Poincaré Inequalities

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μ satisfies a Poincaré inequality wrt some "gradient" $\iff$ (T_t) ergodic Markov semigroup converges to μ exponentially fast wrt the same "gradient"

TLDR:

Thm (Poincaré) X reversible, ergodic, $\mu \in \partial_e \mathcal{I}(X)$. TFAE:

1) $\mathrm{Var}_\mu f \leq c\,\mathcal{E}(f,f)$

2) $\| T_t f - \int f d\mu \|_{\mathcal{L}^2_\mu} \leq e^{-t/c} \| f - \int f d\mu \|_{\mathcal{L}^2_\mu}$.

3) $\mathcal{E}(T_t f, T_t f) \leq e^{-2t/c} \mathcal{E}(f,f)$.

Best constant $c^{-1} :=$ spectral gap.

Goal: Show Poincaré's Theorem and Gaussian - Poincaré as corollary.

Recap: Representations of Markov processes.

Motivation: $\begin{cases} \dot{x}(t) = A x(t) \\ x(0) = x_0 \end{cases} \rightsquigarrow_{\exists ! \text{ soln.}} \underbrace{x_0 \mapsto x(t, x_0)}_{\text{Solution Operator}} = e^{tA} x_0 =: T_t x_0$

Kolmogorov Equation

$\frac{d}{dt} e^{tA} x_0 = A e^{tA} x_0 = e^{tA} A x_0 \rightsquigarrow \frac{d}{dt} T_t f = \mathcal{L} T_t f = T_t \mathcal{L} f$

$f \in \mathcal{C}_0(S)$

$f \in \mathcal{D}(\mathcal{L})$

"$T_t f = e^{tA} f$"

Markov Process

$(X_t)_{t \geq 0}$

Semigroup

$(T_t f)(x) := E_x f(X_t)$

"Initial Condition"

$= \int_\Omega f(X_t(\omega)) \, d\mathbb{P}_x(\omega) = E(f(X_t) \mid X_0 = x)$

$T_0 = id$, $T_{t+s} = T_t T_s$, $\|T_t\| \leq 1$

Generator

$\mathcal{L} f := \lim_{t \downarrow 0} \frac{T_t f - f}{t}$ on $\mathcal{D}(\mathcal{L})$

Stationarity $\forall t \geq 0, f \in \mathcal{C}_0(S): \int T_t f \, d\mu = \int f \, d\mu$

μ-average invariant under dynamics $\mu \in \mathcal{I}(X)$

$\cup\!\!\mid \leftarrow$ We take stationarity in def. of ergodicity.

Ergodicity $\int \Big| \underbrace{T_t f}_{\text{Time Average}} - \overbrace{\int f \, d\mu}^{\text{Space Average}} \Big|^2 d\mu \xrightarrow[t \to \infty]{} 0$

Stationarity + Time average $\xrightarrow{L^2_\mu}$ Space average $\mu \in \partial_e \mathcal{I}(X)$

Poincaré: What can we say about rate?

Reversibility T_t $\mathcal{L}^2(\mu)$-selfadjoint.

Forward Dynamics $\stackrel{d}{=}$ Backward Dynamics

Dirichlet Form $\mathcal{E}(f, g) := -\int f \mathcal{L} g \, d\mu$.

"Inner product tailored to the Markov process"

Thm (Poincaré) X reversible, ergodic, $\mu \in \partial_e \mathcal{I}(X)$. TFAE:

Variance bounded by Dirichlet Form

1) $\mathrm{Var}_\mu f \leq c\, \mathcal{E}(f,f)$

Cvgce. to Space Average

$= \int T_t f \, d\mu$ by μ stationary!

2) $\| T_t f - \int f \, d\mu \|_{\mathcal{L}^2_\mu} \leq e^{-t/c} \| f - \int f\, d\mu \|_{\mathcal{L}^2_\mu}$.

