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Stochastic Receding Horizon Control With Bounded Control Inputs: A Vector Space Approach

Debasish Chatterjee, Peter Hokayem, and John Lygeros

***Abstract*—We design receding horizon control strategies for stochastic discrete-time linear systems with additive (possibly) unbounded disturbances while satisfying hard bounds on the control actions. We pose the problem of selecting an appropriate optimal controller on vector spaces of functions and show that the resulting optimization problem has a tractable convex solution. Under marginal stability of the zero-control and zero-noise system we synthesize receding horizon polices that ensure bounded variance of the states while enforcing hard bounds on the controls. We provide examples that illustrate the effectiveness of our control strategies, and how quantities needed in the formulation of the resulting optimization problems can be calculated off-line.**

***Index Terms*—Constrained control, receding horizon control, stochastic control.**

I. Introduction

Receding horizon control is a popular paradigm for designing control policies. In the context of deterministic systems it has received a considerable amount of attention over the last two decades, and significant advancements have been made in terms of its theoretical foundations as well as industrial applications. The motivation comes primarily from the fact that receding horizon control yields tractable control laws for deterministic systems in the presence of constraints, and has consequently become popular in the industry. The counterpart in the context of stochastic systems, however, is a relatively recent development. In this article we solve the problem of stochastic receding horizon control for linear systems subject to additive (possibly) unbounded disturbances and hard norm bounds on the controls, over a class of feedback policies. Methods for guaranteeing hard bounds on the controls, within our context, while ensuring tractability of the underlying optimization problem are, to our knowledge, not available in the current literature. Preliminary results in this direction were reported in [1].

In the deterministic setting, the receding horizon control scheme is dominated by worst-case analysis relying on robust control and robust optimization methods, see, for example, [2]–[9] and the references therein. The central idea is to synthesize a controller based on the bounds of the noise such that a certain target set becomes invariant with respect to the closed-loop dynamics. However, such an approach tends to yield rather conservative controllers and large infeasibility regions. Moreover, assigning an *a priori* bound to the noise seems to demand considerable insight. A stochastic model of the noise is a natural alternative approach to this problem: the conservativeness of worst-case controllers may be reduced, and one may not need to impose any *a*

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priori bound on the maximum magnitude of the noise. In [10], the authors reformulate the stochastic programming problem as a deterministic one with bounded noise and solve a robust optimization problem over a finite horizon, followed by estimating the performance when the noise is unbounded but takes high values with low probability (as in the Gaussian case). In [11] a slightly different problem is addressed in which the noise enters in a multiplicative manner, and hard constraints on the states and controls are relaxed to constraints resembling the integrated chance constraints of [12] or risk measures in mathematical finance. Similar relaxations of hard constraints to soft probabilistic ones have also appeared in [13] for both multiplicative and additive noise, as well as in [14] for additive noise. There are also other approaches, e.g., those employing randomized algorithms as in [15]–[17]. Related lines of research can be found in [18], [19] dealing with constrained model predictive control (MPC) for stochastic systems motivated by industrial applications, in [20]–[22] dealing with stochastic stability. In [23] the authors proposed a Q-design approach to stochastic MPC which features several aspects present our current work, in particular, expressing control functions as sums of basis functions. Hard control bounds were not directly addressed in [23], but occupy the center stage of our present work. The article [24] deal with a formulation that allows probabilistic state constraints but not hard control constraints, and are hence complementary to the approach in the present article, and [25] treats the case of output feedback. Finally, note that probabilistic constraints on the controllers naturally raise difficult questions on what actions to take when such constraints are violated, see [26] for partial solutions to these issues.

The main contributions of the article are as follows: We give a tractable, convex, and globally feasible solution to the finite-horizon stochastic linear quadratic (LQ) problem for linear systems with possibly unbounded additive noise and hard constraints on the elements of the control policy; this is the subject of Section III. Within this framework one has two directions to pursue in terms of controller design, namely, a posteriori bounding the standard LQ controller, or employing certainty-equivalent receding horizon controller. While the former direction explicitly incorporates some aspects of feedback, the synthesis of the latter involves control constraints and implicitly incorporates the notion of feedback. Our choice of feedback policies explores the middle ground between these two choices: we explicitly incorporate both the control bounds and feedback at the design phase.

More specifically, we adopt a policy that is affine in certain bounded functions of the past noise. The optimal control problem is lifted onto general vector spaces of candidate control functions from which the controller can be selected algorithmically by solving a convex optimization problem. The candidate control functions may represent particular types of control functions that are easy or inexpensive to implement, or may be the only ones available for a specific application. For instance, piecewise constant policy elements with finitely many elements in their range may be viewed as an extended version of a bang-bang controller, or as a hybrid controller with a finite control alphabet. Our novel approach does not require artificially relaxing the hard constraints on the controls to soft probabilistic ones (to ensure large feasible sets), and still provides a globally feasible solution to the problem. Minimal assumptions of the noise sequence being i.i.d and having finite second moment are imposed. The effect of the noise appears in the convex optimization problem as certain fixed cross-covariance matrices, which may be computed off-line and stored. Our mechanism for selection of a policy consists of two steps: The first concerns the structure of our policies, and is motivated by preceding work in robust optimization and MPC [27]–[29]. The second concerns the procedure for selection of an optimal policy from a general vector space of candidate control functions, and is inspired by optimization by vector space methods [30]. With respect to the first step, our policies are more general compared to those in [27]–[29]. With respect to the second, the selection procedure of our policies consists of a one-step tractable static optimization program. Once tractability of the optimization problem is ensured, we employ the resulting control policy in a receding horizon scheme. Under our policies the closed-loop system is in general not necessarily Markovian, and as a result stability of the closed-loop system is neither standard nor immediate. Based on recent developments [20] on stochastic stability of marginally stable linear controlled systems, we have developed techniques that ensure bounded closed-loop variance; these are discussed at the end of Section III. We provide examples in Section IV that demonstrate the effectiveness of our policies with respect to standard methods such as certainty-equivalent MPC, saturated and standard unconstrained LQ.

