AGH University of Science and Technology
Faculty of Applied Mathematics

Symmetries, exact solutions and nonlocal conservation laws of nonlinear partial differential equations

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A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics

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Declaration of the author of this dissertation:

Aware of legal responsibility for making untrue statements I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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List of Abbreviations

dKP Eq.	dispersionless Kadomtsev-Petviashvili equation
FHZ system	Ferapontov-Huard-Zhang system, see system (2.12)
GT Eq.	Gibbons-Tsarev equation, see equation (2.3)
ISTM	inverse scattering transform method
KdV Eq.	Korteweg-deVries equation, see equation (1.13)
KhZ Eq.	Khokhlov-Zabolotskaya equation, see equation (2.32)
KP Eq.	Kadomtsev-Petviashvili equation
ODE	ordinary differential equation
mVw Eq.	modified Veronese web equation, see equation (4.10)
PDE	partial differential equation
rdDym Eq.	r-th dispersionless Dym equation, see equation (4.6)
UH Eq.	universal hierarchy equation, see equation (4.8)
Vw Eq.	Veronese web equation, see equation (4.12)

Introduction

Results of this thesis feature group-invariant solutions, nonlocal symmetries and nonlocal conservation laws for Lax–integrable PDEs – objects studied within the geometrical approach to differential equations. The geometric methods in differential equations belong to one of the several fields of mathematics which emerged from work of Sophus Lie, a Norwegian mathematician living in the end of the 19th century. The essential idea in his mathematical research is that of a continuous group of transformations. At the time a *group* was a brand new term associated with discrete sets and Galois theory, a fact that in some part determined the direction of Lie's scientific activity. His original motivation behind studying continuous symmetry groups in the context of differential equations was to integrate the latter and thus to make the symmetries play in the field of differential equations the role Galois theory plays in algebraic equations. He observed that seemingly unrelated methods of integrating various types of ODEs all appeared to be particular cases of a general method based on the existence of symmetry transformations for a given ODE. Recall that permutations belonging to a Galois group of a given polynomial equation permute its roots, therefore transforming solutions of the equation into solutions just as Lie's symmetry groups do with solutions of differential equations. The general problem dominating a great part of Lie's research can be posed as follows: given a differential equation, ordinary or partial, determine what information about its integration can be deduced from its symmetry group. His ideas and results founded a cornerstone of what is nowadays called geometric methods in differential equations or „Application of Lie Groups to Differential Equations“, as a classical monograph in the subject by Olver (2000), extensively cited in this thesis, is entitled.

The part of mathematical community, whose research was related to Lie's work, concentrated at first more on the abstract theory of Lie groups and algebras. Renewed interest in applications to differential equations came with the book by Birkhoff (1950), which contains symmetry invariant solutions to PDEs arising in fluid mechanics. Another important impulse came from Ovsyannikov (1958). Both authors examined a possibility of finding solutions to differential equations using groups of symmetry transformations. Let us examine in more detail what a symmetry group is capable of in this matter.

First, consider an ODE of n-th order and suppose that it admits an r-parameter group of transformations. Given one exact solution to the ODE it is possible to transform it (under the action of the group) to a more general one, which depends on at most r additional parameters (the same holds for PDEs). Moreover, if $r = 1$, the order of a given ODE can be reduced by one and solutions of the reduced (and hopefully simpler) equation can be then transformed to the solutions of the original ODE. In particular, this procedure enables integration of first order ODEs, which admit one-parameter symmetry group. Analogous result holds for multi-parameter groups if they are solvable.

In the case of n-th order ODEs, providing a solution depending on n constants means finding a general solution. The consequences of finding a symmetry group of a PDE are naturally more modest, but still very useful. For instance, the symmetry group can indicate whether a PDE is transformable to a linear one. The most popular application of a symmetry group of a PDE is to find exact solutions of the PDE. A symmetry group allows to reduce the equation to an equation depending on fewer independent variables (possibly to an ODE). Solutions of the reduced equation lead to solutions of the original PDE, which form a class of exact solutions called symmetry- or group-invariant solutions. The symmetry-invariant solutions include, for example, similarity solutions (invariant with respect to the scaling group) and travelling wave solutions (invariant with respect to translations).

As mentioned, the theory of continuous groups gave rise to several branches of mathematics not directly related to differential equations. An important example is the theory of abstract Lie groups (and Lie algebras), without reference to transformations, which in the first decades of the 20th century dominated in the articles of Lie's students and collaborators. The correspondence between finite Lie groups and finite Lie algebras, which is also due to Lie, is of fundamental importance in this theory. Eventually we will use another framework for geometric methods, and for this reason the basic concepts of the Lie group of transformations and Lie algebra in the vein of the original works by Lie are only briefly sketched in Section 1.1, according to the modern polished formulation as in (Olver, 2000).

A modern geometrical approach to nonlinear differential equations in a predominant part owes its development to a pivotal paper by Gardner, Green, Kruskal and Miura, (Gardner et al., 1967) (which should be considered together with a few other continuation papers which appeared afterwards, see (Miura, 1976) for a review). The paper contained a procedure for solving a Cauchy problem for the Korteweg–deVries (KdV) equation, called the inverse scattering transform method (ISTM), and gave rise to what is now the domain of integrability of PDEs. We discuss briefly the concept of integrability of nonlinear PDEs in Section 1.7. Among other things, examination of the KdV equation revealed that it admits infinite hierarchy of higher order symmetries and conservation laws, a feature that soon appeared to be quite common for integrable equations. In fact, even though it is hard to provide a universal definition of integrability for an arbitrary PDE, the one based on the existence of infinite hierarchies of higher order symmetries or conservation laws is fairly broadly applied, especially when enhanced to include nonlocal symmetries or conservation laws[1].

In the course of the development of the theory of integrable equations it became clear that it is indispensable to study differential equations together with various nonlocal structures related to them, such as Bäcklund, Miura and Cole-Hopf transformations, Lax pairs, zero-curvature representations and Wahlquist-Estabrook prolongation structures. The need for nonlocal theory of nonlinear PDEs appeared also in the context of recursion operators

[1]In general the hierarchies are nonlocal for multi-dimensional equations.

which often happen to be of an integro-differential type (as is the case of Lenard's operator for KdV equation if we stick to the foundational equation in the field). A unifying background for all these objects is provided by the theory of differential coverings developed by Vinogradov and Krasil′shchik (1984; 1989), and in a monograph (Krasil′shchik et al., 1986). Their approach allows not only for rigorous, but also natural definitions of nonlocal objects appearing in the study of PDEs. For instance, nonlocal symmetries and conservation laws for a given equation (or system) in fact can be seen as local symmetries and conservation laws for a certain related system. We apply the framework of differential coverings in this thesis.

The thesis is based on four articles written together with Dr hab. Oleg I. Morozov, who posed all the problems solved in the papers. The first chapter, Preliminaries, introduces basic definitions and concepts used in the thesis. It is based on the monograph (Krasil′shchik and Vinogradov, 1999) and the article (Krasil′shchik and Verbovetsky, 2011). In order to find symmetry algebras and nonlocal conservation laws, which contributes to the original results of the thesis, we used *Jets* package for *Maple*, (Baran and Marvan, 2012). The last section of the Preliminaries, Section 1.10, presents a simple example of calculation of a symmetry group, to give an idea of more technical aspects of computations.

Chapter 2 tackles group-invariant solutions. In Section 2.1 we present the algorithm for finding group-invariant solutions. In Section 2.2 we explain the need for classification of the symmetry algebra into optimal system of subalgebras and introduce the concept of the adjoint representation. The chapter is based on the articles (Lelito and Morozov, 2016) and (Lelito and Morozov, 2018a), which we describe below.

(1) (Lelito and Morozov, 2016).
The Gibbons—Tsarev equation: symmetries, invariant solutions, and applications,
Journal of Nonlinear Mathematical Physics, 23:2 (2016), 243-255.

Results of the paper are presented in Section 2.3. The article provides a classical application of Lie groups to PDEs which is to find exact solutions of the equation in question. The scheme is to find local symmetries, classify them via the adjoint representation of the symmetry group, and then perform the reductions to ODEs which may or may not be integrable by quadratures. In the case of the Gibbons–Tsarev (GT) Eq. (2.3) it was possible to solve all the ODEs resulting from the symmetry-reduction procedure. It was straightforward to apply the obtained symmetry-invariant solutions to Pavlov's Eq. (2.23), two-component reductions of the Benney moments chain (2.4), and to perform the reductions of the Ferapontov–Huard–Zhang (FHZ) system (2.12).

(2) (Lelito and Morozov, 2018a).
Invariant solutions to the Khokhlov–Zabolotskaya singular manifold equation and their application,
Reports on Mathematical Physics, 81:1 (2018), 65-79.

Results of this paper are presented in Section 2.4. While in a great part the computations from the article fall under the same scheme and exploit the same machinery as in

(Lelito and Morozov, 2016), a nonlocal link between two PDEs is essential to find new results. More specifically, first we found symmetry group of the KhZ singular manifold Eq. (2.36), classified it into one-dimensional subalgebras, performed the reductions and studied symmetry-invariant solutions of the KhZ singular manifold Eq. (2.36). Then, using a Miura-type transformation (2.42), which relates solutions of the KhZ singular manifold (2.36) to solutions of the KhZ Eq. (2.32), we obtained new exact solutions of the latter equation.

Chapter 3 is based on the article (Lelito and Morozov, 2018b). In the chapter we study nonlocal symmetries of Plebański's second heavenly equation and their Lie algebra structure. We give more details below.

(3) (Lelito and Morozov, 2018b)
Nonlocal symmetries of Plebański's second heavenly equation,
Journal of Nonlinear Mathematical Physics, 25:2 (2018), 188-197.

Results of the paper are presented in Section 3.2. We studied nonlocal symmetries of Plebański's second heavenly equation (3.1) in an infinite-dimensional covering (3.9) associated to a Lax pair (3.2) with a non-removable spectral parameter. We showed that all local symmetries of the equation admit lifts to full-fledged nonlocal symmetries in the infinite-dimensional covering. We found two new infinite hierarchies of commuting nonlocal symmetries in the covering (3.9) and described the structure of the Lie algebra of the obtained nonlocal symmetries.

Chapter 4 is based on the article (Lelito and Morozov, 2018c), which examines nonlocal conservation laws for several multi–dimensional PDEs.

(4) (Lelito and Morozov, 2018c)
Three-Component Nonlocal Conservation Laws For Lax-Integrable 3D Partial Differential Equations,
Journal of Geometry and Physics, 131 (2018), 89–100.

In the paper we studied three-component nonlocal conservation laws for five three-dimensional Lax-integrable equations: Pavlov's (4.2), the r-th dispersionless Dym (rd-Dym) (4.6), the modified Veronese web (mVw) (4.10), the universal hierarchy (UH) (4.8), and the Veronese web (Vw) (4.12) equation. The conservation laws for Pavlov's Eq. (4.2) and the potential KhZ Eq. (4.15) were found in (Makridin and Pavlov, 2017). We added a proof of their nontriviality. The five equations are related via Bäcklund transformations and we examined the resulting correspondences between the nonlocal conservation laws. In particular, we proved that the nonlocal conservation laws that depend on one pseudopotential are generated from a local conservation law of the Vw equation via appropriate superpositions of the Bäcklund transformations. Also, we proved nontriviality of the conservation laws found in the paper.

In Conclusions we recapitulate the main results of the thesis and discuss them in more detail. The thesis have three appendices. Appendix A contains scripts of some (the shortest) of the computations performed in *Maple* with the use of *Jets* package. In Appendix B we give a full proof of Theorem 3.5.1 regarding structure of the Lie algebra of nonlocal symmetries of Plebański's second heavenly equation. In Appendix C we produce additional two examples of infinite hierarchies of nonlocal conservation laws obtained from the nonlocal conservation laws found in Chapter 4.

Chapter 1

Preliminaries

1.1 Basic notions

The framework for nonlinear differential equations that we employ relies to a great extent on basic notions from differential geometry. However, we do not need this notions in their most general forms and so the definitions will be simplified. The definitions in this section are formulated with a prevalent use of (Olver, 2000) and (Lee, 2009). We begin with a definition of a smooth manifold.

Definition 1.1.1 (Manifold)**.** An *m-dimensional (smooth) manifold* M is a set M, together with a countable collection of subsets $U_\alpha \subset M$ called coordinate charts, and one-to-one functions $\chi_\alpha \colon U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$ called local coordinate maps, which satisfy the following properties:

(a) The coordinate charts cover M:

$$\cup_\alpha U_\alpha = M.$$

(b) On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map

$$\chi_\beta \circ \chi_\alpha^{-1} \colon \chi_\alpha \left(U_\alpha \cap U_\beta\right) \to \chi_\beta \left(U_\alpha \cap U_\beta\right)$$

is a smooth function. ⋄

Definition 1.1.2 (Diffeomorphism)**.** A smooth map between manifolds, which is one-to-one and has a smooth inverse is called a *diffeomorphism*. ⋄

Definition 1.1.3 (Vector bundle)**.** Let M and E be smooth manifolds and V a vector space over a field $\mathbb{F}$. Furthermore, let pr_1 be a projection onto the first factor, and $\pi \colon E \to M$ a smooth surjective map. The quadruple (E, π, M, V) is a smooth (locally trivial) $\mathbb{F}$*–vector bundle* if

(a) for every $p \in M$ there exists an open neighbourhood $U_p \subset M$ and a diffeomorphism $\phi \colon \pi^{-1}(U_p) \to U_p \times V$, called local trivialization of E over U_p, such that the diagram below commutes.

It can be shown that ϕ must be of the form $\phi = (\pi|_{\pi^{-1}(U)}, \Phi)$, where $\Phi \colon \pi^{-1}(U) \to V$ is a smooth map.

$$\begin{tikzcd} \pi^{-1}(U_p) \arrow[r, "\phi"] \arrow[dr, "\pi"'] & U_p \times V \arrow[d, "\mathrm{pr}_1"] \\ & U_p \end{tikzcd}$$

(b) for every $p \in M$ the fiber $E_p := \pi^{-1}(p)$ has the structure of a vector space over the field $\mathbb{F}$, isomorphic to the vector space V,

(c) for every $q \in M$ in the domain U_p of some diffeomorphism $\phi = (\pi, \Phi)$, the map $\Phi|_{E_q} : E_q \to V$ is a vector space isomorphism.

A vector bundle of the form $(M \times V, \mathrm{pr}_1, M, V)$ is called a *trivial bundle*. By abuse of notation we will refer to a vector bundle (E, π, M, V) by π. The exception is the tangent bundle $(TM, \mathrm{pr}_1, M, \mathbb{R}^m)$ (where m is the dimension of M) to which we will refer by TM. The manifold TM is defined as $TM = \cup_p T_pM$, where T_pM is a vector space consisting of vectors tangent to M at a point p. ⋄

Definition 1.1.4 (Section)**.** A (global) smooth section of a vector bundle (E, π, M, V) is a smooth map $\sigma : M \to E$, such that $\pi \circ \sigma = id_M$, (in other words $\sigma(p) \in E_p$). A local smooth section over an open set $U \subset M$ is a smooth map $\sigma : U \to E$, such that $\pi \circ \sigma = id_U$. The set of smooth sections of the bundle π is denoted by $\Gamma(\pi)$. Vector fields are sections of the tangent bundle. ⋄

Definition 1.1.5 (Distribution)**.** A rank k *distribution* on a manifold M is a map that to each point $p \in M$ assigns a k-dimensional vector space $E_p \subset T_pM$. Moreover, we assume that for every $p \in M$ there exists a neighbourhood U_p and a set of vector fields $\{X_1, ..., X_k\}$ defined on U_p, such that for all $q \in U_p$ the vectors $\{X_1(q), ..., X_k(q)\}$ are linearly independent and span E_q. ⋄

Definition 1.1.6 (Integral manifold)**.** A submanifold $N \subset M$ is an integral manifold of the distribution $M \ni p \mapsto E_p \subset T_pM$, if $T_pN = E_p$ for every $p \in N$.

Since all our considerations are local in the sense that we are ready to shrink the domains if necessary, we simplify the exposition of the theory. Instead of functions defined on some m-dimensional manifold M we consider functions defined on open subsets of $\mathbb{R}^m$, and instead of a locally trivial bundle we will consider simply a trivial bundle. Moreover, our focus is on definitions in local coordinates.

Definition 1.1.7 (Differential)**.** Let f be a smooth map $f : M \to N$ between manifolds, with local coordinates on M and N given by $x = (x^1, \dots, x^m)$ and $\mathbf{y} = (y^1, \dots, y^r)$, respectively. Its *differential* $df : TM_x \to TN_{f(x)}$ is a linear map between their tangent spaces. A vector

$X(x) = \sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i}$, where ξ^i are smooth functions, tangent to M at the point x is mapped to a vector tangent to N at a point $\mathbf{y} = f(x)$ according to the formula

$$df\,(X(x)) = \sum_{j=1}^{r}\left(\sum_{i=1}^{m}\xi^i(x)\frac{\partial f^j}{\partial x^i}\right)\frac{\partial}{\partial y^j}.$$

The matrix representation of the differential df is the Jacobian matrix of f. ◇

Definition 1.1.8 (Lie group)**.** An r-parameter *Lie group* is a group $(G, \circ)$ such that G is an r-dimensional manifold and both the group multiplication:

$$m\colon\ G \times G \to G, \qquad m(g,h) := g \circ h, \qquad g,\, h\,\in\, G$$

and the inversion

$$i\colon\ G \to G, \qquad i(g) = g^{-1}, \qquad g \in G$$

are smooth maps between manifolds. ◇

Local Lie group is a subgroup of a Lie group, which consists of elements close to the identity element e of the group. Considering a local Lie group G allows to use the infinitesimal techniques (we give more details after the Definition 1.1.10). From now on, when we say Lie group we mean a *local* Lie group. Furthermore, since in the field of differential equations Lie groups arise as groups of transformations, we specify what is an r-parameter (local) Lie group of transformations.

Definition 1.1.9 (Lie group of transformations)**.** Let M be a smooth manifold. A local r-parameter *Lie group of transformations* acting on M is given by a local r-parameter Lie group G, an open set $U \subset G \times M$ (the domain of the definition of a group action), such that $\{e\} \times M \subset U$, and a map $\Psi\colon\, U \ni (h,x) \mapsto \Psi(h,x) =: h \cdot x \in M$ with the following properties.

a) If $(h,x) \in U$, $(g, h\cdot x) \in U$ and $(g \circ h, x) \in U$ then $g \cdot (h \cdot x) = (g \circ h) \cdot x$.

b) For every $x \in M$, the equality $e \cdot x = x$ holds.

c) If $(g,x) \in U$, then $(g^{-1}, g \cdot x) \in U$ and $g^{-1} \cdot (g \cdot x) = x$. ◇

Another term for the above defined object is *group of symmetry transformations*. In the field of differential equations the manifold M is the set of solutions to a given system of equations (a precise definition will be given later), hence if the equation involves m functions u^j in n independent variables x^i, then $M \subset \mathbb{R}^n \times \mathbb{R}^m$. Let us introduce the notation $u = (u^1,\, \dots,\, u^m)$ and $x = (x^1,\, \dots,\, x^n)$. The map Ψ defines an action of the group G on the solution manifold. In particular, if $u = f(x)$ is a solution to the equation and $\Gamma_f := \{(x,u)\colon\ u = f(x),\, x \in \mathrm{dom}(f)\}$ is its graph, then a graph $\Gamma_{\bar{f}} := \{g \cdot (x,u)\colon\ u = f(x),\, x \in \mathrm{dom}(f)\}$ is a graph of a solution as well.

Let us fix some point $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m = M$ and consider a one–parameter Lie group G of point transformations, the parameter being $\epsilon \in \mathbb{R}$. The orbit of this point is a smooth curve given by a map $\Psi : U \to M$, $\Psi(\epsilon, x, u) = (\bar{x}, \bar{u})$. In the neighbourhood of $\epsilon = 0$, and hence in the neighbourhood of (x, u), this orbit is the integral curve of the following vector field:

$$V(x,u) = \dot{\Psi}^1(0,x,u)\frac{\partial}{\partial x^1} + \ldots + \dot{\Psi}^n(0,x,u)\frac{\partial}{\partial x^n} + \dot{\Psi}^{n+1}(0,x,u)\frac{\partial}{\partial u^1} + \ldots + \dot{\Psi}^{n+m}(0,x,u)\frac{\partial}{\partial u^m},$$

where $\dot{\Psi}^j$ stands for $\frac{\partial}{\partial \epsilon}\Psi^j$. On the other hand, given a smooth vector field

$$V(x,u) = \xi^1(x,u)\frac{\partial}{\partial x^1} + \ldots + \xi^n(x,u)\frac{\partial}{\partial x^n} + \eta^1(x,u)\frac{\partial}{\partial u^1} + \ldots + \eta^m(x,u)\frac{\partial}{\partial u^m},$$

we can look for its integral curve $(\bar{x}, \bar{u}) = \Psi(\epsilon, x, u)$ (flow generated by this vector field). This is done by solving the Cauchy problem for first order system of ordinary differential equations:

$$\begin{cases} \frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}, \bar{u}), \\ \frac{d\bar{u}^j}{d\epsilon} = \eta^j(\bar{x}, \bar{u}), \\ (\bar{x}(0), \bar{u}(0)) = (x, u), \end{cases} \tag{1.1}$$

which by the virtue of the Picard theorem (and smoothness of ξ^i, η^j) has a unique (local) solution. The vector field V is called the *infinitesimal generator* of the action of a Lie group. The possibility of working with infinitesimal generators of symmetry transformations instead of transformations themselves is essential in applications to differential equations. One reason is that equations determining infinitesimal generators are linear (see Section 1.5) and so in practice, given a differential equation, its infinitesimal generators of symmetries are found first and then a corresponding Lie group of transformations is reconstructed from it (if at all). These considerations lead to a concept of a Lie algebra.

Definition 1.1.10 (Lie algebra)**.** A *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinear operation $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, which is skew–symmetric, i.e. $[V, W] = -[W, V]$, and satisfies the *Jacobi identity*

$$[U, [V, W]] + [W, [U, V]] + [V, [W, U]] = 0.$$

$\diamond$

The set of infinitesimal generators of a group G is closed with respect to the operation of taking the commutator of vector fields, defined as

$$[V, W] := V \circ W - W \circ V.$$

Commutator satisfies conditions of the Lie bracket and hence introduces the structure of a Lie algebra into the set of infinitesimal generators of a group G. In the case of finite-dimensional Lie algebras one can always reconstruct from them a corresponding local group of transformations. The so-called first and second Lie's fundamental theorems establish the one-to-one correspondence between (local) Lie groups and Lie algebras, see e.g. (Stormark, 2000).

1.2 Manifold of infinite jets

We treat a differential equation as a submanifold in a jet space - an indispensable concept in geometrical approach to differential equations, whose precise definition we owe to Ehresmann (1951).

We think of a smooth, m-component vector function $u = f(x)$ defined on an open subset M of $\mathbb{R}^n$ as of a section of a trivial bundle $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The coordinates are $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^m)$, and we write $\pi(x, u) = x$. We say that two sections f_1, f_2 are *tangent with order* k over some point x if partial derivatives of functions $u = f_1(x)$, $u = f_2(x)$ coincide up to order k at point x. To identify sections tangent with each other with order k at point x is to introduce an equivalence relation in the set $\Gamma(\pi)$ of sections of the bundle π. We denote a corresponding equivalence class by $[f]^k_x$. The *space of k-jets* of the bundle π is denoted by $J^k(\pi)$ and it is a union over points x of sets of equivalence classes with respect to this tangency relation, that is $J^k(\pi) = \{\, [f]^k_x : x \in M,\ f \in \Gamma(\pi)\}$. The k-th jet $j_k(f)$ is a section of $J^k(\pi)$ such that $j_k(f)(x) = [f]^k_x$. The local coordinates on $J^k(\pi)$ are $(x^i, u^\alpha, u^\alpha_I)$, where $I = (i_1, ..., i_n)$ is a multi-index such that $0 < |I| \leq k$[1]. The coordinates u^α_I correspond to partial derivatives of $|I|$-th order. More precisely, for every local section f we have

$$u^\alpha_I(j_k(f)) = \frac{\partial^{|I|} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\ldots+i_n} f^\alpha}{(\partial x^1)^{i_1} \ldots (\partial x^n)^{i_n}}.$$

Here, we will work with the manifold of infinite jets $J^\infty(\pi)$, which has coordinates $(x^i, u^\alpha, u^\alpha_I)$, where $|I| \in \mathbb{N}$. The manifold $J^\infty(\pi)$ is the inverse limit of the chain of projections

$$\cdots \to J^{k+1}(\pi) \xrightarrow{\pi_{k+1,\,k}} J^k(\pi) \to \cdots \to J^1(\pi) \xrightarrow{\pi_{1,\,0}} J^0(\pi) = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,$$

where $\pi_{k+1,\,k}\colon J^{k+1}(\pi) \to J^k(\pi)$. We denote by π_∞ a vector bundle $\pi_\infty\colon J^\infty(\pi) \to \mathbb{R}^n$. For every local section $f\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f)\colon \mathbb{R}^n \to J^\infty(\pi)$.

1.3 Cartan distribution

With every point $\theta \in J^\infty(\pi)$ we may associate an n-dimensional plane $\mathcal{C}_\theta$, tangent to graphs of all sections passing through this point. Such a plane is called *Cartan plane* and the correspondence $\mathcal{C}\colon \theta \mapsto \mathcal{C}_\theta$ is called an n-dimensional *Cartan distribution*. Let us consider the map:

$$\frac{\partial}{\partial x^k} \mapsto D_k := \frac{\partial}{\partial x^k} + \sum_{I,\alpha} u^\alpha_{I+1_k} \frac{\partial}{\partial u^\alpha_I},$$

with $(i_1, \ldots, i_k, \ldots, i_n) + 1_k = (i_1, \ldots, i_k + 1, \ldots, i_n)$. The vector fields D_k are referred to as *total derivatives*. The Cartan distribution is a geometrical structure, which from all

[1] However, if $u^\alpha_{(0,\ldots,0)}$ occurs it should be understood as u^α.

submanifolds of $J^\infty(\pi)$ distinguishes the ones corresponding to smooth functions. Indeed, a submanifold in $J^\infty(\pi)$ is a maximal integral manifold of $\mathcal{C}$ if and only if it is a graph of $j_\infty(s)$ for some section s of π. The total derivatives $\{D_1, ..., D_n\}$ all pairwise commute on $J^\infty(\pi)$, i.e. $[D_i, D_j] = D_i \circ D_j - D_j \circ D_i = 0$, which makes the distribution involutive (i.e. closed with respect to the Lie bracket $[\,\cdot,\cdot\,]$).

1.4 Equations

We will denote a system of PDEs given by equations

$$\begin{cases} F_1(x^1,\ ...,\ x^n,\ [u^1_I],\ ...,\ [u^m_I]) = 0, \\ \vdots \\ F_R(x^1,\ ...,\ x^n,\ [u^1_I],\ ...,\ [u^m_I]) = 0, \end{cases}$$

where $[u^\alpha_I]$ should be understood as some finite set of partial derivatives of u^α (in general distinct for each function F_R and possibly including u^α itself), by $\{F_r(x^i, u^\alpha_I) = 0\}$ or sometimes even by $F = 0$, where $F = (F_1, ..., F_R)$. A system of s-th order PDEs given by equations $\{F_r(x^i, u^\alpha_I) = 0\}$ involving derivatives of u^α up to order $s \geq 1$, with $R \geq 1$ and $F_r \colon J^s(\pi) \to \mathbb{R}$ being smooth functions, defines a submanifold

$$\mathcal{E}^s = \{(x^i, u^\alpha_I) \in J^s(\pi)\colon\ F_r(x^i, u^\alpha_I) = 0\}$$

in $J^s(\pi)$. It is convenient to consider a given system together with all its differential consequences from the very beginning. We say that

$$\mathcal{E} = \{(x^i, u^\alpha_{I+K}) \in J^\infty(\pi)\colon\ D_K\left(F_r(x^i, u^\alpha_I)\right) = 0,\ |K| \geq 0\} \tag{1.2}$$

is an equation, this way identifying the system $\{F_r(x^i, u^\alpha_I) = 0\}$ with its infinite prolongation. The total derivatives restricted to $\mathcal{E}$ span the Cartan distribution $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}$. A maximal integral manifold of this restricted Cartan distribution is called a *solution* of the equation $\mathcal{E}$.

Restriction of total derivatives to $\mathcal{E}$ is in practice realised through introduction of *internal coordinates*. For the latter to exist it is necessary to impose some technical conditions on the system $\{F_r(x^i, u^\alpha_I) = 0\}$. First, we assume that at any point $\theta \in \mathcal{E}$ the differentials $dF_r(\theta)$ are linearly independent. It follows that the equations $\{F_r(x^i, u^\alpha_I) = 0\}$ can be solved for some partial derivatives in the neighbourhood of any point $\theta \in \mathcal{E}$. Hence, equation $\mathcal{E}$ is defined by the system $\{u_{I_r} = f_r(x^i, u^\alpha_I)\}$. If this system is in a passive orthonomic form (see below), then restriction of an object (operator, vector field or function) to $\mathcal{E}$ is performed by substituting the derivatives u_{I_r} (or their differential consequences) by functions f_r (or their differential consequences). The coordinates appearing in $f_r(x^i, u^\alpha_I)$ are called *internal coordinates*. For the system $\{u_{I_r} = f_r(x^i, u^\alpha_I)\}$ to be in a passive orthonomic form it means that each u_{I_r} appears on the left–hand side of the system only once, and none of the derivatives u_{I_r} appears on any right–hand side. Moreover, the compatibility conditions of the system do not impose additional dependencies between coordinates appearing on the right–hand side.

In other words, the internal coordinates are truly independent of each other. A rigorous discussion about integrability conditions of orthonomic systems and references are found in (Marvan, 2009). A concise recapitulation of the results obtained therein and relevant here, together with discussion about other technical assumptions about $\{F_r(x^i, u_I^\alpha) = 0\}$ can be found in (Krasil′shchik et al., 2017). All equations considered here are in the forms that enable easy introduction of internal variables, see Example 1.6.1.

1.5 Local symmetries

Symmetries will be introduced through the notion of infinitesimal automorphisms of Cartan distribution. These automorphisms are completely described by generating functions corresponding to evolutionary derivations on the algebra $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$. The aforementioned algebra is filtered and defined as $\mathcal{F}(\pi) = \cup_k \mathcal{F}_k(\pi)$, where $\mathcal{F}_k(\pi)$ is the algebra of smooth functions on $J^k(\pi)$. Vector fields on $J^\infty(\pi)$ are derivations of the algebra $\mathcal{F}(\pi)$ and form a set denoted by $\mathcal{X}(\pi)$. In local coordinates they are represented as infinite sums of the form

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i} + \sum_I a_I^\alpha \frac{\partial}{\partial u_I^\alpha}, \qquad a_i,\, a_I^\alpha \in \mathcal{F}(\pi). \tag{1.3}$$

The set $\mathcal{X}(\pi)$ together with the Lie bracket (commutator) $[X, Y] = X \circ Y - Y \circ X$, for $X, Y \in \mathcal{X}(\pi)$, forms a Lie algebra. Vector fields from $\mathcal{X}(\pi)$ lying in the Cartan distribution form a Lie subalgebra of $\mathcal{X}(\pi)$, which we denote by $\mathcal{C}\mathcal{X}(\pi)$. Moreover, we have $\mathcal{X}(\pi) = \mathcal{X}^v(\pi) \oplus \mathcal{C}\,\mathcal{X}(\pi)$, where $\oplus$ is a direct sum and $\mathcal{X}^v(\pi)$ is the Lie algebra of vertical vector fields. A vector field $X \in \mathcal{X}(\pi)$ is vertical if $X(x^i) = 0$, $i = 1, \ldots, n$.

A vector field $X \in \mathcal{X}(\pi)$ is a *symmetry* of the Cartan distribution if $[X, Z] \in \mathcal{C}\,\mathcal{X}(\pi)$ for any $Z \in \mathcal{C}\,\mathcal{X}(\pi)$. In other words, a symmetry X is an infinitesimal automorphism of the Cartan distribution $\mathcal{C}$. The space of symmetries is denoted by $\mathcal{X}_\mathcal{C}(\pi)$, and it is a Lie algebra as well. Symmetries from $\mathcal{C}\,\mathcal{X}(\pi)$ are called trivial. The Lie algebra of nontrivial symmetries is defined as $\mathrm{sym}(\pi) = \mathcal{X}_\mathcal{C}(\pi)/\mathcal{C}\mathcal{X}(\pi)$ and it is identified with $\mathcal{X}_\mathcal{C}(\pi) \cap \mathcal{X}^v(\pi)$.

Example 1.5.1. *Let us see how the above considerations manifest in local coordinates and in the case of $n = m = 1$. That is, consider $\pi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and put $D_1 = D$. Let $X \in \mathcal{X}(\pi)$ be a vector field of the form*

$$X = a\frac{\partial}{\partial x} + \sum_{k=0}^{\infty} b_k \frac{\partial}{\partial u_k}, \qquad a,\, b_k \in \mathcal{F}(\pi), \qquad k = 0, 1, 2, \ldots.$$

If X is a symmetry, then from $[X, D] \in \mathcal{C}\mathcal{X}(\pi)$ it follows that the equation $b_{k+1} = D(b_k) - u_{k+1}\, D(a)$ holds, and so X can be decomposed as

$$X = a\, D + \sum_{k=0}^{\infty} D^k(b_0 - a\, u_1)\frac{\partial}{\partial u_k}.$$

The first summand in the above formula belongs to $\mathcal{CX}(\pi)$, the second is a vertical field from $\mathcal{X}^v(\pi)$ called an evolutionary derivation associated to the generating section $\varphi = b_0 - a\,u_1$. Now it is easy to make a very important observation: every symmetry is uniquely determined by its restriction to the subalgebra $\mathcal{F}_0(\pi) = C^\infty(J^0(\pi))$. $\diamond$

As illustrated by the above example, the nontrivial part of a symmetry $X \in \mathcal{X}(\pi)$ is of the form

$$\mathbf{E}_\varphi = \sum_{|I|\geq 0} \sum_{\alpha=1}^{m} D_I(\varphi^\alpha)\, \frac{\partial}{\partial u_I^\alpha} \tag{1.4}$$

for some smooth function $\varphi\colon J^\infty(\pi) \to \mathbb{R}^m$, with $D_I = D_{(i_1,\ldots,i_n)} = D_{x^1}^{i_1} \circ \ldots \circ D_{x^n}^{i_n}$. The vector field $\mathbf{E}_\varphi$ is called *evolutionary derivation* and φ is called a *generating section* of $\mathbf{E}_\varphi$. In general, there is a one-to-one correspondence between vector fields from $\mathrm{sym}(\pi)$ and generating sections of evolutionary vector fields, see (Krasil′shchik and Vinogradov, 1999, Chapter 4, § 2, Theorem 2.5).

We, however, are interested in symmetries of an equation $\mathcal{E}$ given by (1.2), rather than in the general set $\mathcal{X}_C(\pi)$ (which can by seen as symmetries of a null equation). For this reason, we consider vector fields (1.3) with a_i, $a_I^\alpha \in \mathcal{F}(\mathcal{E})$, where $\mathcal{F}(\mathcal{E})$ denotes the algebra of smooth functions on $\mathcal{E}$. The set of symmetries of $\mathcal{E}$ is defined as $\mathrm{sym}(\mathcal{E}) = \mathcal{X}_C(\mathcal{E})/\mathcal{CX}(\mathcal{E})$. A function $\varphi\colon \mathcal{E} \to \mathbb{R}^m$ is called a (*generating section of an infinitesimal*) *symmetry* of $\mathcal{E}$ when $\mathbf{E}_\varphi(F) = 0$ on $\mathcal{E}$. The symmetries of a given equation are found by solving the (linear) *defining system*

$$\ell_{\mathcal{E}}(\varphi) = 0, \tag{1.5}$$

where $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$ with the matrix differential operator

$$(\ell_F)_{r,\alpha} := \sum_{|I|\geq 0} \frac{\partial F_r}{\partial u_I^\alpha}\, D_I, \tag{1.6}$$

which is also called a Fréchet derivative, see (Olver, 2000, § 5.2). Solutions to (1.5) constitute a Lie algebra with respect to the *Jacobi bracket* $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$. This Lie algebra is denoted by $\mathrm{Sym}(\mathcal{E})$. We distinguish three types of local symmetries:

a) **contact symmetries**, which form a subalgebra $\mathrm{Sym}_1(\mathcal{E}) := \mathrm{Sym}(\mathcal{E}) \cap C^\infty(J^1(\pi), \mathbb{R}^m)$, with a subclass of **point symmetries**, which consists of generating functions of the form $\varphi = (\varphi^1, \ldots, \varphi^m)$, where $\varphi^\alpha \;=\; \sum_{i=1}^{n} \eta^\alpha(x,u) -\; \xi^{i,\alpha}(x,u)\, u_i^\alpha$.

b) **higher order symmetries**, which are symmetries from $\mathrm{Sym}(\mathcal{E}) \setminus C^\infty(J^1(\pi), \mathbb{R}^m)$.

The higher symmetries are also called generalized symmetries. In this thesis we will tackle only contact symmetries. Moreover, we will abuse terminology (as we have already done) and write *symmetry* when referring to a vector field from $\mathrm{sym}(\mathcal{E})$ as well as when referring to a generating section from $\mathrm{Sym}(\mathcal{E})$. All symmetries on $J^\infty(\pi)$ which possess corresponding one-parameter group of transformations (such vector fields are called Lie fields) are in fact liftings (prolongations) of contact transformations (when $m = 1$) or point transformations (when $m > 1$), see (Krasil′shchik and Vinogradov, 1999, Chapter 3, § 3, Theorem 3.1 and Theorem 3.3).

1.6 Differential coverings

Nonlocal objects appearing in the study of differential equations are rigorously introduced using the concept of a differential covering, which is due to Krasil′shchik and Vinogradov (1984, 1989).

The general idea is to construct a certain locally trivial bundle $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$, and a corresponding Cartan distribution $\tilde{\mathcal{C}}$ on $\tilde{\mathcal{E}}$ in such a way that $\tilde{\mathcal{C}}$ is integrable (involutive) and for any $\tilde{\theta} \in \tilde{\mathcal{E}}$, the differential $d\tau|_{\mathcal{C}_{\tilde{\theta}}}$ induces a one to one correspondence between $\tilde{\mathcal{C}}_{\tilde{\theta}}$ and $\mathcal{C}_{\tau(\tilde{\theta})}$. Dimension of the covering is the dimension of the space of the new variables. We will work with infinite-dimensional differential coverings, for which the definition in local coordinates is as follows. Denote the space of nonlocal variables as $\mathcal{W} = \mathbb{R}^\infty$ with coordinates w^s, $s \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) *differential covering* of $\mathcal{E}$ is a trivial bundle $\tau\colon \mathcal{E} \times \mathcal{W} \to \mathcal{E}$ equipped with the *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} W_k^s(x^i, u_I^\alpha, w^j)\,\frac{\partial}{\partial w^s} \tag{1.7}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u_I^\alpha) \in \mathcal{E}$ (note that the total derivatives D_{x^k} are considered as restricted to $\mathcal{E}$). We define the partial derivatives of w^s by $w^s_{x^k} = \tilde{D}_{x^k}(w^s)$. This yields the over-determined system of PDEs called *covering equations*

$$w^s_{x^k} = W^s_k(x^i, u^\alpha_I, w^j), \tag{1.8}$$

compatible whenever $(x^i, u_I^\alpha) \in \mathcal{E}$. A submanifold in $\mathcal{E} \times \mathcal{W}$ determined by equations (1.8) is denoted by $\tilde{\mathcal{E}}$.

As for notation, in this thesis we will consider mostly scalar equations in at most four independent variables. In this case, we write for example $x^1 = t$, $x^2 = x$, $x^3 = y$, $x^4 = z$ and $u^1_{(i,j,k,l)} = u_{t\ldots tx\ldots xy\ldots yz\ldots z}$ with t, x, y and z appearing in the subscript i, j, k, and l times, respectively.

Example 1.6.1. *Let us consider the modified Veronese web Eq.* (4.10)*.*

$$\mathcal{E}\colon\; u_{ty} = u_t\,u_{xy} - u_y\,u_{tx}.$$

The internal coordinates on $\mathcal{E}$ *are*

$$t,\ x,\ y,\ u,\ u_{t\ldots t},\ u_{x\ldots x}, u_{y\ldots y},\ u_{t\ldots t\,x\ldots x},\ u_{x\ldots x\,y\ldots y},$$

or, respectively:

$$t,\ x,\ y,\ u,\ u_{(k,0,0)},\ u_{(0,k,0)}, u_{(0,0,k)},\ u_{(k,l,0)},\ u_{(0,k,l)}, \qquad k \geq 1,\ l \geq 1.$$

In the new notation, $\mathcal{E}$ *is defined by*

$$u_{(1,0,1)} = u_{(1,0,0)}\,u_{(0,1,1)} - u_{(0,0,1)}\,u_{(1,1,0)}.$$

The total derivatives restricted to $\mathcal{E}$ are:

$$\begin{aligned} D_t = \frac{\partial}{\partial t} &+ \sum_{k\geq 0} u_{(k+1,0,0)} \frac{\partial}{\partial u_{(k,0,0)}} + \sum_{k\geq 1} u_{(1,k,0)} \frac{\partial}{\partial u_{(0,k,0)}} + \sum_{k\geq 1, l\geq 1} u_{(k+1,l,0)} \frac{\partial}{\partial u_{(k,l,0)}} \\ &+ \sum_{k\geq 1} D_y^{k-1} \left(u_{(1,0,0)}\, u_{(0,1,1)} - u_{(0,0,1)}\, u_{(1,1,0)} \right) \frac{\partial}{\partial u_{(0,0,k)}} \\ &+ \sum_{k\geq 1, l\geq 1} D_x^k D_y^{l-1} \left(u_{(1,0,0)}\, u_{(0,1,1)} - u_{(0,0,1)}\, u_{(1,1,0)} \right) \frac{\partial}{\partial u_{(0,k,l)}}, \end{aligned}$$

$$\begin{aligned} D_x = \frac{\partial}{\partial x} &+ \sum_{k\geq 0} u_{(k,1,0)} \frac{\partial}{\partial u_{(k,0,0)}} + \sum_{k\geq 1} u_{(0,k+1,0)} \frac{\partial}{\partial u_{(0,k,0)}} + \sum_{k\geq 1, l\geq 1} u_{(0,k+1,l)} \frac{\partial}{\partial u_{(0,k,l)}} \\ &+ \sum_{k\geq 1} u_{(0,1,k)} \frac{\partial}{\partial u_{(0,0,k)}} + \sum_{k\geq 1, l\geq 1} u_{(k,l+1,0)} \frac{\partial}{\partial u_{(k,l,0)}}, \end{aligned}$$

$$\begin{aligned} D_y = \frac{\partial}{\partial y} &+ \sum_{k\geq 0} u_{(0,k,1)} \frac{\partial}{\partial u_{(0,k,0)}} + \sum_{k\geq 1} u_{(0,0,k+1)} \frac{\partial}{\partial u_{(0,0,k)}} + \sum_{k\geq 1, l\geq 1} u_{(0,k,l+1)} \frac{\partial}{\partial u_{(0,k,l)}} \\ &+ \sum_{k\geq 1, l\geq 1} D_x^l D_t^{k-1} \left(u_{(1,0,0)}\, u_{(0,1,1)} - u_{(0,0,1)}\, u_{(1,1,0)} \right) \frac{\partial}{\partial u_{(k,l,0)}} \\ &+ \sum_{k\geq 1} D_t^{k-1} \left(u_{(1,0,0)}\, u_{(0,1,1)} - u_{(0,0,1)}\, u_{(1,1,0)} \right) \frac{\partial}{\partial u_{(k,0,0)}}. \end{aligned}$$

The equation admits a covering defined by the system

$$\tilde{\mathcal{E}}\colon \begin{cases} q_t &= (1+\lambda)^{-1} u_t\, q_x, \\ q_y &= \lambda^{-1}\, u_y\, q_x. \end{cases}$$

The covering is infinite–dimensional, since the nonlocal internal variables are $q_i = q\underbrace{x\ldots x}_{i\,times}$, $i \in \mathbb{N}$. *The extended total derivatives on* $\tilde{\mathcal{E}}$ *are*

$$\tilde{D}_t = D_t + \sum_i D_x^i \left((1+\lambda)^{-1} u_{(1,0,0)}\, q_1 \right) \frac{\partial}{\partial q_i}, \quad \tilde{D}_x = D_x + \sum_i q_{i+1} \frac{\partial}{\partial q_i},$$

$$\tilde{D}_y = D_y + \sum_i D_x^i \left(\lambda^{-1}\, u_{(0,0,1)}\, q_1 \right) \frac{\partial}{\partial q_i}.$$

$\diamond$

The concept of a differential covering can be used to construct concise definitions in nonlocal theory of PDEs. For instance, the notion of a Bäcklund transformation, which is important in Chapter 4, can be formulated as follows.

Definition 1.6.1 (Bäcklund transformation)**.** Bäcklund transformation is a pair of coverings $\tau_1\colon \tilde{\mathcal{E}} \to \mathcal{E}_1$, $\tau_2\colon \tilde{\mathcal{E}} \to \mathcal{E}_2$ between PDEs $\mathcal{E}_1$ and $\mathcal{E}_2$, with the same equation $\tilde{\mathcal{E}}$, as presented on the diagram below.

$$\mathcal{E}_1 \xleftarrow{\tau_1} \tilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2$$

If $\mathcal{E}_1 = \mathcal{E}_2$, then we say that it is a Bäcklund auto-transformation. $\diamond$

Example 1.6.2 (The sine-Gordon equation)**.** *Let us consider the sine-Gordon equation*

$$u_{xy} = \sin(u). \tag{1.9}$$

It admits a covering

$$\begin{cases} v_y &= u_y + 2\,\lambda\,\sin\left(\frac{v+u}{2}\right), \\ v_x &= -u_x + \frac{2}{\lambda}\,\sin\left(\frac{v-u}{2}\right). \end{cases} \tag{1.10}$$

Cross-differentiating the system (1.10) *leads to*

$$\begin{cases} v_{xy} &= u_{xy} + 2\,\sin\left(\frac{v-u}{2}\right)\,\cos\left(\frac{v+u}{2}\right), \\ v_{xy} &= -u_{xy} + 2\,\sin\left(\frac{v+u}{2}\right)\,\cos\left(\frac{v-u}{2}\right). \end{cases} \tag{1.11}$$

The above equations added to each other give

$$v_{xy} = \sin(v),$$

while their difference gives (1.9)*. We conclude that the covering* (1.10) *defines a Bäcklund auto-transformation of the sine-Gordon Eq.* (1.9)*.* ◇

Another term which will be of use in Chapter 4 is *Whitney product* of differential coverings.

