CONVERGENCE TO EQUILIBRIUM

Entropy Methods in Partial Differential Equations and Stochastic Analysis

Alejandro Morera Alvarez

MA8111: Projekt mit Kolloquium (Bachelor)

CONTENTS AND MAIN QUESTIONS

Entropy: Motivation and Problem Statement

- **What is entropy? What is an entropy functional?**

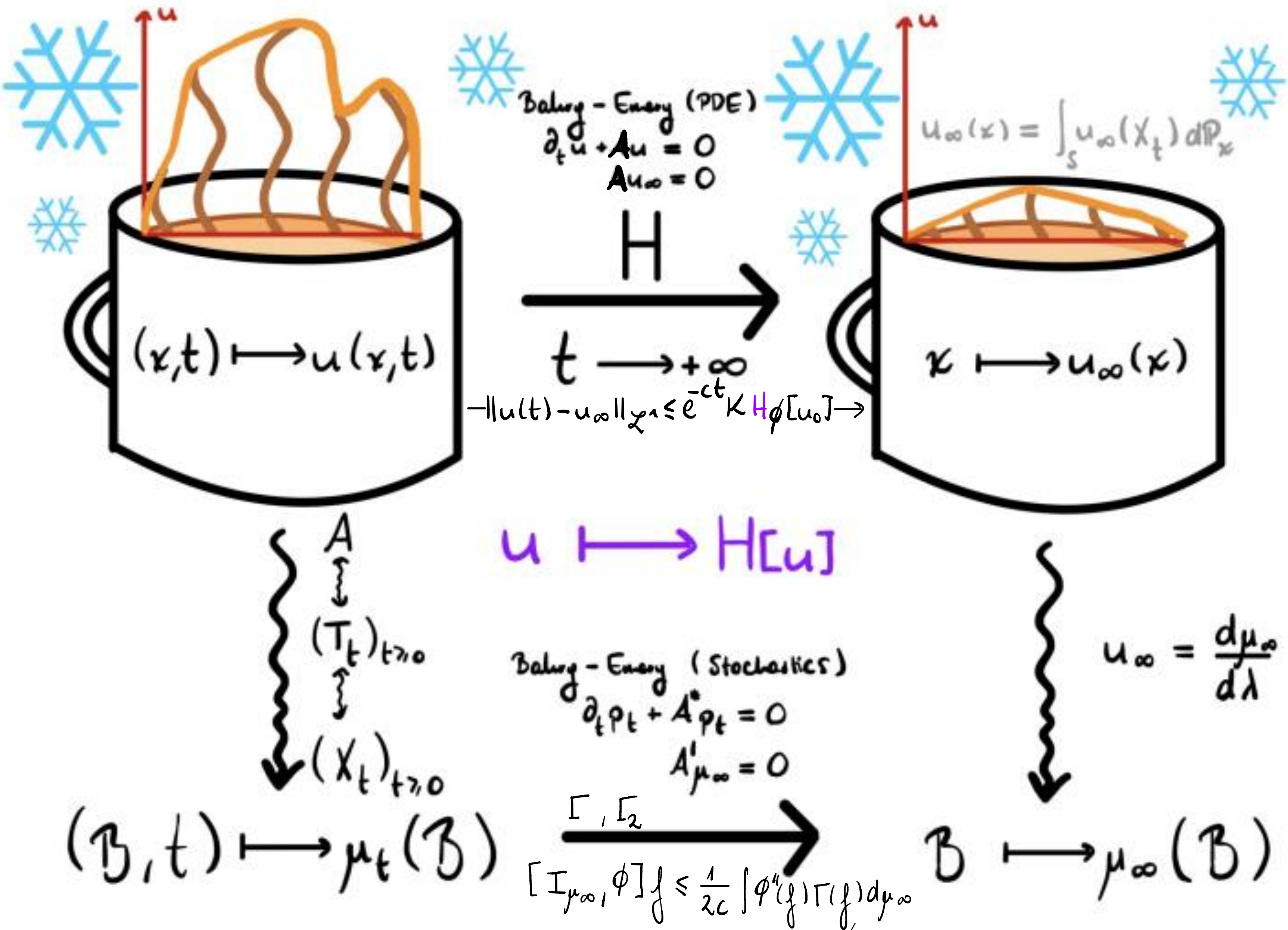
Entropy Methods: Bakry-Emery

- **How can we use entropy functionals in order to determine the rate of convergence?**

Stationarity: Function-Measure Duality

- **Why are stationary solutions of interest in both**

PDEs and stochastic processes?

