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The Patterns of Change in Implied Index Volatilities

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SUMMARY

In this note, we report the results of an empirical study aimed at characterizing the changes over time in the implied volatilities of options on both the S&P 500 and the Nikkei 225 index. The implied volatilities for options of different strikes and expirations form a two-dimensional surface. We have found that most variations in the shape of the surface for either index consist of three patterns that have simple, intuitive interpretations. These collective patterns of motion of the entire surface are called “volatility modes” and were uncovered by analyzing the principal components of changes in each surface over time. In this analysis, we make no assumptions about what drives the changes in implied volatilities.

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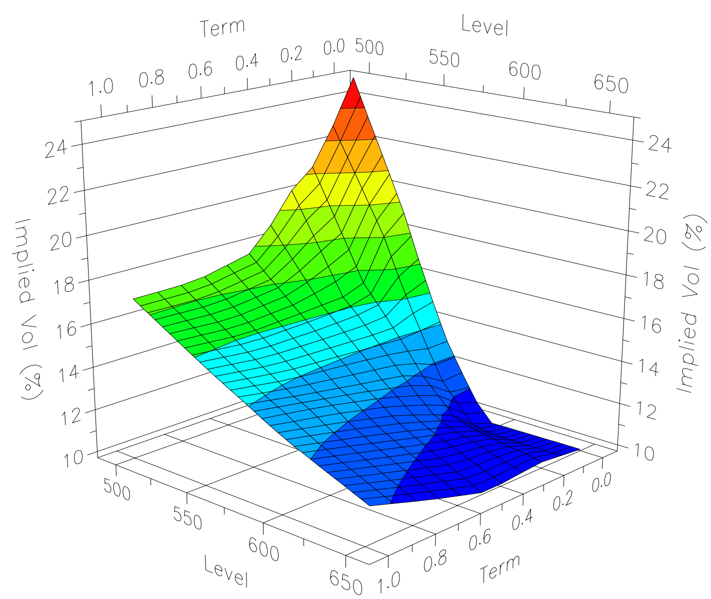
INTRODUCTION
The Patterns of Change in Implied Volatilities

Participants in index options have become accustomed to describing their market in terms of the implied volatility surface[1]– the collection of Black-Scholes implied volatilities that reflect current market prices for options of different strikes and expirations. Option values vary dramatically with market level and strike; implied volatility provides a measure of relative value that is less dependent than option prices on market level, strike, dividend yield and interest rates. Implied volatilities and the implied volatility surface are the parameters that indicate option value, much as yields and yield curves are the parameters that indicate bond value. Figure 1 shows the implied volatility for S&P 500 index options on September 27, 1995.

FIGURE 1. The implied volatility for S&P 500 index options as a function of strike level and term to expiration on September 27, 1995.



1. See *Riding on a Smile* by E. Derman and I. Kani, RISK, Vol. 7, No. 2, 1994 and *The Local Volatility Surface: Unlocking the Information in Index Option Prices* by E. Derman, I. Kani and J. Zou, Financial Analysts Journal, Vol. 52, No. 4, 1996

In recent years, the implied volatility surface of liquid global index options markets has displayed systematic patterns of variation with strike and expiration.

The strike structure. At any given expiration, implied volatilities vary with strike level. Almost always, implied volatilities increase with decreasing strike, that is, out-of-the-money puts trade at higher implied volatilities than out-of-the-money calls. This feature is often referred to as a "negative" skew.

The term structure. For any given strike level, implied volatilities vary with time to expiration. Often, long-term implied volatilities exceed short-term volatilities.

Methodology

Accurate implied volatility data is notoriously hard to obtain, especially for out-of-the-money options. For this study, we have used Goldman Sachs traders' proprietary databases consisting of (1) weekly, mid-market volatility of S&P 500 index options from May 1994 to September 1997 and (2) daily Nikkei 225 index volatility from September 1994 to May 1997.

In both of these databases, the implied volatility of an option is recorded as a function of its term to expiration and its Black-Scholes delta. Traders often choose delta as a second variable, rather than the strike itself because they feel that charting implied volatilities as a function of delta removes many of the inessential and transient features of the data that arise in changing markets. In short, our traders think abstractly about volatility and record in their databases their estimates of the mid-market implied volatilities of options with specific deltas and terms to expiration, and use this information to construct a smooth volatility surface. Again, this is analogous to the way fixed income market participants quote the yields of on-the-run bonds as a function of maturity or duration, using them to interpolate a smooth yield curve.

