Quantum Computing: The primary difference between classical and quantum computing is about "states". While in classical states are either "0" or "1", in quantum the states can be in superposition, assuming "0" and "1" simultaneously. (The analogous of bit in quantum computing: qubit.)

Using quantum algorithm may allow an exponentially speed-up, however, once a superposition is measured it collapse to a value (one of the status)
→ that is why it is hard to design quantum algorithms.
But we can use interference effects to design them.

How to describe Quantum states?

Dirac Notation: Let $a, b \in \mathbb{C}^2$

— ket: $|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \therefore \begin{pmatrix} a_1 + i b_1 \\ a_2 + i b_2 \end{pmatrix}$

— bra: $\langle b|$ → is the transposed complex conjugated of ket
$\therefore \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 \end{pmatrix}^* = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \therefore \begin{pmatrix} a_1 - ib_1 & a_2 - ib_2 \end{pmatrix}$
$\begin{pmatrix} b_1 & b_2 \end{pmatrix}$

Good to mention that $|\varphi_1\rangle$ is a vector and $\langle \varphi_2|$ is the linear functional in quantum mechanics. A linear functional may be described as an operation $\langle \varphi_2|$ applied in some vector $|\varphi_1\rangle$. And the vector complex vector "lives" in a Hielbert space.

— bra-ket → $\langle b|a\rangle$: In this case, is the inner product →
→ $a_1 . b_1 + a_2 . b_2$ → from complex numbers, it is the same as $\langle a|b\rangle^*$. $\in \mathbb{C}^2$

— ket - bra : $|a\rangle\langle b|$ → In this case is the external product →

$$\rightarrow |a\rangle\langle b| = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

→ the states are defined $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

state $|0\rangle$ and $|1\rangle$ are ~~ortogonal~~:

$$\langle 0|1\rangle = (1 \cdot 0) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.0 + 0.1 = \boxed{0}$$

$$\langle \psi|\psi\rangle = 1$$

To ensure that the summation of all probabilities of all possible states is equal 1, all quantum state is normalized, dividing by the vector norm. The norm of vetor $|a\rangle$ is calculated as follows:

$$\| |a\rangle \| = \sqrt{\langle a|a\rangle}.$$

so, to normalize vector $|e_1\rangle$, for example, it should be: $|e_1\rangle = \dfrac{|e'_1\rangle}{\sqrt{\langle e'_1|e}}$

Ex. $|\psi\rangle = \dfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Mesurements

using orthogonal bases to describe and measure quantum ~~states~~

When we measure onto a ~~vector~~ bases, for example, $\{|0\rangle, |1\rangle\}$, during the measurement, the state will colapse into either $|0\rangle$ or $|1\rangle$, because they are eingenstates of σ_z (Pauli-Z operator, the single-qubit quantum gate). It is called **Z-measurement**

There are infinite many different bases.
The most common ones:

$$\left\{ |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right\} \qquad \text{and}$$

$$\left\{ |+i\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i\, |1\rangle) \qquad |-i\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i\, |1\rangle) \right\}$$

corresponding to eingenstates of σ_x and σ_y respectevely.

We call an <u>X measurement</u> if we are doing a measurement onto bases <u>$|+\rangle, |-\rangle$</u>.
We call a <u>Y measurement</u> if we are doing a measurement onto bases <u>$|+i\rangle, |-i\rangle$</u>.
We call a <u>Z measurement</u> if we are doing a measurement onto bases <u>$|0\rangle, |1\rangle$</u>.

Probability of a measurement yields into a result is given by the Born's rule. For instance, the probability that a state ψ collapse during a projective measurement onto bases $\{|x\rangle, |x^+\rangle\}$ to the state $|x\rangle$ is given by

$$P(x) = |\langle x | \psi \rangle|^2, \qquad \boxed{\sum_i P(x_i) = 1}$$

normalized vectors.

Ex $\quad |\psi\rangle = \frac{1}{\sqrt{3}} (|0\rangle + \sqrt{2}\, |1\rangle)$ measure in the bases $\{|0\rangle, |1\rangle\}$

$P(0)$?

$$P(0) = \left| \langle 0 | \frac{1}{\sqrt{3}} (|0\rangle + \sqrt{2}\, |1\rangle) \right|^2$$

Property of inner product

$\rightarrow \langle \alpha | b\beta + c\gamma \rangle = b\langle \alpha|\beta\rangle + c\langle \alpha|\gamma\rangle$

remembering: The inner product of orthogonal vectors is always $\boxed{0}$

then...

$$P(0) = \left|\langle 0| \tfrac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}|1\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{3}}\overset{1}{\cancel{\langle 0|0\rangle}} + \frac{\sqrt{2}}{\sqrt{3}}\overset{0}{\cancel{\langle 0|1\rangle}}\right|^2 =$$

remembering: the inner product of a vector with itself is always $\boxed{1}$

$$= \left|\frac{1}{\sqrt{3}}\right|^2 = \boxed{\frac{1}{3}}$$

$$P(1) = \left|\langle 1| \tfrac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}|1\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{3}}\overset{0}{\cancel{\langle 1|0\rangle}} + \frac{\sqrt{2}}{\sqrt{3}}\overset{1}{\cancel{\langle 1|1\rangle}}\right|^2 = \left|\frac{\sqrt{2}}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