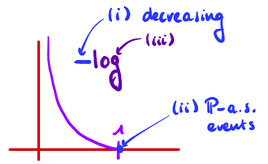

u
Bakry - Emery (PDE)
$\partial_t u + Au = 0$
$Au_\infty = 0$
H
$(x,t) \longmapsto u(x,t)$
$t \longrightarrow +\infty$
$\|u(t) - u_\infty\|_{L^1} \leq e^{-ct} K H_\phi[u_0]$
u
$u_\infty(x) = \int_S u_\infty(X_t)\, d\mathbb{P}_x$
$x \longmapsto u_\infty(x)$
$u \longmapsto H[u]$
A
$(T_t)_{t \geq 0}$
$(X_t)_{t \geq 0}$
Bakry - Emery (Stochastics)
$\partial_t \rho_t + A^* \rho_t = 0$
$A' \mu_\infty = 0$
$u_\infty = \frac{d\mu_\infty}{d\lambda}$
Γ, Γ_2
$(B,t) \longmapsto \mu_t(B)$
$B \longmapsto \mu_\infty(B)$
$[I_{\mu_\infty}, \phi] f \leq \frac{1}{2c} \int \phi''(f) \Gamma(f)\, d\mu_\infty$

MOTIVATION FOR ENTROPY

Physics $\Longrightarrow$ Stochastics: $\mathcal{S} := E_{\mathbb{P}} I(X) := \int_{\Omega} I(\overset{\text{density}}{f}(X(\omega)))\, \mathrm{d}\mathbb{P}(\omega)$

where $f: S \longrightarrow [0,1]$
$\qquad x \longmapsto \mathbb{P}(X = x)$

$I: [0,1] \longmapsto [0,\infty) \rightsquigarrow$ Information Function $\overset{\text{Axioms}}{\rightsquigarrow} \underbrace{C}_{\substack{<0 \\ \text{decreasing}}} \cdot \log$



$\Downarrow$

PDEs

$$\partial_t u(x,t) + A u(x,t) = 0 \text{ for } x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x) \text{ for } x \in \mathbb{R}^d$$

$A: D(A) \longrightarrow X'$ where $D(A) \subseteq X$ Banach space.

Lyapunov functional - $H : D(A) \to \mathbb{R}$

Entropy functional - Convex Lyapunov functional.

$$H[f] = \int_{\mathbb{R}^d \times \mathbb{R}^d} f \log f \, \mathrm{d}\lambda\, \mathrm{d}\nu \qquad H_1[u] := \int_{\mathbb{T}^d} u(\log(u) - 1)\, \mathrm{d}\lambda$$

ENTROPY METHODS

Convergence to equilibrium with various distance notions and nonlinear evolution equations

- **How can we use entropy functionals in order to determine the rate of convergence?**

GENERAL ENTROPY METHODS

1. Compute the entropy production

Recall A maps into X'.

E.g. $\langle f,g\rangle = \int fg\,d\lambda$ for $p\geq 1$, $f\in\mathcal{L}^p$, $g\in\mathcal{L}^q$, $p^{-1}+q^{-1}=1$

$$-\frac{d}{dt}H[u(t)] = \langle A(u(t)),\ H'[u(t)]\ \rangle$$

where the braces $\langle\cdot,\cdot\rangle$ denote some kind of Banach space product.

This structure will arise naturally by using the evolution equation itself.

2. Derive an inequality of the form

$$cH[u] \leq \langle A(u(t)), H'[u(t)]\rangle$$

($H'[u(t)]$ denotes the Fréchet derivative for general normed spaces of H). exist. This implies $dH/dt \leq -cH$.

3. Use Gronwall's lemma to conclude that

$$H[u(t)] \leq H[u_0]\exp(-ct)\,.$$

?

Bakry-Emery: One step by bounding second derivative of entropy functional.