Definition 1.6.2 (Whitney product)**.** Let τ_1 and τ_2 be a pair of coverings $\tau_1\colon \tilde{\mathcal{E}}_1 \to \mathcal{E}$, $\tau_2\colon \tilde{\mathcal{E}}_2 \to \mathcal{E}$. Consider the direct product $\tilde{\mathcal{E}}_1 \times \tilde{\mathcal{E}}_2$ and take the subset $\tilde{\mathcal{E}}_1 \oplus \tilde{\mathcal{E}}_2$ consisting of the points $(\tilde{\theta}_1, \tilde{\theta}_2)$, such that $\tau_1(\tilde{\theta}_1) = \tau_2(\tilde{\theta}_2)$. Then the projection $\tau_1 \oplus \tau_2\colon \tilde{\mathcal{E}}_1 \oplus \tilde{\mathcal{E}}_2 \to \mathcal{E}$ is defined, for which $\tau_1 \oplus \tau_2\,(\tilde{\theta}_1, \tilde{\theta}_2) = \tau_1(\tilde{\theta}_1) = \tau_2(\tilde{\theta}_2)$. The projection $\tau_1 \oplus \tau_2$ is the *Whitney product* of the bundles τ_1, τ_2. With τ_{12} and τ_{21} being projections on the left and right factor, respectively, the diagram below is commutative.

$$\begin{tikzcd}
 & \tilde{\mathcal{E}}_1 \oplus \tilde{\mathcal{E}}_2 \arrow[dl, "\tau_{12}"'] \arrow[dr, "\tau_{21}"] \arrow[dd, "\tau_1 \oplus \tau_2"] & \\
\tilde{\mathcal{E}}_1 \arrow[dr, "\tau_1"'] & & \tilde{\mathcal{E}}_2 \arrow[dl, "\tau_2"] \\
 & \mathcal{E} &
\end{tikzcd}$$

In particular, each fiber $(\tau_1 \oplus \tau_2)^{-1}(\theta)$ for $\theta \in \mathcal{E}$ is a direct sum of fibers $\tau_1^{-1}(\theta)$ and $\tau_2^{-1}(\theta)$. In order to define Whitney product of coverings τ_1, τ_2, we need to discuss a covering structure of the Whitney product of the bundles τ_1, τ_2. In local coordinates, let the Cartan distribution $\mathcal{C}_1$ on τ_1 be given by the extended total derivatives

$$\tilde{D}^{(1)}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} W^s_k(x^i, u^\alpha_I, w^j)\,\frac{\partial}{\partial w^s}$$

and the Cartan distribution $\mathcal{C}_2$ on τ_2 be given by the extended total derivatives

$$\tilde{D}^{(2)}_{x^k} = D_{x^k} + \sum_{l=0}^{\infty} V^l_k(x^i, u^\alpha_I, v^j)\,\frac{\partial}{\partial v^l}.$$

Then, the Cartan distribution $\mathcal{C}$ on the Whitney product $\tau_1 \oplus \tau_2$ is given by the extended total derivatives

$$\tilde{D}^{(1,2)}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} W^s_k(x^i, u^\alpha_I, w^j)\,\frac{\partial}{\partial w^s} + \sum_{l=0}^{\infty} V^l_k(x^i, u^\alpha_I, v^j)\,\frac{\partial}{\partial v^l}. \tag{1.12}$$

The projection $\tau_1 \oplus \tau_2$ and the Cartan distribution $\mathcal{C}$ define a covering structure over $\mathcal{E}$, which is called a *Whitney product of coverings* τ_1 and τ_2. See (Krasil′shchik and Vinogradov, 1999, Chapter 6, § 1.6) for a more detailed discussion in a coordinate-free approach. ◇

1.7 Lax-integrable equations

The common feature of all equations examined in this thesis, namely their integrability, demands some specification, since it is quite a complex concept in the field of nonlinear PDEs and can be addressed via different approaches. In the framework of the theory of differential coverings we adopt the following definition.

Definition 1.7.1 (Lax-integrability)**.** We say that equation is *Lax-integrable* if it admits a differential covering. ◇

A prototype equation in the field of integrable nonlinear PDEs and the one which satisfies all definitions of integrability is the Korteweg-de Vries (KdV) equation:

$$u_t + 6\,u\,u_x + u_{xxx} = 0. \tag{1.13}$$

A method for solving the associated Cauchy problem for a wide class of initial conditions is presented in the fundamental for the theory of nonlinear integrable equations article (Gardner et al., 1967). It was soon recognised that the essential structure behind this method is a Lax pair of differential operators (Lax, 1968), and the method, called inverse scattering transform method (ISTM), can be applied to other nonlinear evolution equations in (1+1)-dimensions, see e.g. (Mikhailov, 2009) or (Ablowitz and Clarkson, 1991) and references therein. Let us examine the original Lax pair in more detail. The overdetermined and linear in v system

$$\begin{aligned} &v_{xx} + u\,v = \lambda\,v,\\ &v_t = u_x\,v - (2\,u + 4\,\lambda)\,v_x \end{aligned} \tag{1.14}$$

is compatible for an arbitrary value of the spectral parameter $\lambda = \text{const.}$, that is, the equality $(v_{xx})_t = (v_t)_{xx}$ holds whenever u satisfies (1.13). Setting $L = \frac{\partial^2}{\partial x^2} + u$ and $M = u_x - (2\,u + 4\,\lambda)\,\frac{\partial}{\partial x}$, system (1.14) can be expressed as $\{L\,v = \lambda\,v,\, v_t = M\,v\}$, and the compatibility condition as

$$L_t + [L,\, M] = 0. \tag{1.15}$$

Here L_t is the operator $v \mapsto u_t v$. A pair of linear differential operators L and M for which the condition (1.15) is equivalent to an evolution $(1+1)$-dimensional equation $u_t = F(t, x, u_k)$, where u_k is the k-th order derivative of u with respect to x, is a Lax pair for this equation. The typical solutions obtained in this case (and in various generalizations) are solitons, and equations possessing such solutions are called soliton equations. The definition of a Lax pair was soon generalized for an arbitrary equation $F(t, x, u_k) = 0$ in $(1+1)$ dimensions, which led to the concept of a zero-curvature representation (see (Ablowitz and Clarkson, 1991) and references therein) and later for equations in more than two dimensions. In those generalizations the equation under consideration arises as a compatibility condition for some overdetermined system, although this compatibility condition is not of the form (1.15). To be more specific, the Lax pair may not involve spectral parameter (as e.g. (2.40)) or can be nonlinear. For instance, a paper by Sergyeyev (2018) gives a number of examples of integrable equations in four dimensions, which are constructed using nonlinear Lax pairs.

Nowadays, a definition that would encompass all usages of the term *Lax pair* should only claim that it is an overdetermined system whose compatibility condition coincides with the equation under consideration, i.e. it defines a differential covering over a given equation. It does not necessarily mean that there is an appropriate generalization of ISTM that can be applied to this Lax pair.

The KdV equation possesses several features the presence of which is considered a hallmark of integrability. It admits a Lax pair, soliton solutions, an infinite hierarchy of higher order symmetries (study of which is called symmetry approach in integrability), an infinite hierarchy of conservation laws, it may be obtained as a reduction of a certain hydrodynamic chain and it satisfies the Painlevé property. Moreover, it admits a Bäcklund auto-transformation, bi-Hamiltonian structure, and a certain bi-linear form used in Hirota method. There are, for example, two books (Zakharov, 1991) and (Mikhailov, 2009), which provide a comprehensive survey of the various definitions of integrability. Relation between various definitions can be obtained for particular types of the given object (e.g. a class of Lax pairs which can be interpreted as an Abelian covering corresponds to an infinite hierarchy of conservation laws), but not in general.

A typical case is that a given integrable equation satisfies more than one definition of integrability, but not all of them. For instance, equations examined in this thesis belong to an important class of integrable dispersionless PDEs. Other term for such equations is *equations of hydrodynamic type* and they are intensively studied within the theory of integrable hydrodynamic chains, see e.g. (Ferapontov and Fordy, 1997), (Pavlov, 2003), (Martínez Alonso and Shabat, 2003), (Ferapontov and Khusnutdinova, 2004a). Another suitable framework is provided by twistor theory, see (Dunajski, 2010). Each of the equations we examined in this thesis, that is the GT Eq. (2.3), the KhZ singular manifold Eq. (2.36), Plebański's second heavenly Eq. (3.1), Pavlov's Eq. (4.2), the UH Eq. (4.8), the rdDym

Eq. (4.6), the mVw Eq. (4.10) and the Vw Eq. (4.12), admits a Lax representation.

1.8 Nonlocal symmetries

Nonlocal symmetry of the equation $\mathcal{E}$ in the covering τ is a local symmetry of $\tilde{\mathcal{E}}$. A naïve approach (or a preliminary step) to define nonlocal symmetries is to substitute $\tilde{D}_{x^k}$ for D_{x^k} in (1.4) and to consider a function $\varphi \in C^\infty(\tilde{\mathcal{E}}, \mathbb{R}^m)$ instead of $\varphi \in C^\infty(\mathcal{E}, \mathbb{R}^m)$. Observe that the resulting vector field $\tilde{\mathbf{E}}_\varphi$ is a restriction of evolutionary derivation on $\mathcal{E} \times \mathcal{W}$ to evolutionary derivation on $\mathcal{E}$, therefore it does not properly modify the concept of local symmetry to nonlocal one. However, such a vector field is still an interesting object, called τ-shadow of a nonlocal symmetry[2]. To sum up, a *shadow of a nonlocal symmetry* of $\mathcal{E}$ in the covering τ with the extended total derivatives (1.7), or simply *τ-shadow*, is a function $\varphi \in C^\infty(\tilde{\mathcal{E}}, \mathbb{R}^m)$, such that

$$\tilde{\mathbf{E}}_\varphi(F) = \sum_{|I|\geq 0} \sum_{\alpha=1}^{m} \tilde{D}_I(\varphi^\alpha) \frac{\partial F}{\partial u_I^\alpha} = 0 \qquad \text{on } \tilde{\mathcal{E}}. \tag{1.16}$$

In its turn, a *nonlocal symmetry* of $\mathcal{E}$ corresponding to the covering τ (also *τ-symmetry* or *full-fledged* nonlocal symmetry) is a vector field

$$\tilde{\mathbf{E}}_{\varphi,A} = \tilde{\mathbf{E}}_\varphi + \sum_{s=0}^{\infty} A^s \frac{\partial}{\partial w^s}, \tag{1.17}$$

with $A^s \in C^\infty(\tilde{\mathcal{E}})$, such that φ satisfies (1.16) and

$$\tilde{D}_{x^k}(A^s) = \tilde{\mathbf{E}}_{\varphi,A}(W_k^s) \tag{1.18}$$

for W_k^s from (1.7), see (Krasil′shchik and Vinogradov, 1999, Chapter 6, § 3.2). Nonlocal symmetries with shadows equal to zero are called *invisible symmetries*.

Remark 1.8.1. *In general, not every τ-shadow corresponds to a τ-symmetry, since equations* (1.18) *provide an obstruction for existence of A^s in* (1.17)*. But for any τ-shadow φ there exists a covering τ_φ and a τ_φ-symmetry whose τ_φ-shadow coincides with φ, see (Krasil′shchik and Vinogradov, 1999, Chapter 6, § 5.8).* ◇

The study of higher local symmetries constitutes a very important part of the domain of integrability of PDEs, allowing, for example, to obtain classification results of integrable equations, see review in (Mikhailov, 2009, Chapter 1). Equations possessing infinite hierarchy of higher local symmetries are called symmetry-integrable. The main advantage of the symmetry approach is that there are necessary conditions for the existence of higher symmetries (or conservation laws) that can be in principle verified for an arbitrary equation, and hence provide an easily applicable criterion for integrability. However, it appears that equations in more than two dimensions in general do not admit infinite hierarchies of higher order symmetries, even if they are integrable in other sense (for example the KhZ Eq. 2.32

[2]It is quite common in the literature that τ-shadows are referred to as nonlocal symmetries.

has no higher order symmetries as showed by Sharomet (1989)), see discussion in (Vinogradov, 1989, § 6). Thus in the field of multidimensional PDEs the need for a more general concept of a symmetry is evident. Applications of nonlocal symmetries are not limited to the multidimensional case though. The notion of a nonlocal symmetry was rigorously introduced first in (Krasil′shchik and Vinogradov, 1984). The motivational example on which the authors present the need and usefulness of nonlocal symmetries is Burgers equation. It is showed in particular that considerations of solutions invariant with respect to a nonlocal symmetry of Burgers equation lead to a famous Cole-Hopf transformation.

There is a great number of works devoted to methods of studying nonlinear PDEs that admit nonlocal symmetries. For the case of potential symmetries, that is, nonlocal symmetries corresponding to Abelian coverings, see (Bluman and Kumei, 2013, Chapter 7), (Bluman et al., 2010), and references therein. As for applications of nonlocal symmetries in non-Abelian coverings, the nonlocal symmetries were used to find the corresponding invariant solutions in e.g. (Reyes, 2006), (Hernández-Heredero and Reyes, 2012, 2013), see also references therein. In (Leo et al., 2001) authors re-obtained soliton solutions of the KdV equation and the Dym equation, as well as found Bäcklund transformations for these equations through considerations of solutions invariant with respect to nonlocal symmetries. An algorithm for linearising PDEs, based on the existence of nonlocal symmetries is presented in (Bluman and Kumei, 1990). Other examples of application of nonlocal symmetries to nonlinear PDE can be found in e.g. (Błaszak, 2002), (Sergyeyev, 2009), (Krasil′shchik and Verbovetsky, 2011), (Bies et al., 2012).

1.9 Nonlocal conservation laws

The fundamental benefit of a knowledge of conservation laws of a differential equation is that conservation laws provide additional information about the dynamics of solutions to the equation. In the case of equations of mathematical physics it is sometimes possible to identify some of the conservation laws as energy, (angular) momentum or mass conservation law. There are many nontrivial applications of conservation laws, though. In Bluman et al. (2010) authors present how local conservation laws can be used to construct nonlocally related PDEs by creating potential systems and to determine whether a nonlinear PDE can be mapped to a linear PDE. Moreover, conservation laws are crucial for a number of numerical schemes for nonlinear PDEs, such as finite element or discontinuous Galerkin method. Knowledge of the conservation laws appears to be particularly helpful in elasticity, examination of existence, uniqueness, stability and global behaviour of solutions. Conservation laws are not sensitive to a chosen set of coordinates, since point transformations map conservation law into conservation law, and they hold for arbitrary boundary conditions – therefore providing a valuable insight into the dynamics of a PDE. Other than that, existence of an infinite hierarchy of conservation laws is used as a definition of integrability of nonlinear PDEs and is a necessary ingredient in the ISTM (see Section 1.7). However, there are integrable equations which do not satisfy this definition, one example is Burgers equation, which possesses only one conservation law. There is a conjecture that in analogy to symmetry-based definition of integrability, the definition based on existence of infinite hierarchy of conservation laws is corrected by allowing for nonlocal ones, see (Vinogradov, 1989). Finally, it should be mentioned that conservation laws and symmetries are, at least for some types of DEs, related objects. The most famous theorem in this subject is due to Noether, who established a one-to-one correspondence between variational symmetries and conservation laws of a given PDE. We know now that in general, this correspondence holds for self-adjoint PDEs, see (Olver, 2000) and (Bluman et al., 2010) for detailed discussions.

The conservation laws will be defined as $(n-1)$ cohomology classes with respect to the horizontal de Rham differential d_h on $\mathcal{E}$ (and on $\tilde{\mathcal{E}}$ in the nonlocal case). In local coordinates this is equivalent to the usual definition based on vanishing of divergence expressions.

Let us consider a space $\bigwedge_h^{(n-1)}(\mathcal{E})$ of horizontal $(n-1)$-forms on $\mathcal{E}$. In local coordinates $\bigwedge_h^{(n-1)}(\mathcal{E})$ is generated by elements of the form

$$\Omega = A\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{n-1}}, \qquad A \in \mathcal{F}(\mathcal{E}).$$

In particular, if $x = (t, x, y)$ as is the case in Chapter 4, any two-form on $\mathcal{E}$ reads

$$A_1\, dx \wedge dy + A_2\, dy \wedge\, dt + A_3\, dt \wedge\, dx, \qquad A_i \in \mathcal{F}(\mathcal{E}).$$

For any $q \in \{0, 1, \ldots, n-1\}$, the horizontal de Rham differential d_h acts from $\bigwedge_h^q(\mathcal{E})$ to $\bigwedge_h^{q+1}(\mathcal{E})$. In local coordinates it acts according to the formula

$$d_h \left(A\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q} \right) = \sum_{i=1}^{n} D_i(A)\, dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q}.$$

Again, if $x = (t, x, y)$, the above formula reads

$$d_h \left(A_1\, dx \wedge dy + A_2\, dy \wedge\, dt + A_3\, dt \wedge\, dx \right) = \left(D_t(A_1) + D_x(A_2) + D_y(A_3) \right) dt \wedge dx \wedge dy.$$

A horizontal $(n-1)$-form Ω is called *closed*, if $d_h(\Omega) = 0$. A conservation law of $\mathcal{E}$ is defined by a closed horizontal $(n-1)$-form on $\mathcal{E}$. Locally, it boils down to vanishing of a divergence expression $\mathrm{Div}(A_1, \ldots, A_n)$ on $\mathcal{E}$, cf. (Olver, 2000). For any $\Omega \in \bigwedge_h^q(\mathcal{E})$, the equality $d_h \circ d_h(\Omega) = 0$ holds. If for a q-form Ω there exist a $(q-1)$-form ω such that $\Omega = d_h(\omega)$, then we say that Ω is *exact*. Clearly any exact form is closed regardless of the equation $\mathcal{E}$ and hence in the case $q = n-1$ gives rise to a trivial conservation law. In the case $x = (t, x, y)$ it is equivalent to the fact that a divergence of a total curl vanishes for any 3-dimensional vector. Conservation laws should be defined more rigorously by identifying those which differ only by a trivial conservation law. Thus it is said that conservation laws are elements of the cohomology group $CL(\mathcal{E}) := \ker d_h / \operatorname{im} d_h =: H_h^{n-1}(\mathcal{E})$. In practice, one works with representatives of equivalence classes of conservation laws and verification of their triviality is in general a nontrivial task. We used a method of generating functions discussed below.

If we impose two technical conditions on the system $\{F_r(x^i, u_I^\alpha) = 0\}$, which are satisfied for most of the equations, then it follows that for any conservation law

$$\Omega = \sum_{i=1}^{n} (-1)^{(i-1)} A_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$

we have

$$\mathrm{Div}(A_1, \, \ldots, A_n) = \sum_{r=1}^{R} H_r \, F_r,$$

where $H_r \in \mathcal{F}(\pi)$. The first condition is called regularity condition on $\mathcal{E}$ and it implies that the differential consequences of $\{F_r(x^i, u_I^\alpha) = 0\}$ form a complete set. More specifically it implies that if a function G vanishes on $\mathcal{E}$, then there exists an operator in total derivatives (a $\mathcal{C}$-differential operator) Δ, such that $G = \Delta(F)$. The second condition is called normality condition and it demands, roughly speaking, that the equation can be put into Cauchy–Kovalevskaya form, see (Vinogradov, 1989) and (Olver, 2000). The functions H_r are called *generating functions* or *characteristics* of the conservation law Ω. Let us use the notation $H = (H_1, \, \ldots, H_R)$, and recall that $F = (F_1, \, \ldots, F_R)$. Functions H_r must satisfy

$$\ell_F^*(H) + \ell_H^*(F) = 0$$

on $J^\infty(\pi)$, where $(\ell_F^*)_{r,\alpha} := \sum_{|I| \geq 0} (-1)^{|I|} D_I \circ \frac{\partial F_r}{\partial u_I^\alpha}$ is the adjoint to ℓ_F. In particular, the functions H_r satisfy

$$\ell_{\mathcal{E}}^*(H) = 0.$$

Define *cosymmetries* as solutions to the above equation. Most importantly, if the functions H_r depend solely on internal coordinates, then the conservation law Ω is trivial if and only if all the functions H_r vanish identically, see (Vinogradov, 1989, § 8) and (Olver, 2000, § 4.3, § 5.3)

In the case when $x = (t, x, y)$, a *nonlocal conservation law* for $\mathcal{E} = \{F_r(x^i, u_I^\alpha) = 0\}$ in the covering τ, that is, a conservation law for $\tilde{\mathcal{E}}$, is a horizontal two-form

$$\Omega = A_1 \, dx \wedge dy + A_2 \, dy \wedge \, dt + A_3 \, dt \wedge \, dx,$$

closed on $\tilde{\mathcal{E}}$ with respect to the horizontal differential d_h, which means

$$d_h \Omega = \left(\tilde{D}_t(A_1) + \tilde{D}_x(A_2) + \tilde{D}_y(A_3) \right) \, dt \wedge dx \wedge dy = 0,$$

see (Krasil′shchik and Vinogradov, 1989, § 6) or (Krasil′shchik and Vinogradov, 1999, Chapter 6, § 1.8) for details. Functions A_i are smooth functions on $\tilde{\mathcal{E}}$. The vector (A_1, A_2, A_3) is referred to as a nonlocal conserved current for the equation $\mathcal{E}$ (and simply a conserved current for $\tilde{\mathcal{E}}$). Suppose that the covering τ is determined by equations

$$w^s_{x^k} = W^s_k(x^i, u_I^\alpha, w^j).$$

Then, finding a representation

$$d_h\,\Omega = \left(\sum_r H_r\,F_r + \sum_{s\geq 0}\sum_{k=1}^{3} H_{k,\,s}\cdot(w^s_{x^k} - W^s_k)\right)\,dt\wedge dx\wedge dy,$$

where the *generating functions* H_r, $H_{k,\,s}$ are defined on $\tilde{\mathcal{E}}$ and are not equal to zero simultaneously, ensures that Ω is nontrivial, (Vinogradov, 1989, § 8).

1.10 Computations

We close preliminaries with an example of calculation of point symmetries for a simple PDE. There are two objectives of giving this example. The first one is to explain why using a computer algebra software is indispensable. The second objective is to roughly explain how the tools we used work.

The procedure of computing symmetries of a differential equation is to a great extent algorithmic and consists of: calculating the formula $\ell_F(\varphi)$ (1.6), restricting it to the equation in question and equating to zero to obtain equation (1.5), performing the *procedure of splitting* to obtain an overdetermined system of linear equations for φ (cf. (1.22)–(1.28)) and finally integrating the overdetermined system to find a generating section φ.

Let us calculate a group of point symmetries of a scalar heat equation $\mathcal{E}$ defined by equation

$$F := u_t - u_{xx} = 0 \tag{1.19}$$

and all its differential consequences. We hence need to solve

$$\ell_F(\varphi) = \left(\sum_{|I|\geq 0}\frac{\partial F}{\partial u_I}\,D_I\,(\varphi)\right) = 0 \qquad \text{on } \mathcal{E},$$

where $\varphi(t,x,u,u_t,u_x) = \eta(t,x,u) - \tau(t,x,u)\,u_t - \xi(t,x,u)\,u_x$. We have

$$\begin{aligned}\ell_F(\varphi) \;&= \left(\sum_{|I|\geq 0}\tfrac{\partial F}{\partial u_I}\,D_I\,(\varphi)\right) = D_t(\varphi) - D^2_x(\varphi) = -\eta_{xx} + \eta_t + u_t\,u_x^2\,\tau_{uu} + 2\,u_t\,u_x\,\tau_{xu} +\\ &+u_t\,u_{xx}\,\tau_u + 2\,u_{tx}\,u_x\,\tau_u + 3\,u_x\,u_{xx}\,\xi_u - u_t\,u_x\,\xi_u - u_{xx}\,\eta_u - 2\,u_x\,\eta_{xu} - u_t\,\tau_t -\\ &-u_t^2\,\tau_u - u_x\,\xi_t + u_t\,\tau_{xx} + 2\,u_{tx}\,\tau_x + u_x^3\,\xi_{uu} + 2\,u_x^2\,\xi_{xu} + u_x\,\xi_{xx} + 2\,u_{xx}\,\xi_x -\\ &-u_x^2\,\eta_{uu} - u_{tt}\,\tau - u_{tx}\,\xi + u_{txx}\,\tau + u_{xxx}\,\xi + u_t\,\eta_u.\end{aligned} \tag{1.20}$$

The above expression, restricted to $\mathcal{E}$, must be equal to zero if φ is a symmetry. The restriction is acquired by substituting u_{xx} for u_t. Therefore, for φ to be a point symmetry of a heat equation, the following equation must hold.

$$\begin{aligned}\ell_{\mathcal{E}}(\varphi) \;&= u_{xx}\,(\tau_{xx} - \tau_t + 2\,\xi_x) + u_x\,(\xi_{xx} - \xi_t - 2\eta_{ux}) + 2\,u_x\,u_{xx}\,(\xi_u + 2\tau_{ux}) +\\ &+2\,u_x\,u_{xxx}\,\tau_u + 2\,u_{xxx}\,\tau_x + u_x^3\,\xi_{uu} + u_x^2\,(2\,\xi_{ux} - \eta_{uu}) + u_x^2\,u_{xx}\,\tau_{uu} +\\ &+\eta_t - \eta_{xx} = 0.\end{aligned} \tag{1.21}$$

The next step is to observe that equation (1.21) can be seen as a polynomial in u_x, u_{xx} and u_{xxx}. Indeed, the functions τ, ξ and η do not depend on u_x, u_{xx}, u_{xxx}. Furthermore, since the restriction to $\mathcal{E}$ is already performed, the derivatives u_x, u_{xx}, u_{xxx} are independent jet variables. We split the equation (1.21) into the system

$$\tau_{xx} - \tau_t + 2\,\xi_x = 0, \tag{1.22}$$
$$\xi_{xx} - \xi_t - 2\eta_{ux} = 0. \tag{1.23}$$
$$\xi_u + 2\tau_{ux} = 0, \tag{1.24}$$
$$\tau_u = \tau_{uu} = \tau_x = 0, \tag{1.25}$$
$$\xi_{uu} = 0, \tag{1.26}$$
$$2\,\xi_{ux} - \eta_{uu} = 0, \tag{1.27}$$
$$\eta_t - \eta_{xx} = 0. \tag{1.28}$$

From (1.25) it follows that $\tau = \tau(t)$, which together with (1.24) gives $\xi = \xi(t,x)$. The latter fact together with (1.27) leads to $\eta_{uu} = 0$. It follows that $\eta = \alpha(t,x)\,u + \beta(t,x)$, for some functions α and β. The equations are further simplified and integrated, leading to the conclusion that the general form of a (generating section of) point symmetry of the heat equation is $\varphi = \eta - \tau\,u_t - \xi\,u_x$, where

$$\begin{aligned} \xi &= c_1 + c_4\,x + 2\,c_5\,t + 4\,c_6\,x\,t, \\ \tau &= c_2 + 2\,c_4\,t + 4\,c_6\,t^2, \\ \eta &= \Big(c_3 - c_5\,x - c_6(2\,t + x^2)\Big)\,u + \beta, \qquad \beta_t = \beta_{xx},\ c_i \in \mathbb{R},\ i \in \{1,\dots,6\}. \end{aligned}$$

Observe that even in this very simple example computations of the expression (1.20) and then (1.21) are very tedious. For this reason, the renewed interest in the applications of Lie groups to differential equations is correlated with the development of computer algebra software. From a number of available tools we have chosen the *Jets* package for *Maple*, see (Baran and Marvan, 2012) for documentation. In Appendix A.1 we present an example of how the software works in the case of looking for point symmetries. The scripts of other computations we performed would not give more insight into the usage of the *package* and in addition they are significantly longer. Roughly speaking, *Jets* package enables to obtain the overdetermined system resulting from the procedure of splitting effortlessly. The integration of the overdetermined system is due to a user. The package was designed to facilitate differential calculus on jet spaces in general, and hence we used it not only to find symmetries, but also to find nonlocal conservation laws.

Chapter 2

Symmetry invariant solutions

2.1 Introduction

This section is based on two articles: (Lelito and Morozov, 2016) and (Lelito and Morozov, 2018a). They both present a classical application of Lie groups to PDEs, which is to find exact solutions of the equations in question. The scheme is to find local symmetries, classify the corresponding algebras via the adjoint representation of the symmetry group and then perform reductions to equations in fewer independent variables. The purpose of classification of symmetry algebras and the concept of the adjoint representation are discussed in Section 2.2.

Section 2.3 is based on the article (Lelito and Morozov, 2016). We find symmetry-invariant solutions to the Gibbons-Tsarev (GT) Eq. (2.3) and use them to find explicit solutions of Pavlov's Eq. (4.2) in Section 2.3.4, and two-component reduction (2.5) of the Benney moments chain in Section 2.3.3. Another application of exact solutions of the GT Eq. (2.3) is to reduce the Ferapontov-Huard-Zhang (FHZ) system (2.12) to a scalar linear second order PDE. We study the reductions of the FHZ system (2.12) in Section 2.3.5. As mentioned later in the text, implicit solution to one of the reductions (with respect to ϕ_2) was noticed by M. Pavlov. It was straightforward to apply the solutions to Pavlov's Eq. (4.2) and two-component reductions (2.5) of the Benney moments chain. We also used the solutions of the GT Eq. (2.3) to reduce the FHZ system (2.12) to scalar linear equations, which we then analysed regarding their integrability in quadratures.

While in a great part the computations from the article (Lelito and Morozov, 2018a), presented in Section 2.4, fall under the same scheme and exploit the same machinery as in (Lelito and Morozov, 2016), the paper demonstrates in particular the applicability of the Miura-type transformation[1] (2.42) between the Khokhlov-Zabolotskaya (KhZ) Eq. (2.32) and the KhZ singular manifold Eq. (2.36). With the use of the transformation (2.42), the symmetry-invariant solutions of the KhZ singular manifold Eq. (2.36) are mapped to new

[1] The original Miura transformation, which relates the KdV Eq. (1.13) with the modified KdV equation, and other similar transformations are nonlocal transformations forming a particular class of Bäcklund transformations.

exact solutions of the KhZ Eq. (2.32). We found symmetry algebra of the KhZ singular manifold Eq. (2.36), classified its one-dimensional subalgebras into optimal system and performed the reductions of the KhZ singular manifold (2.36).

We consider scalar equations ($m = 1$) in two or three dimensions. For the GT Eq. (2.3) the dimension is $n = 2$ and $x = (x^1, x^2) = (x, y)$. For the KhZ singular manifold Eq. (2.36) the dimension is $n = 3$ and $x = (x^1, x^2, x^3) = (t, x, y)$.

Remark 2.1.1. *All contact symmetries of the GT Eq.* (2.3) *and all contact symmetries of the KhZ singular manifold Eq.* (2.36) *appear to be point symmetries, that is, they correspond to vector fields on* $\mathbb{R}^3$ *with coordinates* (x, y, u) *and to vector fields on* $\mathbb{R}^4$ *with coordinates* (t, x, y, w)*, respectively.* ◇

Recall that a generating section $\varphi \in \mathrm{Sym}(\mathcal{E})$ of a point symmetry (or a generating section which is a point symmetry) is of the form

$$\varphi(x, u, u_i) = \eta(x, u) - \sum_{i=1}^{n} \xi^i(x, u) u_i,$$

where η and ξ^i are smooth functions on $J^0(\pi)$, $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. A corresponding evolutionary vector field $\mathbf{E}_\varphi \in \mathrm{sym}(\mathcal{E})$ has the form

$$\mathbf{E}_\varphi = \sum_{|I|\geq 0} D_I(\varphi) \frac{\partial}{\partial u_I},$$

and is a prolongation of the evolutionary derivation

$$\mathbf{E}_\varphi|_{\mathcal{F}^0(\pi)} = \left(\eta(x, u) - \sum_{i=1}^{n} \xi^i(x, u) u_i\right) \frac{\partial}{\partial u} = \varphi(x, u, u_i) \frac{\partial}{\partial u}$$

on $\mathcal{F}^0(\pi)$. If $u = f(x)$ is a solution of $\mathcal{E}$, then its evolution $\tilde{f}(x; \epsilon)$ under the one-parameter group of transformations generated by $\mathbf{E}_\varphi$ is governed by the equation

$$\frac{\partial \tilde{f}}{\partial \epsilon} = \varphi|_{j_1(\tilde{f}(x,\epsilon))}, \qquad \tilde{f}(x, 0) = f(x).$$

Solutions invariant with respect to φ satisfy $\frac{\partial \tilde{f}}{\partial \epsilon} = 0$ and $\tilde{f}(x, \epsilon) = f(x)$ for all ϵ. We conclude as follows. If $\varphi \in \mathrm{Sym}(\mathcal{E})$, then a *$\varphi$-invariant solution to* $\mathcal{E}$ is a solution to the system

$$F_r(x^i, u_I) = 0, \qquad \varphi(x^i, u, u_i) = 0, \qquad r \in \{1, \ldots, R\}. \tag{2.1}$$

The group-invariant solutions form an important class of exact solutions of differential equations (ordinary and partial). Their advantage is that they are obtained in an algorithmic way and the procedure is directly applicable to a given equation (there is no need to recognize the type of the equation or to put it in some special form). In particular, when the considered symmetry is a point symmetry, the equation $\varphi(x^i, u, u_i) = 0$ is a linear first order PDE: it is reasonable to expect that it will be possible to solve it. The solution is substituted into $F_r(x^i, u_I) = 0$, leading to a *reduced* equation, which involves one independent variable less. In the case of the GT Eq. (2.3) the reduced equation is an ODE, in the case of the KhZ singular manifold Eq. (2.36) it is a PDE in two variables. It is not guaranteed that the resulting reduced equation will be solvable in quadratures, but even a reduction to an ODE can be useful, since, for example, it facilitates numerical integration. Once the explicit group-invariant solution is found, a question of its physical importance and stability emerges, but this is a separate problem that we do not tackle in this thesis.

2.2 Classification of Lie subalgebras

Any linear combination of symmetries is again a symmetry, which makes the task of finding all group-invariant solutions rather daunting. When a given Lie algebra is finite dimensional, the quest can be in principle completed. A necessity for some sort of classification becomes more apparent when the symmetry algebra depends on arbitrary functions (is infinite-dimensional) as it is the case with the KhZ singular manifold Eq. (2.36) examined in Section (2.4). The exposition of the problem is based on (Olver, 2000, § 3.3).

The following observations indicate how to approach this problem systematically. Let us consider a Lie group G of transformations acting on $X \times U \subset \mathbb{R}^n \times \mathbb{R}$, a subgroup H of G, and some function $f\colon X \to U$.

Definition 2.2.1 (Group–invariant set)**.** A set $S \subset X \times U$ is H*–invariant* if whenever $s \in S$, and $h \in H$ is such that the element $h \cdot s$ is defined, then $h \cdot s \in S$. ◇

Definition 2.2.2 (Group–invariant function)**.** The function $f(x)$ is H*-invariant*, where H is a subgroup of G, if its graph $\Gamma_f = \{(x, f(x)) \in X \times U \subset \mathbb{R}^n \times \mathbb{R} \colon x \in \mathrm{dom}(f)\}$ is an H–invariant set. ◇

It is easy to see that for any $g \in G$ the function $(g \cdot f)(x)$ is $g\,H\,g^{-1}$–invariant. This observation indicates the need for classification of the group G into non-conjugate subgroups, or equivalently, of the corresponding Lie algebra $\mathfrak{g}$ into non-conjugate subalgebras. Solutions invariant with respect to non-conjugate subalgebras of the full symmetry algebra form an optimal system in the following sense: any symmetry-invariant solution can be obtained via the action of a symmetry group on a solution invariant with respect to one of the non-conjugate subalgebras. The terms defined below will be of use.

Definition 2.2.3 (Group and algebra representation)**.** A *group representation* ρ of a group G on a vector space V is a mapping from G to the general linear group $\mathrm{GL}(V)$ on V, which is a group homomorphism, i.e. $\rho(g\,g') = \rho(g) \cdot \rho(g')$, where $\cdot$ denotes the matrix multiplication and g, g' are elements of G. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a mapping ρ from $\mathfrak{g}$ to the general linear algebra $\mathrm{gl}(V)$ on V, which is an algebra homomorphism, i.e. $\rho([X, Y]) = \rho(X) \cdot \rho(Y) - \rho(Y) \cdot \rho(X)$, where $[X, Y]$ is a commutator of X and Y in $\mathfrak{g}$, and $X, Y \in \mathfrak{g}$, $\rho(X), \rho(Y) \in \mathrm{gl}(V)$. ◇

Definition 2.2.4 (Automorphism)**.** An *automorphism* $f\colon G \to G$ ($f\colon \mathfrak{g} \to \mathfrak{g}$) of a group G (algebra $\mathfrak{g}$) is a bijective group (algebra) homomorphism from G into itself ($\mathfrak{g}$ into itself). The automorphisms arising from conjugations are called *inner automorphisms*. ◇

In order to introduce appropriate tools for classification of a given symmetry algebra $\mathfrak{g}$[2], we study the conjugacy map on the corresponding Lie group G in more detail. For an element $g \in G$, a conjugacy map $C_g\colon G \to G$ defined by the formula $C_g(h) = g\, h\, g^{-1}$ is a diffeomorphism on G and an inner automorphism of G. The differential $dC_g(h)\colon TG|_h \to TG|_{(ghg^{-1})}$ of this diffeomorphism is a linear map between tangent spaces of G. In particular, for the identity element e we have $dC_g(e)\colon TG|_e \to TG|_e$, where $TG|_e$ is identified with $\mathfrak{g}$. The mapping $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, defined as $\mathrm{Ad}\colon g \mapsto dC_g(e)$, where $\mathrm{GL}(\mathfrak{g})$ is the general linear group on the vector space $\mathfrak{g}$, determines a group representation of G called *adjoint representation of a group* G. The notation is $\mathrm{Ad}_g := \mathrm{Ad}(g)$. We have $\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$. The image $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$ of G under the adjoint representation Ad is called the adjoint group of G.

The adjoint representation of a Lie algebra is obtained from considerations regarding the differential of Ad at identity. We have $d\mathrm{Ad}(e)\colon \mathfrak{g} \to \bar{\mathfrak{g}}$, where $\bar{\mathfrak{g}} \subset \mathrm{gl}(\mathfrak{g})$. The mapping $\mathrm{ad} := d\mathrm{Ad}(e)$ determines a Lie algebra representation called *adjoint representation of a Lie algebra* $\mathfrak{g}$. The notation is $\mathrm{ad}_X := \mathrm{ad}(X)$. It can be shown that for a given $X, Y \in \mathfrak{g}$ the map ad_X acts by the formula $\mathrm{ad}_X(Y) = -[X, Y]$.

The formula for the adjoint representation Ad is as follows. For a given vector $X \in \mathfrak{g}$ denote by $\exp(\epsilon X)$ a group element from a one-dimensional subalgebra of G generated by X. Then, $\mathrm{Ad}_{\exp(\epsilon X)}$ is a linear map on the Lie algebra $\mathfrak{g}$ and an inner automorphism of $\mathfrak{g}$, defined for every vector $Y \in \mathfrak{g}$ as follows:

$$\mathrm{Ad}_{\exp(\epsilon X)}Y = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\mathrm{ad}^n_X(Y) = Y - \epsilon\,[X,\, Y] + \frac{\epsilon^2}{2!}\,[X,\, [X,\, Y]] - \frac{\epsilon^3}{3!}\,[X,\, [X,\, [X,\, Y]]] + \dots \quad (2.2)$$

For the purpose of the classification of the algebra $\mathfrak{g}$ into non-conjugate subalgebras, note that the adjoint representation $\mathrm{Ad}_{\exp(\epsilon X)}$ transforms a vector Y generating a subgroup H to the vector $\mathrm{Ad}_{\exp(\epsilon X)}\, Y$ generating the subgroup $g\, H\, g^{-1}$, where $g = \exp(\epsilon X)$.

Definition 2.2.5 (Optimal systems)**.** Let G be a Lie group. An *optimal system of s-parameter subgroups* of G is a list of conjugacy inequivalent s-parameter subgroups with the property that any other s-parameter subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of conjugacy inequivalent s-parameter subalgebras of $\mathfrak{g}$ forms an *optimal system* if every s-parameter subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is equivalent to a unique member $\bar{\mathfrak{h}}$ of the list under the action of some inner automorphism of $\mathfrak{g}$. That is, for every $\mathfrak{h}$ there exists only one $\bar{\mathfrak{h}}$ such that $\mathfrak{h} = \mathrm{Ad}_g(\bar{\mathfrak{h}})$ for some $g \in G$. ◇

[2]The Lie algebra $\mathfrak{g}$ of symmetries of equation $\mathcal{E}$ is precisely $\mathrm{Sym}(\mathcal{E})$ or equivalently $\mathrm{sym}(\mathcal{E})$. The symbol $\mathfrak{g}$ is introduced to emphasize that considerations regarding adjoint representations belong to the theory of abstract Lie groups and algebras and do not need the environment of jets and differential equations.

Although the above definition (and preceding considerations) rely on the correspondence between Lie algebras and Lie groups, there is no such correspondence in the case of infinite-dimensional Lie algebras. In fact, the theory of infinite-dimensional Lie algebras and their connection with Lie pseudogroups is in its early stages, see (Schmid, 2010), (Olver, 2010). The infinite-dimensional symmetry algebra still can be classified into an optimal system of subalgebras with the help of formula (2.2), since its right-hand side is well defined regardless of the existence of the corresponding Lie (pseudo)group. There are, though, certain nontrivial technical difficulties in the course of the classification of infinite-dimensional Lie algebras, which we discuss in the proof of Theorem 2.4.1.

Definition 2.2.6 (Optimal system of group-invariant solutions)**.** An *optimal system of s-parameter group-invariant solutions* to a system of differential equations is a list of solutions $u = f(x)$ with the following properties:

- Each solution in the list is invariant with respect to some s-parameter symmetry group of the system of differential equations.
- If $u = \bar{f}(x)$ is any other s-parameter group-invariant solution, then there exist $g \in G$ such that $\bar{f} = g \cdot f$, where $u = f(x)$ is a solution from the list. $\diamond$

Solutions invariant with respect to an optimal system of s-parameter subgroups (subalgebras) form an optimal system of s-parameter group-invariant solutions. In the next sections we perform a classification of symmetry algebras into optimal systems of one-dimensional subalgebras and find the corresponding group-invariant solutions.

2.3 The Gibbons-Tsarev equation

The Gibbons-Tsarev equation

$$u_{yy} = (u_y + y)\, u_{xx} - u_x\, u_{xy} - 2, \tag{2.3}$$

was obtained in (Gibbons and Tsarev, 1996) as a 2-component reduction of the Benney moments chain (Benney, 1973),

$$A_{n,t} + A_{n+1,x} + n\, A_{n-1}\, A_{0,x} = 0, \qquad n \in \mathbb{N} \cup \{0\}, \tag{2.4}$$

the latter being a classical example of a chain of hydrodynamic type. Namely, suppose that A_2 and A_3 depend functionally on $p := A_0$ and $q := A_1$, that is, $A_2 = R(p, q)$ and $A_3 = S(p, q)$ for some functions R and S. Then for all $n \geq 4$ the moments A_n also depend functionally on A_0 and A_1, thus we can write $A_n = Q_n(p, q)$, where all the functions Q_n may be expressed recurrently in terms of R and S. Substituting $A_2 = R(p, q)$, $A_3 = S(p, q)$ into (2.4) yields an over-determined system

$$\left\{ \begin{array}{lcl} S_q &=& R_p + R_q^2, \\ S_p &=& R_q\,(R_p + p) - 2\,q, \end{array} \right. \tag{2.5}$$

which is compatible whenever $R_{pp} = (R_p + p)\, R_{qq} - R_q\, R_{pq} - 2$. The last equation coincides with (2.3) after renaming $(q, p, R) \mapsto (x, y, u)$. This origin of the GT equation connects it directly with a model (also presented in the above-mentioned Benney's work) meant to describe behaviour of long waves on a shallow, inviscid and incompressible fluid.

Integrability properties of the equation. Actually, in (Gibbons and Tsarev, 1996) the Eq. (2.3) is expressed in terms of the function $z = u + \frac{y^2}{2}$ and it reads

$$z_{yy} + z_x\, z_{xy} - z_y\, z_{xx} + 1 = 0. \tag{2.6}$$

It is related to the non-homogeneous system of hydrodynamic type

$$w_t = w\, v_x - \frac{1}{w - v}, \qquad v_t = w\, v_x + \frac{1}{w - v} \tag{2.7}$$

via the substitution $w = \left(z_y + \sqrt{z_y^2 - 4\, z_x}\right)/2$, $v = \left(z_y - \sqrt{z_y^2 - 4\, z_x}\right)/2$. Moreover, (2.7) admits the conservation law $(w{+}v)_t = (w\, v)_x$ and leads to (2.6) upon $z_x = w{+}v$ and $z_t = w\, v$, cf. (Odesskii and Sokolov, 2013). Integrability of the GT equation can be thus discussed in terms of integrability of non-homogeneous systems of hydrodynamic type (Ferapontov and Fordy, 1997). In (Odesskii and Sokolov, 2013) such systems are called integrable if they admit a Lax pair of the form

$$\psi_x = f(u, \lambda)\psi_\lambda, \qquad \psi_t = g(u, \lambda)\, \psi_\lambda,$$

where $u = (u^1, \ldots, u^m)$. In particular, a Lax pair for the system (2.7) can be expressed in terms of function z as a nonisospectral Lax pair

$$\psi_t = -\frac{1}{z_y + z_x\, \tilde{\lambda} - \tilde{\lambda}^2}\psi_{\tilde{\lambda}}, \qquad \psi_y = \frac{z_x - \tilde{\lambda}}{z_y + z_x\, \tilde{\lambda} - \tilde{\lambda}^2}\psi_{\tilde{\lambda}} \tag{2.8}$$

for (2.6). The GT equation in the form (2.3) was obtained in (Baran et al., 2014) as a symmetry reduction of Pavlov's Eq. (2.23) which we examine in Chapter 4. Moreover, the Lax pair (2.8) is related to the reduction of Pavlov's equation Lax pair (4.3), see (Baran et al., 2015a). The nonlocal symmetries of the GT Eq. (2.3) in the covering

$$\begin{array}{rcl} w_{i,x} & = & w_{i-1,y} + u_x\, w_{i-1,x}, \\ w_{i,y} & = & (u_y + y)\, w_{i-1,x} - (i+1)\, w_{i-2}, \end{array} \qquad i \in \mathbb{Z} \tag{2.9}$$

and the covering itself were found in (Holba et al., 2017), where it is shown that the Lie algebra of these nonlocal symmetries is isomorphic to the Witt algebra. The covering (2.9) can be obtained via substituting $w = \sum \lambda^k\, w_k$ into

$$\begin{array}{rcl} w_x & = & \lambda(w_y + u_x\, w_x), \\ w_y & = & \lambda\, (u_y + y)\, w_x - \partial_\lambda(\lambda^3 w), \end{array} \tag{2.10}$$

which is precisely the Lax pair (2.8), after solving for w_y and w_x, substituting $u = z - \frac{y^2}{2}$, and then a change of variables $\lambda = 1/\tilde{\lambda}$ and $w = \tilde{\lambda}^3 \psi$.