II. Problem Statement

Consider the following discrete-time stochastic dynamical system:[1]

$$x_{t+1} = Ax_t + Bu_t + w_t, \qquad t \in \mathbb{N}_0 \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, u_t is the control input taking values in some given control set $\bar{\mathbb{U}} \subseteq \mathbb{R}^m$ to be defined later, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $(w_t)_{t\in\mathbb{N}_0}$ is a sequence of stochastic noise vectors with $w_t \in \mathbb{W} \subseteq \mathbb{R}^n$. We assume that the initial condition x_0 is given and that, at any time t, x_t is observed perfectly. We do not assume that the components of the noise w_t are uncorrelated, nor that they have zero mean; this effectively means that w_t may be of the form $Fw'_t + b$ for some noise $w'_t \in \mathbb{R}^p$ whose components are uncorrelated or mutually independent, $F \in \mathbb{R}^{n\times p}$, and $b \in \mathbb{R}^n$. Without loss of generality we retain the simpler notation of (1) throughout this article; the results readily extend to the case of $w_t = Fw'_t + b$, as can be seen in [1].

Generally, a *control policy* π is a sequence $(\pi_0, \pi_1, \pi_2, \ldots)$ of Borel measurable maps $\pi_t : (\mathbb{R}^n)^{k(t)} \to \bar{\mathbb{U}}$, $t \in \mathbb{N}_0$. Policies of finite length such as $(\pi_t, \pi_{t+1}, \ldots, \pi_{t+N-1})$ will be denoted in the sequel by $\pi_{t:t+N-1}$. Fix an optimization horizon $N \in \mathbb{N}$ and let us consider the following objective function at time t given the state x_t:

$$V_t \coloneqq \mathbb{E}\left[\sum_{k=0}^{N-1}\Big(x_{t+k}^{\mathsf{T}} Q_k x_{t+k} + u_{t+k}^{\mathsf{T}} R_k u_{t+k}\Big) + x_{t+N}^{\mathsf{T}} Q_N x_{t+N} \,\middle|\, x_t\right] \tag{2}$$

where $Q_t \geqslant 0, R_t \geqslant 0, Q_N \geqslant 0$ are some given symmetric matrices of appropriate dimension. At each time instant t, we are interested in minimizing (2) over the class of causal state feedback strategies Π defined as

$$u_{t+\ell} = \pi_{t+\ell}(x_t, x_{t+1}, \ldots, x_{t+\ell}), \qquad \ell = 0, 1, \ldots, N-1 \tag{3}$$

for some measurable functions $\pi_{t:t+N-1} := \{\pi_t, \ldots, \pi_{t+N-1}\} \in \Pi$, while satisfying $u_t \in \bar{\mathbb{U}}$ for each t. The *receding horizon control* procedure for a given control horizon $N_c \in \{1, \ldots, N\}$ and time t can be described as the successive concatenation of the following steps: measure the state x_t and determine an admissible optimal feedback

[1]Hereafter, $\mathbb{N} := \{1, 2, \ldots\}$ is the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ is the set of all the integers, $\mathbb{R}_{\geqslant 0}$ is the set of nonnegative real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We let $\mathbf{1}_A(\cdot)$ denote the indicator function of a set A, and $\mathbf{I}_{n\times n}$ and $\mathbf{0}_{n\times m}$ denote the n-dimensional identity matrix and $n \times m$-dimensional zeros matrix, respectively. Let $\|\cdot\|$ denote the standard Euclidean norm, and $\|\cdot\|_p$ denote the usual ℓ_p norms. Also, let $\mathbb{E}_{x_0}[\cdot]$ denote the expected value given x_0, and $\mathbf{tr}(\cdot)$ denote the trace of a matrix. For a matrix M we let $\sigma_{\max}(M)$ and $\sigma_{\min}(M)$ denote its maximal and minimal singular values. Finally, for a random vector X let Σ_X denote the matrix $\mathbb{E}\Big[XX^{\mathsf{T}}\Big]$ and μ_X denote the vector $\mathbb{E}\Big[X\Big]$.

control policy, say $\pi^*_{t:t+N-1} \in \Pi$, that minimizes (2) starting from time t, apply the first N_c elements $\pi^*_{t:t+N_c-1}$ of the policy $\pi^*_{t:t+N-1}$, increase t to $t+N_c$. In this context, if $N_c = 1$ then this is usual MPC, and if $N_c = N$, then it is usually known as rolling horizon control. For $t = 0, N_c, 2N_c, \ldots$, consider

$$\min_{\pi_{t:t+N-1}\in\Pi} \left\{ V_t \,\middle|\, (1) \text{ and } u_t,\ldots,u_{t+N-1} \in \bar{\mathbb{U}} \right\}. \tag{4}$$

If feasible, problem (4) yields optimal feedback control laws $\pi^* = \{\pi^*_t, \ldots, \pi^*_{t+N-1}\}$.

The evolution of the system (1) over a single optimization horizon N can be described as

$$X_t = \mathcal{A}x_t + \mathcal{B}U_t + \mathcal{D}W_t \tag{5}$$

where $X_t := \left[x_t^\top, \ldots, x_{t+N}^\top\right]^\top$, $U_t := \left[u_t^\top, \ldots, u_{t+N-1}^\top\right]^\top$, $W_t := \left[w_t^\top, \ldots, w_{t+N-1}^\top\right]^\top$, $\mathcal{A} := \begin{bmatrix} \mathbf{I}_{n\times n} \\ A \\ \vdots \\ A^N \end{bmatrix}$

$$\mathcal{B} := \begin{bmatrix} \mathbf{0}_{n\times m} & \cdots & \cdots & \mathbf{0}_{n\times m} \\ B & \ddots & & \vdots \\ AB & B & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0}_{n\times m} \\ A^{N-1}B & \cdots & AB & B \end{bmatrix},$$

$$\mathcal{D} := \begin{bmatrix} \mathbf{0}_{n\times n} & \cdots & \cdots & \mathbf{0}_{n\times n} \\ \mathbf{I}_{n\times n} & \ddots & & \vdots \\ A & \mathbf{I}_{n\times n} & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0}_{n\times n} \\ A^{N-1} & \cdots & A & \mathbf{I}_{n\times n} \end{bmatrix}.$$