EXAMPLE: LINEAR FOKKER-PLANCK

$$\partial_t u = \nabla \cdot (\nabla u + u \nabla V) =: A^* u, \qquad t > 0, \qquad u(0) = u_0 \text{ in } \mathbb{R}^d \qquad (2.1)$$

Now, let $\phi : [0, \infty) \to \mathbb{R}$ be a smooth, strictly convex function with $\phi(1) = \phi'(1) = 0$ and $1/\phi''$ concave. Define an entropy functional by

$$H_\phi[u] := \int \phi(u/u_\infty)\, d\lambda\,.$$

Theorem 2.2.3 (Exponential Decay of Fokker-Planck, [J], Thm. 2.1) Let $\phi \in \mathcal{C}^4([0, \infty))$ be as above, assume $H_\phi[u_0] < \infty$ and let V be a potential satisfying the Bakry-Emery condition: there exists some $\lambda > 0$ with

$$\boxed{\nabla^2 V - \lambda \cdot \mathrm{id} \geq 0\,.} \tag{2.2}$$

Uniformly convex potential

Then, any normalized, smooth solution to (2.1) converges exponentially fast to the steady state in L^1, i.e. for all $t > 0$, we have

$$||u(t) - u_\infty||_{L^1} \leq \exp(-t\lambda) K_\phi H_\phi[u_0]^{1/2}$$

$$\int \phi\left(\frac{u}{u_\infty}\right) u_\infty\, d\lambda \leq \frac{1}{2\lambda} \int \phi''\left(\frac{u}{u_\infty}\right) \left|\left|\nabla \frac{u}{u_\infty}\right|\right|^2 u_\infty\, d\lambda$$

KEY STEP: UPPER BOUND SECOND DERIVATIVE

1. Compute the entropy production

Lebesgue

$$-\frac{d}{dt}H[u(t)] = -\frac{d}{dt}\int \phi(\overbrace{u/u_\infty}^{=:\rho})\,d\lambda$$

PDE and IBP

$$= \int \phi''(\rho)\,\|\nabla\rho\|^2 u_\infty \,d\lambda$$

★ Assumptions on ϕ

Key Step

2. Bound second time-derivative: $\frac{d^2}{dt^2}H_\phi[u] = -\int \phi'''(\rho)\,\partial_t u\,\|\nabla\rho\|^2 + 2\phi''(\rho)\,\nabla\rho^T\partial_t\nabla\rho\; u_\infty\, d\lambda$

$$= -\int \phi''(\rho)\Big(\nabla\cdot(\nabla^2\rho\,\nabla\rho) - \|\nabla^2\rho\|^2 - \langle\nabla\rho,\nabla V\rangle_{\nabla^2\rho} \underbrace{- \langle\nabla\rho,\nabla\rho\rangle_{\nabla^2 V}}_{\leq -\lambda\|\nabla\rho\|^2}\Big)\, u_\infty\, d\lambda$$

★ $C^0[0,\infty)$

$$+ \int \Big(\phi''''(\rho)\,\|\nabla\rho\|^4 + 2\phi'''(\rho)\,\langle\nabla\rho,\nabla\rho\rangle_{\nabla^2\rho}\Big)\, u_\infty\, d\lambda$$