Exact solutions. In (Kaptsov, 2002) and (Kaptsov and Schmidt, 2005) the method of differential constraints was applied to find solutions of the GT equation that are expressible in terms of solutions of Painlevé equations. Elliptic solutions for some related equations were considered in (Marikhin and Sokolov, 2005).

Applications. As an immediate application of the group-invariant solutions of the GT Eq. (2.3) we get explicit formulas for two-component reductions (2.5) of the Benney moments chain. Two further applications are the following. First, since the GT Eq. (2.3) arises as a symmetry reduction of Pavlov's Eq. (2.23) solutions to the GT Eq. (2.3) provide solutions to Pavlov's Eq. (2.23). Second application follows from the fact that the change of variables

$$z = u + \tfrac{1}{2}\, y^2, \tag{2.11}$$

where $z = z(x, y)$, transforms the GT Eq. (2.3) to the first equation of the FHZ system

$$\left\{ \begin{array}{lcl} z_{yy} & = & z_y\, z_{xx} - z_x\, z_{xy} - 1, \\ w_{yy} & = & z_y\, w_{xx} - z_x\, w_{xy}. \end{array} \right. \tag{2.12}$$

This system was shown in (Ferapontov et al., 2012) to produce two-phase solutions for the dispersionless Kadomtsev–Petviashvili equation[3] (dKP). Namely, if functions $P(r, s)$, $Q(r, s)$ satisfy

$$\left\{ \begin{array}{lcl} P_{ss} & = & P_s\, P_{rr} - P_r\, P_{rs} - 1, \\ Q_{ss} & = & P_s\, Q_{rr} - P_r\, Q_{rs}, \end{array} \right. \tag{2.13}$$

then the system

$$\left\{ \begin{array}{lcl} Q_r & = & x + t\,(r + P_r), \\ Q_s & = & y + t\, P_s, \end{array} \right. \tag{2.14}$$

implicitly defines a solution $r(t, x, y)$, $s(t, x, y)$ to the system

$$\left\{ \begin{array}{lcl} r_t & = & r\, r_x + s_y, \\ r_y & = & s_x, \end{array} \right.$$

which is equivalent to the dKP equation

$$r_{yy} = r_{tx} - (r\, r_x)_x. \tag{2.15}$$

Each solution to the GT Eq. (2.3) yields by substituting (2.11) into the FHZ system (2.12) a linear equation for w. We analyse symmetries of the obtained linear equations. Their corresponding reductions appear to be ordinary differential equations equivalent to Airy's equation,

$$v_{xx} = x\, v,$$

Weber's equation

$$v_{xx} = \left(\frac{1}{4}\, x^2 + \lambda\right)\, v,$$

[3] Which is another term for the KhZ Eq. (2.32), see discussion in Section 2.4

Whittaker's equation

$$v_{xx} = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\,\mu^2 - 1}{4\,x^2}\right) v,$$

and Bessel's equation

$$v_{xx} = \left(\frac{1}{4} + \frac{4\,\mu^2 - 1}{4\,x^2}\right) v,$$

see e.g. (Whittaker and Watson, 1996), (Abramowitz and Stegun, 1972). While Airy's equation is not integrable by quadratures, (Kaplansky, 1957), for Weber's equation, Whittaker's equation and Bessel's equation there exist an infinite number of values of the parameters λ, κ, μ such that these equations are integrable, see (Ritt, 1948), (Kovacic, 1986), and (Ramis and Martinet, 1990). Hence, we obtain an infinite number of cases when FHZ system (2.12) is integrable by quadratures. While the corresponding solutions to (2.13), (2.14) describe two-phase solutions for the dKP Eq. (2.15), their final form appears to be too complicated to write it explicitly.

2.3.1 The symmetry algebra of the Gibbons–Tsarev equation

The symmetry algebra of the GT Eq. (2.3) is five-dimensional and is spanned by symmetries presented in Table (2.1). The commutator table for the symmetry algebra is presented in Table 2.2. The Appendix A.1 contains a script of computing the symmetry algebra of the GT Eq. (2.3) in *Maple* with the help of *Jets* package. The adjoint representation for the symmetry algebra of the GT Eq. (2.3) is presented in Table 2.3.

Symmetry	Generating function	Vector field
ϕ_1	$-y\,u_x + 2\,x$	$y\,\frac{\partial}{\partial x} + 2\,x\,\frac{\partial}{\partial u}$
ϕ_2	$-x\,u_x - \frac{2}{3}\,y\,u_y + \frac{4}{3}\,u$	$x\,\frac{\partial}{\partial x} + \frac{2}{3}\,y\,\frac{\partial}{\partial y} + \frac{4}{3}\,u\,\frac{\partial}{\partial u}$
ϕ_3	$-u_x$	$\frac{\partial}{\partial x}$
ϕ_4	$-u_y - y$	$\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial u}$
ϕ_5	1	$\frac{\partial}{\partial u}$

Table 2.1: Symmetry algebra of the GT Eq. (2.3).

Note that ϕ_2 is a scaling symmetry, while ϕ_3 and ϕ_5 denote invariance of the set of solutions with respect to translations of x and u.

$[\phi_i, \phi_j]$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
ϕ_1	0	$\frac{1}{3}\phi_1$	$-2\phi_2$	$-\phi_3$	0
ϕ_2	$-\frac{1}{3}\phi_1$	0	$-\phi_3$	$-\frac{2}{3}\phi_4$	$-\frac{4}{3}\phi_5$
ϕ_3	$2\phi_2$	ϕ_3	0	0	0
ϕ_4	ϕ_3	$\frac{2}{3}\phi_4$	0	0	0
ϕ_5	0	$\frac{4}{3}\phi_5$	0	0	0

Table 2.2: Commutator table for symmetry algebra of the GT Eq. (2.3).

$\mathrm{Ad}_{\exp(\epsilon\phi_i)}\,\phi_j$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
ϕ_1	ϕ_1	$\phi_2 + \frac{\epsilon}{3}\,\phi_1$	$\phi_3 - 2\,\epsilon\,\phi_5$	$\phi_4 - \epsilon\,\phi_3 + \epsilon^2\,\phi_5$	ϕ_5
ϕ_2	$e^{-\frac{1}{2}\,\epsilon}\,\phi_1$	ϕ_2	$e^{-\epsilon}\,\phi_3$	$e^{-\frac{2}{3}\,\epsilon}\,\phi_4$	$e^{-\frac{4}{3}\,\epsilon}\,\phi_5$
ϕ_3	$\phi_1 + 2\,\epsilon\,\phi_5$	$\phi_2 + \epsilon\,\phi_3$	ϕ_3	ϕ_4	ϕ_5
ϕ_4	$\phi_1 + \epsilon\,\phi_3$	$\phi_2 + \frac{2}{3}\,\epsilon\,\phi_4$	ϕ_3	ϕ_4	ϕ_5
ϕ_5	ϕ_1	$\phi_2 + \frac{4}{3}\,\epsilon\,\phi_5$	ϕ_3	ϕ_4	ϕ_5

Table 2.3: Adjoint representation for symmetry algebra of the GT Eq. (2.3).

Theorem 2.3.1. *The optimal system of one-dimensional subalgebras consists of the subalgebras spanned by the following vectors: ϕ_1, ϕ_2, ϕ_3, $\phi_4 + \alpha\,\phi_1$, $\phi_4 + \alpha\,\phi_5$, where α is an arbitrary constant.*

Proof. Proof is obtained by a standard computation, see e.g. (Olver, 2000, § 3.3). The aim is to show that any vector $v = \alpha_1\,\phi_1 + \alpha_2\,\phi_2 + \alpha_3\,\phi_3 + \alpha_4\,\phi_4 + \alpha_5\,\phi_5$ may be transformed under the adjoint representation from Table 2.3 to one of the vectors generating subalgebras of the optimal system (α_i are arbitrary constants). For example, using Table 2.3 we obtain

$$\mathrm{Ad}_{\exp(\epsilon\,\phi_1)}\,(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5) = \left(\left(1 + \frac{\epsilon}{3}\right)\phi_1 + \phi_2 + (1-\epsilon)\,\phi_3 + \phi_4 + (1-\epsilon)^2\,\phi_5\right).$$

Hence, the choice $\epsilon = 1$ allows to map a symmetry $(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5)$ to a simpler (informally speaking) one $(\frac{4}{3}\phi_1 + \phi_2 + \phi_4)$. In course of the proof, if a linear combination of symmetries ϕ_i with coefficients α_i is considered, we assume that these coefficients are nonzero.

1. We start with trying to simplify vectors which are linear combinations of two vectors from Lie algebra under investigation.

    (a) First, consider vectors of the form $\alpha_2\,\phi_2 + \alpha_i\,\phi_i$, where $i \in \{1,\,3,\,4,\,5\}$. Each of yhe vectors may be transformed into $\alpha_2\,\phi_2$ under $\mathrm{Ad}_{\exp(\epsilon_i\phi_i)}(\cdot)$, where $\epsilon_1 = 3\,\frac{\alpha_1}{\alpha_2}$, $\epsilon_3 = \frac{\alpha_3}{\alpha_2}$, $\epsilon_4 = \frac{3}{2}\,\frac{\alpha_4}{\alpha_2}$ and $\epsilon_5 = \frac{3}{4}\,\frac{\alpha_5}{\alpha_2}$.

    (b) The vectors $\alpha_1\,\phi_1 + \alpha_3\,\phi_3$ and $\alpha_1\,\phi_1 + \alpha_5\,\phi_5$ may be transformed to $\alpha_1\,\phi_1$ - first under $\mathrm{Ad}_{\exp(\epsilon\phi_4)}(\cdot)$ for $\epsilon = \frac{\alpha_3}{\alpha_1}$ and second under $\mathrm{Ad}_{\exp(\epsilon\phi_3)}(\cdot)$ for $\epsilon = \frac{\alpha_5}{2\alpha_1}$.

    (c) We may also simplify $\alpha_3\,\phi_3 + \alpha_5\,\phi_5$ into $\alpha_3\,\phi_3$ acting on it with $\mathrm{Ad}_{\exp(\epsilon\phi_1)}$ when $\epsilon = -\frac{\alpha_5}{2\,\alpha_3}$.

    (d) The vector $\alpha_3\,\phi_3 + \alpha_4\,\phi_4$ may be transformed to $\alpha_4\,\phi_4 + \alpha_5\,\phi_5$ under $\mathrm{Ad}_{\exp(\epsilon\phi_1)}(\cdot)$ with $\epsilon = -\frac{\alpha_3}{\alpha_4}$, where $\alpha_5 = -\frac{\alpha_3^2}{\alpha_4}$. Vectors $\alpha_1\,\phi_1 + \alpha_4\,\phi_4$ and $\alpha_4\,\phi_4 + \alpha_5\,\phi_5$ cannot be further simplified.

    There are no more subcases to consider since we have already examined all $10 = \binom{5}{2}$ of them.

2. Now, consider linear combinations of three vectors from the Lie algebra under investigation.

(a) Every vector of the form $\alpha_2\,\phi_2+\alpha_i\,\phi_i+\alpha_j\,\phi_j$ ($i\neq j$ and $i,j\in\{1,3,4,5\}$) may be transformed into $\alpha_2\,\phi_2$.

i. The vectors $\alpha_2\,\phi_2+\alpha_5\,\phi_5+\alpha_i\,\phi_i$ ($i\in\{1,3,4\}$) are transformed into $\alpha_2\,\phi_2+\alpha_i\,\phi_i$ under $\mathrm{Ad}_{\exp(\epsilon\phi_5)}(\cdot)$ when $\epsilon=\frac{3}{4}\frac{\alpha_5}{\alpha_2}$. We already know by (1a) that vectors $\alpha_2\,\phi_2+\alpha_i\,\phi_i$ are equivalent to $\alpha_2\,\phi_2$ under the action of the adjoint representation.

ii. The vectors $\alpha_2\,\phi_2+\alpha_3\,\phi_3+\alpha_4\,\phi_4$ under $\mathrm{Ad}_{\exp(\epsilon\phi_3)}(\cdot)$ for $\epsilon=\frac{\alpha_3}{\alpha_2}$ are transformed to $\alpha_2\,\phi_2+\alpha_4\,\phi_4$, which again may be transformed to $\alpha_2\,\phi_2$ by (1a).

iii. The vector $\alpha_2\,\phi_2+\alpha_1\,\phi_1+\alpha_3\,\phi_3$ under $\mathrm{Ad}_{\exp(\epsilon\phi_1)}(\cdot)$ for $\epsilon=3\,\frac{\alpha_1}{\alpha_2}$ is transformed to $\alpha_2\,\phi_2+\alpha_5\,\phi_5+\alpha_3\,\phi_3$, where $\alpha_5=6\,\frac{\alpha_1\,\alpha_3}{\alpha_2}$, which may be transformed to $\alpha_2\,\phi_2$ as we showed in (2(a)i).

iv. The vector $\alpha_2\,\phi_2+\alpha_1\,\phi_1+\alpha_4\,\phi_4$ under $\mathrm{Ad}_{\exp(\epsilon\phi_4)}(\cdot)$ for $\epsilon=\frac{3}{2}\frac{\alpha_4}{\alpha_2}$ is transformed to $\alpha_2\,\phi_2+\alpha_1\,\phi_1+\alpha_3\,\phi_3$, where $\alpha_3=-\frac{3}{2}\frac{\alpha_1\,\alpha_4}{\alpha_2}$. As showed above in (2(a)iii), the latter may be transformed into $\alpha_2\,\phi_2$.

(b) The vector $\alpha_1\,\phi_1+\alpha_3\,\phi_3+\alpha_5\,\phi_5$ under $\mathrm{Ad}_{\exp(\epsilon\phi_4)}(\cdot)$ for $\epsilon=\frac{\alpha_3}{\alpha_1}$ is transformed to $\alpha_1\,\phi_1+\alpha_5\,\phi_5$, which may be transformed to $\alpha_1\,\phi_1$ as showed in (1b).

(c) The vectors $\alpha_1\,\phi_1+\alpha_3\,\phi_3+\alpha_4\,\phi_4$ and $\alpha_1\,\phi_1+\alpha_4\,\phi_4+\alpha_5\,\phi_5$ may be transformed to $\alpha_1\,\phi_1+\alpha_4\,\phi_4$ which cannot be further simplified. The first one under $\mathrm{Ad}_{\exp(\epsilon\phi_4)}(\cdot)$ with $\epsilon=\frac{\alpha_3}{\alpha_1}$ and the second one under $\mathrm{Ad}_{\exp(\epsilon\phi_3)}(\cdot)$ with $\epsilon=\frac{\alpha_5}{2\,\alpha_1}$.

(d) The vector $\alpha_3\,\phi_3+\alpha_4\,\phi_4+\alpha_5\,\phi_5$ is transformed to $\alpha_4\,\phi_4+\overline{\alpha_5}\,\phi_5$ under $\mathrm{Ad}_{\exp(\epsilon\phi_1)}(\cdot)$ with $\epsilon=-\frac{\alpha_3}{\alpha_4}$, where $\overline{\alpha_5}=\alpha_5-\frac{\alpha_3^2}{\alpha_4}$.

There are no more cases to consider since we examined all $10=\binom{5}{3}$ of them.

3. We are left with linear combinations of four vectors of the Lie algebra.

(a) All vectors of the form $\alpha_2\,\phi_2+\alpha_i\,\phi_i+\alpha_j\,\phi_j+\alpha_k\,\phi_k$ with i, j, k pairwise distinct and i, j, $k=1$, 3, 4, 5, may be transformed to $\alpha_2\,\phi_2$.

i. The vector $\alpha_2\,\phi_2+\alpha_1\,\phi_1+\alpha_3\,\phi_3+\alpha_4\,\phi_4$ is transformed to $\alpha_2\,\phi_2+\alpha_1\,\phi_1+\overline{\alpha_4}\,\phi_4$ under $\mathrm{Ad}_{\exp(\epsilon\phi_4)}(\cdot)$ with $\epsilon=\frac{\alpha_3}{\alpha_1}$, where $\overline{\alpha_4}=\alpha_4-\frac{2}{3}\frac{\alpha_2\,\alpha_3}{\alpha_1}$. Obtained vector may be transformed to $\alpha_2\,\phi_2$, as showed in (2(a)iv).

ii. The vector $\alpha_2\,\phi_2+\alpha_5\,\phi_5+\alpha_i\,\phi_i+\alpha_j\,\phi_j$ ($i\neq j$ and $i,j\in\{1,3,4\}$) is transformed to $\alpha_2\,\phi_2+\alpha_i\,\phi_i+\alpha_j\,\phi_j$ under $\mathrm{Ad}_{\exp(\epsilon\phi_5)}(\cdot)$ with $\epsilon=\frac{3}{4}\frac{\alpha_5}{\alpha_2}$. Obtained vector may be transformed to $\alpha_2\,\phi_2$, as showed in (2a).

(b) The vector $\alpha_1\,\phi_1+\alpha_3\,\phi_3+\alpha_4\,\phi_4+\alpha_5\,\phi_5$ may be transformed to $\alpha_1\,\phi_1+\alpha_4\,\phi_4+\alpha_5\,\phi_5$ under $\mathrm{Ad}_{\exp(\epsilon\phi_4)}(\cdot)$ with $\epsilon=\frac{\alpha_3}{\alpha_1}$. By (2c), it may be transformed into $\alpha_1\,\phi_1+\alpha_4\,\phi_4$. Obtained vector cannot be further simplified.

There are no more cases to consider since we examined all $5=\binom{5}{4}$ of them.

4. Finally, let us consider a vector of the form $\alpha_1\,\phi_1+\alpha_2\,\phi_2+\alpha_3\,\phi_3+\alpha_4\,\phi_4+\alpha_5\,\phi_5$. Under the action of $\mathrm{Ad}_{\exp(\epsilon\phi_5)}$ with $\epsilon=\frac{3}{4}\frac{\alpha_5}{\alpha_2}$ it is transformed to $\alpha_1\,\phi_1+\alpha_2\,\phi_2+\alpha_3\,\phi_3+\alpha_4\,\phi_4$ which may be further transformed into $\alpha_2\,\phi_2$, as proved in (3(a)i).

□

2.3.2 Reductions and invariant solutions

In this section we find solutions of the GT Eq. (2.3) that are invariant with respect to the optimal system obtained in Theorem 2.3.1.

Reduction with respect to $\boldsymbol{\phi_1}$. The ϕ_1-invariant solutions of the GT Eq. (2.3) satisfy

$$\phi_1=-y\,u_x+2\,x=0.$$

Solving the last equation for u_x and integrating gives $u=x^2\,y^{-1}+W(y)$. Substituting this to GT Eq. (2.3) and solving for unknown function $W(y)$ yields

$$u=\frac{x^2}{y}+\beta\,y^3+\gamma, \tag{2.16}$$

where β, γ are arbitrary constants.

Reduction with respect to $\boldsymbol{\phi_2}$. The ϕ_2-invariant solutions of the GT Eq. (2.3) satisfy

$$\phi_2=-x\,u_x-\frac{2}{3}\,y\,u_y+\frac{4}{3}\,u=0.$$

Solving this we get $u=x^{4/3}\,v(\zeta)$ with $\zeta=y\,x^{-2/3}$. Inserting the outcome into the GT Eq. (2.3) yields the ordinary differential equation

$$v_{\zeta\zeta}=\frac{2\,(\zeta\,v_\zeta^2+(2\,v+3\,\zeta^2)\,v_\zeta-2\,\zeta\,v+9)}{8\,\zeta\,v+4\,\zeta^3-9}. \tag{2.17}$$

The point symmetries of this equation are trivial, so the methods of group analysis can not be applied to its integration. The general solution to (2.17) may be extracted from results of (Pavlov and Tsarev, 2014): this solution may be written in the parametric form

$$\left\{\begin{array}{lcl} v &=& -\dfrac{3^{2/3}}{2\,(1+\epsilon_1+\epsilon_2)^{1/3}}\cdot\dfrac{P_2(t)}{(P_3(t))^{2/3}},\\ \zeta &=& -\dfrac{3^{4/3}}{2\,(1+\epsilon_1+\epsilon_2)^{2/3}}\cdot\dfrac{P_4(t)}{(P_3(t))^{4/3}} \end{array}\right. \tag{2.18}$$

with

$$\begin{aligned} P_2(t) &= (\epsilon_1+\epsilon_2\,t)^2+\epsilon_1+\epsilon_2\,t^2,\\ P_3(t) &= (\epsilon_1+\epsilon_2\,t)^3-\epsilon_1-\epsilon_2\,t^3,\\ P_4(t) &= (1+2\,(\epsilon_1+\epsilon_2))\,(\epsilon_1+\epsilon_2\,t)^4+\epsilon_2\,(2\,(1+\epsilon_1-\epsilon_2^2)+\epsilon_2)\,t^4-4\,\epsilon_1\,\epsilon_2^2\,t^3\\ &\quad -2\,\epsilon_1\,\epsilon_2\,(1+\epsilon_1+\epsilon_2)\,t^2-4\,\epsilon_1^2\,\epsilon_2\,t+\epsilon_1\,(\epsilon_1+2\,(1+\epsilon_2-\epsilon_1^2)), \end{aligned} \tag{2.19}$$

where ϵ_1 and ϵ_2 are arbitrary constants and t is a parameter[4]. Since

$$\det \begin{pmatrix} \dfrac{\partial v}{\partial \epsilon_1} & \dfrac{\partial v}{\partial \epsilon_2} \\ \dfrac{\partial v_\zeta}{\partial \epsilon_1} & \dfrac{\partial v_\zeta}{\partial \epsilon_2} \end{pmatrix} = \frac{3^{5/3}(1+\epsilon_1+\epsilon_2)^{2/3}(t-1)^2(\epsilon_1+(1+\epsilon_2)t)^2(1+\epsilon_1+\epsilon_2 t)^2}{8\,(P_3(t))^{8/3}} \not\equiv 0,$$

system (2.18), (2.19) indeed defines the *general* solution to Eq. (2.17). This fact was not proved in (Pavlov and Tsarev, 2014). This solution is very complicated, so we will not use it in the constructions of Sections 2.3.3, 2.3.4, and 2.3.5. Note that eliminating t from (2.18) yields an algebraic dependence between v and ζ, while it seems to be beyond the capacities of the existing systems of symbolic computations to obtain the explicit form of this dependence for *arbitrary* ϵ_1 and ϵ_2.

Reduction with respect to $\boldsymbol{\phi_3}$. For ϕ_3-invariant solutions of the GT Eq. (2.3) we have

$$\phi_3 = -u_x = 0,$$

so they do not depend on x and thus satisfy $u_{yy} = -2$. Hence these solutions are of the form

$$u = -y^2 + \beta\, y + \gamma \tag{2.20}$$

with $\beta, \gamma = \text{const}$.

Reduction with respect to $\boldsymbol{\phi_4 + \alpha\phi_1}$. Solutions of the GT Eq. (2.3) that are invariant w.r.t. $\phi_4 + \alpha\phi_1$ satisfy

$$\phi_4 + \alpha\,\phi_1 = -\alpha\, y\, u_x - u_y - y + 2\,\alpha\, x = 0.$$

Case $\boldsymbol{\alpha \neq 0}$. When $\alpha \neq 0$, we solve this equation for u_x, substitute the output into the GT Eq. (2.3) and obtain the reduced equation

$$u_{yy} = \frac{2\,(x - \alpha\, y^2)}{y\,(2\,x - \alpha\, y^2)}\, u_y - \frac{4\,\alpha\, x^2}{y\,(2\,x - \alpha\, y^2)}.$$

This is a linear ordinary differential equation with x treated as a parameter. Solutions of this equation are of the form

$$u = 2\,\alpha\, x\, y - \frac{2}{3}\,\alpha^2\, y^3 + W_1(x)\cdot|2\,x - \alpha\, y^2|^{\frac{3}{2}} + W_2(x),$$

where $W_1(x)$ and $W_2(x)$ are arbitrary (smooth) functions of x. By substituting this solution to the GT Eq. (2.3) we obtain that $W_1(x) = \beta = \text{const}$ and $W_2(x) = -\alpha^{-1}\, x + \gamma$, $\gamma = \text{const}$. Finally, solution invariant with respect to symmetry $\phi_4 + \alpha\,\phi_1$ is of the form:

$$u = (2\,\alpha\, y - \alpha^{-1})\, x - \frac{2}{3}\,\alpha^2\, y^3 + \beta\,|\alpha\, y^2 - 2\, x|^{\frac{3}{2}} + \gamma. \tag{2.21}$$

[4] We are grateful to M.V. Pavlov for making this connection and for guiding us through (Pavlov and Tsarev, 2014). The details of the reformulation are too cumbersome for presenting here.

Case $\boldsymbol{\alpha = 0}$. When $\alpha = 0$, we have $u_y = -y$, so $u = -\frac{1}{2}\,y^2 + W_1(x)$. But substituting this into the GT Eq. (2.3) gives a contradiction.

Reduction with respect to $\boldsymbol{\phi_4 + \alpha\,\phi_5}$. Solutions of the GT Eq. (2.3) that are invariant w.r.t. $\phi_5 + \beta\phi_4$ satisfy

$$\phi_4 + \alpha\,\phi_5 = -u_y - y + \alpha = 0.$$

It follows that $u = -\frac{1}{2}\,y^2 + \alpha\,y + W(x)$. Substituting this to the GT Eq. (2.3) and solving for $W(x)$ gives a solution of the form

$$u = \frac{1}{2\,\alpha}\,x^2 - \frac{1}{2}\,y^2 + \alpha\,y + \beta\,x + \gamma \tag{2.22}$$

with β, $\gamma = \text{const}$.

2.3.3 Two-component reductions of the Benney moments chain

Renaming $(q, p, R) \mapsto (x, y, u)$ in system (2.5) and substituting for u a solution of the GT Eq. (2.3) into the resulting system

$$\left\{\begin{array}{lcl} S_x &=& u_y + u_x^2, \\ S_y &=& u_x\,(u_y + y) - 2\,x, \end{array}\right.$$

we obtain a compatible system for S. This system has the following solutions that correspond to the invariant solutions (2.16), (2.20), (2.21), (2.22) of the GT equation, respectively:

$$\begin{array}{rcl}
S &=& \dfrac{x^3}{y^2} + 3\,\beta\,x\,y^2 + \delta, \\
S &=& (\beta - 2\,y)\,x + \delta, \\
S &=& \dfrac{\beta\,(3\,\alpha^2\,y - 2)}{\alpha}\,|\alpha\,y^2 - 2\,x|^{\frac{3}{2}} - \frac{1}{4}\,\alpha^2\,(4\,\alpha + 9\,\beta^2)\,y^4 - 4\,x\,y + \frac{1}{\alpha^2}\,x \\
&+& \frac{4}{3}\,\alpha\,y^3 + (\alpha - 9\,\beta^2)\,x^2 + \frac{1}{2\,\alpha}\,(18\,\alpha^2\,\beta^2\,x + 4\,\alpha^3\,x - 1)\,y^2 + \delta, \\
S &=& \dfrac{x^3}{3\,\alpha^2} + \dfrac{\beta\,x^2}{\alpha} + (\alpha + \beta^2 - y)\,x + \alpha\,\beta\,y + \delta,
\end{array}$$

where δ is an arbitrary constant.

2.3.4 Solutions to Pavlov's equation

The GT Eq. (2.3) itself is a symmetry reduction of Pavlov's equation

$$u_{yy} = u_{tx} + u_y\,u_{xx} - u_x\,u_{xy}. \tag{2.23}$$

More precisely, Pavlov's equation has a class of solutions of the form

$$u(t, x, y) = v(\tau, y) - 2\,t\,x - t^2\,y, \tag{2.24}$$

where $\tau = x + t\,y$ and function $v(\tau, y)$ is a solution of the GT Eq. (2.3) with x replaced by τ. We say more about equation (2.23) in Chapter 4, in which this equation is numbered

Eq. (4.2). Since we know four explicit solutions (2.16), (2.20), (2.21), (2.22) of the GT equation, after substituting them into (2.24) we obtain four explicit solutions of Pavlov's Eq. (2.23). They are, respectively,

$$\begin{aligned}
u &= \frac{x^2}{y} + \beta\, y^3 + \gamma,\\
u &= -y^2 + \beta\, y - 2\,t\,x - t^2\, y + \gamma,\\
u &= (2\,\alpha\, y - 2\,t - \alpha^{-1})\,(x + t\,y) - \frac{2}{3}\,\alpha^2\, y^3 + \beta\,|2\,x - \alpha\, y^2 + t\,y|^{\frac{3}{2}} + t^2\, y + \gamma,\\
u &= \frac{1}{2\,\alpha}\,(x + t\,y)^2 - \frac{1}{2}\,y^2 + \alpha\, y + \beta\,(x + t\,y) - 2\,t\,x - t^2\, y + \gamma.
\end{aligned}$$

2.3.5 Reductions of the FHZ system

In this section we study solutions of the FHZ system (2.12) that correspond to the obtained solutions (2.16), (2.20), (2.21), (2.22) of the GT equation. Each solution of the GT Eq. (2.3) yields by substituting (2.11) into (2.12) a linear equation for w. Any linear equation admits trivial symmetries, that is, symmetries of the form $w_0\, \frac{\partial}{\partial w}$, where w_0 is an arbitrary (but fixed) solution of the equation. We consider nontrivial symmetries of the obtained linear equations. These symmetries allow one to reduce the linear pdes with the unknown function w to ordinary differential equations. For each one of these ODEs we indicate all the cases when the ODE is integrable by quadratures.

Reduction corresponding to the solution (2.16) **invariant w.r.t.** $\boldsymbol{\phi_1}$

For the solution (2.16), the second equation of the FHZ system (2.12) takes the form

$$w_{yy} = (3\,\beta\, y^2 + y - x^2\, y^{-2})\, w_{xx} - 2\,x\,y^{-1}\, w_{xy}.$$

After the change of variables $x = \tilde{x}\,\tilde{y}$, $y = \tilde{y}$, $w = \tilde{w}$ and dropping tildes, the last equation acquires the form $w_{yy} = (3\,\beta\, y{+}1)\, y^{-1}\, w_{xx}$. This equation has a nontrivial symmetry $w_x - \lambda\, w$, where λ is an arbitrary constant. The corresponding reduction $w = e^{\lambda\, x}\, v(y)$ gives an ODE $v_{yy} = \lambda^2\,(3\,\beta\, y + 1)\, y^{-1}\, v$, which after the scaling $\tilde{y} = 2\sqrt{3}\,\lambda\,\beta^{1/2}\, y$ and dropping tildes acquires the form of Whittaker's equation

$$v_{yy} = \left(\frac{1}{4} - \frac{\kappa}{y}\right) v \tag{2.25}$$

with $\kappa = -\frac{1}{6}\sqrt{3}\,\lambda\,\beta^{1/2}$. From the results of (Ramis and Martinet, 1990) it follows that Eq. (2.25) is integrable by quadratures whenever $\kappa \in \mathbb{Z}$. Therefore for each choice of β there exists an infinite number of values for λ such that Eq. (2.25) is integrable by quadratures.

Reduction corresponding to the solution (2.20) **invariant w.r.t.** $\boldsymbol{\phi_3}$

Without loss of generality it is possible to put $\beta = 0$ in solution (2.20). Then, the second equation of the FHZ system (2.12) has the form

$$w_{yy} = -y\, w_{xx}. \tag{2.26}$$

The nontrivial symmetries of this equation are

$$\psi_1 = w_x - \lambda\, w, \qquad \psi_2 = 3\,x\,w_x + 2\,y\,w_y, \qquad \psi_3 = 12\,x\,y\,u_y + 3\,x\,w - (4\,y^3 - 9\,x^2)\,u_x,$$

with $\lambda = const.$

- The ψ_1-invariant solution of Eq. (2.26) is of the form $w = e^{\lambda x}\,v(y)$, where v satisfies $v_{yy} = -\lambda^2\,y\,v$. After rescaling $y = -\lambda^{2/3}\,\tilde{y}$ and dropping tildes, the last equation acquires the form of Airy's equation

$$v_{yy} = y\,v,$$

  which is not integrable by quadratures, (Kaplansky, 1957).

- The ψ_2-invariant solution of Eq. (2.26) is of the form $w = v(\eta)$ with $\eta = x\,y^{-3/2}$, where v is a solution of equation

$$v_{\eta\eta} = -\frac{10\,\eta^2}{4\,\eta^3 + 9}\,v_\eta.$$

  The general solution of this equation is

$$v = c_1 + c_2 \int \frac{d\eta}{(4\,\eta^3 + 9)^{5/6}}.$$

  The last integral can not be expressed in elementary functions, (Ritt, 1948).

- The ψ_3-invariant solution of Eq. (2.26) is of the form

$$w = \frac{y}{(4\,y^3 + 9\,x^2)^{5/6}}\,v(\sigma), \qquad \sigma = \frac{4\,y^3 + 9\,x^2}{y^{3/2}},$$

  where $v(\sigma)$ is a solution to an ODE $v_{\sigma\sigma} = -3^{-1}\,\sigma^{-1}\,v_\sigma$. The last equation is integrable by quadratures, its general solution reads $v = c_1 + c_2\,\sigma^{2/3}$, where c_1, $c_2 = \text{const}$. Therefore we have

$$w = \frac{c_1\,y}{(4\,y^3 + 9\,x^2)^{5/6}} + \frac{c_2}{(4\,y^3 + 9\,x^2)^{1/6}}.$$

Reduction corresponding to the solution (2.21) invariant w.r.t. $\boldsymbol{\phi_4 + \alpha\,\phi_1}$

The solution (2.21) of the GT Eq. (2.3) depends on the parameter β. We examine the cases when $\beta = 0$ and $\beta \neq 0$ separately.

Case $\boldsymbol{\beta \neq 0}$. For the solution (2.21) with $\beta \neq 0$, the second equation of the FHZ system (2.12) acquires the form

$$\begin{aligned} w_{yy} &= -\left(3\,\alpha\,\beta\,y\,|2\,x - \beta\,y^2|^{1/2} - 2\,\beta\,x + 2\,\beta^2\,y^2 - y\right)\,w_{xx} \\ &- \left(3\,\alpha\,|2\,x - \beta\,y^2|^{1/2} + 2\,\beta\,y - \beta^{-1}\right)\,w_{xy}. \end{aligned}$$

After the change of variables $x = \frac{1}{2}\,(\tilde{x}^2 + \beta\,\tilde{y}^2)$, $y = \tilde{y}$, $w = \tilde{w}$ and dropping tildes, we get

$$w_{yy} = \beta\,w_{xx} + \frac{1 - 3\,\alpha\,\beta\,x}{\beta\,x}\,w_{xy}.$$

This equation has a nontrivial symmetry $w_y - \lambda\, w$, $\lambda = \text{const}$. The corresponding reduction $w = e^{\lambda\, y}\, v(x)$ yields an ODE, $v_{xx} = \lambda\,(3\,\alpha\,\beta\,x - 1)\,\beta^{-2}\,x^{-1}\,v_x + \lambda^2\,\beta^{-1}\,v$, which, after the change of variables $v = \frac{\lambda}{2\,\beta}\,(3\,\alpha\,x - \ln x)\,\tilde{v}$ and dropping tildes, takes the form

$$v_{xx} = \frac{\lambda}{4\,\beta^2}\left(\lambda\,(9\,\alpha^2 + 4\,\beta) - \frac{6\,\alpha\,\lambda}{\beta\,x} + \frac{\lambda - 2\,\beta}{\beta^2\,x^2}\right) v. \tag{2.27}$$

Analysis of this equation splits into two branches.

(1) The first one corresponds to the case of $(9\,\alpha^2 + 4\,\beta) \neq 0$. Then, the scaling $\tilde{x} = \lambda\,\beta^{-1}\,(9\,\alpha^2 + 4\,\beta)^{1/2}\,x$, after dropping tildes, transforms (2.27) to Whittaker's equation

$$v_{xx} = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{\lambda\,(\lambda - 2\,\beta)}{4\,\beta^2\,x^2}\right) v \tag{2.28}$$

with $\kappa = 3\,\alpha\,\lambda\,\beta^{-2}\,(9\,\alpha^2 + 4\,\beta)^{-1/2}$ and $\mu = \pm\frac{1}{2}\,(\lambda\,\beta^{-2} - 1)$. As it was shown in (Ramis and Martinet, 1990), this equation is integrable by quadratures whenever $\pm\kappa \pm \mu - \frac{1}{2} \in \mathbb{Z}$. Therefore for each choice of α and β there exists an infinite number of values for λ such that Eq. (2.28) is integrable by quadratures.

(2) The second branch corresponds to the case of $9\,\alpha^2 + 4\,\beta = 0$. Then Eq. (2.27) acquires the form

$$v_{xx} = \left(\frac{A}{x} + \frac{B}{x^2}\right) v$$

with $A = \frac{27}{2}\,\alpha^3\,\tilde{\lambda}^2$, $B = \tilde{\lambda}\,(\tilde{\lambda} - 1)$, and $\tilde{\lambda} = \frac{8}{81}\,\lambda\,\alpha^{-4}$. After the change of variables $v = \frac{1}{2}\,A^{-1/4}\,\tilde{x}^{1/2}\,\tilde{v}$, $x = \frac{1}{16}\,A^{-1}\,\tilde{x}^2$ and dropping tildes we obtain Bessel's equation

$$v_{xx} = \left(\frac{1}{4} + \frac{4\,B + \frac{3}{4}}{x^2}\right) v, \tag{2.29}$$

which is integrable by quadratures whenever $B = -\frac{3}{16} + \frac{1}{2}\left(n + \frac{1}{2}\right)^2$, $n \in \mathbb{Z}$, see (Ritt, 1948). Hence, for each choice of α there exists an infinite number of values for λ such that Eq. (2.29) is integrable by quadratures.

Case $\boldsymbol{\beta = 0}$. The second equation of the FHZ system (2.12) that corresponds to solution (2.21) with $\beta = 0$ has the form

$$w_{yy} = (2\,\alpha\,x - 2\,\alpha^2\,y^2 + y)\,w_{xx} - (2\,\alpha\,y - \alpha^{-1})\,w_{xy}.$$

After the change of variables $x = \frac{1}{8}\,\alpha^{-3}\,(2\,\tilde{x} + \tilde{y}^2 + 2\,\tilde{y} - 1)$, $y = -\frac{1}{2}\,\alpha^{-2}\,\tilde{y}$, $w = \tilde{w}$ and dropping tildes, we get

$$w_{yy} = 2\,(x + y)\,w_{xx} + w_x.$$

This equation has a nontrivial symmetry $w_x + w_y - \lambda\,w$, $\lambda = \text{const}$. The corresponding reduction $w = e^{\lambda\,x}\,v(\tau)$ with $\tau = x + y$ yields an ODE

$$v_{\tau\tau} = -((4\,\lambda\,\tau + 1)\,v_\tau + \lambda\,(2\,\lambda\,\tau + 1)\,v)\,(2\,\tau - 1)^{-1},$$

which, after the change of variables $v = \tilde{x}^{-\lambda/2-1/4}\mathrm{e}^{-\tilde{x}/2}\,\tilde{v}$, $\tau = \frac{1}{2}\,(\lambda^{-1}\,\tilde{x}{+}1)$ and dropping tildes, acquires the form of Whittaker's equation

$$v_{xx} = \left(\frac{1}{4} + \frac{3\,\lambda+1}{4\,x} + \frac{(2\,\lambda+1)^2}{8\,x^2}\right)\,v. \tag{2.30}$$

From the results of (Ramis and Martinet, 1990) it follows that this equation is integrable by quadratures whenever $\lambda = 1 \pm 12\,n \pm (128\,n^2 - 32\,n + 6)^{1/2}$, $n \in \mathbb{Z}$. Therefore for each choice of α there exists an infinite number of values for λ such that Eq. (2.30) is integrable by quadratures.

Reduction corresponding to the solution (2.22) invariant w.r.t. $\phi_4 + \alpha\phi_5$

In the case of solution (2.22) we can put $\beta = 0$ without loss of generality. Then, the second equation of the FHZ system (2.12) takes the form

$$w_{yy} = \alpha\,w_{xx} - \alpha^{-1}\,x\,w_{xy}.$$

The nontrivial symmetry admitted by this equation is $w_y - \lambda\,w$, $\lambda = \text{const}$. It leads to the reduction $w = e^{\lambda\,y}\,v(x)$, where v is a solution of an ODE

$$v_{xx} = \lambda\,\alpha^2\,x\,v_x + \lambda^2\,\alpha\,v.$$

Under the change of variables $v = e^{\frac{1}{4}\,\tilde{x}^2}\,U$, $x = \beta^{-1}\,\lambda^{-1/2}\,\tilde{x}$ and dropping tildes the last equation is transformed to Weber's equation

$$U_{xx} = \left(\tfrac{1}{4}\,x^2 + \mu - \tfrac{1}{2}\right)\,U \tag{2.31}$$

with $\mu = \beta^{-1}\,\lambda^{-1/2}$, which is integrable by quadratures whenever $\mu \in \mathbb{Z}$, (Kovacic, 1986). Therefore for each choice of α there exists an infinite number of values for λ such that Eq. (2.31) is integrable by quadratures.

2.3.6 Conclusion

We found a number of exact solutions to the Gibbons–Tsarev equation. Whereas solutions of this equation obtained in (Marikhin and Sokolov, 2005) and (Kaptsov, 2002), (Kaptsov and Schmidt, 2005) are expressed in terms of the Weierstrass elliptic functions and solutions of the Painlevé equations, respectively, our solutions are rational or algebraic functions, which are easier to use in applications. This allowed us to find exact solutions for Pavlov's Eq. (2.23), integrable reductions of the Ferapontov–Huard–Zhang system, and exact solutions of the two-component reduction of the Benney moments chain.

2.4 The Khokhlov-Zabolotskaya singular manifold equation

The KhZ equation, one of the most popular models in nonlinear acoustics,

$$u_{yy} = u_{tx} + u\,u_{xx} + u_x^2, \tag{2.32}$$

was first derived in (Zabolotskaya and Khokhlov, 1969) from the Navier-Stokes equations in order to describe propagation of sound beams in incompressible materials, see also (Bakhvalov et al., 1982). In the KhZ Eq. (2.32) the unknown function $u = u(t, x, y)$ is the deviation from the equilibrium density of a medium. It is assumed that the wave is weakly nonlinear, almost planar and propagates in an isentropic medium with small viscosity. Substituting s_x for u in the above equation leads to the Lin-Reissner-Tsien[5] equation $s_{yy} = s_{tx} + s_x\, s_{xx}$, which was introduced in (Lin et al., 1948), and therefore provides the link between KhZ equation and the problem of a transonic flow of a compressed gas past a thin airfoil (Cook, 1993). The 40 years long history of KhZ equation has been recently commemorated in (Rudenko, 2010), where the physical phenomena which inspired the emergence of the KhZ Eq. (2.32) and its generalizations are comprehensively discussed. The KhZ equation appeared to be useful also in geophysics (for studying seismic waves) (Koshevaya et al., 2005) as well as in dynamics of liquid metals (Xu, 2009). The KhZ Eq. (2.32) is also known as the dispersionless Kadomtsev-Petviashvili (dKP) equation, see e.g. (Zakharov, 1994) for a discussion about its relation to Kadomtsev-Petviashvili (KP) equation. Finally, the equation is related to two other equations examined in this thesis: it is a reduction of Plebański's second heavenly Eq. (3.1), see (Dunajski et al., 2001), and a reduction of Manakov–Santini system (while the other reduction of the system is Pavlov's Eq. (4.2)), see (Manakov and Santini, 2009).

Integrability properties. The KhZ Eq. (2.32) admits a Lax representation defined by the system

$$q_t = (q^2 - u)\, q_x - u_y - q\, u_x, \tag{2.33}$$
$$q_y = q\, q_x - u_x, \tag{2.34}$$

derived in (Kuz′mina, 1967) and then rediscovered in (Kupershmidt and Manin, 1977) and (Gibbons, 1985).

Remark 2.4.1. *System* (2.33), (2.34) *can be rewritten in the form of nonisospectral Lax representation. Namely, suppose that a function* $\psi(t, x, y, q)$ *defines function* q *implicitly, that is, for a constant* $c \in \mathbb{R}$ *the identity* $\psi(t, x, y, q(t, x, y)) \equiv c$ *holds. Differentiating this identity with respect to* t, x, y *yields the system*

$$\begin{aligned} \psi_t &= (q^2 - u)\,\psi_x + (u_y + q\,u_x)\,\psi_q, \\ \psi_y &= q\,\psi_x + u_x\,\psi_q, \end{aligned} \tag{2.35}$$

[5] The equation is also referred to as the potential KhZ equation, see Example 4.1.1.

see (Dunajski et al., 2001) for an interpretation through twistor theory, (Pavlov et al., 2009) for a connection with the theory of integrable hydrodynamic chains, and (Krasil′shchik, 2016) for geometric interpretation in terms of differential coverings. ◇

The representation (2.35) was used in (Manakov and Santini, 2006) to solve a Cauchy problem for the KhZ Eq. (2.32) by a version of ISTM for vector fields. The KhZ Eq. (2.32) is integrable in the sense of hydrodynamic reductions, (Ferapontov and Khusnutdinova, 2004a).