Using the compact notation above, the optimization Problem (4) can be rewritten as follows:

$$\min_{\pi_{t:t+N-1}\in\Pi} \left\{ \mathbb{E}_{x_t}\left[X_t^\top \mathcal{Q} X_t + U_t^\top \mathcal{R} U_t\right] \,\middle|\, (5) \text{ and } U_t \in \mathbb{U} \right\} \tag{6}$$

where $\mathcal{Q} = \operatorname{diag}\{Q_t, \ldots, Q_{t+N}\}$, $\mathcal{R} = \operatorname{diag}\{R_t, \ldots, R_{t+N-1}\}$, and $\mathbb{U} := (\bar{\mathbb{U}})^N$.

III. Main Result

Let $\mathcal{H}$ be a nonempty finite-dimensional vector space of measurable functions $(\mathfrak{e}^\nu)_{\nu\in\mathcal{I}}$ with the control set $\mathbb{U}$ as their range, and $\mathcal{I}$ is an index set. We are interested in policies of the form

$$U_t = \boldsymbol{\eta}_t + \boldsymbol{\varphi}(W_t) := \begin{bmatrix} \eta_t \\ \eta_{t+1} \\ \vdots \\ \eta_{t+N-1} \end{bmatrix} + \begin{bmatrix} \varphi_0 \\ \varphi_1(w_t) \\ \vdots \\ \varphi_{N-1}(w_t, w_{t+1}, \ldots, w_{t+N-2}) \end{bmatrix} \tag{7}$$

where, w_i is the i-th random noise vector and $\eta_i \in \mathbb{R}^m$ for $i = t, \ldots, t+N-1$, $\varphi_0 = 0 \in \mathbb{R}^m$, $\varphi_j(w_t, \ldots, w_{t+j-1}) = \sum_{i=t}^{t+j-1} \varphi_{j,i}(w_i) \in \mathbb{R}^m$ for $j = 1, \ldots, N-1$, and each function $\varphi_{j,i}$ belongs to the linear span of the basis elements $(\mathfrak{e}^\nu)_{\nu\in\mathcal{I}}$, and thus has a representation as a linear combination $\varphi_{j,i}(\cdot) = \sum_{\nu\in\mathcal{I}} \theta^\nu_{j,i}\mathfrak{e}^\nu(\cdot)$, $j = 1, \ldots, N-1$, $i = t, \ldots, t+j-1$, where $\theta^\nu_{j,i}$ are matrices of coefficients of appropriate dimension. Although this feedback function is directly from the noise, since the state is assumed to be perfectly measured, u_{t+j} is a feedback from all the states $x_t, \ldots, x_{t+j}$. Also by construction, it is causal. Analogous to Fourier coefficients in harmonic analysis, we call the $\theta^\nu_{j,i}$ the ν-th Fourier coefficient of the function $\varphi_{j,i}$. Therefore, for every $j = 1, \ldots, N-1$, we have the representation

$$\begin{aligned} &\varphi_j(w_t, \ldots, w_{t+j-1}) \\ &\quad = \begin{bmatrix} \theta_{j,t} & \theta_{j,t+1} & \ldots & \theta_{j,t+j-1} & 0 & \ldots & 0 \end{bmatrix}_{\mathbb{R}^{m\times n|\mathcal{I}|(N-1)}} \\ &\quad\quad \cdot \begin{bmatrix} \mathfrak{e}(w_t) \\ \mathfrak{e}(w_{t+1}) \\ \vdots \\ \mathfrak{e}(w_{t+N-2}) \end{bmatrix}_{\mathbb{R}^{n|\mathcal{I}|(N-1)\times 1}} \end{aligned} \tag{8}$$

where $\theta_{j,i} := [\,\theta^1_{j,i} \ \cdots \ \theta^{|\mathcal{I}|}_{j,i}\,] \in \mathbb{R}^{m\times n|\mathcal{I}|}$, $\theta^\nu_{j,i} \in \mathbb{R}^{m\times n}$, $\mathfrak{e}(w_i) := [\,\mathfrak{e}^1(w_i) \ \cdots \ \mathfrak{e}^{|\mathcal{I}|}(w_i)\,]^\top$ for $i = t, \ldots, t+N-2$, and $0 \in \mathbb{R}^{m\times n|\mathcal{I}|}$. In this notation the policy (7) can be written as

$$\begin{aligned} U_t &= \boldsymbol{\eta}_t + \boldsymbol{\varphi}(W_t) \\ &= \boldsymbol{\eta}_t + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \theta_{1,t} & 0 & \cdots & 0 \\ \theta_{2,t} & \theta_{2,t+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N-1,t} & \theta_{N-1,t+1} & \cdots & \theta_{N-1,t+N-2} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \mathfrak{e}(w_t) \\ \mathfrak{e}(w_{t+1}) \\ \vdots \\ \mathfrak{e}(w_{t+N-2}) \end{bmatrix} \\ &=: \boldsymbol{\eta}_t + \boldsymbol{\Theta}_t \mathfrak{e}(W_t) \end{aligned} \tag{9}$$

where $\boldsymbol{\Theta}_t$ is now an $Nm \times \left(n(N-1)|\mathcal{I}|\right)$ matrix of Fourier coefficients. The Fourier coefficient matrix $\boldsymbol{\Theta}_t$ and the vector $\boldsymbol{\eta}_t$ depend on x_t and play the role of the optimization parameters in our search for an optimal policy. Note that $\mathfrak{e}(W_t)$ does not include the noise vector w_{t+N-1}, and that $\boldsymbol{\Theta}_t$ is strictly lower block triangular to enforce causality.

