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Is the Volatility Skew Fair?

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SUMMARY

In this note, we explore whether the degree of “skew” in S&P 500 implied option volatilities across different strike prices can be justified by the historical behavior of S&P 500 index returns. We propose two methods for using historical returns, both before and after the 1987 crash, to estimate “fair” options prices and show that, in each era, they can produce skews similar to those observed.

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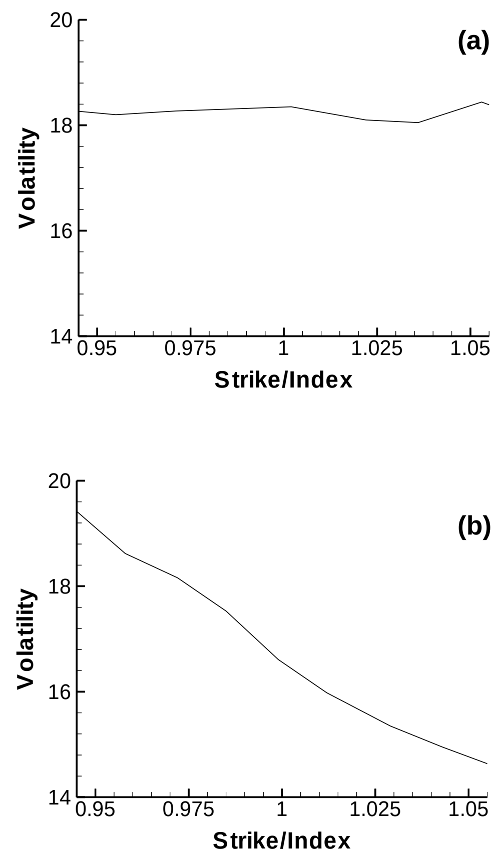
INTRODUCTION

At any instant, imagine taking a snapshot of the current implied volatilities[1] of S&P 500 European options of all strikes. Prior to the stock market crash of 1987, this picture displayed little structure, as shown in Figure 1a. Since the crash, the pattern has altered, as shown in Figure 1b: the options market has developed a pronounced "negative" skew in which the implied volatilities of low-strike options (usually puts) systematically exceed those of higher-strike options (usually calls). A similar post-crash skew is now ubiquitous in most equity index options markets, at all expirations and at most times, despite frequent changes in the absolute level of implied volatility.

FIGURE 1. Representative implied volatility skews of S&P 500 options. (a) Pre-crash. (b) Post-crash. Data taken from M. Rubinstein, "Implied Binomial Trees" ***J. of Finance*****, 69 (1994) pp. 771-818.**



1. In practice, since quoted options prices grow stale as the index moves, we must often rely on a knowledgeable trader's *idea* of a snapshot of current implied volatilities.

Is this skew fair? In this note we begin to investigate this question using historical data on S&P 500 index returns. We divide our data into pre-crash (January 1970 - January 1987) and post-crash (January 1990 - January 1997) eras.

We will restrict our analysis to three-month S&P 500 options, quoting each option price in terms of its Black-Scholes implied volatility Σ and its corresponds strike *K*, expressed as a fraction κ of the current three-month forward level of the index. Analyzing prices in terms of Σ and κ removes much of the *expected* variation in options prices as index level, interest rates, expiration time and strike level change, thus highlighting any unexpected behavior.

All our graphs below will be restricted to the values $0.95 < \kappa < 1.05$. Many options traders like to examine the implied volatilities of options in terms of their associated Black-Scholes hedge ratio Δ. This is analogous to bond market analysts describing bond value in terms of yield to maturity and corresponding duration. For three-month options with an annual volatility of 15% and an interest rate of 6%, the lower (upper) κ values above corresponds to a put (call) Black-Scholes hedge ratio of magnitude -0.18 (0.34).

DEFINING "FAIR"

Our first task is to specify the meaning of the question. We will concentrate on two alternative definitions of a "fair" options price.

"Fair" = Fair risk-neutral expected value

In this approach, we define a fair options price as the price obtained by assuming that future S&P 500 returns will come from the same distribution that produced past returns. More specifically, we use historical S&P 500 daily closing prices to estimate an expected risk-neutral distribution of the index at expiration, and then use that distribution to compute the values of three-month options of various strikes. We discuss this technique in more detail below.