For S&P 500 index option data, the surface is composed of 12 numbers, corresponding to volatilities for options with a put delta of 0.25 (out-of-the-money puts), delta of 0.50 (at-the-money options) and call delta of 0.25 for terms to expiration of one-, three-, six- and 12-months. The daily Nikkei 225 surface has 54 components, corresponding to nine, evenly-spaced delta values for terms to expiration of one-, three-, six-, 12- and 15-months.

For each market, we have analyzed the principal components of the changes in the implied volatility surface. We define the surface change as simply the surface that results from taking the difference of the two successive implied volatility surfaces in our series, component-by-component. Mathematically, every daily or weekly change, $\Delta\sigma(t)$, is expressed in terms of multiples of each of the volatility modes or factors

$$\Delta\sigma_{surface}(t) = \beta_1(t)Z_1 + \beta_2(t)Z_2 \ldots + \beta_N(t)Z_N \qquad \text{(EQ 1)}$$

Note that the coefficients β_i are numbers and are different for each observation time t. In contrast, the factors Z_i are constant *surfaces* that do not vary with time and are determined from an analysis of the entire time series. Here, N refers to the number of components of the surface, which would be 12 for S&P 500 and 54 for Nikkei 225. In order to exactly reproduce every surface change in the series, you need N factors – one for each component of the surface. The aim of principal component analysis is to deduce the factors, or modes, that statistically do the best job of explaining the variability in the series, with the hope that few factors will be needed to explain most of the variation. The analysis identifies the independent, uncorrelated factors $Z_1, \ldots, Z_N$ and estimates their relative importance in explaining the overall variability found in the series.[2]

Modes of Volatility Change

Figures 2 and 3 show the first three principal components for the S&P 500 and Nikkei 225 implied volatility surfaces versus term to expiration, in months, and the delta of a put with corresponding expiration.

It is important to understand that the modes shown do not reflect the relative size and sign of overall change in the volatility surface, but only the pattern.[3]

A word of clarification about the delta axis in Figures 2 and 3: delta values less negative than -0.5 refer to out-of-the-money puts; deltas more negative than -0.5 refer to out-of-the-money calls, with call-delta calculated as $\Delta_{call} = 1 + \Delta_{put}$.

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2. The Z_i are the eigenvectors of the $N \times N$ covariance matrix of surfaces changes $\Delta\sigma$, normalized to unity.
3. In magnitude, each mode is normalized so that the sum of all squared components of individual volatilities within a mode sum to one.

FIGURE 2. Volatility Modes of the S&P 500 Implied Volatility Surface

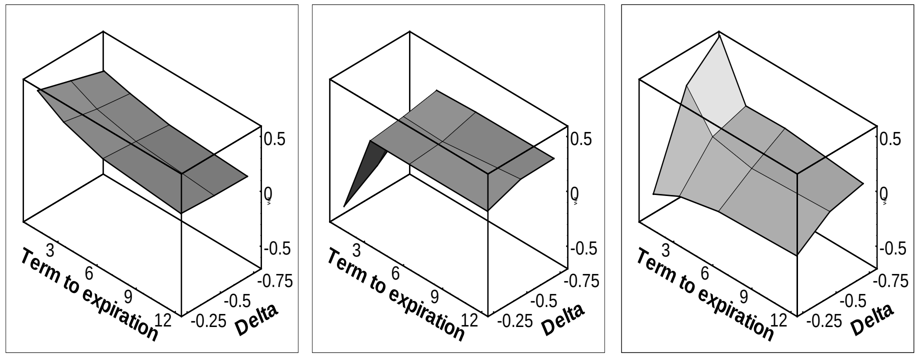
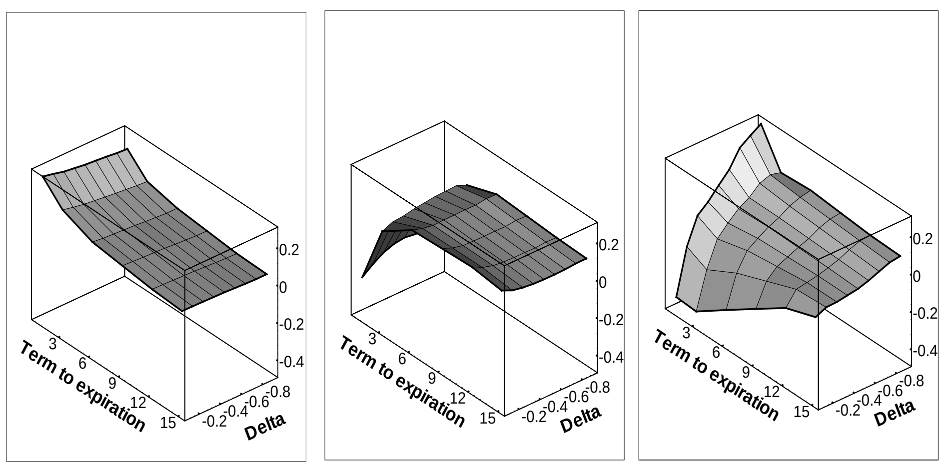