Exact solutions. Due to its extensive utility, KhZ equation is constantly a subject of research. For the purpose of this section, the attempts to find exact solutions are most interesting. In (Kodama, 1988a,b) the Lax representation (2.33), (2.34) was used to find exact solutions to the KhZ Eq. (2.32), while the obtained solutions are invariant with respect to contact symmetries of the KhZ Eq. (2.32) and can be found by means of the method of symmetry reductions. The symmetry-based approach leading to group-invariant solutions of the KhZ Eq. (2.32) was undertaken by Vinogradov and Vorob′ev (1976), Lychagin (1979), and by Sharomet (1989), who classified three- and two-dimensional subalgebras of its symmetry algebra and found corresponding solutions. Moreover, Sharomet proved that the KhZ Eq. (2.32) has no higher symmetries. Later, Ndogmo classified the symmetry algebra of the KhZ Eq. (2.32) into one- and two-dimensional subalgebras and analysed arising reductions, (Ndogmo, 2008, 2009). The symmetry algebra for the KhZ Eq. (2.32) was computed in (Schwarz, 1987). Exact solutions of the KhZ Eq. (2.32) which are polynomial with respect to x were studied in (Xu, 2009). A review of the knowledge of the time about the solutions of the KhZ Eq. (2.32) can be found in (Polyanin and Zaitsev, 2004).

In order to find new exact solutions of the KhZ Eq. (2.32) we study the group-invariant solutions of the equation

$$w_{yy} = w_{tx} + \left(\frac{w_y^2 - w_t\, w_x}{w_x^2}\right) w_{xx}, \tag{2.36}$$

related to the KhZ Eq. (2.32) via the Miura transformation

$$u = \frac{w_y^2}{w_x^2} - \frac{w_t}{w_x}.$$

Then, the above Miura transformation is used to obtain explicit solutions (2.61), (2.62), (2.63), (2.67), (2.72), and implicit solution (2.54), (2.56), (2.57) of the KhZ Eq. (2.32). Among them, solutions (2.61), (2.63), (2.67), (2.72) and (2.54), (2.56), (2.57) are new.

The equation (2.36) can be derived as follows. From (2.34) we conclude that there exists a function $v(t, x, y)$ such that

$$\begin{aligned} v_x &= q, \\ v_y &= \tfrac{1}{2}\, q^2 - u. \end{aligned} \tag{2.37}$$

Excluding q from (2.37) gives

$$u = \tfrac{1}{2}\, v_x^2 - v_y, \tag{2.38}$$

while substituting (2.38) and $q = v_x$ into (2.33) yields equation

$$v_{yy} = v_{tx} + \left(\tfrac{1}{2}\, v_x^2 - v_y\right)\, v_{xx}. \tag{2.39}$$

In its turn Eq. (2.39) admits a Lax representation, (Morozov, 2010),

$$\begin{array}{rcl} w_t &=& \left(\tfrac{1}{2}\, v_x^2 - v_y\right)\, w_x, \\ w_y &=& -v_x\, w_x. \end{array} \tag{2.40}$$

From (2.40) we have

$$\begin{array}{rcl} v_x &=& -\dfrac{w_y}{w_x}, \\ v_y &=& \frac{1}{2}\,\frac{w_y^2}{w_x^2} - \frac{w_t}{w_x}. \end{array} \tag{2.41}$$

Cross-differentiation of (2.41) gives the desired equation (2.36), while substituting (2.41) into (2.38) yields the Miura transformation

$$u = \frac{w_y^2}{w_x^2} - \frac{w_t}{w_x} \tag{2.42}$$

that maps solutions of Eq. (2.36) into solutions of the KhZ Eq. (2.32). Previously, the Miura transformation (2.42) was used in (Morozov, 2008) to find multi-valued solutions for the KhZ Eq. (2.32). Another approach to constructing multi-valued solutions of the KhZ Eq. (2.32) was used in (Kamchatnov and Pavlov, 2016). By means of different considerations, Eq. (2.36) was derived in (Bogdanov and Konopelchenko, 2004) as the dispersionless limit for the KP singular manifold equation, and thus we use the term *the KhZ singular manifold equation* for Eq. (2.36). Singular manifold equations appear in the study of Painlevé property of nonlinear PDEs, see (Weiss, 1983).

Finally, observe that the covering (2.40) provides a Bäcklund transformation between Eq. (2.39) and the KhZ singular manifold Eq. (2.36). When the covering (2.40) is considered together with (2.37) and (2.33), (2.34), the systems determine a Bäcklund transformation between the KhZ Eq. (2.32) and the KhZ singular manifold Eq. (2.36).

2.4.1 Symmetry algebra

The contact symmetry algebra $\mathrm{Sym}_1(\mathcal{E})$ of the KhZ singular manifold Eq. (2.36) is spanned by the following symmetries:

$$\begin{array}{rcl} \phi_0(A_0) &=& -A_0\, w_t - \frac{1}{3}\left(x\, A_0' + \frac{1}{2} y^2\, A_0''\right)\, w_x - \frac{2}{3}\, y\, A_0'\, w_y, \\ \phi_1(A_1) &=& -\frac{1}{2}\, y\, A_1'\, w_x - A_1\, w_y, \\ \phi_2(A_2) &=& -A_2\, w_x, \\ \psi(B) &=& B, \\ \vartheta &=& -2\, x\, w_x - y\, w_y, \end{array} \tag{2.43}$$

where A_0, A_1, A_2 are arbitrary (smooth) functions of t, B is an arbitrary (smooth) function of w, and the prime denotes the derivative w.r.t. the argument of a function. Since the symmetries depend on arbitrary functions, the symmetry algebra spanned by (2.43) is infinite dimensional.

Remark 2.4.2. *The contact symmetry algebras of the KhZ Eq.* (2.32) *and the KhZ singular manifold Eq.* (2.36) *are not isomorphic, since the first algebra depends on three arbitrary functions on* $\mathbb{R}$*, see e.g. (Schwarz, 1987), (Sharomet, 1989), while the second algebra depends on four arbitrary functions on* $\mathbb{R}$*.* $\diamond$

The structure of the algebra spanned by (2.43) and the action of the corresponding adjoint representation of a symmetry group are presented in Table 2.4 and Table 2.5, respectively. Functions G, K, P, Q, R and S appearing in the tables have the following arguments

$$G = G(t), \quad K = K(w), \quad P = P(t,\epsilon), \quad Q = Q(t,\epsilon), \quad R = R(t,\epsilon), \quad S = S(w,\epsilon),$$

and they satisfy equations:

$$P_\epsilon = G_t\,P - G\,P_t, \quad Q_\epsilon = \tfrac{2}{3}\,G_t\,Q - G\,Q_t, \quad R_\epsilon = \tfrac{1}{3}\,G_t\,R - G\,R_t, \quad S_\epsilon = K_w\,S - K\,S_w. \tag{2.44}$$

$[\cdot,\cdot]$	$\phi_0(A_0)$	$\phi_1(A_1)$	$\phi_2(A_2)$	$\psi(B)$	ϑ
$\phi_0(G)$	$\phi_0(A_0'\,G - A_0\,G')$	$\phi_1(A_1'\,G - \frac{2}{3}\,A_1\,G')$	$\phi_2(G\,A_2' - \frac{1}{3}\,A_2\,G')$	0	0
$\phi_1(G)$	$\phi_1(\frac{2}{3}\,A_0'\,G - A_0\,G')$	$\frac{1}{2}\,\phi_2(A_1'\,G - A_1\,G')$	0	0	$\phi_1(G)$
$\phi_2(G)$	$\phi_2(\frac{1}{3}\,G\,A_0' - G'\,A_0)$	0	0	0	$2\,\phi_2(G)$
$\psi(K)$	0	0	0	$\psi(K\,B' - K'B)$	0
ϑ	0	$-\phi_1(A_1)$	$-2\,\phi_2(A_2)$	0	0

Table 2.4: Commutator table for (2.43).

$\mathrm{Ad}_{\exp(\epsilon\phi_i)}\phi_j$	$\phi_0(A_0)$	$\phi_1(A_1)$	$\phi_2(A_2)$	$\psi(B)$	ϑ
$\phi_0(G)$	$\phi_0(P(\epsilon,t))$	$\phi_1(Q(\epsilon,t))$	$\phi_2(R(\epsilon,t))$	$\psi(B)$	ϑ
$\phi_1(G)$	$\star$	$*$	$\phi_2(A_2)$	$\psi(B)$	$\vartheta - \epsilon\phi_1(G)$
$\phi_2(G)$	$\phi_0(A_0) - \epsilon\phi_2(\frac{1}{3}\,GA_0' - G'A_0)$	$\phi_1(A_1)$	$\phi_2(A_2)$	$\psi(B)$	$\vartheta - \epsilon\phi_2(2\,G)$
$\psi(K)$	$\phi_0(A_0)$	$\phi_1(A_1)$	$\phi_2(A_2)$	$\psi(S(\epsilon,w))$	ϑ
ϑ	$\phi_0(A_0)$	$e^{-\epsilon}\phi_1(A_1)$	$e^{-2\epsilon}\phi_2(A_2)$	$\psi(B)$	ϑ

$\star\ =\ \phi_0(A_0) - \epsilon\,\phi_1(\frac{2}{3}\,A_0'\,G - A_0\,G') + \frac{\epsilon^2}{4}\phi_2(\frac{2}{3}\,A_0''G^2 - A_0'G'G - A_0G''G + A_0G'^2)$

$*\ =\ \phi_1(A_1) + \frac{\epsilon}{2}\,\phi_2(A_2'G - A_2G')$

Table 2.5: Adjoint representation for (2.43).

Theorem 2.4.1. *The optimal system of one-dimensional subalgebras of Lie algebra* (2.43) *consists of subalgebras spanned by the following symmetries* ($\alpha \in \mathbb{R}$)*:*

$$\phi_0(1) + \alpha\,\vartheta = w_t - 2\,\alpha\,x\,w_x - \alpha\,y\,w_y,$$

$$\phi_0(1) + \alpha\,\psi(w) = -w_t + \alpha\,w,$$

$$\phi_1(1) + \alpha\,\psi(1) = -w_y + \alpha,$$

$$\phi_2(1) + \alpha\,\psi(1) = -w_x + \alpha,$$

$$\vartheta + \alpha\,\psi(1) = -2\,x\,w_x - y\,w_y + \alpha.$$

Proof. The proof of Theorem 2.4.1 to some extent mimics the proof of Theorem 2.3.1. The difference between them is that while in the case of Theorem 2.3.1 we solve algebraic equations for ϵ, here we need to solve differential equations for the functions G and K appearing in Table 2.5. To give an example of the procedure we show that any symmetry of the form $\phi_0(A_0) + \phi_2(A_2)$ can be transformed under some inner automorphism to $\phi_0(A_0)$. To prove the claim, observe that

$$\mathrm{Ad}_{\exp(\epsilon\,\phi_2(G))}\,(\phi_0(A_0) + \phi_2(A_2)) = \phi_0(A_0) + \phi_2(A_2 - \frac{1}{3}\epsilon\,G\,A_0' + \epsilon\,G' A_0),$$

and so it suffices to choose G in such a way that it solves a linear first order ODE

$$A_2 - \frac{1}{3}\epsilon\,G\,A_0' + \epsilon\,G' A_0 = 0.$$

Indeed, solution to the above equation always exist. The parameter ϵ does not play a role in the existence of the solution and can be chosen arbitrarily. This reasoning is repeated for other combinations of symmetries to the point where it is showed that any symmetry from the Lie algebra spanned by (2.43) is conjugate to a symmetry from the following list

$$\phi_0(A_0)+\alpha\,\vartheta, \qquad \phi_0(A_0)+\alpha\,\psi(B), \qquad \phi_1(A_1)+\alpha\,\psi(B), \qquad \phi_2(A_2)+\alpha\,\psi(B), \qquad \vartheta+\alpha\,\psi(B).$$

We need to make some remarks regarding action of Ad_g on infinite-dimensional Lie algebras of symmetries. Consider for a moment finite-dimensional Lie algebra of vector fields defined on manifold M. Then, for vector fields V and W the following formula holds

$$Ad_{\exp(\epsilon\,V)}(W) = d\Psi_\epsilon(W),$$

where Ψ_ϵ is a local flow of V. In general we can only assume that the flow $\Psi_\epsilon(x)$, where $x \in M$, is defined in an open set of M, and $d\Psi_\epsilon(W)$ is only defined on the image of Ψ_ϵ. In our considerations we will use the right hand side of formula (2.2) in order to compute $Ad_{\exp(\epsilon\,V)}(W)$, since it is always well defined even when the Lie group G does not exist. This choice allows us to avoid discussion about Lie groups of infinite–dimensional Lie algebras.

A proof of the claim that functions A_i and B can be set as $A_i \equiv 1$, $B \equiv 1$ demands more examination. The articles which tackle this problem include (David et al., 1986), (Martina and Winternitz, 1989) and (Ndogmo, 2009). More detailed discussion is provided by De Matteis and Manno (2014), and we prepared our explanation in the vein of their article.

We consider symmetry $\phi_0(A)$ and prove that it can be transformed to $\phi_0(1)$ by an inner automorphism of $\mathrm{Sym}_1(\mathcal{E})$ (that is, by an automorphism of the form $\mathrm{Ad}_{\exp(\epsilon\phi_i(G))}$, see Table 2.5). Function $\phi_0(A)$ is the characteristic of the vector field

$$A(t)\,\partial_t + \frac{1}{3}\left(x\,A'(t) + \frac{1}{2}\,y^2\,A''(t)\right)\,\partial_x + \frac{2}{3}\,y\,A'(t)\,\partial_y. \tag{2.45}$$

Sometimes it will be useful to write the vector fields in the vector form, so that e.g. the vector field (2.45) is denoted as $\left(A(t), \frac{1}{3}\left(x\,A'(t)+\frac{1}{2}\,y^2\,A''(t)\right), \frac{2}{3}\,y\,A'(t)\right)$. First, we take its projection $A(t)\,\partial_t$ on the t-axis. The question is whether for a given symmetry $A(t)\,\partial_t$, there exists a symmetry $G(t)\,\partial_t$, with a local flow $\tau = \tau(t,\epsilon)$, such that $A(t)$ is mapped under this flow to 1. Consider the diffeomorphism $\Phi\colon U(0) \to \tilde{U}(0)$ of the neighbourhood $U(0)$ to another neighbourhood $\tilde{U}(0)$ of $t = 0$ defined by

$$\Phi(t) := \int_0^t \frac{ds}{A(s)}. \tag{2.46}$$

We have $d\Phi(t)\ (A(t)) = 1$, so this diffeomorphism rectifies the vector field $A(t)\,\partial_t$. The next step is to verify whether this diffeomorphism can be embedded into the flow of some vector field $G(t)\partial_t$. The problem of embedding a diffeomorphism into the flow belongs to the theory of functional equations, see discussion in (De Matteis and Manno, 2014, § 3.4).

Definition 2.4.1 (Embedding into a flow)**.** (Zhang, 1992). Let M be a smooth manifold. Let Φ be a diffeomorphism on M. We say that Φ can be *embedded into a flow*, if there exists a flow $\tau\colon M \times \mathbb{R} \to M$ such that $\tau(\cdot, 1) = \Phi$. If Φ is embedded into the flow τ generated by a vector field X, then $\tau(t,\epsilon)$ satisfies

$$\begin{cases} \dfrac{\partial \tau}{\partial \epsilon}(t,\epsilon) = X(\tau(t,\epsilon)), \\ \tau(t,0) = t. \end{cases}$$

$\diamond$

Without loss of generality we make two assumptions about $A(s)$. First, assume that $A(t) > 0$ in some (small) neighbourhood $U(0)$ of $t = 0$. Then $\Phi'(t) > 0$ and Φ is therefore orientation-preserving – this is a basic requirement for a diffeomorphism to embed it into the flow. Moreover, we assume $A(0) > 1$, since otherwise we can consider symmetry $\alpha\,\phi_0(A)$ for positive α big enough to satisfy $\alpha\,A(0) > 1$. The next proposition states that (under the above mentioned assumptions regarding the function $A(s)$), the diffeomorphism $\Phi(t)$ can be embedded into the flow of some vector field.

Proposition 2.4.1. *There exists a vector field $G(t)\,\partial_t$ on some sub-neighbourhood $V(0) \subseteq U(0)$ such that for all $t \in V(0)$ we have*

$$\Phi(t) = \tau(t,1), \tag{2.47}$$

where $\tau(t,\epsilon)$ is the flow of $G(t)\,\partial_t$, in other words, τ is a solution to system

$$\begin{cases} \dfrac{\partial \tau}{\partial \epsilon} = G(\tau), \\ \tau(t,0) = t. \end{cases} \tag{2.48}$$

Proof. System (2.48) yields

$$\int_t^{\tau(t,\epsilon)} \frac{ds}{G(s)} = \epsilon.$$

for $t \neq 0$. Then (2.47) is equivalent to

$$\int_t^{\Phi(t)} \frac{ds}{G(s)} = 1.$$

Differentiating this we have

$$\Phi'(t)\, G(t) = G(\Phi(t)). \tag{2.49}$$

To prove existence of the solution G of this functional equation in the case when A is an analytic function[6] denote $\Phi_0(t) \equiv t$, $\Phi_1(t) \equiv \Phi(t)$, and for $k \geq 1$ define $\Phi_{k+1}(t) \equiv \Phi(\Phi_k(t))$. Recall that $\Phi(0) = 0$ and $\Phi'(0) = (A(0))^{-1} \in (0,1)$. In (McKiernan, 1963), (Zhang, 1992) it was proved that under above conditions the series

$$G(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \Phi_k(t)$$

converges uniformly in some neighbourhood of $t = 0$ and its sum G satisfies (2.47) and (2.49). This finishes the proof of the Proposition 2.4.1. □

Now for function G defined in the Proposition 2.4.1 take the vector field

$$G(t)\,\partial_t + \frac{1}{3}\left(x\, G'(t) + \frac{1}{2}\, y^2\, G''(t)\right) \partial_x + \frac{2}{3}\, y\, G'(t)\, \partial_y \tag{2.50}$$

with the characteristics $\phi_0(G)$. Let $\Psi(t,x,y,\epsilon) = (\tau(t,\epsilon), \xi(t,x,y,\epsilon), \eta(t,y,\epsilon))$ be the flow of (2.50). We claim that $\Psi(t,x,y,1)$ rectifies the vector field (2.45). Indeed, functions τ, ξ, η are solutions to system

$$\begin{cases} \dfrac{\partial \tau}{\partial \epsilon} &=& G(\tau), \\ \dfrac{\partial \xi}{\partial \epsilon} &=& \frac{1}{3}\left(\xi\, G'(\tau) + \frac{1}{2}\,\eta^2\, G''(\tau)\right), \\ \dfrac{\partial \eta}{\partial \epsilon} &=& \frac{2}{3}\,\eta\, G'(\tau) \end{cases}$$

with initial conditions

$$\begin{cases} \tau(t,0) &=& t, \\ \xi(t,x,y,0) &=& x, \\ \eta(t,y,0) &=& y. \end{cases}$$

Therefore, we have

$$\begin{aligned} \xi(t,x,y,\epsilon) &= x\left(\frac{G(\tau)}{G(t)}\right)^{1/3} + \frac{y^2}{6}\,\frac{(G(\tau))^{1/3}}{(G(t))^{4/3}}\,(G'(\tau) - G'(t)), \\ \eta(t,y,\epsilon) &= y\left(\frac{G(\tau)}{G(t)}\right)^{2/3}, \end{aligned}$$

[6] The proof when A is smooth is more tricky, it is given in (Beyer and Channell, 1985).

where we write τ instead of $\tau(t,\epsilon)$. Then, using (2.46), (2.47), and (2.49) we obtain

$$\begin{aligned}\xi(t,x,y,1) &= \frac{x}{(A(t))^{1/3}} - \frac{y^2}{6}\frac{A'(t)}{(A(t))^{4/3}},\\ \eta(t,y,1) &= \frac{y}{(A(t))^{2/3}}.\end{aligned}$$

Now for the Jacobian matrix J of $\Psi(t,x,y,1)$ we have

$$J\cdot\begin{pmatrix} A(t) \\ \frac{1}{3}\left(x\,A'(t)+\frac{1}{2}\,y^2\,A''(t)\right) \\ \frac{2}{3}\,y\,A'(t)\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

In other words, we showed that there exists an inner automorphism of the Lie algebra of symmetries (2.43), which satisfies $Ad_{\exp(\epsilon\,\phi_0(G))}(\phi_0(A)) = (1,0,0)$, as desired.

The argument is repeated almost without change to prove that the vector

$$\frac{1}{2}\,y\,A'(t)\partial_x + A(t)\partial_y$$

with characteristic $\phi_1(A)$ can be mapped under an inner automorphism of $\mathrm{Sym}_1(\mathcal{E})$ to ∂_y. We define a diffeomorphism Φ as

$$\Phi(t) := \int_0^t \frac{ds}{A(s)^{3/2}}$$

and keep the assumptions about $A(t)$ as before. Then, the flow $\Psi(t,x,y,\epsilon)$ of vector (2.50) satisfies

$$\Psi(t,x,y,1) = \left(\Phi(t),\; x\,\Phi'(t)^{-2/3}\,\Phi''(t),\; y\,\Phi'(t)^{2/3}\right).$$

For the Jacobian matrix J of $\Psi(t,x,y,1)$ we have

$$J\cdot\begin{pmatrix}0\\ \frac{1}{2}\,y\,A'(t)\\ A(t)\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

In the case of vector $A(t)\partial_x$ with characteristic $\phi_2(A)$ the diffeomorphism Φ should be defined as

$$\Phi(t) := \int_0^t \frac{ds}{A(s)^3}.$$

Then, the flow $\Psi(t,x,y,\epsilon)$ of vector (2.50) satisfies

$$\Psi(t,x,y,1) = \left(\Phi(t),\; \frac{x}{A(t)} - \frac{y^2}{2}\frac{A'}{A(t)^2},\; \frac{y}{A^2(t)}\right),$$

and the diffeomorphism $\Psi(t,x,y,1)$ rectifies the vector with characteristic $\phi_2(A)$ to the vector with characteristic $\phi_2(1)$.

It remains to show that $B(w)\partial_w$ is mapped under $Ad_{\exp(\epsilon\psi(K))}$ to ∂_w. We define

$$\Phi(w) = \int_0^w \frac{ds}{B(s)}$$

and embed it into the flow $\omega(w,\epsilon)$ of a vector field $K(w)\partial_w$, i.e. $\omega(w,1) = \Phi(w)$ and ω satisfies

$$\begin{cases} \dfrac{\partial\omega}{\partial\epsilon} = K(\omega), \\ \omega(t,0) = w. \end{cases}$$

The existence of such a function K is guaranteed by Proposition 2.4.1. The flow $\Psi(t,x,y,w,\epsilon)$ of the vector field $K(w)\partial_w$ satisfies $\Psi(t,x,y,w,1) = (t,x,y,\Phi(w))$. The diffeomorphism $\Psi(t,x,y,w,1)$ rectifies the vector field $B(w)\partial_w$ to ∂_w.

□

We analyse the reductions corresponding to the optimal system from Theorem 2.4.1. Note that the KhZ singular manifold Eq. (2.36) is trivially satisfied by functions of the form $w(t,x,y) = \alpha\, t\, y + \beta\, x + \gamma\, y + \delta\, x\, y + \mu$, and we refer to them as trivial solutions. The reduction with respect to the fourth symmetry, namely $\phi_2(1) + \alpha\,\psi(1) = -w_x + \alpha$, is omitted since it yields solutions linear with respect to x and hence linear with respect to y.

2.4.2 Reduction w.r.t. $\phi_0(1) + \alpha\, v$

The $(\phi_0(1) + \alpha\,\vartheta)$-invariant solutions of the KhZ singular manifold Eq. (2.36) satisfy Eq. (2.36) and

$$w_t + 2\,\alpha\, x\, w_x + \alpha\, y\, w_y = 0. \tag{2.51}$$

Case $\boldsymbol{\alpha = 0}$. If $\alpha = 0$, we look for solutions that do not depend on t. Hence, solutions of this type satisfy

$$w_{yy} = \frac{w_y^2}{w_x^2}\, w_{xx}. \tag{2.52}$$

We note that this equation admits a Bäcklund auto-transformation $q_x = (w_y)^{-1}$, $q_y = (w_x)^{-1}$, that is, function q is a solution to (2.52) whenever function w is. This is one sense in which Eq. (2.52) is integrable. Moreover, Eq. (2.52) can be mapped to the linear equation

$$v_{\xi\xi} = \left(\frac{\eta}{\xi}\right)^2 v_{\eta\eta} \tag{2.53}$$

by Legendre's transformation

$$x = v_\xi, \qquad y = v_\eta, \tag{2.54}$$

$$w = -v + \xi\, v_\xi + \eta\, v_\eta, \tag{2.55}$$

see e.g. (Bluman and Kumei, 2013). In its turn Eq. (2.53) can be mapped to the wave equation $\tilde{v}_{\tilde{\xi}\tilde{\xi}} = \tilde{v}_{\tilde{\eta}\tilde{\eta}}$ via the point transformation $v = \mathrm{e}^{\tilde{\xi}+\tilde{\eta}}\tilde{v}$, $\xi = \mathrm{e}^{\tilde{\xi}}$, $\eta = \mathrm{e}^{\tilde{\eta}}$. Since the wave equation is integrable by quadratures, we get the general solution to Eq. (2.53) in the form

$$v = \xi\,\eta\,\left(f(\xi\,\eta) + g(\xi\,\eta^{-1})\right), \tag{2.56}$$

where f and g are arbitrary functions of their arguments. Thus (2.55), (2.54), and (2.56) give a parametric form of the (local) general solution to Eq. (2.52). Substituting this solution for w into (2.42) yields a t-independent solution to the KhZ Eq. (2.32) in the implicit form

$$u = \eta^2\xi^{-2} \tag{2.57}$$

combined with (2.54) and (2.56).

Case $\boldsymbol{\alpha \neq 0}$. When $\alpha \neq 0$, we look for solutions of the form $w(t,x,y) = W(\xi,\eta)$, where $\xi = \mathrm{e}^{-2\,\alpha\,t}\,x$ and $\eta = \mathrm{e}^{-\alpha\,t}\,y$. Then function $W(\xi,\eta)$ satisfies the equation:

$$W_{\eta\eta} = \frac{W_\eta}{W_\xi^2}(W_\eta + \alpha\,\eta\,W_\xi)\,W_{\xi\xi} - \alpha\,\eta\,W_{\xi\eta} - 2\,\alpha\,W_\xi. \tag{2.58}$$

This equation possesses four symmetries:

$$\chi_1 = 2\,\xi\,W_\xi + \eta\,W_\eta, \quad \chi_2 = \alpha\,\eta\,W_\xi + 2\,W_\eta, \quad \chi_3 = W_\xi \quad \text{and} \quad \chi_4(B) = B, \tag{2.59}$$

where B is an arbitrary function of W. The commutator table and the adjoint representation for Lie algebra (2.59) are given in Tables 3 and 4. The optimal system of one-dimensional subalgebras for this Lie algebra consists of:

$$\chi_1 + \beta\,\chi_4(1), \qquad \chi_2 + \beta\,\chi_4(1), \qquad \chi_3 + \beta\,\chi_4(1), \qquad \chi_2 + \chi_3 + \beta\,\chi_4(1).$$

$[\cdot,\cdot]$	χ_1	χ_2	χ_3	$\chi_4(\tilde{B})$
χ_1	0	χ_2	$2\chi_3$	0
χ_2	$-\chi_2$	0	0	0
χ_3	$-2\chi_3$	0	0	0
$\chi_4(B)$	0	0	0	$\chi_4(B\tilde{B}' - B'\tilde{B})$

Table 2.6: Commutator table for (2.59).

1. Reduction w.r.t. $\chi_1 + \beta\chi_4(1)$.
   (a) When $\beta = 0$, solutions of Eq. (2.58) that are $(2\,\xi\,W_\xi + \eta\,W_\eta)$-invariant, are of the form $W(\xi,\eta) = F(\zeta)$, where $\zeta = \xi\,\eta^{-2}$. Substitution of $F(\zeta)$ for $W(\xi,\eta)$ in Eq. (2.58) gives $f' = 0$. Hence, it leads to trivial solution of Eq. (2.58) and then, to trivial solution of the KhZ singular manifold Eq. (2.36).

$\mathrm{Ad}_{\exp(\epsilon\chi_i)}\chi_j$	χ_1	χ_2	χ_3	$\chi_4(\tilde{B})$
χ_1	χ_1	$(1-\epsilon)\chi_2$	$(1-2\epsilon)\chi_3$	$\chi_4(\tilde{B})$
χ_2	$\chi_1+\epsilon\chi_2$	χ_2	χ_3	$\chi_4(\tilde{B})$
χ_3	$\chi_1+2\epsilon\chi_3$	χ_2	χ_3	$\chi_4(\tilde{B})$
$\chi_4(B)$	χ_1	χ_2	χ_3	$\chi_4(\hat{B}(\epsilon,w))$

Table 2.7: Adjoint representation for (2.59).

(b) When $\beta \neq 0$, solutions of Eq. (2.58) that are $(2\,\xi\,W_\xi + \eta\,W_\eta + \beta)$-invariant, are of the form $W(\xi,\eta) = -\frac{\beta}{2}\ln(\xi) + F(\zeta)$, where $\zeta = \xi\,\eta^{-2}$. Substituting of $-\frac{\beta}{2}\ln(\xi) + F(\zeta)$ for $W(\xi,\eta)$ in Eq. (2.58) and denoting $F'(\zeta) = G(\zeta)$ gives us

$$G' = -\frac{24\,\zeta^4\,G^3 - 32\,\beta\,\zeta^3\,G^2 + 2\,\alpha\,\beta^2\,\zeta\,G + 6\,\beta^2\,\zeta^2\,G - \alpha\beta^3}{2\,\beta\,\zeta^2\,(2\,\alpha\,\zeta\,G - 8\,\zeta^2\,G + 2\,\beta\,\zeta - \alpha\beta)}.$$

This equation is integrable by quadratures since it admits a rational first integral

$$\frac{(2\,\zeta\,(\zeta-\alpha)\,G + \alpha\,\beta)^2}{(2\,\zeta\,G - \beta)\,(6\,\zeta\,(2\,\zeta-\alpha)\,G - \beta\,(2\,\zeta - 3\,\alpha))} = C, \tag{2.60}$$

$C = \text{const}$. Substituting $w(t,x,y) = -\frac{\beta}{2}\ln(\xi) + F(\zeta) = \alpha\,\beta\,t - \frac{\beta}{2}\ln(x) + F(x\,y^{-2})$ into the Miura transformation (2.42) yields the t-independent solution of the KhZ Eq. (2.32) in the form

$$u = \frac{2\,x\,(8\,x^3\,y^{-6}\,G(x\,y^{-2})^2 - \alpha\,\beta\,(2\,x\,y^{-2}\,G(x\,y^{-2}) - \beta))}{(2\,x\,y^{-2}\,G(x\,y^{-2}) - \beta)^2}, \tag{2.61}$$

where

$$G(\zeta) = \frac{\beta}{2\,\zeta}\,\frac{(\alpha + 4\,C)\,\zeta - \alpha\,(\alpha + 3\,C) \pm \zeta\,(C\,(2\,\zeta + \alpha + 4\,C))^{1/2}}{\zeta^2 - (\alpha + 3\,C)\,(2\,\zeta - \alpha)}.$$

2. Reduction w.r.t. $\chi_2 + \beta\chi_4(1)$.

(a) When $\beta = 0$, solutions of Eq. (2.58), which are $(\alpha\,\eta\,W_\xi + 2\,W_\eta)$-invariant, are of the form $W(\xi,\eta) = F(\zeta)$, where $\zeta = \eta^2 + 4\,\alpha^{-1}\,\xi$. Substitution of $F(\zeta)$ for $W(\xi,\eta)$ in Eq. (2.58) gives us $F' = 0$. Hence, it leads to trivial solution of Eq. (2.58) and then, to trivial solution of the KhZ singular manifold Eq. (2.36).

(b) When $\beta \neq 0$ and assuming that both α and η are positive, $(\alpha\,\eta\,W_\xi - 2\,W_\eta + \beta)$-invariant solutions of Eq. (2.58) are of the form $W(\xi,\eta) = -\frac{1}{2}\beta\,\eta + F(\zeta)$, where $\zeta = \eta^2 - 4\,\alpha^{-1}\,\xi$. Substituting of $-\frac{1}{2}\beta\,\eta + F(\zeta)$ for $W(\xi,\eta)$ in Eq. (2.58) and denoting $F'(\zeta)$ by $G(\zeta)$ leads to:

$$G' = -\frac{24}{\beta^2}\,G^3.$$

The solution is:

$$G(\zeta) = \pm\frac{\beta}{4\sqrt{C\,\beta^2 + 3\,\zeta}}, \qquad C = \text{const.}$$

Substitution of $w(t,x,y) = -\frac{\beta}{2}\,y\,\mathrm{e}^{-\alpha\,t} + F\left(\mathrm{e}^{-2\,\alpha\,t}\,(y^2 - 4\,\alpha^{-1}\,x)\right)$ into the Miura transformation (2.42) gives the following solution of the KhZ Eq. (2.32)

$$u(t,x,y) = \frac{\alpha}{4}\left(2\,\alpha\,y^2 - 4\,x + C\,\alpha\,\beta^2\mathrm{e}^{2\,\alpha\,t}\right), \tag{2.62}$$

which can be found in a more general form in (Xu, 2009, Theorem 3.2).

3. Reduction w.r.t. $\chi_3 + \beta\chi_4(1)$.
   Both cases, $\beta \neq 0$ and $\beta = 0$, lead immediately to trivial solutions.

4. Reduction w.r.t. $\chi_2 + \chi_3 + \beta\chi_4(1)$.
   (a) When $\beta = 0$, $((1+\alpha\,\eta)\,W_\xi + 2\,W_\eta)$-invariant solutions of Eq. (2.58) are of the form $W(\xi,\eta) = F(\zeta)$, where $\zeta = \frac{1}{2}\left(2\,\xi - \frac{\alpha}{2}\,\eta^2 - \eta\right)$. Substitution of $F(\zeta)$ for $W(\xi,\eta)$ in Eq. (2.58) gives $F' = 0$. Hence, it leads to trivial solution of Eq. (2.58) and then, to trivial solution of the KhZ singular manifold Eq. (2.36).

   (b) When $\beta \neq 0$, $((1+\alpha\,\eta)\,W_\xi + 2\,W_\eta + \beta)$-invariant solutions of Eq. (2.58) are of the form $W(\xi,\eta) = -\frac{\beta}{2}\,\eta + F(\zeta)$, where $\zeta = \frac{1}{2}\left(2\,\xi - \frac{\alpha}{2}\,\eta^2 - \eta\right)$. Substituting of $-\frac{\beta}{2}\,y + F(\zeta)$ for $W(\xi,\eta)$ in Eq. (2.58) and denoting $F'(\zeta)$ by $G(\zeta)$ leads to the equation:

$$G' = \frac{6\,\alpha\,G^3}{\beta\,(2\,G + \beta)}.$$

   The solution is:

$$G(\zeta) = \frac{-1 \pm \beta\sqrt{1 - 3\,\alpha\,(\zeta + C)}}{6\,\alpha\,(\zeta + C)}, \qquad C = \text{const.}$$

   Through substituting $w(t,x,y) = -\frac{\beta}{2}\mathrm{e}^{\alpha\,t}\,y + F\left(\frac{1}{2}\,\mathrm{e}^{\alpha\,t}\left(\mathrm{e}^{\alpha\,t}\,(2\,x - \frac{\alpha}{2}\,y^2) - y\right)\right)$ into the Miura transformation (2.42), we obtain:

$$u(t,x,y) = 2\,\alpha\,x - \frac{\alpha^2}{4}\,y^2 + \frac{\mathrm{e}^{2\,\alpha\,t}}{4} + \frac{\mathrm{e}^{2\,\alpha\,t}\beta\,(2\,G + \beta)}{G^2}.$$

   This solution of the KhZ Eq. (2.32) can be written in the form

$$u(t,x,y) = 2\,\alpha\,x - \frac{\alpha^2}{4}\,y^2 + \frac{1}{4}\,\mathrm{e}^{2\,\alpha\,t}\left(1 \pm \mathrm{e}^{-\alpha\,t}\,R^{1/2}\right)^2, \tag{2.63}$$

   where $R = 3\,(\alpha\,y + \mathrm{e}^{\alpha\,t})^2 - 12\,\alpha\,x + C\,\mathrm{e}^{2\,\alpha\,t}$. Solution (2.63) has a form similar to the expression numbered (48) in (Ndogmo, 2009), but it is impossible to transform this expression to the right hand side of (2.63) by any choice of function $f(t)$ appearing in expression (48) and any symmetry transformation of Eq. (2.32).

2.4.3 Reduction w.r.t. $\phi_0(1) + \alpha\,\psi(w)$

The $\phi_0(1) + \alpha\psi(w)$-invariant solutions satisfy the KhZ singular manifold Eq. (2.36) and $w_t = \alpha\, w$. Thus we look for solutions of the form $w(t,x,y) = \mathrm{e}^{\alpha\, t}\, v(x,y)$, where the function $v(x,y)$ satisfies the equation:

$$v_{yy} = \frac{v_y^2 - \alpha\, v\, v_x}{v_x^2}\, v_{xx} + \alpha\, v_x. \tag{2.64}$$

This equation has symmetry $\chi = y\, v_y + \lambda\, x\, v_x + (\lambda - 2)\, v\, \ln(v) + \mu\, v$. We analyse the reduction with respect to this symmetry in the case when $\lambda \neq 2$ and $\lambda = 2$ separately. It should be noted that solution of the KhZ singular manifold Eq. (2.36) of the form $\mathrm{e}^{\alpha\, t}\, v(x,y)$ leads to solutions of KhZ equation which do not depend on t.

Case $\boldsymbol{\lambda} \neq \mathbf{2}$. When $\lambda \neq 2$ we may put $\mu = 0$ w.l.o.g. Then we get

$$v(x,y) = \exp\left(y^{2-\lambda}\, G(\zeta)\right), \qquad \zeta = x\, y^{-\lambda}. \tag{2.65}$$

Substituting this into Eq. (2.64) gives the following ODE of second order:

$$G'' = \frac{(\lambda - 1)\,(G')^2\,(3\,\lambda\,\zeta\, G' + (\lambda - 2)\, G)}{(2\,\lambda\,(\lambda - 2)\,\zeta - \alpha)\, G' + (\lambda - 2)^2\, G^2}. \tag{2.66}$$

In the case $\lambda = 1$ this ODE leads to $G = C_1\,\zeta + C_2$ with $C_1,\ C_2 = \text{const}$, but the corresponding solution to the KhZ Eq. (2.32) appears to be a constant. Eq. (2.66) admits symmetry $\zeta\,\partial_\zeta - G\,\partial_G$ and hence can be reduced to an ODE of the first order. Indeed, introducing new variables $z = \zeta\, G$, $\tau = \ln(\zeta)$ and putting $\zeta_\tau = \Phi(z)$ transforms Eq. (2.66) to

$$\Phi_z = \frac{3\,\lambda\,(\lambda - 1)\,\Phi^3 - (2\,(\lambda^2 + 3\,\lambda - 1)\, z - 3\,\alpha)\,\Phi^2 + z\,((5\,\lambda + 8)\, z + 5\,\alpha)\,\Phi - 2\, z^2\,(3\, z - \alpha)}{\Phi\,((2\,\lambda\,(\lambda - 2)\, z - \alpha)\,\Phi - z\,((\lambda^2 - 4)\, z - \alpha))}.$$

We did not find any cases when this ODE is integrable by quadratures.

Case $\boldsymbol{\lambda} = \mathbf{2}$. In this case, the symmetry χ is of the form $\chi = y\, u_y + 2\, x\, u_x + \mu\, u$. Solving $\chi = 0$ gives a substitution $v(x,y) = y^{-\mu}\, F(\zeta)$ with $\zeta = x\, y^{-2}$ in Eq. (2.64). The reduced equation for $F(\zeta)$ is

$$F'' = \frac{F'^2}{F}\left(1 + \frac{6\,\zeta\, F' + \mu\, F}{(4\mu\,\zeta - \alpha)\, F' + \mu^2\, F}\right).$$

By substituting above $F = e^G$ and then $G' = H$, the equation for F can be further simplified to

$$H' = H^2\left(\frac{6\,\zeta\, H + \mu}{(4\,\mu\,\zeta - \alpha)\, H + \mu^2}\right),$$

which has a first integral

$$\frac{H\,(3\,(2\,\mu\,\zeta - \alpha)\, H + 2\,\mu^2)}{(2\,(\mu\,\zeta - \alpha)\, H + \mu^2)} = C, \qquad C = \text{const}.$$

The general solution of the last equation is of the form

$$H(\zeta) = \frac{\mu^2 \left(2\,C\,\left(\mu\,\zeta - \alpha\right) - 1 \pm \left(C\,(2\,\mu\,\zeta + \alpha) + 1\right)^{1/2}\right)}{4\,\mu^2\,C\,\zeta^2 - (4\,\alpha\,C + 3)\,(2\,\mu\,\zeta - \alpha)}.$$

The corresponding solution to the KhZ Eq. (2.32) is

$$u = 4\,x^2\,y^{-2} + (4\,\mu\,x - \alpha\,y^2)\,\left(H(xy^{-2})\right)^{-1} + \mu^2\,y^2\,\left(H(xy^{-2})\right)^{-2}. \qquad (2.67)$$

This solution belongs to the class of solutions of the KhZ (2.32) of the form $u = x\,r(y\,x^{-1/2}, t)$ considered in (Polyanin and Zaitsev, 2004, § 7.1.2, 5°), where the authors wrote down the PDE for the function r, but did not find any (exact) solutions for this reduced equation.

2.4.4 Reduction w.r.t. $\phi_1(1) + \alpha\,\psi(1)$

The $(\phi_1(1) + \alpha\,\psi(1))$-invariant solutions satisfy the KhZ singular manifold Eq. (2.36) and $w_y = \alpha$. Hence they are of the form $w(t, x, y) = \alpha\,y + v(t, x)$, from which follows that the corresponding solutions to the KhZ Eq. (2.32) are independent of y. Such solutions of the KhZ Eq. (2.32) can be obtained from the first order PDE $u_t + u\,u_x = F(t)$ with an arbitrary function F.

2.4.5 Reduction w.r.t. $\vartheta + \alpha\,\psi(1)$

The $(\vartheta + \alpha\,\psi(1))$-invariant solutions satisfy the KhZ singular manifold Eq. (2.36) and

$$-2\,x\,w_x - \,y\,w_y + \alpha = 0. \qquad (2.68)$$

Case $\boldsymbol{\alpha = 0}$. We look for solutions of the KhZ singular manifold (2.36), which are of the form $w(t, x, y) = W(t, s)$, where $s = y\,x^{-1/2}$. Equation for the function W is

$$W_{ts} = \frac{W_t}{W_s}\,W_{ss} + 6\,s\,W_s. \qquad (2.69)$$

This equation can be rewritten in the form

$$\frac{W_{ss}}{W_s} - \frac{W_{ts}}{W_t} + 6\,s\,\frac{W_s}{W_t} = 0,$$

or

$$\left(\ln\left(\frac{W_s}{W_t}\right)\right)_s + 6\,s\,\frac{W_s}{W_t} = 0. \qquad (2.70)$$

Now, if we put $\ln\left(\frac{W_s}{W_t}\right) = v$, we get $v_s + 6\,s\,e^v = 0$. The general solution of this equation is given in the form $-e^{-v} + 3\,s^2 = h(t)$, where $h(t)$ is an arbitrary function. Going back to function W through the substitution $\ln\left(\frac{W_s}{W_t}\right) = v$ gives the following equation:

$$W_t = \left(3\,s^2 - h(t)\right)\,W_s.$$

Solving this equation with the method of characteristics requires solving $s_t = -3\,s^2 + h(t)$. This is Riccati's equation and cannot be integrated by quadratures for arbitrary $h(t)$. Still, there exists an infinite number of cases when this equation can be integrated, see e.g. (Ritt, 1948, Chapter VI).

Case $\boldsymbol{\alpha \neq 0}$. Solutions to the KhZ singular manifold Eq. (2.36) invariant with respect to $(-2\,x\,w_x - y\,w_y + \alpha)$ are of the form $w(t,x,y) = \frac{\alpha}{2}\ln(x) + W(t,s)$, where $s = x\,y^{-2}$. Function W satisfies the equation

$$W_{ts} = \frac{4\,s^2\,(4\,\alpha\,s + W_t)\;W_s + 2\,\alpha\,s\,(2\,\alpha\,s + W_t)}{4\,s\,W_s\,(s\,W_s + \alpha) + \alpha^2} W_{ss} + \frac{8\,s^3\,(3\,s\,W_s + 4\,\alpha)\;W_s^2 + 2\,\alpha\,s\;(3\,\alpha\,s - W_t)\;W_s - \alpha^2\,W_t}{s\;(4\,s\,W_s\,(s\,W_s + \alpha) + \alpha^2)}. \tag{2.71}$$

The above equation admits the infinite dimensional Lie algebra of symmetries $\chi(A) = A\,W_t - (A's - \frac{1}{6}A'')\,W_s + \frac{\alpha}{12\,s}\,A'' + \frac{\alpha}{6}\,A'$, where $A = A(t)$.

Reduction with respect to the symmetry $\chi(1) = w_t$ leads to the solution of Eq. (2.71) of the form $W(t,s) = F(s)$. If we denote $F'(s) = G(s)$, the following equation holds:

$$G' = -\frac{G\,(12\,s^2\,G^2 + 16\,\alpha\,s\,G + 3\,\alpha^2)}{2\,\alpha\,s\;(4\,s\,G + \alpha)}.$$

The above equation is integrable by quadratures, since it has a first integral

$$\frac{s^3\,G^2}{12\,s^2\,G^2 + 8\,\alpha\,s\,G + \alpha^2} = C, \qquad C = \text{const}.$$

The general solution is of the form

$$G(s) = \frac{\alpha\,(4\,C \pm \sqrt{C\,(s + 4\,C)}}{s\,(s - 12\,C)}.$$

Substituting $w(t,x,y) = \frac{\alpha}{2}\ln(x) + F\,(x\,y^{-2})$ into the Miura transformation (2.42), leads to the following solution of the KhZ Eq. (2.32)

$$u(t,x,y) = \frac{16\,x^4\,G\,(x\,y^{-2})^2}{\left(\alpha\,y^2 + 2\,x\,G\,(x\,y^{-2})\right)^2\,y^2},$$

Its explicit form is

$$u(t,x,y) = \left(\frac{4\,x\,y\,\left(\sqrt{C(x + 4\,C\,y^2)} \mp 4\,C\,y\right)}{2\,y\,\sqrt{C(x + 4\,C\,y^2)} \pm 4\,C\,y^2 \mp x}\right)^2. \tag{2.72}$$

2.4.6 Conclusion

We have studied symmetry-invariant solutions to the KhZ singular manifold Eq. (2.36) and used them to produce exact solutions to the KhZ Eq. (2.32). Solutions (2.61), (2.63), (2.67), (2.72), and the solution defined implicitly by (2.57), (2.54), (2.56) are not presented in the cited literature. Therefore, we have shown that the Bäcklund transformation (2.33), (2.34), (2.37), (2.40), can be applied to find new exact solutions to the KhZ Eq. (2.32).

