Technical University of Munich

Department of Mathematics

School of Computation, Information and Technology

Borel Functional Calculus and the Spectral Theorem

Bachelor Thesis

by

Alejandro Morera Alvarez

Advisor:	Prof. Dr. Folkmar Bornemann
Due Date:	Mathematics [03. 07. 2023]
	Engineering Science [01. 10. 2023]

Mathematik: Hiermit erkläre ich, dass ich die Bachelorarbeit selbstständig und nur mit den angegebenen Hilfsmitteln angefertigt habe.

Ingenieurwissenschaften: Hiermit versichere ich, dass ich die Abschlussarbeit im Studiengang B.Sc. Ingenieurwissenschaften selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Garching bei München, den 29. Juni 2023.

Special Thanks

I would like to thank my advisor, Prof. Bornemann, for giving me an interesting topic, as well as leading me through the way during my Engineering Science/Mathematics Bachelor Thesis. I have not only learned new results while writing this work, but also to be more efficient in my mathematical writing. In addition, I would like to thank all my professors in mathematics, for inspiring my change in direction towards more abstraction. Last but not least, I would like to thank my family and friends for all their support before and during my studies: their unconditional presence has helped me push through every obstacle that has been put in my way.

Zusammenfassung

Wir geben einen Beweis für den Spektralsatz für beschränkte Operatoren wie in *Mathematical Quantization* von N. Weaver ([W]) mithilfe weiteren Quellen. Um Letzteres zu machen, definieren wir ein Funktionalkalkül für beschränkte, Borel-messbare Funktionen basierend auf einem für Polynome; und bauen zusätzlich ein Integral bezüglich eines Spektralmaßes. Hierfür zeigen wir andere wichtige Resultate aus der Funktionalanalysis und Topologie.

Abstract

We present the proof of the spectral theorem for bounded operators as in *Mathematical Quantization* by N. Weaver ([W]) with the help of other sources. In order to do this, we define a functional calculus for bounded, Borel-measurable functions based on one for polynomials; and we also construct an integral with respect to a spectral measure. To achieve these goals, we prove other important results from the branches of functional analysis and topology.

Introduction and Problem Setting

Given a self-adjoint operator $M = M^* \in \mathcal{L}(H)$ over a separable Hilbert space H and a bounded, Borel-measurable function $f : (\Omega, \Sigma) \to (\mathbb{K}, \mathcal{B}(\mathbb{K}))$, the goal of this thesis is to arrive at a spectral decomposition for $f(M) \in \mathcal{L}(H)$ of the form

$$f(M) = \int_{\sigma(M)} f \, dP = \mathcal{U}\mathcal{M}_f\mathcal{U}^* \tag{0.1}$$

which in the particular case $f = \mathrm{id}$ yields

$$M = \mathcal{U}\mathcal{M}_{\mathrm{id}}\mathcal{U}^*$$

where $\mathcal{M}_f : L^2(\sigma(M); \chi) \to L^2(\sigma(M); \chi)$, $\mathcal{M}_f g(\lambda) = f(\lambda)g(\lambda)$ is the corresponding multiplication operator of f, the map $\mathcal{U} : L^2(\sigma(M); \chi) \to H$ is an isometric isomorphism, and χ is a Hilbert bundle. In short, the goal is to build and understand the following commuting diagram:

$$\begin{tikzcd}
L^2(\sigma(M); \chi) \arrow[r, "\mathcal{M}_f"] \arrow[d, "\mathcal{U}"'] & L^2(\sigma(M); \chi) \arrow[d, "\mathcal{U}"] \\
H \arrow[r, "f(M)"] & H
\end{tikzcd}$$

This is called the **spectral theorem** for bounded, self-adjoint operators.
In order to construct such a decomposition, we first build a functional calculus, i.e. a mapping $f \mapsto f(M)$, starting from one for polynomials $p \in \mathbb{K}[x]\big|_{\sigma(M)}$, followed by one for continuous functions $f \in \mathcal{C}(\sigma(M))$, and finally one for bounded, Borel-measurable functions $f \in \mathcal{B}(\sigma(M))$. The latter is what we call **Borel functional calculus**. The extension procedure is summarized in the following diagram:

$$\begin{tikzcd}
 & \mathbb{K}[x]\big|_{\sigma(M)} \arrow[d, hook] \arrow[dr, "\gamma"] & & \\
 & \mathcal{C}(\sigma(M)) \arrow[r, "\Gamma"] \arrow[d, hook] & \mathcal{L}(H) \arrow[d, hook] & \\
\mathcal{B}(\sigma(M)) \arrow[r, hook, "\Phi"] \arrow[rrr, bend right, "\hat{\Gamma}"'] & \mathcal{C}(\sigma(M))'' \arrow[r, "\Gamma''"] & \mathcal{L}(H)'' \arrow[r, two heads, "P_{\mathcal{L}(H)}"] & \mathcal{L}(H)
\end{tikzcd}$$

Then, by using the above result in the special case $f = \mathbb{1}_S \mapsto \mathbb{1}_S(M)$, we shall construct a so-called spectral measure. After developing theory on the latter notion, we will build up a representation of $f(M)$ in integral form. In the last chapter, we prove the spectral theorem and arrive at the expression (0.1).
Every notion that was unfamiliar to the reader in this brief overview will be defined within the actual content of this thesis. As the reader is (hopefully) excited to read and understand the above, let us proceed without further ado!

Contents

1. Finite-Dimensional Spectral Theorem

From linear algebra, we know that for a finite-dimensional Hilbert space H over $\mathbb{C}$, we can identify $H \cong \ell^2(\{1, ..., n\})$. The well-known **spectral theorem** for self-adjoint operators over finite-dimensional Hilbert spaces (cf. version in [K], Cor. 19.15b) states that for any self-adjoint operator $M = M^* \in \mathcal{L}(H)$, there exists an isometric isomorphism $U \in \mathcal{L}(H)$ and a diagonal operator $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_n) \in \mathcal{L}(H)$ where $\lambda_1, ..., \lambda_n \in \mathbb{R}$ (not necessarily pairwise distinct), such that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\Lambda"] \arrow[d, "U"'] & H \arrow[d, "U"] \\ H \arrow[r, "M"'] & H \end{tikzcd}$$

The set $\sigma(M) := \{\lambda_1, ..., \lambda_n\}$ is called the **spectrum** of M.
We can look at the above result from a different point of view. For $\lambda \in \sigma(M)$, the map $\Lambda_\lambda := \operatorname{diag}(\llbracket \Lambda_{kk} = \lambda \rrbracket)_{k=1}^n \in \mathcal{L}(H)$ is clearly an orthogonal projection, since $\Lambda_\lambda^2 = \Lambda_\lambda$ and $\Lambda_\lambda^* = \Lambda_\lambda$. The family $(\Lambda_\lambda)_{\lambda \in \sigma(M)}$ induces three important decompositions. Firstly, let $H_\lambda := \operatorname{ran}(\Lambda_\lambda)$. By the spectral theorem, the set $\{H_\lambda \mid \lambda \in \sigma(M)\}$ forms a partition of H in the sense that $H_\lambda \cap H_\mu = \{0\}$ for $\lambda \neq \mu$, and $H = \langle H_\lambda \mid \lambda \in \sigma(M) \rangle$. Thus,

$$H = \bigoplus_{\lambda \in \sigma(M)} H_\lambda \tag{1.1}$$

Secondly, we can rewrite the diagonal operator Λ from the spectral theorem in the form of a block decomposition,

$$\Lambda = \sum_{\lambda \in \sigma(M)} \lambda \Lambda_\lambda$$

Thirdly, conjugating each Λ_λ under the isometric isomorphism U yields another orthogonal projection, $P_\lambda := U \Lambda_\lambda U^*$. This gives a different way to write the decomposition $M = U \Lambda U^*$ as

$$M = \sum_{\lambda \in \sigma(M)} \lambda P_\lambda$$

The last equation opens the door to a **functional calculus**, i.e. a map $f \mapsto f(M)$ for a chosen function space. For the next theorem, we need a notion which will be introduced in the next chapter (cf. Def. 2.10). However, at this point of the thesis, it suffices for the reader to simply appreciate the structure-preserving nature of the map stated in the following claim.

Theorem 1.1 (Finite-Dimensional Functional Calculus). *Given a self-adjoint operator $M \in \mathcal{L}(H)$ over a finite-dimensional Hilbert space H, the map*

$$\varphi : \ell^\infty(\sigma(M)) \to \mathcal{L}(H), f \mapsto \varphi(f) = f(M) := \sum_{\lambda \in \sigma(M)} f(\lambda) P_\lambda$$

defines a unital, algebra $\star$-homomorphism.

Proof. The map φ is clearly an algebra homomorphism, since for any $\alpha \in \mathbb{C}$ and $f, g \in \ell^\infty(\sigma(M))$ we have $(f + \alpha g)(\lambda) = f(\lambda) + \alpha g(\lambda)$ which implies the linearity of the whole map, and multiplicativity comes from $(fg)(\lambda) = f(\lambda)g(\lambda)$ together with $P_\lambda = P_\lambda^2$. The unital property is also simple:

$$\varphi(1) = \sum_{\lambda \in \sigma(M)} P_\lambda = U\Big(\sum_{\lambda \in \sigma(M)} \Lambda_\lambda\Big)U^* = U\mathrm{id}U^* = UU^* = \mathrm{id}$$

Finally, the $\star$-property follows from orthogonal projections being self-adjoint and the (anti)linearity of taking adjoints,

$$\varphi(\overline{f}) = \sum_{\lambda \in \sigma(M)} \overline{f(\lambda)} P_\lambda^* = \Big(\sum_{\lambda \in \sigma(M)} f(\lambda) P_\lambda\Big)^* = \varphi(f)^*$$

□

Furthermore, consider the disjoint union induced by (1.1):

$$\chi = \bigcup_{\lambda \in \sigma(M)} \{\lambda\} \times H_\lambda$$

We will generalize the above construction in the next chapter as so-called Hilbert bundles (cf. Def. 2.16). Moreover, define

$$\ell^2(\sigma(M); \chi) := \{f : ||f||_2^2 := \sum_{\lambda \in \sigma(M)} ||f(\lambda)||_{H_\lambda}^2 < \infty \text{ and } f(\lambda) \in H_\lambda, \forall \lambda \in \sigma(M)\}$$

where the first condition was only included here so that the reader can compare this definition to the general one given in Def. 2.16. Additionally, observe how the structure of χ is reflected in the second condition imposed on the images of the elements of $\ell^2(\sigma(M); \chi)$. Also, we note there is a natural isometric isomorphism $\mathcal{U}$ for $\ell^2(\sigma(M); \chi) \cong H$ established by

$$f \mapsto \sum_{\lambda \in \sigma(M)} f(\lambda) e_\lambda$$

where the set $(e_\lambda)_{\lambda \in \sigma(M)}$ is the (ordered) basis induced by the columns of the operator $U : H \to H$. Finally, with the multiplication operator corresponding to f, i.e. the operator $\mathcal{M}_f : \ell^2(\sigma(M); \chi) \to \ell^2(\sigma(M); \chi), g \mapsto \mathcal{M}_f g = fg$ defined by $(fg)(\lambda) := f(\lambda)g(\lambda)$, we get another representation of the functional calculus defined in Thm. 1.1,

$$f(M) = \sum_{\lambda \in \sigma(M)} U f(\lambda) \Lambda_\lambda U^* = \mathcal{U}\mathcal{M}_f\mathcal{U}^*$$

thereby yielding the following commutative diagram:

$$\begin{tikzcd}
\ell^2(\sigma(M); \chi) \arrow[r, "\mathcal{M}_f"] \arrow[d, "\mathcal{U}"'] & \ell^2(\sigma(M); \chi) \arrow[d, "\mathcal{U}"] \\
H \arrow[r, "f(M)"] & H
\end{tikzcd}$$

In particular, for $f = \mathrm{id}$, we get the decomposition:

$$M = \mathcal{U}\mathcal{M}_{\mathrm{id}}\mathcal{U}^*$$

We generalize this new vantage point of the spectral theorem in Chapter 7.

2. Ouverture

This chapter serves the purpose of introducing certain notions and results which will be central to the development of the rest of this thesis. As can be seen in Appendix B, this section marks the beginning of several entrance points to the road towards the bounded spectral theorem.
Let X be a nonempty set. Throughout this thesis, we shall denote by $\mathcal{N}_x \subseteq \mathcal{O}$ the neighborhood system of an element $x \in X$ with respect to a given topology $\mathcal{O}$. Moreover, any Hilbert space H will be taken over $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If a specific field $\mathbb{K}$ is not chosen, one can either take $\mathbb{R}$ or $\mathbb{C}$, whereby the real case will follow as a special case of the complex one. Finally, we shall denote the predual of a dual space X as $'X$.

2.1. Topology

We begin with some basic definitions and gradually get to more meaningful results that will play an important role in later chapters.

Definition 2.1 (Weaker and Stronger Topologies)**.** Let $(X, \mathcal{O})$ and $(X, \mathcal{O}')$ be two topological spaces over the same ground set X. If $\mathcal{O} \subseteq \mathcal{O}'$, the topology $\mathcal{O}$ is said to be **weaker** than $\mathcal{O}'$, and $\mathcal{O}'$ is said to be **stronger** than $\mathcal{O}$.

If we endow X with the discrete topology, i.e. the *strongest* topology on X, $\mathcal{O}_X := \mathcal{P}(X)$, then all functions $f \in Y^X$ are continuous. A more interesting concept would be: given a family of functions $\mathcal{F} \subseteq Y^X$, we would like to construct the *weakest* topology on X such that every $f \in \mathcal{F}$ is continuous. Does it even exist? If so, which properties does it have?
For a given subset $\varsigma \subseteq \mathcal{P}(X)$, we shall denote by $\langle \varsigma \rangle$ the weakest topology containing the sets in ς. In that sense, the topology $\langle \varsigma \rangle$ is generated by the collection ς. Choosing an appropriate ς yields an answer to our first question. The second one will be answered throughout the rest of this section.

Proposition 2.2 (Existence and Uniqueness of the Initial Topology)**.** Let X be a nonempty set and $(Y, \mathcal{O}_Y)$ be a topological space. Given a family of functions $\mathcal{F} \subseteq Y^X$, there is a smallest topology on X such that each $f \in \mathcal{F}$ is continuous.

Proof. ([M], Prop. 2.4.1.) *Existence:* Since we want all of the $f \in \mathcal{F}$ to be continuous, we take $\varsigma := \{f^{-1}(U) \mid f \in \mathcal{F}, U \in \mathcal{O}_Y\}$ as a subbasis, and let $\mathcal{O}_{\mathcal{F}} := \langle \varsigma \rangle$ be its corresponding generated topology. Because $\varsigma \subseteq \mathcal{O}_{\mathcal{F}}$, we have that all elements $f \in \mathcal{F}$ are continuous.
Minimality: Assume that $\mathcal{O}$ is a topology on X such that all the $f \in \mathcal{F}$ are continuous. Then, $\varsigma \subseteq \mathcal{O}$, and hence, $\mathcal{O}_{\mathcal{F}} = \langle \varsigma \rangle \subseteq \mathcal{O}$.

$\square$

Definition 2.3 (Initial Topology)**.** Let X be a nonempty set and $(Y, \mathcal{O}_Y)$ a topological space. Given a family of functions $\mathcal{F} \subseteq Y^X$, we define the **initial topology** on X with respect to $\mathcal{F}$ as the weakest topology which makes all of the functions $f \in \mathcal{F}$ continuous.

We define two special instances of initial topologies for a normed space. Regardless of the name, the important fact is that we now have *less* open sets than in the topology induced by the norm, to which we shall refer to as the **strong topology**. Some consequences of having weaker topologies are highlighted in the introduction of Sec. 4.1.

Definition 2.4 (Weak and Weak* Topology - [M], Def. 2.5.1, 2.6.1)**.** The **weak topology** on a normed space X is the initial topology with respect to the collection of all its functionals, i.e. $\mathcal{F} = X'$. Similarly, the **weak* topology** on a normed space X is the initial topology with respect to the collection of functions $\mathcal{F} = {}'X$ interpreted as a subset of the bidual of ${}'X$, i.e. ${}'X \subseteq X'$ via the canonical embedding $\iota : {}'X \hookrightarrow X'$ (cf. [B], Thm. 3.28).
Note: In order for a normed space X to have a weak* topology, it must be a dual space.

For our purposes, we require an important characterization fulfilled by initial topologies. In order to state this, we will introduce nets, which are a generalization of sequences. Recall that sequences are simply maps $a : \mathbb{N} \to X$. An idea to abstract them is to take *any* directed set $(I, \leq)$ instead of $(\mathbb{N}, \leq_{\mathbb{N}})$ with $\leq_{\mathbb{N}}$ being the usual ordering of $\mathbb{N}$.

