École Polytechnique Fédérale de Lausanne

Section de Mathématiques

Fyodorov-Bouchaud Formula and BPZ Equations

Projet de Master

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Preface

This semester project explores Gaussian Multiplicative Chaos (GMC) and Conformal Field Theory (CFT), linking advanced ideas from probability theory, mathematical physics, and algebra. We begin by carefully building up from foundational concepts toward intricate mathematical structures, revealing fascinating connections and applications along the way.

Chapter 1 serves as our entry point, discussing Gaussian Free Fields (GFF) and Gaussian Multiplicative Chaos from both mathematicians' and physicists' perspectives. We introduce the required mathematical notions alongside their respective construction, and discuss some of their properties. The chapter also defines what we mean by Liouville Conformal Field Theory and provide with the essential tools for what follows. In Chapter 2, we reach the main focus of the project: the Fyodorov-Bouchaud formula. We present the statement and offer some key insights of the proof, emphasizing on the so-called BPZ equations, geometric intuition and the use of hypergeometric functions. Chapter 3 addresses uniform integrability and study rare events in the semiclassical limit as an application of the Fyodorov-Bouchaud formula. We then come back to other aspects of the BPZ equations, in particular a relation to Schramm–Loewner Evolutions (SLEs) and finally touch upon the algebra behind the BPZ equations. The appendices contain supplementary proofs which are not crucial for the understanding of the rest of the project.

On a personal note, this semester was particularly challenging beyond academic responsibilities. It was undoubtedly one of the most testing periods of my life and made me reconsider several aspects of both my academic and personal life. However, thanks to the incredible positivity, strength, and support from my family and close friends, I found the resilience to persist and successfully continue my double master's degree without interruption. I would therefore like to thank and dedicate this project to my family and friends, who have stood by me and encouraged me throughout this journey. Their presence, humor, and constant reassurance played an key role in keeping my motivation alive during challenging times.

Last but not least, I am grateful to my advisor, Baptiste Cercle, for his guidance, feedback, and insightful discussions. Our meetings greatly enriched my understanding of these complex subjects, continually inspiring my curiosity and research enthusiasm.

Contents

Chapter 1

Foundations

1.1 Gaussian Free Fields and Gaussian Multiplicative Chaos: The Mathematicians' Playground

The goal of this chapter is to introduce the foundational notions that will be used in the main result of this report. We begin by presenting the existence and uniqueness of the GFF over some bounded, regular domain $U \subseteq \mathbb{R}^d$, as well as available modifications and mollifications for this process. We then specialize to the GFF on the unit disk and on the upper half-plane with Neumann boundary and mean-zero conditions. This chapter is mainly based on results presented in [BP] and [HRV].

1.1.1 Gaussian Free Fields: Existence, Uniqueness, and Regularization

Brownian motion with multi-dimensional time: $h \sim \mathcal{N}(0, G)$. Rough, random, correlated.

Construction of the Gaussian Free Field

The goal of this section is to present, without proof, the results required for the construction of the GFF in the continuum. Roughly speaking, the GFF associated to U is the centered Gaussian process indexed by measures of finite Green energy taking values in the space of distributions with covariance equal to the Green function associated to the domain U. More precisely, let us consider the distributional partial differential equation (PDE)

$$\Delta G(x, \cdot) = \delta_x(\cdot) \tag{1.1}$$

together with some boundary condition, typically either the Dirichlet boundary condition $G(x, \cdot) = 0$ for $x \in \partial U$ or the Neumann boundary condition $\partial_n G(x, \cdot) = 0$ for $x \in \partial U$ together with the mean-zero condition, i.e. that the average of $G(x, \cdot)$ on ∂U is zero. There are some details regarding the latter conditions which we shall skip in the following discussion. For a more detailed treatment of the Neumann boundary conditions, see, for instance ([BP], Ch. 6).

We would like to set $G(x, y)$ to be the covariance function of a Gaussian process X. It is easy to see that in any case, G is symmetric in its entries: $G(x, y) = G(y, x)$ for all $x \neq y$ – so far, so good: x influences y in the same amount as y influences x. The problem is, however, that in dimensions two and higher, one often has that "$G(x, x) = \infty$" (for example, in a two-dimensional setting, this divergence occurs logarithmically) and can thus not be the variance of "$X(x)$" (or "$X(\delta_x)$" in the notation that follows). To circumvent this problem, let $\mathcal{R}(U)$ be the set of all Radon measures over U and define the set

$$\mathcal{M}_G(U) := \left\{ \rho \in \mathcal{R}(U) \mid \int \mathrm{d}\rho(x)\, \mathrm{d}\rho(y) G(x, y) < \infty \right\}$$

of all such measures with finite Green energy. Observe that by the above remark that $\delta_x \notin \mathcal{M}_G(U)$ for many two-dimensional settings. We interpret this as the field X not being defined pointwise.

In view of setting

$$\Gamma_G(\rho_1, \rho_2) := \int \mathrm{d}\rho_1(x)\, \mathrm{d}\rho_2(y) G(x, y)$$

to be the covariance of our distributional-valued Gaussian process X, one can show that Γ_G is positive in the sense that

$$\forall \lambda \in \mathbb{R}^d, \rho_1, ..., \rho_n \in \mathcal{M}_G(U) : \sum_{i=1}^{d} \lambda_i \lambda_j \Gamma_G(\rho_i, \rho_j) \geq 0. \tag{1.2}$$

Therefore, using Kolmogorov's extension theorem, Γ_G defines a valid covariance function for a Gaussian process X indexed by the set $\mathcal{M}_G(U)$. In other words:

Theorem 1.1.1 (Existence and Uniqueness of the GFF - [BP], Thm. 1.29) There exists a unique stochastic process $(X_\rho)_{\rho\in\mathcal{M}_G(U)}$ such that for all $\rho_1, ..., \rho_n \in \mathcal{M}_G(U)$ we have

$$(X_{\rho_1}, ..., X_{\rho_n}) \sim \mathcal{N}(0, (\Gamma_G(\rho_i, \rho_j))_{i,j=1}^n). \tag{1.3}$$

Definition 1.1.2 ([BP], Def. 1.30)
The **Gaussian free field** associated to U with given boundary conditions is the Gaussian process h indexed by $\mathcal{M}_G(U)$ which satisfies (1.3). We write

$$X_\rho =: (X, \rho) \tag{1.4}$$

for $\rho \in \mathcal{M}_G(U)$.

The problem with the above representation is that it is not yet clear where the process lives, as Kolmogorov's extension theorem only guarantees the existence of a probability measure on $\mathbb{R}^{\mathcal{M}_G}$, but makes no claims regarding the regularity of the values taken by the process. Moreover, note that the above theorem only makes a claim regarding countably many measures, yet variables constructed by taking the supremum over a larger set need not be measurable anymore – unless we can show the existence of a certain modification in a some space, whose topology allows for such maps to be measurable. The following result states that the process X admits a version which lies in the space of bounded linear functionals over smooth and compactly supported test functions $\mathcal{D}(U)$. Let $H_0^{-1}(U)$ be the closure of $\mathcal{D}(U)$ under the norm $||\rho||_\nabla^2 := \int ||\nabla\rho(x)||^2 \, \mathrm{d}x$.

Theorem 1.1.3 (GFF as a Distribution-Valued Process - [BP], Cor. 1.48) There exists a version of the Gaussian free field as a stochastic process $(h, \rho)_{\rho\in H_0^{-1}(U)}$ that is an element of $H_0^1(U)$ $\mathbb{P}$-a.s.

Remark 1.1.4 The notation (1.4) is justified by Thm. 1.1.3, where we view the GFF h as a distribution acting on a measure ρ of finite Green energy, interpreted as a smooth function. ◇

We know that distributions, or generalized functions, exhibit a rather irregular behavior. Roughly speaking, they may have spikes that diverge in $\mathbb{R}$ when evaluated at a "point". One can imagine it as a mountain range full of rough peaks and valleys. In order to understand expressions such as the *exponential* of a GFF, we consider for every $\epsilon > 0$ approximations X_ϵ of the random distribution X which converge, in some sense, to the original field X. One can imagine having a really sharp image being the GFF, and blurred versions of the image as the X_ϵ, where the smaller the $\epsilon > 0$, the more similar the image looks as the original GFF. For instance, on the unit disk, we can consider the **circle-average regularization**: for $x \in \mathbb{D}$ and $\epsilon \in (0, d(x, \partial\mathbb{D}))$

$$X_\epsilon(x) = \frac{1}{2\pi} \int_0^{2\pi} X\left(x + \epsilon e^{i\theta}\right) \mathrm{d}\theta$$

or, for $x \in \partial\mathbb{D}$ we have that the integral in $X_\epsilon(x)$ is only over the sector of the half-circle which lies inside U (for $\epsilon > 0$ small enough) and we divide over π instead of 2π.

Similarly, on the upper half-plane, we can take some smooth density $\rho : [0, +\infty) \mapsto [0, +\infty)$ with support in $[0, 1]$ and such that $\pi \int_0^\infty \rho(t) \, \mathrm{d}t = 1$, and take for $z \in \mathbb{H}$

$$X_\epsilon(z) := (\rho_\epsilon * X)(z) = \int_{\mathbb{H}} d^2x X(x) \rho_\epsilon(z - x), \tag{1.5}$$

and for $s \in \partial\mathbb{H} = \mathbb{R}$

$$X_\epsilon(s) := 2(\rho_\epsilon * X)(s) = 2 \int_{\mathbb{H}} d^2x X(x) \rho_\epsilon(s - x), \tag{1.6}$$

the idea being that the density ρ_ϵ approximates a point evaluation "$X(x)$" (or rather "$X(\delta_x)$") of the field X at $x \in \overline{U}$ as $\epsilon \searrow 0$.

Remark 1.1.5 Observe that thanks to Thm. 1.1.3, these regularizations, and their derivatives with respect to z and s, respectively, can be viewed as distributional pairings (X, ρ_ϵ) and $(X, \partial_z\rho_\epsilon)$. ◇

Gaussian Free Field: Unit Disk and Half Plane

We will now construct the GFF X on the unit disk $\mathbb{D}$ with Neumann boundary conditions together with the mean-zero condition. For this, we consider the problem

$$\begin{cases} \Delta G^{\mathbb{D}}(x,\cdot) = \delta_x(\cdot), \\ \partial_n G^{\mathbb{D}}(x,\cdot)|_{\partial\mathbb{D}} = -1. \end{cases}$$

with the further condition that the mean of $G(x,\cdot)$ vanishes along the boundary

$$\frac{1}{2\pi}\int_{\partial\mathbb{D}} G^{\mathbb{D}}(x,y)\,\mathrm{d}y = 0$$

for all $x \in \partial\mathbb{D}$. One can show (cf. [BP], Exa. 6.12) that the function

$$G^{\mathbb{D}}(x,y) = \ln\frac{1}{|x-y||1-x\bar{y}|} \tag{1.7}$$

is the unique solution that satisfies the distributional PDE together with the boundary and mean conditions. We define the corresponding GFF X as the centered Gaussian process with covariance kernel G, that is $\mathbb{E}\left[X(x)X(y)\right] = G^{\mathbb{D}}(x,y)$.

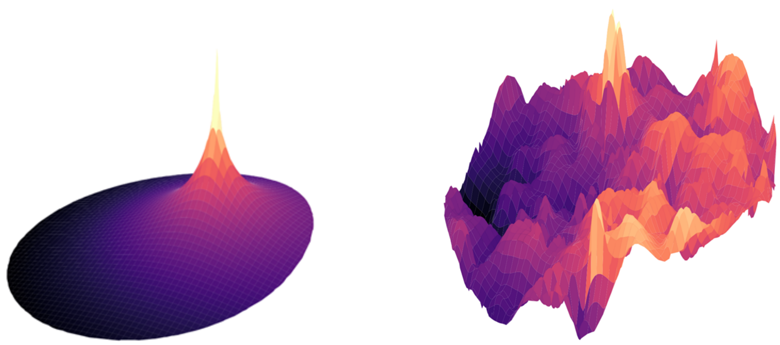

Figure 1.1: Green Function on the Unit Disk and its corresponding GFF.

In the next chapter, it will turn out to be simpler to work on the upper half-plane $\mathbb{H}$, perform some calculations on that domain , and then go to the unit disk. To move back and forth between these conformally equivalent domains, we need the following result:

Proposition 1.1.6 ([BP], Prop. 1.14) If ψ is a conformal isomorphism from U onto its image $\psi(U)$. Then for all $x, y \in U$ with $x \neq y$:

$$G^{\psi(U)}(\psi(x),\psi(y)) = G^U(x,y).$$

Example 1.1.7 The map $\varphi_1 := -i\frac{z+i}{z-i}$ is a conformal isomorphism from the unit disk $\mathbb{D}$ to the upper half-plane $\mathbb{H}$. Its inverse is often referred to as the **Cayley transform**. Using Prop. 1.1.6 applied to φ_1 yields the corresponding Green function of the Neumann problem on the upper half plane (up to a normalization constant)

$$G^{\mathbb{H}}(x,y) = \ln\frac{1}{|x-y||x-\bar{y}|} + \ln|x+i|^2 + \ln|y+i|^2 - 2\ln 2. \tag{1.8}$$

$\triangle$

1.1.2 Gaussian Multiplicative Chaos

Divide, stretch, repeat.

The formal definition of the Gaussian Multiplicative Chaos (GMC) is a random measure $\mathcal{M}$ defined as

$$\mathcal{M}(\mathrm{d}z) = \exp\left(\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}\left[X(z)^2\right]\right)\sigma(\mathrm{d}z) \tag{1.9}$$

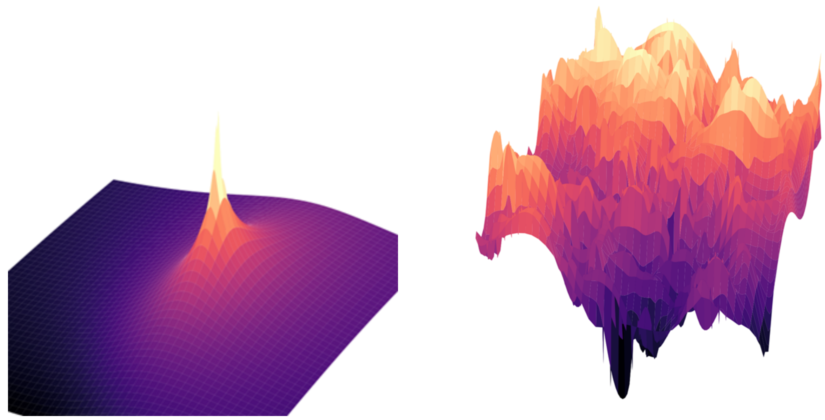

Figure 1.2: Green Function on the Upper Half-Plane and its corresponding GFF.

where $\gamma \in (0, 2)$ is called the **coupling constant**, σ is a reference measure, and X is a centered, logarithmically correlated Gaussian field, that is, X is Gaussian and has a covariance kernel of the form

$$K(x, y) = \log |x - y|^{-1} + g(x, y)$$

where g is continuous over $\overline{U} \times \overline{U}$. Without any context, this definition may seem suspicious and daunting for three main reasons:

1. Why do we consider such kernels?
2. Why do we even take the exponential of such a Gaussian field?
3. Why is this object even well-defined?

The first question is motivated by the special case of the GFF in a two dimensional setting $U \subseteq \mathbb{R}^2$, since in this case, the corresponding Green function has the following behavior

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|^{-1} + \mathcal{O}(1)$$

as $y \to x$ (cf. [BP] Prop. 1.18 for a proof and (1.7), as well as (1.8) for examples).

