Part 1: Functional Analysis

Basic Measure Theory We need some basic definitions and theorems:

Definitions

"For almost all $n \in \mathbb{N}$" $:\iff$ "For all $n \in \mathbb{N}$ but finitely many exceptions".
We say $f \in L^p :\iff [f] \in \mathcal{L}^p$.
Measureable functions are the ones whose preimage of measurable sets are measurable: $f^{-1}(\mathcal{A}') \subseteq \mathcal{A}$
Integrable functions are those whose integral of both positive and negative part respectively are finite.

Inequalities

Each of the following inequalities' proofs are based on the previous one.

Jensen: Let $(X, \mathcal{A}, \mathbb{P})$ be a probability space (i.e. X is a set, $\mathcal{A}$ is a σ-field over X and $\mathbb{P}(X) = 1$), $\phi : I \to \mathbb{R}$ a convex mapping and $f : X \to \bar{\mathbb{R}}$ an integrable function with $f(X) \subseteq I$ a.e.
Then,

$$\phi \circ \int_X f d\mu \leq \int_X \phi \circ f d\mu$$

Hölder: For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (we call p and q Hölder conjugates) $f, g : X \to \bar{\mathbb{R}}$ measureable functions. Then,

$$||fg||_1 \leq ||f||_p ||g||_q$$

Minkowski: For $\in [1, \infty)$ and $f, g : X \to \bar{\mathbb{R}}$ measurable we have

$$||f + g||_p \leq ||f||_p + ||g||_p$$

which some simply refer to as the "triangle inequality" (in L^p).

The Main Space $\mathbb{K}^{\mathbb{N}} := \{(a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{K}\}$ is a vector space with respect to pointwise addition and scalar multiplication, that is, for $x := (x_n)_{n \in \mathbb{N}}$, $y := (y_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ and $\alpha \in \mathbb{K}$:

$$(x + y)_n := x_n + y_n, \quad (\alpha x)_n := \alpha x_n, \quad \forall n \in \mathbb{N}$$

Proof: The fulfillment of the vector space axioms are all quite clear:
$\mathbb{K}^{\mathbb{N}}$ is an abelian group, with neutral element $0 := (0, 0, 0, ...)$ and the

inverse of $x \in \mathbb{K}^{\mathbb{N}}$ would be $-x := (-x_0, -x_1, -x_2, ...)$
Sequence-field-multiplication is clearly distributive with respect to both sequence addition and field addition. Field multiplication is associative with field-sequence multiplication.
Finally, neutral multiplicative element of the field acts as a neutral element with respect to sequence-field multiplication.
This gives us a vector space structure.

From now on, if we want to prove that if some subset of $\mathbb{K}^{\mathbb{N}}$ is a vector space, it suffices that we show it is a subvector space of $\mathbb{K}^{\mathbb{N}}$.

The set $\{e_i \mid i \in \{1, ..., n\}\}$ is linearly independent Let's recall why this is the case: Let $\sum_{i=1}^{n} \lambda_i e_i = 0$ for $\lambda_i \in \mathbb{K}$. Then, clearly $\lambda_i = 0, \forall i \in \{1, ..., n\}\}$

The set $\{e_i \mid i \in \{1, ..., n\}\}$ is linearly independent Recall from linear algebra that a set is called linearly independent if all finite subsets of it are linearly independent.
So, linear independence of the set $\{e_i \mid i \in \mathbb{N}\}$ follows from the linear independence of the finite case.

Generating system of $\{e_i \mid i \in \mathbb{N}\}$ Here things start getting interesting.
Claim: $\langle\{e_i \mid i \in \mathbb{N}\}\rangle = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{K} : a_n = 0 \text{ for almost all } n \in \mathbb{N}\}$
Proof: Recall that the generation of a set is equal to the set of all (finite) linear combinations of the elements of the set. This gives us the claim.
Define $c_{00} := \langle\{e_i \mid i \in \mathbb{N}\}\rangle$ and note that it is clearly isomorphic to the set of all polynomials with coefficients in $\mathbb{K}$, with an isomorphism given by

$$\Phi : c_{00} \to \mathbb{K}[x], (a_n)_{n\in\mathbb{N}} \mapsto \sum_{i=0}^{\infty} a_i x^i$$

So one can, for instance, equip c_{00} with the Cauchy-Product of the polynomials to get an inner multiplication in c_{00}.
Example: $(3, 1, 0, 0, ...)^2 = (9, 6, 1, 0, 0, ...)$