Remark 1: The policy in (9) is flexible in the sense that not all block elements of $\boldsymbol{\Theta}$ need to be utilized. Note that significant reduction of complexity may be achieved by setting certain block elements of $\boldsymbol{\Theta}$ to zero or to some arbitrary matrix. In particular, one may opt to set all block elements of $\boldsymbol{\Theta}$ except $\theta_{1,t}, \theta_{2,t+1}, \ldots, \theta_{N-1,t+N-2}$ to zero, in which case, the total number of optimization variables is proportional to N instead of N^2 as in (9). This matter is further illustrated in Example 6 below, where we demonstrate that this blocking may employed to improve computational speed with minimal loss in performance with respect to the fully general policy (9). $\triangleleft$

In what follows, as a matter of notation, by $(\boldsymbol{\Theta}_t)_j$ we shall denote the j-th block-row of the matrix $\boldsymbol{\Theta}_t$ in (9), i.e., $(\boldsymbol{\Theta}_t)_j := [\,\theta_{j,t} \ \cdots \ \theta_{j,t+j-1} \ 0 \ \cdots \ 0\,]$, for $j = 0, \ldots, N-1$, with $(\boldsymbol{\Theta}_t)_0$ being the identically 0 (block) row. We require:

Assumption 2: The sequence $(w_t)_{t\in\mathbb{N}_0}$ of noise vectors is i.i.d with $\Sigma = \mathbb{E}\left[w_t w_t^\top\right]$. $\diamondsuit$

So far we have not stipulated any boundedness properties on the elements of the vector space $\mathcal{H}$. This means that the control policy elements may be unbounded maps. We stipulate the following structure on the control sets: For a given $p \in [1, \infty]$, the control input vector u_t

is bounded in p-norm at each instant of time t, i.e., for $p \in [1, \infty]$ let $U_{\max}^{(p)} > 0$ be given, with

$$u_t \in \bar{\mathbb{U}}_p := \left\{ \xi \in \mathbb{R}^m \,\middle|\, \|\xi\|_p \leqslant U_{\max}^{(p)} \right\} \quad \text{for } t \in \mathbb{N}_0, \quad \text{and}$$
$$\mathbb{U}_p := (\bar{\mathbb{U}}_p)^N. \tag{10}$$

One could easily include more general constraint sets $\mathbb{U}_p$, for instance, to capture bounds on the rate of change of controls. Our basic result is the next Theorem.

Theorem 3: Consider the system (1). *Suppose that Assumption 2 holds, and every component of the basis functions* $\mathfrak{e}^\nu$ *is bounded by* $\mathcal{E} > 0$ *in magnitude. Then for all times* $t = 0, N_c, 2N_c, \ldots$, *the problem* (6) *under the policy* (7) *and control sets* (10) *for* $p \in [1, \infty]$ *is convex with respect to the decision variables* $(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)$ *defined in* (9). *For* $p = 1, 2,$ *and* ∞ *it admits tractable, convex, and globally feasible versions with tighter domains of* $(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)$, *given by*

$$\begin{aligned}\min_{(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)} \ & \mathbf{tr}\Big(\boldsymbol{\Theta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\Theta}_t \Sigma_{\mathfrak{e}}\Big) + 2\mathbf{tr}\Big(\boldsymbol{\Theta}_t^\top B^\top Q D \Sigma_{\mathfrak{e}}'\Big) \\ & + \boldsymbol{\eta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\eta}_t \\ & + 2\Big(x_t^\top A^\top Q B \boldsymbol{\eta}_t + \boldsymbol{\eta}_t^\top B^\top Q D \mu_W + x_t^\top A^\top Q B \boldsymbol{\Theta}_t \mu_{\mathfrak{e}}\Big) \\ & + 2\boldsymbol{\eta}_t^\top \Big(R + B^\top Q B\Big)\boldsymbol{\Theta}_t \mu_{\mathfrak{e}} + c_t \end{aligned}$$

$$\text{sbj. to } \boldsymbol{\Theta}_t \text{ strictly lower block triangular as in (9),}$$

$$\begin{cases} p = 1: & \|\eta_{t+j}\|_1 + \mathcal{E} j \, \|(\boldsymbol{\Theta}_t)_j\|_1 \leqslant U_{\max}^{(1)} \\ & \text{for } j = 0, \ldots, N-1, \\ p = \infty: & |(\eta_{t+j})_i| + \mathcal{E}\, \|((\boldsymbol{\Theta}_t)_j)_i\|_1 \leqslant U_{\max}^{(\infty)} \\ & \text{for } i = 1, \ldots, m, \text{ and } j = 0, \ldots, N-1, \\ p = 2: & \|[\eta_{t+j} \ (\boldsymbol{\Theta}_t)_j]\|_2 \sqrt{1 + \mathcal{E} j} \leqslant U_{\max}^{(2)} \\ & \text{for } j = 0, \ldots, N-1 \end{cases} \tag{11}$$

where $\Sigma_{\mathfrak{e}} := \mathbb{E}\Big[\mathfrak{e}(W_t)\mathfrak{e}(W_t)^\top\Big]$, $\Sigma_{\mathfrak{e}}' := \mathbb{E}\Big[W_t \mathfrak{e}(W_t)^\top\Big]$, $\mu_W := \mathbb{E}[W_t]$, $\mu_{\mathfrak{e}} := \mathbb{E}[\mathfrak{e}(W_t)]$, *and* $c_t := x_t^\top A^\top Q A x_t + 2 x_t^\top A^\top Q D \mu_W + \mathbf{tr}\Big(D^\top Q D \Sigma_W\Big)$.