Definition 2.5 (Net and Net Convergence - [Ba], Def. 5.12)**.** A **net** is a map $x : I \to X, a \mapsto x_a$, where $(I, \leq)$ is a directed set and $(X, \mathcal{O})$ is a topological space. We shall denote nets by $(x_a)_{a\in I}$ or simply as (x_a).
A net $(x_a) \subseteq X$ is said to **converge** to $x \in X$, if for any neighborhood $U \in \mathcal{N}_x$ of x there is an element $A \in I$, such that whenever $a \geq A$ holds, it follows $x_a \in U$. One writes $x_a \to x$ or $x = \lim_a x_a$.

Remark 2.6. Instead of having to check the whole neighborhood system $\mathcal{N}_x$ of a given point $x \in X$ for the convergence condition in Def. 2.5, it suffices to show the statement holds for those sets S in a subbasis ς of the topology which contain x (cf. [M], Prop. 2.1.15).

Lemma 2.7 (Net Characterization for Continuity - [M], Prop. 2.1.21)**.** Let $f : X \to Y$ be a function between two topological spaces, $(x_a) \subseteq X$ a net, and $x \in X$. Then, f is continuous on X if and only if $f(x_a) \to f(x)$ whenever (x_a) is a net in X converging to an x in X.

Note: The above result is *not* true if "net" is changed to "sequence." Indeed, in general topological spaces, **continuity** of a function f, i.e. $\forall U \in \mathcal{O}_Y : f^{-1}(U) \in \mathcal{O}_X$ *implies* **sequential continuity**, i.e. $f(x_n) \to f(x)$ whenever (x_n) is a sequence in X converging to an x in X; but the converse is not true. Other properties that sequences lack are closures being precisely sequential-closures and Hausdorff spaces being characterized by nets having unique limits (cf. Rmk. 4.7). The absence of these properties comes from a space not being first-countable (cf. [Ba], Exa. 5.6). These and other phenomena motivate the use of nets rather than sequences.
Finally, using the definition of the initial topology, we get a result that *characterizes* the convergence of a certain net $(x_a) \subseteq X$ by using the convergence of the nets $(f(x_a)) \subseteq Y$ for all $f \in \mathcal{F} \subseteq Y^X$.

Theorem 2.8 (Initial Topology and Nets)**.** *Let X be a nonempty set, $(Y, \mathcal{O}_Y)$ be a topological space and $\mathcal{F} \subseteq Y^X$ be a family of functions. Endow X with $\mathcal{O}_{\mathcal{F}}$, the initial topology on X with respect to $\mathcal{F}$. Then,*

$$x_a \to x \iff \forall f \in \mathcal{F} : f(x_a) \to f(x)$$

for any $x \in X$.

Proof. ([M], Prop. 2.4.4). The forward direction follows easily: given a convergent net $x_a \to x$, since X is endowed with the initial topology, all of the $f \in \mathcal{F}$ are continuous and thus, by L. 2.7, we have for all $f \in \mathcal{F}$ that $f(x_a) \to f(x)$.
For the converse direction, take some $f \in \mathcal{F}$ and $U \in \mathcal{N}_{f(x)}$. By the convergence assumption, there is some $A_{f,U} \in I$ with the property $x_a \in f^{-1}(U)$ whenever $a \geq A_{f,U}$. Since ς from Prop. 2.2 is a subbasis for the initial topology, it follows by Rmk. 2.6 that $x_a \to x$.

□

The above results have been developed in order to understand the following example and special cases of it in Chapter 4.

Example 2.9. Let X be a normed space. In ([B], Def. 3.33), weak* convergence of a *sequence* $(x'_n) \subseteq X'$ is *defined* as $x'_n \xrightarrow{w*} x' \iff \forall x \in X : x'_n(x) \to x'(x)$. With Thm. 2.8, we see how the analogous convergence notion for nets instead *defines* the weak* topology itself:
We again identify X as a subset of X''. The **weak* topology** on $X' \subseteq \mathbb{K}^X$ is the initial topology on X' with respect to $\mathcal{F} := X$. This means that this topology is the smallest topology that makes all the $x \in X \subseteq X''$ continuous. By Thm. 2.8, it holds that the weak* topology is *precisely* the one induced by the convergence notion

$$x'_a \xrightarrow{w*} x' \iff \forall x \in X : x'_a(x) \to x'(x)$$

Analogously, the **weak topology** on X is the one induced by the net-equivalent of the weak convergence definition seen in ([B], Def. 3.33).

2.2. Functional Analysis

In order to appropriately connect two algebraic structures, we need special maps between them so that the fundamental properties of the former are preserved in the latter. Such maps are called homomorphisms. Confusion may arise due to the fact that their definition depends on whether we have groups, algebras, modules, etc. Note that regardless of the underlying algebraic structure, there is always a common ground, which we shall adopt as a metadefinition:

A homomorphism is a map between two algebraic structures of the same type which is compatible with the structure's axioms; i.e. its image preserves the algebraic essence of the domain in the codomain.

Definition 2.10 (Algebra Homomorphism - [W], pp. 53, 54)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two conjugation-closed algebras with unit over a field $\mathbb{K}$. A map $\Phi : \mathcal{A} \to \mathcal{B}$ is called an **algebra $\star$-homomorphism**, if it is an algebra homomorphism (i.e. if it is linear and preserves multiplication) and it preserves the action of conjugation: $\Phi(f^*) = \Phi(f)^*$. We shall refer to the latter property as the **$\star$-property**. An algebra homomorphism is called **unital**, if $\Phi(1) = 1$. We say that the map Φ has the **unital property**.

We present three approximation results which will be useful in later chapters. Firstly, in Chapter 4 we will need a lemma that lets us approximate any compact operator over a Hilbert space with a sequence of finite-dimensional range operators. The proof relies on Hilbert spaces having projections onto any of its closed subspaces.

Lemma 2.11 (Approximation of Compact Operators)**.** Let H be a Hilbert space. Then,

$$\mathcal{K}(H) = \overline{\mathcal{F}(H)}$$

Proof. ([F], Ex. 32c.) The reverse inclusion follows from ([B], Cor. 2.15), because H is a special case of a Banach space.
Now, let $M \in \mathcal{K}(H)$ and take $\epsilon > 0$. By compactness of M, there exist $\{y_1, ..., y_n\} \subseteq H$ with the property

$$\overline{M(\mathcal{B}_H)} \subseteq \cup_{i=1}^n B_\epsilon(y_i)$$

Define the orthogonal projection $P_\epsilon : H \twoheadrightarrow \text{span}\{y_1, ..., y_n\}$. Since its range is finite-dimensional, so is the one of $P_\epsilon M$. Take an element $x \in \mathcal{B}_H$. Then, there exists some $i \in \{1, ..., n\}$ such that $||Mx - y_i|| < \epsilon$. We conclude,

$$\begin{aligned} ||P_\epsilon Mx - Mx|| &= ||(P_\epsilon - \text{id})(Mx - y_i + y_i)|| \\ &\leq ||(P_\epsilon - \text{id})(Mx - y_i)|| + ||(P_\epsilon - \text{id})y_i|| \leq ||\text{id} - P_\epsilon||\,||Mx - y_i|| < \epsilon \end{aligned}$$

□

Secondly, we state a powerful density result which will be used to build up an integral for the Borel-measurable functions in Chapter 6.

Proposition 2.12 (Approximation of L^p - [R], Sec. 7.4, Prop. 7.9)**.** Let (Ω, Σ) be a measurable space, $S \in \Sigma$, $p \in (1, \infty]$ and $\epsilon > 0$. Then, for any $f \in L^p(S)$, there is a simple function φ with the property
$$||f - \varphi||_p < \epsilon$$

Note: Recall the Sombrero lemma gives for any non-negative *measurable* function, some sequence of simple functions which converges to the given function *almost everywhere* monotonically from below. The above proposition gives a more powerful conclusion (convergence in norm), but also has stronger assumptions (the function is in $L^p(S)$ for some $1 < p \leq \infty$).
Thirdly, we present an important density result which will be used to uniquely extend operators while preserving the norm. The proof is based on approximating any element in the domain by a sequence in the given dense set, hence the title of the following proposition.

Proposition 2.13 (Approximation of the Extension - [M], Thm. 1.9.1)**.** Let X be a normed space, $A \subseteq X$ be a dense subset, and Y be a Banach space. Assume $\gamma : A \to Y$ is an operator. Then, there exists a unique operator $\Gamma : X \to Y$ with the properties $\Gamma\big|_A = \gamma$ and $||\Gamma|| = ||\gamma||$.

Finally, we would like to generalize the set χ from Chapter 1 (cf. Eqn. 1) and its natural function space $\ell^2(\sigma(M); \chi)$ (cf. Eqn. 1). Not only this, but also in Chapter 7, we will require an important result regarding the isometric-isomorphic relation between two objects of these type. For this, we require two preliminary results. The reader should focus on the second statement, because the first one only provides us with the tools to conclude surjectivity in the other one.

Lemma 2.14. Let H_1, H_2 be two Hilbert spaces and $U \in \mathcal{L}(H_1, H_2)$ be an isometry. Then, the range of U is closed. In particular, if the range of an isometry U is dense, then U is surjective.

Proof. By the subspace criterion ([B], 1.7), it suffices to show that the range of U is complete. Let $(Ux_n) \subseteq H_2$ be a Cauchy sequence. Since U is an isometry, (x_n) is also a Cauchy sequence. By completeness of H_1, we have that there exists some $x \in H_1$ with the property $x = \lim_{n\to\infty} x_n$. By continuity of U, we have $Ux = \lim_{n\to\infty} Ux_n$, as was to be shown. The second claim is clear: assuming U is an isometry whose image is dense in H_2, we get $U(H_1) = \overline{U(H_1)} = H_2$. □

Proposition 2.15. Let (Ω, Σ, μ) be a σ-finite measure space and let H be a separable Hilbert space of dimension $n \in \mathbb{N} \cup \{\infty\}$. Then,

$$L^2(\Omega; H) \cong \bigoplus_{j=1}^{n} L^2(\Omega; \mathbb{K})$$

Proof. ([W], Prop. 2.4.7). Assume that $n = \infty$; the finite-dimensional case is similar and simpler.
Step 1: Define the candidate isometric isomorphism $\hat{U}$ over a dense set. By the separability of H, using ([B], Thm. 5.28a), there exists a countable, orthonormal basis $(e_j)_{j\in\mathbb{N}}$. For a sequence $(f_j) \subseteq L^2(\Omega; \mathbb{K})$ where almost all f_j are zero, we define $U : \bigoplus_{j\in\mathbb{N}} L^2(\Omega; \mathbb{K}) \to L^2(\Omega; H)$ by

$$U\Bigg(\bigoplus_{j\in\mathbb{N}} f_j\Bigg)(x) := \sum_{j\in\mathbb{N}} f_j(x) e_j$$

Step 2: U is measurable and an isometric isomorphism.
Step 2.1: The measurability of U is clear. Moreover, by the Pythagorean theorem,

$$\left|\left|\sum_{j\in\mathbb{N}} f_j e_j\right|\right|^2 = \sum_{j\in\mathbb{N}} ||f_j||^2 = \left|\left|\bigoplus_{j\in\mathbb{N}} f_j\right|\right|^2$$

and thus U is an isometry over the set of finitely-supported sequences $(f_j) \subseteq L^2(\Omega;\mathbb{K})$. Using that this domain is dense in $\bigoplus_{j\in\mathbb{N}} L^2(\Omega;\mathbb{K})$, we have by Prop. 2.13 that U extends to an isometry $\hat{U}$ over the whole space $\bigoplus_{j\in\mathbb{N}} L^2(\Omega;\mathbb{K})$. Note since $\hat{U}$ is an isometry, it follows that it is also injective.
Step 2.2: $\hat{U}$ is surjective. For any $f \in L^2(\Omega; H)$ define $f_j(x) := (f(x), e_j)$, for all $x \in \Omega$ and each $j \in \mathbb{N}$. Then, by ([B], Cor. 5.33), we have $f = \sum_{j\in\mathbb{N}} f_j e_j$. Note that each of the terms of the sum belongs to the range of $\hat{U}$. Since $\hat{U}$ is an isometry, its range is closed by L. 2.14 and hence the *limit* of the partial sums of the above series belongs to the range of $\hat{U}$ as well. By the arbitrariness of f, we conclude that $\hat{U}$ is surjective.

□

The last result motivates the following definition.

Definition 2.16 (Hilbert bundle and Hilbert space of L^2 sections - [W], Def. 2.4.8)**.** Let (Ω, Σ, μ) be a σ-finite measure space. A **measurable Hilbert bundle** over Ω is a disjoint union

$$\chi = \coprod_{j=1}^{\infty} (\Omega_j \times H_j)$$

where $(\Omega_j) \subseteq \Sigma$ is a measurable partition of Ω, and (H_j) are Hilbert spaces of dimension $j \in \mathbb{N} \cup \{\infty\}$.
The **Hilbert space of L^2 sections** of a measurable Hilbert bundle χ is the direct sum

$$L^2(\Omega;\chi) = \bigoplus_{j=1}^{n} L^2(\Omega_j; H_j) = \{f : f|_{\Omega_j} \in L^2(\Omega_j; H_j)\}$$

and its norm is given by

$$||f||^2 := \sum_{j=1}^{\infty} \int_{\Omega_j} ||f(x)||^2_{H_j}\, d\mu(x)$$

3. Continuous Functional Calculus

The main goal for this chapter is to define a particular algebra $\star$-homomorphism $f \mapsto f(M)$ for continuous functions, which agrees with the natural definition of a functional calculus $p \mapsto p(M)$ for polynomials.
Throughout this chapter, let H be a Hilbert space over $\mathbb{K}$, $M \in \mathcal{L}(H)$ be a self-adjoint operator, and $\sigma(M) \subseteq \mathbb{C}$ its spectrum.[1] By ([B], Cor. 6.4) the spectrum of any operator is compact. Therefore, by the extreme value theorem, we can endow the space of polynomials and that of continuous functions over $\sigma(M)$ with the infinity norm. We shall use this norm on these spaces from now on.

3.1. Polynomial Functional Calculus

Let $p = \sum_{n=0}^{N} a_n x^n \in \mathbb{K}[x]\big|_{\sigma(M)}$ be a polynomial restricted to the spectrum of M, where $a_n \in \mathbb{K}$, $\forall n \in \{0, ..., N\}$ and $N \in \mathbb{N}$. We define

$$p(M) := \sum_{n=0}^{N} a_n M^n \tag{3.1}$$

By the triangle inequality and the submultiplicativity of the operator norm, we have that $p(M)$ is an operator. Moreover, if the polynomial has real coefficients, it follows by the (anti)linearity of taking adjoints $M \mapsto M^*$ and $\overline{a_n} = a_n$ that $p(M)$ is also self-adjoint:

$$(p(M))^* = \left(\sum_{n=0}^{N} a_n M^n\right)^* = \sum_{n=0}^{N} a_n M^n = p(M)$$

We will generalize this remark for real-valued, bounded, Borel-measurable functions $f \in \mathcal{B}(\sigma(M))$ once we have refined our functional calculus in Chapters 5 and 6 (cf. Rmk. 6.3).

Lemma 3.1 (Commuting Property of Polynomials)**.** Given a polynomial $p \in \mathbb{K}[x]$ and an operator $T \in \mathcal{L}(H)$, it holds

$$\sigma(p(T)) = p(\sigma(T)) \tag{3.2}$$

Proof. ([W], L. 3.2.7). Assume p is not the zero polynomial, since otherwise the claim is trivial. By the fundamental theorem of algebra, we can write $\lambda - p$ in factored form as $\alpha \prod_{n=1}^{N}(x - b_n)$, where the $b_n \in \mathbb{C}$ are the complex roots of $\lambda - p$ and $\alpha \in \mathbb{C}^\times$. Using (3.1), we get

$$\lambda \cdot \mathrm{id} - p(T) = \alpha \prod_{n=1}^{N} (T - b_n \cdot \mathrm{id})$$

which is a product of commuting operators. Therefore, $\lambda \cdot \mathrm{id} - p(T)$ is invertible if and only if all the factors are invertible, which is precisely the case whenever all the roots b_n are not in the spectrum of T. This, in turn, is equivalent to $p(x) \neq \lambda$ for all $x \in \sigma(T)$. Hence, $\lambda \notin \sigma(p(T))$ if and only if $\lambda \notin p(\sigma(T))$.

$\square$

[1] To be more precise, $\sigma(M) \subseteq \mathbb{R}$ by the self-adjointness of M using ([B], L. 6.8).

Lemma 3.2 (Commuting Property of Inverses)**.** Let $T \in \mathcal{L}(H)$ be an invertible operator over a Hilbert space H. Then,

$$\sigma(T^{-1}) = (\sigma(T))^{-1}$$

Proof. First, note that because T is invertible, we have $0 \notin \sigma(T)$. Now, let $\lambda \in \mathbb{C}^{\times}$. Then, we rewrite

$$T^{-1} - \lambda \cdot \mathrm{id} = T^{-1}(\mathrm{id} - \lambda T) = -\lambda T^{-1}(T - \lambda^{-1} \cdot \mathrm{id})$$

Thus, we see that $\lambda \in \sigma(T^{-1}) \iff \lambda \in (\sigma(T))^{-1}$.