ℓ^p-Spaces are Vector Spaces We now get to the main definition. For $p \in [1, \infty)$

$$\ell^p := \{(a_n)_{n\in\mathbb{N}} : \sum_{i=0}^{\infty} |a_n|^p < \infty\}$$

and

$$\ell^\infty := \{(a_n)_{n\in\mathbb{N}} : |a_n| < \infty, \forall n \in \mathbb{N}\}$$

Claim: For $p \in [1, \infty]$ the sets ℓ^p are Vector Spaces.
Proof: We simply show they are subvector spaces (the reason is remarked

previously). Clearly, $0 \in \ell^p$. Further, let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^p$, $\alpha \in \mathbb{K}$, and $p \in [1, \infty)$. Then,

$$\sum_{i=0}^{\infty} |a_n + \alpha b_n|^p \leq \sum_{i=0}^{\infty} |a_n|^p + |\alpha|^p \sum_{i=0}^{\infty} |b_n|^p < \infty$$

whereby Minkowski's inequality was used.
For $p = \infty$ the claim is clear.

ℓ^p-Spaces are *Normed* Vector Spaces Claim: $||x||_p := (\sum_{i=0}^{\infty} |a_n|^p)^{1/p}$ for $p \in [1, \infty)$ and $||x||_\infty := \sup_{n\in\mathbb{N}} |a_n|$ for $p = \infty$ defines a norm.
Proof: For the $p = \infty$ case, the claim is clear. For $p \in [1, \infty)$ definiteness is clear, as in measure theory we define $0 \cdot \infty := 0$. Homogeneity is also clear. Finally, the triangle inequality follows from Minkowski's Inequality.

Are the following vector spaces? We already saw that c_{00} and ℓ^p for $p \in [1, \infty]$ are vector spaces.

$c := \{(a_n)_{n\in\mathbb{N}} : (a_n) \text{ is convergent}\}$ is a vector space: the zero sequence is vacuously convergent, we know from Analysis 1 that the sum of convergent sequences is convergent. Finally, sequence convergence is preserved under scalar multiplication.

$c_x := \{(a_n)_{n\in\mathbb{N}} : a_n \to x\}$ is a vector space $\iff x = 0$
Let $x = 0$. Then, the zero sequence is in c_0, as well as $a + \beta b \in c_0$ for $a, b \in c_0, \beta \in \mathbb{K}$.
Conversly, if $x \neq 0$, then, for $a, b \in c_x$ we have $(a + b)_n \to 2x \neq x$ as $n \to \infty$.

ℓ^p-Spaces are onions Let $p < q$ for $q \in (1, \infty]$. Then $\ell^p \subset \ell^q$
Proof: Let $x \in \ell^p$. Then, $\exists N \in \mathbb{N} : |x_n| < \epsilon$ with some $\epsilon \in (0, 1)$ for all $n \geq N$, as it would otherwise not even be a null-sequence and its $p-$norm would diverge. Thus,

$$||x||_q^q = \sum_{i=0}^{\infty} |x_i|^q = \sum_{i=0}^{N} |x_i|^q + \sum_{i=N+1}^{\infty} |x_i|^q \leq \sum_{i=0}^{N} |x_i|^q + \sum_{i=N+1}^{\infty} |x_i|^p$$

$$\leq \sum_{i=0}^{N} |x_i|^q + \sum_{i=0}^{\infty} |x_i|^p = \sum_{i=0}^{N} |x_i|^q + ||x||_p^p < \infty$$

because $x \in \ell^p$ Indeed, the inclusion is strict, since $1 < q/p$, so $\sum_{n=1}^{\infty} \frac{1}{n^{q/p}} < \infty$ which means $(n^{-1/p}) \in \ell^q$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^{q/q}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, as it is the harmonic series, implying $(n^{-1/p}) \notin \ell^p$

Interesting Isomorphisms We have seen $\langle \{e_i \mid i \in \mathbb{N}\} \rangle \cong \mathbb{K}[x]$.

Now, analogously, we have that the set of all sequences over a field is isomorphic to the set of all formal power series over that same field:

$$\{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{K}\} \cong \{\sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{K}\}$$

So, similar as to what we did before for c_{00}, we can equip the set of all sequences in $\mathbb{K}$ with the inner multiplicative structure gotten from the (formal) Cauchy-Product, thereby getting things like

$$(1, 1, 1, ...) \cdot (1, -1, 0, 0, 0, ...) = (1, 0, 0, 0, ...)$$

Which is simply the geometric series multiplied by its inverse.