As the other two questions require more detailed answers, we shall divide these into two subsections:

Motivation of GMC: String Theory and Turbulence

It turns out that measures of the form (1.9) appear in branches ranging finance, mathematical physics, and the study of turbulence. In fact, the probabilistic formalism offered by defining such a measure allows for a representation of a Lie algebra, called the Virasoro algebra, which plays a central role in free bosonic theory (cf. Ch. 3 and [Ton], [Pol]).

Moreover, it turns out that Kahane's motivation to study measures such as (1.9) was to get a model for turbulence (cf. [BP], Ch. 3). In order to get such a chaotic model, one can consider Mandelbrot's multiplicative cascade model (MCM). This model involves random measures based on an elementary multiplicative construction. It was introduced to simulate the energy dissipation in intermittent turbulence. Visually, one can imagine dividing an interval into smaller sub-intervals and recursively assigning each segment a random weight, represented as the height of the segment. Alternatively, the construction can be visualized as a "divisible square", which undergoes multiplicative transformations via a family of independent, mean-one random variables (cf. Fig. 1.3).

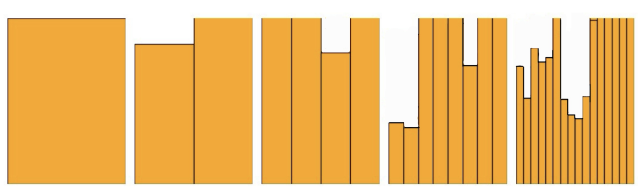

Figure 1.3: First five steps of the MCM.

The idea of the GMC is to generalize this construction in the continuous case and to study, for instance, non-degeneracy conditions, as well as moments of the partition function, as we shall see in the next chapter.

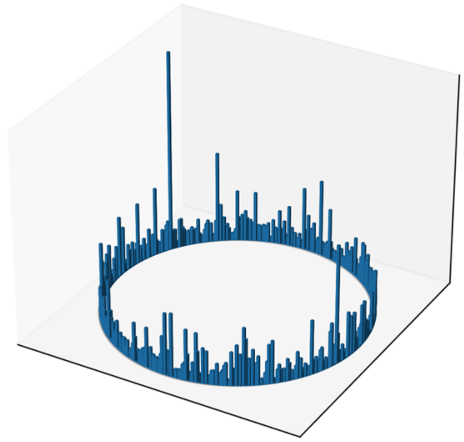

Figure 1.4: GMC Measure on the Unit Circle.

Well-Definedness of the GMC

To address the problem of constructing the exponential of a distribution-valued random variable, one defines the GMC measure $\mathcal{M}$ as the limit of an appropriate regularization $\mathcal{M}_\varepsilon$ of $\mathcal{M}$. We shall use

$$\mathcal{M}_\varepsilon(S) := \int_S e^{\gamma X_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(z)^2]} \sigma(\mathrm{d}z)$$

for a Borel set $S \subseteq U$. The approximations $X_\varepsilon(z)$ are often chosen to be convolutions with smooth and compactly supported kernels (cf. Sec. 1.1.1, and [BP] Sec. 1.12). All in all, together with ([BP], Thm. 3.2), we get

Definition 1.1.8 (Gaussian Multiplicative Chaos - [BP], Thm. 3.2)
Let $\gamma \in (0, 2)$. Then, there exists a non-degenerate random measure $\mathcal{M}$, independent of the chosen regularization kernel, which is the limit in probability in the topology of weak convergence of measures on U

$$\lim_{\varepsilon \to 0} \int_{(\cdot)} e^{\gamma X_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(z)^2]} \sigma(\mathrm{d}z)$$

We call this limit **Gaussian multiplicative chaos**.

There is of course a lot more to say regarding this object. The interested reader is referred to ([BP], Ch. 3).

1.2 Liouville Conformal Field Theory: The Physicists' Playground

Bosons, Strings, Polyakov: Action!

In the previous section, we introduced the probabilistic framework required for the main result, which is the notion of the GMC measure in some domain $U \subseteq \mathbb{C}$. We now specialize to the case where $U = \mathbb{H}$ is the half-plane with the idea to later use conformal equivalence of $\mathbb{H}$ and the unit disk $\mathbb{D}$, and we turn to the mathematical physical part of the setting. Most of the results which will be used in the next two chapters concern the special case $U = \mathbb{H}$, as we will then map this domain conformally to the unit disk $\mathbb{D}$.

1.2.1 Conformal Field Theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space. Note we do not assume that $\mathbb{P}$ is a probability – for instance, in our context, we will set $\mathbb{P}$ to be the GMC measure on a region conformally equivalent to the unit disk $\mathbb{D}$.

Definition 1.2.1 (Field Theory - [BP], p. 151) Let A be an arbitrary indexing set. A **statistical field theory** is a family of random field $(\psi_\alpha(z))_{\alpha\in A, z\in\mathbb{C}}$. The random fields are also referred to as **primary fields**.

The Polyakov action S associates to a random field ψ its corresponding energy $S(\psi)$. It is named after the physicist Alexander Polyakov, who in [Pol] introduced Liouville Quantum Field Theory (LCFT) as a model for quantizing the bosonic string in the conformal gauge and gravity in two space-time dimensions ([DKRV]).

Definition 1.2.2 (Polyakov Action - [HRV], p. 2)
The **Polyakov action** S in the setting of Liouville CFT on a two-dimensional surface D with boundary ∂D and reference metric $\hat{g}$ is given by the functional on maps $X : D \to \mathbb{R}$

$$S(X, g) := \frac{1}{4\pi}\int_D \left(|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}\right)\lambda_g + \frac{1}{2\pi}\int_{\partial D}\left(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right)\lambda_{\partial g}$$

where γ is called the **coupling constant**, Q is called the **background charge**, $\mu, \mu_\partial \geq 0$ are so-called **cosmological constants**, and $\partial^{\hat{g}}, R_{\hat{g}}$ and $\lambda_{\hat{g}}$ respectively stand for the gradient, Ricci scalar curvature and volume form in the metric $\hat{g}$.

Though it seems to be a complicated expression (and indeed it is), it is worth remarking that it admits the same symmetries as an equivalent action, called the Nambu-Goto action, whilst adding further symmetries, like the Weyl invariance (cf. Def. 1.2.4), as well as removes a square-root appearing in the Nambu-Goto action, which makes it difficult to quantize using path integrals (cf. [Ton]).

Example 1.2.3 In the case of LCFT on the upper half plane $\mathbb{H}$ equipped with the **Poincare metric** $\hat{g} = \frac{4}{|z+i|^4}$, the Polyakov action S_L is given by the expression

$$S_L(X, \hat{g}) = \frac{1}{4\pi}\int_{\mathbb{H}}\left|\partial^{\hat{g}} X\right|^2 \hat{g}(z)d^2z + \frac{1}{2\pi}\int_{\mathbb{R}}\left(QX + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right)\hat{g}(s)^{1/2}ds$$

where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ is the **background charge**, and $\mu_\partial > 0$. △

The primary fields can be multiplied with one another, giving rise to the notion of a **correlation function**, which is simply an expression of the form

$$\langle\psi_{\alpha_1}(z_1)...\psi_{\alpha_n}(z_n)\rangle := \int \psi_{\alpha_1}(z_1)...\psi_{\alpha_n}(z_n)\,\mathrm{d}\mathbb{P}.$$

Though there is no formal definition of what a conformal field theory is, we shall take to be a statistical field theory which satisfies certain invariance properties under conformal mappings:

Definition 1.2.4 (Conformal Field Theory - [BP], p. 152, 153)
A **conformal field theory** is a family of random field $(\psi_\alpha(z))_{\alpha\in A, z\in U}$ which satisfies a global conformal invariance property: for any conformal isomorphism $f : U \to U'$, we have

$$\langle\psi_{\alpha_1}(f(z_1))\dots\psi_{\alpha_k}(f(z_k))\rangle = \left(\prod_{i=1}^k \left|f'(z_i)\right|^{-2\Delta_{\alpha_i}}\right)\langle\psi_{\alpha_1}(z_1)\dots\psi_{\alpha_k}(z_k)\rangle \tag{1.10}$$

for some numbers $\Delta_\alpha, \alpha \in A$ called the **conformal weights** associated to the primary fields; and a local conformal invariance property, which is described via the **Weyl invariance**: given a metric g on U and a smooth function $\rho : U \to \mathbb{R}$, we have

$$\langle\psi_{\alpha_1}(z_1)\dots\psi_{\alpha_k}(z_k)\rangle_{e^\rho g} = \langle\psi_{\alpha_1}(z_1)\dots\psi_{\alpha_k}(z_k)\rangle_g.$$

Example 1.2.5 Let X be the GFF on $\mathbb{H}$ with Neumann boundary conditions, and take $A = \mathbb{R}$. In Liouville conformal field theory (LCFT) on $\mathbb{H}$, we are interested in the random fields

$$V_\alpha(z) = e^{\alpha\left(X(z) + \frac{Q}{2}\ln\hat{g}(z)\right)}$$

for weights $\alpha \in \mathbb{R}$ and points in the bulk $z \in \mathbb{H}$, as well as

$$V_\beta(s) = e^{\frac{\beta}{2}\left(X(s) + \frac{Q}{2}\ln\hat{g}(s)\right)}$$

for weights $\beta \in \mathbb{R}$ and points in the boundary $s \in \partial\mathbb{H}$. These fields are known as **vertex operators** with **insertions** at (z, α) and (s, β), respectively. Of course, to make sense of these expressions, one needs to perform appropriate approximations X_ϵ of the distributional-valued process X, and let the functional of the GFF be the limit (in some appropriate sense) as $\epsilon \searrow 0$.

More precisely, given insertions (z_i, α_i) and (s_j, β_j), we would like to make sense of the following **n-point correlations**

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} V_{\beta_j}(s_j) \right\rangle_{\mathbb{H}} := \int_\Sigma D_{\hat{g}} X \prod_{i=1}^{N} e^{\alpha_i\left(X(z_i)+\frac{Q}{2}\ln \hat{g}(z_i)\right)} \prod_{j=1}^{M} e^{\frac{\beta_j}{2}\left(X(s_j)+\frac{Q}{2}\ln \hat{g}(s_j)\right)} e^{-S_L(X,\hat{g})}. \tag{1.11}$$

In order to do so, we use the approximation procedure presented in the first section (cf. (1.5) and (1.6)) and

$$\begin{aligned}\left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} V_{\beta_j}(s_j) \right\rangle_{\mathbb{H},\epsilon} = \int_{\mathbb{R}} dce^{-Qc} \Bigg[\mathbb{E}\Bigg[&\prod_{i=1}^{N} \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i\left(X_\epsilon(z_i)+\frac{Q}{2}\ln \hat{g}(z_i)+c\right)} \prod_{j=1}^{M} \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}\left(X_\epsilon(s_j)+\frac{Q}{2}\ln \hat{g}(s_j)+c\right)} \\ &\times \exp\left(-\mu_\partial \int_{\mathbb{R}} \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}\left(X_\epsilon(s)+\frac{Q}{2}\ln \hat{g}(s)+c\right)} ds\right)\Bigg]\Bigg],\end{aligned}$$

where the weights α_i, β_i satisfy the so-called **Seilberg bounds**

$$\sum_{i=1}^{N} \alpha_i + \frac{1}{2}\sum_{j=1}^{M} \beta_j > Q \quad \text{and } \forall j, \quad \beta_j < Q.$$

It is shown in ([HRV], Thm. 3.1 on Seiberg Bounds) that under these conditions, the limit as $\epsilon \searrow 0$ exists, thus defining (1.11) as this limit. Note the extra powers in the exponent are there so as to compensate for the logarithmic divergence of the diagonal term $\mathbb{E}X_\epsilon^2(x)$.

The conformal dimension of the vertex operator $V_\alpha(z)$ is equal to $\Delta_\alpha = \frac{\alpha}{2}\left(Q - \frac{\alpha}{2}\right)$. This is a consequence of how the GFF acts under precomposition with a conformal isomorphism. △

Remark 1.2.6 In fact, if one takes a closer look at the proof of ([HRV], Thm. 3.1), one observes that the convergence of the above regularization to some object does not depend on the power $\frac{\gamma}{2}$ (though the limiting object does in general depend on this parameter). We shall exploit this fact in the next chapter, where we shall tweak some parameters of this mollification:

$$\int_{\mathbb{R}} dce^{-Qc}\mathbb{E}\left[\prod_{i=1}^{N} \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i\left(X_\epsilon(z_i)+\frac{Q}{2}\ln \hat{g}(z_i)+c\right)} \prod_{j=1}^{M} \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}\left(X_\epsilon(s_j)+\frac{Q}{2}\ln \hat{g}(s_j)+c\right)} \exp\left(-\mu_\partial \int_{\mathbb{R}} \epsilon^{\frac{\gamma^2}{4}} e^{\eta\left(X_\epsilon(s)+\frac{Q}{2}\ln \hat{g}(s)+c\right)} ds\right)\right], \tag{1.12}$$

which we shall denote as $\left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} V_{\beta_j}(s_j) \right\rangle_{\mathbb{H},\epsilon,\eta}$ for short, where $\eta > 0$ is some parameter. ◇

Chapter 2

Fyodorov-Bouchaud Formula: Statement and Proof

Moments of the random partition function.

The goal of this chapter is to present the derivation of the moments of the partition function of the GMC measure on the unit circle

$$Y_\gamma = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X\left(e^{i\theta}\right)}d\theta \tag{2.1}$$

that is, X is the GFF on the unit circle and $\gamma \in (0,2)$. The expression 2.1 should be understood as a limit in probability in the topology of weak convergence of some mollification of the GFF X as discussed in the previous chapter.

2.1 Main Result

One of the main goals of this manuscript is to present the proof of the following result due to G. Remy, using techniques from LCFT:

Theorem 2.1.1 (Fyodorov-Bouchaud Formula - [Rem], Thm. 1.1)
Let $\gamma \in (0,2)$. For all real p such that $p < \frac{4}{\gamma^2}$ the following identity holds:

$$\mathbb{E}\left[Y_\gamma^p\right] = \frac{\Gamma\left(1 - p\frac{\gamma^2}{4}\right)}{\Gamma\left(1-\frac{\gamma^2}{4}\right)^p}. \tag{2.2}$$

Equivalently, we can state that Y_γ follows the law

$$Y_\gamma \overset{\mathrm{d}}{=} \frac{1}{\Gamma\left(1-\frac{\gamma^2}{4}\right)}\mathcal{E}(1)^{-\frac{\gamma^2}{4}}, \tag{2.3}$$

where $\mathcal{E}(1)$ is an exponential law of parameter 1.

Remark 2.1.2 The condition $p < \frac{4}{\gamma^2}$ is necessary and sufficient for the finiteness of the p-th moment of Y_γ. ◇

In a nutshell, the strategy of the proof is as follows: We will link the negative moments of the partition function with a **hypergeometric function**, by relating the former with a solution to a so-called **Belavin-Polyakov-Zamolodchikov (BPZ) equation**. From the properties of the asymptotics of hypergeometric functions and the relation between their coefficients, we will derive a formula for the negative *integer* moments of the partition function, and then use Fourier theory to extend this formula to all negative moments. From there, it is simply a matter of using the diffeomorphism $x \mapsto 1/x$ over the positive real numbers, to conclude the proof of Thm. 2.1.1.

Set

$$U(\gamma,p) = \mathbb{E}\left[\left(\int_0^{2\pi} e^{\frac{\gamma}{2}X\left(e^{i\theta}\right)}d\theta\right)^p\right]$$

and note this is the sought after expression up to a factor of $1/(2\pi)^p$, i.e. $U(\gamma,p) = (2\pi)^p\mathbb{E}\left[Y_\gamma^p\right]$.