□

We now get to the origins of the most important homomorphism in this thesis. The following proposition merely defines the map and proves some of its properties. Its domain will be further extended in two steps in later sections and chapters.

Proposition 3.3 (Polynomial Functional Calculus)**.** The map

$$\gamma : \mathbb{K}[x]\big|_{\sigma(M)} \to \mathcal{L}(H),\ p \mapsto p(M)$$

is a unital, isometric algebra $\star$-homomorphism.

Proof. ([W], Thm. 3.2.8). First, assume the polynomial p has real coefficients. Since $M = M^*$, it follows by an introductory remark that $p(M)$ is also a self-adjoint operator. These are also normal. We can thus apply ([B], Thm. 6.7) in order to get $|\sigma(p(M))| = ||p(M)||$, where $|\sigma(T)|$ is the spectral radius of $T \in \mathcal{L}(H)$. By Gelfand's theorem ([B], Thm. 6.6) and L. 3.1,

$$||p(M)|| = \max_{\lambda \in \sigma(M)} |p(\lambda)| = ||p||_\infty$$

Now, for $p \in \mathbb{C}[x]\big|_{\sigma(M)}$, we have that the polynomial $|p|^2 = p\overline{p}$ is real. Applying the above result yields

$$||p||_\infty^2 = ||p(M)p(M)^*|| = ||p(M)||^2$$

where we used ([B], Thm. 5.37a) in the second equal sign.
Finally, the fact that γ is a unital, algebra $\star$-homomorphism follows directly from the definition of $p(M)$.

□

3.2. Distance between Spectra

The following definition allows us to compare two spectra in terms of the distance between their elements.

Definition 3.4 (Distance between Spectra - [L], Ch. 31, Thm. 5)**.** The **distance between the spectra** of two operators $S, T \in \mathcal{L}(H)$ is defined as

$$\mathrm{dist}(\sigma(S), \sigma(T)) := \max\big\{ \max_{\nu \in \sigma(T)} \min_{\mu \in \sigma(S)} |\nu - \mu|,\ \max_{\mu \in \sigma(S)} \min_{\nu \in \sigma(T)} |\nu - \mu| \big\} \tag{3.3}$$

If it is the first time the reader sees such an expression, it is worth to take a look at Fig. 3.1 and understand the arguments inside the outermost maximum. For instance, consider

$$\max_{\nu \in \sigma(T)} \min_{\mu \in \sigma(S)} |\nu - \mu|$$

In order to calculate this expression, we first fix some $\nu \in \sigma(T)$ and run through all $\mu \in \sigma(S)$ with the goal of *minimizing* the amount $|\nu - \mu|$. We denote the set of $\mu \in \sigma(S)$ such that the minimum is achieved by $Q_\nu \subseteq \sigma(S)$. We then go through all $\nu \in \sigma(T)$ with the aim of *maximizing* the distance from it to its corresponding set Q_ν. Note that both steps are well-defined, since the spectrum of any operator is compact and the absolute value is continuous. More explicitly,

- $Q_\nu \neq \emptyset$ as a consequence of the extreme value theorem.
- Q_ν is compact: for a fixed $\nu \in \sigma(T)$ define $f_\nu : \sigma(S) \to [0, \infty), \mu \mapsto f_\nu(\mu) := |\nu - \mu|$ and $r := f_\nu(\mu_\star)$ where $\mu_\star \in Q_\nu$. Then, $Q_\nu = f_\nu^{-1}(\{r\}) \subseteq \sigma(S)$ is closed by continuity of f_ν and therefore, by compactness of $\sigma(S)$, the set Q_ν is also compact.
- The map $g : \sigma(T) \to [0, \infty), \nu \mapsto \inf_{\mu \in Q_\nu} |\nu - \mu| = \min_{\mu \in Q_\nu} |\nu - \mu|$ is therefore well-defined, continuous and by compactness of $\sigma(T)$, it follows by the extreme value theorem that g attains its maximum.

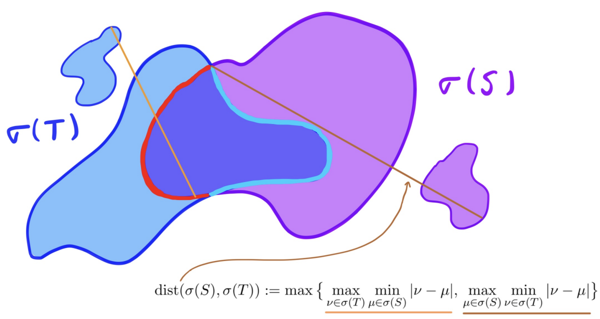

Figure 3.1.: Distance between two spectra on the complex plane.

The next proposition will enable us to show the continuous function analogue of L. 3.1 by providing a simple upper bound for (3.3). Note that neither in the definition above nor in the upcoming proposition do we assume that either S or T are self-adjoint.

Proposition 3.5 (Distance between Spectra)**.** Given two operators $S, T \in \mathcal{L}(H)$ over a Hilbert space H, the following inequality holds:

$$\operatorname{dist}(\sigma(S), \sigma(T)) \leq ||S - T||$$

Proof. ([L], Ch. 31, Thm. 5). Without loss of generality, assume $\delta := \operatorname{dist}(\sigma(S), \sigma(T)) = \max_{\nu \in \sigma(T)} \min_{\mu \in \sigma(S)} |\nu - \mu|$, otherwise, swap the roles of both spectra.
Now, assume, by way of contradiction, that $d := ||S - T|| < \delta$. Then, there exists some $\nu \in \sigma(T)$ with the property

$$\min_{\mu \in \sigma(S)} |\mu - \nu| > d \tag{3.4}$$

Note that $\nu \in \mathbb{C} \backslash \sigma(S)$, since we would otherwise have by definition of our distance $0 = \delta > d \geq 0$. Hence, $S - \nu \cdot \mathrm{id}$ is invertible. Using (3.4), we have in particular, $|\sigma(S - \nu \cdot \mathrm{id})| > d$. Applying L. 3.2 together with the preceding inequality yields,

$$|\sigma((S - \nu \cdot \mathrm{id})^{-1})| = |\sigma(S - \nu \cdot \mathrm{id})|^{-1} < d^{-1}$$

By ([B], Cor. 6.4), the spectral radius is bounded by the norm of the corresponding operator, i.e. $|\sigma(S)| \leq ||S||$ and hence, taking the inverse,

$$||(S - \nu \cdot \mathrm{id})^{-1}|| \leq |\sigma(S - \nu \cdot \mathrm{id})|^{-1} < d^{-1}$$

Thus,

$$||(S - \nu \cdot \mathrm{id})^{-1}|| < d^{-1} \tag{3.5}$$

Now, we rewrite

$$T - \nu \cdot \mathrm{id} = T - \nu \cdot \mathrm{id} + S - S = (S - \nu \cdot \mathrm{id})(\mathrm{id} + (S - \nu \cdot \mathrm{id})^{-1}(T - S))$$

and define $K := (S - \nu \cdot \mathrm{id})^{-1}(T - S)$. Using submultiplicativity of norms, inequality (3.5), and $d = ||T - S||$, we get,

$$||K|| \leq ||(S - \nu \cdot \mathrm{id})^{-1}|| \, ||T - S|| < d^{-1}d = 1$$

which by the Neumann series theorem ([B], Thm. 2.10) implies that $\mathrm{id} + K$ is invertible. So, since both factors $S - \nu \cdot \mathrm{id}$ and $\mathrm{id} + K$ of the operator $T - \nu \cdot \mathrm{id}$ are invertible, the latter is invertible as well. But this contradicts $\nu \in \sigma(T)$.

□

3.3. Extension to Continuous Functions

We now have all the ingredients to show the main result of this chapter: the extension of the map from Prop. 3.3 to continuous functions.

Theorem 3.6 (Continuous Functional Calculus)**.** *Let $M \in \mathcal{L}(H)$ be a self-adjoint operator and γ as in Prop. 3.3. Then, there exists a unique, unital, isometric algebra $\star$-homomorphism Γ which makes the following diagram commute:*

$$\begin{tikzcd} \mathbb{K}[x]\big|_{\sigma(M)} \arrow[r, "\gamma"] \arrow[d, hook] & \mathcal{L}(H) \\ \mathcal{C}(\sigma(M)) \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

i.e. Γ extends γ to continuous functions over the spectrum of M and is the unique extension preserving the properties in Prop. 3.3.
Moreover, setting $f(M) := \Gamma(f)$ for $f \in \mathcal{C}(\sigma(M))$ yields

$$\sigma(f(M)) = f(\sigma(M)) \tag{3.6}$$

Proof. ([W], Thm. 3.2.8 and [L], Ch. 31, Thm. 6iii).
Step 1: Construct the extension with the desired properties. First, note that since $\sigma(M)$ is compact, we get by the Stone-Weierstrass theorem that the space $\mathbb{K}[x]\big|_{\sigma(M)}$ is dense in $\mathcal{C}(\sigma(M))$. Using Prop. 2.13, we get that γ extends uniquely to an operator

$$\Gamma : \mathcal{C}(\sigma(M)) \to \mathcal{L}(H), \, f \mapsto \Gamma(f)$$

with $f(M) := \Gamma(f) = \lim_{n\to\infty} \gamma(p_n)$, where $(p_n) \subseteq \mathbb{K}[x]\big|_{\sigma(M)}$ is some sequence of polynomials which converges uniformly to f, and the limit in the assignment is with respect to the operator norm. Moreover, the isometry property follows directly from the continuity of norms and the fact that γ is itself an isometry:

$$||\Gamma(f)|| = \lim_{n\to\infty} ||\gamma(p_n)|| = \lim_{n\to\infty} ||p_n||_\infty = ||f||_\infty$$

Similarly, the unital, algebra $\star$-homomorphism properties of γ are passed on to Γ.
Step 2: Show the commuting property (3.6). Using the inequality in Prop. 3.5, we get

$$\mathrm{dist}(\sigma(f(M)) - \sigma(p_n(M))) \leq ||f(M) - p_n(M)|| \xrightarrow{n\to\infty} 0$$

and hence,

$$\sigma(f(M)) = \lim_{n\to\infty} \sigma(p_n(M))$$

Taking the polynomial p_n out of the spectrum using L. 3.1 and subsequently taking the limit yields the desired result.

$\square$

4. Advanced Tools from Topology and Functional Analysis

Before we go on to extend the functional calculus defined in the previous chapter to bounded, Borel-measurable functions, we need to equip ourselves with stronger machinery: weaker topologies. These will be induced by the initial topology with respect to a well-chosen function set. In addition, we will show Goldstine's theorem, which is essential for the uniqueness of our last extension step in Thm. 5.7. Finally, we will show that $\mathcal{L}(H)$ is the dual space of $L_1(H)$, the space of trace class operators.
Throughout this chapter, let H be a separable Hilbert space.

4.1. Weaker Notions of Convergence

The choice of open sets for a topology is a compromise between continuous functions and compact sets. Indeed, the more open sets we have, the more chances there are for a function to have open preimages of open sets, i.e. to be continuous. At the same time, though, we get more possible open covers for a given set, which lessens the chances of a set being compact (cf. [Z2], Rmk. 2.31). A classical example of this behaviour is seen in Riesz's theorem ([B], Thm. 1.24), which states that the closed unit ball is *not* compact with respect to the strong topology precisely when we are dealing with infinite-dimensional spaces. This section is all about *relaxing* topologies and focusing on the *essential* maps which we want to be continuous so that, intuitively speaking, we do not lose a lot of compactness while keeping the continuity we need.
In Chapter 2, we defined the weak and weak* topologies for a normed space X. In this section, we start constructing other topologies which are smaller than the one induced by the norm. A natural instance of such a topology is exemplified by the following definition and will be used in the next chapter. It relies on the well-known Riesz representation theorem,

$$\mathcal{C}(K)' \cong \mathcal{M}(K)$$

for a compact metric space K, where $\mathcal{M}(K)$ is the set of all signed/complex ($\mathbb{R}/\mathbb{C}$) Borel measures on K (cf. [B], Thm. 2.24).

Definition 4.1 (Initial $\mathcal{M}(K)$-Topology - [W], pp. 55-56)**.** Let K be a compact metric space. The initial topology on the set of all bounded, Borel-measurable functions $\mathcal{B}(K)$ with respect to $\mathcal{F} := \mathcal{M}(K)$ is called the **initial $\mathcal{M}(K)$-topology**. Equivalently, it is the weakest topology such that for all $\mu \in \mathcal{M}(K)$ the map $\mathcal{B}(K) \ni f \mapsto \int_K f\, d\mu$ is continuous.
In terms of nets, we have using Thm. 2.8, that this topology is induced by the net convergence

$$(f_a) \xrightarrow{\mu} f \iff \forall \mu \in \mathcal{M}(K) : \int_K f_a\, d\mu \to \int_K f\, d\mu$$

It is also possible to weaken the convergence notion for *operators* in both the weak- and weak*-sense. We shall do this once we have shown that $\mathcal{L}(H)$ is a dual space, so as to introduce both the weak- and weak*-operator topologies at the same time.
Since there are several topologies we could endow a space with, we need a definition which clarifies *in what sense* we mean we have a certain net-convergence.

Definition 4.2 (Convergence and Continuity)**.** Let $(X, \mathcal{O}_\alpha)$ and $(Y, \mathcal{O}_\beta)$ be topological spaces. For a net $(x_a) \subseteq X$, we write

$$x_a \xrightarrow{\alpha} x$$

whenever (x_a) converges to $x \in X$ in the topology $\mathcal{O}_\alpha$. Moreover, a map $L : (X, \mathcal{O}_\alpha) \to (Y, \mathcal{O}_\beta)$ is said to be **α-β-continuous**, if

$$x_a \xrightarrow{\alpha} x \implies Lx_a \xrightarrow{\beta} Lx$$

The following proposition yields the continuity of the adjoint of an operator L in the weak* topologies of its domain and codomain. This is interesting because although L might not be continuous in a smaller topology than the strong one, its adjoint does have this property. The result will be used in the next chapter in order to show the *continuity* of the extension of the functional calculus from Thm. 3.6.

Proposition 4.3. Let X and Y be two normed spaces, and $L \in \mathcal{L}(X, Y)$. Then, its adjoint, $L' \in \mathcal{L}(Y', X')$, is w*-w*-continuous.

Proof. Let $(y'_a) \subseteq Y'$ be a net and $y' \in Y'$ with $y'_a \xrightarrow{w*} y'$, and let $x \in X$. We need to show $(L'y'_a, x) \to (L'y', x)$. Note that $Lx \in Y$ and hence, by the weak*-convergence of (y'_a), we have $(y'_a, Lx) \to (y', Lx)$. Therefore, using the definition of the adjoint twice,

$$(L'y'_a, x) = (y'_a, Lx) \to (y', Lx) = (L'y', x)$$

$\square$

4.2. Goldstine's Theorem and Weak* Density

This section is dedicated to Goldstine's theorem, which, together with Prop. 2.13, will yield the *uniqueness* of the extension for our continuous functional calculus.
In order to be well-equipped for our main claim, we require L. 4.6, which is another version of the geometric Hahn-Banach theorem given in ([B], Thm. 3.26). In contrast to the latter, this lemma takes place in the context of a topology which is not necessarily metrizable and yields the separation of other kinds of sets. Now, even though we will not have a notion of distance, notice that the essence of the proof is trying to mend the lack of a metric by using the continuity of certain operations and geometric concepts, thus yielding a distance-like behaviour.
Let us define our special playground in which our lemma will take place.

Definition 4.4 (Locally Convex Space - [M], Def. 2.2.1)**.** A topological space $(X, \mathcal{O})$ is called a **locally convex space** (short: LCS), if X is a linear space endowed with a topology $\mathcal{O}$, where vector addition and scalar-vector multiplication are continuous operations, and $\mathcal{O}$ has a basis consisting of convex sets.

We have already encountered instances of such structures without explicitly mentioning it.

Example 4.5 (Weak* Topology on X')**.** Given a normed space X, take some $x' \in X'$, a positive scalar $\epsilon > 0$, and a finite set $A := \{x_1, ..., x_n\} \in X$. Define

$$W^*_x(A; \epsilon) := \{y' \in X' : \, |(y' - x')(x)| < \epsilon, \forall x \in A\}$$

The collection of all subsets of the above form is a basis for the weak* topology of X'. Therefore, X' endowed with the weak* topology is a locally convex space.