Chapter 3

Nonlocal symmetries

3.1 Introduction

This chapter is based on the article (Lelito and Morozov, 2018b). We study nonlocal symmetries of Plebański's second heavenly equation (3.1) in an infinite-dimensional covering (3.9) associated to a Lax pair (3.2) with a non-removable spectral parameter. We show that all local symmetries of the equation admit lifts to full-fledged nonlocal symmetries in the infinite-dimensional covering. We find two new infinite hierarchies of commuting nonlocal symmetries in the covering (3.9). Existence of nonlocal commuting flows is one of the more important features of integrable equations. Moreover, we describe the structure of the Lie algebra of the obtained nonlocal symmetries. The following definitions will be of use.

Definition 3.1.1 (Solvable Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_{k-1}]$ for $k > 1$. The algebra $\mathfrak{g}$ is *solvable* if there is k such that $\mathfrak{g}_k = 0$. $\diamond$

Definition 3.1.2 (Ideal)**.** Let $\mathfrak{g}$ be a Lie algebra with a Lie bracket $[\cdot, \cdot]$ and let $\mathfrak{a}$ be its Lie subalgebra. Then $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ if $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$. $\diamond$

Definition 3.1.3 (Direct sum of Lie algebras)**.** Let $\mathfrak{g}$, $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras. Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, where $\oplus$ denotes sum as vector spaces. The Lie bracket

$$[(A_1, B_1), (A_2, B_2)] := ([A_1, A_2], [B_1, B_2]), \qquad A_i \in \mathfrak{a},\ B_i \in \mathfrak{b}, i = 1, 2,$$

defines a Lie algebra structure on $\mathfrak{g}$ called *direct sum* of Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$. In particular, we have $[\mathfrak{a}, \mathfrak{b}] = 0$, where we identified $\mathfrak{a} \oplus 0$ and $0 \oplus \mathfrak{b}$ with $\mathfrak{a}$ and $\mathfrak{b}$, respectively. $\diamond$

Definition 3.1.4 (Semi-direct product of Lie algebras)**.** Let $\mathfrak{g}$, $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras. Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, where $\oplus$ denotes sum as vector spaces. Furthermore, let $\mathfrak{a}$ be an ideal of the algebra $\mathfrak{g}$. The Lie bracket

$$[(A_1, B_1), (A_2, B_2)] := ([A_1, A_2] + [B_1, A_2] - [B_2, A_1], [B_1, B_2]), \quad A_i \in \mathfrak{a},\ B_i \in \mathfrak{b}, i = 1, 2,$$

defines a Lie algebra structure on $\mathfrak{g}$ called *semi-direct product* of Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$, denoted by $\mathfrak{a} \rtimes \mathfrak{b}$ $\diamond$

Definition 3.1.5 (Center)**.** Let $\mathfrak{g}$ be a Lie algebra with a Lie bracket $[\cdot,\cdot]$. The set

$$C := \{A \in \mathfrak{g} \colon [A,B] = 0 \quad \forall\, B \in \mathfrak{g}\}$$

is the *center* of the Lie algebra $\mathfrak{g}$. ◇

3.2 Plebański's second heavenly equation

Plebański's second heavenly equation

$$u_{xz} - u_{ty} - u_{xx}\,u_{yy} + u_{xy}^2 = 0 \tag{3.1}$$

is of importance in general relativity, since it defines metrics, which are solutions of Einstein equations under certain assumptions. Originally Plebański's Eq. (3.1) was derived in (Plebański, 1975) as an equation for a single potential governing anti-self-dual Ricci-flat metric on four dimensional complex manifold, but it also corresponds to a special case of anti-self-dual null-Kähler metric, see (Dunajski, 2010). Plebański's Eq. (3.1) is an example of a nonlinear integrable equation in four independent variables. For instance, it corresponds to one of four types of Monge-Ampére type system in four dimensions, which are all Lax-integrable, see (Doubrov et al., 2017). The Lax pair for Plebański's Eq. (3.1) is

$$\begin{cases} q_t &= (u_{xy} + \lambda)\,q_x - u_{xx}\,q_y, \\ q_z &= u_{yy}\,q_x - (u_{xy} - \lambda)\,q_y, \end{cases} \tag{3.2}$$

with a non-removable parameter λ, and was found by Plebański.

The Lax pair (3.2) was used in (Manakov and Santini, 2009) to solve a Cauchy problem for Plebański's Eq. (3.1) via ISTM. Another integrability property of Plebański's Eq. (3.1) is its tri-Hamiltonian structure, admitted by a first-order nonlinear evolutionary system obtained from Plebański's Eq. (3.1), see (Neyzi et al., 2005). For other classes of four-dimensional integrable equations see for instance (Kruglikov and Morozov, 2015) and (Sergyeyev, 2018).

Expanding the pseudopotential q in the covering (3.2) into a Taylor series $q = \sum_{k=0}^{\infty} \lambda^k\, q_k$ yields a new covering (3.9) with pseudopotentials q_k. The goal of this chapter is to study nonlocal symmetries of Plebański's Eq. (3.1) in this covering. Plebański's Eq. (3.1) has the infinite-dimensional Lie algebra $\mathfrak{s}$ of local contact symmetries and, as it was shown in (Kruglikov and Morozov, 2012), is uniquely defined by this algebra.

We show that all local symmetries of Plebański's Eq. (3.1) have lifts to nonlocal symmetries, that is, to symmetries of the system (3.1), (3.9). Not every integrable equation allows for lifts of local symmetries, see (Krasil′shchik and Vinogradov, 1999, Ch. 6). Also, we find two infinite hierarchies Υ and Γ of nonlocal symmetries such that

$$[\Upsilon, \Upsilon] = 0, \qquad [\Gamma, \Gamma] = 0. \tag{3.3}$$

The existence of an infinite hierarchy of commuting flows is one of the most important properties of integrability, see (Fuchssteiner and Fokas, 1981/82; Krasil′shchik et al., 1986;

Olver, 2000; Ablowitz et al., 1993; Błaszak, 1998) and references therein. In addition, we find the structure of the Lie algebra[1] $\mathfrak{s} \oplus \Upsilon \oplus \Gamma$ (the sum of vector spaces).

3.3 Local symmetries

The local contact symmetries of Plebański's Eq. (3.1) are solutions $\varphi = \varphi(t, x, y, z, u, u_t, u_x, u_y, u_z)$ to the equation

$$\ell_{\mathcal{E}}(\varphi) = D_x\, D_z(\varphi) - D_t\, D_y(\varphi) - u_{xx}\, D_y^2(\varphi) - u_{yy} D_x^2(\varphi) + 2\, u_{xy}\, D_x\, D_y(\varphi) = 0.$$

Direct computations show that the Lie algebra $\mathfrak{s} = \mathrm{Sym}_1(\mathcal{E})$ is generated by the following functions

$$\begin{array}{rcl}
\varphi_0(A) & = & -A_z\, u_t - (x\, A_{tz} + y\, A_{zz})\, u_x + (x\, A_{tt} + y\, A_{tz})\, u_y + A_t\, u_z \\
& & -\frac{1}{6}\, (x^3\, A_{ttt} + 3\, x^2\, y\, A_{ttz} + 3\, x\, y^2\, A_{tzz} + y^3\, A_{zzz}), \\
\varphi_1(A) & = & -A_z\, u_x + A_t\, u_y - \frac{1}{2}\, (x^2\, A_{tt} + 2\, x\, y\, A_{tz} + y^2\, A_{zz}), \\
\varphi_2(A) & = & -x\, A_t - y\, A_z, \\
\varphi_3(A) & = & -A, \\
\psi_1 & = & 3\, u - x\, u_x - y\, u_y, \\
\psi_2 & = & -3\, t\, u_t - x\, u_x - y\, u_y - 3\, z\, u_z, \\
\psi_3 & = & -t\, u_x - z\, u_y,
\end{array}$$

where $A = A(t, z)$, and $B = B(t, z)$ below, are arbitrary functions of t and z. The structure of $\mathfrak{s}$ is given by equations

$$\{\varphi_i(A), \varphi_j(B)\} = \begin{cases} \varphi_{i+j}(A_t B_z - A_z B_t), & i + j \le 4, \\ 0, & i + j > 4, \end{cases} \tag{3.4}$$

$$\{\psi_1, \varphi_j(A)\} = j\, \varphi_j(A), \tag{3.5}$$

$$\{\psi_2, \varphi_j(A)\} = \varphi_j(2\, (3 - j)\, A - 3\, (t\, A_t + z\, A_z)), \tag{3.6}$$

$$\{\psi_3, \varphi_j(A)\} = \begin{cases} \varphi_{j+1}((2 - j)\, A - t\, A_t - z\, A_z), & j \le 2, \\ 0, & j = 3, \end{cases} \tag{3.7}$$

$$\{\psi_1, \psi_2\} = 0, \qquad \{\psi_1, \psi_3\} = \psi_3, \qquad \{\psi_2, \psi_3\} = -2\, \psi_3. \tag{3.8}$$

Remark 3.3.1. *We have $\mathfrak{s} = \mathfrak{s}_\infty \rtimes \mathfrak{s}_\diamond$, where $\mathfrak{s}_\infty$ is generated by $\varphi_i(A)$ and $\mathfrak{s}_\diamond$ is generated by ψ_j. The algebra $\mathfrak{s}_\infty$ admits the following description. Consider the (commutative associative) algebra of truncated polynomials $\mathbb{R}_4[s] = \mathbb{R}[s]/(s^4)$ and the Lie algebra $\mathfrak{h}$ of Hamiltonian vector fields on $\mathbb{R}^2$, (Fuks, 1986). Then $\mathfrak{s}_\infty$ is isomorphic to the Lie algebra $\mathbb{R}_4[s] \otimes \mathfrak{h}$ with the bracket $[f \otimes V, g \otimes W] = fg \otimes [V, W]$ for $f, g \in \mathbb{R}_4[s]$ and $V, W \in \mathfrak{h}$. The isomorphism is given by $\phi_i(A) \mapsto s^i \otimes (A_z \partial_t - A_t \partial_z)$.* ◇

[1] We identify here the Lie algebra $\mathfrak{s}$ of local symmetries with its lift to nonlocal symmetries in the covering (3.9).

3.4 Infinite-dimensional covering, shadows, and nonlocal symmetries

Substituting

$$q = \sum_{k=0}^{\infty} \lambda^k q_k$$

in the system (3.2) yields new (infinite-dimensional) covering

$$\left\{ \begin{array}{lcl} q_{0,t} & = & u_{xy}\, q_{0,x} - u_{xx}\, q_{0,y}, \\ q_{0,z} & = & u_{yy}\, q_{0,x} - u_{xy}\, q_{0,y}, \\ q_{m,t} & = & u_{xy}\, q_{m,x} - u_{xx}\, q_{m,y} + q_{m-1,x}, \\ q_{m,z} & = & u_{yy}\, q_{m,x} - u_{xy}\, q_{m,y} + q_{m-1,y}, \qquad m \ge 1. \end{array} \right. \tag{3.9}$$

Direct computations prove:

Proposition 3.4.1. *Function $v = q$ is a shadow in the covering (3.2).* $\diamond$

Then we have:

Corollary 3.4.1. *Functions $v_k = q_k$, $k \ge 0$, are shadows in the covering (3.9).* $\diamond$

Remark 3.4.1. *To simplify notation, here and below we use φ_{xz} for $\tilde{D}_x \tilde{D}_z(\varphi)$, etc.* $\diamond$

A nonlocal symmetry of Plebański's Eq. (3.1) is an infinite sequence $(\varphi, Q_0, Q_1, \dots, Q_m, \dots)$, where $\varphi = \varphi(x^i, u, u_{x^i}, \dots, q_j, q_{j,x}, q_{j,y}, \dots)$ and $Q_m = Q_m(x^i, u, u_{x^i}, \dots, q_j, q_{j,x}, q_{j,y}, \dots)$, with $m \ge 0$, are solutions to the equation

$$\varphi_{xz} - \varphi_{ty} - u_{yy}\, \varphi_{xx} - u_{xx}\, \varphi_{yy} + 2\, u_{xy}\, \varphi_{xy} = 0 \tag{3.10}$$

and to the linearisation

$$\left\{ \begin{array}{lcl} Q_{0,t} & = & u_{xy}\, Q_{0,x} + q_{0,x}\, \varphi_{xy} - u_{xx}\, Q_{0,y} - q_{0,y}\, \varphi_{xx}, \\ Q_{0,z} & = & u_{yy}\, Q_{0,x} + q_{0,x}\, \varphi_{yy} - u_{xy}\, Q_{0,y} - q_{0,y}\, \varphi_{xy}, \\ Q_{m,t} & = & u_{xy}\, Q_{m,x} + q_{m,x}\, \varphi_{xy} - u_{xx}\, Q_{m,y} - q_{m,y}\, \varphi_{xx} + Q_{m-1,x}, \\ Q_{m,z} & = & u_{yy}\, Q_{m,x} + q_{m,x}\, \varphi_{yy} - u_{xy}\, Q_{m,y} - q_{m,y}\, \varphi_{xy} + Q_{m-1,y}, \qquad m \ge 1, \end{array} \right. \tag{3.11}$$

of the system (3.9).

The nonlocal symmetries of Plebański's Eq. (3.1) which arise as lifts of local symmetries are described by the following theorem.

Theorem 3.4.1. *The local symmetries $\varphi_0(A)$, ..., $\varphi_3(A)$, ψ_1, ψ_2, ψ_3 have the lifts $\Phi_0(A)$, ..., $\Phi_3(A)$, Ψ_1, Ψ_2, Ψ_3 to the nonlocal symmetries in the covering (3.9) defined as $\Phi_i(A) = (\varphi_i(A), \Phi_{i,0}(A), \Phi_{i,1}(A), \dots, \Phi_{i,k}(A), \dots)$, $\Psi_j = (\psi_j, \Psi_{j,0}, \Psi_{j,1}, \dots, \Psi_{j,k}, \dots)$, with*

$$\begin{array}{rcl} \Phi_{0,k}(A) & = & -A_z\, q_{k,t} - (x\, A_{tz} + y\, A_{zz})\, q_{k,x} + (x\, A_{tt} + y\, A_{tz})\, q_{k,y} + A_t\, q_{k,z}, \\ \Phi_{1,k}(A) & = & -A_z\, q_{k,x} + A_t\, q_{k,y}, \\ \Phi_{2,k}(A) & = & \Phi_{3,k}(A) = 0, \\ \Psi_{1,k} & = & -x\, q_{k,x} - y\, q_{k,y} - k\, q_k, \\ \Psi_{2,k} & = & -3\, t\, q_{k,t} - x\, q_{k,x} - y\, q_{k,y} - 3\, z\, q_{k,z} + 2\, k\, q_k, \\ \Psi_{3,k} & = & -t\, q_{k,x} - z\, q_{k,y} + (k+1)\, q_{k+1}. \end{array}$$

Proof. The proof is performed by combining induction and direct calculation. We write down the scheme below and refer to Appendix A.2 for an example of computations in Maple performed for Ψ_2. We use everywhere (3.1) and (3.9).

- Verifying that Φ_0 is a nonlocal symmetry:
  Step 1: Direct substitution of $(\varphi_0(A), \Phi_{0,0}(A))$ for (φ, Q_0) in (3.11).
  Step 2: We assume $(\varphi_0(A), \Phi_{0,0}(A), ..., \Phi_{0,k-1})$ is a solution $(\varphi, Q_0, ..., Q_{k-1})$ of (3.11) for $m = 1, ..., k-1$.
  Step 3: Substitution of $\Phi_{0,k}$ for Q_k in (3.11) and direct calculation leads to identity.

- Verifying that Φ_1 is a nonlocal symmetry:
  Step 1: Direct substitution of $(\varphi_1(A), \Phi_{1,0}(A))$ for (φ, Q_0) in (3.11).
  Step 2: We assume $(\varphi_0(A), \Phi_{1,0}(A), ..., \Phi_{1,k-1})$ is a solution $(\varphi, Q_0, ..., Q_{k-1})$ of (3.11) for $m = 1, ..., k-1$.
  Step 3: Substitution of $\Phi_{1,k}$ for Q_k in (3.11) and direct calculation leads to identity.

- Verifying that Ψ_1 is a nonlocal symmetry:
  Step 1: Direct substitution of $(\psi_1, \Psi_{1,0})$ for (φ, Q_0) in (3.11).
  Step 2: We assume $(\psi_1, \Psi_{1,0}, ..., \Psi_{1,k-1})$ is a solution $(\varphi, Q_0, ..., Q_{k-1})$ of (3.11) for $m = 1, ..., k-1$.
  Step 3: Substitution of $\Psi_{1,k}$ for Q_k in (3.11) and direct calculation leads to identity..

- Verifying that Ψ_2 is a nonlocal symmetry:
  Step 1: Direct substitution of $(\psi_2, \Psi_{2,0})$ for (φ, Q_0) in (3.11).
  Step 2: We assume $(\psi_2, \Psi_{2,0}, ..., \Psi_{2,k-1})$ is a solution $(\varphi, Q_0, ..., Q_{k-1})$ of (3.11) for $m = 1, ..., k-1$.
  Step 3: Substitution of $\Psi_{2,k}$ for Q_k in (3.11) and direct calculation leads to the equation
  $$q_{k,x,z} = q_{k,t,y} + u_{yy}q_{k,x,x} + u_{xx}q_{yy} - 2u_{xy}q_{k,x,y},$$
  which is satisfied since q_k as a shadow must satisfy (3.10).

- Verifying that Ψ_3 is a nonlocal symmetry:
  Step 1: Direct substitution of $(\psi_3, \Psi_{3,0})$ for (φ, Q_0) in (3.11).
  Step 2: We assume $(\psi_3, \Psi_{3,0}, ..., \Psi_{3,k-1})$ is a solution $(\varphi, Q_0, ..., Q_{k-1})$ of (3.11) for $m = 1, ..., k-1$.
  Step 3: Substitution of $\Psi_{3,k}$ for Q_k in (3.11) and direct calculation leads to identity.

□

The first hierarchy of nonlocal commuting symmetries is obtained from shadows.

Theorem 3.4.2. *The shadows $\upsilon_k = q_k$, $k \geq 0$, have the lifts Υ_k to the nonlocal symmetries in the covering* (3.9) *defined as $\Upsilon_k = (q_k, \Upsilon_{k,0}, \Upsilon_{k,1}, \ldots, \Upsilon_{k,m}, \ldots)$ with*

$$\Upsilon_{k,m} = \sum_{s=0}^{m} \langle q_s, q_{k+m+1-s} \rangle,$$

where $\langle a, b \rangle := a_x\, b_y - a_y\, b_x$.

Proof. For a given Υ_k we need to show that $\Upsilon_{k,0}$, which is equal to $\langle q_0, q_{k+1}\rangle$, satisfies the first two equations of the system (3.11). This is a fairly easy and direct calculation and so we omit this. We assume that Υ_{k-1} is a nonlocal symmetry. Consider $\Upsilon_{k,m}$ for $m \geq 1$ and observe that the equation for $Q_{m,t}$ in the system (3.11) can be written as $Q_{m,t} = \langle Q_m, u_x\rangle + \langle q_m, \varphi_x\rangle + Q_{m-1,x}$. We claim that [2]

$$\Upsilon_{k,m,t} = \langle \Upsilon_{k,m}, u_x\rangle + \langle q_m, q_{k,x}\rangle + \Upsilon_{k,m-1,x}.$$

Indeed,

$$\begin{aligned}
\Upsilon_{k,m,t} &= \sum_{s=0}^{m}\langle q_s,\, q_{k+m+1-s}\rangle_t = \sum_{s=0}^{m}\langle q_{s,t},\, q_{k+m+1-s}\rangle + \sum_{s=0}^{m}\langle q_s,\, q_{k+m+1-s,t}\rangle \\
&= \langle\langle q_0, u_x\rangle, q_{k+m+1}\rangle + \sum_{s=1}^{m}\langle\langle q_s, u_x\rangle + q_{s-1,x}, q_{k+m+1-s}\rangle \\
&+ \langle q_0, \langle q_{k+m+1}, u_x\rangle + q_{k+m,x}\rangle + \sum_{s=1}^{m}\langle q_s, \langle q_{k+m+1-s}, u_x\rangle + q_{k+m-s,x}\rangle \\
&= \langle\langle q_0, u_x\rangle, q_{k+m+1}\rangle + \sum_{s=1}^{m}\langle\langle q_s, u_x\rangle, q_{k+m+1-s}\rangle + \underbrace{\sum_{s=1}^{m}\langle q_{s-1,x}, q_{k+m+1-s}\rangle}_{=\sum_{s=0}^{m-1}\langle q_{s,x}, q_{k+m-s}\rangle} \\
&+ \langle q_0, \langle q_{k+m+1}, u_x\rangle\rangle + \underbrace{\langle q_0, q_{k+m,x}\rangle + \sum_{s=1}^{m}\langle q_s, q_{k+m-s,x}\rangle\rangle}_{=\langle q_m, q_{k,x}\rangle + \sum_{s=0}^{m-1}\langle q_s, q_{k+m-s,x}\rangle} + \sum_{s=1}^{m}\langle q_s, \langle q_{k+m+1-s}, u_x\rangle\rangle \\
&= \langle q_m, q_{k,x}\rangle + \Upsilon_{k,m-1,x} + \sum_{s=0}^{m}\underbrace{\langle\langle q_s, u_x\rangle, q_{k+m+1-s}\rangle}_{=\langle\langle q_s, q_{k+m+1-s}\rangle, u_x\rangle - \langle\langle q_{k+m+1-s}, u_x\rangle, q_s\rangle} + \langle\langle q_{k+m+1-s}, u_x\rangle, q_s\rangle \\
&= \langle \Upsilon_{k,m}, u_x\rangle + \langle q_m, q_{k,x}\rangle + \Upsilon_{k,m-1,x}.
\end{aligned}$$

Proof of the equality $\Upsilon_{k,m,z} = \langle \Upsilon_{k,m}, u_y\rangle + \langle q_m, q_{k,y}\rangle + \Upsilon_{k,m-1,y}$ is analogous. □

With the use of the bracket $\langle\cdot,\cdot\rangle$, introduced in Theorem 3.4.2, the equations (3.1), (3.2), (3.9) and (3.11) can be written as

$$u_{xz} = u_{ty} + \langle u_x, u_y\rangle, \qquad \begin{cases} q_t &= \lambda\, q_x + \langle q, u_x\rangle, \\ q_z &= \lambda\, q_y + \langle q, u_y\rangle, \end{cases}$$

$$\begin{cases} q_{0,t} &= \langle q_0, u_x\rangle, \\ q_{0,z} &= \langle q_0, u_y\rangle, \\ q_{m,t} &= \langle q_m, u_x\rangle + q_{m-1,x}, \\ q_{m,z} &= \langle q_m, u_y\rangle + q_{m-1,y}, \end{cases} \qquad \begin{cases} Q_{0,t} &= \langle Q_0, u_x\rangle + \langle q_0, \varphi_x\rangle, \\ Q_{0,z} &= \langle Q_0, u_y\rangle + \langle q_0, \varphi_y\rangle, \\ Q_{m,t} &= \langle Q_m, u_x\rangle + \langle q_m, \varphi_x\rangle + Q_{m-1,x}, \\ Q_{m,z} &= \langle Q_m, u_y\rangle + \langle q_m, \varphi_y\rangle + Q_{m-1,y}. \end{cases}$$

Remark 3.4.2. *The bracket $\langle\cdot,\cdot\rangle$ is a Lie bracket. It endows the space $C^\infty(\mathbb{R}^2)$ of smooth functions on $\mathbb{R}^2$ with the structure of a Lie algebra with the noncentral part isomorphic to the*

[2]See Remark 3.4.1 for notation.

algebra of Hamiltonian vector fields on $\mathbb{R}^2$ *and the center generated by the constant functions. The Hamiltonian vector fields on* $\mathbb{R}^2$ *are of the form* $f_x\,\partial_y - f_y\,\partial_x$, *with* $f \in C^\infty(\mathbb{R}^2)$. *The above-mentioned isomorphism is given by the map* $f \mapsto \langle f, \cdot\rangle$. ◇

We note that the nonlocal symmetries Υ_k are similar to the nonlocal symmetries found in (Morozov and Sergyeyev, 2014) for the four-dimensional Martínez Alonso–Shabat equation, but the Lie brackets on $C^\infty(\mathbb{R}^2)$ here are different from the ones in the constructions of (Morozov and Sergyeyev, 2014).

There is yet another hierarchy of nonlocal commuting symmetries, this time invisible.

Theorem 3.4.3. *Plebański's Eq.* (3.1) *has invisible symmetries (symmetries with the zero shadow) in the covering* (3.9) *defined as*

$$\Gamma_k = (\underbrace{0,\dots,0}_{k}, q_0, q_1, q_2, \dots, q_m, \dots), \qquad k \geq 1.$$

Proof. We verify that

$$\begin{aligned} \Gamma_{k,0,t} &= \langle \Gamma_{k,0}, u_x\rangle + \langle q_0, 0\rangle, \\ \Gamma_{k,m,t} &= \langle \Gamma_{k,m}, u_x\rangle + \langle q_m, 0\rangle + \Gamma_{k,m-1,x}, \qquad m \geq 1. \end{aligned}$$

The function $\Gamma_{1,0}$ is equal to q_0 and the first equation is satisfied by the virtue of (3.9). Now we consider only $k \geq 2$, and so we have $\Gamma_{k,0} \equiv 0$. In general, we have $\Gamma_{k,m} \equiv 0$ for $m \in \{0,1,2,\dots,k-2\}$. Another observation is that $\Gamma_{k,k-1+j} = q_j$, with $j \in \{0,1,2,\dots\}$, and for $\Gamma_{k,k-1+j}$ the equation $\Gamma_{k,m,t} = \langle \Gamma_{k,m}, u_x\rangle + \langle q_m, 0\rangle + \Gamma_{k,m-1,x}$ acquires the form

$$\begin{aligned} q_{0,t} &= \langle q_0, u_x\rangle & j &= 0, \\ q_{j,t} &= \langle q_j, u_x\rangle + q_{j-1,x} & j &\geq 1, \end{aligned}$$

which is again satisfied by the virtue of (3.9). □

3.5 The structure of the algebra of nonlocal symmetries

The structure of the algebra $\tilde{\mathfrak{s}}$ of nonlocal symmetries $\Phi_m(A)$, Ψ_k, Υ_i, Γ_j of Plebański's Eq. (3.1) in the covering (3.9) is described by the following theorem.

Theorem 3.5.1. *The Jacobi brackets of lifts of the local symmetries are lifts of the Jacobi brackets of the corresponding local symmetries, that is, the commutators for* $\Phi_m(A)$, Ψ_k *satisfy equations* (3.4)—(3.8) *with* $\varphi_m(A)$, ψ_k *replaced by* $\Phi_m(A)$, Ψ_k, *respectively. The other Jacobi brackets are*

$$\begin{aligned} \{\Phi_m(A), \Upsilon_k\} &= \{\Phi_m(A), \Gamma_i\} = \{\Upsilon_k, \Upsilon_l\} = \{\Gamma_i, \Gamma_j\} = 0, \qquad 0 \leq m \leq 3, \quad k,l \geq 0, \quad i,j \geq 1, \\ \{\Psi_1, \Upsilon_k\} &= -(k+3)\,\Upsilon_k, \qquad \{\Psi_2, \Upsilon_k\} = 2\,k\,\Upsilon_k, \qquad \{\Psi_3, \Upsilon_k\} = (k+1)\,\Upsilon_{k+1}, \\ \{\Psi_1, \Gamma_i\} &= -(i-1)\,\Gamma_i, \qquad \{\Psi_2, \Gamma_i\} = 2\,(i-1)\,\Gamma_i, \qquad \{\Psi_3, \Gamma_i\} = (i-1)\,\Gamma_{i-1}, \\ \{\Gamma_i, \Upsilon_k\} &= \begin{cases} \Upsilon_{k-i+1}, & k \geq i-1, \\ 0, & k < i-1. \end{cases} \end{aligned}$$

Proof. We will prove that $\{\Phi_0(A), \Upsilon_k\} = 0$. The proofs for the other Jacobi brackets is similar and therefore is presented in Appendix B.

We start with writing down explicitly the formula for computing the Jacobi bracket in the case of some infinite-dimensional vectors Θ and Ω. In order to facilitate applying this formula to vectors $\Phi_0(A)$ and Υ_k, we enumerate coordinates of the vectors Θ and Ω in the following way: $\Theta = (\Theta_{-1}, \Theta_0, \dots, \Theta_m, \dots)$, $\Omega = (\Omega_{-1}, \Omega_0, \dots, \Omega_m, \dots)$. Then the Jacobi bracket is of the form $\{\Theta, \Omega\} = (\{\Theta, \Omega\}_{-1}, \{\Theta, \Omega\}_0, \{\Theta, \Omega\}_1, \dots)$, where

$$\begin{aligned}\{\Theta,\Omega\}_j &= \sum_I \Bigg(\tilde{D}_I(\Theta_{-1}) \frac{\partial}{\partial u_I}(\Omega_j) - \tilde{D}_I(\Omega_{-1}) \frac{\partial}{\partial u_I}(\Theta_j) \\ &+ \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Theta_m) \frac{\partial}{\partial q_{m,I}}(\Omega_j) - \tilde{D}_I(\Omega_m) \frac{\partial}{\partial q_{m,I}}(\Theta_j) \right) \Bigg), \qquad j \geq -1.\end{aligned}$$

As for the coordinates $q_{m,I}$, the functions $\Phi_{0,j}(A)$ depend at most on $q_{j,t}$, $q_{j,x}$, $q_{j,y}$, $q_{j,z}$, q_j or q_{j+1}, and $\Upsilon_{k,j}$ depend at most on $q_{m,x}$, $q_{k+j+1-m,x}$, $q_{m,y}$, $q_{k+j+1-m,y}$ for $m = 0, \dots, j$. The following observations will be used frequently:

$$\begin{aligned}\sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m) \frac{\partial}{\partial q_{m,I}}(\Phi_{0,j}) &= \sum_{l=t,x,y,z} \tilde{D}_l(a_j) \frac{\partial}{\partial q_{j,l}}(\Phi_{0,j}), \qquad j \geq -1,\ i = 0,1,2,3. \\ \sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m) \frac{\partial}{\partial q_{m,I}}(\Upsilon_{k,j}) &= \sum_{m=0}^{j} \Big(\tilde{D}_x(a_m) q_{k+j+1-m,y} - \tilde{D}_x(a_{k+j+1-m}) q_{m,y} \\ &- \tilde{D}_y(a_m) q_{k+j+1-m,x} + \tilde{D}_y(a_{k+j+1-m}) q_{m,x} \Big) \\ &= \sum_{m=0}^{j} \langle a_m, q_{k+j+1-m} \rangle - \langle a_{k+j+1-m}, q_m \rangle.\end{aligned}$$

We have

$$\begin{aligned}\{\Phi_i(A), \Upsilon_k\}_{-1} &= \sum_I \tilde{D}_I(\varphi_i(A)) \underbrace{\frac{\partial}{\partial u_I}(q_k)}_{=0} - \sum_I \tilde{D}_I(q_k) \frac{\partial}{\partial u_I}(\varphi_i(A)) \\ &+ \underbrace{\sum_{m=0}^{\infty} \tilde{D}_I(\Phi_{i,m}(A)) \frac{\partial}{\partial q_{m,I}}(q_k)}_{=\Phi_{i,k}(A)} - \sum_{m=0}^{\infty} \tilde{D}_I(\Upsilon_{k,m}) \underbrace{\frac{\partial}{\partial q_{m,I}}(\varphi_i(A))}_{=0}, \qquad 0 \leq i \leq 3.\end{aligned}$$

It is easy to calculate that $\sum\limits_I \tilde{D}_I(q_k) \frac{\partial}{\partial u_I}(\varphi_i(A)) = \Phi_i^k(A)$ for $i = 0,1,2,3$, and hence $\{\Phi_i(A), \Upsilon_k\}_{-1} = 0$. Then for $j \geq 0$ we get

$$\begin{aligned}\{\Phi_i(A), \Upsilon_k\}_j &= \sum_I \tilde{D}_I(\varphi_i(A)) \underbrace{\frac{\partial}{\partial u_I}(\Upsilon_{k,j})}_{=0} - \sum_I \tilde{D}_I(q_k) \underbrace{\frac{\partial}{\partial u_I}(\Phi_{i,j}(A))}_{=0} \\ &+ \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Phi_{i,m}(A)) \frac{\partial}{\partial q_{m,I}}(\Upsilon_{k,j}) \right) - \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Upsilon_{k,m}) \frac{\partial}{\partial q_{m,I}}(\Phi_{i,j}(A)) \right).\end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
&\{\Phi_0(A), \Upsilon_k\}_j = \\
&= \sum_{l=x,y}\sum_{m=0}^{\infty}\left(\tilde{D}_l\left(\Phi_{0,m}(A)\right)\frac{\partial}{\partial\, q_{m,l}}\left(\Upsilon_{k,j}\right)\right) - \sum_{l=t,x,y,z}\sum_{m=0}^{\infty}\left(\tilde{D}_l\left(\Upsilon_{k,m}\right)\frac{\partial}{\partial\, q_{m,l}}\left(\Phi_{0,j}(A)\right)\right) \\
&= \sum_{m=0}^{j}\Big(\tilde{D}_x\left(\Phi_{0,m}(A)\right)q_{k+j+1-m,y} - \tilde{D}_x\left(\Phi_{0,k+j+1-m}(A)\right)q_{m,y} - \tilde{D}_y\left(\Phi_{0,m}(A)\right)q_{k+j+1-m,x} \\
&+ \tilde{D}_y\left(\Phi_{0,k+j+1-m}(A)\right)q_{m,x}\Big) - \sum_{l=t,x,y,z}\tilde{D}_l\left(\Upsilon_{k,j}\right)\frac{\partial}{\partial\, q_{j,l}}\left(\Phi_{0,j}(A)\right) \\
&= \sum_{m=0}^{j}\left(\langle\Phi_{0,m}(A), q_{k+j+1-m}\rangle - \langle\Phi_{0,k+j+1-m}(A), q_m\rangle\right) \\
&- \sum_{l=t,x,y,z}\tilde{D}_l\left(\sum_{m=0}^{j}\langle q_m, q_{k+j+1-m}\rangle\right)\frac{\partial}{\partial\, q_{j,l}}\left(\Phi_{0,j}(A)\right) \\
&= \sum_{m=0}^{j}\Bigg(\langle\Phi_{0,m}(A), q_{k+j+1-m}\rangle - \langle\Phi_{0,k+j+1-m}(A), q_m\rangle \\
&- \sum_{l=t,x,y,z}\left(\langle q_{m,l}, q_{k+j+1-m}\rangle + \langle q_m, q_{k+j+1-m,l}\rangle\right)\frac{\partial}{\partial\, q_{j,l}}\left(\Phi_{0,j}(A)\right)\Bigg) \\
&= \sum_{m=0}^{j}\left(\langle\Phi_{0,m}(A), q_{k+j+1-m}\rangle - \langle\Phi_{0,k+j+1-m}(A), q_m\rangle\right. \\
&+ \left.\langle\Phi_{0,k+j+1-m}(A), q_m\rangle - \langle\Phi_{0,m}(A), q_{k+j+1-m}\rangle\right) = 0,
\end{aligned}
$$

as claimed. □

Denote by Υ the Lie algebra generated by Υ_i, and by Γ the Lie algebra generated by Γ_j. Identify $\mathfrak{s}_\infty$ and $\mathfrak{s}_\diamond$ with the Lie algebra generated by $\Phi_m(A)$ and the Lie algebra generated by Ψ_k, respectively. Finally, denote by $\tilde{\mathfrak{s}}$ the sum as vector spaces $\tilde{\mathfrak{s}} = \mathfrak{s}_\infty \oplus \mathfrak{s}_\diamond \oplus \Upsilon \oplus \Gamma$. We recapitulate the commutator relations of the Lie algebra $\tilde{\mathfrak{s}}$.

$$
\begin{array}{rclrclrcl}
[\mathfrak{s}_\infty, \mathfrak{s}_\infty] &=& \mathfrak{s}_\infty, & [\mathfrak{s}_\diamond, \mathfrak{s}_\infty] &=& \mathfrak{s}_\infty, & [\mathfrak{s}_\diamond, \mathfrak{s}_\diamond] &=& \mathrm{span}\{\Psi_3\} \subset \mathfrak{s}_\diamond, \\
[\Upsilon, \Upsilon] &=& 0 & [\Gamma, \Gamma] &=& 0 & [\mathfrak{s}_\infty, \Upsilon] &=& [\mathfrak{s}_\infty, \Gamma] = 0 \\
[\mathfrak{s}_\diamond, \Upsilon] &=& \Upsilon, & [\mathfrak{s}_\diamond, \Gamma] &=& \Gamma, & [\Upsilon, \Gamma] &=& \Upsilon.
\end{array}
$$

The algebra of lifts of local symmetries is the semi-direct product $\mathfrak{s} = \mathfrak{s}_\infty \rtimes \mathfrak{s}_\diamond$, where $\mathfrak{s}_\infty = [[\mathfrak{s}, \mathfrak{s}], [\mathfrak{s}, \mathfrak{s}]]$ is an infinite-dimensional ideal of $\mathfrak{s}$. Moreover, since $[[\mathfrak{s}_\diamond, \mathfrak{s}_\diamond], [\mathfrak{s}_\diamond, \mathfrak{s}_\diamond]] = 0$, the subalgebra $\mathfrak{s}_\diamond$ is a three-dimensional solvable Lie algebra. The subalgebra Υ is an ideal of $\mathfrak{s}_\diamond \oplus \Upsilon \oplus \Gamma$, while Γ is an ideal of $\mathfrak{s}_\diamond \oplus \Gamma$ (with $\oplus$ being sum of vector spaces).

Remark 3.5.1. *The Lie algebra of nonlocal symmetries of Plebański's Eq.* (3.1) *has the structure*

$$\tilde{\mathfrak{s}} = (\mathfrak{s}_\infty \oplus (\Upsilon \rtimes \Gamma)) \rtimes \mathfrak{s}_\diamond,$$

where $\oplus$ denotes a direct sum, and $\rtimes$ denotes a semi-direct product of Lie algebras. ◇

Conclusion

The second heavenly equation possesses an interesting structure of the Lie algebra of its nonlocal symmetries in the covering (3.9). All local symmetries can be lifted to nonlocal ones, to constitute a subalgebra $\mathfrak{s}$. There are two infinite hierarchies of commuting nonlocal symmetries, namely Υ_k with $k \geq 0$ and Γ_i with $i \geq 1$. The detailed structure of the Lie algebra is given by Theorem 3.5.1.

Chapter 4

Nonlocal conservation laws

4.1 Introduction

In this chapter, based on the article (Lelito and Morozov, 2018c), we study five three-dimensional Lax-integrable equations: Pavlov's (4.2), r-th dispersionless Dym (rdDym) (4.6), modified Veronese web (mVw) (4.10), universal hierarchy (UH) (4.8) and Veronese web (Vw) (4.12) equation[1]. They all belong to a class of second order quasilinear equations of the form

$$f_{11}\, u_{tt} + f_{22}\, u_{xx} + f_{33}\, u_{yy} + 2\, f_{12}\, u_{tx} + 2\, f_{13}\, u_{ty} + 2\, f_{23}\, u_{xy} = 0, \tag{4.1}$$

where f_{ij} are some functions depending on the first derivatives of u. Moreover, they all possess Lax pairs of the form[2]

$$\psi_t = F(\psi_x, u_t, u_x, u_y), \qquad \psi_y = G(\psi_x, u_t, u_x, u_y),$$

with non-removable parameters. Admitting such a Lax pair by an equation of the form (4.1) is equivalent to integrability in the hydrodynamic sense, see (Ferapontov and Khusnutdinova, 2004b) and (Burovskiy et al., 2010). If the equation is in addition nondegenerate and hyperbolic, then the Lax-integrability is equivalent to integrability considered within twistor theory. More specifically, as shown in (Calderbank and Kruglikov, 2016), a conformal structure associated to the equation is of Einstein–Weyl type on its solutions.

All the aforementioned equations are linearly degenerate and hence possess infinitely many global solutions, see (Ferapontov et al., 2011), and (Ferapontov and Moss, 2015) for classification of integrable linearly degenerate systems of second order in three dimensions. Moreover, they all can be obtained as symmetry reductions of the five-dimensional equation $u_{yz} = u_{ts} + u_s\, u_{xz} - u_z\, u_{xs}$, see (Baran et al., 2015b). Below we discuss the applications of the equations in question, their origin and roles played in various field of mathematics, both pure and applied.

[1] As well as the KhZ equations(2.32) and (4.15).

[2] The authors of (Burovskiy et al., 2010) adopt a term *dispersionless Lax pairs* for this kind of Lax pairs.

In (Martínez Alonso and Shabat, 2004) hydrodynamic reductions of infinite hierarchy of integrable systems, called universal hierarchy, are considered. It is showed that this universal hierarchy includes several multidimensional integrable equations. In particular, authors derived rdDym Eq. (4.6), UH Eq. (4.8), and four-dimensional Martínez Alonso–Shabat equation. Then, in (Morozov and Sergyeyev, 2014), authors showed how rdDym Eq. (4.6), UH Eq. (4.8) and mVw Eq. (4.10) arise as reductions of Martínez Alonso–Shabat equation. In the same paper a Bäcklund transformation between mVw Eq. (4.10) and Vw Eq. (4.12) is presented.

The five equations we examine in this chapter were thoroughly studied in the series of papers by Baran, Krasil′shchik, Morozov and Vojčák. It follows from (Baran et al., 2015a) that equations (4.2), (4.6), (4.8), and (4.12) have higher local conservation laws. Symmetry reductions and exact solutions of equations (4.2), (4.6), (4.8), and (4.10) were studied in (Baran et al., 2014). Furthermore, as demonstrated in (Baran et al., 2016b), their algebras of nonlocal symmetries possess similar structures.

Pavlov's equation

Pavlov's equation

$$u_{yy} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy}, \tag{4.2}$$

is considered together with the covering

$$\left\{ \begin{array}{lcl} q_t & = & (\lambda^2 - \lambda\, u_x - u_y)\, q_x, \\ q_y & = & (\lambda - u_x)\, q_x, \end{array} \right. \tag{4.3}$$

found in (Pavlov, 2003) and (Dunajski, 2004). Pavlov's Eq. (4.2), obtained e.g. in (Kuz′mina, 1967), (Mikhalev, 1992), (Pavlov, 2003), (Dunajski, 2004), is a symmetry reduction of Plebański's second heavenly Eq. (3.1) (Dunajski, 1998), and has the GT Eq. (2.3) as its own symmetry reduction (Baran et al., 2014). Moreover, it is a $u = 0$ reduction of the Manakov-Santini system (Manakov and Santini, 2006)

$$\left\{ \begin{array}{lcl} u_{tx} + u_{yy} & = & -(u\, u_x)_x - v_x\, u_{xy} + v_y\, u_{xx}, \\ v_{tx} + v_{yy} & = & -u\, v_{xx} - v_x\, v_{xy} + v_y\, v_{xx}, \end{array} \right. \tag{4.4}$$

while the $v = 0$ reduction of (4.4) is the KhZ Eq. (2.32). Integrability of Pavlov's Eq. (4.2) within the twistor approach, where it is called hyper-CR (Cauchy-Riemann) equation, is established in (Dunajski, 2004). This terminology for Pavlov's equation follows from the fact that all Lorentzian hyper-CR Einstein-Weyl structures are locally of the form

$$h = (dy - v\, dt)^2 - 4(dx - v\, dy + w\, dt)\, dt, \qquad \omega = v_x\, dy + (v\, v_x + 2\, v_y)\, dt,$$

where w and v satisfy

$$v_t + w_y + v\, w_x - w\, v_x = 0, \qquad v_y + w_x = 0, \tag{4.5}$$

(Dunajski, 2004) and the above system is equivalent to Pavlov's Eq. (4.2) upon introducing the potential u such that $v = u_x$ and $w = -u_y$. Further discussion about Einstein-Weyl structures corresponding to the system (4.5) can be found in (Dunajski, 2008).

Yet another context for Pavlov's equation comes from the fact that it belongs to a class of r-th dispersionless Kadomtsev–Petviashvili equations (Błaszak, 2002). Pavlov's Eq. (4.2) was also obtained from other considerations, related to looped cotangent Virasoro algebra, in (Ovsienko and Roger, 2007), where it is showed that, coupled with another equation, Eq. (4.2) admits nonlocal bi-Hamiltonian form and infinite series of first integrals in involution.

The Lax pair (4.3) was used in (Grinevich et al., 2015) to solve a Cauchy problem for Pavlov's equation. See also (Manakov and Santini, 2009) for a discussion about long-time behaviour of some solutions. In (Morozov, 2012a) a recursion operator and shadows of nonlocal symmetries for Pavlov's Eq. (4.2) were found.

The rdDym equation

The rdDym equation

$$u_{ty} = u_x\, u_{xy} - u_y\, u_{xx}, \tag{4.6}$$

belongs to a class of r-th dispersionless $(2+1)$-dimensional Harry Dym equations (Błaszak, 2002), (Pavlov, 2003). We consider it in the covering

$$\begin{cases} q_t &= (u_x - \lambda)\, q_x, \\ q_y &= \lambda^{-1}\, u_y\, q_x, \end{cases} \tag{4.7}$$

found in (Morozov, 2009). It admits a bi-Hamiltonian structure (Ovsienko, 2010). A recursion operator and shadows of nonlocal symmetries were found in (Morozov, 2012a). In (Morozov, 2012b) a covering and a Bäcklund transformation for a generalization of (4.6) are obtained. The rdDym Eq. (4.6) belongs to an infinite set of quasilinear (2+1) dimensional equations related to extended Kupershmidt's lattice, (Pavlov, 2006).