⋮

IBP and Convexity

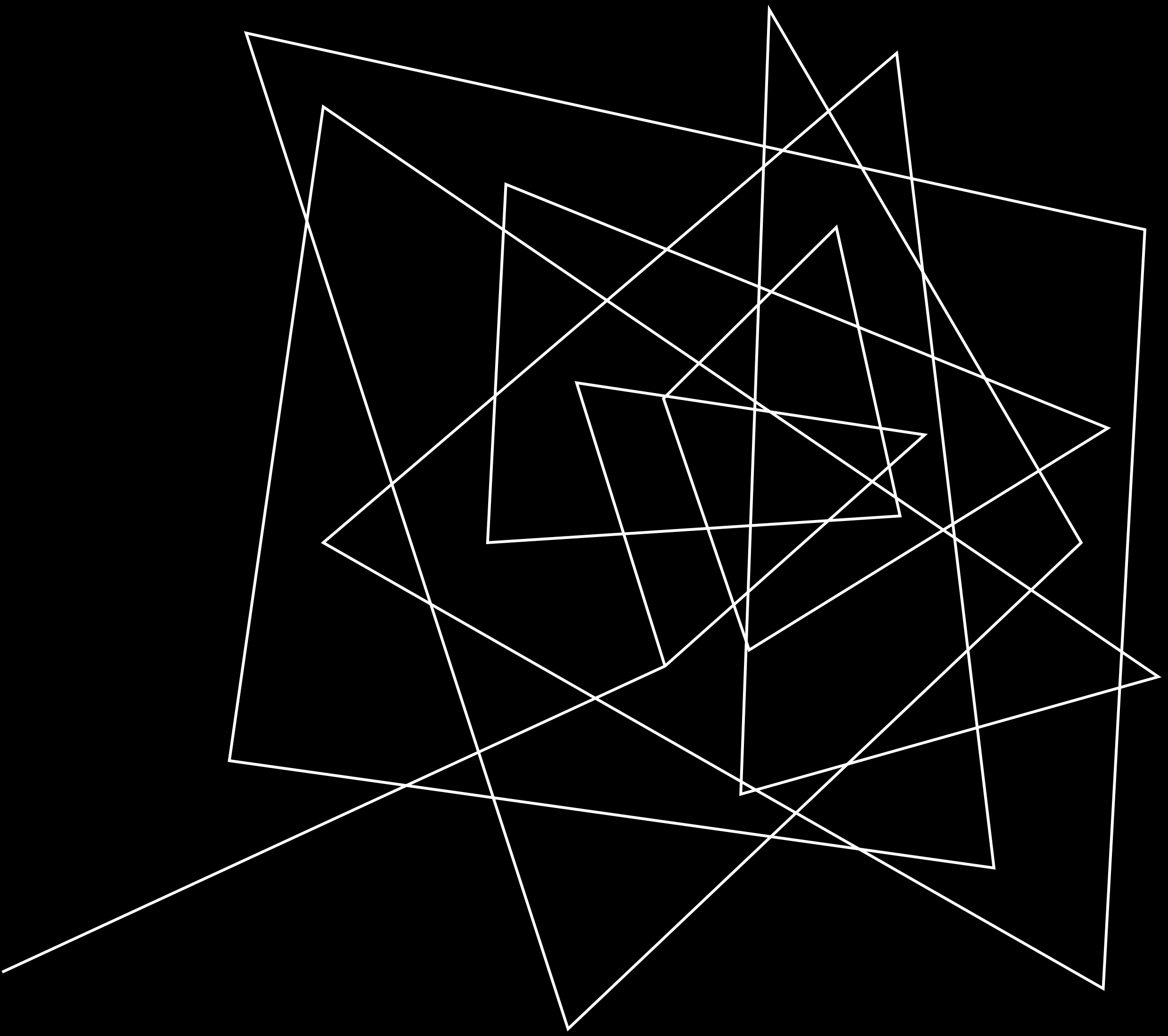
Assumption ★

$$\geq 2\int \phi''(\rho)\,\|\nabla^2\rho + \phi'''(\rho)/\phi''(\rho)\,\nabla\rho\otimes\nabla\rho\|^2 u_\infty\, d\lambda + \int \underbrace{\Big(\phi''''(\rho) - 2\phi'''(\rho)^2/\phi''(\rho)\Big)}_{= -(\phi'')^2\cdot\underbrace{(1/\phi'')''}_{\leq 0} \geq 0} \underbrace{\|\nabla\rho\|^4}_{\geq 0} u_\infty\, d\lambda$$

$$-2\lambda\frac{d}{dt}H[u]$$

$$\Longrightarrow \boxed{\frac{d^2}{dt^2}H_\phi[u] \geq -2\lambda H_\phi[u]}$$

3. Conclude with Gronwall and a special inequality (Csiszar-Kullback-Pinsker).



STATIONARITY

Time-independent solutions to evolution equations

Why are stationary solutions of interest in both PDEs and stochastic processes?

WHY STATIONARITY?

1. Convergence to time-independent solutions (cf. Linear Fokker-Planck).

$$||u(t) - u_\infty||_{L^1} \leq \exp(-t\lambda) K_\phi H_\phi[u_0]^{1/2}$$

2. Dynkin's Equation: Mean Value Representation of stationary solutions.

$$u_\infty(x) = \int_S u_\infty(X_t)\, \mathrm{d}\mathbb{P}_x$$

3. Function-Measure Correspondence.

$$\begin{tikzcd}
A \arrow[r, leftrightarrow, "\text{Hille-Yosida}"] \arrow[d, rightsquigarrow] & (X_t)_{t\geq 0} \arrow[d, rightsquigarrow] \\
u_\infty \arrow[r, leftrightarrow, "u_\infty = \frac{\mathrm{d}\mu_\infty}{\mathrm{d}\lambda}"] & \mu_\infty
\end{tikzcd}$$

