Computational Logic - Assignment 3

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Exercise 2

Let's break down the proof in two parts:

1. R is symmetric $\Rightarrow \varphi \Rightarrow \Box \diamond \varphi$
2. $\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ R is symmetric

R is symmetric $\Rightarrow \varphi \Rightarrow \Box \diamond \varphi$:
Let's assume that φ is true in a world w, $w \vDash \varphi$. We need to show that $w \vDash \Box \diamond \varphi$. By definition of modal operators, since R is symmetric, if v is accessible from w, then w is accessible from v. Therefore, if φ is true in v, then φ is also true in w. This implies that $\Box\varphi$ is true in w and $\diamond\varphi$ is true in w. Therefore $\Box \diamond \varphi$ holds in w. Since this holds for an arbitrary world w where φ is true, we can conclude that $\varphi \Rightarrow \Box \diamond \varphi$.
$\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ R is symmetric:
To show symmetry, we need to show that if wRv then vRw for all worlds w and v. Let's assume wRv, this means that v is accessible form w. Let's also consider the formula $\varphi = \neg\Box\neg \diamond \bot$, which is not necessarily false. By assuming $\varphi \Rightarrow \Box \diamond \varphi$, we have $\neg\Box\neg \diamond \bot \Rightarrow \Box \diamond \neg\Box\neg \diamond \bot$. In the world w:

1. $\neg\Box\neg \diamond \bot$ is true in w since $\diamond\bot$ is true in vv, which is accessible from w.
2. $\Box \diamond \neg\Box\neg \diamond \bot$ is true in w by assumption.

This implies that $\neg\Box\neg \diamond \bot$ is true in all worlds accessible from w, including v. Now, considering formula $\diamond\bot$, we have that this is true in v since $\neg\Box\neg \diamond \bot$ is true in v. Therefore vRw. Since this is true in an arbitrary world, we can conclude that R is symmetric.

Exercise 3

As in Exercise 2, we will break down this proof in two parts:

1. F is euclidean $\Rightarrow F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi$
2. $F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ F is euclidean

F is euclidean $\Rightarrow F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi$:
Let w be a world in W, Suppose $w \vDash \diamond\varphi$, meaning there exists a world v such that wRv and $v \vDash \varphi$. We now need to show that $w \vDash \Box \diamond \varphi$. Thanks to the euclidean property, if wRv and wRu then vRu for any u in W. Since wRv is reflexive, vRw, then vRv. This means that $\diamond\varphi$ is true at v. Now $\Box \diamond \varphi$ is ture at w because $\diamond\varphi$ is ture at all worlds accessible from w.

$\boldsymbol{F} \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ **F is euclidean**:

Let's take any $\boldsymbol{x, y, z}$ in W such that $\boldsymbol{xRy}$ and $\boldsymbol{xRz}$. We need to show that $\boldsymbol{yRz}$. Let's consider the formula $\boldsymbol{\phi = \diamond\varphi}$, where $\boldsymbol{\phi}$ is an arbitrary formula. Since $\boldsymbol{xRy}$ there exists $\boldsymbol{u}$ such that $\boldsymbol{xRu}$ and $\boldsymbol{u \vDash \varphi}$. Now, $\boldsymbol{xRz}$ and, by assuming $\boldsymbol{F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi}$, we have $\boldsymbol{xRz \Rightarrow zRw}$ for any w such that $\boldsymbol{xRu}$. Therefore we have $\boldsymbol{zRw}$ for some w such that $\boldsymbol{xRu}$. Since y is accessible from x, y is also accessible from u, which implies $\boldsymbol{yRw}$. Given $\boldsymbol{yRw}$ and $\boldsymbol{zRw}$, by transitiviy we have $\boldsymbol{yRz}$. Since this holds for arbitrary $\boldsymbol{x, y, z}$ in W, we can conclude that F is euclidean.