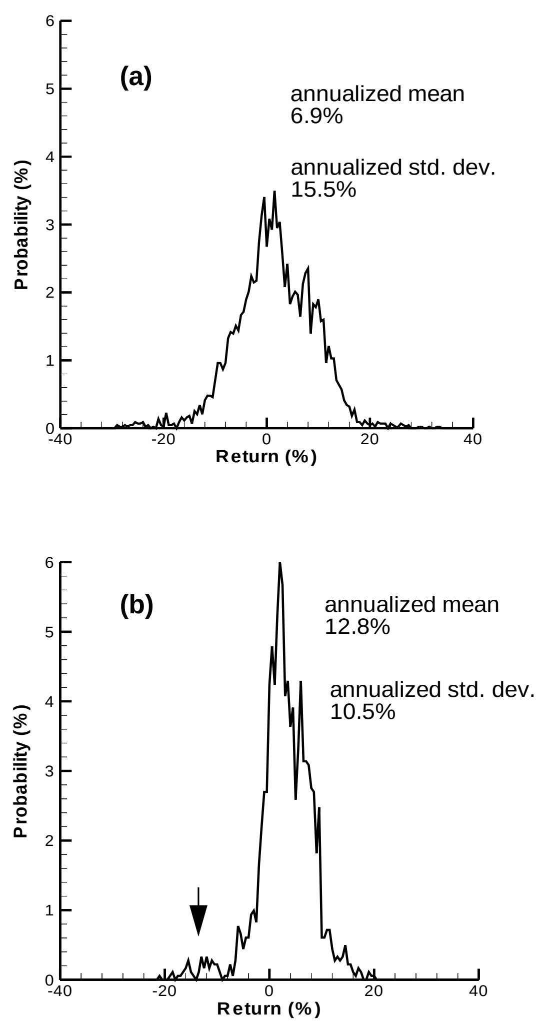
"Fair" = Fair replication value

Alternatively, we define a fair options price as the cost of using the index and a riskless bond to replicate the option dynamically over the actual subsequent evolution of the index, neglecting all transactions costs. We describe the details of the replication assumptions below.

ESTIMATING THE FAIR RISK-NEUTRAL EXPECTED SKEW

Consider the historical daily series of index levels S_i, where the subscript i denotes each successive trading day. We can construct the rolling series of continuously compounded index returns for a period of N trading days by calculating $R_i = \log[S_{i+N}/S_i]$. Figure 2 shows the historical distribution of actual three-month S&P 500 returns for both the pre-crash and post-crash periods. The pre-crash return distribution is characterized by a 6.9% annualized mean return and a 15.5% annualized standard deviation or volatility, and appears approximately symmetric and lognormal. In contrast, the post-crash distribution has a higher mean return (12.8%) and a lower volatility (10.5%), as well as an asymmetric secondary peak at about a return of -12%.

FIGURE 2. The historical distribution of rolling three-month S&P 500 returns. (a) Pre-crash. (b) Post-crash. Arrow indicates secondary peak.



If these actual distributions were used to compute expected options payoffs, it is clear that the secondary peak would raise the implied volatilities of out-of-the-money puts relative to out-of-the-money calls. However, options theory dictates that the theoretical value of an option is obtained by replicating the option using the index and a riskless bond at every instant during the stochastic evolution of the index. The resultant option value is given by the expected present value of the payoff over a “risk-neutral distribution” whose mean return is the riskless rate, not the actual expected return, and whose distribution depends on the assumed stochastic evolution process.

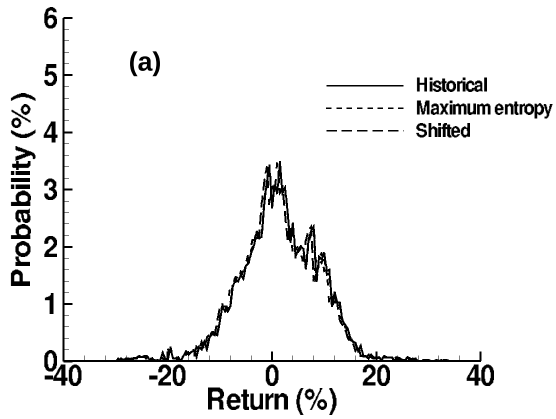
In theory, the risk-neutral index distribution has the same shape as the expected actual distribution if the actual evolution is assumed to be lognormal. But we do not know the theoretical nature of the actual index evolution, though Figure 2b indicates that it does deviate from lognormal. In short, we do not know with any certainty how to derive the risk-neutral distribution we need for options valuation from the past sequence of rolling three-month returns.