Proof. ([M], Prop. 2.4.12). Note that for each $x' \in X'$, $\epsilon > 0$, and each finite subset $A \subseteq X'$, the sets $W^*_x(A;\epsilon)$ are open. Take some $x_1, ..., x_n \in X$ and view X as a subset of X'' as we have done previously. Choose some $U_1, ..., U_n \in \mathcal{O}_{\mathbb{K}}$ and take an element $x' \in \cap_{i=1}^n x_i^{-1}(U_i)$. Next, because the sets $(U_i)_{i=1}^n$ are open, we can choose $\epsilon > 0$ so small such that it fulfills $\{\alpha \in \mathbb{K} : |x_i(x') - \alpha| < \epsilon\} \subseteq U_i$, $\forall i \in \{1, ..., n\}$. Then, this $\epsilon > 0$ satisfies

$$W^*_{x'}(\{x_1, ..., x_n\}; \epsilon) \subseteq \cap_{i=1}^n x_i^{-1}(U_i)$$

Since ς as in the proof of Prop. 2.2 is a subbasis for the initial topology with respect to $\mathcal{F} = X$, i.e. the weak* topology, we are done.

□

Note: The geometric Hahn-Banach theorem ([B], Thm. 3.26) can be shown for a locally convex space X. The instances in which ([B], Thm. 3.26) and its previous steps use the notion of a norm can be appropriately adapted to the more general context of a locally convex space. For a proof of the LCS version, we refer the reader to ([M], Thm. 2.2.19), since restating it would digress from the main purpose of this thesis.

Lemma 4.6 (Hahn-Banach Separation - LCS Version)**.** Let X be a locally convex space, $C \subseteq X$ a nonempty, closed, convex subset, and $x_0 \notin C$. Then, there exists a functional $x' \in X'$ such that

$$\Re(x'(x_0)) < \inf\{\Re(x'(y)) \mid y \in C\}$$

Proof. ([M], Thm. 2.2.26 and [M], Thm. 2.2.28). We prove this in two steps.
Step 1: For $U \subseteq X$ nonempty, open, convex subset disjoint from C, we show there exists an element $x' \in X'$ and a real number s such that for all $y \in U$ and $z \in C$, we have

$$\Re(x'(y)) < s \leq \Re(x'(z))$$

Indeed, by ([M], Thm. 2.2.19) there is an element $x' \in X'$ such that for each $y \in U$ and $z \in C$,

$$\Re(x'(y)) < \Re(x'(z)) \tag{4.1}$$

Therefore, there is some $s \in \mathbb{R}$ with the property

$$\sup\{\Re(x'(y)) \mid y \in U\} \leq s \leq \inf\{\Re(x'(z)) \mid z \in C\} \tag{4.2}$$

Now, fix some $y \in U$ and $z \in C$. Then, by the continuity of the vector space operations, we have some $t \in (0, 1)$ such that $tz + (1-t)y \in U$. Hence, using (4.2) and then (4.1), we get

$$\begin{aligned} \Re(x'(z)) \quad &\geq s \\ &\geq \Re(x'(tz + (1-t)y)) \\ &= t\Re(x'(z)) + (1-t)\Re(x'(y)) \\ &> t\Re(x'(y)) + (1-t)\Re(x'(y)) \\ &= \Re(x'(y)) \end{aligned}$$

Step 2: We show the claim in the lemma. By assumption, $0 \notin C - \{x_0\}$. Since X is locally convex and $X \setminus (C - \{x_0\})$ is open, there exists a convex neighborhood of zero $U \in \mathcal{N}_0$ such that $U \subseteq X \setminus (C - \{x_0\})$, meaning $U \cap (C - \{x_0\}) = \emptyset$, i.e. $(U + \{x_0\}) \cap C = \emptyset$. In other words, $U + \{x_0\}$ is a nonempty, open, convex set, which is disjoint from C. Thus, using $x_0 = 0 + x_0 \in U + \{x_0\}$, it follows from Step 1 that there is some $x' \in X'$ such that

$$\Re(x'(x_0)) < \inf\{\Re(x'(y)) \mid y \in C\}$$

□

Remark 4.7. From ([M], Prop. 2.1.17) we know that Hausdorff spaces are characterized by nets having unique limits. As a consequence, the weak* topology on X'' is Hausdorff. The preceding statement together with the fact that compact sets in Hausdorff spaces are closed (cf. [Ba], Thm. 6.13), shows that compact sets of X'' are closed in the weak* topology of X''.

Theorem 4.8 (Goldstine's Theorem)**.** *Let X be a normed space. Then X is weakly* dense in X''.*

Proof. ([BK], Thm. 5.40 and [M], Thm. 2.6.26). We shall again interpret X as a subspace of X'' and denote the weak*-closure of a subset $U \subseteq X''$ as $\overline{U}^{w^*}$. We need to show that

$$\overline{\mathcal{B}}_X^{w^*} = \mathcal{B}_{X''}$$

Step 1: "$\subseteq$". By the Banach-Alaoglu theorem ([M], Thm. 2.6.18), the set $\mathcal{B}_{X''}$ is compact in the weak* topology of X'' and hence, by Rmk. 4.7, $\mathcal{B}_{X''}$ is also closed in the same topology, i.e. $\overline{\mathcal{B}}_X^{w^*} = \mathcal{B}_X$. By $X \subseteq X''$, it follows that $\mathcal{B}_X \subseteq \mathcal{B}_{X''}$ and thus, we get the first inclusion.
Step 2: "$\supseteq$". For the converse, assume there exists some $x_0'' \in \mathcal{B}_{X''} \setminus \overline{\mathcal{B}}_X^{w^*}$. Now, because $\overline{\mathcal{B}}_X^{w^*}$ is nonempty, closed and convex in the weak* topology, we can apply L. 4.6 to get an element $x' \in X'$ with the property

$$\Re(x_0''(x')) > \sup\{\Re(x''(x')) \mid x'' \in \overline{\mathcal{B}}_X^{w^*}\}$$

Therefore,

$$\begin{aligned} |x_0''(x')| &\geq \Re(x_0''(x')) \\ &> \sup\{\Re(x''(x')) \mid x'' \in \overline{\mathcal{B}}_X^{w^*}\} \\ &\geq \sup\{\Re(x''(x')) \mid x'' \in \mathcal{B}_X\} = ||\Re(x')|| = ||x'|| \end{aligned}$$

i.e. $||x_0''|| > 1$, which contradicts $x_0'' \in \mathcal{B}_{X''}$.

□

4.3. Trace Class Operators and Weak(*)-Operator Topologies

The main goal for this section is to show that $\mathcal{L}(H)$ is a dual space. This will allow us to define the weak* topology on $\mathcal{L}(H)$. We construct and refer several results back to ([L], Ch. 30) and ([W], Sec. 6.3) on trace class.
In linear algebra, we define the *trace* of an operator $M \in \mathbb{K}^{n\times n}$ as $\text{tr}(M) := \sum_{i=1}^n M_{ii} = \sum_{i=1}^n (Me_i, e_i)$, for the standard basis vectors $(e_i|_j) = \delta_{ij}$. A well-known result states that the trace is invariant under conjugation, i.e. if $U \in \mathbb{K}^{n\times n}$ is some invertible operator, we have that $\text{tr}(UMU^{-1}) = \text{tr}(M)$. In other words, the trace is invariant under a change of basis. We want to generalize the notion of trace and its invariance under a basis change in infinite dimensions. But first, we need the following:

Definition 4.9 (Positive Operator)**.** An operator $M \in \mathcal{L}(H)$ is called **positive**, if

$$\forall x \in H : (Mx, x) \geq 0$$

Lemma 4.10 (Polar Decomposition - [W], L. 5.4.7)**.** For any $M \in \mathcal{L}(H)$, there exists a **polar decomposition**, i.e. a representation of M in factorized form,

$$M = U|M|$$

where $|M|$ is a self-adjoint, positive operator, and $U^*U = \text{id}$ on the range of $|M|$ and zero elsewhere. We call $|M|$ the **absolute value** of M.

We shall not go into the proof of this result. However, it is worth remarking that the construction of this decomposition uses the continuous functional calculus from the previous chapter.
The special case when $M \in \mathcal{K}(H)$ is compact is an interesting one. Indeed, for such operators, $|M|$ is compact as well (cf. [L], p. 330). Moreover, since $|M|$ is also self-adjoint, it is normal. Hence, by ([B], Thm. 6.16), there is an orthonormal basis $(e_n) \subseteq H$ of eigenvectors of $|M|$ and a corresponding sequence of eigenvalues $(\lambda_n) \in c_0$ such that for any $x \in H$ we can write

$$|M|x = \sum_{n\in\mathbb{N}} \lambda_n (x, e_n) e_n$$

Since $|M|$ is positive, we have that $(\lambda_m) \subseteq \mathbb{R}_{\geq 0}$: for all $m \in \mathbb{N}$,

$$0 \leq (|M|e_m, e_m) = \left(\sum_{n\in\mathbb{N}} \lambda_n (e_m, e_n) e_n, e_m \right) = \sum_{n\in\mathbb{N}} \lambda_n \delta_{nm} = \lambda_m$$

With the purpose of mimicking the finite-dimensional trace, we use the above calculation to define

$$\mathrm{tr}(|M|) := \sum_{m\in\mathbb{N}} (|M|e_m, e_m) = \sum_{m\in\mathbb{N}} \lambda_m$$

However, the above series is not necessarily convergent, because the sequence (λ_m) is only a nullsequence. Nonetheless, in the case that the series *does* converge, because $\lambda_m \geq 0$, it is also unconditionally convergent, and therefore, the series is invariant under permutations $\pi \in \mathcal{S}_{\mathbb{N}}$. This means that the value $\mathrm{tr}(|M|)$ is independent on the order in which the basis elements are put in.
More generally, we show that the above expression is also invariant under a change of basis (cf. [W], L. 6.3.1): Let (e_n) and (f_n) be orthonormal bases of H and M as above. Then, by ([B], A. 51) there exists a unique, positive operator $\sqrt{|M|} \in \mathcal{L}(H)$ with the property $(\sqrt{|M|})^2 = |M|$. Then,

$$\begin{aligned}\sum_{n\in\mathbb{N}} (|M|e_n, e_n) = \sum_{n\in\mathbb{N}} (\sqrt{|M|}e_n, \sqrt{|M|}, e_n) &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} (\sqrt{|M|}e_n, f_m)(f_m, \sqrt{|M|}e_n) \\ &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |(\sqrt{|M|}e_n, f_m)|^2\end{aligned}$$

where we used ([B], Cor. 5.33). Similarly, we get

$$\sum_{m\in\mathbb{N}} (|M|f_m, f_m) = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |(\sqrt{|M|}f_m, e_n)|^2$$

and hence, by the self-adjointness of $\sqrt{|M|}$, both sums are equal. The above results motivate the following definitions.

Definition 4.11 (Trace Class)**.** Let $(e_n)_{n\in\mathbb{N}} \subseteq H$ be an orthonormal basis of a separable Hilbert space H and let $M \in \mathcal{K}(H)$ be a compact operator. We define the **trace** of the absolute value of M as

$$\mathrm{tr}(|M|) := \sum_{n\in\mathbb{N}} (|M|e_n, e_n)$$

The set

$$L_1(H) := \{M \in \mathcal{K}(H) \mid \mathrm{tr}(|M|) < \infty\}$$

is called the ideal of **trace class operators**, and the map $M \mapsto ||M||_1 := \mathrm{tr}(|M|)$ is called the **trace norm**.

Remark 4.12. The names *ideal* and *norm* are justified by ([L], Ch. 30, Thm. 2), and it turns out that $(L_1(H), ||\cdot||_1)$ is also a Banach space (cf. [L], Ch. 30, Ex. 4).

To show the most important results in this section, we state two short lemmata and one remark regarding a density result:

Lemma 4.13 ([W], L. 6.3.2)**.** For any separable Hilbert space H, we have $\mathcal{F}(H) \subseteq L_1(H)$.

Lemma 4.14 ([W], L. 6.3.4)**.** For $M \in \mathcal{L}(H)$, we have

$$\operatorname{tr}(|M|) = \sup_{F \in \mathcal{F}(H), ||F|| \leq 1} \{|\operatorname{tr}(MF)|\}$$

Remark 4.15 ([W], L. 6.3.8)**.** Finite-dimensional range operators are spanned by rank-1 operators of the form $C_{u,v} : H \to H, x \mapsto (x,v)u$, for $u, v \in H$: Let $F \in \mathcal{F}(H)$ and $U := \operatorname{ran}(F)$ be its range with $n := \dim U$. Then, there is a basis $\{e_1, ..., e_n\} \subseteq H$ of U. Take $C \in \mathcal{T} := \operatorname{span}\{C_{u,v} \mid u, v \in H\}$ with $Cx := \sum_{i=1}^n \lambda_i (x, v) e_i$, where λ_i, v can be chosen so that $Fx = Cx$ for all $x \in H$. Also, note that

$$\operatorname{tr}(C_{u,v}) = \sum_{n \in \mathbb{N}} ((e_n, v)u, e_n) = \sum_{n \in \mathbb{N}} (u, e_n)(e_n, v) = (u, v)$$

and thus $\operatorname{tr}(|C_{u,v}|) = ||u|| \, ||v||$.
Moreover, the set $\mathcal{T}$ is dense in $L_1(H)$: By L. 4.13 and the above, we have $\mathcal{F}(H) \subseteq \mathcal{T} \subseteq L_1(H)$. Therefore, we get the inclusions $\mathcal{F}(H) \subseteq \mathcal{T} \subseteq L_1(H) \subseteq \mathcal{K}(H)$. By L. 2.2, we know that the finite-dimensional range operators are dense in the trace class operators. Therefore, we conclude,

$$\overline{\mathcal{T}} = L_1(H)$$

Spoiler alert: This result will enable us to test what we shall call weak*-operator convergence of a net (cf. Def. 4.18) with the operators $C_{u,v} \in \mathcal{T}$. However – and this is a big however – this is only possible when the given net of operators is *bounded* (cf. L. 4.19 and Exa. 4.21).

We are now ready to show that $\mathcal{L}(H)$ is a dual space.

Theorem 4.16 ($L_1(H)' \cong \mathcal{L}(H)$)**.** *For $M \in \mathcal{L}(H)$, we have that*

$$\Phi_M : L_1(H) \to \mathbb{K}, L \mapsto \Phi_M(L) := \operatorname{tr}(LM)$$

defines a functional over $L_1(H)$, i.e. an element of $L_1(H)'$.
The mapping

$$\Phi : \mathcal{L}(H) \to L_1(H)', \, M \mapsto \Phi_M$$

is an isometric isomorphism between $\mathcal{L}(H)$ and $L_1(H)'$.

Proof. ([W], Thm. 6.3.9).
Step 1: Show Φ is an isometry.
Step 1.1: $||\Phi_M|| \leq ||M||$. Let $M \in \mathcal{L}(H)$ and $L \in L_1(H)$. Then,

$$|\Phi_M(L)| = |\operatorname{tr}(LM)| \leq \operatorname{tr}(|LM|) \leq ||L||_1 ||M||$$

where we used the following properties in the given order: $L_1(H) \trianglelefteq \mathcal{L}(H)$ for the well-definedness of the trace of the product LM, then ([L], Ch. 30, Thm. 4i), and finally ([L], Ch. 30, Thm. 2iii). Thus, we get $||\Phi_M|| \leq ||M||$.

Step 1.2: $||\Phi_M|| \geq ||M||$. We stress test the above inequality. For $u, v \in H$, define the maps $C_{u,v} \in \mathcal{T}$ as in Rmk. 4.15. We have

$$|\Phi_M(C_{u,v})| = |\mathrm{tr}(MC_{u,v})| = |\sum_{n\in\mathbb{N}}(M(e_n, v)u, e_n)| = |\sum_{n\in\mathbb{N}}(Mu, e_n)(e_n, v)| = |(Mu, v)| \quad (4.3)$$

for some orthonormal basis $(e_n) \subseteq H$. Taking the supremum over $u, v \in \mathcal{B}_H$ and using L. 4.14, we get $||\Phi_M|| \geq ||M||$. Hence, overall, Φ is isometric and thus also injective.
Step 2: Φ is surjective. Let $f \in L_1(H)'$. The map $(u, v) \mapsto f(C_{u,v})$ is a well-defined, bounded, sesquilinear form: we see

$$|f(C_{u,v})| \leq ||f||\,||u||\,||v||$$

from which the boundedness follows, and sesquilinearity comes from the scalar product in $C_{u,v}$ and linearity of f. Thus, by the Lax-Milgram theorem (cf. [B], A. 40), there exists some $M \in \mathcal{L}(H)$ with the property $(Mu, v) = f(C_{u,v})$. Comparing this with the calculations in (4.3) without absolute values yields $\Phi_M(C_{u,v}) = f(C_{u,v})$. Now, by the density conclusion of Rmk. 4.15, we get that $\Phi_M = f$.