The universal hierarchy equation

The UH equation

$$u_{yy} = u_t\, u_{xy} - u_y\, u_{tx} \tag{4.8}$$

appeared in (Pavlov, 2003) and (Martínez Alonso and Shabat, 2004). It has the following differential covering:

$$\begin{cases} q_t &= \lambda^{-2}(\lambda\, u_t - u_y)\, q_x, \\ q_y &= \lambda^{-1} u_y\, q_x. \end{cases} \tag{4.9}$$

The modified Veronese web equation

The mVw equation

$$u_{ty} = u_t\, u_{xy} - u_y\, u_{tx}, \tag{4.10}$$

appeared in (Martínez Alonso and Shabat, 2004). Some particular solutions for this equation were found in (Adler and Shabat, 2007). It is considered together with the covering

$$\begin{cases} q_t &= (1+\lambda)^{-1} u_t\, q_x, \\ q_y &= \lambda^{-1}\, u_y\, q_x, \end{cases} \tag{4.11}$$

found in (Ferapontov and Moss, 2015) as a Bäcklund transformation between the mVw Eq. (4.10) and the Vw Eq. (4.12).

The Veronese web equation

The Vw equation is of the form

$$u_{ty} = \lambda^{-1}\, u_x^{-1}\, \left((\lambda+1)\, u_t\, u_{xy} - u_y\, u_{tx}\right). \tag{4.12}$$

The concept of a Veronese web (a kind of a one-parameter family of foliations) was developed in order to study odd-dimensional bi-Hamiltonian systems, (Gelfand and Zakharevich, 1991). Then, in (Zakharevich, 2000) the author considered a PDE determining a Veronese web in three dimensions, which is precisely (4.12), found its Lax pair

$$\begin{cases} q_t &= \dfrac{\mu\,(\lambda+1)}{\lambda\,(\mu+1)}\, \dfrac{u_t}{u_x}\, q_x, \\[2ex] q_y &= \dfrac{\mu}{\lambda}\, \dfrac{u_y}{u_x}\, q_x, \end{cases} \qquad \mu \neq \lambda \tag{4.13}$$

and showed a connection with the nonlinear Riemann problem using twistor transform. The Vw Eq. (4.12) appears in the literature, e.g in (Zakharevich, 2000), as the ABC equation $A\, u_x\, u_{ty} + B\, u_t\, u_{xy} + C\, u_y\, u_{tx} = 0$ with $A + B + C = 0$. In the case of $A + B + C \neq 0$, the ABC equation was examined by Krasil′shchik et al. (2016), who found an infinite hierarchy of two-component nonlocal conservation laws that it admits.. Yet another term, see e.g. (Burovskiy et al., 2010), (Marvan and Sergyeyev, 2012) and (Dunajski and Kryński, 2014), is dispersionless Hirota equation.

Conservation laws

Unlike the three-component case, there are many results regarding two-component nonlocal conservation laws for Lax-integrable three-dimensional PDEs. The fact follows from the relation between Abelian coverings and conservation laws, (Krasil′shchik and Vinogradov, 1989), (Krasil′shchik and Vinogradov, 1999), (Krasil′shchik and Verbovetsky, 2011).

Definition 4.1.1. A covering is said to be an *Abelian covering* when system (1.8) can be mapped to the form

$$w^0_{x^k} = T^0_k(x^i, u^\alpha_I), \qquad w^j_{x^k} = T^s_k(x^i, u^\alpha_I, w^0, \ldots, w^{j-1}), \quad j \in \mathbb{N} \tag{4.14}$$

by a change of variables x^i, u^α_I, w^s. Otherwise the covering is non-Abelian.

Two-component nonlocal conservation laws for some PDE $\mathcal{E}$ in an Abelian covering arise as compatibility conditions of the system (4.14). For example, two infinite hierarchies of such conservation laws are found for the rdDym Eq. (4.6) in (Baran et al., 2016a). Other examples of such hierarchies, obtained in the same vein, include results for the *ABC* equation (4.12), (Krasil′shchik et al., 2016), constant astigmatism equation (Hlaváč and Marvan, 2017) and some (3+1)-dimensional systems (Sergyeyev, 2018), see also (Krasil'shchik and Sergyeyev, 2015). In a different context, two-component nonlocal conservation laws were obtained from nonlocal symmetries for the Chaplygin gas system and used to obtain exact (and at the same time not symmetry-invariant) solutions, (Ibragimov, 2012), (Ibragimov, 2014).

It was shown in (Burovskiy et al., 2010) that any equation of the form (4.1) possesses four local first-order conservation laws with coefficients independent of t, x, and y. Among the examples presented in (Burovskiy et al., 2010), there is one three-component local conservation law for the ABC equation and one for dispersionless Toda singular manifold equation. As for the *nonlocal* three-component conservation laws, some examples can be extracted from (Makridin and Pavlov, 2017) for the Khokhlov–Zabolotskaya equation and Pavlov's equation. Indeed, the recent results from (Makridin and Pavlov, 2017) together with the ones from (Morozov and Pavlov, 2017) constitute the main motivation for our research. Another instance of three-component nonlocal conservation law concerns the Kadomtsev–Petviashvili equation and comes from (Ibragimov, 2011).

For each of the equations (4.6), (4.8), (4.10), (4.12) we found two nonlocal three-component conservation laws. One nonlocal conservation law from each pair depends on one pseudopotential and is generated from a local conservation law of the covering equation. We show that these nonlocal conservation laws are generated from a local conservation law of the Veronese web equation by superpositions of the Bäcklund transformations. The second conservation law from each pair depends on two pseudopotentials, that is, it is defined on the Whitney product (see Definition 1.6.2) of two isomorphic coverings over the PDE under consideration. This nonlocal conservation law is generated from a nonlocal conservation law of the covering equation. We prove that all of the nonlocal conservation laws are nontrivial since they have non-zero generating functions.

Coverings (4.3), (4.7), (4.9), (4.11), (4.13), and the nonlocal conservation laws in these coverings can be expanded into Laurent series with respect to the spectral parameter. This yields two series of infinite-dimensional coverings and two series of nonlocal conservation laws for each of the equations (4.2), (4.6), (4.8), (4.10), (4.12). One of the series coincides with the series of nonlocal conservation laws for Pavlov's Eq. (4.2) considered in (Makridin and Pavlov, 2017). All the series of nonlocal conservation laws can be proven to be nontrivial. The same construction can be applied to the Whitney products of the covering and the nonlocal conservation laws that depend on two pseudopotentials.

We present results for Pavlov's equation, the rdDym equation, the universal hierarchy equation, the mVw equation, and the Vw equation in Sections 4.2.1, 4.2.2, 4.2.3, 4.2.4, and 4.2.5, respectively. In Section 4.2.6 we give an example of producing infinite hierarchies of three-component nonlocal conservation laws for the mVw Eq. (4.10) via expansions of its nonlocal conservation law Ω_4 (4.21) into Laurent series. In Section 4.3 we study correspondences between the nonlocal conservation laws under the Bäcklund transformations from (Morozov and Pavlov, 2017). We show that the nonlocal conservation laws with one pseudopotential are generated from a local conservation law (4.54) of the Vw Eq. (4.12) by superpositions of the Bäcklund transformations.

The following example provides a recapitulation of the outcomes corresponding to the Khokhlov–Zabolotskaya equation obtained in (Makridin and Pavlov, 2017) and relevant to this section. We add a proof of nontriviality of the conservation laws.

Example 4.1.1. *The Khokhlov-Zabolotskaya equation*

$$u_{yy} = u_{tx} + u_x\, u_{xx}, \tag{4.15}$$

in its potential form admits the infinite dimensional covering defined by the system

$$\left\{ \begin{array}{lcl} q_t & = & \frac{1}{3}\, q_x^3 - u_x\, q_x - u_y, \\ q_y & = & \frac{1}{2}\, q_x^2 - u_x. \end{array} \right. \tag{4.16}$$

The space of nonlocal variables have coordinates $(q,\, q_i)$, $i > 0$, *where* $q_i = q\underbrace{x\ldots x}_{i\text{ times}}$. *Eq.* (4.15) *has three three-component nonlocal conservation laws in the covering* (4.16)

$$\begin{array}{rcl} \Phi_1 & = & q_x^2\, dx \wedge dy + u_x^2\, dy \wedge dt + 2\, q_x \left(u_x - \frac{1}{3} q_x^2\right) dt \wedge dx, \\ \Phi_2 & = & q_x\, (q_x^2 + 6\, u_x)\, dx \wedge dy - (q_x^4 + 6\, (q_x\, u_y - u_x^2))\, dt \wedge dx \\ & & - \left(\frac{1}{5} q_x^5 - u_x\, q_x^3 + 3\, u_y\, q_x^2 + 6\, u_x\, (q_x\, u_x - u_y)\right) dy \wedge dt, \\ \Phi_3 & = & (q_x - s_x)^3 \left(dx \wedge dy + \left(\frac{1}{5}(q_x^2 + 3\, q_x\, s_x + s_x^2) + u_x\right) dy \wedge dt \right. \\ & & \left. -(q_x + s_x)\, dt \wedge dx\right), \end{array}$$

where the second pseudopotential s *satisfies the same system* (4.16) *as* q

$$\left\{ \begin{array}{lcl} s_t & = & \frac{1}{3}\, s_x^3 - u_x\, s_x - u_y, \\ s_y & = & \frac{1}{2}\, s_x^2 - u_x. \end{array} \right.$$

This means that Φ_3 *is defined on the Whitney product of two copies of the covering defined* *by system* (4.16). *We have*

$$\begin{aligned}
d_h\Phi_1 &= 2\,(q_x^2-u_x)\left(q_y-\tfrac{1}{2}\,q_x^2+u_x\right)\,dt\wedge dx\wedge dy,\\
d_h\Phi_2 &= 2\,\left(3\,q_{xx}\,(u_{yy}-u_{tx}-u_x\,u_{xx})+(2\,q_x^3-3\,u_y)_x\,\left(q_y-\tfrac{1}{2}\,q_x^2+u_x\right)\right)\,dt\wedge dx\wedge dy,\\
d_h\Phi_3 &= 2\,\left(((q_x-s_x)^2\,(2\,q_x+s_x))_x\,\left(q_y-\tfrac{1}{2}\,q_x^2+u_x\right)\right.\\
&- \left.((q_x-s_x)^2\,(q_x+2\,s_x))_x\,\left(s_y-\tfrac{1}{2}\,s_x^2+u_x\right)\right)\,dt\wedge dx\wedge dy.
\end{aligned}$$

Therefore Φ_1, Φ_2, *and* Φ_3 *are nontrivial nonlocal conservation laws for the potential KhZ Eq.* (4.15). ◇

4.2 Three-component nonlocal conservation laws

Below, notation $d_h\Omega \equiv \Theta$ means the following statement: there exists a 2-form Ψ on $J^\infty(\pi)\times\mathcal{W}$ such that $d_h(\Omega+\Psi)=\Theta$ and $\Psi|_{\tilde{\mathcal{E}}}=0$.

4.2.1 Pavlov's equation

Equations determining $\tilde{\mathcal{E}}$ for Pavlov's Eq. (4.2) and the covering (4.3) are labelled as follows.

$$F_{1,0}=u_{yy}-u_{tx}-u_y\,u_{xx}+u_x\,u_{xy}=0,$$

$$\left\{\begin{array}{lcl}
F_{1,1} &=& q_t-(\lambda^2-\lambda\,u_x-u_y)\,q_x=0,\\
F_{1,2} &=& q_y-(\lambda-u_x)\,q_x=0.
\end{array}\right.$$

Two nonlocal conservation laws of Pavlov's Eq. (4.2) in the covering (4.3) were found in (Makridin and Pavlov, 2017). We show below that they are nontrivial. The first conservation law

$$\Omega_1 \;=\; q_x^2\,(dx\wedge dy+(u_y+u_x^2-2\,\lambda\,u_x+\lambda^2)\,dy\wedge dt+(u_x-2\lambda)\,dt\wedge dx)\,, \tag{4.17}$$

depends on one pseudopotential q, while the second conservation law

$$\Upsilon_1 \;=\; q_x\,s_x\,(dx\wedge dy+(u_y+u_x^2-(\lambda+\mu)\,u_x+\lambda\,\mu)\,dy\wedge dt+(u_x-\lambda-\mu)\,dt\wedge dx)$$

depends also on the second pseudopotential s, which satisfies the copy of the system (4.3) with another value μ of the non-removable parameter:

$$\left\{\begin{array}{lcl}
F_{1,3} &=& s_t-(\mu^2-\mu\,u_x-u_y)\,s_x=0,\\
F_{1,4} &=& s_y-(\mu-u_x)\,s_x=0.
\end{array}\right. \tag{4.18}$$

More precisely, Υ_1 is defined on the Whitney product of the coverings defined by systems (4.3) and (4.18). Note that Ω_1 is a result of a substitution of q for s and accordingly λ for μ in Υ_1.

We have

$$\begin{aligned}
d_h\Omega_1 &\equiv -2\,\left(q_{xx}\,F_{1,1}+(q_x\,(u_x-2\,\lambda))_x\,F_{1,2}\right)\,dt\wedge dx\wedge dy,\\
d_h\Upsilon_1 &\equiv -\left(s_{xx}\,F_{1,1}+(s_x\,(u_x-\lambda-\mu))_x\,F_{1,2}+q_{xx}\,F_{1,3}\right.\\
&+ \left.(q_x\,(u_x-\lambda-\mu))_x\,F_{1,4}\right)\,dt\wedge dx\wedge dy.
\end{aligned}$$

Therefore, nonlocal conservation laws Ω_1 and Υ_1 for equation (4.2) are nontrivial.

4.2.2 The rdDym equation

Equations determining $\tilde{\mathcal{E}}$ for the rdDym Eq. (4.6) and the covering (4.7) are labelled as follows.

$$F_{2,0} = u_{ty} - u_x\, u_{xy} + u_y\, u_{xx} = 0,$$

$$\left\{ \begin{array}{lcl} F_{2,1} & = & q_t - (u_x - \lambda)\, q_x = 0, \\ F_{2,2} & = & q_y - \lambda^{-1}\, u_y\, q_x = 0. \end{array} \right.$$

The first nonlocal conservation law is

$$\Omega_2 = q_x^2\, \Big(u_y\, dx \wedge dy + u_y\, (2\lambda - u_x)\, dy \wedge\, dt - \lambda^2\, dt \wedge\, dx\Big)\,, \tag{4.19}$$

and the second one is

$$\Upsilon_2 = q_x\, s_x\, \left(u_y\, dx \wedge dy + u_y\, (\lambda + \mu - u_x)\, dy \wedge\, dt - \lambda\, \mu\, dt \wedge\, dx\right).$$

Similarly to the case of Pavlov's Eq. (4.2) the second conservation law is defined on the Whitney product of the covering (4.7) and its copy

$$\left\{ \begin{array}{lcl} F_{2,3} & = & s_t - (u_x - \mu)\, s_x = 0, \\ F_{2,4} & = & s_y - \mu^{-1} u_y\, s_x = 0. \end{array} \right.$$

The nonlocal conservation laws are nontrivial by the virtue of

$$\begin{array}{c} d_h\, \Omega_2 \equiv q_x\, \left(q_x\, F_{2,0} - 2\, u_{xy}\, F_{2,1}\right)\, dt \wedge dx \wedge dy, \\ d_h\, \Upsilon_2 \equiv (q_x\, s_x\, F_{2,0} + \lambda\, \mu\, (s_{xx}\, F_{2,2} + q_{xx}\, F_{2,4}) - (u_y\, s_x)_x\, F_{2,1} \\ -(u_y\, q_x)_x\, F_{2,3})\, dt \wedge dx \wedge dy. \end{array}$$

4.2.3 The UH equation

Equations determining $\tilde{\mathcal{E}}$ for the UH Eq. (4.8) and the covering (4.9) are labelled as follows.

$$F_{3,0} = u_{yy} - u_t\, u_{xy} + u_y\, u_{tx} = 0.$$

$$\left\{ \begin{array}{lcl} F_{3,1} & = & q_t - \lambda^{-2}(\lambda\, u_t - u_y)\, q_x = 0, \\ F_{3,2} & = & q_y - \lambda^{-1} u_y\, q_x = 0. \end{array} \right.$$

There are two nonlocal conservation laws of of the UH Eq. (4.8):

$$\Omega_3 = q_x^2\, \Big(\lambda^2\, u_y\, dx \wedge\, dy - u_y^2\, dy \wedge dt - \lambda\, (\lambda\, u_t - 2\, u_y)\, dt \wedge\, dx\Big) \tag{4.20}$$

and

$$\Upsilon_3 = q_x\, s_x\, \Big(\lambda\, \mu\, u_y\, dx \wedge\, dy - u_y^2\, dy \wedge dt - (\lambda\, u_t - (\lambda + \mu)\, u_y)\, dt \wedge\, dx\Big)\,.$$

The second one is defined on the Whitney product of the covering (4.9) and its copy

$$\left\{ \begin{array}{lcl} F_{3,3} & = & s_t - \mu^{-2}(\mu\, u_t - u_y) s_x = 0, \\ F_{3,4} & = & s_y - \mu^{-1} u_y\, s_x = 0. \end{array} \right.$$

The conservation laws are nontrivial since

$$\begin{array}{rcl} d_h \Omega_3 & \equiv & 2\, \lambda\, \Big(q_x^2\, F_{3,0} - \lambda^2\, (q_x\, u_y)_x\, F_{3,1} + \lambda\, (q_x\, (\lambda\, u_t - 2\, u_y))_x\, F_{3,2}\Big)\, dt \wedge dx \wedge dy, \\ d_h \Upsilon_3 & \equiv & \Big((\lambda + \mu)\, q_x^2\, F_{3,0} + \lambda\, \mu\, ((s_x\, u_y)_x\, F_{3,1} + (q_x\, u_y)_x\, F_{3,3}) \\ & - & (s_x\, (\lambda\, \mu\, u_t - (\lambda + \mu)\, u_y)\, F_{3,2} - (q_x\, (\lambda\, \mu\, u_t - (\lambda + \mu)\, u_y)\, F_{3,4}\Big)\, dt \wedge dx \wedge dy. \end{array}$$

4.2.4 The mVw equation

Equations determining $\tilde{\mathcal{E}}$ for the mVw Eq. (4.10) and the covering (4.11) are labelled as follows.

$$F_{4,0} = u_{ty} - u_t\, u_{xy} + u_y\, u_{tx} = 0.$$

$$\begin{cases} F_{4,1} &= q_t - (1+\lambda)^{-1} u_t\, q_x = 0, \\ F_{4,2} &= q_y - \lambda^{-1}\, u_y\, q_x = 0. \end{cases}$$

There are two nonlocal conservation laws of the mVw Eq. (4.10):

$$\begin{aligned} \Omega_4 &= q_x^2 \left((\lambda+1)^2\, u_y\, dx \wedge dy - u_t\, u_y\, dy \wedge\, dt - \lambda^2\, u_t\, dt \wedge\, dx\right), \\ \Upsilon_4 &= q_x\, s_x\, \left((\lambda+1)\,(\mu+1)\, u_y\, dx \wedge dy - u_t\, u_y\, dy \wedge\, dt - \lambda\, \mu\, u_t\, dt \wedge\, dx\right). \end{aligned} \tag{4.21}$$

The second one is defined on the Whitney product of the covering (4.11) and its copy

$$\begin{cases} F_{4,3} &= s_t - (1+\mu)^{-1} u_t\, s_x = 0, \\ F_{4,4} &= s_y - \mu^{-1}\, u_y\, s_x = 0. \end{cases}$$

The nontriviality of the conservation laws follows from

$$\begin{aligned} d_h\Omega_4 &\equiv \left((1+2\,\lambda)\, q_x^2\, F_{4,0} - 2\,(\lambda+1)^2\,(u_y\, q_x)_x\, F_{4,1} + 2\,\lambda^2\,(u_t\, q_y)_x\, F_{4,2}\right) dt \wedge dx \wedge dy, \\ d_h\Upsilon_4 &\equiv \Big((1+\lambda+\mu)\, q_x\, s_x\, F_{4,0} - (\lambda+1)\,(\mu+1)\,((u_y\, s_x)_x\, F_{4,1} + (u_y\, q_x)_x\, F_{4,3}) \\ &+ \lambda\, \mu\,((u_t\, s_y)_x\, F_{4,2} + (u_t\, q_x)_x\, F_{4,4})\Big)\, dt \wedge dx \wedge dy. \end{aligned}$$

4.2.5 The Vw equation

After solving (4.11) for u_t and u_y and renaming q by u in the resulting identity $(u_t)_y = (u_y)_t$ one obtains the Vw Eq. (4.12)

$$F_{5,0} = u_{ty} - \lambda^{-1}\, u_x^{-1}\,((\lambda+1)\, u_t\, u_{xy} - u_y\, u_{tx}) = 0.$$

Therefore system (4.11) defines a Bäcklund transformation between the mVw Eq. (4.10) and the Vw Eq. (4.12). In its turn, the Vw Eq. (4.12) has a covering (4.13), which is labelled as

$$\begin{cases} F_{5,1} &= q_t - \dfrac{\mu\,(\lambda+1)}{\lambda\,(\mu+1)}\, \dfrac{u_t}{u_x}\, q_x = 0, \\ F_{5,2} &= q_y - \dfrac{\mu}{\lambda}\, \dfrac{u_y}{u_x}\, q_x = 0, \end{cases} \qquad \mu \neq \lambda.$$

There are two nonlocal conservation laws of the Vw Eq. (4.12):

$$\begin{aligned} \Omega_5 &= q_x^2\, u_x^{-2} \left(\lambda\,(\mu+1)^2\, u_x\, u_y dx \wedge dy - \mu^2\,(\lambda+1)\, u_t\, u_y\, dy \wedge dt \right.\\ &- \left.\lambda\,(\lambda+1)\, u_t\, u_x\, dt \wedge dx\right) \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \Upsilon_5 &= q_x\, s_x\, u_x^{-2}\, \left(\lambda\,(\mu+1)\,(\kappa+1)\, u_x\, u_y dx \wedge dy - \mu\,\kappa\,(\lambda+1)\, u_t\, u_y\, dy \wedge dt \right.\\ &- \left.\lambda\,(\lambda+1)\, u_t\, u_x\, dt \wedge dx\right). \end{aligned}$$

The last 2-form is defined on the Whitney product of the covering (4.13) and its copy

$$\begin{cases} F_{5,3} &= s_t - \dfrac{\kappa\,(\lambda+1)}{\lambda\,(\kappa+1)}\,\dfrac{u_t}{u_x}\,s_x = 0, \\ F_{5,4} &= s_y - \dfrac{\kappa}{\lambda}\,\dfrac{u_y}{u_x}\,s_x = 0, \end{cases}$$

where $\kappa \neq \lambda$, $\kappa \neq \mu$. The nonlocal conservation laws are nontrivial since

$$\begin{aligned} d_h\Omega_5 &\equiv \lambda\,q_x\,u_x^{-1}\,\Big((\mu^2+2\,\mu-\lambda)\,q_x\,F_{5,0} - 2\,(\mu+1)^2\,u_{xy}\,F_{5,1} + 2\,(\lambda+1)\,u_{tx}\,F_{5,2}\Big)\,dt\wedge dx\wedge dy, \\ d_h\Upsilon_5 &\equiv \Big(\lambda\,(\kappa-\lambda+\mu\,(\kappa+1))\,q_x\,s_x\,u_x^{-1}\,F_{5,0} \\ &- \lambda\,(\mu+1)\,(\kappa+1)\,\Big((s_xu_yu_x^{-1})_x\,F_{5,1} + (q_xu_yu_x^{-1})_x\,F_{5,3}\Big) \\ &- \lambda\,(\lambda+1)\,\Big((s_xu_tu_x^{-1})_x\,F_{5,2} + (q_xu_tu_x^{-1})_x\,F_{5,4}\Big)\Big)\,dt\wedge dx\wedge dy. \end{aligned}$$

4.2.6 Infinite hierarchies of nonlocal conservation laws

The covering equations (4.3), (4.7), (4.9), (4.11), (4.13), and the nonlocal conservation laws Ω_i, Υ_i, $i \in \{1,\dots,5\}$, can be expanded into Laurent series with respect to the spectral parameters λ and μ. This yields series of infinite-dimensional coverings and series of nonlocal conservation laws for each of the equations (4.2), (4.6), (4.8), (4.10), (4.12). One of the series coincides with the series of nonlocal conservation laws for equation (4.2) considered in (Makridin and Pavlov, 2017). We give an example of such an expansion for the mVw Eq. (4.10)[3].

Example 4.2.1. *Consider the covering over the mVw Eq.* (4.10) *defined by system* (4.11) *and the nonlocal conservation law Ω_4 defined by* (4.21)*. Substitute for q its Laurent series expansion $q = \sum_{i\in\mathbb{Z}} q_i\lambda^i$. This gives the following system*

$$\begin{cases} q_{i-1,t} &= u_t\,q_{i,x} - q_{i,t}, \\ q_{i-1,y} &= u_y\,q_{i,x}. \end{cases}$$

This system defines an infinite-dimensional covering over the mVw Eq. (4.10) *with pseudopotentials q_i, $i \in \mathbb{Z}$. After setting $q_0 = y$, $q_i = 0$ for $i \le -1$, we get the positive covering which acquires the form*

$$\begin{cases} q_{1,t} &= u_t\,u_y^{-1}, \\ q_{1,x} &= u_y^{-1}, \\ q_{i,t} &= u_t\,u_y^{-1}\,q_{i-1,y} - q_{i-1,t}, \\ q_{i,x} &= u_y^{-1}\,q_{i-1,y} \end{cases}$$

with $i \ge 2$. The internal nonlocal variables in this covering are q_i, $q_{i,y}$, $q_{i,yy}$ and so on. Consider the expansion $\Omega_4 = \sum_{i\ge 0} \lambda^i\,\Omega^+_{4,i}$. We have $\Omega^+_{4,0} = \Omega^+_{4,1} = 0$,

$$\begin{aligned} \Omega^+_{4,2} &= u_y^{-1}\,(dx\wedge dy - u_t\,dy\wedge dt), \\ \Omega^+_{4,3} &= u_y^{-1}\,\left((1+q_{1,y})\,dx\wedge dy - u_t\,q_{1,y}\,dy\wedge dt\right), \end{aligned}$$

[3] We have chosen this equation because its Lie algebras of nonlocal symmetries in the positive and the negative covering have the most interesting structures in comparison with the other equations, see (Baran et al., 2016a)

and for $m \geq 2$ we get

$$
\begin{aligned}
\Omega^{+}_{4,2m} &= u_y^{-1}\left[q^2_{m-2,y} + q^2_{m-1,y} + 2\left(\sum_{k=0}^{m-3} q_{k,y}\, q_{2m-4-k,y}\right.\right.\\
&+ \left.\left. 2\sum_{k=0}^{m-2} q_{k,y}\, q_{2m-3-k,y} + \sum_{k=0}^{m-2} q_{k,y}\, q_{2m-2-k,y}\right)\right] dx \wedge dy\\
&- u_t\, u_y^{-1}\left(q^2_{m-1,y} + 2\sum_{k=0}^{m-2} q_{k,y}\, q_{2m-2-k,y}\right) dy \wedge dt\\
&- u_t\, u_y^{-2}\left(q^2_{m-2,y} + 2\sum_{k=0}^{m-3} q_{k,y}\, q_{2m-4-k,y}\right) dt \wedge dx,
\end{aligned}
$$

$$
\begin{aligned}
\Omega^{+}_{4,2m+1} &= u_y^{-1}\left[q^2_{m-1,y} + \left(\sum_{k=0}^{m-2} q_{k,y}\, q_{2m-3-k,y}\right.\right.\\
&+ \left.\left. 2\sum_{k=0}^{m-2} q_{k,y}\, q_{2m-2-k,y} + \sum_{k=0}^{m-1} q_{k,y}\, q_{2m-1-k,y}\right)\right] dx \wedge dy\\
&- u_t\, u_y^{-1}\left(\sum_{k=0}^{m-1} q_{k,y}\, q_{2m-1-k,y}\right) dy \wedge dt\\
&- u_t\, u_y^{-2}\left(\sum_{k=0}^{m-2} q_{k,y}\, q_{2m-3-k,y}\right) dt \wedge dx.
\end{aligned}
$$

Likewise, after setting $q_j = 0$ for $j \geq 1$ and relabelling $r_i := q_{-i-1}$, where $r_{-1} := q_0 = x$, $r_0 := q_{-1} = u$, the negative covering acquires the form

$$
\left\{
\begin{aligned}
r_{1,t} &= u_t\,(u_x - 1),\\
r_{1,y} &= u_x\, u_y,\\
r_{i,t} &= u_t\, r_{i-1,x} - r_{i-1,t},\\
r_{i,y} &= u_y\, r_{i-1,x}.
\end{aligned}
\right.
$$

Then for expansion $\Omega_4 = \sum_{i\geq -2} \lambda^{-i}\,\Omega^{-}_{4,i}$ we have $\Omega^{-}_{4,-2} = u_y\, dx \wedge dy - u_t\, dy \wedge dt$, which is a trivial conservation law, and then

$$
\begin{aligned}
\Omega^{-}_{4,-1} &= u_y\,(u_x + 1)\, dx \wedge dy - u_t\, u_x\, dy \wedge dt\\
\Omega^{-}_{4,0} &= u_y\,(u_x^2 + 1 + 2\, r_{1,x} + 4\, u_x)\, dx \wedge dy - u_t\, u_y\, dy \wedge dt - u_t(u_x^2 - 2\, r_{1,x})\, dt \wedge dx,\\
\Omega^{-}_{4,1} &= u_y\,(u_x^2 + r_{2,x} + u_x\, r_{1,x} + 2\, r_{1,x} + u_x)\, dx \wedge dy - u_t\, u_x\, u_y\, dy \wedge dt\\
&- u_t(r_{2,x} + u_x\, r_{1,x})\, dt \wedge dx,
\end{aligned}
$$

while for $i \geq 2$ we have

$$
\Omega^{-}_{4,i} = P^{-}_i\, dx \wedge dy + Q^{-}_i\, dy \wedge dt + R^{-}_i\, dt \wedge dx,
$$

where

$$
\begin{aligned}
P^{-}_{2m} &= u_y\left[r^2_{m,x} + r^2_{m-1,x} + 2\left(\sum_{k=-1}^{m-1} r_{k,x}\, r_{2m-k,x} + 2\sum_{k=-1}^{m-1} r_{k,x}\, r_{2m-1-k,x} + \sum_{k=-1}^{m-2} r_{k,x}\, r_{2m-2-k,x}\right)\right],\\
Q^{-}_{2m} &= -u_t\, u_y\left(r^2_{m-1,x} + 2\sum_{k=-1}^{m-2} r_{k,x}\, r_{2m-2-k,x}\right),\\
R^{-}_{2m} &= -u_t\left(r^2_{m,x} + 2\sum_{k=-1}^{m-1} r_{k,x}\, r_{2m-k,x}\right),
\end{aligned}
$$

$$
\begin{aligned}
P^{-}_{2m+1} &= u_y\left[r^2_{m,x} + 2\left(\sum_{k=-1}^{m} r_{k,x}\, r_{2m+1-k,x} + 2\sum_{k=-1}^{m-1} r_{k,x}\, r_{2m-k,x} + \sum_{k=-1}^{m-1} r_{k,x}\, r_{2m-1-k,x}\right)\right],\\
Q^{-}_{2m+1} &= -u_t\, u_y \sum_{k=-1}^{m-1} r_{k,x}\, r_{2m-1-k,x},\\
R^{-}_{2m+1} &= -u_t \sum_{k=-1}^{m} r_{k,x}\, r_{2m+1-k,x}
\end{aligned}
$$

for $m \geq 1$. Note that both hierachies contain one non-trivial local conservation law, while all their other non-trivial conservation laws are nonlocal. $\diamond$

In the same manner it is possible to expand the nonlocal conservation laws Ω_i for $i \in \{1, 2, 3, 5\}$ into series with respect to spectral parameters of their coverings (we provide the hierarchies for rdDym and universal hierarchy equation in appendices C.1 and C.2, respectively) as well as to expand the nonlocal conservation laws Υ_i into double series with respect to λ and μ, to obtain the associated hierarchies of nonlocal conservation laws, and then to prove their nontriviality.

4.3 Bäcklund transformations and conservation laws

As it was shown in (Morozov and Pavlov, 2017), equations (4.2), (4.6), (4.8), (4.10) are related by Bäcklund transformations. The Vw Eq. (4.12) is related to the mVw Eq. (4.10) by the Bäcklund transformation (4.11). These relations can be presented by the following diagram:

$$
\begin{tikzcd}
 & & & \mathcal{E}_5 \arrow[d, "(4.11)"] \\
 & \widetilde{\mathcal{E}}_2 \arrow[r, equal, "(4.36)"] \arrow[d, "(4.7)"] & \widetilde{\mathcal{E}}_3 \arrow[r, equal, "(4.41)"] \arrow[d, "(4.9)"] & \mathcal{E}_4 \\
\widetilde{\mathcal{E}}_1 \arrow[r, equal, "(4.27)"] \arrow[d, "(4.3)"] & \mathcal{E}_2 & \mathcal{E}_3 & \\
\mathcal{E}_1 & & &
\end{tikzcd}
$$

In this diagram $\mathcal{E}_1$, ..., $\mathcal{E}_5$ denote Pavlov's Eq. (4.2), the rdDym Eq. (4.6), the UH Eq. (4.8), the mVw Eq. (4.10), and the Vw Eq. (4.12), respectively, while $\tilde{\mathcal{E}}_1$, $\tilde{\mathcal{E}}_2$, $\tilde{\mathcal{E}}_3$ denote equations (4.26), (4.34), and (4.37), respectively. Whereas in general an action of a Bäcklund transformation on a conservation law is not defined, the specific form of the nonlocal conservation laws (4.17), (4.19), (4.20), (4.21), (4.22) and the Bäcklund transformations between equations (4.2), (4.6), (4.8), (4.10), (4.12) allows one to find correspondences between these nonlocal conservation laws and some local conservation laws of the considered equations. In this section we present the results of computations of such correspondences.

4.3.1 Pavlov's equation and the rdDym equation

Consider Pavlov's equation written as

$$\tilde{u}_{\tilde{y}\tilde{y}} = \tilde{u}_{\tilde{t}\tilde{x}} + \tilde{u}_{\tilde{y}}\, \tilde{u}_{\tilde{x}\tilde{x}} - \tilde{u}_{\tilde{x}}\, \tilde{u}_{\tilde{x}\tilde{y}} \tag{4.23}$$

and its covering

$$\begin{cases} \tilde{q}_{\tilde{t}} &= (\lambda^2 - \lambda\,\tilde{u}_{\tilde{x}} - \tilde{u}_{\tilde{y}})\;\tilde{q}_{\tilde{x}}, \\ \tilde{q}_{\tilde{y}} &= (\lambda - \tilde{u}_{\tilde{x}})\;\tilde{q}_{\tilde{x}}. \end{cases} \tag{4.24}$$

From (4.24) we have

$$\begin{cases} \tilde{u}_{\tilde{x}} &= \lambda - \tilde{q}_{\tilde{y}}\,\tilde{q}_{\tilde{x}}^{-1}, \\ \tilde{u}_{\tilde{y}} &= (\lambda\,\tilde{q}_{\tilde{y}} - \tilde{q}_{\tilde{t}})\,\tilde{q}_{\tilde{x}}^{-1}. \end{cases} \tag{4.25}$$

Then identity $(\tilde{u}_{\tilde{x}})_{\tilde{y}} = (\tilde{u}_{\tilde{y}})_{\tilde{x}}$ gives

$$\tilde{q}_{\tilde{y}\tilde{y}} = \tilde{q}_{\tilde{t}\tilde{x}} + \frac{\lambda\,\tilde{q}_{\tilde{y}} - \tilde{q}_{\tilde{t}}}{\tilde{q}_{\tilde{x}}}\,\tilde{q}_{\tilde{x}\tilde{x}} + \frac{\tilde{q}_{\tilde{y}} - \lambda\,\tilde{q}_{\tilde{x}}}{\tilde{q}_{\tilde{x}}}\,\tilde{q}_{\tilde{x}\tilde{y}}. \tag{4.26}$$

Therefore, (4.24) defines a Bäcklund transformation between (4.23) and (4.26). In its turn Eq. (4.26) is isomorphic to the rdDym Eq. (4.6) and the exact form of the map between these two equations is

$$\tilde{t} = t, \quad \tilde{x} = \lambda\,x - u, \quad \tilde{y} = x, \quad \tilde{q} = y. \tag{4.27}$$

The nonlocal conservation law Ω_1 (4.17) written as

$$\Omega_1 = \tilde{q}_{\tilde{x}}^2\left(d\tilde{x}\wedge d\tilde{y} + \left((\tilde{u}_{\tilde{x}}+\lambda)^2 + \tilde{u}_{\tilde{y}}\right)d\tilde{y}\wedge d\tilde{t} + (2\lambda + \tilde{u}_{\tilde{x}})\,d\tilde{t}\wedge d\tilde{x}\right)$$

after expressing $\tilde{u}_{\tilde{x}}$ and $\tilde{u}_{\tilde{y}}$ via (4.25) gives a local conservation law

$$\tilde{q}_{\tilde{x}}^2\,d\tilde{x}\wedge d\tilde{y} + \left(\tilde{q}_{\tilde{x}}(\lambda\,\tilde{q}_{\tilde{y}} - \tilde{q}_{\tilde{t}}) + \tilde{q}_{\tilde{y}}^2\right)d\tilde{y}\wedge d\tilde{t} - \tilde{q}_{\tilde{x}}\,(\lambda\,\tilde{q}_{x} + \tilde{q}_{\tilde{y}})\,d\tilde{t}\wedge d\tilde{x} \tag{4.28}$$

of Eq. (4.26). Then, transformation (4.27) maps (4.28) to a local conservation law

$$\frac{1}{u_y}\,dx\wedge dy - \frac{u_x}{u_y}\,dy\wedge dt$$

of the rdDym Eq. (4.6).

Now let us consider the copy

$$\begin{cases} q_t &= (u_x - \mu)\,q_x, \\ q_y &= \mu^{-1}\,u_y\,q_x, \end{cases} \tag{4.29}$$

of the system (4.7) and the nonlocal conservation law Ω_2 (4.19) of the rdDym Eq. (4.6) written as

$$\Omega_2 = q_x^2\left(u_y\,dx\wedge dy + u_y(2\mu - u_x)\,dy\wedge dt - \mu^2 dt\wedge dx\right). \tag{4.30}$$

We take the inverse transformation for (4.27) and extend it to the pseudopotential q identically

$$t = \tilde{t}, \quad x = \tilde{y}, \quad y = \tilde{q}, \quad u = \lambda\tilde{y} - \tilde{x}, \quad q = s. \tag{4.31}$$

Then superposition of Bäcklund transformations (4.29) and (4.24) maps the rdDym Eq. (4.6) to (4.26) and covering (4.29) to covering

$$\begin{cases} s_{\tilde{t}} &= ((\lambda-\mu)^2 - (\lambda-\mu)\,\tilde{u}_{\tilde{x}}) - \tilde{u}_{\tilde{y}})\;s_{\tilde{x}}, \\ s_{\tilde{y}} &= (\lambda - \mu - \tilde{u}_{\tilde{x}})\,s_{\tilde{x}} \end{cases} \tag{4.32}$$

of (4.23). Also, this superposition maps the nonlocal conservation law (4.30) to the nonlocal conservation law

$$s_{\tilde{x}}^2 \left(d\tilde{x} \wedge d\tilde{y} + \left((\tilde{u}_{\tilde{x}} + \lambda - \mu)^2 + \tilde{u}_{\tilde{y}}\right) d\tilde{y} \wedge d\tilde{t} + (\tilde{u}_{\tilde{x}} - 2\,(\lambda - \mu))\; d\tilde{t} \wedge d\tilde{x}\right) \tag{4.33}$$

of (4.23). Note that coverings (4.32) and (4.3), as well as conservation laws (4.33) and Ω_1 (4.17), differ only by notation. Therefore the Bäcklund transformation between the rdDym Eq. (4.6) and Pavlov's Eq. (4.2) maps the nonlocal law Ω_2 (4.19) to Ω_1 (4.17).

4.3.2 The rdDym equation and the UH equation

The covering (4.29) defines a Bäcklund transformation between the rdDym Eq. (4.6) and the following one:

$$q_{ty} = \frac{\mu\, q_x + q_t}{q_x}\, q_{xy} - \frac{\mu\, q_y}{q_x}\, q_{xx}. \tag{4.34}$$

The nonlocal conservation law Ω_2 (4.30) of the rdDym equation corresponds to a local conservation law of (4.34) of the form

$$q_x\, q_y\, dx \wedge dy + q_y\, (\mu\, q_x - q_t)\, dy \wedge dt - \mu\, q_x^2\, dt \wedge dx. \tag{4.35}$$

Transformation

$$t = \lambda^{-1}\, \mu^{-1}\, \tilde{t}, \quad x = \tilde{y}, \quad y = \tilde{x}, \quad q = \tilde{q} \tag{4.36}$$

maps (4.34) to equation

$$\tilde{q}_{\tilde{y}\tilde{y}} = \frac{\lambda\, \tilde{q}_{\tilde{t}} + \tilde{q}_{\tilde{y}}}{\tilde{q}_{\tilde{x}}}\, \tilde{q}_{\tilde{x}\tilde{y}} - \frac{\lambda\, \tilde{q}_{\tilde{y}}}{\tilde{q}_{\tilde{x}}}\, \tilde{q}_{\tilde{x}\tilde{x}}, \tag{4.37}$$

which is related via the Bäcklund transformation

$$\left\{ \begin{array}{lcl} \tilde{q}_{\tilde{t}} & = & \lambda^{-2}\, (\lambda\, \tilde{u}_{\tilde{t}} - \tilde{u}_{\tilde{y}})\; \tilde{q}_{\tilde{x}}, \\ \tilde{q}_{\tilde{y}} & = & \lambda^{-1} \tilde{u}_{\tilde{y}}\, \tilde{q}_{\tilde{x}}, \end{array} \right. \tag{4.38}$$

with the UH equation

$$\tilde{u}_{\tilde{y}\tilde{y}} = \tilde{u}_{\tilde{t}}\, \tilde{u}_{\tilde{x}\,\tilde{y}} - \tilde{u}_{\tilde{y}} \tilde{u}_{\tilde{t}\tilde{x}}. \tag{4.39}$$

Under the transformation (4.36) and the Bäcklund transformation (4.38) the conservation law (4.35) is mapped to the nonlocal conservation law

$$\tilde{q}_{\tilde{x}}^2 \left(\lambda^2\, \tilde{u}_{\tilde{y}}\, d\tilde{x} \wedge d\tilde{y} - \tilde{u}_{\tilde{y}}^2\, d\tilde{y} \wedge d\tilde{t} - \lambda\, (\lambda\, \tilde{u}_{\tilde{t}} - 2\tilde{u}_{\tilde{y}})\; d\tilde{t} \wedge d\tilde{x}\right) \tag{4.40}$$

of the UH Eq. (4.39) in the covering (4.38). The conservation law (4.40) differs from the conservation law Ω_3 (4.20) by notation.

4.3.3 The UH equation and the mVw equation

Transformation

$$\tilde{t} = -\lambda\, x, \quad \tilde{x} = t, \quad \tilde{y} = u, \quad \tilde{q} = y, \tag{4.41}$$

maps Eq. (4.37) to the mVw Eq. (4.10). In its turn, the nonlocal conservation law (4.40) can be expressed in terms of the pseudopotential $\tilde{q}$ as a local conservation law of (4.37) by substituting for $\tilde{u}_{\tilde{t}}$ and $\tilde{u}_{\tilde{y}}$ their expressions obtained from (4.38).

$$\tilde{\Omega}_3 = \lambda\, \tilde{q}_{\tilde{x}}\, \tilde{q}_{\tilde{y}}\, d\tilde{x} \wedge d\tilde{y} - \tilde{q}_{\tilde{y}}^2\, d\tilde{y} \wedge d\tilde{t} - \tilde{q}_{\tilde{x}}(\lambda\, \tilde{q}_{\tilde{t}} - \tilde{q}_{\tilde{y}})\, d\tilde{t} \wedge d\tilde{x}. \tag{4.42}$$

This conservation law is mapped under the transformation (4.41) to a local conservation law

$$\frac{u_t}{u_y}\, dy \wedge dt - \frac{1}{u_y}\, dx \wedge dy$$

of the mVw Eq. (4.10).

Now let us consider the covering of the mVw Eq. (4.10) defined by the system

$$\begin{cases} q_t &= (1+\mu)^{-1} u_t\, q_x, \\ q_y &= \mu^{-1}\, u_y\, q_x, \end{cases} \tag{4.43}$$

and the nonlocal conservation law Ω_4 (4.21) written in this covering as

$$\Omega_4 = q_x^2\, \left((\mu+1)^2\, u_y\, dx \wedge dy - u_t\, u_y\, dy \wedge\, dt - \mu^2\, u_t\, dt \wedge\, dx\right). \tag{4.44}$$

We consider the inverse transformation to the map (4.41) and extend its action to the pseudopotential q by identity:

$$t = \tilde{x}, \quad x = -\lambda^{-1}\, \tilde{t}, \quad y = \tilde{q}, \quad u = \tilde{y}, \quad q = s. \tag{4.45}$$

Then, the covering (4.43) is mapped to

$$\begin{cases} s_{\tilde{t}} &= \alpha^{-2}\, (\alpha\, \tilde{u}_{\tilde{t}} - \tilde{u}_{\tilde{y}})\; s_{\tilde{x}}, \\ s_{\tilde{y}} &= \alpha^{-1} \tilde{u}_{\tilde{y}}\, s_{\tilde{x}}, \end{cases} \tag{4.46}$$

where $\alpha = \lambda\, (1+\mu)^{-1}$, and the conservation law (4.44) is mapped to

$$s_{\tilde{x}}^2\, \left(\alpha^2\, \tilde{u}_{\tilde{y}}\, d\tilde{x} \wedge d\tilde{y} - \tilde{u}_{\tilde{y}}^2\, d\tilde{y} \wedge d\tilde{t} - \alpha\, \left(\alpha\, \tilde{u}_{\tilde{t}} - 2\tilde{u}_{\tilde{y}}\right) d\tilde{t} \wedge d\tilde{x}\right).$$

This nonlocal conservation law differs from Ω_3 (4.20) by notation.

4.3.4 The rdDym equation and the mVw equation

Transformation

$$t = -\mu^{-1}\, \tilde{x}, \quad x = \tilde{u}, \quad y = \tilde{t}, \quad q = \tilde{y} \tag{4.47}$$

maps Eq. (4.34), which is related to the rdDym Eq. (4.6) via the Bäcklund transformation (4.29), to the mVw equation written as

$$\tilde{u}_{\tilde{t}\tilde{y}} = \tilde{u}_{\tilde{t}}\, \tilde{u}_{\tilde{x}\tilde{y}} - \tilde{u}_{\tilde{y}}\, \tilde{u}_{\tilde{t}\tilde{x}}. \tag{4.48}$$

The nonlocal conservation law (4.30) of rdDym Eq. (4.6), which corresponds to the local conservation law (4.35) of Eq. (4.34), is transformed under (4.47) to a local conservation law

$$\frac{1}{\tilde{u}_{\tilde{y}}}\, d\tilde{x} \wedge d\tilde{y} - \frac{\tilde{u}_{\tilde{t}}}{\tilde{u}_{\tilde{y}}}\, d\tilde{y} \wedge d\tilde{t}.$$

of Eq. (4.48).