Proof of Theorem 3: It is clear that $X_t^\top Q X_t + U_t^\top R U_t$ is convex quadratic, and both X_t and U_t are affine functions of the design parameters $(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)$ for any realization of the noise W_t. It follows [31, Section 3.2] that V_t is convex in $(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)$. Moreover, the control constraint sets in (10) are convex in $(\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t)$. This settles the first claim. The objective function (2) is given by

$$\begin{aligned} & \mathbb{E}_{x_t}\Big[\Big(Ax_t + BU_t + DW_t\Big)^\top Q \Big(Ax_t + BU_t + DW_t\Big)\Big] \\ & \quad + \mathbb{E}_{x_t}\Big[U_t{}^\top R U_t\Big] \\ & = \mathbf{tr}\Big(\boldsymbol{\Theta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\Theta}_t \Sigma_{\mathfrak{e}}\Big) + 2\mathbf{tr}\Big(\boldsymbol{\Theta}_t^\top B^\top Q D \Sigma_{\mathfrak{e}}'\Big) \\ & \quad + \boldsymbol{\eta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\eta}_t \\ & \quad + 2\Big(x_t^\top A^\top Q B \boldsymbol{\eta}_t + \boldsymbol{\eta}_t^\top B^\top Q D \mu_W + x_t^\top A^\top Q B \boldsymbol{\Theta}_t \mu_{\mathfrak{e}}\Big) \\ & \quad + 2\boldsymbol{\eta}_t^\top \Big(R + B^\top Q B\Big)\boldsymbol{\Theta}_t \mu_{\mathfrak{e}} + c_t. \end{aligned}$$

Since the matrix $\Sigma_{\mathfrak{e}}$ is positive semidefinite, it can be expressed as a finite nonnegative linear combination of matrices of the type $\sigma\sigma^\top$, for vectors σ of appropriate dimension [32, Theorem 1.10]. Accordingly, if $\Sigma_{\mathfrak{e}} = \sum_{i=1}^k \sigma_i \sigma_i^\top$, k being the rank of $\Sigma_{\mathfrak{e}}$, then $\mathbf{tr}\Big(\boldsymbol{\Theta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\Theta}_t \Sigma_{\mathfrak{e}}\Big) = \sum_{i=1}^k \Big(\sigma_i^\top \boldsymbol{\Theta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\Theta}_t \sigma_i\Big)$. Defining $\widehat{\boldsymbol{\Theta}}_{t,i} := \boldsymbol{\Theta}_t \sigma_i$ and adjoining these equalities to the constraints of the optimization program (11), we arrive at the optimization program

$$\begin{aligned} \underset{(\boldsymbol{\Theta}_t, \widehat{\boldsymbol{\Theta}}_{t,1}, \ldots, \widehat{\boldsymbol{\Theta}}_{t,k})}{\text{minimize}} \ & \sum_{i=1}^k \widehat{\boldsymbol{\Theta}}_{t,i}^\top \Big(B^\top Q B + R\Big)\widehat{\boldsymbol{\Theta}}_{t,i} \\ & + 2\mathbf{tr}\Big(\boldsymbol{\Theta}_t^\top B^\top Q D \Sigma_{\mathfrak{e}}'\Big) + \boldsymbol{\eta}_t^\top \Big(B^\top Q B + R\Big)\boldsymbol{\eta}_t \\ & + 2\Big(x_t^\top A^\top Q B \eta + \boldsymbol{\eta}_t^\top B^\top Q D \mu_W \\ & + x_t^\top A^\top Q B \boldsymbol{\Theta}_t \mu_{\mathfrak{e}}\Big) \\ & + 2\boldsymbol{\eta}_t^\top \Big(R + B^\top Q B\Big)\boldsymbol{\Theta}_t \mu^{\mathfrak{e}} + c_t \\ \text{subject to} \ & \boldsymbol{\Theta}_t \text{ strictly lower block triangular as in (9),} \\ & \widehat{\boldsymbol{\Theta}}_{t,i} = \boldsymbol{\Theta}_t \sigma_i \text{ for } i = 1, \ldots, k.. \end{aligned} \tag{12}$$

We see immediately that (12) is a convex program in the parameters $\boldsymbol{\eta}_t, \Theta_t$ and $\widehat{\boldsymbol{\Theta}}_{t,i}$, and the cost is equivalent to the cost in (11).

It only remains to consider the last triplet of constraints in (11). First we consider the cases of $p = 1, \infty$. Using the notation above, we see that the constraints can be written as

$$\begin{cases} p = 1: & \|\eta_{t+j}\|_1 + j\mathcal{E}\, \|(\boldsymbol{\Theta}_t)_j\|_1 \leqslant U_{\max}^{(1)} \\ & \text{for } j = 0, \ldots, N-1, \\ p = \infty: & |(\eta_{t+j})_i| + \mathcal{E}\, \|((\boldsymbol{\Theta}_t)_j)_i\|_1 \leqslant U_{\max}^{(\infty)} \\ & \text{for } i = 1, \ldots, m, \text{ and } j = 0, \ldots, N-1 \end{cases} \tag{13}$$

where *for* $p = \infty$ *the new constraints are tight*[1]. It follows that the objective function in (12) is quadratic and the constraints in (12)–(13) are affine in the optimization parameters $\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t$, and $\widehat{\boldsymbol{\Theta}}_{t,i}$. As such, for $p = 1, \infty$ our problem is a *quadratic program* (QP).