□

Remark 4.17. The same proof with different spaces yields that $L_1(H)$ is the dual space of the compact operators $\mathcal{K}(H)$. As a consequence, $\mathcal{L}(H)$ is a *bidual* space.
Additionally, note that the mapping $(M, L) \mapsto \mathrm{tr}(LM)$ canonically defines the action of $\mathcal{L}(H)$ on its predual, $L_1(H)$, as in $\langle M, L\rangle = \mathrm{tr}(LM)$. This is completely analogous to the isometric isomorphism seen in ([B], Sec. 2.22), $\Phi : \ell^q \to (\ell^p)'$, $x \mapsto \Phi_x$ with $\Phi_x(y) = \sum_{n\in\mathbb{N}} x_n y_n$, which induces the canonical action bracket for a Hölder-conjugate pair $(p, q) \in [1, \infty) \times (1, \infty]$.

As mentioned in Sec. 4.1, we can also define weaker topologies over the space of operators.

Definition 4.18 (Weak- and Weak*-Operator Topologies - [W], Def. 3.3.1)**.** The **weak-operator topology** on $\mathcal{L}(H)$ is the initial topology with respect to $\mathcal{F} := \{f_{x,y} \in \mathcal{L}(H)' \mid M \mapsto f_{x,y}(M) = (Mx, y),\ x, y \in H\}$. In terms of nets,

$$(M_a) \xrightarrow{w} M \iff \forall x, y \in H : (M_a x, y) \to (Mx, y)$$

The **weak*-operator topology** on $\mathcal{L}(H)$ is the initial topology with respect to $\mathcal{F} := L_1(H)$ interpreted as a subspace of $\mathcal{L}(H)'$. In other words, it is the weak* topology on $\mathcal{L}(H)$. Writing the action bracket as the trace, we see that this topology is induced by the net-convergence notion,

$$(M_a) \xrightarrow{w*} M \iff \forall L \in L_1(H) : \mathrm{tr}(M_a L) \to \mathrm{tr}(ML)$$

We now want to show that both topologies agree whenever the given set is *bounded* in norm. The next result gives a hint on why we need boundedness.

Lemma 4.19 (Dense Testing)**.** Let U be a dense subset of a Banach space X and $(x'_a) \subseteq X$ be a net in X'. Then

$$x'_a \xrightarrow{w*} x' \iff (x'_a) \text{ is bounded and } \forall u \in U : x'_a(u) \to x'(u)$$

Proof. For the forward direction, we have for all $x \in X$ that $x'_a(x) \to x'(x)$, i.e. we have a converging net of *scalars*. Hence, we get that $(x'_a(x))_{a\in I}$ is pointwise bounded, which by the Banach-Steinhaus theorem implies that (x'_a) is bounded. Moreover, since $U \subseteq X$, it follows for all $u \in U$ that $x'_a(u) \to x'(u)$.
For the converse direction, let $x \in X$ and take some $\epsilon > 0$. Note that it suffices to show the claim for the case $x' = 0$. By the boundedness assumption, we have $B := 1 + \sup_{a\in I} ||x'_a|| \in [1, \infty)$. Moreover, by density of U in X, there exists some $u \in U$ with the property $||x - u|| < \epsilon/2B$.

Lastly, by the convergence assumption, there is some $A \in I$ such that if $a \geq A$ we get $|x'_a(u)| < \epsilon/2$. We conclude that for all $a \geq A$,

$$|x'_a(x)| \leq |x'_a(u)| + |x'_a(u) - x'_a(x)| \leq B\,||x-u|| + |x'_a(u)| < \epsilon$$

□

Proposition 4.20 (Weak-Weak* Agreement)**.** The weak*-operator topology agrees with the weak-operator topology on bounded subsets of $\mathcal{L}(H)$. In other words,

$$M_a \xrightarrow{w*} M \iff (M_a) \subseteq \mathcal{L}(H) \text{ bounded and } M_a \xrightarrow{w} M$$

Proof. ([W], Thm. 6.3.9). Assume $(M_a) \subseteq \mathcal{L}(H)$ is a bounded net and $M \in \mathcal{L}(H)$. Then, by definition, we have that $M_a \xrightarrow{w*} M$ if and only if $\forall L \in L_1(H) : \operatorname{tr}(M_a L) \to \operatorname{tr}(ML)$. From Rmk. 4.15, we know $\overline{\mathcal{T}} = L_1(H)$, and hence, since the net is bounded, we can use L. 4.19 in order to see that the latter convergence is equivalent to

$$\forall u, v \in H : \operatorname{tr}(M_a C_{u,v}) = \langle C_{u,v}, M_a \rangle \to \langle C_{u,v}, M \rangle = \operatorname{tr}(M C_{u,v})$$

From Step 1.2 in the proof of Thm. 4.16, we also see that $\operatorname{tr}(MC_{u,v}) = (Mu, v)$. Thus, $\operatorname{tr}(M_a C_{u,v}) \to \operatorname{tr}(MC_{u,v})$ is equivalent to $(M_a u, v) \to (Mu, v)$ for $u, v \in H$, which is precisely the convergence in the weak-operator topology.

□

Example 4.21. Note that the boundedness of the net from L. 4.19 is necessary. Consider $H = \ell^2$ and $U := \{e_1, e_2, ...\}$ the set of unit sequences. By ([B], Exa. 1.29b), we know that U is dense in ℓ^2. Define a sequence of functionals by

$$x'_n(x) := \sum_{m=1}^{n} (x, e_m)$$

We have for all $m \in \mathbb{N}$ that

$$x'_n(e_m) = \sum_{m=1}^{n} \delta_{nm} = 1$$

Moreover, we see by taking $x = \sum_{k \in \mathbb{N}} \frac{1}{k} e_k \in H$ that the sequence is unbounded

$$x'_n(x) = \sum_{m=1}^{n} (x, e_m) = \sum_{m=1}^{n} \sum_{k \in \mathbb{N}} \frac{1}{k} \delta_{km} = \sum_{m=1}^{n} \frac{1}{m}$$

since these are precisely the partial sums of the harmonic series. Thus, in particular, the sequence (x'_n) cannot converge weakly* to 1.

5. Borel Functional Calculus

In the following, let H be a separable Hilbert space, $M \in \mathcal{L}(H)$ be a self-adjoint operator, and $\Gamma(H)$ be the collection of all closed subspaces of H.
The extension task in Chapter 3 proved to be relatively simple. The main goal for this chapter is to enlarge the domain again, now to bounded, Borel-measurable functions. Moreover, we will show that this map is continuous when choosing special initial topologies.
We shall define the extension by constructing and using the following commutative diagram:

$$\begin{tikzcd}
& \mathcal{C}(\sigma(M)) \arrow[r, "\Gamma"] \arrow[d, hook] & \mathcal{L}(H) \arrow[d, hook] & \\
\mathcal{B}(\sigma(M)) \arrow[r, hook, "\Phi"] \arrow[rrr, bend right, "\hat{\Gamma}"'] & \mathcal{C}(\sigma(M))'' \arrow[r, "\Gamma''"] & \mathcal{L}(H)'' \arrow[r, "P_{\mathcal{L}(H)}"] & \mathcal{L}(H)
\end{tikzcd}$$

which we shall refer to as the *Borel extension diagram.*

5.1. Borel Extension Diagram

Before diving into the main theorem of this chapter, we prove some simple facts about the structure and well-definedness of the above diagram.

Lemma 5.1. The space $(\mathcal{B}(\sigma(M)), ||\cdot||_\infty)$ is Banach.

Proof. We identify $\mathcal{B}(\sigma(M))$ as a subspace of $\ell^\infty(\sigma(M))$. Given that $(\ell^\infty(\sigma(M)), ||\cdot||_\infty)$ is a Banach space (cf. [B], Exa. 1.6), by the subspace criterion ([B], 1.7), it suffices to show that $(\mathcal{B}(\sigma(M)), ||\cdot||_\infty)$ is a closed subspace. To that end, let $(f_n) \subseteq \mathcal{B}(\sigma(M))$ be a sequence with $(f_n) \to f \in \ell^\infty(\sigma(M))$ with respect to $||\cdot||_\infty$. By assumption, f is bounded. Finally, uniform limits are also pointwise limits, so we conclude using that pointwise limits of measurable functions are measurable (cf. [Z3], Cor. 2.11), that $f \in \mathcal{B}(\sigma(M))$.

□

Lemma 5.2. The embedding $\Phi : \mathcal{B}(\sigma(M)) \hookrightarrow \mathcal{C}(\sigma(M))''$ is well-defined and continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{C}(\sigma(M))''$ with the weak* topology.

Proof. By the Riesz representation theorem, we know $\mathcal{M}(\sigma(M)) \cong \mathcal{C}(\sigma(M))'$. Given any Borel-measurable function $f \in \mathcal{B}(\sigma(M))$, we claim that the map $\Phi_f : \mathcal{M}(\sigma(M)) \cong \mathcal{C}(\sigma(M))' \to \mathbb{K}$ defined by $\mu \mapsto \int_{\sigma(M)} f \, d\mu$ is a functional, meaning we need to show linearity and continuity with respect to the given topologies. The former comes from the integral being linear. For the latter, let $(f_a) \subseteq \mathcal{B}(\sigma(M))$ be a net with $f_a \xrightarrow{\mu} f \in \mathcal{B}(\sigma(M))$. This means that for all $\mu \in \mathcal{M}(\sigma(M))$

$$\Phi_{f_a}(\mu) = \int_{\sigma(M)} f_a \, d\mu \to \int_{\sigma(M)} f \, d\mu = \Phi_f(\mu)$$

Using the definition of weak*-convergence, the latter is equivalent to $\Phi_{f_a} \xrightarrow{w*} \Phi_f$. Therefore, the map $\Phi : \mathcal{B}(\sigma(M)) \to \mathcal{C}(\sigma(M))''$, $f \mapsto \Phi_f$ defines the desired embedding.

□

The above lemma together with Goldstine's theorem yield a result worth mentioning, since it may not be entirely obvious at first glance.

Proposition 5.3 (μ-Approximation)**.** For any bounded, Borel-measurable function $f \in \mathcal{B}(\sigma(M))$, there is a net of continuous functions $(f_a) \subseteq \mathcal{C}(\sigma(M))$ with the property $f_a \xrightarrow{\mu} f$.

Proof. By Goldstine's theorem (Thm. 4.8), we have for each $f \in \mathcal{C}(\sigma(M))''$ some net $(f_a) \subseteq \mathcal{C}(\sigma(M))$ with $f_a \xrightarrow{w*} f$. Recalling that weak* convergence is tested by the predual, in this case by $\mathcal{C}(\sigma(M))' \cong \mathcal{M}(\sigma(M))$, we have $f_a \xrightarrow{\mu} f$. By L. 5.2, the set $\mathcal{B}(\sigma(M))$ embeds to $\mathcal{C}(\sigma(M))''$ with the desired continuity, thus yielding the claim.

□

Remark 5.4. Let us take a closer look at the center of the Borel extension diagram. From ([B], L. 3.49a), we know the double adjoint Γ'' of any operator $\Gamma : \mathcal{C}(\sigma(M)) \to \mathcal{L}(H)$ makes the following diagram commute:

$$\begin{tikzcd} \mathcal{C}(\sigma(M)) \arrow[r, "\Gamma"] \arrow[d, hook, "\iota_{\mathcal{C}(\sigma(M))}"'] & \mathcal{L}(H) \arrow[d, hook, "\iota_{\mathcal{L}(H)}"] \\ \mathcal{C}(\sigma(M))'' \arrow[r, "\Gamma''"] & \mathcal{L}(H)'' \end{tikzcd}$$

i.e. it holds

$$\Gamma'' \iota_{\mathcal{C}(\sigma(M))} = \iota_{\mathcal{L}(H)} \Gamma$$

Moreover, ([B], A. 28b) shows that because Γ is isometric, so is Γ''. Because Γ is an operator, it follows from Prop. 4.3 that Γ'' is w*-w*-continuous.

So far, we have constructed the composition $\Gamma''\Phi$ from the Borel extension diagram. We now need a way back from $\mathcal{L}(H)''$ to $\mathcal{L}(H)$. We do this by using a canonical map which is available to us because $\mathcal{L}(H)$ is a dual space.

Proposition 5.5. Let X be a normed space. Then, X' is closed and complemented in X''' by means of a canonical projection $P_{X'} : X''' \twoheadrightarrow X'$.

Proof. Let $\iota : X \hookrightarrow X''$ be the canonical embedding of a normed space into its bidual. Consider its adjoint $\iota' : X''' \to X'$, $x''' \mapsto x'''|_{X \subseteq X''}$, which is a restriction (cf. [B], A. 22b). It is clearly a projection, because restricting twice is the same as restricting once to the same subspace. Finally, by ([B], Sec. 7.5 L. D, b), the image $\iota'(X''') \cong X'$ is closed and complemented. So ι' yields the desired projection.

□

The following remark is crucial and is thus worth reading mindfully with the Borel extension diagram at hand.

Remark 5.6. By Prop. 5.5, there is a canonical projection $L_1(H)''' \twoheadrightarrow L_1(H)'$ and by Thm. 4.16, the space $\mathcal{L}(H)$ is the dual of $L_1(H)$. Hence, rewriting $\mathcal{L}(H) = L_1(H)'$, we get a projection

$$P_{\mathcal{L}(H)} : \mathcal{L}(H)'' \twoheadrightarrow \mathcal{L}(H)$$

Recall that the above map is the adjoint of the canonical embedding. The latter is an element of $\mathcal{L}('\mathcal{L}(H), \mathcal{L}(H)')$. Thus, by Prop. 4.3, its adjoint $P_{\mathcal{L}(H)}$ is w*-w*-continuous. Now, by definition, the weak* topology on $\mathcal{L}(H)$ is the weak*-operator topology, so the projection $P_{\mathcal{L}(H)}$ is continuous from $\mathcal{L}(H)''$ with the weak* topology into $\mathcal{L}(H)$ with the weak*-operator topology. Next, we can use Prop. 4.20 in order to conclude that the projection is even continuous into $\mathcal{L}(H)$ with the weak-operator topology.
Finally, postcomposing $\Gamma''\Phi$ with $P_{\mathcal{L}(H)}$, and recalling the former is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{C}(\sigma(M))''$ with the weak* topology, yields the map

$$\hat{\Gamma} := P_{\mathcal{L}(H)} \Gamma'' \Phi$$

which is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology. From now on, when we refer to the continuity of $\hat{\Gamma}$, we mean with respect to the topologies just mentioned.

5.2. Extension to Bounded, Borel-Measurable Functions

One should probably take a deep breath after having read the previous remark. Done? Good, because now it is time to harvest all the hard work we have done so far.

Theorem 5.7 (Borel Functional Calculus)**.** *The map $\hat{\Gamma}$ from Rmk. 5.6 is the unique, unital algebra $\star$-homomorphism which extends Γ from Thm. 3.6 as a continuous map from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology.*

Proof. We divide this proof into two steps.
Step 1: $\hat{\Gamma}$ is the unique extension of Γ. Let $f \in \mathcal{C}(\sigma(M))$. Then,

$$P_{\mathcal{L}(H)}\Gamma''(f) = P_{\mathcal{L}(H)}\Gamma(f) = \Gamma(f)$$

which means that $P_{\mathcal{L}(H)}\Gamma'' : \mathcal{C}(\sigma(M))'' \to \mathcal{L}(H)$ coincides with Γ over the continuous functions, and we also get $||P_{\mathcal{L}(H)}\Gamma''|| \geq 1$. Moreover, by submultiplicativity of norms, we have $||P_{\mathcal{L}(H)}\Gamma''|| \leq 1$. So, overall, the map $P_{\mathcal{L}(H)}\Gamma''$ preserves the norm of Γ and coincides with it over $\mathcal{C}(\sigma(M))$. Further, by Goldstine's theorem, Thm. 4.8, we know $\mathcal{C}(\sigma(M))$ is weakly* dense in its bidual.
Therefore, by Prop. 2.13, we conclude that $P_{\mathcal{L}(H)}\Gamma''$ is the unique extension of Γ with domain $\mathcal{C}(\sigma(M))''$. Precomposing with the inclusion $\Phi : \mathcal{B}(\sigma(M)) \hookrightarrow \mathcal{C}(\sigma(M))''$ simply restricts the domain. Therefore, $\hat{\Gamma}$ is the desired unique extension of Γ which is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology.
Step 2: $\hat{\Gamma}$ is a unital, algebra $\star$-homomorphism. By Prop. 5.3, we have that for any $f \in \mathcal{B}(\sigma(M))$ there is some net $(f_a) \subseteq \mathcal{C}(\sigma(M))$ with the property $f_a \xrightarrow{\mu} f$. Similar to the proof of Thm. 3.6, we can write

$$f(M) := \hat{\Gamma}(f) = \lim_a \Gamma(f_a) \tag{5.1}$$

where the limit is meant in the weak-operator topology on $\mathcal{L}(H)$. From (5.1), we clearly see that $\hat{\Gamma}$ is a unital algebra $\star$-homomorphism, since limits preserve the desired properties of Γ.

