to the "open loop" strategy that one would obtain from the solution of problem (5). Therefore, we repeatedly solved both problem (IV-C) and problem (5), for 14 increasing values of $\sigma_{\max}$, and we plotted, for each problem, the optimal portfolio return at the final stage, versus the associated variance. These results are graphically reported in Figure 2(a). The numerical computations have been performed on a Windows machine under Matlab, using the YALMIP parser and the SeDuMi solver, see [9].

As it should be expected, the linear recourse action provides portfolios that always dominate those obtained by the open loop strategy. Moreover, the improvement in the achievable expected return is not negligible, peaking at about 30% in the low variance zone, see Figure 2(b).

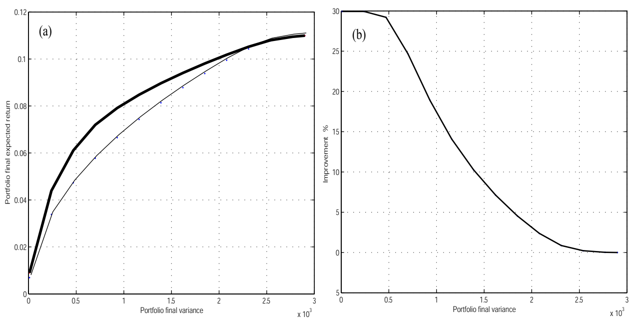

Fig. 2. (a) Each point on the plot represents the maximum portfolio return at the final stage, achievable for the corresponding variance limit in abscissa. Bold line is the frontier obtained solving problem (IV-C), while the light line is obtained via problem (5). (b) Percent improvement in expected return, upon the no-recourse strategy.

For a specific value of $\sigma^2_{\max}$, say $\sigma^2_{\max} = 0.001$, the solutions of problems (5) and (IV-C) look as follows. Problem (5) yields the a-priori portfolio adjustments

$$\begin{aligned} u(0) &= [0.484\ \ 0.083\ \ 0.000\ \ 0.063\ \ 0.000\ \ 0.066\ \ -0.696]^T \\ u(1) &= [0.030\ \ 0.006\ \ 0.000\ \ -0.009\ \ 0.000\ \ 0.002\ \ -0.029]^T \end{aligned}$$

to which it corresponds a maximum portfolio return at end of second period equal to 0.069. In the same setup, problem (IV-C) yields instead the here-and-now adjustment

$$u(0) = [0.759\ \ 0.157\ \ 0.000\ \ 0.000\ \ 0.000\ \ 0.084\ \ -1]^T$$

and the recourse parameters

$$\theta = [-0.325\ \ -0.094\ \ 0.000\ \ 0.036\ \ 0.000\ \ -0.040\ \ 0.424]^T$$

$$\Theta = \begin{bmatrix} -4.048 & -0.805 & -0.543 & -0.808 & -0.104 & -0.381 & 0 \\ 0.252 & -0.149 & -0.193 & -0.287 & -0.037 & 0.021 & 0 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0 \\ 0.269 & 0.067 & 0.052 & 0.077 & 0.01 & 0.031 & 0 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.000 & 0.000 & 0 \\ 0.363 & 0.082 & 0.044 & 0.061 & 0.007 & -0.049 & 0 \\ 3.164 & 0.805 & 0.641 & 0.957 & 0.124 & 0.378 & 0 \end{bmatrix},$$

providing a maximum portfolio return at end of second period equal to 0.081.

Remark 1 (Interpretation of the recourse parameters): Looking at (9) it is natural to give the following interpretation to the optimal recourse parameters θ, Θ returned by problem (IV-C). Vector θ represents the adjustment actions that the investor would perform on the portfolio at $k = 1$, if the gains over the first period were exactly as expected (i.e. if the outcome of $g(1)$ were equal to $\bar{g}(1)$). If this is not the case, the actual portfolio adjustment $u(1)$ is corrected by the term $\Theta(g(1) - \bar{g}(1))$, and hence the element Θ_{ij} in position row i and column j of Θ is interpreted as the sensitivity of $u_i(1)$ to deviations of $g_j(1)$ from its expectation.

For instance, referring to the previous numerical example, by observing θ and Θ we may conclude that the second stage decision $u(1)$ is rather sensitive to the deviation of the first period gain of AUTS.MI (the first asset) from its expectation. Indeed, if this gain is much higher than expected, (i.e. $g_1(1) - \bar{g}_1(1) \gg 0$), the recourse policy (9) will prescribe to aggressively sell this asset, and safeguard the profits by essentially putting them all into cash (notice the first and last coefficients in θ and in the first column of Θ).

V. Conclusions

In this paper, we proposed a technique for two-stage portfolio allocation based on affine parameterization of the recourse policy. In this setting, the sub-optimal portfolio adjustments can be found exactly and in a numerically efficient way, by solving an appropriate convex second order program. Several research issues are now open. First, we are elaborating the method in order to extend it to the general multi-period case. Second, while our method appears to be more simple, insightful and scalable than multi-stage stochastic programming, yet it has to be compared in performance, at least numerically, with some of the implementations of stochastic programming. Finally, since a major issue in Markowitz portfolio optimization is that of sensitivity to uncertainty in the data (for instance, in reality, the expected returns and covariances are not exactly known, see e.g. [7], [8]), a promising research direction is that of adding robustness to uncertainty in the multi-stage allocation problem.

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