Now let us consider the covering

$$\left\{\begin{array}{lcl} \tilde{q}_{\tilde{t}} & = & (1+\lambda)^{-1}\tilde{u}_{\tilde{t}}\,\tilde{q}_{\tilde{x}}, \\ \tilde{q}_{\tilde{y}} & = & \lambda^{-1}\,\tilde{u}_{\tilde{y}}\,\tilde{q}_{\tilde{x}}, \end{array}\right. \tag{4.49}$$

of Eq. (4.48) and the nonlocal conservation law Ω_4 (4.21) written as

$$\tilde{q}_{\tilde{x}}^2\,\Big((\lambda+1)^2\,\tilde{u}_{\tilde{y}}\,d\tilde{x}\wedge d\tilde{y}-\tilde{u}_{\tilde{t}}\,\tilde{u}_{\tilde{y}}\,d\tilde{y}\wedge\,d\tilde{t}-\lambda^2\,\tilde{u}_{\tilde{t}}\,d\tilde{t}\wedge\,d\tilde{x}\Big)\,. \tag{4.50}$$

They are mapped under the transformation

$$\tilde{t}=y,\quad \tilde{x}=\mu\,t,\quad \tilde{y}=q,\quad \tilde{u}=x,\quad \tilde{q}=s \tag{4.51}$$

to the covering

$$\left\{\begin{array}{l} s_t=(u_x-\alpha)q_x, \\ s_y=\alpha^{-1}u_y\,q_x, \end{array}\right. \tag{4.52}$$

with $\alpha=\mu\,(1+\lambda)$ and the nonlocal conservation law

$$s_x^2\,\Big(u_y\,dx\wedge dy+u_y(2\alpha-u_x)\,dy\wedge dt-\alpha^2\,dt\wedge dx\Big)$$

of the rdDym Eq. (4.6). This conservation law differs from Ω_2 (4.19) by notation.

4.3.5 The Vw equation

The Vw Eq. (4.12), written as

$$q_{ty}=\lambda^{-1}\,q_x^{-1}\,((\lambda+1)\,q_t\,q_{xy}-q_y\,q_{tx}), \tag{4.53}$$

has a local conservation law

$$(\lambda+1)\,q_x\,q_y\,dx\wedge dy-\lambda\,q_t\,q_y\,dy\wedge dt-q_t\,q_x\,dt\wedge dx. \tag{4.54}$$

The Bäcklund transformation (4.11) relates Eq. (4.53) to the mVw Eq. (4.10). Expressing q_t, q_y in (4.54) with the use of Bäcklund transformation (4.11) yields Ω_4 (4.20). Therefore the nonlocal conservation law Ω_4 (4.21) of the mVw equation is generated from the local conservation law (4.54) of Eq. (4.53) by the Bäcklund transformation (4.11).

Finally, the nonlocal conservation law Ω_5 (4.22) of the Vw Eq. (4.12) can be obtained from (4.54) by expressing q_t, q_y via the Bäcklund transformation (4.13).

4.4 Conclusion

The results of this chapter can be summarized in form of the theorem.

Theorem 4.4.1. *Equations* (4.2), (4.6), (4.8), (4.10), (4.12) *have nontrivial nonlocal conservation laws* (4.17), (4.19), (4.20), (4.21), (4.22) *in the coverings* (4.3), (4.7), (4.9), (4.11), (4.12)*, respectively. The nonlocal conservation laws are generated from the local conservation law* (4.54) *of Eq.* (4.53) *by the appropriate combinations of the Bäcklund transformations between these equations.*

Conclusions

The thesis is based on several new results concerning particular, relevant in physics as well as some areas of mathematics, nonlinear integrable PDEs. The applied machinery belongs to the geometrical approach to differential equations which has its roots in the theory of Lie symmetry transformations. In Chapter 2 we used classical tools from the theory of Lie groups to find exact solutions of Gibbons-Tsarev equation and Khokhlov-Zabolotskaya equation (while in the second case we in addition demonstrated application of Miura-type transformation to the procedure). The remaining chapters concern nonlocal symmetries and conservation laws, objects which are most rigorously introduced within the framework developed by Krasil′shchik and Vinogradov (1984, 1989). In Chapter 3 we found full–fledged nonlocal symmetries for Plebański's second heavenly equation and, which is still a rare situation, described the structure of their Lie algebra. In Chapter 4 we found three-dimensional nonlocal conservation laws for five integrable PDEs. We showed how to use them as sources of infinite hierarchies of conservation laws of this type. Moreover, we presented the relations between them, which boil down to the fact that all the nonlocal conservation laws we found can be derived via Bäcklund transformations from one local conservation law. There are many examples (in the field of integrable equations) of two-dimensional nonlocal or three-dimensional local conservation laws, but the examination of three-dimensional nonlocal conservation laws is a rather underdeveloped subject. An important instance of research into the three-dimensional nonlocal conservation laws is provided by a relatively recent paper by Makridin and Pavlov (2017). We discuss the results in more details below.

The Gibbons-Tsarev equation

In Section 2.3 we looked for optimal system of one-parameter group-invariant solutions to the GT Eq. (2.3), which (the GT Eq. (2.3)) was obtained as a 2-component reduction of Benney's moment equations (2.4) describing behaviour of long waves on a shallow, inviscid and incompressible fluid. This research is a part of the large and important problem of finding solutions to the multi-component reductions of Benney's system. An extensive literature devoted to this problem includes e.g. (Kupershmidt and Manin, 1977), (Zakharov, 1980, 1981), (Krichever, 1988), (Gibbons and Kodama, 1994), (Gibbons and Tsarev, 1999), (Yu and Gibbons, 2000), (Baldwin and Gibbons, 2003, 2004, 2006), (Bogdanov and Konopelchenko, 2004), (Kokotov and Korotkin, 2007), (Pavlov, 2008), (England and Gibbons, 2009), (Gibbons et al., 2009), (Gibbons, 2012). Searching for exact solutions to the multi-component reductions of Benney's system by means of the methods of Lie group analysis is an interesting and promising direction for the further research. Whereas solutions of this equation obtained in (Marikhin and Sokolov, 2005) and (Kaptsov, 2002), (Kaptsov and Schmidt, 2005) are expressed in terms of the Weierstrass elliptic functions and solutions of the Painlevé equations, respectively, our solutions are rational or algebraic functions, which are easier to use in applications. This allowed us to find exact solutions for Pavlov's Eq. (4.2) (having itself important applications, see Section 4.1) and integrable reductions of the FHZ system (2.12), which in its turn provides solutions to the KhZ Eq. (2.32).

The Khokhlov–Zabolotskaya equation

We found a number of new exact solutions to the KhZ Eq. (2.32). These are explicit solutions (2.61), (2.63), (2.67), (2.72), and the solution defined implicitly by (2.54), (2.56) and (2.57). The KhZ equation is a fundamental model in nonlinear acoustics, but established its role in geophysics and dynamics of liquid metal as well. It is related to the Lin-Reissner-Tsien equation and to the dKP equation, which both have their own range of applications in mathematical modelling. The solutions were found according to the following scheme. We considered another equation, nonlocally related to the KhZ equation via Bäcklund (and in this case Miura-type) transformation, called the KhZ singular manifold equation (2.36). We found its symmetry algebra, which happened to be, naturally, infinite-dimensional. This fact leads to some technical difficulties concerning classification of subalgebras into optimal system of one-dimensional subalgebras, which we discuss in the course of the proof of Theorem 2.4.1. Next, we found optimal system of symmetry-invariant solutions to the KhZ singular manifold Eq. (2.36) and mapped them to exact solutions of the KhZ Eq. (2.32) via the Bäcklund transformation (2.38). Therefore, we have shown that the Bäcklund transformation (2.33), (2.34), (2.37), (2.40), can be applied to find new exact solutions to the KhZ Eq. (2.32). It is natural to ask whether other nonlocal structures associated with the KhZ equation can be used to study its exact solutions. For instance, one could consider nonlocal conservation laws of the KhZ Eq. (2.32) from (Makridin and Pavlov, 2017) and apply to them the technique from (Ibragimov and Avdonina, 2013).

Another challenge is concerned with the fact that the KhZ Eq. (2.32) is the dispersionless limit of the KP equation, see (Gibbons, 1985) and references therein. The contact symmetry algebra $\mathrm{Sym}_1(\mathrm{KhZ})$ of the KhZ equation is one-dimensional right extension, (Fuks, 1986, § 1.4.4), of the contact symmetry algebra $\mathrm{Sym}_1(\mathrm{KP})$ of the KP equation, i.e. $\mathrm{Sym}_1(\textsc{KhZ}) = \mathrm{Sym}_1(\mathrm{KP}) \rtimes \mathbb{R}\, V$, where V is the scaling symmetry of the KhZ Eq. (2.32). Therefore, all the invariant solutions of the KP equation studied in (David et al., 1986) survive in the dispersionless limit. The corresponding invariant solutions of the KhZ equation were studied in (Ndogmo, 2008, 2009). The open problem is to study how the dispersionless limit acts on the solutions of the KP equation that are obtained either by Hirota's direct method, (Hirota, 2004), or by using Darboux's transformation, (Matveev and Salle, 1991).

Nonlocal symmetries of Plebański's second heavenly equation

In Chapter 3 we examined nonlocal symmetries of Plebański's second heavenly equation (3.1) in the covering (3.9). The equation is of importance in general relativity, since it defines metrics solving Einstein equations under certain assumptions. We obtained a number of interesting results. First, it appears that all local symmetries of the equation can be lifted to nonlocal ones. Moreover, the equation admits two infinite hierarchies of nonlocal commuting symmetries, and one of these hierarchies can be employed to construct new solutions using the techniques presented e.g. in (Krasil′shchik et al., 1986), (Olver, 2000), (Krasil′shchik and Vinogradov, 1999), (Reyes, 2006), (Hernández-Heredero and Reyes, 2012, 2013), (Bies et al., 2012). Finally, we described the structure of the resulting Lie algebra of nonlocal symmetries. We emphasize that finding an explicit form of nonlocal symmetries and the commutator relations for an infinite-dimensional symmetry algebra of nonlocal symmetries (rather than just shadows) is quite rare. We know just a number of similar results in the literature, (Ablowitz et al., 1993), (Finley III and McIver, 2004), (Hernández-Heredero and Reyes, 2012, 2013), (Bies et al., 2012), (Morozov and Sergyeyev, 2014), (Igonin and Marvan, 2014), (Baran et al., 2016a,b). We think that it would be very interesting to study how the infinite-dimensional Lie algebra of nonlocal symmetries reflects the algebraic structure behind the integrability properties of the considered equation, cf. (Morozov, 2017, 2018).

Nonlocal conservation laws

In Chapter 4 we studied three-component nonlocal conservation laws for five three-dimensional Lax-integrable equations. These are Pavlov's (4.2), the rdDym Eq. (4.6), the mVw Eq. (4.10), the UH Eq. (4.8) and the Vw Eq. (4.12). Pavlov's equation arises in the theory of hydrodynamic chains and with regard to Einstein-Weyl structures, Vw equation has applications to odd-dimensional bi-Hamiltonian systems, while all the equations are nonlocally interrelated. The conservation laws for Pavlov's Eq. (4.2) and the potential KhZ Eq. (4.15) were found in (Makridin and Pavlov, 2017). We added a proof of their nontriviality. For each of the equation we found two three-dimensional nonlocal conservation laws. One conservation law of each pair depends on one pseudopotential, while the other depends on two pseudopotentials and therefore is defined on a Whitney product of coverings. We proved nontriviality of all the conservation laws found in Chapter 4. Three-dimensional nonlocal conservation laws are rarely examined in the literature – our motivation for looking for such objects follows from a recent article by Makridin and Pavlov (2017). One of the application of the conservation laws we found is to use them as a source of infinite hierarchy of three-dimensional conservation laws, as demonstrated in Section 4.2.6 and Appendix C. Existence of such a hierarchy imposes a strong constraint for the dynamics of the system. Another possible application of the conservation laws we found is to use a method developed by (Ibragimov, 2012) to find exact solutions of the examined equations.

All the examined equations are related via Bäcklund transformations and we demonstrated the resulting correspondences between the nonlocal conservation laws. In particular, we proved that the nonlocal conservation laws that depend on one pseudopotential are generated from a local conservation law of the Vw equation via appropriate superpositions of Bäcklund transformations, as stated in Theorem 4.4.1. This procedure is purely computational, i.e. in general an action of a Bäcklund transformation on a conservation law is not defined. The computations were possible due to special forms of nonlocal conservation laws and Bäcklund transformations. In particular, the derivatives appearing in the nonlocal conservation laws could always be substituted with right hand sides of equations defining Bäcklund transformations.

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Appendix A

Maple scripts

We do not intend to discuss here the usage of neither Maple, nor *Jets* in particular. The manual and documentation are available at the webpage *http://jets.math.slu.cz/*. Below, the inputs are printed in black, with comments in italics, and the outputs are printed in blue.

A.1 Computation of symmetries and their algebra structure for the GT equation

```
> read("Jets.s");
                    `JETS 5.0`
   `Differential calculus on jet spaces and diffieties`
                 `for Maple 6, 7, 8, 9`
                 `as of 6 December 2005`

`Blimit = 25000  ressize = 500  putsize = 40  maxsize = 20`
> coordinates([x, y], [u], 3):
> GibbonsTsarev := equation(u_yy = (a*y+u_y)*u_xx-u_x*u_xy-2*a):
> dependence(xi[1](x, y, u), xi[2](x, y, u), eta(x, y, u), a()):  nonzero(a):
```

// The function U is a generating section of a point symmetry of the Gibbons–Tsarev equation. For this reason we assume that it is in a special form. Computations for a general contact symmetry are longer but do not yield new symmetries.

```
> U := -u_x*xi[1]-u_y*xi[2]+eta:
> unknowns(xi[1], xi[2], eta):
```

// The command below produces the expression $S := \ell_{\mathcal{E}}(U)$*, cf.* (1.21).

```
> S := simplify(symmetries(u = U)):
```

// The run(S) command splits the equation S=0, cf. (1.22)–(1.28).

```
> run(S);
```

‘<0>‘, ‘Put: ‘ $$\frac{d}{du}\xi_2 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d^2}{du^2}\xi_1 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d}{dx}\xi_2 = 2\frac{d}{du}\xi_1$$

‘<0>‘, ‘Put: ‘ $$\frac{d^3}{du^3}\eta = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d^2}{du^2}\eta = 2\frac{d^2}{dxdu}\xi_1$$

‘<0>‘, ‘Put: ‘ $$\frac{d}{du}\eta = 2\frac{d}{dx}\xi_1 - \frac{d}{dy}\xi_2$$

‘<0>‘, ‘Put: ‘ $$\frac{d^3}{dudx^2}\xi_1 = 0$$

$$\frac{d^3}{dudxdy}\xi_1 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d^2}{dx^2}\xi_1 = 2\frac{d^2}{dydu}\xi_1$$

$$\frac{d^2}{dy^2}\xi_2 = 2\frac{d^2}{dxdy}\xi_1 - 2\left(\frac{d^2}{dxdu}\xi_1\right)ay$$

‘<0>‘, ‘Put: ‘ $$\frac{d}{dx}\eta = -4\,ay\frac{d}{du}\xi_1 + 2\frac{d}{dy}\xi_1$$

$$\frac{d}{dy}\eta = 2\left(\frac{d}{dx}\xi_1\right)ay - a\xi_2 - 2\left(\frac{d}{dy}\xi_2\right)ay$$

$$\frac{d^3}{dudy^2}\xi_1 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d^2}{dy^2}\xi_1 = a\frac{d}{du}\xi_1$$

$$\frac{d^2}{dxdu}\xi_1 = 0$$

$$\frac{d^2}{dydu}\xi_1 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d^2}{dxdy}\xi_1 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d}{du}\xi_1 = 0$$

‘<0>‘, ‘Put: ‘ $$\frac{d}{dy}\xi_2 = 2/3\frac{d}{dx}\xi_1$$

‘<0>‘, ‘Success! ‘

```
> Z := clear(pds); dependence();
```

$$\left\{\frac{d}{du}\eta - 4/3\frac{d}{dx}\xi_1, \frac{d}{dx}\eta - 2\frac{d}{dy}\xi_1, \frac{d}{dy}\xi_2 - 2/3\frac{d}{dx}\xi_1, \frac{d}{dy}\eta - 2/3\left(\frac{d}{dx}\xi_1\right)ay + a\xi_2, \frac{d^2}{dx^2}\xi_1, \frac{d^2}{dxdy}\xi_1, \frac{d^2}{dy^2}\xi_1\right\}$$

$$a = \{\}, \eta = \{u, x, y\}, \xi_1 = \{x, y\}, \xi_2 = \{y\}$$

```
> xi[1] := y*a[1]+A[2]:
> dependence(a[1](), A[2](x)):
```

```
> unknowns(a[1], A[2], xi[2], eta):
> run(Z);
```

'<0>', 'Put: '

$$\frac{d^2}{du^2}\eta = 0$$
$$\frac{d^2}{dx\,du}\eta = 0$$
$$\frac{d^2}{dy\,du}\eta = 0$$
$$\frac{d^2}{dx^2}A_2 = 0$$
$$\frac{d}{dy}\xi_2 = 2/3\,\frac{d}{dx}A_2$$

'<0>', 'Put: '

$$\frac{d}{du}\eta = 4/3\,\frac{d}{dx}A_2$$
$$\frac{d}{dx}\eta = 2\,a_1$$
$$\frac{d}{dy}\eta = 2/3\left(\frac{d}{dx}A_2\right)ay - a\xi_2$$

'<0>', 'Success! '

```
> Z := clear(pds); dependence();
```

$$\left\{\frac{d}{du}\eta - 4/3\,\frac{d}{dx}A_2, \frac{d}{dx}\eta - 2\,a_1, \frac{d}{dy}\xi_2 - 2/3\,\frac{d}{dx}A_2, \frac{d}{dy}\eta - 2/3\left(\frac{d}{dx}A_2\right)ay + a\xi_2, \frac{d^2}{dx^2}A_2\right\}$$

$$a = \{\}\,,\ \eta = \{u, x, y\}\,,\ A_2 = \{x\}\,,\ a_1 = \{\}\,,\ \xi_2 = \{y\}$$

```
> A[2] := x*a[2]+a[3]:
> dependence(a[2](), a[3]()):
> unknowns(a[1], a[2], a[3], xi[2], eta):
> run(Z);
```

'<0>', 'Put: '

$$\frac{d^2}{du^2}\eta = 0$$
$$\frac{d^2}{du\,dx}\eta = 0$$
$$\frac{d^2}{du\,dy}\eta = 0$$
$$\frac{d}{dy}\xi_2 = 2/3\,a_2$$

'<0>', 'Put: '

$$\frac{d}{du}\eta = 4/3\,a_2$$
$$\frac{d}{dx}\eta = 2\,a_1$$
$$\frac{d}{dy}\eta = 2/3\,a_2ay - a\xi_2$$

'<0>', 'Success! '

```
> Z := clear(pds);  dependence();
```

$$\left\{\frac{d}{du}\eta - 4/3\,a_2, \frac{d}{dx}\eta - 2\,a_1, \frac{d}{dy}\xi_2 - 2/3\,a_2, \frac{d}{dy}\eta - 2/3\,a_2ay + a\xi_2\right\}$$

$$a = \{\},\ \eta = \{u, x, y\},\ a_1 = \{\},\ a_2 = \{\},\ a_3 = \{\},\ \xi_2 = \{y\}$$

```
> xi[2] := (2/3)*a[2]*y+a[4]:
> dependence(a[4]()):
> eta := 2*x*a[1]+A[3]:
> dependence(A[3](y, u)):
> unknowns(a[1], a[2], a[3], a[4], A[3]):
> run(Z):
```

'<0>', 'Put: '

$$\frac{d^2}{du^2}A_3 = 0$$
$$\frac{d^2}{du dy}A_3 = 0$$

'<0>', 'Put: '

$$\frac{d}{du}A_3 = 4/3\, a_2$$
$$\frac{d}{dy}A_3 = -a a_4$$

'<0>', 'Success! '

```
> Z := clear(pds);
```

$$\left\{a a_4 + \frac{d}{dy}A_3, \frac{d}{du}A_3 - 4/3\, a_2\right\}$$

```
> A[3] := (4/3)*a[2]*u+A[4]:
> dependence(A[4](y)):
> unknowns(a[1], a[2], a[3], a[4], A[4]):
> run(Z);
```

'<0>', 'Put: '

$$\frac{d}{dy}A_4 = -a a_4$$

'<0>', 'Success! '

```
> A[4] := -a*y*a[4]+a[5]:
> dependence(a[5]());
```

$$a = \{\},\ a_1 = \{\},\ a_2 = \{\},\ a_3 = \{\},\ a_4 = \{\},\ a_5 = \{\}$$

```
> U;
```

$$-(x\, a_2 + y\, a_1 + a_3)\, u_x - (2/3\, a_2 y + a_4)\, u_y + 2\, a_1 x + 4/3\, a_2 u - a\, a_4 y + a_5$$

```
> phi[1] := subs(a[2] = 0, a[3] = 0, a[4] = 0, a[5] = 0, a[1] = 1, U);
```

$$-y\, u_x + 2\, x$$

```
> phi[2] := subs(a[1] = 0, a[3] = 0, a[4] = 0, a[5] = 0, a[2] = 1, U);
```

$$-x\, u_x - 2/3\, y\, u_y + 4/3\, u$$

```
> phi[3] := subs(a[2] = 0, a[1] = 0, a[4] = 0, a[5] = 0, a[3] = 1, U);
```

$$-u_x$$

```
> phi[4] := subs(a[2] = 0, a[3] = 0, a[1] = 0, a[5] = 0, a[4] = 1, U);
```

$$-u_y - a\, y$$

```
> phi[5] := subs(a[2] = 0, a[3] = 0, a[4] = 0, a[1] = 0, a[5] = , U);
```

$$1$$

// The Jacobi(,) command computes the Jacobi bracket of two functions. If φ_1 is a generating section of a vector field V, and φ_2 is a generating section of a vector field W, then $\{\varphi_1, \varphi_2\}$ is a generating section of a vector field $[V, W]$.

```
> expand(Jacobi([phi[1]], [phi[2]])[1]);
```

$$-1/3\,y\,u_x + 2/3\,x$$

```
> expand(Jacobi([phi[1]], [phi[3]])[1]);
```

$$-2$$

```
> expand(Jacobi([phi[1]], [phi[4]])[1]);
```

$$u_x$$

```
> expand(Jacobi([phi[1]], [phi[5]])[1]);
```

$$0$$

```
> expand(Jacobi([phi[2]], [phi[3]])[1]);
```

$$u_x$$

```
> expand(Jacobi([phi[2]], [phi[4]])[1]);
```

$$2/3\,u_y + 2/3\,a\,y$$

```
> expand(Jacobi([phi[2]], [phi[5]])[1]);
```

$$-4/3$$

```
> expand(Jacobi([phi[3]], [phi[4]])[1]);
```

$$0$$

```
> expand(Jacobi([phi[3]], [phi[5]])[1]);
```

$$0$$

```
> expand(Jacobi([phi[4]], [phi[5]])[1]);
```

$$0$$

// If φ_1 is a generating section of a vector field V, and φ_2 is a generating section of a vector field W, then the ad_n(φ_1, φ_2,k) function computes $\mathrm{ad}_V^k(W)$, and the function Ad(φ_1, φ_2) returns first five terms of the series $\mathrm{Ad}_{\exp(\epsilon V)}(W)$, see formula (2.2).

```
> ad_n := proc(v, w, n)
    if n = 0 then
      w;
    elif n = 1 then
      expand(-Jacobi([v], [w]));
    else
      expand(-Jacobi([v], ad_n(v, w, n - 1)));
    end if;
  end proc;

> Ad := proc(v, w)
  local A, i;
    A := w;
    for i while not(ad_n(v, w, i)[1] = 0) and i < 6 do
      A := A + ε^i/i! * ad_n(v, w, i)[1];
    end do;
    return A
  end proc;
```

```
> expand(Ad(phi[1], phi[2])-phi[2]+(1/3)*epsilon*phi[1])
```

$$0$$

```
> expand(Ad(phi[3], phi[1])-phi[1]+2*epsilon*phi[5])
```

$$0$$

```
> expand(Ad(phi[3], phi[2])-phi[2]+epsilon*phi[3])
```

$$0$$

```
> expand(Ad(phi[4], phi[1])-phi[1]+epsilon*phi[3])
```

```
> expand(Ad(phi[4], phi[2])-phi[2]+(2/3)*epsilon*phi[4])
```
$$0$$

```
> expand(Ad(phi[5], phi[2])-phi[2]+(4/3)*epsilon*phi[5])
```
$$0$$

```
> expand(Ad(phi[1], phi[3])-phi[3]-2*epsilon*phi[5])
```
$$0$$

```
> expand(Ad(phi[1], phi[4])-phi[4]-epsilon*phi[3]-epsilon^2*phi[5])
```
$$0$$

```
> collect(expand(Ad(phi[2], phi[1])), [u_x, x, y])
```
$$\left(-1-1/3\,\epsilon-1/18\,\epsilon^2-\frac{\epsilon^3}{162}-\frac{\epsilon^4}{1944}-\frac{\epsilon^5}{29160}\right)y\,u_x+\left(2+2/3\,\epsilon+1/9\,\epsilon^2+\frac{\epsilon^3}{81}+\frac{\epsilon^4}{972}+\frac{\epsilon^5}{14580}\right)x$$

```
> expand(series(exp((1/3)*epsilon), epsilon = 0, 6)*phi[1])
```
$$-\left(1+\epsilon/3+1/18\,\epsilon^2+\frac{\epsilon^3}{162}+\frac{\epsilon^4}{1944}+\frac{\epsilon^5}{29160}+O\left(\epsilon^6\right)\right)y\,u_x+\left(2+2/3\,\epsilon+1/9\,\epsilon^2+\frac{\epsilon^3}{81}+\frac{\epsilon^4}{972}+\frac{\epsilon^5}{14580}+O\left(\epsilon^6\right)\right)x$$

```
> collect(Ad(phi[2], phi[3]), [u_x])
```
$$\left(-1-\epsilon-1/2\,\epsilon^2-1/6\,\epsilon^3-1/24\,\epsilon^4-\frac{\epsilon^5}{120}\right)u_x$$

```
> expand(series(exp(epsilon), epsilon = 0, 6)*phi[3])
```
$$-\left(1+\epsilon+1/2\,\epsilon^2+1/6\,\epsilon^3+1/24\,\epsilon^4+\frac{\epsilon^5}{120}+O\left(\epsilon^6\right)\right)u_x$$

```
> collect(Ad(phi[2], phi[4]), [u_y, y, epsilon])
```
$$\left(-2/3\,\epsilon-2/9\,\epsilon^2-\frac{4\,\epsilon^3}{81}-1-\frac{2\,\epsilon^4}{243}-\frac{4\,\epsilon^5}{3645}\right)u_y+\left(-2/9\,\epsilon^2a-a-2/3\,\epsilon\,a-\frac{2\,\epsilon^4a}{243}-\frac{4\,\epsilon^3a}{81}-\frac{4\,\epsilon^5a}{3645}\right)y$$

```
> expand(series(exp((2/3)*epsilon), epsilon = 0, 6)*phi[4])
```
$$-\left(1+2/3\,\epsilon+2/9\,\epsilon^2+\frac{4\,\epsilon^3}{81}+\frac{2\,\epsilon^4}{243}+\frac{4\,\epsilon^5}{3645}+O\left(\epsilon^6\right)\right)u_y-\left(1+2/3\,\epsilon+2/9\,\epsilon^2+\frac{4\,\epsilon^3}{81}+\frac{2\,\epsilon^4}{243}+\frac{4\,\epsilon^5}{3645}+O\left(\epsilon^6\right)\right)ay$$

```
> Ad(phi[2], phi[5])
```
$$1+4/3\,\epsilon+\frac{8\,\epsilon^2}{9}+\frac{32\,\epsilon^3}{81}+\frac{32\,\epsilon^4}{243}+\frac{128\,\epsilon^5}{3645}$$

```
> series(exp((4/3)*epsilon), epsilon = 0, 6)*phi[5]
```
$$1+4/3\,\epsilon+\frac{8\,\epsilon^2}{9}+\frac{32\,\epsilon^3}{81}+\frac{32\,\epsilon^4}{243}+\frac{128\,\epsilon^5}{3645}+O\left(\epsilon^6\right)$$

A.2 Computations for the proof of Theorem 3.4.1

```
> read("Jets.s"):
> coordinates([t, x, y, z], [u, q0, qk1, qk], 3):
> dependence(k()):
```

// Below we define Plebański's Eq. (3.1) *and its covering equations*(3.9).

```
> HE2 := convert([equation(u_xz = u_xx*u_yy - u_xy^2 + u_ty, q0_t = q0_x*u_xy -
```

```
q0_y*u_xx, q0_z = q0_x*u_yy - q0_y*u_xy, qk_t = qk_x*u_xy - qk_y*u_xx + qk1_x, qk_z
= qk_x*u_yy - qk_y*u_xy + qk1_y)], Vector);
```

$$\begin{bmatrix} u\_xz = u\_xx\, u\_yy - u\_xy^2 + u\_ty \\ q0\_t = u\_xy\, q0\_x - u\_xx\, q0\_y \\ q0\_z = u\_yy\, q0\_x - u\_xy\, q0\_y \\ qk\_t = u\_xy\, qk\_x - u\_xx\, qk\_y + qk1\_x \\ qk\_z = u\_yy\, qk\_x - u\_xy\, qk\_y + qk1\_y \end{bmatrix}$$

// Below we define linearisation of Plebański's Eq. (3.1) *and equations* (3.11).

```
> lin_sys := convert([TD(U, x, z)-TD(U, t, y)-u_yy*TD(U, x, x)-u_xx*TD(U, y, y)
+ 2*u_xy*TD(U, x, y) = 0, TD(Q, t)-u_xy*TD(Q, x)-q0_x*TD(U, x, y)+u_xx*TD(Q, y)+q0_y*TD(U,
x, x) = 0, TD(Q, z)-u_yy*TD(Q, x)-q0_x*TD(U, y, y)+u_xy*TD(Q, y)+q0_y*TD(U, x, y) =
0, TD(Qk, t)-u_xy*TD(Qk, x)-qk_x*TD(U, x, y)+u_xx*TD(Qk, y)+qk_y*TD(U, x, x)-TD(Qk1,
x) = 0, TD(Qk, z)-u_yy*TD(Qk, x)-qk_x*TD(U, y, y)+u_xy*TD(Qk, y)+qk_y*TD(U, x, y)-TD(Qk1,
y) = 0], Vector);
```

$$\begin{bmatrix} TD(U,xz) - TD(U,ty) - u\_yy\, TD(U,x^2) - u\_xx\, TD(U,y^2) + 2\, u\_xy\, TD(U,xy) = 0 \\ TD(Q,t) - u\_xy\, TD(Q,x) - q0\_x\, TD(U,xy) + u\_xx\, TD(Q,y) + q0\_y\, TD(U,x^2) = 0 \\ TD(Q,z) - u\_yy\, TD(Q,x) - q0\_x\, TD(U,y^2) + u\_xy\, TD(Q,y) + q0\_y\, TD(U,xy) = 0 \\ TD(Qk,t) - u\_xy\, TD(Qk,x) - qk\_x\, TD(U,xy) + u\_xx\, TD(Qk,y) + qk\_y\, TD(U,x^2) - TD(Qk1,x) = 0 \\ TD(Qk,z) - u\_yy\, TD(Qk,x) - qk\_x\, TD(U,y^2) + u\_xy\, TD(Qk,y) + qk\_y\, TD(U,xy) - TD(Qk1,y) = 0 \end{bmatrix}$$

```
> psi_2 := -3*t*u_t-u_x*x-u_y*y-3*u_z*z:
> Psi2[0] := simplify(-3*q0_t*t-q0_x*x-q0_y*y-3*q0_z*z):
```

// Below we verify that ψ_2 and $\Psi_{2,0}$ solve the first three equations of lin_sys.

```
> simplify(evalTD(subs(U = psi_2, Q = Psi2[0], lin_sys[1 .. 3])));
```

$$\begin{bmatrix} 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{bmatrix}$$

```
> Psi2[k-1] := simplify(-3*t*qk1_t-x*qk1_x-y*qk1_y-3*z*qk1_z+(2*(k-1))*qk1):
> Psi2[k] := simplify(2*k*qk-3*qk_t*t-qk_x*x-qk_y*y-3*qk_z*z):
```

// We assume, that $(U, Q_0, ..., Q_{k-1}) = (\psi_2, \Psi_{2,0}, ..., \Psi_{2,k-1})$ solves the system (3.11) *for $m \le k-1$. We verify below that then $\Psi_{2,k}$ satisfies th equations for Q_k in the system* (3.11) *(these are the last two equations of lin_sys).*

```
> result := simplify(evalTD(subs(U = psi_2, Qk = Psi2[k], Qk1 = Psi2[k-1], lin_sys[4
.. 5])));
```

$$\left[\begin{array}{c} -3\,qk1\_\_xx\,u\_\_yy\,z + 6\,u\_\_xy\,qk1\_\_xy\,z - 3\,u\_\_xx\,qk1\_\_yy\,z - 3\,qk1\_\_ty\,z + 3\,z\,qk1\_\_xz = 0 \\ 3\,qk1\_\_xx\,u\_\_yy\,t - 6\,u\_\_xy\,qk1\_\_xy\,t + 3\,u\_\_xx\,qk1\_\_yy\,t + 3\,t\,qk1\_\_ty - 3\,qk1\_\_xz\,t = 0 \end{array}\right]$$

```
> factor(result[1])
```

$$-3\,z\,(qk1\_\_xx\,u\_\_yy - 2\,u\_\_xy\,qk1\_\_xy + u\_\_xx\,qk1\_\_yy + qk1\_\_ty - qk1\_\_xz) = 0$$

```
> factor(result[2])
```

$$3\,t\,(qk1\_\_xx\,u\_\_yy - 2\,u\_\_xy\,qk1\_\_xy + u\_\_xx\,qk1\_\_yy + qk1\_\_ty - qk1\_\_xz) = 0$$

// The above two equations are satisfied whenever qk1, which corresponds to q_{k-1} *in the system* (3.11)*, is a shadow. Which means always, see Corollary 3.4.1.*

Appendix B

Proof of Theorem 3.5.1.

To facilitate reading we recall Theorem 3.5.1 and initial observations from its (partial) proof from Section 3.5.

Theorem 3.5.1. *The Jacobi brackets of lifts of the local symmetries are lifts of the Jacobi brackets of the corresponding local symmetries, that is, the commutators for* $\Phi_m(A)$*,* Ψ_k *satisfy equations* (3.4)—(3.8) *with* $\varphi_m(A)$*,* ψ_k *replaced by* $\Phi_m(A)$*,* Ψ_k*, respectively. The other Jacobi brackets are*

$$
\begin{aligned}
\{\Phi_m(A), \Upsilon_k\} &= \{\Phi_m(A), \Gamma_i\} = \{\Upsilon_k, \Upsilon_l\} = \{\Gamma_i, \Gamma_j\} = 0, \qquad 0 \le m \le 3, \quad k, l \ge 0, \quad i, j \ge 1,\\
\{\Psi_1, \Upsilon_k\} &= -(k+3)\,\Upsilon_k, \qquad \{\Psi_2, \Upsilon_k\} = 2\,k\,\Upsilon_k, \qquad \{\Psi_3, \Upsilon_k\} = (k+1)\,\Upsilon_{k+1},\\
\{\Psi_1, \Gamma_i\} &= -(i-1)\,\Gamma_i, \qquad \{\Psi_2, \Gamma_i\} = 2\,(i-1)\,\Gamma_i, \qquad \{\Psi_3, \Gamma_i\} = (i-1)\,\Gamma_{i-1},\\
\{\Gamma_i, \Upsilon_k\} &= \begin{cases} \Upsilon_{k-i+1}, & k \ge i-1,\\ 0, & k < i-1. \end{cases}
\end{aligned}
$$

Proof. We already proved in Section 3.5 that $\{\Phi_0(A), \Upsilon_k\} = 0$. Below we present the formula for computing the Jacobi bracket in the case of some infinite-dimensional vectors Θ and Ω. For convenient application of the formula to vectors $\Phi_0(A)$ and Υ_k we enumerate coordinates of the vectors Θ and Ω in the following way: $\Theta = (\Theta_{-1}, \Theta_0, \ldots, \Theta_m, \ldots)$, $\Omega = (\Omega_{-1}, \Omega_0, \ldots, \Omega_m, \ldots)$. Then the Jacobi bracket is of the form $\{\Theta, \Omega\} = (\{\Theta, \Omega\}_{-1}, \{\Theta, \Omega\}_0, \{\Theta, \Omega\}_1, \ldots)$, where

$$
\begin{aligned}
\{\Theta, \Omega\}_j &= \sum_I \Bigg(\tilde{D}_I(\Theta_{-1}) \frac{\partial}{\partial u_I}(\Omega_j) - \tilde{D}_I(\Omega_{-1}) \frac{\partial}{\partial u_I}(\Theta_j)\\
&+ \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Theta_m) \frac{\partial}{\partial q_{m,I}}(\Omega_j) - \tilde{D}_I(\Omega_m) \frac{\partial}{\partial q_{m,I}}(\Theta_j) \right) \Bigg), \qquad j \ge -1.
\end{aligned}
$$

As for coordinates $q_{m,I}$, the functions $\Phi_{0,j}(A)$ depend at most on $q_{j,t}$, $q_{j,x}$, $q_{j,y}$, $q_{j,z}$, q_j or q_{j+1}, and $\Upsilon_{k,j}$ depend at most on $q_{m,x}$, $q_{k+j+1-m,x}$, $q_{m,y}$, $q_{k+j+1-m,y}$ for $m = 0, \ldots, j$. The following observations will be used frequently:

$$
\begin{aligned}
\sum_I \sum_{m=0}^{\infty} \tilde{D}_I(a_m) \tfrac{\partial}{\partial q_{m,I}}(\Upsilon_{k,j}) &= \sum_{m=0}^{j} \Big(\tilde{D}_x(a_m) q_{k+j+1-m,y} - \tilde{D}_x(a_{k+j+1-m}) q_{m,y}\\
&- \tilde{D}_y(a_m) q_{k+j+1-m,x} + \tilde{D}_y(a_{k+j+1-m}) q_{m,x} \Big)\\
&= \sum_{m=0}^{j} \langle a_m, q_{k+j+1-m} \rangle - \langle a_{k+j+1-m}, q_m \rangle, \qquad a_m \in C^\infty(\mathcal{E} \times W).
\end{aligned}
\tag{B.1}
$$

We have

$$\begin{aligned}\{\Phi_i(A),\Upsilon_k\}_{-1} &= \sum_I \tilde{D}_I\left(\phi_i(A)\right)\underbrace{\frac{\partial}{\partial u_I}\left(q_k\right)}_{=0}-\sum_I \tilde{D}_I\left(q_k\right)\tfrac{\partial}{\partial u_I}\left(\varphi_i(A)\right)\\ &+\sum_{m=0}^{\infty}\underbrace{\tilde{D}_I\left(\Phi_{i,m}(A)\right)\frac{\partial}{\partial q_{m,I}}\left(q_k\right)}_{=\Phi_{i,k}(A)}-\sum_{m=0}^{\infty}\tilde{D}_I\left(\Upsilon_{k,m}\right)\underbrace{\frac{\partial}{\partial q_{m,I}}\left(\varphi_i(A)\right)}_{=0},\qquad 0\le i\le 3.\end{aligned}\tag{B.2}$$

It is easy to see that $\sum_I \tilde{D}_I\left(q_k\right)\frac{\partial}{\partial u_I}\left(\varphi_i(A)\right)=\Phi_i^k(A)$ for $i=0,1,2,3$, hence

$$\{\Phi_i(A),\Upsilon_k\}_{-1}=0,\qquad i=0,1,2,3.\tag{B.3}$$

Then for $j\ge 0$ we get

$$\begin{aligned}\{\Phi_i(A),\Upsilon_k\}_j &= \sum_I \tilde{D}_I\left(\varphi_i(A)\right)\underbrace{\frac{\partial}{\partial u_I}\left(\Upsilon_{k,j}\right)}_{=0}-\sum_I \tilde{D}_I\left(q_k\right)\underbrace{\frac{\partial}{\partial u_I}\left(\Phi_{i,j}(A)\right)}_{=0}\\ &+\sum_I\sum_{m=0}^{\infty}\Big(\tilde{D}_I\left(\Phi_{i,m}(A)\right)\tfrac{\partial}{\partial q_{m,I}}\left(\Upsilon_{k,j}\right)\Big)-\sum_I\sum_{m=0}^{\infty}\Big(\tilde{D}_I\left(\Upsilon_{k,m}\right)\tfrac{\partial}{\partial q_{m,I}}\left(\Phi_{i,j}(A)\right)\Big).\end{aligned}\tag{B.4}$$

(1) We verify that $\{\Phi_1(A),\Upsilon_k\}=0$, with $k\ge 0$. Due to (B.3) it is enough to consider the expression below for $j\ge 0$.

$$\begin{aligned}\{\Phi_1(A),\Upsilon_k\}_j &= \sum_{m=0}^{j}\Big(\langle\Phi_1^m(A),q_{k+j+1-m}\rangle-\langle\Phi_1^{k+j+1-m}(A),q_m\rangle\Big)\\ &- \sum_{l=x,y}\tilde{D}_l\left(\sum_{m=0}^{j}\langle q_m,q_{k+j+1-m}\rangle\right)\tfrac{\partial}{\partial q_{j,l}}\Phi_1^j(A)\\ &= \sum_{m=0}^{j}\Big(\langle\Phi_1^m(A),q_{k+j+1-m}\rangle-\langle\Phi_1^{k+j+1-m}(A),q_m\rangle\\ &- \sum_{l=x,y}\left(\langle q_{m,l},q_{k+j+1-m}\rangle+\langle q_m,q_{k+j+1-m,l}\rangle\right)\tfrac{\partial}{\partial q_{j,l}}\Phi_1^j(A)\Big)\\ &= \sum_{m=0}^{j}\Big(\langle\Phi_1^m(A),q_{k+j+1-m}\rangle-\langle\Phi_1^{k+j+1-m}(A),q_m\rangle\\ &+ \langle\Phi_1^{k+j+1-m}(A),q_m\rangle-\langle\Phi_1^m(A),q_{k+j+1-m}\rangle\Big)=0.\qquad\diamond\end{aligned}\tag{B.5}$$

(2) We verify that $\{\Psi_1,\Upsilon_k\}=-(k+3)\,\Upsilon_k$, with $k\ge 0$. For $j=-1$ we have

$$\begin{aligned}\{\Psi_1,\Upsilon_k\}_{-1} &= \sum_I\left(\tilde{D}_I\left(\psi_1\right)\underbrace{\frac{\partial}{\partial u_I}\left(q_k\right)}_{=0}-\tilde{D}_I\left(q_k\right)\tfrac{\partial}{\partial u_I}\left(\psi_1\right)\right)\\ &+ \sum_I\sum_{m=0}^{\infty}\left(\tilde{D}_I\left(\Psi_1^m\right)\tfrac{\partial}{\partial q_{m,I}}\left(q_k\right)-\tilde{D}_I\left(\Upsilon_k^m\right)\underbrace{\frac{\partial}{\partial q_{m,I}}\left(\psi_1\right)}_{=0}\right)\\ &= -\sum_I\tilde{D}_I\left(q_k\right)\tfrac{\partial}{\partial u_I}\left(\psi_1\right)+\Psi_1^k=-(k+3)\,q_k.\end{aligned}\tag{B.6}$$

For $j \geq 0$ we have

$$
\begin{aligned}
\{\Psi_1, \Upsilon_k\}_j &= \sum_I \left(\tilde{D}_I(\psi_1) \underbrace{\frac{\partial}{\partial u_I}\left(\Upsilon_k^j\right)}_{=0} - \tilde{D}_I(q_k) \underbrace{\frac{\partial}{\partial u_I}\left(\Psi_1^j\right)}_{=0} \right) \\
&+ \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Psi_1^m) \tfrac{\partial}{\partial q_{m,I}} \left(\Upsilon_k^j\right) - \tilde{D}_I(\Upsilon_k^m) \tfrac{\partial}{\partial q_{m,I}} \left(\Psi_1^j\right) \right) \\
&= \sum_{m=0}^{j} \left(\tilde{D}_x(\Psi_1^m)\, q_{k+j+1-m,y} - \tilde{D}_x\left(\Psi_1^{k+j+1-m}\right) q_{m,y} - \tilde{D}_y(\Psi_1^m)\, q_{k+j+1-m,x} \right. \\
&+ \left. \tilde{D}_y\left(\Psi_1^{k+j+1-m}\right) q_{m,x} \right) - \sum_I \left(\tilde{D}_I\left(\Upsilon_k^j\right) \tfrac{\partial}{\partial q_{j,I}} \Psi_1^j \right) \\
&= \sum_{m=0}^{j} \Big(-(q_{m,x} + x\, q_{m,x,x} + y\, q_{m,y,x} + m\, q_{m,x})\, q_{k+j+1-m,y} \\
&+ (q_{k+j+1-m,x} + x\, q_{k+j+1-m,x,x} + y\, q_{k+j+1-m,y,x} + (k+j+1-m)\, q_{k+j+1-m,x}) q_{m,y} \\
&+ (x\, q_{m,x,y} + q_{m,y} + y\, q_{m,y,y} + m\, q_{m,y})\, q_{k+j+1-m,x} \\
&- (x\, q_{k+j+1-m,x,y} + q_{k+j+1-m,y} + y\, q_{k+j+1-m,y,y} + (k+j+1-m)\, q_{k+j+1-m,y})\, q_{m,x} \Big) \\
&+ j\, \Upsilon_k^j + x \sum_{m=0}^{j} (\langle q_{m,x}, q_{k+j+1-m} \rangle + \langle q_m, q_{k+j+1-m,x} \rangle) \\
&+ y \sum_{m=0}^{j} (\langle q_{m,y}, q_{k+j+1-m} \rangle + \langle q_m, q_{k+j+1-m,y} \rangle) \\
&= \sum_{m=0}^{j} -x \langle q_{m,x}, q_{k+j+1-m} \rangle - y \langle q_{m,y}, q_{k+j+1-m} \rangle - \langle q_m, q_{k+j+1-m} \rangle \\
&+ x \langle q_{k+j+1-m,x}, q_m \rangle + x \langle q_{k+j+1-m,x}, q_m \rangle + y \langle q_{k+j+1-m,y}, q_m \rangle \\
&+ \langle q_{k+j+1-m}, q_m \rangle - m \langle q_m, q_{k+j+1-m} \rangle - (k+j+1-m) \langle q_m, q_{k+j+1-m} \rangle \\
&+ x (\langle q_{m,x}, q_{k+j+1-m} \rangle + \langle q_m, q_{k+j+1-m,x} \rangle) + y (\langle q_{m,y}, q_{k+j+1-m} \rangle + \langle q_m, q_{k+j+1-m,y} \rangle) \\
&+ j\, \Upsilon_k^j = j\, \Upsilon_k^j - (k+j+1)\Upsilon_k^j - 2\, \Upsilon_k^j = -(k+3)\Upsilon_k^j. \qquad \diamond
\end{aligned}
$$