□

Remark 5.8. Another way to prove the existence of $f \mapsto f(M)$ is by inducing it using Lax-Milgram's theorem (cf. [B], A. 40b) through an appropriately chosen sesquilinear form (cf. [W], Thm. 3.3.3).

From now on, we shall employ both Borel sets *and* bounded, Borel-measurable functions. We will use the same notation for both sets, that is $\mathcal{B}(\sigma(M))$, and infer from context whether one refers to either a set or a function.
Theorem 5.7 extends our functional calculus vastly; in particular, we can now apply it to the class of LEGO® – I mean indicator – functions.

Corollary 5.9 (Indicator Calculus)**.** Given a self-adjoint operator M over a separable Hilbert space H, the map

$$\mathbb{1} : \mathcal{B}(\sigma(M)) \to \mathcal{L}(H),\ S \mapsto \mathbb{1}_S(M) := \hat{\Gamma}(\mathbb{1}_S)$$

is well-defined.

In the next section, we will see that the images of the above map are *orthogonal projections*, which will be the building blocks of an integral we will define in the upcoming chapter. The range of each image element is a closed subspace of H by ([B], Sec. 7.5, L. b), and the map

$$\Theta : (\sigma(M), \mathcal{B}(\sigma(M))) \to (H, \Gamma(H)),\ S \mapsto \Theta(S) := \operatorname{ran}(\mathbb{1}_S(M))$$

will turn out to be a so-called *spectral measure*. Such maps have analogous properties to those of probability measures.

5.3. Spectral Measures

In order to start building an integral which represents the Borel functional calculus, we begin to analyze measure-like maps called spectral measures. To define these, we first need to introduce a new setting in which the rest of this chapter will unfold. We shall combine the algebraic and order versions from [G] into one definition, so as to make the following more intuitive.

Definition 5.10 (Partially-Ordered Lattice - [G], Sec. 1.8, pp. 11-12)**.** Let X be a nonempty set. A **semilattice** is a tuple $(X, \star)$, where $\star : X \times X \to X$ is a binary, idempotent, commutative and associative operation. A **partially-ordered lattice** is a tuple $(X, \wedge, \vee, \leq)$, where $(X, \leq)$ is a partial order, and $(X, \wedge)$, $(X, \vee)$ are semilattices which satisfy the absorption identities:

$$a \wedge (a \vee b) = a \vee (a \wedge b) = a$$

For the rest of the chapter, we will be working solely with the following examples of partially-ordered lattices.

Example 5.11.

- Let H be a Hilbert space. If we endow $\Gamma(H)$ with the set inclusion as a partial order $\subseteq$ and the operations $U \wedge V := U \cap V$ and $U \vee V := \overline{U + V}$, then $(\Gamma(H), \wedge, \vee, \subseteq)$ is a partially-ordered lattice. It is called a **Hilbert lattice**. We shorten its notation by $(H, \Gamma(H))$.
- Similarly, given a measurable space (Ω, Σ), the σ-field $\Sigma \subseteq \mathcal{P}(\Omega)$ can also be made into a partially-ordered lattice $(\Sigma, \cap, \cup, \subseteq)$. We shall call it a **measure lattice** and simply write the measurable space itself (Ω, Σ) to refer to it.

As in the case for algebras, we need a notion for what it means for a map between lattices to preserve their structure, i.e. we need to define what a lattice homomorphism is. Using the metadefinition at the beginning of Sec. 2.2, the following should not come as a surprise.

Definition 5.12 (Lattice Homomorphism - [G], Sec. 3.2, p. 30)**.** Let (Ω, Σ) be a measure lattice and $(H, \Gamma(H))$ be a Hilbert lattice each endowed with the operations shown in Exa. 5.11. A map between lattices $\Theta : (\Omega, \Sigma) \to (H, \Gamma(H))$ is said to be a **lattice homomorphism**, if for all $S, T \in \Sigma$

$$\Theta(S \cap T) = \Theta(S) \wedge \Theta(T) \qquad \Theta(S \cup T) = \Theta(S) \vee \Theta(T)$$

A lattice homomorphism Θ is said to be **σ-closed**, if the image of countable intersections is the countable intersection of the images,

$$\Theta(\cap_{n\in\mathbb{N}} S_n) = \wedge_{n\in\mathbb{N}} \Theta(S_n)$$

Moreover, Θ is said to be **unital**, if the image of the ground set Ω equals the Hilbert space H,

$$\Theta(\Omega) = H$$

Finally, it is said to be **complemented**, if the image of the complement of a set is equal to the complemented subspace of the image of that same set,

$$\Theta(S^c) = \Theta(S)^{\perp}$$

Remark 5.13. For a unital, complemented, lattice homomorphism between a measure lattice (Ω, Σ) and a Hilbert lattice $(H, \Gamma(H))$, it holds $\Theta(\emptyset) = \Theta(\Omega^c) = \Theta(\Omega)^\perp = H^\perp = \{0\}$.

With these definitions in mind, we can now introduce a critical notion for the last two chapters.

Definition 5.14 (Spectral Measure)**.** A **spectral measure** Θ is a σ-closed, unital, complemented, lattice homomorphism

$$\Theta : (\Omega, \Sigma) \to (H, \Gamma(H)),\ S \mapsto \Theta(S)$$

whose images fulfill the following compatibility condition

$$\forall S, T \in \Sigma : [P_{\Theta(S)}, P_{\Theta(T)}] = 0 \tag{5.2}$$

Remark 5.15. Recall from measure theory that a probability measure $\mu : \Sigma \to \mathbb{R}_{\geq 0}$ is a map with the following properties:

$(M1)$ Σ is a σ-field.

$(M2)$ $\mu(\emptyset) = 0, \quad \mu(\Omega) = 1.$

$(M3)$ For a pairwise disjoint sequence of measurable sets $(U_n) \subseteq \Sigma$:

$$(U_n) \subseteq \Sigma \implies \mu(\coprod_{n\in\mathbb{N}} U_n) = \sum_{n\in\mathbb{N}} \mu(U_n)$$

Note the resemblance of probability measures to spectral measures: $(M1)$ can be interpreted as Θ having a measure lattice (Ω, Σ) as a domain. By Rmk. 5.13 together with the unital property, we get the analogue of $(M2)$. Finally, the σ-closedness is similar to $(M3)$.

The compatibility condition (5.2) is the quantum-propositional logic equivalent of the following:

If two events U and V are mutually exclusive, then U implies the opposite of V.

The above is the classical-propositional logic formulation of the *tertium non datur* principle: either the proposition is true or its negation. We show they are indeed equivalent. In order to do this, we need an important lemma.

Lemma 5.16 (Projection Equivalences)**.** For $U, V \in \Gamma(H)$, the following are equivalent:

i) $P_U P_V \in \mathcal{L}(H)$ is an orthogonal projection $\quad$ *ii)* $P_U P_V = P_{U\cap V}$ $\quad$ *iii)*$[P_U, P_V] = 0$

Proof. We show $ii) \implies i) \implies iii) \implies ii)$.
$ii) \implies i)$: This is clear.
$i) \implies iii)$: If $P_U P_V$ is an orthogonal projection, we have by its self-adjointness and that of P_U and P_V,

$$P_U P_V = (P_U P_V)^* = P_V^* P_U^* = P_V P_U$$

$iii) \implies ii)$: We show $P_U P_V x \in U \cap V$ and $(P_U P_V x, z) = (x, z)$ for all $x \in H$ and $z \in U \cap V$. To show the former claim, take some $x \in H$ and note that $P_U x \in U$ and $P_V x \in V$. Hence, given that $P_U P_V x = P_V P_U x$, we must have $P_U P_V x \in U \cap V$.
For the latter claim, we see by using $P_U z = P_V z = z$ for $z \in U \cap V$ and the self-adjointness of orthogonal projections that

$$(P_U P_V x, z) = (P_V x, z) = (x, z)$$

Since $P_{U\cap V}$ is characterized by the two properties above, we get $P_U P_V = P_{U\cap V}$.

$\square$

Proposition 5.17 (Tertium non datur)**.** For $U, V \in \Gamma(H)$ with $U \cap V = \{0\}$, we have the equivalence

$$U \perp V \iff [P_U, P_V] = 0$$

Proof. Take $U, V \in \Gamma(H)$ with $U \cap V = \{0\}$. We use the equivalences of L. 5.16 to conclude that

$$[P_U, P_V] = 0 \iff P_U P_V = P_{U \cap V} = P_{\{0\}} = 0$$
$$\iff \forall u \in U, v \in V : (u, v) = (P_U u, P_V v) = (u, P_{U \cap V} v) = (u, 0) = 0 \iff U \perp V$$

□

We now show that the map from Cor. 5.9 induces a spectral measure.

Theorem 5.18 (Spectral Measure)**.** *Given a self-adjoint operator M over a separable Hilbert space H, the map*

$$\mathbb{1} : \mathcal{B}(\sigma(M)) \to \mathcal{L}(H),\ S \mapsto \mathbb{1}_S(M)$$

has orthogonal projections as images, i.e. for each Borel set $S \in \mathcal{B}(\sigma(M))$, the map

$$\mathbb{1}_S(M) : H \to H, x \mapsto (\mathbb{1}_S(M))(x)$$

is an orthogonal projection.
Furthermore, the map

$$\Theta : (\sigma(M), \mathcal{B}(\sigma(M))) \to (H, \Gamma(H)),\ S \mapsto \Theta(S) := \operatorname{ran}(\mathbb{1}_S(M))$$

is a spectral measure.

Proof. The proof is given in great detail. However, the key idea to keep in mind is that all these properties follow from $f \mapsto f(M)$ being an algebra $\star$-homomorphism (cf. Thm. 5.7). Loosely speaking, this means that if $\mathbb{1}_S \in \mathcal{B}(\sigma(M))$ has some property, it will translate into $\mathbb{1}_S(M) \in \mathcal{L}(H)$ having it as well.
Step 1: The map $\mathbb{1}_S(M)$ is an orthogonal projection. Indeed, let $S \in \mathcal{B}(\sigma(M))$ be a Borel set. Since $\mathbb{1}_S^2 = \mathbb{1}_S = \overline{\mathbb{1}_S}$, it follows

$$\mathbb{1}_S(M)^2 = \mathbb{1}_S(M) = \mathbb{1}_S(M)^*$$

We conclude using ([B], Sec. 5.46 C).
Step 2: The map Θ is well-defined, because ranges of projections are closed by ([B], Sec. 7.5, L. b), i.e. $\operatorname{ran}(\mathbb{1}_S(M)) \in \Gamma(H)$.
Step 3: The map Θ is a spectral measure.
The images of Θ are compatible: note that $\mathbb{1}_S \mathbb{1}_T = \mathbb{1}_T \mathbb{1}_S$. Hence, for the induced operators, we also have

$$[\mathbb{1}_S(M), \mathbb{1}_T(M)] = 0 \tag{5.3}$$

Moreover, Θ is $\wedge$-preserving: by $\mathbb{1}_{S \cap T} = \mathbb{1}_S \mathbb{1}_T$, we have

$$\operatorname{ran}(\mathbb{1}_{S \cap T}(M)) = \operatorname{ran}(\mathbb{1}_S(M)) \wedge \operatorname{ran}(\mathbb{1}_T(M)) \tag{5.4}$$

Clearly, Θ is unital: since $\mathbb{1}_{\sigma(M)} = 1$ as a function in $\mathcal{B}(\sigma(M))$, we have

$$\operatorname{ran}(\mathbb{1}_{\sigma(M)}(M)) = H$$

Furthermore, Θ is complemented: note that because $\mathbb{1}_{S^c} + \mathbb{1}_S = 1$ and $S \cap S^c = \emptyset$, we get

$$\operatorname{ran}(\mathbb{1}_S(M)) \oplus \operatorname{ran}(\mathbb{1}_{S^c}(M)) = H \tag{5.5}$$

and thus, $\mathrm{ran}(\mathbb{1}_{S^c}(M)) = \mathrm{ran}(\mathbb{1}_S(M))^\perp$.
We also have that Θ is $\vee$-preserving: one sees using (5.4) and (5.5) together with De Morgan's law that

$$\begin{aligned}\mathrm{ran}(\mathbb{1}_{S\cup T}(M)) = \mathrm{ran}(\mathbb{1}_{(S^c\cap T^c)^c}(M)) = \mathrm{ran}(\mathbb{1}_{S^c\cap T^c}(M))^\perp = (\mathrm{ran}(\mathbb{1}_{S^c}(M)) \wedge \mathrm{ran}(\mathbb{1}_{T^c}(M)))^\perp \\ = (\mathrm{ran}(\mathbb{1}_S(M))^\perp \wedge \mathrm{ran}(\mathbb{1}_T(M))^\perp)^\perp = \mathrm{ran}(\mathbb{1}_S(M)) \vee \mathrm{ran}(\mathbb{1}_T(M))\end{aligned}$$

in short,

$$\mathrm{ran}(\mathbb{1}_{S\cup T}(M)) = \mathrm{ran}(\mathbb{1}_S(M)) \vee \mathrm{ran}(\mathbb{1}_T(M))$$

where we used the identity $\overline{U+V} = (U^\perp \cap V^\perp)^\perp$ for closed subspaces $U, V \in \Gamma(H)$ (cf. L. A.1).
Finally, Θ is σ-closed: since we have shown Θ is a lattice homomorphism, it preserves infima, i.e. for a sequence $(S_n) \subseteq \mathcal{B}(\sigma(M))$, we get

$$\mathrm{ran}(\mathbb{1}_{\cap_{n\in\mathbb{N}} S_n}(M)) = \wedge_{n\in\mathbb{N}}\mathrm{ran}(\mathbb{1}_{S_n}(M))$$

$\square$

6. Integration with Spectral Measures

In the following, let H be a separable Hilbert space, $M \in \mathcal{L}(H)$ be a self-adjoint operator, and (Ω, Σ) be a measurable space.
Although the preceding chapter already provided us with a mapping $f \mapsto f(M)$ for $f \in \mathcal{B}(\sigma(M))$, we would like to use the measure-like properties of spectral measures (cf. Rmk. 5.15) in order to define an integral which agrees with our functional calculus. In our case, we want a map

$$\mathrm{I} : \mathcal{B}(\Omega) \to \mathcal{L}(H), f \mapsto \mathrm{I}(f) := \int_\Omega f \, dP$$

which has the properties of a unital algebra $\star$-homomorphism. Indeed, we will show that such a map has the property that when using the ground set $\Omega = \sigma(M)$, we retrieve the Borel functional calculus $f \mapsto f(M)$ from the previous chapter, i.e.

$$f(M) = \int_{\sigma(M)} f \, dP$$

The idea of *how* to construct such an integral will be to mimic the essence of the construction of the Lebesgue integral for bounded, Borel-measurable functions.[1] We shall use the following steps as a guideline:

Proof Strategy 6.1 (Integral Construction)**.** We define an integral using the spectral measure from Thm. 5.18 constructively for the following functions in the given order:

I) Indicator functions.

II) Simple functions.

III) Bounded, Borel-measurable functions.

The key is that simple functions are linear combinations of indicator functions, and that for every bounded, Borel-measurable function f, there exists a sequence of simple functions which converges uniformly to f (cf. Prop. 2.12). In other words, the above classes of functions are ordered in such a way that the latter can be generated using the former.

6.1. Spectral Integral

In the upcoming theorem, the properties we will show are: well-definedness and being a non-expansive, unital algebra $\star$-homomorphism. During the proof, we will indicate which class of functions we are currently constructing the integral for, so as to make the proof more digestible.

Theorem 6.2 (Spectral Integral)**.** *The mapping*

$$\mathrm{I} : \mathcal{B}(\Omega, \Sigma) \to \mathcal{L}(H), \ f \mapsto \int_\Omega f \, dP$$

[1] The classical construction goes from *indicator* functions to non-negative *simple* functions; next to non-negative *measurable* functions; then to *measurable* functions; and finally to *integrable* functions. Mind that we are constructing the integral for a different class of functions and hence, we take another (but still analogous) approach.

defined constructively by

$$\int_\Omega \mathbb{1}_S \, dP := \mathbb{1}_S(M) \tag{6.1}$$

is a nonexpansive, unital algebra $\star$-homomorphism.
In particular, setting $\Omega = \sigma(M)$ yields the Borel functional calculus from Thm. 5.7,

$$f(M) = \int_{\sigma(M)} f \, dP$$

Proof. We divide the proof into two main parts: Part 1, in which we show the integral is a nonexpansive, unital $\star$-homomorphism, and this will be in turn subdivided into roman numerals referring to each function class in Proof Strategy 6.1; and Part 2, in which we show the integral coincides with the map from Thm. 5.7.