For the case of $p = 2$, note that $\eta_{t+j} + (\boldsymbol{\Theta}_t)_j \mathfrak{e}(W_t) = [\eta_{t+j} \quad (\boldsymbol{\Theta}_t)_j] \begin{bmatrix} 1 \\ \mathfrak{e}(W_t) \end{bmatrix}$, and by definition of $\mathcal{E}$ it is clear that $\left\| [\eta_{t+j} \quad (\boldsymbol{\Theta}_t)_j] \begin{bmatrix} 1 \\ \mathfrak{e}(W_t) \end{bmatrix} \right\|_2 \leqslant \|[\eta_{t+j} \quad (\Theta_t)_j]\|_2 \sqrt{1 + \mathcal{E} j}$. This immediately translates to the third constraint in Problem 11, which makes the problem a *semidefinite program* (SDP). For $p = 1$ and $p = 2$ the constraints in (13) are more restrictive than the original ones in (12) due to application of the triangle inequality. ∎

The optimization problem (11) simplifies if we assume that $\mu_{\mathfrak{e}} = \mathbb{E}[\mathfrak{e}(W_t)] = 0$. Note that $\mathbb{E}[\mathfrak{e}(W_t)] = 0$ if and only if $\mathbb{E}\Big[\mathfrak{e}_i^\nu((w_{t+j})_i)\Big] = 0$ for all $\nu \in \mathcal{I}$, $j = 0, \ldots, N-1$, and $i = 1, \ldots, n$. At an intuitive level this translates to the condition that the functions $\mathfrak{e}_i^\nu \in \mathcal{H}$ should be "centered" with respect to the random variables $(w_{t+j})_i$. In particular, this simply means that for noise distributions that are symmetric about 0, the functions $\mathfrak{e}^\nu$ should be centered at 0 and be antisymmetric. The matrices $\Sigma_{\mathfrak{e}}$ and $\Sigma_{\mathfrak{e}}'$ and the vectors μ_W and $\mu_{\mathfrak{e}}$ in Theorem 3 are all constants independent of x_t, and can be computed off-line. As such, even if closed-form expressions for the entries of the matrices do not exist, they can be numerically computed to desired precision. The optimization problem (11) is a quadratic program [31, p. 152] for $p = 1, \infty$, and a semidefinite program [31, p. 169] for $p = 2$, in the optimization parameters $\Big\{\boldsymbol{\eta}_t, \boldsymbol{\Theta}_t, \Big\{\widehat{\boldsymbol{\Theta}}_{t,i}, i = 1, \ldots, k\Big\}\Big\}$, and can be easily coded in standard software packages such as cvx [33] or YALMIP [34]. Note that the optimization problem (11) is always feasible (simply set $\boldsymbol{\Theta}_t = 0$ and $\boldsymbol{\eta}_t = 0$ to see this). Finally, note that the last triplet of constraints in

Problem (11) for various values of p, is a result of robustly satisfying the constraints posed by the various control sets (10) for any realization of the noise W_t. In general, the total number of decision variables in the optimization program (11) is $mN\Big(1 + (1/2)n(N-1)|\mathcal{I}|\Big)$. This number can be substantially reduced, e.g., by choosing $\mathcal{H}$ to be 1-dimensional, or by fixing certain elements of the Fourier coefficient matrix $\mathbf{\Theta}_t$ to 0.

Mean-Square Boundedness of the States: In contrast to the deterministic setting, under unbounded noise (e.g., Gaussian,) it is not possible to ensure a robustly invariant bounded subset of $\mathbb{R}^n$ for strictly unstable systems. The only known case where bounded closed-loop variance of (1) can be ensured with bounded controls is when A is Lyapunov stable and one has sufficient control authority; see [20] for details. In the context of receding horizon control, it was demonstrated in [35] that by adjoining an appropriate constraint in the optimization program under the assumptions of stabilizability of the pair (A, B) with a reachability index $\kappa := \min\{k \mid \operatorname{rank} \mathfrak{R}_k(A,B) = n_o\}$, $\mathfrak{R}_k(A,M) := [A^{k-1}M \ \cdots \ AM \ M]$, that $C_4 := \sup_{t\in\mathbb{N}_0} \mathbb{E}[\|w_t\|^4] < \infty$, and, without loss of generality, that $\mathbb{E}[w_t] = 0$, it is indeed possible to ensure bounded variance in closed-loop. One key feature of the technique in [35] was that a control horizon of κ steps was required.

Computational Aspects: Our method generally requires the solution of an SDP whereas simpler certainty-equivalent (CE) MPC methods require solving relatively simpler types of convex optimization problems, QP or LP, depending on the norm used. In addition, our method generally includes more decision variables, $mN\Big(1 + (1/2)n(N-1)|\mathcal{I}|\Big)$ of them, as opposed to only Nm for CE methods. Even though numerical studies suggest that many of these additional decision variables can be set to a constant or even zero without a significant impact on performance, (see Example 6 below,) the computational burden with the proposed method is in general high. For the numerical results presented in Section IV, the mean computation times were 130% to 150% higher for the method proposed here over CE; the difference may be even larger if one leverages on numerically efficient methods for implementing standard CE-MPC schemes that have been developed over the years. On the other hand, our simulation results also indicate that in some cases substantial benefits may be obtained in terms of the achieved cost (more than 100% in Example 6) for this additional computational effort. Moreover, Example 6 also indicates that because of the recourse nature of the policies used, the optimization problem does not need to be solved at every time step to achieve this improvement. The choice of which method to implement therefore depends on the problem at hand (importance of achieved cost, available computational resources, etc).

IV. Numerical Examples

In this section we present numerical examples to illustrate our results: a comparison of the performance of our policy (9) against that of the standard finite horizon LQ controller under unconstrained controls (Example 4), performance of our policy (9) against a saturated/clipped LQ controller (Example 5), and the effectiveness of our policy (9) compared against certainty-equivalent receding horizon control (Example 6).