by the way: $P(0) + P(1) \therefore \frac{1}{3} + \frac{2}{3} = \boxed{1}$

<u>Other example</u>: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$, in basis $\{|+\rangle, |-\rangle\}$

$$P(+) = |\langle + |\psi\rangle|^2 \therefore \left|\left[\frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)\right] \cdot \left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right]\right|^2 \therefore$$

$$\therefore \left|\frac{1}{2}\left[\left(\langle 0| + \langle 1|\right) \cdot \left(|0\rangle - |1\rangle\right)\right]\right|^2 = \frac{1}{4}\left|\overset{1}{\cancel{\langle 0|0\rangle}} - \overset{0}{\cancel{\langle 0|1\rangle}} + \overset{0}{\cancel{\langle 1|0\rangle}} - \overset{1}{\cancel{\langle 1|1\rangle}}\right|^2 \therefore$$

$$\therefore \frac{1}{4}\left(1 - 1\right)^2 = \boxed{0}$$

$$P(-) = \left|\langle - | \psi \rangle\right|^2 \therefore \left| \frac{1}{\sqrt{2}} \left(\langle 0| - \langle 1|\right) \cdot \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right)\right|^2 \therefore$$

$$\therefore \left| \frac{1}{2}\left(\overset{1}{\cancel{\langle 0|0\rangle}} - \overset{0}{\cancel{\langle 0|1\rangle}} - \overset{0}{\cancel{\langle 1|0\rangle}} + \overset{1}{\cancel{\langle 1|1\rangle}} \right)\right|^2 = \left|\frac{1}{\cancel{2}} \cdot \cancel{2}\right|^2 = \boxed{1}$$

Representation of states

Bloch Sphere

Any normalized (pure) state as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
where $\phi \in [0, 2\pi]$ describes the relative phase
and $\theta \in [0, \pi]$ determine the probability to measure $|0\rangle$ or $|1\rangle$

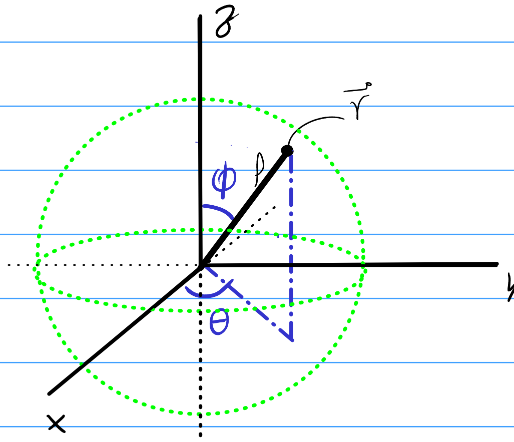
$p(0) = \cos^2\frac{\theta}{2}$, $p(1) = \sin^2\frac{\theta}{2}$ $\therefore$

REF: $\boxed{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1}$

All normalized pure state can be illustrated on the surface of a sphere with radius $|\vec{r}| = 1$, which is called **Bloch Sphere.**

just remembering:

$$\boxed{\begin{aligned} x &= \rho \,\mathrm{sen}\,\theta \cos\phi \\ y &= \rho \,\mathrm{sen}\,\theta \,\mathrm{sen}\,\phi \\ z &= \rho \cos\theta \end{aligned}}$$

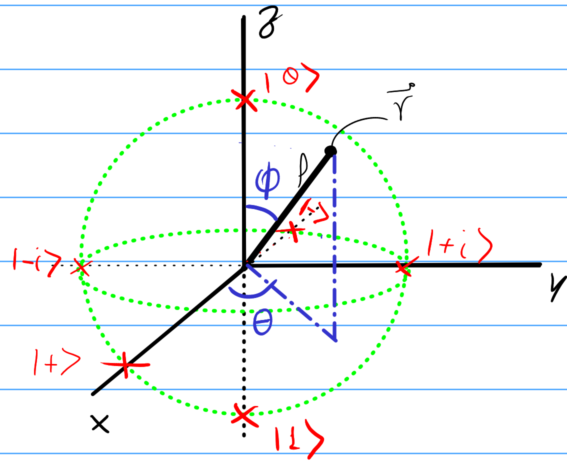


$\vec{r} = (\rho, \theta, \phi)$
"x, y, z"

then Bloch vector

$$\vec{r} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$

Ex $|0\rangle$: $\theta = 0$, any $\phi \rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



$|1\rangle$: $\theta = \pi$; any $\phi \rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$|+\rangle$: $\theta = \pi/2$; $\phi = 0 \rightarrow \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$|-\rangle$: $\theta = \pi/2$; $\phi = \pi \rightarrow \vec{r} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$|+i\rangle$: $\theta = \pi/2$; $\phi\ \pi/2 \rightarrow \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$|-i\rangle$: $\theta = \pi/2$; $\phi\ 3\pi/2 \rightarrow \vec{r} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

On the Bloch sphere, angles are twice as big as in Hilbert space

$|0\rangle$ and $|1\rangle$ are orthogonal, but on the Bloch sphere their angle is $180°$.

From the general state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle \ldots$

$\rightarrow$ θ is the angle in the Bloch sphere, while $\frac{\theta}{2}$ is the angle in the Hilbert space.

$\rightarrow$ Z-measurement corresponds to a projection onto the z-axis. analogously for x and y.