Part 1

I) First, let $f = \mathbb{1}_S$ for some measurable set $S \in \Sigma$. Then, we have $\int_\Omega f \, dP = \mathbb{1}_S(M)$ by assumption (6.1). By Thm. 5.7, the integral is a unital algebra $\star$-homomorphism on the set of indicator functions and has a norm less or equal to one.
II) Second, take a simple function over Ω, i.e. $f = \sum_{i=1}^N \lambda_i \mathbb{1}_{S_i}$, where $(\lambda_i) \subseteq \mathbb{C}$ and the sets $(S_i) \subseteq \Sigma$ form a finite measurable partition of Ω. Because we want the integral to be linear, we define

$$\int_\Omega f \, dP := \sum_{i=1}^N \lambda_i \mathbb{1}_{S_i}(M) \tag{6.2}$$

Step II.1: The integral is well-defined. To show this, we need to prove that the above definition agrees with the indicator function case (which is clear) and that if we have two different representations of f, we still get the same integral. In order to see the latter, consider $f = \sum_{i=1}^N \lambda_i \mathbb{1}_{S_i} = \sum_{j=1}^M \mu_j \mathbb{1}_{T_j}$ for scalars $(\mu_j) \subseteq \mathbb{C}$ and a finite measurable partition $(T_j) \subseteq \Sigma$. We need to show that

$$\sum_{i=1}^N \lambda_i \mathbb{1}_{S_i}(M) = \sum_{j=1}^M \mu_j \mathbb{1}_{T_j}(M) \tag{6.3}$$

Similar to ([Z1], L. 14.4), we first consider the case in which the partition (T_j) is a one-point refinement of (S_i), meaning that there exists some $k \in \{1, ..., N\}$ such that $S_k = T_k \cup T_{k+1}$, and for $i \neq k$, we have $S_i = T_i$ if $i < k$, and $S_i = T_{i+1}$ if $i > k$ (cf. Fig. 6.1). Then, $M = N + 1$ and

$$\sum_{j=1}^M \mu_j \mathbb{1}_{T_j}(M) = \sum_{i=1, i\neq k}^N \lambda_i \mathbb{1}_{S_i}(M) + \lambda_k \mathbb{1}_{T_k}(M) + \lambda_k \mathbb{1}_{T_{k+1}}(M) = \sum_{i=1}^N \lambda_i \mathbb{1}_{S_i}(M)$$

Since f is a simple function, the general case follows by repeating the above procedure *finitely many* times. Indeed, for two arbitrary, finite, measurable partitions $(S_i), (T_j) \subseteq \Sigma$, we have that the set

$$\langle (S_i), (T_j) \rangle := \{U = S, T, S \setminus T, T \setminus S, S \cap T \neq \emptyset \mid S \in (S_i), T \in (T_j)\}$$

generates a common measurable refinement of (S_i) and (T_j), and thus (6.3) holds.
Step II.2: The integral is a unital algebra $\star$-homomorphism. We have that our map is linear by the way we defined it in (6.2), and unital, since $\int_\Omega 1 \, dP = \int_\Omega \mathbb{1}_\Omega \, dP = \mathrm{id}$. Moreover, it is

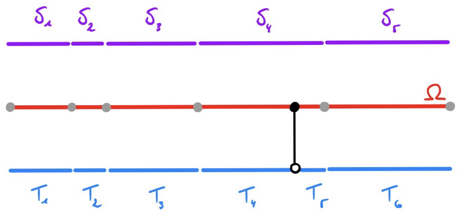

Figure 6.1.: Example of a one-point refinement for $k = 4$.

multiplicative: for any two simple functions $f = \sum_{i=1}^{N} \lambda_i \mathbb{1}_{S_i}$ and $g = \sum_{j=1}^{M} \mu_j \mathbb{1}_{T_j}$, we have

$$\begin{aligned}\int_\Omega fg\, dP &= \int_\Omega \sum_{i=1}^{N}\sum_{j=1}^{M} \lambda_i\mu_j \mathbb{1}_{S_i}\mathbb{1}_{T_j}\, dP = \sum_{i=1}^{N}\sum_{j=1}^{M} \lambda_i\mu_j \mathbb{1}_{S_i}(M)\mathbb{1}_{T_j}(M) \\ &= \sum_{i=1}^{N} \lambda_i \mathbb{1}_{S_i}(M) \cdot \sum_{j=1}^{M} \mu_j \mathbb{1}_{T_j}(M) = \Big(\int_\Omega f\, dP\Big) \cdot \Big(\int_\Omega g\, dP\Big)\end{aligned}$$

where we used the compatibility condition as in (5.3).
The $\star$-property follows straightforwardly:

$$\int \overline{f}\, dP = \sum_{i=1}^{N} \overline{\lambda_i} \mathbb{1}_{S_i}(M)^* = \Big(\sum_{i=1}^{N} \lambda_i \mathbb{1}_{S_i}\Big)^* = \Big(\int_\Omega f\, dP\Big)^*$$

where in the first equal sign, we used that $\hat{\Gamma}$ from Thm. 5.7 has the $\star$-property.
Step II.3: The integral is nonexpansive:

$$\Big|\Big| \int_\Omega f\, dP \Big|\Big| = \Big|\Big| \sum_{i=1}^{N} \lambda_i \mathbb{1}_{S_i}(M) \Big|\Big| \leq \max_{0\leq i\leq N} |\lambda_i| \cdot 1 = ||f||_\infty$$

III) Thirdly, take a bounded, Borel-measurable function $f \in \mathcal{B}(\Omega, \Sigma)$. By Prop. 2.12, there exists a sequence of simple functions (f_n) which converges uniformly to f. Again, since we want the integral to be linear, we set

$$\int_\Omega f\, dP := \lim_{n\to\infty} \int_\Omega f_n\, dP \tag{6.4}$$

Step III.1: The integral is well-defined. Since the elements in the approximating sequences (f_n) are simple functions, the above definition coincides with the one in Step II. Now, take two sequences of simple functions $(g_n), (h_n)$ which converge uniformly to f. We must show that

$$\lim_{n\to\infty} \int_\Omega g_n\, dP = \lim_{n\to\infty} \int_\Omega h_n\, dP$$

This is quite straightforward to see using the nonexpansiveness from Step II.3 and taking the limit:

$$\Big|\Big| \int_\Omega (g_n - h_n)\, dP \Big|\Big| \leq ||g_n - h_n||_\infty \to 0$$

Step III.2: The integral is a unital algebra $\star$-homomorphism. The unital property is clear; the $\star$-property follows from the (anti)linearity of conjugation, the $\star$-property from Step II and taking limits; and linearity comes from how we defined the integral in (6.4) and using that limits themselves are linear as well. It is also multiplicative: let (f_n) and (g_n) be two sequences of simple functions which converge uniformly to f and g, respectively. Then $(f_n g_n)$ converges uniformly to fg. So, using the multiplicative property from Step II and that of limits,

$$\int_\Omega fg\,dP = \lim_{n\to\infty}\int_\Omega f_n g_n\,dP = \lim_{n\to\infty}\int_\Omega f_n\,dP \cdot \lim_{n\to\infty}\int_\Omega g_n\,dP = \int_\Omega f\,dP \cdot \int_\Omega g\,dP$$

Step III.3: The integral is nonexpansive:

$$\left|\left|\int_\Omega f\,dP\right|\right| = \left|\left|\lim_{n\to\infty}\int_\Omega f_n\,dP\right|\right| = \lim_{n\to\infty}\left|\left|\int_\Omega f_n\,dP\right|\right| \le \lim_{n\to\infty}||f_n||_\infty = ||f||_\infty$$

where we used the continuity of norms to swap the limit out and then back inside the norms, and the inequality follows from Step II.3. We are done constructing the integral.

Part 2

We now have to show that

$$f(M) = \int_{\sigma(M)} f\,dP$$

i.e., that the integral coincides with the Borel functional calculus we defined in the previous chapter whenever we set $\Omega = \sigma(M)$.

It suffices to show that the map $\mathcal{B}(\sigma(M)) \to \mathcal{L}(H), f \mapsto \int_{\sigma(M)} f\,dP$ is a unital, $\star$-homomorphism which is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology and that it coincides with the map from Thm. 5.7 on the set of Borel-measurable *indicator* functions. The fact that both expressions coincide for *all* Borel-measurable functions $f \in \mathcal{B}(\sigma(M))$ follows by the density result in Prop. 2.12 and the to-be-shown continuity of the integral.

Step 1: Setting $\Omega = \sigma(M)$ and using Part 1, we get that the map $f \mapsto \int_{\sigma(M)} f\,dP$ is a unital algebra $\star$-homomorphism.

Step 2: The fact that $f \mapsto f(M)$ and the integral coincide for indicator functions follows by assumption (6.1).

Step 3: The integral is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology: Let $(f_a) \subseteq \mathcal{B}(\sigma(M))$ be a net with $f_a \xrightarrow{\mu} f \in \mathcal{B}(\sigma(M))$. Note that for each $x, y \in H$, we have that the following map is a functional over $\mathcal{C}(\sigma(M))$:

$$\mathcal{C}(\sigma(M)) \ni g \mapsto l_{x,y}(g) = \left(\int_{\sigma(M)} g\,dPx, y\right)$$

and hence, by the Riesz representation theorem, there exists a measure $\mu_{x,y} \in \mathcal{M}(\sigma(M))$ with

$$\int_{\sigma(M)} g\,d\mu_{x,y} = \left(\int_{\sigma(M)} g\,dPx, y\right)$$

Thereby, we get by definition of (f_a) converging to f in the initial $\mathcal{M}(\sigma(M))$-topology,

$$\left(\int_{\sigma(M)} f_a\,dPx, y\right) = \int_{\sigma(M)} f_a\,d\mu_{x,y} \to \int_{\sigma(M)} f\,d\mu_{x,y} = \left(\int_{\sigma(M)} f\,dPx, y\right)$$

which yields the desired continuity and concludes our proof.

□

Remark 6.3. Note that if f is real-valued, then the operator $\int_\Omega f\,dP$ is self-adjoint:

$$\left(\int_\Omega f\,dP\right)^* = \int_\Omega \overline{f}\,dP = \int_\Omega f\,dP$$

since the integral map I has the $\star$-property.

7. Bounded Spectral Theorem

We are ready to generalize the spectral theorem from linear algebra into an infinite-dimensional setting. We follow the construction in [W].

Theorem 7.1 (Bounded Spectral Theorem)**.** *Let H be a separable Hilbert space, $M = M^* \in \mathcal{L}(H)$ a self-adjoint operator, and $f : (\Omega, \Sigma) \to (\mathbb{K}, \mathcal{B}(\mathbb{K}))$ a bounded, Borel-measurable function. Then, there exists*

- *a probability measure $\mu \in \mathcal{M}(\sigma(M))$,*
- *a measurable Hilbert bundle χ over $\sigma(M)$,*
- *and an isometric isomorphism $\mathcal{U} : L^2(\sigma(M); \chi) \to H$*

such that the following diagram commutes

$$\begin{array}{ccc} L^2(\sigma(M);\chi) & \xrightarrow{\mathcal{M}_f} & L^2(\sigma(M);\chi) \\ \mathcal{U}\big\downarrow & & \big\downarrow\mathcal{U} \\ H & \xrightarrow{\quad f(M) \quad} & H \end{array}$$

where $\mathcal{M}_f : L^2(\sigma(M); \chi) \to L^2(\sigma(M); \chi)$, $\mathcal{M}_f g(\lambda) = f(\lambda)g(\lambda)$ is the multiplication operator corresponding to f.

In order to show the above theorem, we require a strong preliminary result. We show that any given spectral measure induces a probability measure and an isometric isomorphism between H and a well-chosen space with a special property. One could say the next result provides the cherry to the spectral cake – I mean spectral theorem, Thm. 7.1 – since the latter will simply be the feast – I mean conclusion – of all our work. But hang on, because this cherry is two and a half pages long! We encourage the reader to take a short break and grab a snack before reading this thoroughly.

Lemma 7.2. Let (Ω, Σ) be a measure lattice, $(H, \Gamma(H))$ be a Hilbert lattice over a separable Hilbert space H, and $\Theta : (\Omega, \Sigma) \to (H, \Gamma(H))$ be a spectral measure. Then, there exist

- a probability measure μ on (Ω, Σ) and
- an isometric isomorphism $\mathcal{U} : L^2(\Omega; \chi) \to H$

such that for all $S \in \Sigma$

$$\Theta(S) = \mathcal{U}(L^2(S; \chi)) \tag{7.1}$$

Proof. ([W], Thm. 3.4.2, Cor. 3.4.3.)
Step 1: Construct a special family of vectors with help of the spectral measure. For any $v \in H$, define $E_v := \{P_{\Theta(S)}v \mid S \in \Sigma\}$ and let $(v_k) \subseteq \mathcal{B}_H$ be a maximal family of unit vectors such that for $k \neq j$ we have $E_{v_k} \perp E_{v_j}$ $(\star)$. Since H is a separable Hilbert space, by ([B], Thm. 5.28), the family (v_k) is at most countably infinite. So, without loss of generality, assume $(v_k)_{k \in \mathbb{N}}$. Moreover, the family (E_k) with $E_k := E_{v_k}$ generates H: otherwise, there would exist some unit

vector $v \in H$ such that for all $k \in \mathbb{N}$ it holds $v \perp E_k$. This in turn would imply for all $k \in \mathbb{N}$ and $S, T \in \Sigma$,

$$(P_{\Theta(S)}v, P_{\Theta(T)}v_k) = (v, P_{\Theta(S)}P_{\Theta(T)}v_k) = (v, P_{\Theta(S\cap T)}v_k) = 0$$

where we used the self-adjointness of orthogonal projections and the $\wedge$-compatibility of spectral measures. But then, the family $(v_k) \cup \{v\} \supsetneq (v_k)$ also fulfills the condition $(\star)$, contradicting the maximality of (v_k). Thus, $\bigoplus_{k\in\mathbb{N}} E_k = H$.

Step 2: Construct the probability measure μ and induce a natural family of Radon-Nikodym derivatives $(f_k)_{k\in\mathbb{N}}$.

Step 2.1: For all $k \in \mathbb{N}$, define

$$\mu_k : \Sigma \to \mathbb{R},\ S \mapsto \mu_k(S) := ||P_{\Theta(S)}v_k||^2$$

We claim these are probability measures. We check the conditions from Rmk. 5.15. Property $(M1)$ is clearly fulfilled. Moreover, for all $k \in \mathbb{N}$,

$$\mu_k(\emptyset) = ||P_{\Theta(\emptyset)}v_k||^2 = ||0 \cdot v_k||^2 = 0 \qquad \mu_k(\Omega) = ||P_{\Theta(\Omega)}v_k||^2 = ||\mathrm{id} \cdot v_k||^2 = 1$$

where we used Rmk. 5.13, as well as the unital property of spectral measures together with $v_k \in \mathcal{B}_H$. This gives $(M2)$. Finally, choose a pairwise disjoint sequence $(S_n) \subseteq \Sigma$. Then,

$$\mu_k(\coprod_{n\in\mathbb{N}} S_n) = ||P_{\Theta(\coprod_{n\in\mathbb{N}} S_n)}v_k||^2 = ||P_{\vee_{n\in\mathbb{N}}\Theta(S_n)}v_k||^2 = ||\sum_{n\in\mathbb{N}} P_{\Theta(S_n)}v_k||^2 = \sum_{n\in\mathbb{N}} ||P_{\Theta(S_n)}v_k||^2$$

where we used σ-closedness, the complement-property of spectral measures, and in the last step, we used the Pythagorean theorem since the summands are pairwise orthogonal by construction. This proves the claim.

Step 2.2: We claim that the map

$$\mu : \Sigma \to \mathbb{R},\ S \mapsto \mu(S) := \sum_{k\in\mathbb{N}} 2^{-k}\mu_k(S)$$

is the desired probability measure. Indeed, the fact that μ fulfills the requirements seen in Rmk. 5.15 would hold for *any* family of probability measures indexed by the natural numbers $(\mu_k)_{k\in\mathbb{N}}$. Again, it has a σ-field as a domain, so $(M1)$ is fulfilled, and $(M2)$ follows from the measure-theoretic convention $0 \cdot \infty = 0$ and the geometric series, $\sum_{k\in\mathbb{N}} 2^{-k} \cdot 1 = 1$. Finally, $(M3)$ follows by using the measure property $(M3)_k$ of each μ_k, and interchanging the order of summation, which is possible since we are summing over non-negative numbers (cf. [Mas], Ch. 7, pp. 60-61). The fact that μ fulfills the requirements of our lemma will be seen in the next steps.

Step 2.3: We have that $\mu_k \ll \mu$, since if $\mu(S) = 0$ it directly follows from the definition of μ and the non-negativity of measure values, that $\mu_k(S) = 0$. Thus, for each $k \in \mathbb{N}$, by the Radon-Nikodym theorem, we have $d\mu_k = f_k d\mu$, for some non-negative, μ-integrable function $f_k \in L^1(\Omega, \Sigma, \mu)$.