FIGURE 3. Volatility Modes of the Nikkei 225 Implied Volatility Surface



The shapes of the modes for the S&P 500 and Nikkei 225 are strikingly similar, especially since the data itself was recorded by independent traders separated by hours and oceans.

Mode 1: ***The level of volatility mode.***
In this pattern, all volatilities move in the same direction, with little dependence on the delta of the option, but with short-term volatilities moving more than those of longer term.

Mode 2: *The term structure of volatility mode.*
In this pattern, one-month expiration volatilities move in the opposite direction to all longer-term volatilities, again roughly independent of delta. Furthermore, all longer-term volatilities move by approximately equal amounts.

Mode 3: *The skew mode.*
This is the first mode that exhibits dependence on delta. In this pattern, the volatilities of out-of-the-money puts move in the opposite direction to those of out-of-the-money calls. In addition, short-term puts exhibit greater skew movement than longer-term puts.

These patterns or modes are consistent with the experience and terminology of market observers who use expressions such as “the term structure flattened” or “the skew steepened” to describe typical volatility changes. It is encouraging that the three modes affecting level, term structure and skew can be so clearly identified from historical data of two different markets. This suggests that the two markets themselves regard these patterns as independent and uncorrelated. If this were not the case, modes incorporating more complicated patterns would have emerged from the principal component analysis.

The relative importance of each volatility mode is displayed in Table 1 and is defined as the percentage of the total variance of surface changes explained by that particular mode.[4] The volatility level mode clearly dominates the surface variation, representing 82% of the movement of the S&P 500 surface and 86% of the movement of the Nikkei 225 surface. Taken together, the first three modes account for 91% of the total variation of the S&P 500 surface and 96% of the Nikkei 225 surface.

The ability to reduce the complexity of the motion of a full surface to just three factors with this degree of accuracy is an important conceptual and practical simplification. Regarding the importance of higher modes, incorporating modes beyond the third did not seem worthwhile for a number of reasons listed below.

4. The variance of surface change, in turn, is the sum of the variances of each of the N components of the surface.

TABLE 1. Relative Importance of Each Volatility Mode

Mode	S&P 500	Nikkei 225
Mode 1: ***Volatility Level***	**81.6**	**85.6**
Mode 2: ***Term Structure***	**5.0**	**7.9**
Mode 3: ***Skew***	**4.1**	**2.4**
Total % of Variance explained by Modes 1-3	**90.7**	**95.9**

First, the incremental variance explained by each successive mode falls slowly after the third mode, especially for the S&P 500 data, where modes 4, 5 and 6 explain 2.1%, 1.7% and 1.6% of the total variance, respectively. For the Nikkei 225 data, the variance explained is 0.9%, 0.8% and 0.7%.

Second, and more importantly, the higher modes appear to be less stable than the first three. This was established by performing an independent principal component analysis on each half of our data set. The higher modes, unlike the first three, showed significant differences between the two periods.

Third, and perhaps related to the second reason, the higher modes did not correspond in any simple way to our intuitive understandings of skew and term structure motions. Instead, they seemed to involve complicated mixtures of both.