2.2 BPZ Equation

One of the main chunks to be shown is the following result:

Theorem 2.2.1 ([Rem], Thm. 2.1) Let $\gamma \in (0,2)$ and $\alpha > Q + \frac{\gamma}{2}$. Then $(z_1, z) \mapsto \left\langle V_{-\frac{\gamma}{2}}(z) V_\alpha(z_1) \right\rangle_{\mathbb{H}}$ is $\mathcal{C}^2$ on the set $\{z_1, z \in \mathbb{H} \mid z_1 \neq z\}$ and is a solution of the following PDE

$$\left(\frac{4}{\gamma^2}\partial_{zz} + \frac{\Delta_{-\frac{\gamma}{2}}}{(z-\bar{z})^2} + \frac{\Delta_\alpha}{(z-z_1)^2} + \frac{\Delta_\alpha}{(z-\overline{z_1})^2} + \frac{1}{z-\bar{z}}\partial_{\bar{z}} + \frac{1}{z-z_1}\partial_{z_1} + \frac{1}{z-\overline{z_1}}\partial_{\bar{z}_1} \right) \left\langle V_{-\frac{\gamma}{2}}(z) V_\alpha(z_1) \right\rangle_{\mathbb{H}} = 0.$$

The statement in Thm. 2.2.1 feels a bit like a magic guess: why do we choose the weight $-\frac{\gamma}{2}$ in the first field and $\eta = \frac{\gamma}{2}$ for the parameter in the exponent in (1.12)? In order to reveal the analytic witchery, we have decided to work with the tweaked regularization for the 2-point correlation function from (1.12) with general insertions and instead show:

Theorem 2.2.2 (BPZ Equation)
Let the pair $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ satisfy $\alpha_1 + \alpha_2 > Q$. The BPZ equation

$$\left(\frac{1}{\alpha_1^2}\partial_{z_1 z_1} + \frac{\Delta_{\alpha_1}}{(z_1-\bar{z_1})^2} + \frac{\Delta_{\alpha_2}}{(z_1-z_2)^2} + \frac{\Delta_{\alpha_2}}{(z_1-\overline{z_2})^2} + \frac{1}{z_1-\bar{z_1}}\partial_{\bar{z_1}} + \frac{1}{z_1-z_2}\partial_{z_2} + \frac{1}{z_1-\overline{z_2}}\partial_{\bar{z}_2} \right) \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle_{\mathbb{H},\eta} = 0$$

for the tweaked two-point correlation function $\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle_{\mathbb{H},\eta}$ holds, *if and only if*

$$(\alpha_1, \alpha_2, \eta) \in \left\{ \left(\frac{-\gamma}{2}, \alpha_2, \frac{\gamma}{2} \right) \mid \gamma \in (0,2), \alpha_2 \in \mathbb{R} \right\} \cup \left\{ \left(\frac{-2}{\gamma}, \alpha_2, \frac{2}{\gamma} \right) \mid \gamma \in (2,\infty), \alpha_2 \in \mathbb{R} \right\}.$$

We extend the notation in (1.12) to more general integrals, such as

$$\langle X(f) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle_{\mathbb{H},\epsilon,\eta} := \int_{\mathbb{R}} dc e^{-Qc} \mathbb{E} \left[X(f) \epsilon^{\frac{\alpha_1^2}{2}} e^{\alpha_1 \left(X_\epsilon(z_1) + \frac{Q}{2}\ln \hat{g}(z_1) + c \right)} \epsilon^{\frac{\alpha_2^2}{4}} e^{\alpha_2 \left(X_\epsilon(z_2) + \frac{Q}{2}\ln \hat{g}(z_2) + c \right)} \right.$$
$$\left. \times \exp\left(-\mu_\partial \int_{\mathbb{R}} \epsilon^{\frac{\gamma^2}{4}} e^{\eta \left(X_\epsilon(s) + \frac{Q}{2}\ln \hat{g}(s) + c \right)} ds \right) \right],$$

where $X(f)$ denotes the distributional pairing of the Neumann GFF with a test function $f \in \mathcal{D}$ over $\mathbb{H}$.

The proof of the BPZ formula is a repeated application of Girsanov's theorem, getting rid of the metric-dependent terms, calculating specific point-wise covariance structures of the GFF on the upper half-plane, and carefully taking partial derivatives.

More precisely, the Gaussian structure is mostly exploited in the calculation-based proofs of BPZ equations in the form of a corollary of **Girsanov's theorem**. Roughly speaking, this result tells us what happens if we act on a Gaussian field with a continuous, bounded functional. It states that under such actions, we recover a measure which is equivalent to the distribution of the original field and gives us the corresponding density.

Theorem 2.2.3 (Girsanov, [Var]) Let X be a GFF and let $X(\phi)$ be a Gaussian random variable measurable with respect to this field. Suppose F is a bounded, continuous function. Then the following identity holds:

$$\mathbb{E}\Big[\exp\Big(X(\phi) - \tfrac{1}{2}\mathbb{E}[X(\phi)^2]\Big) F\big(X(\cdot)\big)\Big] = \mathbb{E}\Big[F\Big(X(\cdot) + \mathbb{E}[X(\phi)]\Big)\Big].$$

From this, we derive the useful formula

Corollary 2.2.4 (Integration By Parts) The following equality holds:

$$\begin{aligned}\langle X(f)V_{\alpha_1}(z_1)\,V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\epsilon,\eta} &= \alpha_1\mathbb{E}\left[X(f)X_\epsilon(z_1)\right]\langle V_{\alpha_1}(z_1)\,V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\epsilon,\eta} + \alpha_2\mathbb{E}\left[X(f)X_\epsilon(z_2)\right]\langle V_{\alpha_1}(z_1)\,V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\epsilon,\eta}\\ &- \mu_\partial\eta\int_{\mathbb{R}}\mathbb{E}\left[X(f)X_\epsilon(s)\right]\langle V_{2\eta}(s)V_{\alpha_1}(z_1)\,V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\epsilon,\eta}\,ds.\end{aligned}$$

Proof: Note that $F : X \mapsto \langle X(f)V_{\alpha_1}(z_1)\,V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\epsilon,\eta}$ can be viewed as a continuous, bounded functional of X. Now, take the scaled version $uX(f)$ of the distributional action and apply Thm. 2.2.3 to get

$$\mathbb{E}\left[\exp(uX(f)) - \frac{u^2}{2}\mathbb{E}\left[X(f)\right]F((X(x))_{x\in\mathbb{H}})\right] = \mathbb{E}\left[F((X(x)+u\mathbb{E}\left[X(x)X(f)\right])_{x\in\mathbb{H}})\right]$$

for any continuous functional F of the GFF X. Now, take the derivative with respect to u, and evaluate at $u=0$ to get the desired expression. □

When we start taking derivatives, we will start getting metric-dependent terms, and terms which include derivatives of the GFF in a covariance structure. The former terms will (luckily) cancel out thanks to the following result, and we can get an explicit expression for the latter covariance structures thanks to L. 2.2.6:

Lemma 2.2.5 (Vanishing Metric-Dependence)

$$\mu_\partial\eta\int_{\mathbb{R}}ds\left\langle V_{2\eta}(s)\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}V_{\beta_j}(s_j)\right\rangle_{\mathbb{H},\eta} = \left(\sum_{i=1}^{N}\alpha_i + \sum_{j=1}^{M}\frac{\beta_j}{2} - Q\right)\left\langle\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}V_{\beta_j}(s_j)\right\rangle_{\mathbb{H},\eta}$$

Proof: This follows from the affine change of variable $c \mapsto \frac{2}{\gamma}\log(\mu_\partial) + c$ applied to $\langle V_{\alpha_1}(z_1)\,V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\epsilon,\eta}$, differentiating with respect to μ_∂, and then taking the limit. □

Lemma 2.2.6 (Covariance Structure) The following limits hold:

$$\begin{aligned}\lim_{\epsilon\searrow 0}\mathbb{E}\left[\partial_{z_1}X_\epsilon(z_1)X_\epsilon(z_2)\right] &= \frac{-1}{2}\left(\frac{1}{z_1-z_2}+\frac{1}{z_1-\overline{z_2}}\right) - \frac{1}{2}\partial_{z_1}\log\hat{g}(z_1)\\ \lim_{\epsilon\searrow 0}\mathbb{E}\left[\partial_{z_1}X_\epsilon(z_1)X_\epsilon(z_1)\right] &= \frac{-1}{2}\frac{1}{z_1-\overline{z_1}} - \frac{1}{2}\partial_{z_1}\log\hat{g}(z_1)\\ \lim_{\epsilon\searrow 0}\mathbb{E}\left[\partial_{z_1}X_\epsilon(z_1)X_\epsilon(s)\right] &= -\frac{1}{z_1-s} - \frac{1}{2}\partial_{z_1}\log\hat{g}(z_1)\end{aligned}$$

Proof: We compute the following quantities:

$$
\begin{aligned}
\mathbb{E}\left[\partial_{z_1} X_\epsilon(z_1) X_\epsilon\left(z_2\right)\right] &= \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \mathbb{E}[X(x)X(y)]\, \partial_{z_1}\rho_\epsilon(z_1-x)\, \rho_\epsilon\left(z_2-y\right) \\
&= \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \partial_x \mathbb{E}[X(x)X(y)]\, \rho_\epsilon(z_1-x)\, \rho_\epsilon\left(z_2-y\right) \\
&= -\frac{1}{2} \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y \left(\frac{1}{x-y}+\frac{1}{x-\bar{y}}\right) \rho_\epsilon(z_1-x)\, \rho_\epsilon\left(z_2-y\right) - \frac{1}{2}\int_{\mathbb{H}} d^2x\, \partial_x \ln \hat{g}(x)\, \rho_\epsilon(z_1-x), \\
\mathbb{E}\left[\partial_{z_1} X_\epsilon(z_1) X_\epsilon(z_1)\right] &= \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \mathbb{E}[X(x)X(y)]\, \partial_{z_1}\rho_\epsilon(z_1-x)\, \rho_\epsilon(z_1-y) \\
&= \partial_{z_1} \frac{1}{2} \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \ln \frac{1}{|x-y|}\, \rho_\epsilon(z_1-x)\, \rho_\epsilon(z_1-y) \\
&\quad + \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \partial_x \left(\ln \frac{1}{|x-\bar{y}|} - \frac{1}{2}\ln \hat{g}(x)\right) \rho_\epsilon(z_1-x)\, \rho_\epsilon(z_1-y) \\
&= -\frac{1}{2} \int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \frac{1}{x-\bar{y}}\, \rho_\epsilon(z_1-x)\, \rho_\epsilon(z_1-y) - \frac{1}{2}\int_{\mathbb{H}} d^2x\, \partial_x \ln \hat{g}(x)\, \rho_\epsilon(z_1-x), \\
\mathbb{E}\left[\partial_{z_1} X_\epsilon(z_1) X_\epsilon(s)\right] &= 2\int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y\, \mathbb{E}[X(x)X(y)]\, \partial_{z_1}\rho_\epsilon(z_1-x)\, \rho_\epsilon(s-y) \\
&= -\int_{\mathbb{H}} \int_{\mathbb{H}} d^2x\, d^2y \left(\frac{1}{x-y}+\frac{1}{x-\bar{y}}\right) \rho_\epsilon(z_1-x)\, \rho_\epsilon(s-y) - \frac{1}{2}\int_{\mathbb{H}} d^2x\, \partial_x \ln \hat{g}(x)\, \rho_\epsilon(z_1-x)
\end{aligned}
$$

and take the limit as $\epsilon \searrow 0$ to conclude. □

Putting everything together, we get:

Theorem 2.2.7 The first partial derivative of a general two-point correlation function on the upper-half plane is

$$
\begin{aligned}
\partial_1 \left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} &= \left(\frac{-\alpha_1^2}{2}\frac{1}{z_1-\overline{z_1}} - \frac{\alpha_1\alpha_2}{2}\left(\frac{1}{z_1-z_2}+\frac{1}{z_1-\overline{z_2}}\right)\right)\left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \\
&\quad + \mu_\partial \eta \alpha_1 \int_{\mathbb{R}} \frac{1}{z_1-s}\left\langle V_{2\eta}\left(s\right) V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \mathrm{d}s
\end{aligned}
$$

Proof: Taking the derivative ∂_1 of the generalized approximation of the two-point correlation function yields

$$
\partial_1 \left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\epsilon,\eta} = \alpha_1 \left\langle (\partial_1 X_\epsilon(z_1)) V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\epsilon} + \alpha_1 \frac{Q}{2}\partial_1 \log \hat{g}(z_1)\left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\epsilon,\eta}.
$$

We can apply L. 2.2.4 to the first term on the right-hand side with f equal to the first partial derivative of the regularization kernel ρ_ϵ. Using this observation, together with L. 2.2.6, we

$$
\begin{aligned}
\partial_1 \left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} &= \left(\frac{-\alpha_1^2}{2}\frac{1}{z_1-\overline{z_1}} - \frac{\alpha_1\alpha_2}{2}\left(\frac{1}{z_1-z_2}+\frac{1}{z_1-\overline{z_2}}\right)\right)\left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \\
&\quad + \mu_\partial \eta \alpha_1 \int_{\mathbb{R}} \frac{1}{z_1-s}\left\langle V_{2\eta}\left(s\right) V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \mathrm{d}s \\
&\quad + \partial_1 \log \hat{g}(z_1)\Bigg[\left(-\frac{\alpha_1^2}{2} - \frac{\alpha_1\alpha_2}{2} + \frac{\alpha_1 Q}{2}\right)\left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \\
&\quad + \frac{\mu_\partial \eta \alpha_1}{2} \int_{\mathbb{R}} \frac{1}{z_1-s}\left\langle V_{2\eta}\left(s\right) V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \mathrm{d}s\Bigg].
\end{aligned}
$$

We conclude by observing that the term

$$
\frac{-\alpha_1}{2}\left((\alpha_1+\alpha_2-Q)\left\langle V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} - \mu_\partial \eta \int_{\mathbb{R}} \frac{1}{z_1-s}\left\langle V_{2\eta}\left(s\right) V_{\alpha_1}\left(z_1\right) V_{\alpha_2}\left(z_2\right)\right\rangle_{\mathbb{H},\eta} \mathrm{d}s\right)
$$

multiplying the metric-dependent summand in the expression for the first derivative cancels out for any value of η due to L. 2.2.5. □

Remark 2.2.8 It is worth noticing that the calculation of higher-order derivatives involve expected values with at most $n+1$ factors, involving derivatives of up to n-th order of the regularization of the GFF. Alternatively, one can recursively calculate the n-th derivatives of the correlation function using the previous order derivative. This procedure will indeed involve an application of Girasnov's theorem, though circumvents the calculation of the above-mentioned expectations (which is manageable, but can become tedioius whenever many terms are inside the expected value). This procedure is analogous to the fact that the BPZ equations of higher-order can be recursively obtained starting from the lower-order ones. ◇

Lemma 2.2.9 Taking derivatives and applying the limits of the Gaussian covariance kernels calculated in [Rem], one gets

$$\begin{aligned}\partial_1 \langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} &= \frac{-\alpha_1^2}{2}\left(\frac{1}{z_1-\bar{z_1}}\right)\langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} \\ &+ \frac{-\alpha_1\alpha_2}{2}\left(\frac{1}{z_1-z_2}+\frac{1}{z_1-\bar{z_2}}\right)\langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} \\ &+ \alpha_1\eta\left(\frac{1}{z_1-s}\right)\langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta}\end{aligned}$$