Step 3: Construct a corresponding Hilbert bundle χ. For each $k \in \mathbb{N}$ and for $n \in \mathbb{N}_0 \cup \{\infty\}$, define

$$S_k := \{x \in \Omega \mid f_k(x) > 0\} \qquad \Omega_n := \{x \in \Omega \mid x \in S_k \text{ for exactly } n \text{ values of } k\}$$

Thus, $(\Omega_n) \subseteq \Sigma$ is a measurable partition of Ω. For each n, let H_n be an n-dimensional Hilbert space, and let $\chi := \coprod_{n\in\mathbb{N}_0\cup\{\infty\}}(\Omega_n \times H_n)$. Using Prop. 2.15, we have $L^2(\Omega;\chi) \cong \bigoplus_{k\in\mathbb{N}} L^2(S_k, \mu|_{S_k})$.

Step 4: Construct the desired isometric isomorphism. This will turn out to be a composition of two maps, which we shall refer to as the *right* and the *left* map.

Step 4.1: Define the *right* map.
Step 4.1.1: We start with the summands for the *right* map. We claim that for each $k \in \mathbb{N}$ the assignment

$$V_k : L^2(S_k, \mu|_{S_k}) \to L^2(S_k, \mu_k|_{S_k}),\ g \mapsto g/\sqrt{f_k}$$

is an isometric isomorphism. Surjectivity is clear. The isometry follows by the definition of the Radon-Nikodym derivative: note that by the definition of S_k, the expression $d\mu|_{S_k} = \frac{d\mu_k}{f_k}$ is well-defined, and hence,

$$\left|\left|\frac{g}{\sqrt{f_k}}\right|\right|^2_{L^2(S_k,\mu_k|_{S_k})} = \int_{S_k} \left(\frac{g}{\sqrt{f_k}}\right)^2 d\mu_k = \int_{S_k} g^2\, d\mu = ||g||^2_{L^2(S_k,\mu|_{S_k})}$$

from which we also get injectivity.
Step 4.1.2: To get the *right* map, we first take the direct sum $W := \bigoplus_{k\in\mathbb{N}} V_k$ and then use Prop. 2.15 twice to get

$$V : L^2(\Omega;\chi) \cong \bigoplus_{k\in\mathbb{N}} L^2(S_k, \mu|_{S_k}) \xrightarrow{W} \bigoplus_{k\in\mathbb{N}} L^2(S_k, \mu_k|_{S_k}) \cong \bigoplus_{k\in\mathbb{N}} L^2(\Omega, \mu_k) \tag{7.2}$$

which is the desired *right* map, as we shall see.
Step 4.2: Construct the *left* map.
Step 4.2.1: For each $k \in \mathbb{N}$, consider the following map over the set of measurable indicator functions,

$$\mathbb{1}_S \mapsto U_k(\mathbb{1}_S) := P_{\Theta(S)} v_k$$

and extend it linearly to their span in $L^2(\Omega, \Sigma, \mu_k)$. The reader should feel a density argument of Prop. 2.12 coming up. We have that each U_k is a well-defined, isometric operator, since

$$\begin{aligned}\Big(U_k(\sum_{i=1}^N a_i \mathbb{1}_{S_i}), U_k(\sum_{j=1}^M b_j \mathbb{1}_{S_j})\Big) &= \sum_{i,j=1}^{N,M} a_i\overline{b_j}(P_{\Theta(S_i)}v_k, P_{\Theta(S_j)v_k}) \\ &= \sum_{i,j=1}^{N,M} a_i\overline{b_j}\mu_k(S_i \cap S_j) \\ &= \Big(\sum_{i=1}^N a_i \mathbb{1}_{S_i}, \sum_{j=1}^M b_j \mathbb{1}_{S_j}\Big) \end{aligned} \tag{7.3}$$

where in the second equal sign we used

$$\begin{aligned}\mu_k(S_i \cap S_j) = ||P_{\Theta(S_i\cap S_j)}v_k||^2 &= (P_{\Theta(S_i\cap S_j)}v_k, P_{\Theta(S_i\cap S_j)}v_k) \\ &= (v_k, P_{\Theta(S_i\cap S_j)}v_k) = (P_{\Theta(S_i)}v_k, P_{\Theta(S_j)v_k})\end{aligned}$$

Thus, from (7.3), we see that the map is indeed an isometry, and hence bounded and injective. Moreover, by the above calculations, $\sum_{i=0}^N a_i \mathbb{1}_{S_i} = 0$ implies $U_k\big(\sum_{i=0}^N a_i \mathbb{1}_{S_i}\big) = 0$, and thus U_k is well-defined over the extended domain. Using Prop. 2.12, we have that U_k extends to an isometric embedding $\hat{U}_k : L^2(\Omega, \mu_k) \hookrightarrow H$, whose range is E_k by construction. Finally, because $\bigoplus_{k\in\mathbb{N}} E_k = H$, the map $T := \bigoplus_{k\in\mathbb{N}} \hat{U}_k$ is an isometric isomorphism,

$$T : \bigoplus_{k\in\mathbb{N}} L^2(\Omega, \mu_k) \xrightarrow{\cong} H \tag{7.4}$$

Step 4.2.2: We show that for all $S \in \Sigma$, the map T fulfills $\Theta(S) = T\big(\bigoplus_{k\in\mathbb{N}} L^2(S, \mu_k|_S)\big)$. Take some measurable sets $B, S \in \Sigma$. Note that $B \subseteq S$ implies by the order-preserving property of spectral measures, that $\hat{U}_k(\mathbb{1}_B) \in \Theta(S)$. Thus, by linearity and continuity also $\hat{U}_k(L^2(S, \mu_k|_S)) \subseteq \Theta(S)$ and thereby,

$$T\left(\bigoplus_{k\in\mathbb{N}} L^2(S, \mu_k|_S)\right) \subseteq \Theta(S)$$

The converse subset relation follows by taking complements on both sides and the relation $A \subseteq B \iff A^c \supseteq B^c$. Therefore,

$$T\left(\bigoplus_{k\in\mathbb{N}} L^2(S, \mu_k|_S)\right) = \Theta(S)$$

Step 5: We conclude the construction by taking the right map V from (7.2) and the left map T from (7.4), and setting

$$\mathcal{U} := TV : L^2(\Omega; \chi) \xrightarrow{\cong} H$$

This yields the desired map and concludes the proof.

□

We are now ready to show the bounded spectral theorem (Thm. 7.1). All the hard work has already been done in the preceding result and previous chapters. The following is a ride to enjoy.

Proof. (Bounded Spectral Theorem, ([W], Thm. 3.5.1))
Take the spectral measure Θ from Thm. 5.18, and let μ, χ and $\mathcal{U}$ be as in the previous result.
Claim: The resolution map

$$R : \mathcal{B}(\sigma(M)) \to \mathcal{L}(H),\ f \mapsto \mathcal{U}\mathcal{M}_f\mathcal{U}^*$$

is a untial, algebra $\star$-homomorphism, which is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology.
Step 1: R is a unital algebra $\star$-homomorphism. The map R is clearly well-defined and linear. Next, R has the unital property, because $R(1) = \mathcal{U}\mathcal{M}_1\mathcal{U}^* = \mathcal{U}\mathcal{U}^* = \mathrm{id}$. The $\star$-property follows from $R(\overline{f}) = \mathcal{U}\mathcal{M}_{\overline{f}}\mathcal{U}^* = \mathcal{U}\mathcal{M}_f^*\mathcal{U}^* = (\mathcal{U}\mathcal{M}_f\mathcal{U}^*)^* = R(f)^*$. Further, the multiplicative property is seen as $R(fg) = \mathcal{U}\mathcal{M}_{fg}\mathcal{U}^* = \mathcal{U}\mathcal{M}_f\mathcal{M}_g\mathcal{U}^* = (\mathcal{U}\mathcal{M}_f\mathcal{U}^*)(\mathcal{U}\mathcal{M}_g\mathcal{U}^*) = R(f)R(g)$.
Step 2: R is continuous from $\mathcal{B}(\sigma(M))$ with the initial $\mathcal{M}(\sigma(M))$-topology into $\mathcal{L}(H)$ with the weak-operator topology. Take a net $(f_a) \subseteq \mathcal{B}(\sigma(M))$ with $f_a \xrightarrow{\mu} f \in \mathcal{B}(\sigma(M))$. Similar to Part 2, Step 3 of the proof of Thm. 6.2, for each $x, y \in H$, we have a measure $\nu_{x,y} \in \mathcal{M}(\sigma(M))$ with

$$\int_{\sigma(M)} f\, d\nu_{x,y} = \big(R(f)x, y\big)$$

and thereby,

$$\big(R(f_a)x, y\big) = \int_{\sigma(M)} f_a\, d\nu_{x,y} \to \int_{\sigma(M)} f\, d\nu_{x,y} = \big(R(f)x, y\big)$$

meaning $R(f_a) \xrightarrow{w} R(f)$.
Step 3: Since R agrees with the Borel functional calculus from Thm. 5.7 on the set of measurable *indicator* functions, we conclude by the continuity of R shown in Step 2 and the density provided by Prop. 2.12 that

$$f(M) = \int_{\sigma(M)} f\, dP = \mathcal{U}\mathcal{M}_f\mathcal{U}^*$$

The particular case $f = \mathrm{id}$ yields the spectral resolution of M,

$$M = \mathcal{U}\mathcal{M}_{\mathrm{id}}\mathcal{U}^*$$

□

8. References

- [B] F. Bornemann: Functional Analysis, Lecture Notes 2023.
- [Ba] U. Bauer: Topology, Lecture Notes 2022.
- [BK] A. Bowers, N.J. Kalton: An Introductory Course in Functional Analysis, Springer, 2014.
- [F] G. Friesecke: Functional Analysis, Lecture Notes 2022.
- [G] G. Grätzer: Lattice Theory: Foundation, Birkhäuser, Springer Basel AG 2011.
- [K] G. Kemper: Lineare Algebra 1 und 2, Skript 2021.
- [L] P.D. Lax, Functional Analysis, Wiley, 2002.
- [M] R.E. Megginson, An Introduction to Banach Space Theory, Springer, 1998.
- [Mas] P. Massopust: Analysis 3, Lecture Notes 2022.
- [R] H. L. Royden: Real Analysis, 4th Edition, 2010.
- [W] N. Weaver: Mathematical Quantization, CRC Press, 2001.
- [Z1] M. Brokate and J. Zimmer: Analysis 1, Lecture Notes 2021/2022.
- [Z2] M. Brokate and J. Zimmer: Analysis 2, Lecture Notes 2021.
- [Z3] M. Brokate and J. Zimmer: Analysis 3, Lecture Notes 2022/2023.

A. Bonus Material

This section was included to avoid distracting the reader from the main purpose of Thm. 5.18. In that result, specifically in Step 3.4, we used the following lemma:

Lemma A.1 ($\wedge$-$\vee$-$\perp$ Lemma - [W], Cor. 2.2.6)**.** For closed subspaces U, V of a Hilbert space H, we have

$$(U^\perp \wedge V^\perp)^\perp = U \vee V$$

Proof. We need to show $(U^\perp \cap V^\perp)^\perp = \overline{U+V}$. We first have by ([Z3], Ü. 11.2) that $U^{\perp\perp} = U$ for closed subspaces $U \in \Gamma(H)$. Next, assume that $x \in \overline{U+V}$. Then, $x \perp U, V$, so $x \in U^\perp \cap V^\perp$. Conversely, if $x \in U^\perp \cap V^\perp$, then x is orthogonal to every element of U and V, and thus by continuity of the scalar product, it is also orthogonal to the closed span of the sum of U and V. Overall, $(U+V)^\perp = U^\perp \cap V^\perp$. We conclude using the last two results:

$$(U \cap V)^\perp = (U^{\perp\perp} \cap V^{\perp\perp})^\perp = \overline{(U^\perp + V^\perp)}^{\perp\perp} = \overline{(U^\perp + V^\perp)}$$

which can be rewritten by changing the spaces U, V for their respective complements

$$(U^\perp \cap V^\perp)^\perp = \overline{U+V}$$

□

B. Proof Map

The following map is meant for the reader who wishes to see how the most important results in each chapter are connected.

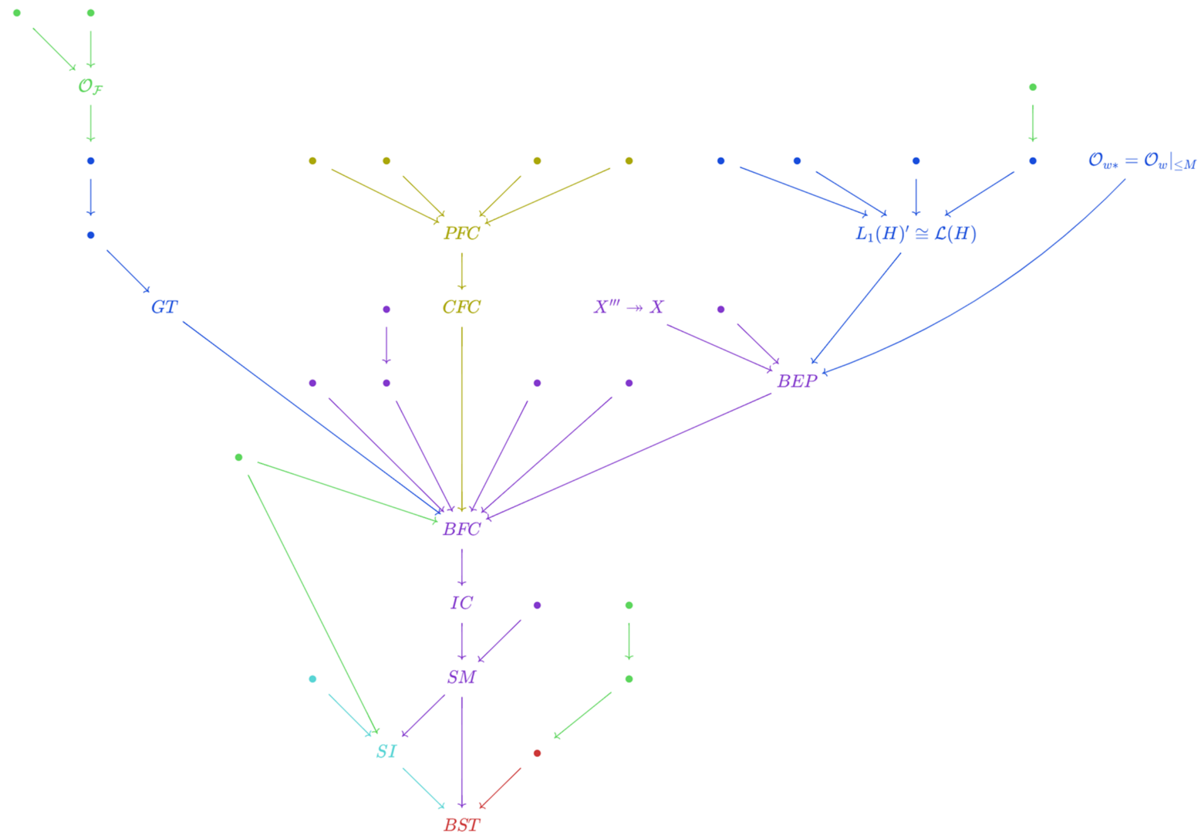

Figure B.1.: Proof Map.

Firstly, each node is a statement which was proven somewhere in the thesis. In addition, each color represents a different chapter: light green stands for Chapter 2, yellow for Chapter 3, dark blue for Chapter 4, purple for Chapter 5, light blue for Chapter 6, and red for Chapter 7. Finally, the most important results were given an abbreviation instead of leaving them as a bullet in the map: $\mathcal{O}_{\mathcal{F}}$ stands for Thm. 2.8; *PFC* for *Polynomial Functional Calculus*, Prop. 3.3; *CFC* for *Continuous Functional Calculus*, Thm. 3.6; *GT* for *Goldstine's Theorem*, Thm. 4.8; $L_1(H)' \cong \mathcal{L}(H)$ for Thm. 4.16; $\mathcal{O}_{w^*} = \mathcal{O}_w|_{\leq M}$ for Prop. 4.20; $X''' \twoheadrightarrow X'$ for Prop. 5.5; *BEP* for the *Bidual Extension Property*, Rmk. 5.6; *BFC* for the *Borel Functional Calculus* 5.7; *IC* for *Indicator Calculus*, Cor. 5.9; *SM* for *Spectral Measure*, Thm. 5.18; *SI* for *Spectral Integral*, Thm. 6.2; and *BST* for the *Bounded Spectral Theorem*, Thm. 7.1.
Remark: Although $\mathcal{O}_{\mathcal{F}}$ may not seem as connected as other results, recall that it was this result that gave us the way into other notions of continuity and therefore, the connectivity of the map understates its importance. To mend this, I have positioned it as the highest and leftmost result with label in the map.