Although there are undoubtedly cases in which the nuances of surface motion captured by the higher modes are important, a three-mode description of surface changes may provide a useful framework for addressing volatility trading and risk management issues.

The Magnitude of Observed Volatility Movements

As we have explained already, the volatility modes do not convey how much the surface has moved, only the manner in which it has moved. The sign and size of the motion depend on the coefficients bi that multiply the mode surfaces as defined in Equation (1). To get a sense of the typical interweek volatility change for the S&P 500, and interday volatility change for the Nikkei 225, we have displayed fluctuations associated with a one-standard-deviation event for each of the modes in Figures 4 and 5.

FIGURE 4. Motions corresponding to a one-standard-deviation *weekly* move for each mode of the S&P 500. The primary two-dimensional cross sections of each mode's surface are shown.

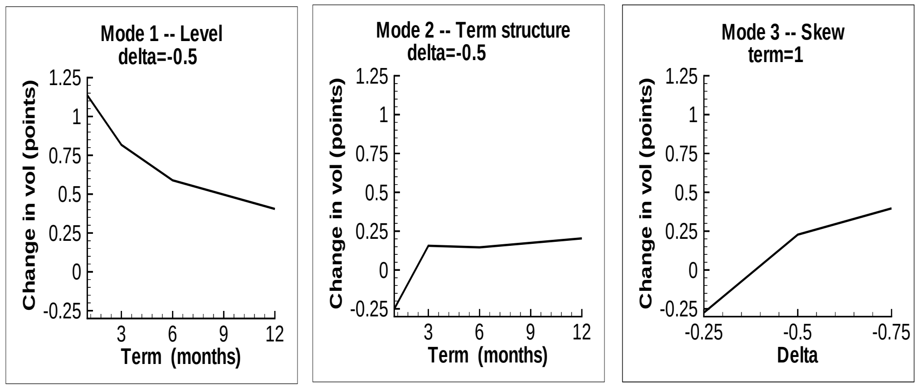
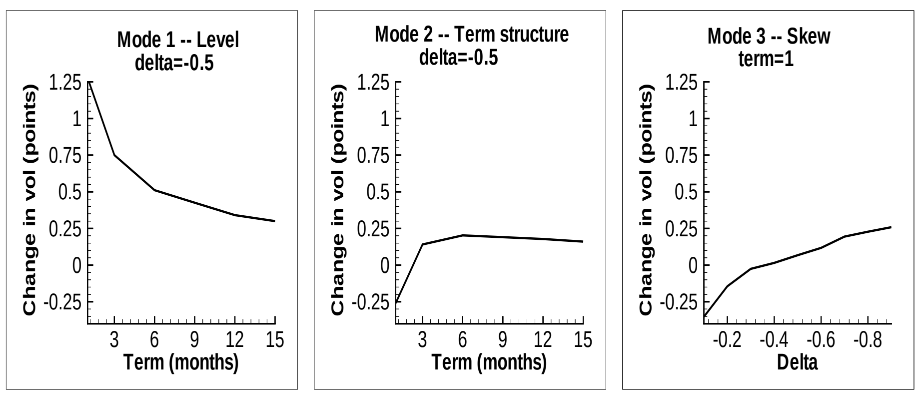


FIGURE 5. Motions corresponding to a one-standard-deviation *daily* move for each mode of the Nikkei 225. The primary two-dimensional cross sections of each mode's surface are shown



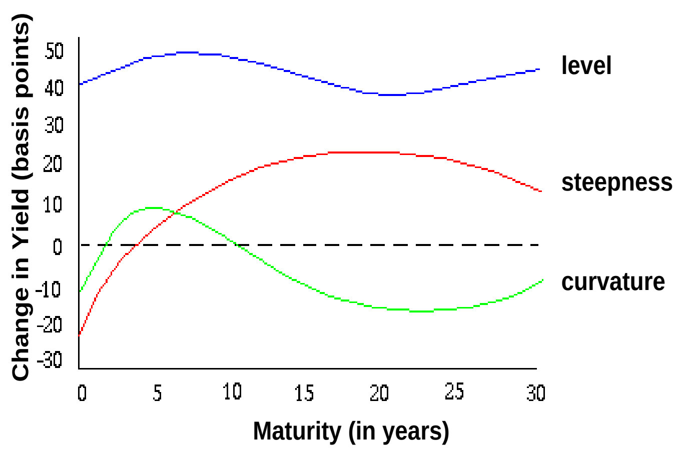
For greater clarity, we have chosen not to plot the whole surface, but instead the most important two-dimensional cross section of each mode. The fact that the typical sizes for the S&P 500 *weekly* and Nikkei 225 *daily* moves are quite close in size is just a coincidence and is a reflection of the overall higher variability in the Nikkei 225 volatility compared to the S&P 500.