Example 4 (Unconstrained Controls): We compared in simulation our policy against that of the LQ problem for the following two controllable 3-D linear systems:

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.4 & 0.5 & -0.25 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k + w_k, \tag{14}$$

$$x_{k+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k + w_k. \tag{15}$$

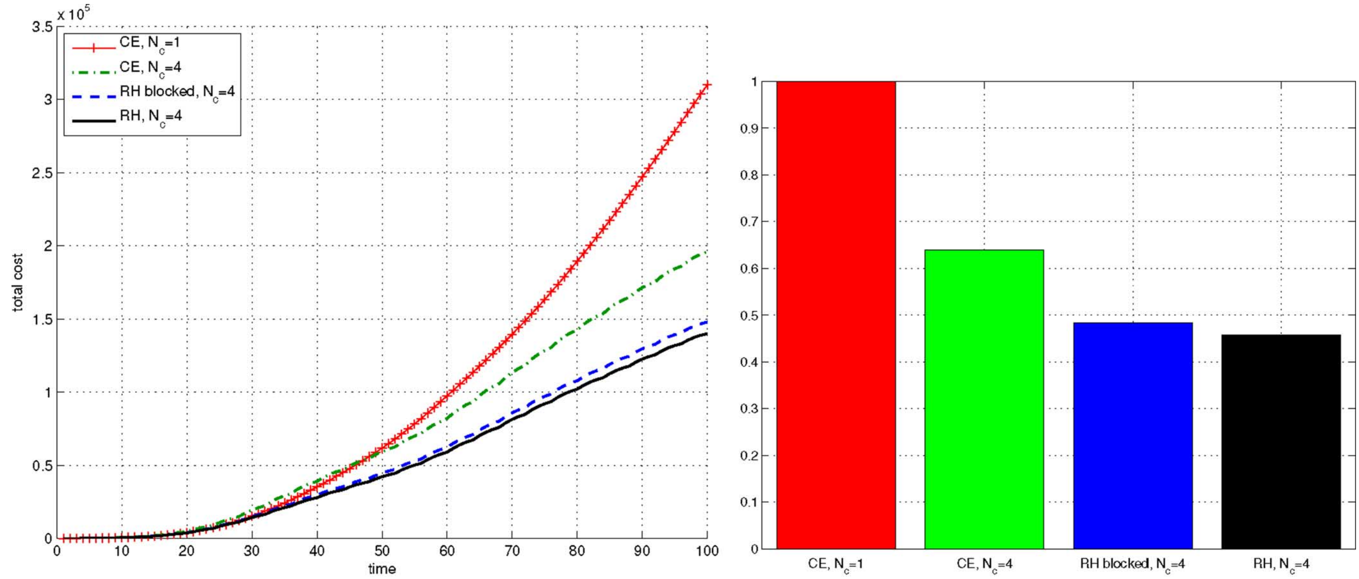
In each case we solved an unconstrained finite-horizon LQ optimal control problem corresponding to states and control weights $Q_t = 3\,\mathbf{I}_{3\times3}$ and $R_t = 1$ for every t. We selected an optimization horizon $N = 50$, and simulated the system responses starting from 10^3 different initial conditions x_0 selected at random uniformly from the cube $[-100, 100]^3$, and noise sequences w_t corresponding to i.i.d Gaussian noise of mean 0 and (randomly chosen) variance $\Sigma_w = \begin{bmatrix} 2.8304 & 5.4915 & 3.6123 \\ 5.4915 & 11.5549 & 6.8967 \\ 3.6123 & 6.8967 & 4.6260 \end{bmatrix}$. We selected the nonlinear bounded term $\mathfrak{e}(W_t)$ in our policy to be a vector of scalar sigmoidal functions $\varphi(\xi) := 0.2\xi/\sqrt{1+0.04\xi^2}$ applied to each coordinate of the noise vector. The covariance matrices $\Sigma_{\mathfrak{e}}$ and $\Sigma_{\mathfrak{e}'}$ that are required to solve the optimization problem (9) were computed empirically via classical Monte Carlo methods [36, Section 3.2] using 10^6 i.i.d samples. The test results for (14) showed that the mean of the ratio of the cost of LQ to the cost corresponding of our policy is 0.99916, and the standard deviation of this ratio is 0.003619. The test results for (15) showed that the mean of the ratio of the cost of LQ against the cost of our policy is 0.99673 and the corresponding standard deviation is 0.008045. Computations for determining our policy in the above two cases were carried out in the MATLAB-based software package cvx. In the case of the system (15) the solver utilized by cvx reported numerical problems in five different runs, for which it gave values of the aforementioned ratio below 0.96. Note that we have not discarded these five cases from the mean and variance figures reported above. The close-to-optimal performance of our policy is surprising in view of the fact that the vector-space $\mathcal{H}$ is the linear span of one bounded function, and does not contain the theoretically optimal linear (in the current state) controller. △

Example 5 (Saturated LQ and Receding Horizon): We compare the performance of saturated LQ against our policy (9) for the system (15) in this example. We fixed the optimization horizon $N = 2$, the control horizon $N_c = 1$, and the weight matrices for the states and the control to be $Q_t = \mathbf{I}_{3\times3}$ and $R_t = 0.01$ for all t, respectively. The control bounds in both cases was $[-2, 2]$, the nonlinear bounded term $\mathfrak{e}(W_t)$ in our policy (9) was a vector of scalar standard saturation functions applied to each coordinate of the nosie vector, and the LQ control input was saturated at ±2. The covariance matrices $\Sigma_{\mathfrak{e}}$ and $\Sigma_{\mathfrak{e}'}$ required to solve the optimization problem (11) were computed empirically via standard Monte Carlo integration methods using 10^6 i.i.d samples. We simulated the system starting from the same initial condition $x_0 = [0\ 0\ 0]^{\mathsf{T}}$ for 100 different independent realizations of the noise sequence w_t over a horizon of 200. The simulations were coded in MATLAB and the optimization programs were coded in the software package cvx. The average total cost incurred at the end of the simulation horizon when using the saturated LQ scheme above was 1.790×10^{12} units, whereas the average total cost incurred at the end of the simulation horizon ($t = 200$) using our policy (9) in a receding horizon fashion was 4.486×10^8 units. △

Example 6 (Constrained Controls): Consider the 2-dimensional linear stochastic system