Proof: This is a consequence of Thm. 2.2.3 and taking the limit $\epsilon \searrow 0$ to conclude. □

Using Thm. 2.2.7 and L. 2.2.9 we can get a closed expression for:

Theorem 2.2.10 (Second Derivative)

$$\begin{aligned}\partial_1^2 \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} &= \left(\frac{-\alpha_1^2}{2}\frac{1}{z_1-\overline{z_1}} - \frac{\alpha_1\alpha_2}{2}\left(\frac{1}{z_1-z_2}+\frac{1}{z_1-\overline{z_2}}\right)\right)\partial_1\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} \\ &+ \left(\frac{\alpha_1^2}{2(z_1-\bar{z}_1)^2} + \frac{\alpha_1\alpha_2}{2}\left(\frac{1}{(z_1-z_2)^2}+\frac{1}{(z_1-\bar{z}_2)^2}\right)\right)\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} \\ &+ \mu_\partial\eta\alpha_1\int_{\mathbb{R}}\left(-\frac{1}{(z_1-s)^2}\langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta} + \frac{1}{z_1-s}\partial_1\langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta}\right)\mathrm{d}s\end{aligned}$$

Proof: We are now ready to prove Thm. 2.2.2: plugging in everything to the general, second order BPZ operator,

$$\frac{1}{\alpha_1^2}\partial_{z_1z_1} + \frac{\Delta_{\alpha_1}}{(z_1-\bar{z_1})^2} + \frac{\Delta_{\alpha_2}}{(z_1-z_2)^2} + \frac{\Delta_{\alpha_2}}{(z_1-\overline{z_2})^2} + \frac{1}{z_1-\bar{z_1}}\partial_{\bar{z_1}} + \frac{1}{z_1-z_2}\partial_{z_2} + \frac{1}{z_1-\overline{z_2}}\partial_{\bar{z}_2}.$$

we get a polynomial system of quadratic equations in α_1 and η, if we assume α_2 to be arbitrary. This system will essentially reduce to being quadratic in α_1 and linear in η. More precisely, we can start calculating the term without μ_∂ with order α_2^0:

$$\left(\frac{\alpha_1^2}{4}+\frac{1}{2}+\Delta_{\alpha_1}+\frac{\alpha_1^2}{2}\right)\frac{1}{(z_1-\bar{z_1})^2}\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta}$$

which vanishes if and only if α_1 satisfies

$$0 = 1 + Q\alpha_1 + \alpha_1^2,$$

that is, if and only if $0 = 1 + \frac{2\alpha_1}{\gamma} + \frac{\alpha_1\gamma}{2} + \alpha_1^2$, which is the case if and only if $\alpha_1 \in \left\{\frac{-\gamma}{2}, \frac{-2}{\gamma}\right\}$. In other words, *if* the BPZ equations hold, then we necessarily have $\alpha_1 \in \left\{\frac{-\gamma}{2}, \frac{-2}{\gamma}\right\}$. [1] The first value corresponds to the primary solution, for which we get from integrability theory that $\gamma \in (0,2)$. The second value corresponds to the dual solution, for which we require $\gamma \in (2,\infty)$.

Similarly, the other terms without μ_∂ with the order α_2^1 cancel out under the condition $0 = Q + 1/\alpha_1 + \alpha_1$, which leads to the same solution set for α_1. The terms without μ_∂ with the order α_2^2 do not require any specific value of α_1 to cancel out.

Finally, we can proceed similarly with the terms with μ_∂: in order for them to cancel out, we can use the same symmetrization techniques from [Rem] together with Girsanov's theorem. After some algebraic manipulation, we see that these terms cancel out if and only if $\eta = -\alpha_1$. For instance, if we consider the terms which have $\frac{1}{(z-s)^2}$-integrals in the second derivative of the partition function. Indeed, after symmetrizing the term

$$\mu_\partial\eta\alpha_1\int_{\mathbb{R}}\frac{-1}{(z-s)^2}\langle V_{2\eta}(s) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2)\rangle_{\mathbb{H},\eta}\,\mathrm{d}s,$$

one gets the condition $\mu_\partial\eta^2\alpha_1^2 + \mu_\partial\eta\alpha_1^3 = 0$, which is satisfied for $\eta = -\alpha_1$. □

Remark 2.2.11 Algebraically speaking, it is not a coincidence that Thm. 2.2.2 holds. Roughly speaking, the calculations we have presented come from a representation of the so-called **Virasoro algebra**. We can tweak the corresponding dimension parameter of the representation such that it becomes **degenerate**. Under these degeneracy values, the so-called conformal blocks can be computed as the solution of certain linear differential equations. Using that any correlation function is built up from these conformal blocks, a general BPZ equation follows. For more details, see Sec. 4.2.

[1] For this reason, in LCFT, vertex operators with such a weight are called **degenerate field insertions**.

In fact, one can see, for instance in ([KRV], Thm. 2.1) or ([Zhu], Thm. 1.14) that all we need is *one* degenerate field insertion inside a $n+1$-point correlation function to get a BPZ equation. The further complications in the latter result arise because the upper half-plane has a boundary, whereas the Riemann sphere does not have a boundary. In fact, one can already start seeing the complications of having a boundary in the Polyakov action functional from Def. 1.2.2. $\diamond$

By following the same steps as the proof of Thm. 2.2.2, we get

Theorem 2.2.12 (BPZ Equation for n-Point Correlation Function)
Suppose that the vector $\alpha \in \mathbb{R}^n$ satisfies $\sum_{j=1}^n \alpha_i > Q$. The BPZ equation

$$\left(\frac{1}{\alpha_1^2}\partial_{z_1 z_1} + \frac{\Delta_{\alpha_1}}{(z_1-\bar{z_1})^2} + \sum_{j=2}^{n}\left[\frac{\Delta_{\alpha_j}}{(z_1-z_j)^2} + \frac{\Delta_{\alpha_j}}{(z_1-\overline{z_j})^2}\right]\right.$$
$$\left. + \frac{1}{z_1-\bar{z_1}}\partial_{\bar{z_1}} + \sum_{j=2}^{n}\left[\frac{1}{z_1-z_j}\partial_{z_j} + \frac{1}{z_1-\overline{z_j}}\partial_{\bar{z}_j}\right]\right)\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle_{\mathbb{H},\eta} = 0$$

for the tweaked n-point correlation function $\langle V_{\alpha_1}(z_1),\ldots,V_{\alpha_n}(z_n)\rangle_{\mathbb{H},\eta}$ holds, *if and only if*

$$(\alpha_1,\ldots,\alpha_n,\eta) \in \left\{\left(\frac{-\gamma}{2},\alpha_2,\ldots,\alpha_n,\frac{\gamma}{2}\right) \mid \gamma\in(0,2),(\alpha_j)_{j=2}^n\subseteq\mathbb{R}\right\} \cup \left\{\left(\frac{-2}{\gamma},\alpha_2,\ldots,\alpha_n,\frac{2}{\gamma}\right) \mid \gamma\in(2,\infty),(\alpha_j)_{j=2}^n\subseteq\mathbb{R}\right\}.$$

2.3 Change of Variable: A Geometric Excursion

We temporarily come back to our exploration of the general two-point correlation function with η in the exponential of the regularization. Recall that the BPZ equations are used to get an asymptotic behaviour for the two-point correlation function, and therefore also one for the function G in [Rem], which itself relates directly to the negative moments of the partition function U. Whether or not a BPZ equation holds, we can still make the calculations which gets us from our two-point correlation function to a generalized version of G. The paragraphs to come follow some ideas and results from both [Rem] and [HRV2].

The so-called **KPZ formula** allows to get an explicit expression of the conformal factor in (1.10) and can be viewed as a conformal Jacobi theorem for the partition function. For instance, the **Cayley transform** $\psi_1 : x \mapsto \frac{x-i}{x+i}$ is a conformal isomorphism from the upper half plane $\mathbb{H}$ with the Poincare metric $\hat{g}$ to the unit disk $\mathbb{D}$ with the Euclidean metric. For visualization purposes, it is worth imagining a point "$i\infty$" on the imaginary axis which is infinitely far away from the origin. The line $(0,i\infty)$ gets mapped to the horizontal axis of the unit disk $(-1,+1)$, and i gets mapped to its midpoint – the origin. Fig. 4.1 helps in viewing how a finite rectangle deforms into the unit disk.

The first KPZ formula ([HRV], Prop. 3.7) yields for the two-point correlation function with the vertex operators with marked points $z, z_1 \in \mathbb{H}$ with $z_1 \neq z_2$:

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle_{\mathbb{H}} = \left|\psi_1'(z_1)\right|^{2\Delta_{\alpha_1}}\left|\psi_1'(z)\right|^{2\Delta_{\alpha_2}}\langle V_{\alpha_1}(\psi_1(z_1))V_{\alpha_2}(\psi_1(z_2))\rangle_{\mathbb{D}}.$$

For our purposes, we will need to conformally map the unit disk onto itself in such a way that $\psi_1(z_1)$ is mapped to the positive, real part of the disk, i.e. on $t\in(0,1)$ and $\psi_1(z_2)$ is mapped to 0. Such a conformal automorphism can be explicitly given by a **Moebius transformation** ψ_2 as $\psi_2(x) = e^{i\theta}\frac{x-\psi_1(z_1)}{1-x\psi_1(z_1)}$ for $\theta = -\arg\left(\frac{\psi_1(z_1)-\psi_1(z_2)}{1-\psi_1(z_1)\psi_1(z_2)}\right)$. Indeed, from the numerator, it is easy to see that z_2 is indeed mapped to the origin, and the complex exponential rotates the line connecting the new origin with the image of z_1 under the composition map such that it lies on the positive real line whilst staying on the unit disk (cf. Fig. 4.1), and the value of t can be explicitly calculated to be

$$t = \left|\frac{z_1-z_2}{z_1-\overline{z_2}}\right|$$

a quantity which can be constructed using only projective geometry (cf. Fig. 4.2). This explicit representation allows us to analytically see that $t \searrow 0$ corresponds to z_1 and z_2 getting arbitrarily close, whereas $t \nearrow 1$ corresponds to z_1 and z_2 getting infinitely far away from one another.

We use the other KPZ formula ([HRV], Thm. 3.5) to obtain

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle_{\mathbb{H}} = \frac{1}{|z_2-\bar{z}_2|^{2\Delta_{\alpha_2}-2\Delta_{\alpha_1}}}\frac{1}{|z_1-\bar{z}_2|^{4\Delta_{\alpha_1}}}\langle V_{\alpha_1}(t)V_{\alpha_2}(0)\rangle_{\mathbb{D}}.$$

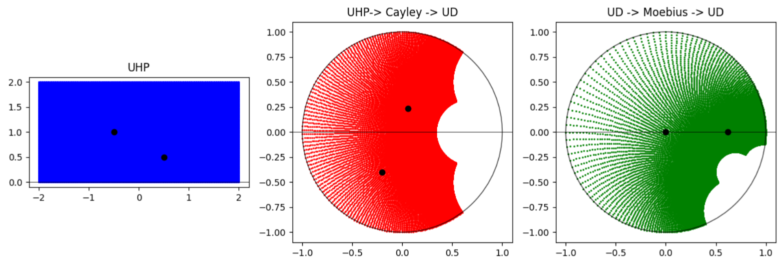

Figure 2.1: Positioning

Another key observation in [Rem] is the connection between $\langle V_{\alpha_1}(t)\, V_{\alpha_2}(0)\rangle_{\mathbb{D}}$ and the partition function presented in [HRV2], Cor. 3.10. Using their same notation, note that, on the one hand $\mathcal{Z} = \langle V_{\alpha_1}(t)\, V_{\alpha_2}(0)\rangle_{\mathbb{D}}$. On the other hand, the normalization constant $\mathcal{Z}$ can be explicitly calculated by setting $F = 1$. By doing so, one sees that the function G defined in the beginning is indeed equal to the two-point correlation function $\langle V_{\alpha_1}(t)\, V_{\alpha_2}(0)\rangle_{\mathbb{D}}$ up to some function in t. More precisely, recall from that same result that

$$\mathcal{Z} = \int_{\mathbb{R}} u^{\frac{1}{\gamma}(\alpha_1+\alpha_2-Q)-1}\mathbb{E}\left[Z_0(\mathbb{D})^{-\frac{2}{\gamma}(\alpha_1+\alpha_2-Q)}\right]e^{-\mu u}\,\mathrm{d}u,$$

where

$$Z_0^{\partial}(\partial\mathbb{D}) = \int_{\partial\mathbb{D}} \exp\left(\frac{\gamma}{2}(\alpha_1 G(x,z_1)+\alpha_2 G(x,z_2))\right)\exp\left(\eta X\right)\,\mathrm{d}x.$$

and G here denotes the Green function on the unit disk (not to be confused with the G defined below, which has three inputs instead of two). Using the substitution

$$u(c) = \mu_{\partial}\exp\left(\frac{\gamma}{2}c\right)\int_{\partial\mathbb{D}}\exp\left(\epsilon^{\frac{\gamma^2}{4}}\exp\left(\eta X_{\epsilon}(s)\right)\right)\,\mathrm{d}s$$

we get for the expectation

$$\mathbb{E}\left[Z_0(\mathbb{D})^{-\frac{2}{\gamma}(\alpha_1+\alpha_2-Q)}\right] = t^{-\alpha_1\alpha_2}(1-t^2)^{\frac{1}{2}\eta\alpha_2}\mathbb{E}\left[\left(\int_0^{2\pi}|e^{i\theta}-t|^{-\gamma\alpha_1}|e^{i\theta}-0|^{-\gamma\alpha_2}e^{\eta X(e^{i\theta})}\right)^{-\frac{2}{\gamma}(\alpha_1+\alpha_2-Q)}\right],$$

which does not depend on u, so that the rest of the integral evaluates to a gamma function up to a multiplicative factor:

$$\frac{2}{\gamma}\mu_{\partial}^{-\frac{2}{\gamma}(\alpha_1+\alpha_2-Q)}\Gamma\left(-\frac{2}{\gamma}\left(\alpha_1+\alpha_2-Q\right)\right).$$

From the first expression, we see a first instance of why we chose the Moebius transformation to map $\psi_1(z_2)$ to zero: it is precisely this condition that makes the original correlation function on the upper half-plane, which depended on two parameters, to concentrate this dependence in the single parameter $t \in (0,1)$, as the modulus of the exponential under a purely imaginary number is 1. A second instance will be seen when we change the BPZ equations into an ordinary differential equation with respect to t.

All in all, we conclude that

$$\begin{aligned}\langle V_{\alpha_1}(t)\, V_{\alpha_2}(0)\rangle_{\mathbb{D}} = \frac{2}{\gamma}\mu_{\partial}^{-\frac{2}{\gamma}(\alpha_1+\alpha_2-Q)}\Gamma\left(-\frac{2}{\gamma}\left(\alpha_1+\alpha_2-Q\right)\right)\\ \cdot t^{-\alpha_1\alpha_2}(1-t^2)^{\frac{1}{2}\eta\alpha_2}\mathbb{E}\left[\left(\int_0^{2\pi}|e^{i\theta}-t|^{-\gamma\alpha_1}e^{\eta X(e^{i\theta})}\right)^{-\frac{2}{\gamma}(\alpha_1+\alpha_2-Q)}\right]\end{aligned}$$

Finally, we recall that it is precisely for the values $\alpha_1 = \frac{-\gamma}{2}$ and $\eta = \frac{\gamma}{2}$ that the two-point correlation function solves the (primal) BPZ equation (cf. Thm. 2.2.2). Moreover, under these same values, the expectation also coincides with the function G from [Rem] defined as

$$G(\gamma,p,t) := \mathbb{E}\left[\left(\int_0^{2\pi}\left|t-e^{i\theta}\right|^{\frac{\gamma^2}{2}}e^{\frac{\gamma}{2}X\left(e^{i\theta}\right)}d\theta\right)^p\right],$$

where we set $-p = \frac{2}{\gamma}\left(\alpha_1+\alpha_2-Q\right)$. We now understand the choice of this function G and its parameters. We see that $p < 0$ is equivalent to the relaxed Seiberg bounds for the well-definedness of the partition function from [HRV2].