HILLE-YOSIDA THEOREM: MOTIVATION

$$X_t : \overbrace{\{1, ..., d\}}^{=:S} \to \mathbb{R}, j \mapsto (x_0)_j = x_0(j)$$

Can we generalize this
for stochastic
processes?

$$T_t x_0 := \exp(tA)x_0 \qquad \dot{x}(t) = Ax(t), \quad x(0) = x_0$$

HILLE-YOSIDA THEOREM: INGREDIENTS

Feller Process $(X_t)_{t\geq 0}$ $(B_t)_{t\geq 0}$

Markov Process
(IC) $\forall x \in S : \mathbb{P}_x(X_0 = x) = 1.$

(MP) $\forall s, t \geq 0 : E_x(f(X_{s+t}) \mid \mathcal{F}_t) = E_{X_t} f(X_s).$

(FP) $\forall f \in \mathcal{C}_0(S), t \geq 0 : x \mapsto E_x f(X_t)$ is in $\mathcal{C}_0(S)$.

Locally Compact Space

$\mathcal{C}_0(S) := \{ f \in \mathcal{C}(S) \mid \forall \varepsilon > 0 \; \exists K \subset\subset S : |f(x)| < \varepsilon, \forall x \in S \setminus K \}$

Probability Semigroup $(T_t)_{t\geq 0}$ $T_t : \mathcal{C}_0(S) \longrightarrow \mathcal{C}_0(S)$

This is partly why we require (FP)

$T_t f := f * g_t \, , \; g_t := \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-\|\cdot\|}{2t}\right)$ $t > 0$

Exponential Properties
(S1) $T_0 = \mathrm{id}.$

(S2) $\forall f \in \mathcal{C}_0(S) : T_t f \xrightarrow{t \searrow 0} f$ with respect to $||\cdot||_\infty$.

(S3) $T_{t+s} = T_t T_s.$

(S4) $f \geq 0 \implies T_t f \geq 0.$

Stochastic Property
(S5) There exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_0(S)$ which is bounded in norm, i.e. $\sup ||f_n|| < \infty$, such that for all $x \in S$ we have $f_n(x) \to 1$ and $T_t f_n(x) \to 1$ as $n \to \infty$. "Sum of rows = 1" (Of A)

(Infinitesimal Probability) Generator A $A = \frac{1}{2}\Delta$

"RHS derivative at zero"
$$A : D(A) \to \mathcal{C}_0(S), f \mapsto Af := \lim_{t \searrow 0} \frac{T_t f - f}{t}$$

HILLE-YOSIDA THEOREM

$$X_t : \{1, ..., d\} \to \mathbb{R}, j \mapsto (x_0)_j$$

$$T_t x_0 := \exp(tA)x_0 \qquad \dot{x}(t) = Ax(t), \quad x(0) = x_0$$

$(B_t)_{t \geq 0}$

$(X_t)_{t\geq 0}$

Riesz' Theorem

Martingale result

$$T_t f(x) \rightsquigarrow (\mu_{t,x})_{t\in\mathbb{Q}, x\in S} \rightsquigarrow (\mathbb{P}_x)_{x\in S} \rightsquigarrow (X_t)_{t\geq 0}$$

$(x \mapsto T_t f(x) := E_x f(X_t)) \in \mathcal{C}_0(S)$

FP

Generalized Exponential Formula

$$T_t f = \lim_{n\to\infty} (\mathrm{id} - \tfrac{t}{n} A)^{-n}$$

$f * g_t$

$(T_t)_{t\geq 0}$

A

$\frac{1}{2}\Delta$

$$A f := \lim_{t\searrow 0} \frac{T_t f - f}{t}$$

$$g_t := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\|\cdot\|^2}{2t}\right)$$

HILLE-YOSIDA THEOREM

Theorem 3.1.9 (Hille-Yosida, [P], Thm. 3.1; [B], Thm. 2.14) A linear operator A is the generator of a probability semigroup $(T_t)_{t\geq 0}$ if and only if the following conditions are met:

(L1) A is a closed operator, i.e. $(f_n) \subseteq D(A)$ with $f_n \xrightarrow{||\cdot||_\infty} f \in D(A)$ and $Af_n \xrightarrow{||\cdot||_\infty} g$ implies $g = Af$.

(L2) $D(A)$ is dense in $\mathcal{C}_0(S)$.

