Computational Logic - Assignment 3

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Exercise 2

Let's break down the proof in two parts:

1. R is symmetric $\Rightarrow \psi \Rightarrow \Box \diamond \psi$
2. $\psi \Rightarrow \Box \diamond \psi \Rightarrow$ R is symmetric

R is symmetric $\Rightarrow \psi \Rightarrow \Box \diamond \psi$:
Let's assume that ψ is true in a world w, $w \vDash \psi$. We need to show that $w \vDash \Box \diamond \psi$. By definition of modal operators, since R is symmetric, if v is accessible from w, then w is accessible from v. Therefore, if ψ is true in v, then ψ is also true in w. This implies that $\Box\psi$ is true in w and $\diamond\psi$ is true in w. Therefore $\Box \diamond \psi$ holds in w. Since this holds for an arbitrary world w where ψ is true, we can conclude that $\psi \Rightarrow \Box \diamond \psi$.
$\psi \Rightarrow \Box \diamond \psi \Rightarrow$ R is symmetric:
To show symmetry, we need to show that if wRv then vRw for all worlds w and v. Let's assume wRv, this means that v is accessible form w. Let's also consider the formula $\psi = \neg\Box\neg \diamond \bot$, which is not necessarily false. By assuming $\psi \Rightarrow \Box \diamond \psi$, we have $\neg\Box\neg \diamond \bot \Rightarrow \Box \diamond \neg\Box\neg \diamond \bot$. In the world w:

1. $\neg\Box\neg \diamond \bot$ is true in w since $\diamond\bot$ is true in vv, which is accessible from w.
2. $\Box \diamond \neg\Box\neg \diamond \bot$ is true in w by assumption.

This implies that $\neg\Box\neg \diamond \bot$ is true in all worlds accessible from w, including v. Now, considering formula $\diamond\bot$, we have that this is true in v since $\neg\Box\neg \diamond \bot$ is true in v. Therefore vRw. Since this is true in an arbitrary world, we can conclude that R is symmetric.

Exercise 3

As in Exercise 2, we will break down this proof in two parts:

1. F is euclidean $\Rightarrow F \vDash \diamond\psi \Rightarrow \Box \diamond \psi$
2. $F \vDash \diamond\psi \Rightarrow \Box \diamond \psi \Rightarrow$ F is euclidean

F is euclidean $\Rightarrow F \vDash \diamond\psi \Rightarrow \Box \diamond \psi$:
Let w be a world in W, Suppose $w \vDash \diamond\psi$, meaning there exists a world v such that wRv and $v \vDash \psi$. We now need to show that $w \vDash \Box \diamond \psi$. Thanks to the euclidean property, if wRv and wRu then vRu for any u in W. Since wRv is reflexive, vRw, then vRv. This means that $\diamond\psi$ is true at v. Now $\Box \diamond \psi$ is ture at w because $\diamond\psi$ is ture at all worlds accessible from w.

$F \models \diamond\psi \Rightarrow \Box\diamond\psi \Rightarrow$ F is euclidean:

Let's take any x, y, z in W such that xRy and xRz. We need to show that yRz. Let's consider the formula $\phi = \diamond\psi$, where ϕ is an arbitrary formula. Since xRy there exists u such that xRu and $u \models \psi$. Now, xRz and, by assuming $F \models \diamond\psi \Rightarrow \Box\diamond\psi$, we have $xRz \Rightarrow zRw$ for any w such that xRu. Therefore we have zRw for some w such that xRu. Since y is accessible from x, y is also accessible from u, which implies yRw. Given yRw and zRw, by transitiviy we have yRz. Since this holds for arbitrary x, y, z in W, we can conclude that F is euclidean.