# Computational Logic - Assignment 3

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## Exercise 2

Let's break down the proof in two parts:

1. R is symmetric $\Rightarrow \varphi \Rightarrow \Box \diamond \varphi$
2. $\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ R is symmetric

**R is symmetric $\Rightarrow \varphi \Rightarrow \Box \diamond \varphi$:**
Let's assume that $\varphi$ is true in a world w, $w \vDash \varphi$. We need to show that $w \vDash \Box \diamond \varphi$. By definition of modal operators, since R is symmetric, if v is accessible from w, then w is accessible from v. Therefore, if $\varphi$ is true in v, then $\varphi$ is also true in w. This implies that $\Box\varphi$ is true in w and $\diamond\varphi$ is true in w. Therefore $\Box \diamond \varphi$ holds in w. Since this holds for an arbitrary world w where $\varphi$ is true, we can conclude that $\varphi \Rightarrow \Box \diamond \varphi$.
**$\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ R is symmetric**:
To show symmetry, we need to show that if $wRv$ then $vRw$ for all worlds w and v. Let's assume $wRv$, this means that v is accessible form w. Let's also consider the formula $\varphi = \neg\Box\neg \diamond \bot$, which is not necessarily false. By assuming $\varphi \Rightarrow \Box \diamond \varphi$, we have $\neg\Box\neg \diamond \bot \Rightarrow \Box \diamond \neg\Box\neg \diamond \bot$. In the world w:

1. $\neg\Box\neg \diamond \bot$ is true in w since $\diamond\bot$ is true in vv, which is accessible from w.
2. $\Box \diamond \neg\Box\neg \diamond \bot$ is true in w by assumption.

This implies that $\neg\Box\neg \diamond \bot$ is true in all worlds accessible from w, including v. Now, considering formula $\diamond\bot$, we have that this is true in v since $\neg\Box\neg \diamond \bot$ is true in v. Therefore $vRw$. Since this is true in an arbitrary world, we can conclude that R is symmetric.

## Exercise 3

As in Exercise 2, we will break down this proof in two parts:

1. F is euclidean $\Rightarrow F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi$
2. $F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi \Rightarrow$ F is euclidean

**F is euclidean $\Rightarrow F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi$:**
Let w be a world in W, Suppose $w \vDash \diamond\varphi$, meaning there exists a world v such that $wRv$ and $v \vDash \varphi$. We now need to show that $w \vDash \Box \diamond \varphi$. Thanks to the euclidean property, if $wRv$ and $wRu$ then $vRu$ for any u in W. Since $wRv$ is reflexive, $vRw$, then $vRv$. This means that $\diamond\varphi$ is true at v. Now $\Box \diamond \varphi$ is ture at w because $\diamond\varphi$ is ture at all worlds accessible from w.

$\boldsymbol{F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi \Rightarrow}$ **F is euclidean**:
Let's take any $x, y, z$ in W such that $xRy$ and $xRz$. We need to show that $yRz$. Let's consider the formula $\phi = \diamond\varphi$, where $\phi$ is an arbitrary formula. Since $xRy$ there exists $u$ such that $xRu$ and $u \vDash \varphi$. Now, $xRz$ and, by assuming $F \vDash \diamond\varphi \Rightarrow \Box \diamond \varphi$, we have $xRz \Rightarrow zRw$ for any w such that $xRu$. Therefore we have $zRw$ for some w such that $xRu$. Since y is accessible from x, y is also accessible from u, which implies $yRw$. Given $yRw$ and $zRw$, by transitiviy we have $yRz$. Since this holds for arbitrary $x, y, z$ in W, we can conclude that F is euclidean.

# Exercise 4

## Model 1

1. $x_1 \Vdash \diamond \diamond \Box\bot$
2. $x_2 \Vdash (\diamond\Box\bot) \wedge \neg(\diamond \diamond \diamond\Box\bot)$
3. $x_3 \Vdash \Box\bot$
4. $x_4 \Vdash \diamond \diamond \diamond\Box\bot$

## Model 2

1. $x_1 \Vdash \diamond \diamond \Box\bot$: $x_1$ is the world where two jumps are needed to reach a world with no accessible neighbours.
2. $x_2, x_4 \Vdash \diamond\Box\bot$: States 2 and 4 cannot be uniquely characterized since they can both directly reach a world with no accessible neighbours.
3. $x_3 \Vdash \Box\bot$: $x_3$ is the world with no accessible neighbours.

# Exercise 5

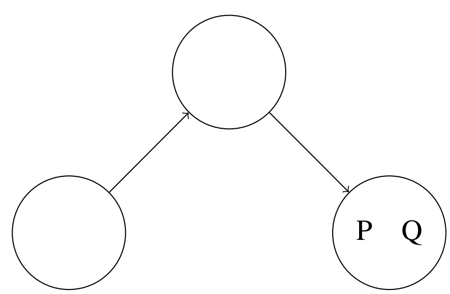
1. $\vdash \diamond(P \rightarrow Q) \rightarrow (\Box P \rightarrow \diamond Q)$

| | | |
|---|---|---|
| 1. | $\diamond(P \to Q)$ | Assumption |
| 2. | $\neg\Box\neg(P \to Q)$ | Rewrite 1 |
| 3. | $\Box P$ | Assumption |
| 4. | $\Box\neg Q$ | Assumption |
| 5. | $P$ | $\Box e3$ |
| 6. | $\neg Q$ | $\Box e4$ |
| 7. | $P \to Q$ | Assumption |
| 8. | $Q$ | $\to e5, 7$ |
| 9. | $\bot$ | $\neg e6, 8$ |
| 10. | $\neg(P \to Q)$ | $\neg i7-9$ |
| 11. | $\Box\neg(P \to Q)$ | $\Box i5, 10$ |
| 12. | $\bot$ | $\neg e2, 11$ |
| 13. | $\neg\Box\neg Q$ | $\neg i4-12$ |
| 14. | $\Box P \to \neg\Box\neg Q$ | $\to i3-13$ |
| 15. | $\Box P \to \diamond Q$ | Rewrite 15 |
| 16. | $\diamond(P \to Q) \to (\Box P \to \diamond Q)$ | $\to i1-15$ |

The box from line 5 to line 10 should be dashed, yet I had no idea how to make one.

$$2.\ \vdash \Box(\Box P \to Q) \vee \Box(\Box Q \to P)$$

To prove the invalidity of this sequent it is sufficient to provide such Kripke structure:




The leftmost world does not satisfy either of the two operands of the given disjunction, we can then conclude that the sequent is not a tautology.