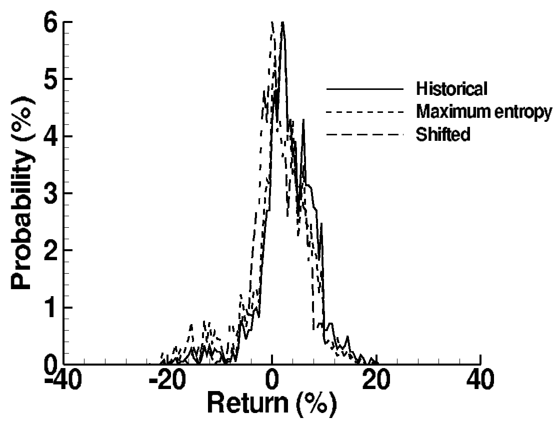
Nevertheless, in order to proceed, we will attempt to transform the historical distributions of Figure 2 into risk-neutral distributions that can be used to estimate the current fair skew. It is imperative that the current risk-neutral distribution satisfy the forward condition – that is, it must lead to an expected index level equal to the current forward index level. The simplest way to achieve this is to shift (without a change in shape) the historical return distribution in each era of Figure 2 until the mean return produces the current forward level. A more sophisticated approach is to modify the historical distribution in the minimal way, using the maximum entropy principle[2], so as to satisfy the forward condition.

Figure 3 shows the pre- and post-crash risk-neutral distributions corresponding to both of these methods, assuming current three-month interest rates are 6% annually. The shape of the risk-neutral distributions of Figure 3a and 3b are largely insensitive to the method used to extract them from the historical distribution. The post-crash distribution has a substantially larger tail at low returns than the pre-crash distribution.

2. M. Stutzer, “A Simple Nonparametric Approach to Derivative Security Valuation” *J. of Finance*, 51 (1996) pp. 1633-1652.

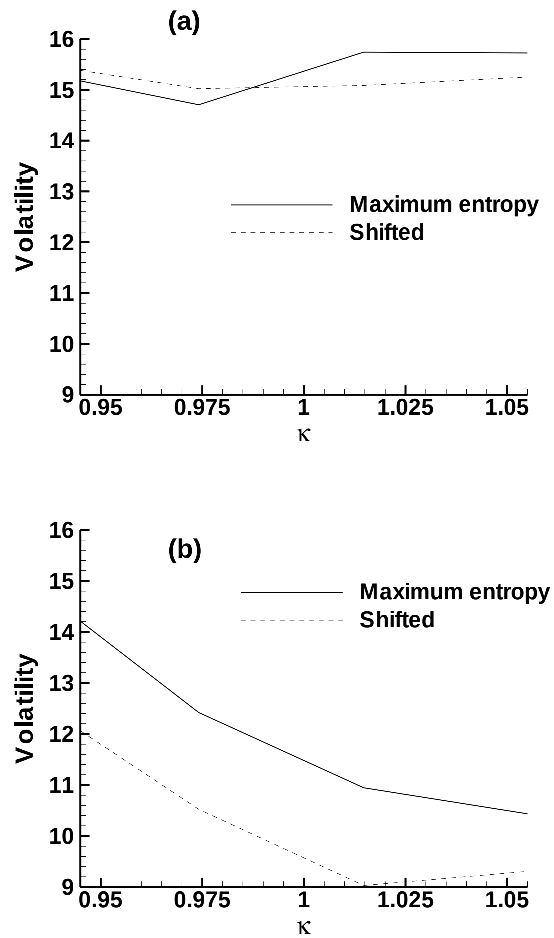
FIGURE 3. The three-month risk-neutral distribution of S&P 500 returns constructed from the historical distributions using a 6% riskless rate. (a) pre-crash (b) post-crash.





We now use these distributions to compute current fair options prices as the expected value of the payoff over the distribution, assuming a riskless annual interest rate of 6%. Figure 4 shows the fair risk-neutral implied volatility skew based on these expected distributions. The pre-crash fair skew is approximately flat, independent of κ. The post-crash volatilities increase for low strikes, with a slope similar to the actual skew of Figure 1. Ignoring absolute levels of implied volatility, which vary dramatically over time, this simple test of fairness seems to indicate that the slope of the market skew is approximately fair in the light of post-crash market behavior.

FIGURE 4. The fair risk-neutral skew based on the distributions of Figure 3. (a) Pre-crash. (b) Post- crash.