Remark 2.3.1 From the integrability conditions of the GMC measure ($\gamma \in (0,2)$), we see that G is continuous in $t \in [0,1]$. ◇

Take $\alpha_2 = \alpha \in \mathbb{R}$ be a real parameter such that the Seiberg bounds are satisfied. Then, the full expression for our new two-point correlation function on the unit disk reads[2]

$$\left\langle V_{\frac{-\gamma}{2}}(t)\, V_\alpha(0) \right\rangle_{\mathbb{D}} = \frac{2}{\gamma} \mu_\partial^p \Gamma(p)\, t^{\frac{\gamma\alpha}{2}} (1-t^2)^{\frac{1}{2}\frac{\gamma\alpha}{2}} G(\gamma, p, t), \tag{2.4}$$

and it satisfies a second order BPZ equation. Observe that all the parameters which uniquely determine whether a BPZ equation is satisfied are inside of the expectation.

Now, using our change of coordinates under $\phi := \psi_2 \circ \psi_1$ which was constructed such that $\phi(z_1) = t$ and $\phi(z_2) = 0$, we can rewrite $\left\langle V_{\frac{-\gamma}{2}}(z_1)\, V_\alpha(z_2) \right\rangle_{\mathbb{H}}$ in terms of t:

$$\left\langle V_{\frac{-\gamma}{2}}(z_1)\, V_\alpha(z_2) \right\rangle_{\mathbb{H}} = \frac{(1-t)^{4\Delta_{\frac{-\gamma}{2}}}}{2^{2(\Delta_{\frac{-\gamma}{2}} + \Delta_\alpha)}} \left\langle V_{\frac{-\gamma}{2}}(t)\, V_\alpha(0) \right\rangle_{\mathbb{D}}. \tag{2.5}$$

Proposition 2.3.2 The BPZ operator from Thm. 2.2.2 in terms of $\left\langle V_{\frac{-\gamma}{2}}(t)\, V_\alpha(0) \right\rangle_{\mathbb{D}}$ reads

$$\left(\frac{t^2}{\gamma^2} \frac{d^2}{dt^2} + \left(-\frac{t}{\gamma^2} + \frac{2t^3 - t}{2(1-t^2)} \right) \frac{d}{dt} + \left(\Delta_\alpha + \Delta_{-\frac{\gamma}{2}} \frac{2t^2 - t^4}{(t^2-1)^2} \right) \right) \left\langle V_{-\frac{\gamma}{2}}(t) V_\alpha(0) \right\rangle_{\mathbb{D}} = 0.$$

Proof: This is "only" a change of variables of the differentiation operators, in other words, an exhaustive application of the chain rule. However, it is worth noticing that the composition ϕ contains a lot of symmetry, which can be seen in how the differential operators are mapped from one coordinate system to another. For instance, one can calculate that

$$\frac{\Delta_{\alpha_1}}{(z_1 - \overline{z_1})^2} = -\Delta_{\alpha_1} \frac{(1-t)^2}{4(1+t)^2} \qquad \frac{\Delta_{\alpha_1}}{(z_1 - z_2)^2} = -\Delta_{\alpha_1} \frac{(1-t)^2}{4t^2} \qquad \frac{\Delta_{\alpha_2}}{(z_1 - \overline{z_2})^2} = -\Delta_{\alpha_2} \frac{(1-t)^2}{4}$$

which are purely geometrical equalities (cf. Fig. 4.2). To see the first identity, we may assume up to a rotation of the unit disk, that the scalar z_1 is of the form

$$z_1 = i\, \frac{1+t}{1-t}$$

for $t \in (0,1)$, so that indeed z_1 will end up being mapped to $t \in (0,1)$ under a composition of the Cayley transform and a Moebius transformation. We calculate

$$z_1 - \overline{z_1} = 2i\, \frac{1+t}{1-t},$$

which is imaginary for *any* choice of z_1, regardless of our initial z_1. Therefore,

$$(z_1 - \overline{z_1})^2 = \left(2i\, \frac{1+t}{1-t} \right)^2 = -4\, \frac{(1+t)^2}{(1-t)^2},$$

showing the desired identity.

Moreover, for the differential operators, employing the chain rule

$$\frac{1}{z_1 - \overline{z_1}} \partial_{z_1} = -\frac{(1-t)^3}{8(1+t)} \partial_t \qquad \frac{1}{z_1 - z_2} \partial_{z_2} = \frac{(1-t)^2 (1+t)}{8t} \partial_t \qquad \frac{1}{z_1 - \overline{z_2}} \partial_{\overline{z_2}} = -(1-t^2)\, \partial_t.$$

More precisely, by inverting the defining relation for z_1 in terms of t (which can be easily done using the fact that Moebius transformations are algebraically isomorphic to $\mathrm{GL}_n(\mathbb{C}, 2)$, and that the objects in the latter group are rather easy to invert) and differentiating, we obtain

$$\partial_{z_1} t = \frac{(1-t)^2}{4i}.$$

Therefore,

$$\frac{1}{z_1 - \overline{z_1}} \partial_{z_1} = \frac{1}{2i\, \frac{1+t}{1-t}} \frac{(1-t)^2}{4i} \partial_t = -\frac{(1-t)^3}{8(1+t)} \partial_t.$$

[2] In our general setting, we would have had

$$\left\langle V_{\alpha_1}(t)\, V_{\alpha_2}(0) \right\rangle_{\mathbb{D}} = \frac{2}{\gamma} \mu_\partial^p \Gamma(p)\, t^{-\alpha_1 \alpha_2} (1-t^2)^{\frac{1}{2}\eta\alpha_2} \mathbb{E}\left[\left(\int_0^{2\pi} |e^{i\theta} - t|^{-\gamma\alpha_1} e^{\eta X(e^{i\theta})} \right)^p \right].$$

but the problem with continuing along this path is that we do not always have a BPZ equation.

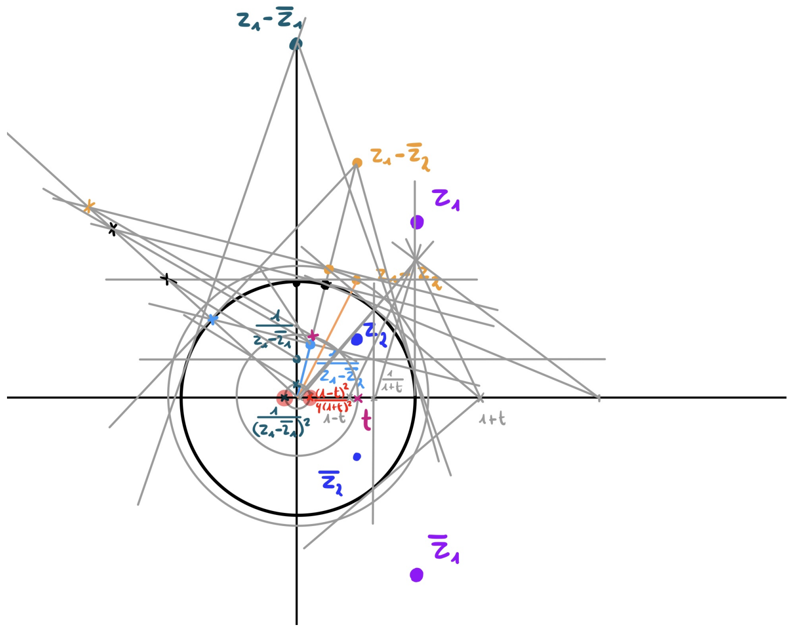

Figure 2.2: Visual Proof of the First Identity using Projective Geometry.

Putting the above relations and (2.5) together into the partial differential equation in Thm. 2.2.2 yields the desired result. □

Remark 2.3.3 The visual proof of the geometric construction follows two principles of projective geometry and the polar representation of complex numbers. More precisely, we have used the geometric constructions presented in [RG] for addition and multiplication on the real line, cf. [RG], Fig. 5.5. ◇

Using (2.4) and Prop. 2.3.2, we conclude:

Corollary 2.3.4 (BPZ for G)

$$\left(t\left(1-t^2\right)\frac{\partial^2}{\partial t^2}+\left(t^2-1\right)\frac{\partial}{\partial t}+2\left(C-(A+B+1)t^2\right)\frac{\partial}{\partial t}-4ABt\right)G(\gamma,p,t)=0$$

with $A=-\frac{\gamma^2 p}{4}$, $B=-\frac{\gamma^2}{4}$, and $C=\frac{\gamma^2}{4}(1-p)+1$.

In other words, we have just shown that the BPZ equation for the two-point correlation function on the upper half-plane can be turned into a BPZ equation for $G(\gamma,p,\cdot)$ on the open interval $(0,1)$. We conclude through the change of variable $t^2=x\in(0,1)$ that $H(x)=G(\gamma,p,t)$ is a hypergeometric function:

$$\left(x(1-x)\frac{\partial^2}{\partial x^2}+(C-(A+B+1)x)\frac{\partial}{\partial x}-AB\right)H(x)=0. \tag{2.6}$$

In the following section, we exploit the properties of such functions.

2.4 Hypergeometric Expansion and Shift Equation

The function $H:(0,1)\to\mathbb{R}, x\mapsto H(x):=G(\gamma,p,\sqrt{x})$ solving (2.6) allows corresponding asymptotic expansions for the values near the boundary of the form (cf. [Rem], Prop. 1.3)

$$G(\gamma,p,t)=C_1F\left(-\frac{\gamma^2 p}{4},-\frac{\gamma^2}{4},\frac{\gamma^2}{4}(1-p)+1,t^2\right)+C_2t^{\frac{\gamma^2}{2}(p-1)}F\left(-\frac{\gamma^2}{4},\frac{\gamma^2}{4}(p-2),\frac{\gamma^2}{4}(p-1)+1,t^2\right)$$

and

$$G(\gamma,p,t)=B_1F\left(-\frac{\gamma^2 p}{4},-\frac{\gamma^2}{4},-\frac{\gamma^2}{2},1-t^2\right)+B_2\left(1-t^2\right)^{1+\frac{\gamma^2}{2}}F\left(1+\frac{\gamma^2}{4},\frac{\gamma^2}{4}(2-p)+1,2+\frac{\gamma^2}{2},1-t^2\right)$$

where F is the standard hypergeometric function, and the expansion coefficients depend on the parameters γ and p. The solution space of (2.6) is two-dimensional, and therefore the coefficients of the expansions are linked through a **change of basis formula** (cf. [Rem] and [AAR])

$$\binom{B_1}{B_2} = \begin{pmatrix} \frac{\Gamma\left(1+\frac{\gamma^2}{2}\right)\Gamma\left(\frac{\gamma^2}{4}(1-p)+1\right)}{\Gamma\left(1+\frac{\gamma^2}{4}\right)\Gamma\left(\frac{\gamma^2}{4}(2-p)+1\right)} & \frac{\Gamma\left(1+\frac{\gamma^2}{2}\right)\Gamma\left(\frac{\gamma^2}{4}(p-1)+1\right)}{\Gamma\left(1+\frac{\gamma^2}{4}\right)\Gamma\left(\frac{\gamma^2}{4}p+1\right)} \\ \frac{\Gamma\left(-1-\frac{\gamma^2}{2}\right)\Gamma\left(\frac{\gamma^2}{4}(1-p)+1\right)}{\Gamma\left(-\frac{\gamma^2}{4}\right)\Gamma\left(-\frac{\gamma^2 p}{4}\right)} & \frac{\Gamma\left(-1-\frac{\gamma^2}{2}\right)\Gamma\left(\frac{\gamma^2}{4}(p-1)+1\right)}{\Gamma\left(-\frac{\gamma^2}{4}\right)\Gamma\left(\frac{\gamma^2}{4}(p-2)\right)} \end{pmatrix} \binom{C_1}{C_2}$$

Remark 2.4.1 (Properties of the Hypergeometric Function)

$$\begin{aligned} F(A,B,C,x) &= \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} F(A,B,A+B-C+1,1-x) \\ &\quad + (1-x)^{C-A-B} \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} F(C-A,C-B,C-A-B+1,1-x) \end{aligned}$$

and

$$F(A,B,C,1) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} \tag{4.11}$$

◇

The strategy is now as follows: Observe that as $p<0$, we have $\frac{\gamma^2}{2}(p-1)<0$ and therefore $C_2=0$. Using that G is continuous in t on $[0,1]$, we see by taking $t=0$, and using the first asymptotic expansion together with $C_2=0$ we get

$$U(\gamma,p) = G(\gamma,p,0) = C_1 \cdot 1 + 0 = C_1.$$

On the other side of the interval, at $t=1$, we have

$$\mathbb{E}\left[\left(\int_0^{2\pi} \left|1-e^{i\theta}\right|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X\left(e^{i\theta}\right)} d\theta\right)^p\right] = G(\gamma,p,1) = B_1 \cdot 1 + 0 = B_1.$$

We would like to show the following result, which is a consequence of the asymptotics of G and comparing coefficients:

Proposition 2.4.2 For all $\gamma \in (0,2) \setminus \left\{\sqrt{2}\right\}$, we have

$$B_2 = 2\pi p \frac{\Gamma\left(-\frac{\gamma^2}{2}-1\right)}{\Gamma\left(-\frac{\gamma^2}{4}\right)^2} U(\gamma,p-1).$$

In order to show Prop. 2.4.2, we need some intermediate results:

Lemma 2.4.3

$$\mathbb{E}\left[\int_0^{2\pi} e^{\frac{\gamma}{2}X\left(e^{iu}\right)} du \left(\int_0^{2\pi} e^{\frac{\gamma}{2}X\left(e^{i\theta}\right)} d\theta\right)^{p-1}\right] = \int_0^{2\pi} \mathbb{E}\left[\left(\int_0^{2\pi} \frac{e^{\frac{\gamma}{2}X\left(e^{i\theta}\right)}}{\left|e^{iu}-e^{i\theta}\right|^{\frac{\gamma^2}{2}}} d\theta\right)^{p-1}\right] du$$

Proof: The above expressions are to be understood as limits of mollifications, as we had for the correlation functions. The key step is to notice that

$$F : X_\epsilon \mapsto \left(\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon\left(e^{i\theta}\right)} d\theta\right)^{p-1}$$

is a continuous, bounded functional of the regularized GFF. This yields

$$\mathbb{E}\left[\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon\left(e^{iu}\right)} du \left(\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon\left(e^{i\theta}\right)} d\theta\right)^{p-1}\right] = \int_0^{2\pi} \mathbb{E}\left[\left(\int_0^{2\pi} e^{\frac{\gamma}{2}\left(X_\epsilon\left(e^{i\theta}\right)+\frac{\gamma}{2}\mathbb{E}\left[X_\epsilon\left(e^{i\theta}\right)X_\epsilon\left(e^{iu}\right)\right]\right)} d\theta\right)^{p-1}\right] du.$$

The limit as $\epsilon \searrow 0$ on the covariance of the regularizations of the GFF yields $\left|e^{iu}-e^{i\theta}\right|^{\gamma^2/2}$ (recall the calculations of Ch. 3). We conclude by taking the limit on both sides. $\square$

Lemma 2.4.4 (Taylor Expansion) For $\epsilon > 0$ and $u \in [0, 2\pi]$ let $h_u(1-\epsilon) = \left|1 - \epsilon - e^{iu}\right|^{\frac{\gamma^2}{2}}$ and let $c : \partial\mathbb{D} \mapsto \mathbb{R}$ be a continuous function defined on the unit circle. Then we have:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \frac{1}{2^{1+\frac{\gamma^2}{2}}} \frac{1}{2\pi} \int_0^{2\pi} \left(h_u(1-\epsilon) - h_u(1) + \epsilon h_u'(1) - \frac{\epsilon^2}{2} h_u''(1) \right) c\left(e^{iu}\right) du = \frac{\Gamma\left(-\frac{\gamma^2}{2} - 1\right)}{\Gamma\left(-\frac{\gamma^2}{4}\right)^2} c(1).$$

Proof Idea: The proof follows two main steps:

Step 1: The integral

$$\frac{1}{\epsilon^{1+\frac{\gamma^2}{2}}} \int_0^{2\pi} \left| h_u(1-\epsilon) - h_u(1) + \epsilon h_u'(1) - \frac{\epsilon^2}{2} h_u''(1) \right| du$$

remains bounded as $\epsilon \searrow 0$.