(3) We verify that $\{\Psi_2, \Upsilon_k\} = 2\,k\,\Upsilon_k$, with $k \geq 0$. To prove that $\{\Psi_2, \Upsilon_k\}_{-1} = 2\,k\,q_k$ is to mimic the computations of (B.6). We proceed to the case of $j \geq 0$.

$$\{\Psi_2, \Upsilon_k\}_j = \sum_I \left(\tilde{D}_I\,(\psi_2) \underbrace{\frac{\partial}{\partial\,u_I} \Upsilon_k^j}_{=0} - \tilde{D}_I\,(q_k) \underbrace{\frac{\partial}{\partial\,u_I} \Psi_1^j}_{=0} \right) + \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I\,(\Psi_2^m)\, \tfrac{\partial}{\partial\,q_{m,l}} \Upsilon_k^j \right) \tag{B.7}$$

$$- \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I\,(\Upsilon_k^m)\, \tfrac{\partial}{\partial\,q_{m,I}} \left(\Psi_2^j\right)\right).$$

$$\begin{aligned}
\{\Psi_2, \Upsilon_k\}_j &= \sum_{m=0}^{j} \Big\{ \tilde{D}_x\,(\Psi_2^m)\, q_{k+j+1-m,y} - \tilde{D}_x \left(\Psi_2^{k+j+1-m}\right) q_{m,y} - \tilde{D}_y\,(\Psi_2^m)\, q_{k+j+1-m,x} \\
&+ \tilde{D}_y \left(\Psi_2^{k+j+1-m}\right) q_{m,x} \Big\} - \sum_I \left(\tilde{D}_I \left(\Upsilon_k^j\right) \tfrac{\partial}{\partial\,q_{j,I}} \left(\Psi_2^j\right)\right) \\
&= \sum_{m=0}^{j} \Big((-3\,t\,q_{m,t,x} - q_{m,x} - x\,q_{m,x,x} - y\,q_{m,y,x} - 3\,z\,q_{m,z,x} + 2m\,q_{m,x})\, q_{k+j+1-m,y} \\
&+ (3\,t\,q_{k+j+1-m,t,x} + q_{k+j+1-m,x} + x\,q_{k+j+1-m,x,x} + y\,q_{k+j+1-m,y,x} \\
&+ 3\,z\,q_{k+j+1-m,z,x} - 2(k+j+1-m)\,q_{k+j+1-m,x})\, q_{m,y} \\
&+ (3\,t\,q_{m,t,y} + x\,q_{m,x,y} + q_{m,y} + y\,q_{m,y,y} + 3\,z\,q_{m,z,y} - 2\,m\,q_{m,y})\, q_{k+j+1-m,x} \\
&- (3\,t\,q_{k+j+1-m,t,y} + x\,q_{k+j+1-m,x,y} + q_{k+j+1-m,y} + y\,q_{k+j+1-m,y,y} \\
&+ 3\,z\,q_{k+j+1-m,z,y} - 2(k+j+1-m)\,q_{k+j+1-m,y})\, q_{m,x} \Big) - 2\,j\,\Upsilon_k^j + 3\,t\,\tilde{D}_t \left(\Upsilon_k^j\right) \\
&+ x\,\tilde{D}_x \left(\Upsilon_k^j\right) + 3\,z\,\tilde{D}_z \left(\Upsilon_k^j\right) = -3\,t\,\tilde{D}_t \left(\Upsilon_k^j\right) - x\,\tilde{D}_x \left(\Upsilon_k^j\right) - 3\,z\,\tilde{D}_z \left(\Upsilon_k^j\right) - 2\,\Upsilon_k^j \\
&+ \sum_{m=0}^{j} 2\,m\,\langle q_m, q_{k+j+1-m}\rangle + 2(k+j+1-m)\langle q_m, q_{k+j+1-m}\rangle \\
&- 2\,j\,\Upsilon_k^j + 3\,t\tilde{D}_t \left(\Upsilon_k^j\right) + x\,\tilde{D}_x \left(\Upsilon_k^j\right) + 3\,z\,\tilde{D}_z \left(\Upsilon_k^j\right) = 2\,k\,\Upsilon_k^j. \qquad \diamond
\end{aligned}$$

(4) We verify that $\{\Psi_3, \Upsilon_k\} = (k+1)\,\Upsilon_{k+1}$, with $k \geq 0$. The computations of the formula below for the case $j=-1$ are analogous to the ones presented in (B.6). Therefore we consider $j \geq 0$.

$$
\begin{aligned}
\{\Psi_3, \Upsilon_k\}_j \; &= \sum_I \left(\tilde{D}_I\,(\psi_3)\,\underbrace{\frac{\partial}{\partial\, u_I}\Upsilon_k^j}_{=0} - \tilde{D}_I\,(q_k)\,\underbrace{\frac{\partial}{\partial\, u_I}\Psi_3^j}_{=0} \right) + \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I\,(\Psi_3^m)\,\tfrac{\partial}{\partial\, q_{m,I}}\Upsilon_k^j \right) \\
&- \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I\,(\Upsilon_k^m)\,\tfrac{\partial}{\partial\, q_{m,I}}\left(\Psi_3^j\right)\right).
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
\{\Psi_3, \Upsilon_k\}_j \; &= \sum_{m=0}^{j} \Big\{ \tilde{D}_x\,(\Psi_3^m)\, q_{k+j+1-m,y} - \tilde{D}_x\left(\Psi_3^{k+j+1-m}\right) q_{m,y} - \tilde{D}_y\,(\Psi_3^m)\, q_{k+j+1-m,x} \\
&+ \tilde{D}_y\left(\Psi_3^{k+j+1-m}\right) q_{m,x} \Big\} - \sum_I \left(\tilde{D}_I\left(\Upsilon_k^{j+1}\right) \tfrac{\partial}{\partial\, q_{j+1,I}}\left(\Psi_3^j\right) - \tilde{D}_I\left(\Upsilon_k^j\right)\tfrac{\partial}{\partial\, q_{j,I}}\left(\Psi_3^j\right)\right) \\
&= \sum_{m=0}^{j} \Big((-t\, q_{m,x,x} - z\, q_{m,y,x} + (m+1)\, q_{m+1,x})\, q_{k+j+1-m,y} \\
&+ (t\, q_{k+j+1-m,x,x} + z\, q_{k+j+1-m,x,y} + (k+j+2-m)\, q_{k+j+2-m,x})\, q_{m,y} \\
&+ (t\, q_{m,x,y} + z\, q_{m,y,y} - (m+1)\, q_{m+1,y})\, q_{k+j+1-m,x} \\
&- (t\, q_{k+j+1-m,x,y} + z\, q_{k+j+1-m,y,y} - (k+j+2-m)\, q_{k+j+2-m,y})\, q_{m,x} \Big) \\
&- (j+1)\Upsilon_k^{j+1} + t\tilde{D}_x\left(\Upsilon_k^j\right) + z\,\tilde{D}_y\left(\Upsilon_k^j\right) \\
&= \sum_{m=0}^{j} (m+1)\langle q_{m+1}, q_{k+j+1-m}\rangle + (k+j+2-m)\langle q_m, q_{k+j+2-m}\rangle - (j+1)\Upsilon_k^{j+1} \\
&= \sum_{s=0}^{j+1} s\langle q_s, q_{(k+1)+j+1-s}\rangle + \sum_{s=0}^{j}(k+j+2-s)\langle q_s, q_{(k+1)+j+1-s}\rangle - (j+1)\Upsilon_k^{j+1} \\
&= \sum_{s=0}^{j} \left(s\langle q_s, q_{(k+1)+j+1-s}\rangle + (k+j+2-s)\langle q_s, q_{(k+1)+j+1-s}\rangle \right) \\
&+ (j+1)\langle q_{j+1}, q_{k+1}\rangle - (j+1)\Upsilon_k^{j+1} \\
&= (k+j+2)\Upsilon_{k+1}^j + (j+1)\langle q_{j+1}, q_{k+1}\rangle - (j+1)\underbrace{\Upsilon_k^{j+1}}_{=\Upsilon_{k+1}^j + \langle q_{j+1}, q_{k+1}\rangle} \\
&= (k+1)\,\Upsilon_{k+1}^j
\end{aligned}
$$

$\diamond$

(5) We verify that $\{\Phi_i(A), \Gamma_k\} = 0$, with $i \in \{0,1,2,3\}$ and $k \geq 1$. Observe that $\Gamma_{k,j} = 0$ for $j \leq k-2$, and $\Gamma_{k,j} = q_{j-k+1}$ for $j \geq k-1$.

$$\begin{aligned}\{\Phi_i(A), \Gamma_k\}_{-1} &= \sum_I \tilde{D}_I(\phi_i(A)) \tfrac{\partial}{\partial u_I}(0) - \sum_I \tilde{D}_I(0) \tfrac{\partial}{\partial u_I}(\varphi_i(A)) \\ &+ \sum_{m=0}^{\infty} \tilde{D}_I(\Phi_{i,m}(A)) \tfrac{\partial}{\partial q_{m,I}}(0) - \sum_{m=0}^{\infty} \tilde{D}_I(\Gamma_{k,m}) \underbrace{\frac{\partial}{\partial q_{m,I}}(\varphi_i(A))}_{=0} = 0. \end{aligned} \tag{B.9}$$

For $j \geq k-1$ we have

$$\begin{aligned}\{\Phi_i(A), \Gamma_k\}_j &= \sum_I \tilde{D}_I(\varphi_i(A)) \underbrace{\frac{\partial}{\partial u_I}(\Gamma_{k,j})}_{=0} - \sum_I \tilde{D}_I(0) \tfrac{\partial}{\partial u_I}(\Phi_{i,j}(A)) \\ &+ \underbrace{\sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Phi_{i,m}(A)) \frac{\partial}{\partial q_{m,I}}(\Gamma_{k,j})\right)}_{=\Phi_{i,j-k+1}} - \underbrace{\sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Gamma_{k,m}) \frac{\partial}{\partial q_{m,I}}(\Phi_{i,j}(A))\right)}_{=\sum_I \tilde{D}_I(q_{j-k+1}) \frac{\partial}{\partial q_{j,I}}(\Phi_{i,j}(A)) = \Phi_{i,j-k+1}} = 0. \end{aligned}$$

$\diamond$

(6) We compute Jacobi brackets of the form $\{\Psi_k, \Gamma_i\}$, with $i \geq 1$. Recall that $\Gamma_{i,j} = 0$ for $j \leq i-2$. We consider only the nonzero terms arising for $j \geq i-1$. For $k = 1, 2$ we have

$$\begin{aligned}\{\Psi_k, \Gamma_i\}_j &= \sum_I \tilde{D}_I(\psi_k) \underbrace{\frac{\partial}{\partial u_I}(\Gamma_{i,j})}_{=0} - \sum_I \tilde{D}_I(0) \tfrac{\partial}{\partial u_I}(\Psi_{k,j}) \\ &+ \underbrace{\sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Psi_{k,m}) \frac{\partial}{\partial q_{m,I}}(\Gamma_{i,j})\right)}_{\Psi_{k,j-i+1}} - \underbrace{\sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Gamma_{i,m}) \frac{\partial}{\partial q_{m,I}}(\Psi_{k,j})\right)}_{=\sum_I \tilde{D}_I(q_{j-i+1}) \frac{\partial}{\partial q_{j,I}}(\Psi_{k,j})}. \end{aligned} \tag{B.10}$$

It is straightforward to verify that $\{\Psi_1, \Gamma_i\}_j = -(i-1)\, q_{j-i+1} = -(i-1)\, \Gamma_{i,j}$ and $\{\Psi_2, \Gamma_i\}_j = 2\,(i-1)\, q_{j-i+1} = 2\,(i-1)\, \Gamma_{i,j}$. For $\{\Psi_3, \Gamma_i\}$ we have

$$\begin{aligned}\{\Psi_3, \Gamma_i\}_j &= \sum_I \tilde{D}_I(\psi_3) \underbrace{\frac{\partial}{\partial u_I}(\Gamma_{i,j})}_{=0} - \sum_I \tilde{D}_I(0) \tfrac{\partial}{\partial u_I}(\Psi_{3,j}) \\ &+ \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Psi_{3,m}) \tfrac{\partial}{\partial q_{m,I}}(\Gamma_{i,j})\right) - \sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Gamma_{i,m}) \tfrac{\partial}{\partial q_{m,I}}(\Psi_{3,j})\right) \\ &= \Psi_{3,j-i+1} - \sum_I \tilde{D}_I(q_{j-i+1}) \tfrac{\partial}{\partial q_{j,I}}(\Psi_{3,j}) + q_{j+1-i+1} \tfrac{\partial}{\partial q_{j+1}}(\Psi_{3,j}) = (i-1)\, \Gamma_{i-1}. \end{aligned}$$

$\diamond$

(7) We verify that $\{\Gamma_i, \Gamma_k\} = 0$. Obviously $\{\Gamma_k, \Gamma_i\}_{-1} = 0$. We have $\Gamma_{i,j} = q_{j-i+1}$ for $j \geq i-1$ and $\Gamma_{k,j} = q_{j-k+1}$ for $j \geq k-1$. Hence, the potential nonzero terms, for $j \geq \max\{i-1, k-1\}$ are

$$\{\Gamma_k, \Gamma_i\}_j = \underbrace{\sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Gamma_{k,m}) \frac{\partial}{\partial q_{m,I}}(\Gamma_{i,j})\right)}_{=\Gamma_{i,j-k+1} = q_{j-k+1-i+1}} - \underbrace{\sum_I \sum_{m=0}^{\infty} \left(\tilde{D}_I(\Gamma_{i,m}) \frac{\partial}{\partial q_{m,I}}(\Gamma_{k,j})\right)}_{=\Gamma_{k,j-i+1} = q_{j-i+1-k+1}} = 0.$$

$\diamond$

(8) We verify that $\{\Upsilon_k, \Upsilon_l\} = 0$.

$$\begin{aligned}\{\Upsilon_k, \Upsilon_l\}_{-1} &= \sum_I \tilde{D}_I(q_k) \tfrac{\partial}{\partial u_I}(q_l) - \sum_I \tilde{D}_I(q_l) \tfrac{\partial}{\partial u_I}(q_k) \\ &+ \underbrace{\sum_{m=0}^{\infty} \tilde{D}_I(\Upsilon_{k,m}) \frac{\partial}{\partial q_{m,I}}(q_l)}_{=\Upsilon_{k,l}} - \underbrace{\sum_{m=0}^{\infty} \tilde{D}_I(\Upsilon_{l,m}) \frac{\partial}{\partial q_{m,I}}(q_k)}_{=\Upsilon_{l,k}} = 0. \end{aligned} \tag{B.11}$$

Without loss of generality we assume that $k < l$. Then

$$\Upsilon_{k,l} - \Upsilon_{l,k} = \sum_{s=k+1}^{l} \langle q_s, q_{k+l+1-s} \rangle = \sum_{s=0}^{l-(k+1)} \langle q_{k+1+s}, q_{l-s} \rangle.$$

Define $a_s := \langle q_{k+1+s}, q_{l-s} \rangle$ and observe that $a_s = -a_{l-(k+1)-s}$ and so the first and last summand cancel each other, as well as the second summand and the next to the last one, and so on. If $l-(k+1)$ is an even number, then the number of summands is odd with the summand in the middle being $a_{(l-(k+1))/2}$. From the equality $a_s = -a_{l-(k+1)-s}$ it follows that $a_{(l-(k+1))/2} = 0$. We conclude that $\Upsilon_{k,l} - \Upsilon_{l,k} = 0$ and hence $\{\Upsilon_k, \Upsilon_l\}_{-1} = 0$. For $j \geq 0$ we have

$$\begin{aligned}\{\Upsilon_k, \Upsilon_l\}_j &= \sum_I \Bigg(\tilde{D}_I(q_k) \underbrace{\frac{\partial}{\partial u_I}(\Upsilon_{l,j})}_{=0} - \tilde{D}_I(q_l) \underbrace{\frac{\partial}{\partial u_I}(\Upsilon_{k,j})}_{=0} \\ &+ \sum_{m=0}^{\infty} \Big(\tilde{D}_I(\Upsilon_{k,m}) \tfrac{\partial}{\partial q_{m,I}}(\Upsilon_{l,j}) - \tilde{D}_I(\Upsilon_{l,m}) \tfrac{\partial}{\partial q_{m,I}}(\Upsilon_{k,j}) \Big) \Bigg). \end{aligned} \tag{B.12}$$

Observe that $\Upsilon_{k,j} = \sum_{s=0}^{j} \langle q_s, q_{k+j+1-s} \rangle = \sum_{s=0}^{\min\{k,j\}} \langle q_s, q_{k+j+1-s} \rangle$. We assume that $j < k < l$.

$$\begin{aligned}\{\Upsilon_k, \Upsilon_l\}_j &= \sum_I \Bigg(\sum_{m=0}^{\infty} \tilde{D}_I(\Upsilon_{k,m}) \tfrac{\partial}{\partial q_{m,I}} \Big(\sum_{s=0}^{j} \langle q_s, q_{l+j+1-s} \rangle \Big) \\ &- \sum_{m=0}^{\infty} \tilde{D}_I(\Upsilon_{l,m}) \tfrac{\partial}{\partial q_{m,I}} \Big(\sum_{s=0}^{j} \langle q_s, q_{k+j+1-s} \rangle \Big) \Bigg) \\ &= \sum_I \sum_{m=0}^{j} \Bigg(\tilde{D}_I(\Upsilon_{k,m}) \tfrac{\partial}{\partial q_{m,I}} \Big(\sum_{s=0}^{j} \langle q_s, q_{l+j+1-s} \rangle \Big) \\ &+ \tilde{D}_I(\Upsilon_{k,l+j+1-m}) \tfrac{\partial}{\partial q_{l+j+1-m,I}} \Big(\sum_{s=0}^{j} \langle q_s, q_{l+j+1-s} \rangle \Big) \\ &- \tilde{D}_I(\Upsilon_{l,m}) \tfrac{\partial}{\partial q_{m,I}} \Big(\sum_{s=0}^{j} \langle q_s, q_{k+j+1-s} \rangle \Big) - \\ &- \tilde{D}_I(\Upsilon_{l,k+j+1-m}) \tfrac{\partial}{\partial q_{k+j+1-m,I}} \Big(\sum_{s=0}^{j} \langle q_s, q_{k+j+1-s} \rangle \Big) \Bigg) \\ &= \sum_{m=0}^{j} \Big(\langle \Upsilon_{k,m}, q_{l+j+1-m} \rangle - \langle \Upsilon_{k,l+j+1-m}, q_m \rangle - \langle \Upsilon_{l,m}, q_{k+j+1-m} \rangle + \langle \Upsilon_{l,k+j+1-m}, q_m \rangle \Big) \\ &= \sum_{m=0}^{j} \Bigg(\sum_{s=0}^{m} \langle \langle q_s, q_{k+m+1-s} \rangle, q_{l+j+1-m} \rangle - \sum_{s=0}^{m} \langle \langle q_s, q_{l+m+1-s} \rangle, q_{k+j+1-m} \rangle \\ &- \sum_{s=k+j+2-m}^{l+j+1-m} \langle \langle q_s, q_{k+l+j+2-m-s} \rangle, q_m \rangle \Bigg). \end{aligned} \tag{B.13}$$

If we fix q_r, with $r \in \{0, 1, ..., j\}$, and show that the terms containing q_r annihilate each other, it follows that the whole expression is equal to zero. That is because every summand in (B.13) contains q_r for some $r \in \{0, 1, ..., j\}$. The terms of (B.13) involving q_r are the following.

$$\begin{aligned}&\sum_{m=r}^{j}\Big(\langle\langle q_r, q_{k+m+1-r}\rangle, q_{l+j+1-m}\rangle - \langle\langle q_r, q_{l+m+1-r}\rangle, q_{k+j+1-m}\rangle\Big)\\ &-\sum_{s=k+j+2-r}^{l+j+1-r}\langle\langle q_s, q_{k+l+j+2-r-s}\rangle, q_r\rangle.\end{aligned}\tag{B.14}$$

We apply Jacobi identity to the bracket $\langle\langle q_r, q_{k+m+1-r}\rangle, q_{l+j+1-m}\rangle$ and obtain

$$\begin{aligned}\text{(B.14)} \quad = &\sum_{m=r}^{j}\Big(-\langle\langle q_{k+m+1-r}, q_{l+j+1-m}\rangle, q_r\rangle - \langle\langle q_{l+j+1-m}, q_r\rangle, q_{k+m+1-r}\rangle\\ &-\langle\langle q_r, q_{l+m+1-r}\rangle, q_{k+j+1-m}\rangle\Big) - \sum_{s=k+j+2-r}^{l+j+1-r}\langle\langle q_s, q_{k+l+j+2-r-s}\rangle, q_r\rangle.\end{aligned}\tag{B.15}$$

We rearrange the expression.

$$\begin{aligned}\text{(B.14)} \quad = &\sum_{m=r}^{j}\Big(\langle\langle q_r, q_{l+j+1-m}\rangle, q_{k+m+1-r}\rangle - \langle\langle q_r, q_{l+m+1-r}\rangle, q_{k+j+1-m}\rangle\Big)\\ &-\sum_{s=k+j+2-r}^{l+j+1-r}\langle\langle q_s, q_{k+l+j+2-r-s}\rangle, q_r\rangle - \sum_{m=r}^{j}\langle\langle q_{k+m+1-r}, q_{l+j+1-m}\rangle, q_r\rangle.\end{aligned}\tag{B.16}$$

Observe that in the first sum the first summand (for $m = r$) and the last summand (for $m = j$) cancel each other, and so we claim that

$$\sum_{m=r}^{j}\Big(\langle\langle q_r, q_{l+j+1-m}\rangle, q_{k+m+1-r}\rangle - \langle\langle q_r, q_{l+m+1-r}\rangle, q_{k+j+1-m}\rangle\Big) = 0.\tag{B.17}$$

The argument is analogous to the one presented in the course of the proof that $\Upsilon_{k,l} - \Upsilon_{l,k} = 0$. Hence the summands involving q_r which are left to analysis, are equal to

$$\begin{aligned}\text{(B.14)} \quad &= -\sum_{s=k+j+2-r}^{l+j+1-r}\langle\langle q_s, q_{k+l+j+2-r-s}\rangle, q_r\rangle - \sum_{m=r}^{j}\langle\langle q_{k+m+1-r}, q_{l+j+1-m}\rangle, q_r\rangle\\ &= \Big\langle -\underbrace{\sum_{s=0}^{l-k-1}\langle q_{k+j+2-r+s}, q_{l-s}\rangle}_{*} - \sum_{m=0}^{j-r}\langle q_{k+m+1}, q_{l+j+1-m-r}\rangle, q_r\Big\rangle.\end{aligned}\tag{B.18}$$

Let us denote $n := k + 1$ for a moment. Then

$$\begin{aligned}* \quad &= \sum_{s=0}^{l-n}\langle q_{n+j+1-r+s}, q_{l-s}\rangle = \underbrace{\sum_{s=0}^{l-n-j-1+r}\langle q_{n+j+1-r+s}, q_{l-s}\rangle}_{=0} + \sum_{s=l-n-j+r}^{l-n}\langle q_{n+j+1-r+s}, q_{l-s}\rangle\\ &= \sum_{s=0}^{j-r}\langle q_{l+1+s}, q_{k+1+j-r-s}\rangle.\end{aligned}\tag{B.19}$$

To see that the sum in (B.19) which is said to be equal to zero indeed vanishes, note that the first summand (for $s = 0$) and the last (for $s = l - n - j - 1 + r$) cancel each other. The same holds for the second summand and the next to the last one, and so on. To sum up, we have

$$\begin{aligned}\text{(B.14)} \quad &= -\Big\langle\sum_{s=0}^{j-r}\langle q_{l+1+s}, q_{k+1+j-r-s}\rangle + \sum_{m=0}^{j-r}\langle q_{k+m+1}, q_{l+j+1-m-r}\rangle, q_r\Big\rangle\\ &= -\Big\langle\sum_{m=0}^{j-r}\langle q_{l+1+m}, q_{k+1+j-r-m}\rangle + \langle q_{k+m+1}, q_{l+j+1-m-r}\rangle, q_r\Big\rangle.\end{aligned}\tag{B.20}$$

The sum in the first argument of the above bracket is equal to zero. This claim follows again from the observation, that the first summand (for $m = 0$) and the last (for $m = j - r$) cancel each other. The same holds for the second summand and the next to the last one, and so on.

$\diamond$

(9) We verify that

$$\{\Gamma_i, \Upsilon_k\} = \begin{cases} \Upsilon_{k-i+1}, & k \geq i-1, \\ 0, & k < i-1. \end{cases}$$

Recall, that $\Gamma_{i,j} = 0$ for $j \leq i-2$ and $\Gamma_{i,j} = q_{j-i+1}$ for $j \geq i-1$.

$$\begin{aligned} \{\Gamma_i, \Upsilon_k\}_{-1} &= \sum_I \Big(\tilde{D}_I(0) \tfrac{\partial}{\partial u_I}(q_k) - \tilde{D}_I(q_k) \tfrac{\partial}{\partial u_I}(0) \\ &+ \sum_{m=0}^{\infty} \Big(\tilde{D}_I(\Gamma_{i,m}) \tfrac{\partial}{\partial q_{m,I}}(q_k) - \tilde{D}_I(\Upsilon_{k,m}) \tfrac{\partial}{\partial q_{m,I}}(0) \Big) \Big) = \Gamma_{i,k} = q_{k-i+1}. \end{aligned} \tag{B.21}$$

For $j \geq 0$ we have

$$\begin{aligned} \{\Gamma_i, \Upsilon_k\}_j &= \sum_I \Big(\tilde{D}_I(0) \tfrac{\partial}{\partial u_I}(\Upsilon_{k,j}) - \tilde{D}_I(q_k) \tfrac{\partial}{\partial u_I}(q_{j-i+1}) \\ &+ \sum_{m=0}^{\infty} \Big(\tilde{D}_I(\Gamma_{i,m}) \tfrac{\partial}{\partial q_{m,I}}(\Upsilon_{k,j}) - \tilde{D}_I(\Upsilon_{k,m}) \tfrac{\partial}{\partial q_{m,I}}(q_{j-i+1}) \Big) \Big) \\ &= \sum_{m=0}^{j} \left(\langle \Gamma_{i,m}, q_{k+j+1-m} \rangle - \langle \Gamma_{i,k+j+1-m}, q_m \rangle \right) - \Upsilon_{k,j-i+1} \\ &= \sum_{m=i-1}^{j} \langle \Gamma_{i,m}, q_{k+j+1-m} \rangle - \sum_{m=0}^{j} \langle \Gamma_{i,k+j+1-m}, q_m \rangle - \Upsilon_{k,j-i+1} \\ &= \sum_{m=i-1}^{j} \langle q_{m-i+1}, q_{k+j+1-m} \rangle - \sum_{m=0}^{j} \langle q_{k-i+1+j+1-m}, q_m \rangle - \Upsilon_{k,j-i+1} \\ &= \sum_{m=0}^{j-i+1} \langle q_m, q_{k+j+1-i+1-m} \rangle + \Upsilon_{k-i+1,j} - \Upsilon_{k,j-i+1} \\ &= \Upsilon_{k,j-i+1} + \Upsilon_{k-i+1,j} - \Upsilon_{k,j-i+1} = \Upsilon_{k-i+1,j}. \qquad \diamond \end{aligned}$$

We verified all the equalities for the Jacobi brackets of Theorem 3.5.1.

$\square$

Appendix C

Infinite hierarchies of nonlocal conservation laws

C.1 The rdDym equation

Consider the covering (4.7) over the rdDym Eq. (4.6) together with its nonlocal conservation law Ω_2 (4.19). Substituting for q its Laurent expansion $\sum\limits_{k=-\infty}^{\infty} q_k\,\lambda^k$ into (4.7) and (4.19) yields

$$\left\{\begin{array}{lcl} q_{m,t} &=& u_x\, q_{m,x} - q_{m-1,x}, \\ q_{m,y} &=& u_y\, q_{m+1,x}, \end{array}\right. \qquad m \in \mathbb{N}, \tag{C.1}$$

and

$$\begin{aligned} \Omega_2 &= u_y \left(\sum_k q_{k,x}^2 \lambda^{2k} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j} \right) dx \,\wedge\, dy \\ &+ \left(2\, u_y \left(\sum_k q_{k,x}^2 \lambda^{2k+1} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j+1} \right) \right. \\ &\left. -u_x\, u_y \left(\sum_k q_{k,x}^2 \lambda^{2k} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j} \right) \right) dy \,\wedge\, dt \\ &- \left(\sum_k q_{k,x}^2 \lambda^{2k+2} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j+2} \right) dt \,\wedge\, dx. \end{aligned} \tag{C.2}$$

The system (C.1) defines an infinite-dimensional covering over the rdDym Eq. (4.6) with pseudopotentials q_i, $i \in \mathbb{Z}$. The positive covering is obtained by putting $q_0 = y$ and $q_j = 0$ for $j \leq -1$. It acquires the form

$$\left\{\begin{array}{lcl} q_{1,t} &=& \frac{u_x}{u_y}, \\ q_{1,x} &=& \frac{1}{u_y}, \\ q_{i,t} &=& \frac{u_x}{u_y}\, q_{i-1,y} - \frac{1}{u_y}\, q_{i-2,y}, \\ q_{i,x} &=& \frac{1}{u_y}\, q_{i-1,y}, \end{array}\right. \qquad i \geq 2. \tag{C.3}$$

Let Ω_i^+ denote the conservation law obtained through collecting coefficients of λ^i. The first two conservation laws

$$\begin{array}{lcl} \Omega_2^+ &=& \frac{1}{u_y}\left(dx \,\wedge\, dy - u_x\, dy \,\wedge\, dt\right), \\ \Omega_3^+ &=& \frac{1}{u_y}\left(q_{1,y}\, dx \,\wedge\, dy + (1 - u_x\, q_{1,y})\; dy \,\wedge\, dt\right) \end{array} \tag{C.4}$$

are local and nontrivial by the virtue of

$$\begin{array}{lcl} d_h\, \Omega_2^+ &=& \frac{1}{u_y^2}\left(u_{ty} - u_x\, u_{xy} + u_y\, u_{xx}\right) dt \wedge dx \wedge dy, \\ d_h\, \Omega_3^+ &=& \frac{q_{1,y}}{u_y^2}\left(u_{ty} - u_x\, u_{xy} + u_y\, u_{xx}\right) dt \wedge dx \wedge dy. \end{array}$$

For $m \geq 2$ we have:

$$
\begin{aligned}
\Omega^+_{2m} &= \tfrac{1}{u_y}\left(q^2_{m-1,y} + 2\sum_{k=1}^{m-1} q_{k-1,y}\, q_{2m-k-1,y}\right) dx \wedge dy \\
&+ \left[\tfrac{4}{u_y}\sum_{k=1}^{m-1} q_{k-1,y}\, q_{2m-2-k,y} - \tfrac{u_x}{u_y}\left(q^2_{m-1,y} + 2\sum_{k=1}^{m-1} q_{k-1,y}\, q_{2m-k-1,y}\right)\right] dy \wedge dt \\
&- \tfrac{1}{u_y^2}\left(q^2_{m-1,y} + 2\sum_{k=1}^{m-2} q_{k-1,y}\, q_{2m-3-k,y}\right) dt \wedge dx, \\
\Omega^+_{2m+1} &= \left(\tfrac{1}{u_y}\sum_{k=1}^{m} q_{k-1,y}\, q_{2m-k,y}\right) dx \wedge dy \\
&+ \left[\tfrac{1}{u_y}\left(q^2_{m-1,y} + 2\sum_{k=0}^{m-1} q_{k-1,y}\, q_{2m-k-1,y}\right) - 2\,\tfrac{u_x}{u_y}\sum_{k=1}^{m} q_{k-1,y}\, q_{2m-k,y}\right] dy \wedge dt \\
&- \tfrac{1}{u_y^2}\left(\sum_{k=1}^{m-1} q_{k-1,y}\, q_{2m-2-k,y}\right) dt \wedge dx.
\end{aligned}
\tag{C.5}
$$

The negative covering is obtained by setting $q_0 = -t$, $q_{-1} = x$, $q_{-2} = u$, $q_j = 0$ for $j \geq 1$, and relabelling $r_i := q_{-2-i}$ for $i \geq 0$. It has the form

$$
\begin{cases}
r_{1,x} = u_x^2 - u_t, \\
r_{1,y} = u_x\, u_y, \\
r_{i,x} = u_x\, r_{i-1,x} - r_{i-1,t}, \\
r_{i,y} = u_y\, r_{i-1,x},
\end{cases}
\qquad i \geq 2.
\tag{C.6}
$$

Note that $r_{i,x} = u_x^{i-1}(u_x^2 - u_t) - \sum_{k=0}^{i-2} u_x^k r_{i-1-k,t}$, so the conservation laws in the negative covering can be expressed solely in terms of the internal nonlocal variables r_i, $r_{i,t}$. Nevertheless, to avoid messy formulas, we keep the variables $r_{i,x}$. We denote by Ω^-_m what is obtained from examining coefficients of λ^{-m} in C.2, with $m \geq -2$. The coefficients of λ^2 and λ^1 are equal to zero. The conservation law corresponding to the coefficient of λ^0 is equal to $1\, dt \wedge dx$. We have

$$
\begin{aligned}
\Omega^-_1 &= u_y\, dy \wedge dt - u_x\, dt \wedge dx, \\
\Omega^-_2 &= u_y\, dx \wedge dy + 3\, u_x\, u_y\, dy \wedge dt - \left(3\, u_x^2 - 2\, u_t\right) dt \wedge dx, \\
\Omega^-_3 &= u_x\, u_y\, dx \wedge dy + 2\, u_y\left(u_x^2 - u_t\right) dy \wedge dt - \\
&- \left(r_{2,x} + u_x\left(u_x^2 - u_t\right)\right) dt \wedge dx.
\end{aligned}
\tag{C.7}
$$

The conservation law Ω^-_1 is trivial. The conservation law Ω^-_2 is local and nontrivial since $d_h \Omega^-_2 = 3(u_{ty} - u_x\, u_{xy} + u_y\, u_{xx})\, dt \wedge dx \wedge dy$. For $i \geq 2$ we have

$$
\Omega^-_i = P^-_i\, dx \wedge dy + Q^-_i\, dy \wedge dt + R^-_i\, dt \wedge dx,
\tag{C.8}
$$

where

$$
\begin{aligned}
P^-_{2m} &= u_y \left(r^2_{m-2,x} + 2 \left(r_{2m-1,x} + u_x\, r_{2m,x} + \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-k-4,x} \right)\right),\\
Q^-_{2m} &= 4\, u_y \left(r_{2m-2,x} + u_x\, r_{2m-3,x} + \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-3-k,x} \right)\\
&\quad - u_x\, u_y \left(r^2_{m-2,x} + 2\,(r_{2m-3} + u_x\, r_{2m-4}) + 2 \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-k-4,x} \right),\\
R^-_{2m} &= -r^2_{m-1,x} - 2(r_{2m-1} + u_x\, r_{2m-2}) - 2 \sum_{k=1}^{m} r_{k,x}\, r_{2m-2-k,x},\\
P^-_{2m+1} &= u_y \left(r_{2m-2,x} + u_x\, r_{2m-3,x} + \sum_{k=1}^{m-2} r_{k,x}\, r_{2m-3-k,x} \right)\\
Q^-_{2m+1} &= u_y \left(r^2_{m-2,x} + 2(r_{2m-3,x} + u_x\, r_{2m-4,x}) + 2 \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-4-k,x} \right)\\
&\quad - 2\, u_x\, u_y \left(r_{2m-2,x} + r_{2m-3,x} + \sum_{k=1}^{m-2} r_{k,x}\, r_{2m-3-k,x} \right),\\
R^-_{2m+1} &= -r_{2m-2,x} - u_x\, r_{2m-3,x} - \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-3-k,x}.
\end{aligned}
\tag{C.9}
$$

C.2 The UH equation

After substituting for q its Laurent expansion $\sum\limits_{k=-\infty}^{\infty} q_k\, \lambda^k$, the covering. (4.9) and nonlocal conservation law Ω_3 (4.20) of the UH Eq. (4.8) transform to

$$
\left\{
\begin{array}{lcl}
q_{i,t} &=& u_t\, q_{i+1,x} - u_y\, q_{i+2,x},\\
q_{i,y} &=& u_y\, q_{i+1,x},
\end{array}
\right.
\tag{C.10}
$$

and

$$
\begin{aligned}
\Omega_3 &= u_y \left(\sum_k q^2_{k,x} \lambda^{2k+2} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j+2} \right) dx \wedge dy\\
&\quad - u_y^2 \left(\sum_k q^2_{k,x} \lambda^{2k} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j} \right) dy \wedge dt\\
&\quad - \Bigg[u_t \left(\sum_k q^2_{k,x} \lambda^{2k+2} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j+2} \right)\\
&\quad - 2\, u_y \left(\sum_k q^2_{k,x} \lambda^{2k+1} + 2 \sum_{k<j} q_{k,x}\, q_{j,x} \lambda^{k+j+1} \right) \Bigg]\; dt \wedge dx.
\end{aligned}
\tag{C.11}
$$

The system (C.10) defines an infinite-dimensional covering over the UH Eq. (4.8) with pseudopotentials q_i, $i \in \mathbb{Z}$. The positive covering is obtained by putting $q_0 = y$ and $q_i = 0$ for $i \le -1$. It acquires the form

$$
\left\{
\begin{array}{lcll}
q_{1,x} &=& \frac{1}{u_y}, &\\
q_{1,y} &=& \frac{u_t}{u_y}, &\\
q_{i,x} &=& \frac{1}{u_y}\, q_{i-1,y}, &\\
q_{i,y} &=& \frac{u_t}{u_y}\, q_{i-1,y} - q_{i-1,t}, & \qquad i \ge 2.
\end{array}
\right.
\tag{C.12}
$$

The internal nonlocal variables in the positive covering are q_i, $q_{i,t}$, $q_{i,tt}$ and so on. The variables $q_{i,y}$ can be expressed in terms of the internal variables via the formula $q_{i,y} = \left(\frac{u_t}{u_y}\right)^i - \sum\limits_{k=1}^{i-1} \left(\frac{u_t}{u_y}\right)^{k-1} q_{i-k,t}$. However, we will keep the variables $q_{i,y}$ for the sake of readability of the formulas. Let Ω_i^+ denote the result of examining coefficients of λ^i in (C.11), with $i \ge 0$. The conservation law corresponding to the coefficient of λ^2 is equal to $1\, dy \wedge dt$. We have

$$
\begin{array}{lcl}
\Omega_3^+ &=& -\frac{u_t}{u_y}\, dy \wedge dt + \frac{1}{u_y}\, dt \wedge dx,\\
\Omega_4^+ &=& \frac{1}{u_y}\, dx \wedge dy - \left(q^2_{1,y} + 2\, q_{2,y} \right) dy \wedge dt - \left(\frac{u_t}{u_y^2} - \frac{4}{u_y}\, q_{1,y} \right) dt \wedge dx.
\end{array}
\tag{C.13}
$$

The conservation law Ω_3^+ is local and nontrivial since $d_h\Omega_3^+ = (u_{yy} - u_t\, u_{xy} + u_y\, u_{tx})\, dt\wedge dx\wedge dy$. The remaining conservation laws are of the form

$$\Omega_i^+ = P_i^+\, dx\wedge dy + Q_i^+\, dy\wedge dt + R_i^+\, dt\wedge dx, \tag{C.14}$$

where for $m\geq 2$

$$\begin{aligned}
P_{2m+1}^+ &= \tfrac{1}{u_y}\left(q_{2m-3,y} + \textstyle\sum_{k=2}^{m-1} q_{k-1,y}\, q_{2m-2-k,y}\right),\\
Q_{2m+1}^+ &= -q_{2m-1,y} - \textstyle\sum_{k=2}^{m} q_{k-1,y}\, q_{2m-k,y},\\
R_{2m+1}^+ &= -\tfrac{u_t}{u_y^2}\left(q_{2m-3,y} + \textstyle\sum_{k=2}^{m-1} q_{k-1,y}\, q_{2m-2-k,y}\right) + \tfrac{1}{u_y}\left(q_{m-1,y}^2 + 2\, q_{2m-2,y} + 2\textstyle\sum_{k=2}^{m-1} q_{k-1,y}\, q_{2m-k-1,y}\right),
\end{aligned} \tag{C.15}$$

and for $m\geq 3$

$$\begin{aligned}
P_{2m}^+ &= \tfrac{1}{u_y}\left(q_{m-2,y}^2 + 2\, q_{2m-4,y} + 2\textstyle\sum_{k=2}^{m-2} q_{k-1,y}\, q_{2m-3-k,y}\right),\\
Q_{2m}^+ &= -\left(q_{m-1,y}^2 + 2\, q_{2m-2,y} + 2\textstyle\sum_{k=2}^{m-1} q_{k-1,y}\, q_{2m-1-k,y}\right),\\
R_{2m}^+ &= -\tfrac{u_t}{u_y^2}\left(q_{m-2,y}^2 + 2\, q_{2m-4,y} + 2\textstyle\sum_{k=2}^{m-2} q_{k-1,y}\, q_{2m-3-k,y}\right) + \tfrac{4}{u_y}\left(q_{2m-3,y} + \textstyle\sum_{k=2}^{m-1} q_{k-1,y}\, q_{2m-2-k,y}\right).
\end{aligned} \tag{C.16}$$

The negative covering is obtained by setting $q_0 = x$, $q_{-1} = u$, $q_j = 0$ for $j\geq 1$, and relabelling $r_i := q_{-i-1}$ for $i\geq 1$. It acquires the form

$$\left\{\begin{array}{rcl}
r_{1,t} &=& u_t\, u_x - u_y,\\
r_{1,y} &=& u_x\, u_y,\\
r_{i,t} &=& u_t\, r_{i-1,x} - u_y\, r_{i-2,x},\\
r_{i,y} &=& u_y\, r_{i-1,x},
\end{array}\right. \qquad i\geq 1. \tag{C.17}$$

The nonlocal internal variables are r_i, $r_{i,x}$, $r_{i,xx}$ and so on. We denote by Ω_m^- what is obtained from examining coefficients of λ^{-m} in (C.11), with $m\geq -2$. Then, the first two conservation laws are

$$\begin{aligned}
\Omega_{-2}^- &= u_y\, dx\wedge dy - u_t\, dt\wedge dx,\\
\Omega_{-1}^- &= u_x\, u_y\, dx\wedge dy - (u_t\, u_x - u_y)\, dt\wedge dx.
\end{aligned} \tag{C.18}$$

The local conservation law Ω_{-2}^- is trivial, while the local conservation law Ω_{-1}^- is not, since $d_h\,\Omega_{-1}^- = (u_{yy} - u_t\, u_{xy} - u_y\, u_{tx})dt\wedge dx\wedge dy$. The remaining conservation laws are nonlocal and constitute the following hierarchy:

$$\begin{aligned}
\Omega_0^- &= u_y\,(u_x^2 + 2\, r_{1,x})\, dx\wedge dy - u_y^2\, dy\wedge dt - \left(u_t\,(u_x^2 + 2\, r_{1,x}) - 4\, u_x\, u_y\right)\, dt\wedge dx,\\
\Omega_1^- &= u_y\,(r_{2,x} + u_x\, r_{1,x})\, dx\wedge dy - u_y^2\, u_x\, dy\wedge dt - \left(u_t\,(r_{2,x} + u_x\, r_{1,x}) - u_y(u_x^2 + 2\, r_{1,x})\right)\, dt\wedge dx.
\end{aligned} \tag{C.19}$$

$$\Omega_i^- = P_i^-\, dx\wedge dy + Q_i^-\, dy\wedge dt + R_i^-\, dt\wedge dx, \qquad i\geq 2, \tag{C.20}$$

where for $m \geq 1$ we have

$$
\begin{aligned}
P^-_{2m} &= u_y \left(r^2_{m,x} + 2(r_{2m+1,x} + u_x\, r_{2m,x}) + 2 \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-k,x} \right), \\
Q^-_{2m} &= -u_y^2 \left(r^2_{m-1,x} + 2(r_{2m-1,x} + u_x\, r_{2m-2,x}) + 2 \sum_{k=1}^{m-2} r_{k,x}\, r_{2m-k,x} \right), \\
R^-_{2m} &= -u_t \left(r^2_{m,x} + 2(r_{2m+1,x} + u_x\, r_{2m,x}) + 2 \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-k,x} \right) \\
&\quad -4\, u_y \left(r_{2m,x} + u_x\, r_{2m-1,x} + \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-1-k,x} \right), \\
P^-_{2m+1} &= u_y \left(r_{2m+2,x} + u_x\, r_{2m+1,x} + \sum_{k=1}^{m} r_{k,x}\, r_{2m+1-k,x} \right), \\
Q^-_{2m+1} &= -u_y^2 \left(r_{2m,x} + u_x\, r_{2m-1,x} + \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-1-k,x} \right), \\
R^-_{2m+1} &= -u_t \left(r_{2m+2,x} + u_x\, r_{2m+1,x} + \sum_{k=1}^{m} r_{k,x}\, r_{2m+1-k,x} \right) \\
&\quad -u_y \left(r^2_{m} + 2(r_{2m+1,x} + u_x\, r_{2m,x}) + 2 \sum_{k=1}^{m-1} r_{k,x}\, r_{2m-k,x} \right).
\end{aligned}
\tag{C.21}
$$