ESTIMATING THE FAIR REPLICATION SKEW

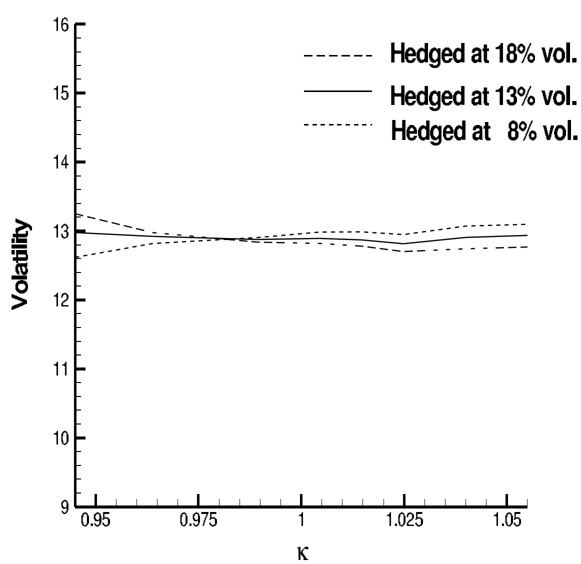
In this section we examine the fair value of the skew when option value is defined as replication value. We intend to replicate three-month options of various strikes (expressed in κ units) over all rolling three-month periods in our sample, rehedging once at the close of each trading day via the Black-Scholes prescription, ignoring transactions costs. For each definite value of κ, we will average all three-month option replication costs and convert that value into an equivalent Black-Scholes fair implied volatility. The set of fair values for all κ determines the fair skew.

We do not know exactly which Black-Scholes "hedge" volatility (or "hedge" interest rate) would have been used while replicating the option during each day of each rolling three-month period. Our simulation of the past will inevitably be mis-specified and less rich than reality. In addition, we have only about 500 historical replication paths on which to test fairness. To ascertain whether this endeavor is feasible, we want to confirm that replication over a limited number of paths with mis-specified rates and volatilities can still provide enough accuracy to identify a fair skew. Therefore, we first test our methodology by (1) specifying an evolution process of our own choice whose fair skew we understand, and (2) then verifying that the average replication cost, over 500 randomly generated histories, can expose the actual skew.

Testing the Replication Method: Simulating in a Lognormal World with Constant Volatility and a Flat Skew

Figure 5 shows the implied volatility skew obtained by calculating the replication cost for options whose underlying index evolves lognormally with a constant 13% volatility. In this world, there should be no skew. To determine the sensitivity to mis-specification, we show the skew obtained by averaging the option cost that results from replicating using the Black-Scholes formula with a constant hedge interest rate of 6% and hedge volatility values of 8%, 13% and 18%. It is clear that, on average, the flat structure of the fair skew is visible, even when we hedge at the wrong volatility and use only a limited number of replication scenarios.

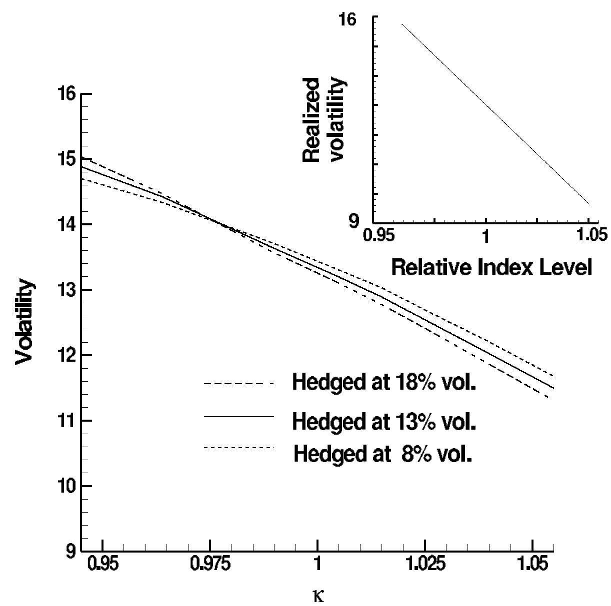
FIGURE 5. The fair replication skew for three different hedge volatilities when the index evolves lognormally with a 6% return and a 13% realized volatility. All implied volatilities at a definite κ are calculated by simulating the option replication cost over each of 500 randomly generated paths, averaging the cost over all paths, and then converting the cost to an implied volatility.