Step 2: For $\gamma \neq \sqrt{2}$, one has

$$\lim_{t \to 1} \frac{1}{(1-t^2)^{1+\frac{\gamma^2}{2}}} \frac{1}{2\pi} \int_0^{2\pi} \left(h_u(t) - h_u(1) - (t-1) h_u'(1) - \frac{(t-1)^2}{2} h_u''(1) \right) du = \frac{\Gamma\left(-\frac{\gamma^2}{2} - 1\right)}{\Gamma\left(-\frac{\gamma^2}{4}\right)^2}.$$

The reason one does not show the above claim for $\gamma = \sqrt{2}$ is that the gamma functions that appear in the expression have a pole. For each of the steps, one has to make a case distinction between $\gamma < 2$ and $\gamma > \sqrt{2}$.[3] To get the shift equation for all $\gamma \in (0, 2)$, we will use the continuity of $\gamma \mapsto U(\gamma, p)$. $\square$

We are ready to show the shift equation:

Theorem 2.4.5 (Shift Equation) For all $\gamma \in (0, 2)$ and for $p \leq 0$, we have the relation

$$U(\gamma, p) = \frac{2\pi\Gamma\left(1 - p\frac{\gamma^2}{4}\right)}{\Gamma\left(1 - \frac{\gamma^2}{4}\right)\Gamma\left(1 - (p-1)\frac{\gamma^2}{4}\right)} U(\gamma, p-1).$$

Proof: With the change of basis formula, plugging in $C_1 = U(\gamma, p)$, $C_2 = 0$, and B_2 from Prop. 2.4.2 we get for $p < 0$ and $\gamma \in (0, 2) \setminus \sqrt{2}$:

$$\frac{U(\gamma, p)}{U(\gamma, p-1)} = \frac{2\pi p \Gamma\left(-\frac{\gamma^2 p}{4}\right)}{\Gamma\left(-\frac{\gamma^2}{4}\right)\Gamma\left(1 - (p-1)\frac{\gamma^2}{4}\right)} = \frac{2\pi\Gamma\left(1 - p\frac{\gamma^2}{4}\right)}{\Gamma\left(1 - \frac{\gamma^2}{4}\right)\Gamma\left(1 - (p-1)\frac{\gamma^2}{4}\right)}. \tag{2.7}$$

Since $p \mapsto U(\gamma, p)$ is continuous at 0, so the above holds for $p \leq 0$ and $\gamma \neq \sqrt{2}$.

Observe that $\gamma \mapsto U(\gamma, p)$ is continuous: the random variable

$$\mathcal{M}_\gamma(\partial\mathbb{D}) = \int_0^{2\pi} e^{\gamma X - \frac{\gamma^2}{2}\mathbb{E}[X^2]} \frac{d\theta}{2\pi}.$$

is continuous almost surely, and bounded away from zero on any compact interval. Therefore, for any fixed $\gamma \in (0, 2)$ such that $p < \frac{4}{\gamma^2}$, we have from integration theory of GMC measures that the integral of $\mathcal{M}_\gamma(\partial\mathbb{D})^p$ is finite and can be bounded by a constant independent of γ. We conclude the continuity of $\gamma \mapsto U(\gamma, p)$ using dominated convergence. In particular, for $\gamma = \sqrt{2}$, we can extend (2.7) to all $p \leq 0$ and $\gamma \in (0, 2)$. $\square$

[3] Technically, we only need to expand up to the second derivative of $h_u(1-\epsilon)$ for the case $\gamma > \sqrt{2}$. This is because in Step 1, if we only expand up to the first derivative, one comes to a point in which we would need that ϵ^2 beats $\epsilon^{-(1+\gamma^2/2)}$ as $\epsilon \searrow 0$, but this only happens in the case $\gamma < \sqrt{2}$. It is only if we expand until the next power of $h_u(1-\epsilon)$ that we get the desired behavior.

2.5 Finishing Touches

From Thm. 2.4.5, recalling that $U(\gamma, p) = (2\pi)^p \mathbb{E}\left[Y_\gamma^p\right]$, we can recursively starting from $U(\gamma, 0) = 1$ get all the non-positive integer moments of Y_γ:

$$\mathbb{E}\left[Y_\gamma^{-n}\right] = \Gamma\left(1 + \frac{n\gamma^2}{4}\right)\Gamma\left(1 - \frac{\gamma^2}{4}\right)^n, \quad \forall n \in \mathbb{N}, \tag{2.8}$$

from which we see that the series $\phi : z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}\mathbb{E}\left[Y_\gamma^{-n}\right]$ has infinite radius of convergence. From Fourier theory (cf. [Dur], Sec. 3.3), we know this means that the law of Y_γ^{-1} is uniquely determined by its moments. Moreover, using the Fourier inversion theorem ([Dur], Thm. 3.3.14.), the random variable Y_γ^{-1} admits a Lebesgue density and we can calculate it explicitly by getting the inverse Fourier transform of ϕ. This yields

$$f_{\frac{1}{Y_\gamma}}(y) = \frac{4}{\beta\gamma^2}\left(\frac{y}{\beta}\right)^{\frac{4}{\gamma^2}-1} e^{-\left(\frac{y}{\beta}\right)\frac{4}{\gamma^2}}\mathbf{1}_{[0,\infty[}(y).$$

The mapping $t \mapsto 1/t$ is a diffeomorphism on $(0, \infty)$ and thus one can use Jacobi's theorem in order to make the above density into one for the random variable Y_γ itself:

$$f_{Y_\gamma}(y) = \frac{4\beta}{\gamma^2}(\beta y)^{-\frac{4}{\gamma^2}-1} e^{-(\beta y)^{-\frac{4}{\gamma^2}}}\mathbf{1}_{[0,\infty[}(y),$$

from which we can now extract all the *real* p-th moments for $p < \frac{4}{\gamma^2}$ from (2.8), which concludes the proof of the Fyodorov-Bouchaud formula in Thm. 2.1.1. $\square$

Chapter 3

Fyodorov-Bouchaud Formula: Consequences

The goal of this chapter is to analyze the behavior of the partition function Y_γ as $\gamma \searrow 0$ with help of the Fyodorov-Bouchaud formula. In particular, we start with some direct corollaries of Thm. 2.1.1 and then move on to a large deviation result.

3.1 Semiclassical Limit

In this chapter, we want to study one of the implications of Thm. 2.1.1, called the **semiclassical limit**. By this, we simply mean setting $p = \frac{p_0}{\gamma^2}$ for some fixed $p_0 < 4$, and taking the limit $\gamma \searrow 0$. Note that in this case, for all $\gamma \in (0, 2)$, the assumptions of Thm. 2.1.1 are met.

3.1.1 Uniform Integrability: A Warm-Up

Let us first calculate the semiclassical limit of (2.2). The right-hand side is of pure analytical nature. The numerator yields $\Gamma\left(1 - \frac{p_0}{4}\right)$. For the denominator, we recall the expansion

$$\Gamma(1+z) = 1 - \gamma_{EM} z + \mathcal{O}(z^2)$$

as $z \to 0$, where γ_{EM} is the Euler-Mascheroni constant (which is known to be approximately 0.5772). Therefore, we get

$$\Gamma\left(1 - \frac{\gamma^2}{4}\right)^{\frac{p_0}{\gamma^2}} = \left(1 + \frac{\gamma_{EM} p_0}{4}\frac{\gamma^2}{p_0} + \mathcal{O}(\gamma^4)\right)^{\frac{p_0}{\gamma^2}} \xrightarrow{\gamma \searrow 0} \exp\left(\frac{\gamma_{EM} p_0}{4}\right).$$

Overall,

$$\lim_{\gamma \searrow 0} \mathbb{E}\left[Y_\gamma^{p_0/\gamma^2}\right] = \Gamma\left(1 - \frac{p_0}{4}\right) \exp\left(-\frac{\gamma_{EM} p_0}{4}\right). \tag{3.1}$$

A more subtle question is to ask ourselves whether we can bring the limit inside of the expectation in (3.1).

On the other hand, from (2.3), we see directly by taking the power $\frac{p_0}{\gamma^2}$ and using Thm. 2.1.1 that

$$Y_\gamma^{p_0/\gamma^2} \xrightarrow{\gamma \searrow 0} (\exp(\gamma_{EM})\mathcal{E}(1))^{-\frac{p_0}{4}}$$

in distribution. Its density is given by $1 - \exp\left(-e^{-\gamma_{EM}} x^{-4/p_0}\right)$ for $x \geq 0$.

We aim to show that

Proposition 3.1.1 For $p_0 \in [0, 4)$ the family $(Y_\gamma^{p_0/\gamma^2})_{\gamma \in (0,2)}$ is bounded in $L^q(\mathbb{P})$ for $q \in (1, \frac{4}{p_0})$. In particular, $(Y_\gamma^{p_0/\gamma^2})_{\gamma \in (0,2)}$ is uniformly integrable for all $p_0 \in (0, 4)$.

Proof: We first note that

$$\mathbb{E}\left[Y_\gamma^{\frac{qp_0}{\gamma^2}}\right] = \frac{\Gamma\left(1 - \frac{qp_0}{4}\right)}{\Gamma\left(1 - \frac{\gamma^2}{4}\right)^{\frac{qp_0}{\gamma^2}}}.$$

On the one hand, since $p_0 < 4$, there exists $q > 1$ such that $q < \frac{4}{p_0}$, i.e. $1 - \frac{qp_0}{4} > 0$. Hence, $\Gamma\left(1 - \frac{qp_0}{4}\right)$ is finite.

On the other hand, for $p_0 \geq 0$, the function $(0,2) \ni \gamma \mapsto \Gamma\Big(1 - \frac{qp_0}{4}\Big)^{\frac{qp_0}{\gamma^2}}$ is positive and increasing as γ increases. Moreover, as $\gamma \searrow 0$, we have

$$\frac{1}{\Gamma\Big(1 - \frac{\gamma^2}{4}\Big)^{\frac{qp_0}{\gamma^2}}} \to \exp\Big\{-\frac{qp_0\,\gamma_{\mathrm{EM}}}{4}\Big\} > 0.$$

We conclude that for $q \in (1, \frac{4}{p_0})$,

$$\sup_{\gamma \in (0,2)} \mathbb{E}\Big[Y_\gamma^{\frac{qp_0}{\gamma^2}}\Big] < \infty.$$

□

It follows by Prop. 3.1.1 and (3.1) that for $p_0 \in [0, 4)$,

$$\mathbb{E}\left[\lim_{\gamma \searrow 0} Y_\gamma^{p_0/\gamma^2}\right] = \Gamma\left(1 - \frac{p_0}{4}\right) \exp\left(-\frac{\gamma_{EM}\, p_0}{4}\right). \tag{3.2}$$

For $p_0 < 0$, the story is different. There, we have that the function $(0,2) \ni \gamma \mapsto \Gamma\Big(1 - \frac{qp_0}{4}\Big)^{\frac{qp_0}{\gamma^2}}$ is positive and decreasing as γ increases. Therefore, the multiplicative inverse of this function has an asymptote at $\gamma = 2^-$. However, as we are interested in the behavior of γ close to 0, we can consider Prop. 3.1.1 for the same family of random variables, restricted to the parameters $\gamma \in (0,1)$. Then, choosing any $q > 1$ yields that

$$\sup_{\gamma \in (0,1)} \mathbb{E}\Big[Y_\gamma^{\frac{qp_0}{\gamma^2}}\Big] < \infty,$$

and hence

Theorem 3.1.2 For all $p_0 < 4$, the family $(Y_\gamma^{p_0/\gamma^2})_{\gamma \in (0,1)}$ is bounded in $L^q(\mathbb{P})$ for some $q > 1$. In particular, $(Y_\gamma^{p_0/\gamma^2})_{\gamma \in (0,1)}$ is uniformly integrable and (3.2) holds.

3.1.2 Large Deviations

In this section, we want to study the rare events of the random variable Y_γ^{p/γ^2} (or some scaling thereof) as $\gamma \searrow 0$. This will be done by establishing a so-called large deviation principle:

Definition 3.1.3 (Large Deviation Principle - [DS]) A family $(\mu_\gamma)_{\gamma \in (0,1)} \subseteq \mathcal{M}_1(\mathbb{R})$ of probability measures is said to satisfy a **large deviation principle** (LDP) with **rate function** $I : \mathbb{R} \to [0, +\infty]$ and **speed** $0 < \alpha_\gamma \nearrow \infty$, if for all Borel-measurable sets $A \in \mathcal{B}(\mathbb{R})$, we have

$$-\inf_{A^o} I \leq \underline{\lim}_{\,\gamma} \alpha_\gamma \log(\mu_\gamma(A)) \leq \overline{\lim}_{\,\gamma} \alpha_\gamma \log(\mu_\gamma(A)) \leq -\inf_{\overline{A}} I,$$

where the limits are taken as $\gamma \searrow 0$.

In order to achieve such estimates, we need to recall some definitions and results from the theory of large deviations. First and foremost:

Definition 3.1.4 (Log Moment Generating Function) Let X be a scalar-valued random variable. Its **log moment generating function** (log-MGF) is given by

$$\Lambda_X(\lambda) = \log \mathbb{E} \exp(\lambda X).$$

Thanks to Thm. 2.1.1, we have

$$\Lambda_{\frac{1}{\beta_\gamma} Y_\gamma^{p/\gamma^2}}(\lambda) = \log \int_0^\infty \exp\left(\frac{\lambda x^{-p/4}}{\beta_\gamma \Gamma\,(1 - \gamma^2/4)^{p/\gamma^2}} - x\right) \mathrm{d}x. \tag{3.3}$$

Definition 3.1.5 (Exponential Tightness) A family $(\mu_\gamma)_{\gamma \in (0,1)} \subseteq \mathcal{M}_1(\mathbb{R})$ of probability measures is said to be **exponentially tight**, if for all $L > 0$, there exists a compact set $K \subseteq \mathbb{R}$ with the property

$$\overline{\lim}_{\,\gamma \searrow 0} \gamma \log \mu_\gamma(\mathbb{R} \setminus K) \leq -L.$$

The idea to achieve a LDP for the measures $\mu_\gamma := (\frac{1}{\beta_\gamma} Y_\gamma^{p/\gamma^2})_* \mathbb{P}$ is to use Gärtner-Ellis theorem:

Theorem 3.1.6 (Gärtner-Ellis - [DZ], Sec. 4.5.3) Denote by Λ_γ the log-MGF of a real-valued random variable X_γ and suppose that the limit

$$\Lambda(\lambda) = \lim_{\gamma \searrow 0} \alpha_\gamma^{-1} \Lambda_\gamma(\alpha_\gamma \lambda) \tag{3.4}$$

exists for some $\lambda \in \mathbb{R}$ and is differentiable for such λ. Moreover, assume that (μ_γ) is exponentially tight. Then, (μ_γ) satisfies a LDP with rate function

$$\Lambda^* := \sup_\lambda (\lambda x - \Lambda(\lambda))$$

given by the **Legendre transform** of Λ (over the values where the limit (3.4) exists in $\mathbb{R}$), with speed α_γ.