Sequences of Sequences Note that we are not limited to working only with sequences, but also with sequences of sequences. One can formally imagine this notion as follows:

We were working with ∞-dimensional vectors, whose entries were simply the sequence entries. Now, to further aid imagination, place this vector horizontally.

For each of the entries, we now add infinitely long columns below them; we get an object from $\mathbb{K}^{\mathbb{N}\times\mathbb{N}}$. Traversing this object vertically represents the evolution of each of the components of the sequence we previously had.

Hence, we end up with an element $x = (x_n^\nu) \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$.

Equiped with this definition, we can proceed to our next important result.

ℓ^p-Spaces are Banach Spaces for $p \in [1, \infty\ ]$ ℓ^p-Spaces are complete.

Proof: Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $p \in [1, \infty)$ and x^α be a Cauchy Sequence in ℓ^p. We show it converges in ℓ^p.

Let $\epsilon > 0$. Since x^α is Cauchy, there exists an $N \in \mathbb{N}$ such that $\forall \nu, \mu \geq N$

$$||x^\nu - x^\mu||_p^p = \sum_{k\in\mathbb{N}} |x_k^\nu - x_k^\mu|^p \leq \epsilon^p$$

i.e. for a fixed $k \in \mathbb{N}$, the sequence $(x_k^\nu)_{\nu\in\mathbb{N}}$ is a Cauchy Sequence. Since $\mathbb{K}$ is complete, there exists a $x_k \in \mathbb{K}$ such that $x_k^\nu \to x_k$ as $\nu \to \infty$. Let $x := (x_k)$

Claim 1: $x \in \ell^p$.
Since x^α is Cauchy, it is also bounded by some constant $C > 0$. Let $K, \nu \in \mathbb{N}$, then

$$\sum_{k=0}^{K} |x_k^\nu|^p \leq ||x^\nu||_p^p \leq C^p$$

Letting $\nu \to \infty$ yields $\sum_{k=0}^{K} |x_k|^p \leq C^p$
Letting $K \to \infty$ yields $\sum_{k=0}^{\infty} |x_k|^p = ||x||_p^p \leq C^p$
Which concludes the proof of the claim.

Claim 2: x^α converges to x in ℓ^p
Let $K \in \mathbb{N}$ and $\nu, \mu \in \mathbb{N}$ sufficiently large. Then,

$$\sum_{k=0}^{K} |x_k^\nu - x_k^\mu|^p \leq ||x^\nu - x^\mu||_p^p \leq \epsilon^p$$

Letting $\nu \to \infty$ and then $K \to \infty$ yields the claim.

For $p = \infty$, let x^α be Cauchy. Then, it is bounded in ℓ^∞, thus $x \in \ell^\infty$. Finally, $|x_k - x_k^\nu| \leq \epsilon \forall k \in \mathbb{N}$ implies $||x - x^\nu||_\infty = \sup_{k \in \mathbb{N}} |x_k - x_k^\nu| \leq \epsilon$.

ℓ^2 is a Hilbert Space Endowed with the canonical scalar product $\langle x, y \rangle := \sum_{i \in \mathbb{N}} x_i y_i$ we get a Hilbert Space (see later seminar).

Part 2: Topology

Some Basic Topology We shall go through some topological concepts we need for our next results.

Topology A topology $(X, \mathcal{O})$ is a tuple that consists of a set X and subset of its power set $\mathcal{O} \subseteq \mathcal{P}(X)$ such that the following hold:
(T1) $\emptyset, X \in \mathcal{O}$
(T2) $U, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
(T3) $(U)_{i\in I} \subseteq \mathcal{O} \implies \cup_{i\in I} U_i \in \mathcal{O}$
We shall call X the ground set and the sets in $\mathcal{O}$ open sets.

Neighborhoods and Neighborhood Basis A subset of the ground set X is called a neighborhood of $x \in X$, if there exists an open set $U : x \in U \subseteq V$.
A subset of the neighborhoods of $x \in X$ is called neighborhood basis of x, if every neighborhood of x contains a neighborhood-basis element.

First Countability A topological space is called first-countable, if for every $x \in X$ has a countable neighborhood basis.