(L3) $(0, \infty) \subseteq \rho(A)$. Resolvent

(L4) $\forall \lambda > 0 : ||(\lambda \cdot \mathrm{id} - A)^{-1}|| \leq 1/\lambda$.

(L5) $\forall \lambda > 0 \exists (f_n)_{n\in\mathbb{N}} \subseteq D(A) : ((\mathrm{id} - \lambda A) f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_0(S)$, $\sup_{n\in\mathbb{N}} ||f_n|| < \infty$, and for all $x \in S$:

$$f_n(x) \to 1 \quad Af_n(x) \to 0.$$

"Sum of rows = 0"

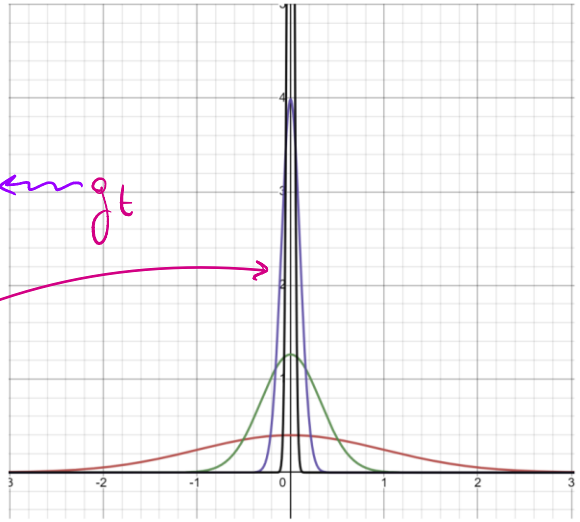
GENERALIZED DYNKIN'S FORMULA

$$M_t := \overbrace{f(X_t)}^{\in \mathcal{L}^1(\mathbb{P}_x)} - \overbrace{\int_{[0,t]} \underset{x}{A} f(X_s)\,\mathrm{d}\lambda(s)}^{\in \mathcal{L}^1(\mathbb{P}_x)}$$

Assumption: "Behaves like $\frac{1}{2}\Delta$"

is a martingale with respect to the filtration induced by the Feller process $(X_t)_{t\geq 0}$.

$T_t = f * g_t \leftarrow g_t$



Corollary 3.2.5 (Martingale Mean Value Property) Consider the setting of Thm. 3.2.3, and let $u_\infty \in \ker A$ be a stationary solution to (1.3) such that $u_\infty(X_t) \in L^1(\mathbb{P}_x)$ for all $x \in S$ and $t \geq 0$. Then,

$$(u_\infty(X_t))_{t\geq 0}$$

is a martingale. In particular, stationary solutions can be written as the expected value of u_∞ evaluated at a process X_t started at $x \in S$:

$$\boxed{u_\infty(x) = \int_S u_\infty(X_t)\,\mathrm{d}\mathbb{P}_x}$$

for any $t \geq 0$.

STATIONARY MEASURES

From Feller process

Where the process starts

$$(B,t) \mapsto \mu_t(B) := \int_S \underbrace{\mathbb{P}_x(X_t \in B)}_{\text{Transition probability}} \, d\mu(x)$$

Prob. Measure

Definition 3.3.1 ([B], Def. 7.1) A non-zero measure $\mu_\infty \in \mathcal{M}(S)$ is said to be **invariant** or **stationary** under the Feller process $(X_t)_{t\geq 0}$, if

$$\forall t \geq 0 : \mu_t = \mu_\infty \, .$$

Theorem 3.3.2 (Stationarity, [B] Thm. 7.2) The following statements are equivalent:

1. The measure $\mu \in \mathcal{M}(S)$ is invariant under $(X_t)_{t\geq 0}$.
2. For all $f \in \mathcal{C}_0(S)$ and $t \geq 0$:
$$\int T_t f \, d\mu = \int f \, d\mu \, .$$
3. For all $f \in D(A)$:
$$\int Af \, d\mu = 0 \, .$$

STATIONARY MEASURES: EXAMPLE

Example 3.3.5 (Parts from [B], H. 5.3) Consider a bounded generator $A \in \mathcal{L}(\ell^2)$ for the Hilbert space $\ell^2 := \ell^2(\mathbb{N})$. We can write the set of stationary probability measures as

$$\mathcal{I}_1 = \ker(A^*) \cap \partial B_1^{||\cdot||_{\ell^1}}(0)$$

constraining ourselves to sequences with non-negative entries (i.e. we do not deal with signed measures with mass 1).