Remark 3.1.7 In the proof of Thm. 3.1.6 on $\mathbb{R}$ (or more generally, in the finite-dimensional case), one can see that one first shows the upper bound in the LDP, and from it and the existence of (3.4), the exponential tightness follows. $\diamond$

In view of applying , we aim to get the asymptotics of $\frac{1}{\alpha_\gamma} \log I(\gamma) := \frac{1}{\alpha_\gamma} \Lambda_{\frac{1}{\beta_\gamma} Y_\gamma^{p/\gamma}}(\alpha_\gamma \lambda)$ for some $\alpha_\gamma \nearrow \infty$ as $\gamma \searrow 0$. We do this by applying Laplace's method (cf. [Olv], Ch. 3, Sec. 7):

Theorem 3.1.8 (Laplace's Method) Let $I(\gamma) := \int_a^b \exp(-\gamma^{-1} P(x)) Q(x) \, \mathrm{d}x$ for P, Q smooth and asymptotic polynomial growth (for the detailed assumptions, cf. [Olv]). Then, as $\gamma \searrow 0$:

$$I(\gamma) \sim \operatorname{sgn}(P'(a)) \frac{\gamma Q(a) \exp(-\gamma^{-1} P(a))}{P'(a)} \quad \text{for } P'(a) \neq 0,\ Q(a) \neq 0 \qquad \text{(Boundary Value Case)}$$

and

$$I(\gamma) \sim \sqrt{2\pi\gamma} \frac{Q(x_*) \exp(-\gamma^{-1} P(x_*))}{\sqrt{P''(x_*)}} \quad \text{for } x_* \text{ simple minimum of } P,\ Q(x_*) \neq 0 \qquad \text{(Inner Extremum Case)}$$

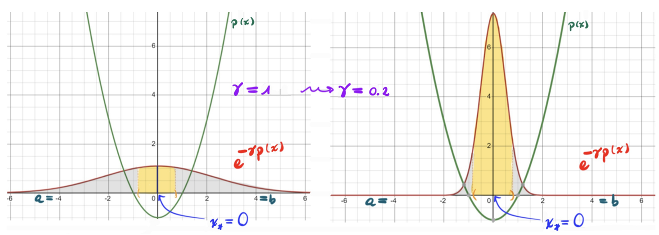

Figure 3.1: Inner Extremum Case of Laplace's Method - Values near Extremal Points contribute more to the Integral in the Limit $\gamma \searrow 0$.

In our case $Q = 1$ and using the asymptotics from the previous subsection, we can write $A := \frac{\alpha_\gamma}{\beta_\gamma} \lambda e^{-p\gamma_{EM}/4} < 0$, we have $P \sim A x^{-p/4} - x$. Take $p \in (0, 4)$. Extremal value candidates for P are given by solutions to the equation $P' \sim -\frac{p}{4} A x^{-p/4-1} - 1 = 0$, that is

$$x_* = \left(-\frac{p\,A}{4}\right)^{\frac{4}{p+4}} > 0,$$

whenever $\lambda < 0$ and $p > 0$. This leads to

$$P(x_*) \sim (-A)^{\frac{4}{p+4}} \left(\frac{p}{4}\right)^{-\frac{p}{p+4}} \frac{4-p}{4}.$$

Since

$$P''(x_*) \sim \frac{p(p+4)}{16} A \left(-\frac{p\,A}{4}\right)^{-\frac{p+8}{p+4}} > 0,$$

we get P attains a minimum at x_*. Applying the inner extremum case of Laplace's method yields as $\gamma \searrow 0$

$$
\begin{aligned}
I(\gamma) &\sim 4\sqrt{\frac{2\pi}{\exp(-\gamma_{EM}p/4)p(p+4)A}}\sqrt{\left(\frac{-4}{pA}\right)^{-\frac{p+8}{p+4}}}\exp\left(-\exp(-\gamma_{EM}p/4)(-A)^{\frac{4}{p+4}}\left(\frac{p}{4}\right)^{-\frac{p}{p+4}}\frac{4-p}{4}\right)\\
&= 2\cdot\exp\left(\frac{\left(-\frac{A}{e^{\frac{\gamma_E p}{4}}}\right)^{\frac{4}{4+p}}(p-4)}{2^{\frac{8}{4+p}}p^{\frac{p}{4+p}}}\right)\cdot\sqrt{-\frac{2^{\frac{p-4}{4+p}}\cdot e^{\frac{\gamma_E p}{4}}}{\left(-\frac{1}{Ap}\right)^{\frac{4}{4+p}}(4+p)}}\cdot\sqrt{\pi}\\
&= 2\cdot\exp\left(\frac{\left(-\frac{\lambda\alpha_\gamma}{\beta_\gamma\exp(\frac{p\gamma_{EM}}{2})}\right)^{\frac{4}{4+p}}(p-4)}{2^{\frac{8}{4+p}}p^{\frac{p}{4+p}}}\right)\cdot\sqrt{-\frac{2^{\frac{p-4}{4+p}}\pi\exp\left(\frac{p\gamma_{EM}}{4}\right)}{\left(-\frac{\beta_\gamma\exp(\frac{p\gamma_{EM}}{4})}{\alpha_\gamma\lambda p}\right)^{\frac{4}{4+p}}(4+p)}}.
\end{aligned}
$$

Taking the logarithm and multiplying by α_γ^{-1}, we see that if $\alpha_\gamma\beta_\gamma^{\frac{4}{p}} \sim \kappa$, we get

$$
\Lambda_{\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2}}(\lambda) \sim \left(\frac{(p-4)(-\lambda)^{\frac{4}{4+p}}}{2^{\frac{8}{4+p}}p^{\frac{p}{4+p}}\exp\left(\frac{2p\gamma_{EM}}{4+p}\right)}\right)\cdot\kappa^{-\frac{p}{4+p}}.
$$

On the other hand, if $\lambda > 0$, there is no extremal value candidate in $(0,\infty)$. We therefore proceed by using the boundary value case of the Laplace method to get

$$
I(\gamma) \sim \exp\left(\frac{\gamma_{EM}p}{4}\right),
$$

thus,

$$
\Lambda_{\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2}}(\lambda) \sim \frac{\gamma_{EM}p}{4\alpha_\gamma} \to 0,
$$

as $\gamma \searrow 0$. This then means that $\Lambda_{\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2}}(\lambda) \sim 0$. Optimizing over the non-trivial values of the expression inside the Legendre transform yields (cf. Thm. 3.1.6):

Theorem 3.1.9 (Semiclassical Large Deviation Principle)
Let $p \in (0,4)$, and choose $(\alpha_\gamma),(\beta_\gamma) \subseteq (0,\infty)$ with the following asymptotic behaviors as $\gamma \searrow 0$:

$$
\alpha_\gamma \nearrow \infty, \qquad \beta_\gamma \searrow 0, \qquad \alpha_\gamma\beta_\gamma^{\frac{4}{p}} \to \kappa
$$

for some $\kappa \in (0,\infty)$. Then, $\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2} \overset{\mathbb{P}}{\sim} \mu_\gamma$ satisfies a LDP with rate function $\Lambda^* = \Lambda^*_\kappa$ as

$$
\Lambda^*_\kappa(x) = \begin{cases} \frac{p}{4}\,x\left(\frac{4(4-p)}{(4+p)x}\,\frac{1}{2^{\frac{8}{4+p}}\,p^{\frac{p}{4+p}}\,\exp\left(\frac{2p\,\gamma_{EM}}{4+p}\right)}\,\kappa^{-\frac{p}{4+p}}\right)^{\frac{4+p}{p}} \sim \frac{1}{\kappa x^{4/p}}, & x \geq 0\\ +\infty, & x < 0 \end{cases}
$$

with speed α_γ.

Interpretation: Heuristically speaking, the rate function describes the asymptotic behavior

$$
\mathbb{P}\left(\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2} \approx x\right) \sim \exp(-\alpha_\gamma\Lambda^*_\kappa(x))
$$

for small values of γ. This means, in other words, that the closer the rate function is to zero, the more likely these values are. Conversely, the closer it is to infinity, the less likely they are. In that sense, negative x values of $\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2}$ remain unlikely as $\gamma \searrow 0$. This is in accordance to the fact that $\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2} \geq 0$ $\mathbb{P}$-a.s. On the other hand, positive x values of $\frac{1}{\beta_\gamma}Y_\gamma^{p/\gamma^2}$ become more typical, the larger x is (cf. Fig. 3.2).

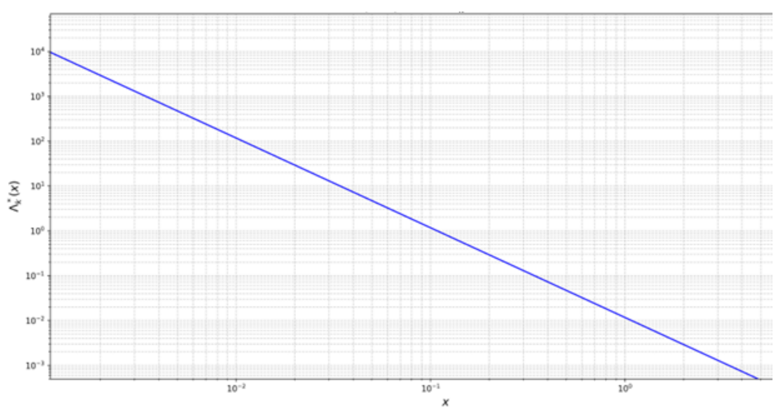

Figure 3.2: Semiclassical Rate Function Λ_κ^* for $\kappa = 2$, and $p = 1$.

Chapter 4

BPZ Equations: A Dynamical and Algebraic Viewpoint

In this chapter, we discuss connections of BPZ equations and other objects, and reveal the algebraic machinery of the BPZ equations. We start with their relation to Schramm-Loewner Evolutions, which describe how random curves develop over time in two-dimensions under a particular random noise; and conclude by giving a brief overview of the algebraic perspective of the BPZ equations.

4.1 BPZ Equations and SLEs

In this section, we connect BPZ equations (cf. Thm. 2.2.2) to so-called Schramm-Loewner Evolution (SLE) on the upper half-plane $\mathbb{H}$. Our setting begins in the classical context of the complex plane, where we derive a general BPZ equation based on the dynamics of the SLE. We then go to more abstract settings, in which we consider other types of noise dynamics. We conclude with a probabilistic interpretation of some correlation functions in LCFT.

Definition 4.1.1 (Loewner Equation, Chordal SLE - [Dub]) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $z \in \mathbb{C}$, $\kappa \geq 0$, and B_t be a standard $\mathbb{P}$-Brownian motion. The random ordinary differential equation (ODE) given by

$$\begin{cases} \frac{\partial}{\partial t} g_t(z) &= \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \\ g_0(z) &= z, \end{cases} \tag{4.1}$$

is called the **Loewner equation**. It is well-defined up to a random time τ_z. We refer to g_t as a **chordal SLE**(κ) **in** $\mathbb{H}$.

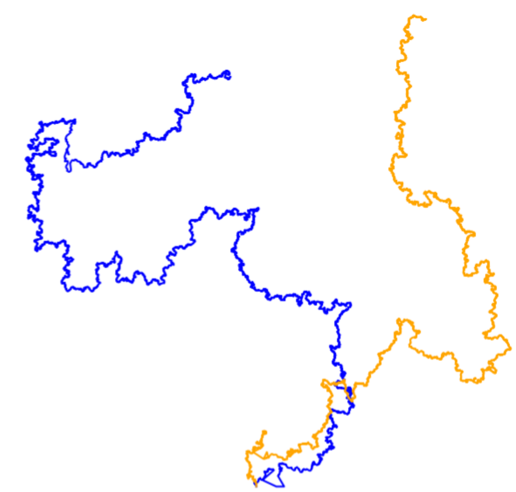

Figure 4.1: SLE (blue) and $\frac{\kappa}{1+x}$-Drifted SLE (orange) with $\kappa = 2$.

For each $t \geq 0$, the solution to (4.1) is a conformal isomorphism (in German, "schlicht", cf. [Loe]). For an introduction to SLEs, the reader is invited to read [Kem]. In this report, we shall consider a more general setting where the noise in the denominator of (4.1) is a semimartingale X defined by the stochastic differential equation (SDE)

$$dX_t = \kappa\, b(X_t)dt + \sqrt{\kappa}\, dB_t, \tag{4.2}$$

with initial condition $X_0 = 0$, where $b \in L^2(\lambda)$. The motivation behind this choice of noise is that it is equivalent to the one considered in (4.1) in the following sense: defining

$$\theta_t := \kappa\, b\big(X_t\big)$$

we can use Girsanov's theorem ([LG], Thm. 5.22) with

$$dW_t = dB_t + \theta_t\, dt,$$

in order to push the semimartgingale X into a (speed-changed) $\mathbb{Q}$-Brownian motion, for some equivalent measure $\mathbb{Q} \approx \mathbb{P}$ by doing a change of measure. Then, the SDE for X can be written as

$$dX_t = \sqrt{\kappa} dW_t$$

for a $\mathbb{Q}$-Brownian motion W, which is the noise considered in (4.1). More precisely, the change of measure is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda_\infty,$$

where

$$\Lambda_t = \exp\Big(-\int_0^t \kappa\, b\big(X_s\big)\, dB_s - \frac{1}{2}\int_0^t \kappa^2\, b\big(X_s\big)^2\, ds\Big)$$

is the stochastic exponential of θ. The integrability condition on b together with Novikov's condition ([LG], Thm. 5.23) ensure that the conditions of Girsanov's theorem are met. For more details, see ([LG], Sec. 5.5). In short, all this means that by using a change of measure, we can make the semimartingale driving noise look as the (scaled-time) Brownian motion considered in (4.1).