While, of course, there are no guarantees that these numbers will continue to be representative of the variability of future implied volatility, they may still be useful as rough yardsticks for volatility risk management. For example, the definition of vega (the sensitivity of an option portfolio's value to changes in volatility) could be enlarged to include sensitivities to a one-standard-deviation, or some other multiple, move in each of the volatility modes.

Analogy with Modes of the Yield Curve

Our results for the implied volatility surface are strongly reminiscent of similar findings describing movements of yield curves in Fixed Income markets. A study carried out at Goldman-Sachs[5] examined weekly changes of the zero-coupon yield curve for the U.S. Treasury market between January 1984 and June 1988. A principal component analysis of the weekly changes revealed three main factors responsible for explaining, on average, 98.4% of the yield changes of zero-coupon instruments of maturities ranging from six months to 18 years. The modes were easily identified as a level, or parallel shift factor, accounting for 89.5%; a steepness factor, contributing 8.5%; and a curvature factor, representing 2.0%. Figure 6 shows the factors and is adapted from that study.

FIGURE 6. Modes of the U.S. yield curve, adapted from Litterman et al., op. cit.



An Application of Mode Analysis: ***Correlations with Index Level***

One of the most obvious applications of volatility mode analysis is to simplify the description of correlations between movements in the underlying index and changes in implied volatilities. Using modes simplifies the complexity of the problem. Rather than looking at the daily correlations of each of the 54 components of volatility changes with the

5. See *Common Factors Affecting Bond Returns* by R. Litterman and J. Scheinkman, Fixed Income Research Note, Goldman, Sachs & Co., September 1988.

daily return of the Nikkei 225 index, or the same weekly correlations involving the 12 components of the S&P 500 surface and its index return, it is easier to calculate correlations between each of the mode coefficients β_i and the index return. This is possible because the same methodology that identifies the modes also arrives at a time series of coefficients $\beta_i(t)$, which is the explicit decomposition of every surface change in the data series into modes as expressed in Equation (1).

The results are shown in Table 2. In both markets, there is a negative correlation of similar magnitude (-0.61 for S&P 500 and -0.67 for Nikkei 225) between index return and volatility "level". This result indicates that volatilities have tended to fall when the index has gone up, and vice-versa. Interestingly, the skew and term structure modes appear to be essentially uncorrelated with index level movements.

TABLE 2. Correlation Between Index Return and Mode Coefficient

Mode	S&P 500	Nikkei 225
Mode 1: *Volatility Level*	**-0.61**	**-0.67**
Mode 2: *Term Structure*	**-0.07**	**-0.05**
Mode 3: *Skew*	**0.07**	**0.04**

Conclusion and Future Direction

Using principal components analysis, we have found that most of the variation in the structure of implied volatilities of the S&P 500 and Nikkei 225 index option markets can be understood in terms of three simple volatility patterns that have an intuitive interpretation as volatility level, term structure and skew factors. The pronounced similarity found between the modes of both markets invites further study of the behavior of other volatility markets.

As volatility markets evolve and mature, issues relating to volatility risk management and trading are likely to become increasingly important. Volatility modes may provide a useful framework to address these issues. A natural first step in this direction may be to study the feasibility of volatility mode hedging, that is, whether it is possible to construct a portfolio of options, either static or dynamic, that is sensitive to only one particular volatility mode and no other.

SELECTED QUANTITATIVE STRATEGIES PUBLICATIONS

June 1990 *Understanding Guaranteed Exchange-Rate Contracts In Foreign Stock Investments*
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