Remark: Closure $cl(\cdot)$ vs. Limit Closure $limcl(\cdot)$ In general topological spaces, the concepts of closure and limit closure are fundamentally different.
In fact, one can prove that $limcl(A) \subseteq cl(A)$ for $A \subseteq X$, meaning that if one can find a sequence that converges to a given point $x \in X$, then, for all neighborhoods V of x, we have that the intersection $V \cap A$ is nonempty. The other way around is not always true.

However, $cl(\cdot) = limcl(\cdot)$ in metric spaces One can show that in metric spaces, $cl(A) = limcl(A), \forall A \subseteq X$.
All the above was in order to justify the use of the limit closure instead of the closure in the following density results.

Density and Separability $A \subseteq X$ is said to be dense in X, if its closure is equal to X.
A topological space is called separable, if the ground set X has a countable dense subset.

c_{00} is dense in ℓ^p with respect to $||\cdot||_p$ for $p \in [1, \infty)$ Proof:
Recall from topology that in metric spaces, the closure of a set is equal to the set of all elements which are a limit of some sequence in the set.
In this case, it is easier to work with the limit closure. We show that for an arbitrary $x \in \ell^p$ we can find a sequence in c_{00} which converge to x in the $||\cdot||_p$-norm.
Define $y^{(i)} := (x_1, x_2, ..., x_i, 0, 0, ...) \in c_{00}$. Let $\epsilon > 0$. Since $x \in \ell^p, \exists N \in$

$\mathbb{N} : \sum_{n=N}^{\infty} |x_n|^p < \epsilon$ by the Cauchy-Criterion from Analysis 1. Then, for $i \geq N$ we have

$$||x - y^{(i)}||_p^p = \sum_{n=i+1}^{\infty} |x_n|^p < \sum_{n=N}^{\infty} |x_n|^p < \epsilon$$

hence, $y^{(i)} \to x$ in ℓ^p.

c_{00} is not dense in ℓ^∞ with respect to $||\cdot||_\infty$ Proof: We find a sequence x in ℓ^∞ such that there is no sequence in c_{00} which converges to x.
The sequence $x = (1, 1, 1, ...) \in \ell^\infty$ as the components are bounded. Let $(y^i) \in c_{00}$. Note that a zero must appear in each of the y^i. This means $||x - y^i||_\infty \geq 1$, meaning (y^i) does not get arbitrarily close to x, hence it does not converge to it.

c_{00} is separable (w.r.t. any p-norm) From now on, we work in $\mathbb{R}$
Proof of separability:
Let q_{00} be the set of all rational sequences, where all but finitely many entries are zero.
q_{00} is countable (left as an exercise). Hint: write it as a countable union/product of countable sets.
q_{00} is dense in c_{00}, as $\mathbb{Q}$ is dense in $\mathbb{R}$, which means that the elements in c_{00} can be approximated by elements in q_{00}.

ℓ^p-Spaces are separable We show for each $p \in [1, \infty)$ that there is a countable, dense subset of ℓ^p.
Let $x \in \ell^p$ and $\epsilon > 0$. We've shown that c_{00} is dense in ℓ^p, meaning that there exists a sequence $y \in c_{00}$ with $||x - y||_p < \epsilon/2$.
By the previous result, we know there exists $z \in q_{00}$ such that $||y - z||_p < \epsilon/2$. Using the triangle inequality,

$$||x - z||_p \leq ||x - y||_p + ||y - z||_p < \epsilon$$

ℓ^∞ is not separable Let $A \subseteq \ell^\infty$ be any dense subset of ℓ^∞.
Consider the subset $S := \{0, 1\}^{\mathbb{N}}$ of all binary sequences. Note it is not countable. Hence, the family $(B_{1/2}(x))_{x \in S}$ is uncountable and disjoint, since for two $x, y \in \ell^\infty, x \neq y$ we have $B_{1/2}(x) \cap B_{1/2}(y) = \emptyset$ (because, by definition $B_r(x) := \{y \in \ell^\infty : ||x - y||_\infty < r\}$).
Claim: Each ball in the family must contain at least one element of A.
Each open ball B is a neighborhood of all its points, and by definition of A being a dense subset, every point is a closure point, meaning $B \cap A \neq \emptyset$.
Thus, from the claim it directly follows that A must be uncountable as well. Therefore, ℓ^∞ is not separable.

Sources:
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Functional Analysis Lecture Prof. Friesecke
Topology Lecture by Prof. Ulrich Bauer
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