However, we did not include the drift $\kappa\, b(X_t)\, dt$ in vane:

Proposition 4.1.2 Let Φ be the unique solution to the parabolic PDE

$$\begin{cases} \partial_t \Phi(t,x,z) + \kappa\, b(x)\partial_x \Phi(t,x,z) + \frac{\kappa}{2}\, \partial_x^2 \Phi(t,x,z) \;=\; \frac{2}{\Phi(t,x,z)-x}, \\ \Phi(0,x,z) = z, \end{cases} \tag{4.3}$$

for $t \geq 0, x \in \mathbb{R}$ and a fixed parameter $z \in \mathbb{C}$. Then, $g_t(z) = \Phi(t, X_t, z)$ solves (4.1). Moreover,

$$dg_t(z) = \frac{2}{g_t(z) - X_t}\, dt + \sqrt{\kappa}\, \partial_x \Phi(t, X_t, z)\, dB_t.$$

Proof: We apply Itô's formula to $\Phi(t, X_t, z)$ on t and X_t, taking z as fixed. We get

$$dg_t(z) = d\Phi(t, X_t, z) = \Big[\partial_t \Phi(t,x,z) + \kappa\, b(x)\partial_x \Phi(t,x,z) + \frac{\kappa}{2}\, \partial_x^2 \Phi(t,x,z)\Big]\, dt + \sqrt{\kappa}\partial_x \Phi(t, X_t, z) dB_t.$$

Using (4.3), we get that the term between the brackets is precisely $\frac{2}{g_t(z)-X_t}$, yielding both claims. □

Remark 4.1.3 The operator on the left-hand side of the PDE (4.3) is the **(infinitesimal) generator** $\mathcal{L}$ of the process (t, X_t). It is precisely the operator which satisfies the property that $f(t, X_t)$ is a martingale if and only if $f \in \ker \mathcal{L}$, for a smooth and compactly supported function f. For more details, see [RY], Ch. VII, Sec. 2. ⋄

We now differenciate (4.1) with respect to z and get

$$\partial_t\big(g_t'(z)\big) = -\frac{2\, g_t'(z)}{\big(g_t(z) - X_t\big)^2}. \tag{4.4}$$

Noticing that the random process X does *not* depend on z, no additional stochastic term comes up in the derivative. Hence, we directly obtain the differential $dg_t'(z)$ by using Schwarz theorem on commuting derivatives

$$dg_t'(z) = -\frac{2\, g_t'(z)}{\big(g_t(z) - X_t\big)^2}\, dt.$$

Therefore, $g_t'(z)$ is a process of finite variation and thus has vanishing quadratic variation. On the other hand, we get using L. 4.1.2

$$d\langle g_t(z_i), g_t(z_j)\rangle = \kappa\,\partial_x\Phi(t, X_t, z_i)\partial_x\Phi(t, X_t, z_j)\,dt.$$

for $z_i, z_j \in \mathbb{C}$.

We exploit the power rule of polynomial differentiation together with a further application of Itô's product rule and (4.4) to get the following result:

Lemma 4.1.4 The process $G_t := \prod_{i=1}^n g_t'(z_i)^{\Delta_{\alpha_i}}$ satisfies

$$dG_t = -2G_t \sum_{i=1}^n \frac{\Delta_{\alpha_i}}{(g_t(z_i) - X_t)^2} dt.$$

In particular, G is a process of finite variation.

In a certain sense, G is self-replicating under the stochastic differentiation. Finally, using the above results, we get a connection between SLEs, local martingales, and BPZ equations:

Theorem 4.1.5 (SLE and BPZ Equation)
Let G_t as in L. 4.1.4, g_t the solution to (4.1), $z_1, ..., z_n \in \mathbb{C}$ points in the complex plane with $y_i = g_t(z_i)$ for $i = 1, ..., n$, let Φ be the solution to (4.3), and h be a smooth function.

Then, the process

$$M_t(z_1, ..., z_n) = G_t h(g_t(z_1), ..., g_t(z_n))$$

is a local martingale, if and only if h satisfies the BPZ equation

$$\left(\frac{\kappa}{2}\sum_{i,j=1}^n \partial_x\Phi(t, x, g_t^{-1}(y_i))\partial_x\Phi(t, x, g_t^{-1}(y_j))\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^n \frac{2}{y_i - x}\frac{\partial}{\partial y_i} - \sum_{i=1}^n \frac{2\Delta_{\alpha_i}}{(y_i - x)^2}\right) h(y_1, ..., y_n) = 0$$

for all $x \in \mathbb{R}$.

Remark 4.1.6 Whenever M is a local martingale, we have

$$\mathrm{d}M_t(z_1, ..., z_n) = \sqrt{\kappa} G_t \sum_{i=1}^n \partial_x\Phi(t, X_t, z_i)(\partial_i h)(g_t(z_1), ..., g_t(z_n))\,\mathrm{d}B_t,$$

which implies that M is a true martingale, if and only if for all $i = 1, ..., n$ we have

$$\int_0^{\tau_z} (\partial_x\Phi(t, X_t, z_i))^2 (\partial_i h)(g_t(z_1), ..., g_t(z_n))^2 \prod_{j=1}^n g_t'(z_j)^{2\Delta_{\alpha_j}}\,\mathrm{d}t < \infty$$

where τ_z is the lifetime of (4.1). ◇

Remark 4.1.7 In the case where $\sqrt{\kappa} = \frac{2}{\gamma}$ and Φ is normalized in the sense that

$$\partial_x\Phi(t, x, g_t^{-1}(y_i)) = \delta_{i,1}$$

we get (up to a multiplicative constant 2)

$$\left(\frac{1}{\gamma^2}\frac{\partial^2}{\partial y_1^2} + \sum_{i=1}^n \frac{1}{y_i - x}\frac{\partial}{\partial y_i} - \sum_{i=1}^n \frac{\Delta_{\alpha_i}}{(y_i - x)^2}\right) h(y_1, ..., y_n) = 0$$

which resembles that of Thm. 2.2.2. Moreover, note that all the right-hand side of (4.1) is doing is inserting singularities at the points $y_i = g_t^{-1}(z_i)$ for $i = 1, ..., n$. In addition, one can see from the calculations that lead to this result, that the mapping $x \mapsto \frac{1}{y-x}$ is a good choice in terms of structure, for its derivatives simply introduce more poles at the same point y (up to a multiplicative factor). ◇

Following the same steps as for the above result, we get:

Theorem 4.1.8 Consider the generalized Loewner evolution

$$\begin{cases} \frac{\partial}{\partial t} g_t(z) &= f(g_t(z), X_t), \\ g_0(z) &= z, \end{cases}$$

where the noise is given by the semimartingale

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

with initial condition $X_0 = 0$. The process

$$M_t(z_1, ..., z_n) = G_t h(g_t(z_1), ..., g_t(z_n))$$

is a local martingale if and only if

$$\Bigg(\frac{\sigma(x)^2}{2} \sum_{i,j=1}^{n} \partial_x \Phi(t, x, g_t^{-1}(y_i)) \partial_x \Phi(t, x, g_t^{-1}(y_j)) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n} f(\Phi(t, x, g_t^{-1}(y_i)), x) \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} \Delta_{\alpha_i} (\partial_1 f)(\Phi(t, x, g_t^{-1}(y_i)), x) \Bigg) h(y_1, ..., y_n) = 0$$

where Φ is the solution to

$$\begin{cases} \partial_t \Phi(t, x, z) + b(x) \partial_x \Phi(t, x, z) + \frac{\sigma^2(x)}{2} \partial_x^2 \Phi(t, x, z) &= f(\Phi(t, x, z), x), \\ \Phi(0, x, z) = z. \end{cases}$$

In this case, we have

$$dM_t = \sigma(X_t)\, G_t \sum_{i=1}^{n} \partial_x \Phi(t, X_t, z_i)(\partial_i h)(g_t(z_1), ..., g_t(z_n)) dB_t.$$

The following is a direct consequence of martingales having constant expectation:

Corollary 4.1.9 (Mean Value Representation of Correlation Functions)
Whenever M is a true martingale, we have the following mean-value representation of h:

$$h(z_1, ..., z_n) = \mathbb{E}\left[h(g_t(z_1), ..., g_t(z_n)) \prod_{i=1}^{n} ((\partial_z \Phi)(t, X_t, z_i))^{\Delta_{\alpha_i}} \right].$$

Finally, a reinterpretation of ([RS], Thm. 2) shows that 3-point correlation functions can be interpreted as probabilities. More precisely, let $z_1 \in \mathbb{H}$ be a point in the upper half plane. We aim to show the following result:

Theorem 4.1.10 (SLE-Probabilistic Interpretation of BPZ - [RS], Thm. 2)
The probability that a SLE from $z_2 = 0$ to $z_3 = \infty$ (as an element of the one-point compactification of the complex plane) passes to the left of z_1 is equal to $\langle V_{-\frac{\gamma}{2}}(z_1) V_{\alpha_2}(0) V_{\alpha_3}(\infty) \rangle_{\mathbb{H}, \frac{\gamma}{2}}$.

Remark 4.1.11 A discrete analogue of the SLEs could be the meromorphic dynamical system

$$\begin{cases} g_{n+1}(z) = f(g_n(z), X_n), \\ g_0(z) = z \end{cases}$$

where $f : \mathbb{C} \to \mathbb{C}$ is a meromorphic function and the noise X satisfies

$$\begin{cases} X_{n+1} = X_n + b(X_n) + \sigma(X_n)\xi_n, \\ X_0 = 0, \end{cases}$$

for an iid sequence (ξ_n). Note that if b and σ are measurable, we can view X as a Markov chain $X_{n+1} = F(X_n, \xi_n)$ for some measurable F. Questions such as determining properties of its (random) trajectories, Fatou and Julia sets, as well as convergence to the continuous system (4.1) could be a potential extension of this report. ◇

4.2 BPZ Equations and Virasoro Algebras

The goal of this section is to explain the algebraic machinery underlying the BPZ equations. This section follows definitions and examples from [Hum], [Sch], and [JZ].

Definition 4.2.1 (Lie Algebra, Central Extension) A **Lie algebra** $\mathfrak{g}$ over a field $\mathbb{K}$ is a linear space equipped with a bilinear map called the **Lie bracket** $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying:

1. Antisymmetry: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.
2. Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

An exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is called a **central extension** of $\mathfrak{g}$ by $\mathfrak{a}$, if $[\mathfrak{a}, \mathfrak{h}] = 0$, that is $[X, Y] = 0$ for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{h}$. Here, we identify $\mathfrak{a}$ with the corresponding subalgebra of $\mathfrak{h}$.

Example 4.2.2 (Witt and Virasoro Algebra) The **Witt algebra** is the Lie algebra spanned by the polynomial vector fields L_n on the unit circle $\mathbb{S}^1$ defined as

$$L_n := z^{1-n} \frac{d}{dz}$$

endowed with the Lie bracket induced by composition. It is easy to see that

$$[L_n, L_m] = (n - m)L_{n+m}, \quad \forall\, n, m \in \mathbb{Z}.$$

The **Virasoro algebra** Vir is the unique central extension of the Witt algebra (cf. [**?**], Thm. 5.1), that is Vir $= \mathrm{W} \oplus \mathbb{C}Z$, with Lie bracket

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0} \frac{n}{12} \left(n^2 - 1\right) Z \tag{4.5}$$

and $[L_n, Z] = 0$ for all integers $n, m \in \mathbb{Z}$. △

Remark 4.2.3 Alternatively (see, for instance [BPZ]), the operators (L_n) are viewed as the coefficients in the formal Laurent expansion of the so-called **stress-energy tensor**

$$T = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}.$$

◇

Definition 4.2.4 A **highest-weight state** of the Virasoro algebra is a vector $|\Delta\rangle$ that is zeroed out by all positive generator modes and diagonalizes L_0. Equivalently, it is the state associated with a primary field V_Δ of conformal dimension Δ, defined by

$$L_n V_\Delta = 0 \text{ for all } n > 0,$$
$$L_0 V_\Delta = \Delta V_\Delta.$$

All other states in the same irreducible representation can be obtained by applying negative generator modes $L_{n<0}$. The **Verma module** $\mathcal{V}_\Delta$ is the representation which is spanned by all **descendant states**, that is, states of the form $L_{-n_1} L_{-n_2} \cdots L_{-n_k} V_\Delta$ with $n_i > 0$. A basis is given by products of the form $L_{-n_1} L_{-n_2} \cdots L_{-n_k} |\Delta\rangle$ with $1 \le n_1 \le n_2 \le \cdots \le n_k$. The **level** N of such a state is given by $N = \sum_{i=1}^{k} n_i$. If $\mathcal{V}_\Delta$ has no proper submodules, it said to be **irreducible**.

Definition 4.2.5 (Null-Vectors and Degeneracy) A **null-vector** V_Δ is a highest-weight descendant state. If there exists such a state in the Verma module $\mathcal{V}_\Delta$, the latter is said to be **reducible**. This implies that $\mathcal{V}_\Delta$ contains a proper submodule, generated by that null-vector. The module $\mathcal{V}_\Delta$ is then said to be **degenerate of type** (m, n), where $m, n \in \mathbb{N}$, if the smallest degree null-vector is of degree $N = mn$. In this case, one can quotient out the submodule generated by V_Δ in order to get an irreducible module.

Example 4.2.6 ([JZ]) The parameters $m, n \in \mathbb{N}$ give rise to two solutions for the conformal weights $\Delta_{m,n}$, which are the zeros of the Kac determinant

$$\Delta_{(m,n)} = \frac{Q^2 - \left(mb + nb^{-1}\right)^2}{4}, \quad m, n \in \mathbb{N}.$$

For instance, if the null-vector is at the level $N = 1$, then

$$L_n \left(L_{-1} V_\Delta\right) = 0, \quad n > 0.$$

For $n \geq 2$, the above condition is trivial, as V_Δ is only level 1, so it gets zeroed out. For $n = 1$, we have

$$0 = L_1 L_{-1} V_\Delta = 2 L_0 V_\Delta = 2\Delta V_\Delta$$

Thus, $V_\Delta = 1$ and has zero conformal dimension.

Suppose that the level $N = 2$ descendant field $\tilde{V} = \left(C_{1,1} L_{-1}^2 + C_2 L_{-2}\right) V_\Delta$ is a null-vector. Then, $L_n \tilde{V} = 0$ for $n \geq 1$. The non-trivial constraints are

$$\begin{aligned} 0 &= L_1 \tilde{V} = \left((4\Delta + 2) C_{1,1} + 3 C_2\right) L_{-1} V_\Delta \\ 0 &= L_2 \tilde{V} = \left(6\Delta C_{1,1} + \left(4\Delta + \frac{c}{2}\right) C_2\right) V_\Delta \end{aligned}$$

Therefore, we require

$$\begin{vmatrix} 4\Delta + 2 & 3 \\ 6\Delta & 4\Delta + \frac{c}{2} \end{vmatrix} = 0$$

which gives two solutions

$$\begin{aligned} \Delta_{(2,1)} &= -\tfrac{1}{2} - \tfrac{3}{4} b^2, \quad & \tilde{V}_{(2,1)} &= \left(\tfrac{1}{b^2} L_{-1}^2 + L_{-2}\right) V_{\Delta_{(2,1)}}, \\ \Delta_{(1,2)} &= -\tfrac{1}{2} - \tfrac{3}{4b^2}, \quad & \tilde{V}_{(1,2)} &= \left(b^2 L_{-1}^2 + L_{-2}\right) V_{\Delta_{(1,2)}}. \end{aligned}$$

Thanks to the **conformal Ward identities**

$$\left\langle T(z) \prod_{i=-1}^{r+1} V_{\Delta_i}(z_i) \right\rangle = \sum_{i=-1}^{r+1} \left(\frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z - z_i)^2} \right) \left\langle \prod_{i=-1}^{r+1} V_{\Delta_i}(z_i) \right\rangle,$$

we can derive the BPZ equation on an $(r + 3)$-point correlation function

$$\left[\frac{1}{b^2} \frac{\partial^2}{\partial z_0^2} + \sum_{i \neq 0} \left(\frac{1}{z_0 - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z_0 - z_i)^2} \right) \right] \left\langle V_{\Delta_{(2,1)}}(z_0) \prod_{i \neq 0} V_{\Delta_i}(z_i) \right\rangle = 0,$$

where $\Delta_0 = \Delta_{(2,1)}$. The above equation corresponds to those of the form in Thm. 2.2.2. $\triangle$

As explained in [Cer], Appendix B.2, one can actually translate the representation-theoretic results into linear differential equations. This procedure boils down to using the *local* Ward identities to express the four-point correlation function in terms of one of its descendants, and then inverting the global Ward identities in order to reduce the number of descendants through a matrix equation.

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