$$A = \lambda \begin{pmatrix} -1 & 1 & 0 & 0 & \dots \\ 1/2 & -1 & 1/2 & 0 & \dots \\ 0 & 1/2 & -1 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\mathcal{I}_1 = \emptyset.$$

<u>Conservation of mass:</u>

$$A = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/3 & -1 & 2/3 \\ 1/2 & 1/2 & -1 \end{pmatrix}$$

Sum of rows is zero, only non-negative entries on the diagonal.

$$\underbrace{\mu_\infty = \frac{1}{27}(8, 9, 10)}$$

$\forall f \in \underset{\{1,2,3\} \to \mathbb{R}}{\mathbb{R}^3} : \int_{\{1,2,3\}} Af \, d\mu_\infty = (\mu_\infty, Af)_{\ell^2}$

$= (A^*\mu_\infty, f)_{\ell^2} \Rightarrow A^*\mu_\infty = 0$

Notice that mass is conserved (cf. PDE case).

$\sum_{i=1}^{3} (\mu_\infty)_i = 1$

STATIONARITY: FUNCTION-MEASURE CORRESPONDENCE

$$0 = \int Af \; \mathrm{d}\mu_\infty = \langle \mu_\infty, Af \rangle = \langle A' \mu_\infty, f \rangle$$

$$d\mu_\infty = u_\infty d\lambda$$

$$\begin{tikzcd}
A \arrow[r, leftrightarrow, "\text{Hille-Yosida}"] \arrow[d, rightsquigarrow] & (X_t)_{t \geq 0} \arrow[d, rightsquigarrow] \\
u_\infty \arrow[r, leftrightarrow, "u_\infty = \frac{\mathrm{d}\mu_\infty}{\mathrm{d}\lambda}"] & \mu_\infty
\end{tikzcd}$$

ORNSTEIN-UHLENBECK $\leftarrow\rightarrow$ FOKKER-PLANCK

$$T_t f(x) = \int_S p_t(x,y) f(y)\, d\mu(y)$$

$$\int_S \partial_t p_t(x,y) f(y)\, d\mu(y) = \frac{d}{dt} T_t f(x) \overset{\text{Kolmogorov}}{=} A_x T_t f(x) = \int_S f(y) A_x p_t(x,y)\, d\mu(y)$$ Primal

One can also consider: $\partial_t p_t(x,y) = A_y^* p_t(x,y)$ Dual (where $\int f A g \, d\mu = \int g A^* f \, d\mu$)

Example 3.5.2 (Dual Operator: Ornstein-Uhlenbeck, [J], Exa. 2.4, p. 28) Consider the generator A defined by

$$Af = \Delta f - \langle \nabla V, \nabla f \rangle$$

$\mathcal{L}^2$ dual through partial integration (assuming the functions are smooth enough and have vanishing support)

$$A^* u = \nabla \cdot (\nabla u + u \nabla V)$$

$$0 = \int_{\mathbb{R}^d} Af \, d\nu \overset{d\nu = u_\infty d\lambda}{=} \int_{\mathbb{R}^d} (A^* u_\infty) f \, d\lambda$$

Rmk: $A := \frac{1}{2}\Delta$ is self-dual wrt $\mathcal{L}^2(\lambda)$.

BARKY-EMERY: STOCHASTIC ANALYSIS

$$\Gamma(f,g) := \frac{1}{2}(A(fg) - fA(g) - gA(f)) \qquad \Gamma_2(f,g) := \frac{1}{2}(A\Gamma(f,g) - \Gamma(Af,g) - \Gamma(f,Ag))$$

Theorem 3.5.3 ([J], Thm. 2.2) Let $\phi \in \mathcal{C}^2([0,\infty))$ be convex with $1/\phi''$ concave. Assume there exists some $\lambda > 0$ such that for all suitable non-negative functions f

$$\Gamma_2(f) \geq \lambda\Gamma(f) \qquad \underbrace{\nabla^2 V \geq \lambda \, id}_{} \tag{3.15}$$

cf. uniform convex potential from before

holds. Then,

$$\int \phi(f)\, d\mu_\infty - \phi\left(\int f\, d\mu_\infty\right) \leq \frac{1}{2\lambda}\int \phi''(f)\Gamma(f)\, d\mu_\infty\,.$$

THANK YOU

Alejandro Morera Alvarez

alexmorera.a@gmail.com