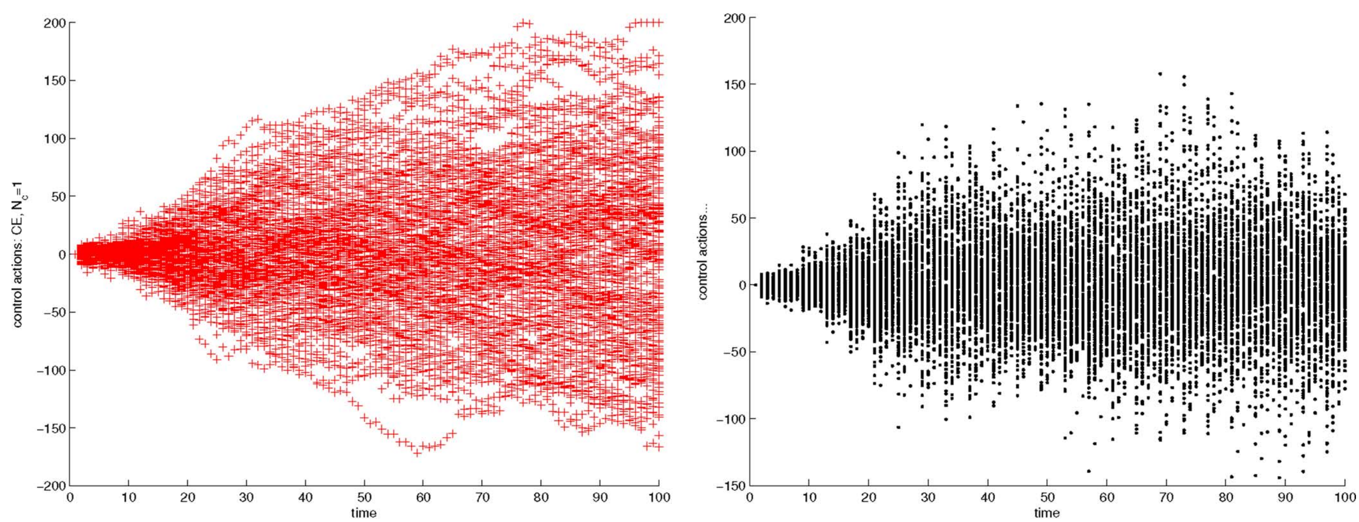
$$x_{t+1} = \begin{bmatrix} 1.23 & -0.15 \\ 0.25 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.14 \\ 0.12 \end{bmatrix} u_t + w_t \tag{16}$$

where $(w_t)_{t\in\mathbb{N}_0}$ is a sequence of i.i.d. Gaussian noise with zero mean and (randomly generated) variance $\begin{bmatrix} 2.7220 & 4.9760 \\ 4.9760 & 9.1026 \end{bmatrix}$. Let the weight matrices corresponding to the states and control be $Q_t = \mathbf{I}_{2\times2}$ and $R_t = 0.8$ for each t. The covariance matrices $\Sigma_{\mathfrak{e}}$ and $\Sigma_{\mathfrak{e}'}$ that



(a) (b)

Fig. 1. Performance comparison corresponding to Example 6 : Certainty-Equivalent MPC with $N_c = 1$, $\boldsymbol{\Theta}_t \equiv 0$, $W_t \equiv 0$, (denoted by "CE, $N_c = 1$"), Certainty-Equivalent controller with $N_c = 4$, $\boldsymbol{\Theta}_{N_c t} \equiv 0$, $W_t \equiv 0$, (denoted by "CE, $N_c = 4$"), Receding Horizon controller with $N_c = 4$, all elements of $\boldsymbol{\Theta}_{N_c t}$ except $\theta_{1,N_c t}, \theta_{2,N_c t+1}, \ldots, \theta_{N-1,N_c t+N-2}$ in (9) set to zero, (denoted by "RH blocked, $N_c = 4$), and fully general Receding Horizon controller with $N_c = 4$ (denoted by "RH, $N_c = 4$"). (a) Plot of total costs for simulation horizon 100. (b) Ratio of total costs at time 100 with the cost for CE, $N_c = 1$, normalized to 1.



(a) (b)

Fig. 2. Control actions corresponding to Example 6. (a) Certainty-Equivalent MPC with $N_c = 1$. (b) Receding Horizon controller with $N_c = 4$.

are required to solve the optimization problem (9) were computed empirically via classical Monte Carlo integration methods [36, Section 3.2] using 10^6 samples. The following simulations were coded in YALMIP and were solved using SDPT-3. We fixed the optimization horizon $N = 7$, the nonlinear saturation $\mathfrak{e}(W_t)$ to be a vector of scalar sigmoidal functions $\varphi(\xi) := 0.2\xi/\sqrt{1 + 0.04\xi^2}$ applied to each coordinate of the noise vector. A comparison of the performances of four representative strategies is given in Fig. 1 (see the caption for a full description). The plots reported here correspond to 150 sample paths. The control constraints in all cases were $u_t \in [-200, 200]$. The four strategies are based on variations of our policy (9) versus traditional certainty-equivalent MPC. The plots clearly illustrate the advantages of utilizing the full capabilities of our policy (9) over traditional stochastic MPC methods. In particular, Fig. 1(b) demonstrates that the cost incurred by certainty-equivalent MPC is more than twice that by the receding horizon controller in (9). Moreover, it is clear from Fig. 1(a) that the certainty-equivalent controller leads to a quadratic growth in the total cost, whereas the corresponding growth for the others is linear. Fig. 2 shows the control actions corresponding to the certainty-equivalent controller with $N_c = 1$ and the receding horizon controller with $N_c = 4$. $\triangle$

Example 6 illustrates that there exist seemingly innocuous problems for which the intuitive application of CE-MPC scheme leads to inferior performance compared to the performance of the proposed policies. Because of the use of feedback policies, the method proposed in this article allows us to deal effectively with such systems by extending the control horizon in the receding horizon implementation. Note that even though a longer control horizon can also be used in the standard CE problem, the results are inferior [Fig. 1(b)] since the resulting open-loop sequence does not take into account the measured states in subsequent steps.

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