Testing the Replication Method: Simulating in World with Index-dependent Volatility and a Negative Skew

Figure 6 shows the implied volatility skew obtained by replication when the underlying index evolves with a volatility that equals 13% when the index is at 100, and that increases (decreases) by 0.7 percentage points for every one percentage point fall (rise) in the index. This linear variation of realized volatility with index leads to a negative fair skew[3]. This negative skew shows up clearly in our simulation results, roughly independent of the value of volatility used to carry out the hedge. This gives us confidence that a fair negative skew can be detected by this technique.

FIGURE 6. The fair replication skew for three different hedge volatilities when the index evolves with an index-dependent realized volatility as shown in inset.

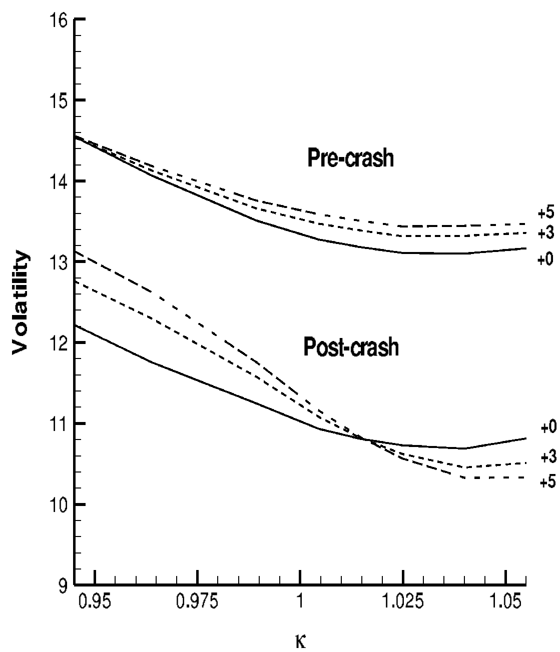


The Fair Historical Replication Skew

Figure 7 shows the fair replication skew for both the pre-crash and the post-crash periods. For each data sample, for a variety of strike values κ, we have replicated the options cost over each rolling three-month historical period under the following assumptions: (1) we use the Black-Scholes replication prescription; (2) we use an annual hedge interest rate fixed at 6% over any option's lifetime; and (3) we use a hedge volatility several points higher than the historically prevailing realized volatility of the index over the life of the option being replicated. This spread is intended to reflect the historical tendency for implied volatilities to exceed realized volatilities by several percentage points.

It would have been more realistic to replicate using a hedge interest rate set to the prevailing historical interest rate. We have not been able to do this, because fair options prices depend on interest rates and it is difficult to aggregate prices over different interest rate environments consistently. We have therefore restricted ourselves to using a hedge interest rate fixed at 6%. We find that the fair skew we extract is relatively insensitive to the assumed fixed interest rate.

FIGURE 7. The fair replication skew for a fixed 6% hedge interest rate and for spreads of 0, 3 and 5 percentage points between the hedge volatility and the subsequent realized volatility.



Observing Figure 7, we notice that, irrespective of hedge volatility, the fair replication skew for the pre-crash sample shows only a small tilt, compatible in shape with both the actual skew in Figure 1a and the fair risk-neutral expected skew of Figure 4a. The post-crash fair replication skew for a hedge volatility equal to the realized volatility resembles the pre-crash skew. However, for a hedge volatility spread of 3 or 5 points over realized volatility, the fair post-crash skew resembles both the actual skew of Figure 1b and the fair risk-neutral expected skew of Figure 4b. As the assumed hedge volatilities in our replication scheme increase, a qualitative difference between the pre- and post-crash fair skews emerges. This difference supports the approximate fairness of actual post-crash skew shapes.

CONCLUSION

We have used historical S&P 500 index data to estimate the fair pre- and post-crash skews for three-month S&P 500 European index options using two different methods, the expected risk-neutral method and the replication method. Based on our limited statistical sample, we draw the following conclusions.

For the pre-crash period, both methods produce no appreciable skew. The methods disagree on the fair level of overall implied volatility. But volatilities changes much more frequently than the shape of the skew, and estimating its fair value may require more realistic assumptions about prevailing market conditions and hedging procedures and costs.

Post-crash, the replication method with a hedge volatility 3 to 5 percentage points above historical realized volatility seems to agree in shape with the fair risk-neutral expected skew, and is qualitatively similar to typically observed post-crash skews.

SELECTED QUANTITATIVE STRATEGIES PUBLICATIONS

June 1990	*Understanding Guaranteed Exchange-Rate Contracts In Foreign Stock Investments* Emanuel Derman, Piotr Karasinski and Jeffrey S. Wecker
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