$\rightsquigarrow$ 2) $\Leftrightarrow$ $\mathrm{Var}_\mu T_t f \leq e^{-2t/c} \mathrm{Var}_\mu f$

Exponential Decay of Variance of the Dynamics

Exponential Contractivity of Dirichlet Form

3) $\mathcal{E}(T_t f, T_t f) \leq e^{-2t/c} \mathcal{E}(f,f)$.

Exercise: (Ornstein-Uhlenbeck) (Hints left in in your slides)

Why? Use OU to derive GP inequality using Poincaré.

$$X_t = e^{-t} X_0 + e^{-t} B_{e^{2t}-1}, \quad X_0 \perp B$$

$$(T_t f)(x) = E\, f(e^{-t}x + \sqrt{1-e^{-2t}}\,\xi), \quad \xi \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

$$\mathcal{L} f = -x D_x f + D_x^2 f$$

$+ \mu = \mathcal{N}(0,1)$ ergodic

Standard Gaussian

$$\mathcal{E}(f,g) = \int D_x f \cdot D_x g \, d\mu, \quad d\mu = g_{0,1}\, d\lambda \implies \mathcal{E}(f,f) = \int (D_x f)^2 d\mu \overset{X \sim \mathcal{N}(0,1)}{=} \boxed{E\, D_x f(X)^2}$$ seems familiar!

Stationarity $\int T_t f \, d\mu = \int f \, d\mu$

Ergodicity $T_t f \longrightarrow \int f \, d\mu$ in $\mathcal{L}^2(\mu)$

Reversibility $\langle f, \mathcal{L} g \rangle_\mu = \langle \mathcal{L} f, g \rangle_\mu$

1) $\mathrm{Var}_\mu f \le c\,\mathcal{E}(f,f)$.

2) $\mathrm{Var}_\mu T_t f \le e^{-2t/c}\,\mathrm{Var}_\mu f$

3) $\mathcal{E}(T_t f, T_t f) \le e^{-2t/c}\,\mathcal{E}(f,f)$.

Cor (Gaussian-Poincaré) $X \sim \mathcal{N}(0,1) \Rightarrow \mathrm{Var}\, f(X) \le E\, D_x f(X)^2$.

Pf: We show 3) holds.

$(T_t f)(x) = \int f(e^{-t}x + \sqrt{1-e^{-2t}}\,\xi)\, dP$

$$D_x T_t f(x) \overset{DCT+}{=} e^{-t}\, T_t D_x f(x) \quad (*)$$

$d\mu = g_{0,1}\, d\lambda$, i.e. $X_* P = \mu$ ✓

$\mathcal{E}(f,g) = \int D_x f \cdot D_x g\, d\mu$

$$\Rightarrow \mathcal{E}(T_t f, T_t f) = \| D_x T_t f \|^2_{\mathcal{L}^2_\mu} \overset{(*)}{=} e^{-2t}\, \| T_t D_x f\|^2_{\mathcal{L}^2_\mu} \overset{\|T_t\|\le 1}{\le} e^{-2t}\, \|D_x f\|^2_{\mathcal{L}^2_\mu} = e^{-2t/1}\,\mathcal{E}(f,f)$$

$\Rightarrow$ 3) holds with $c = 1$. Thus, by Poincaré, the claim follows. □

Prove Poincaré with:

Lemma I $\quad \frac{d}{dt}\mathrm{Var}_\mu T_t f = -2\,\mathcal{E}(T_t f, T_t f)$. $\quad$ Stationarity of μ.

Lemma II $\quad \mathrm{Var}_\mu f = 2\int_0^\infty \mathcal{E}(T_t f, T_t f)\, d\lambda(t)$. $\quad$ Ergodicity and FTC.

Product Rule + $\frac{d}{dt}T_t f = \mathcal{L} T_t f$

$$\text{I)} \quad \frac{d}{dt}\mathrm{Var}_\mu T_t f = \frac{d}{dt}\int (T_t f)^2 - \Big(\underbrace{\int T_t f\, d\mu}_{=\int f d\mu \text{ (Stationarity)} \rightsquigarrow \text{constant}}\Big)^2 d\mu = \frac{d}{dt}\int (T_t f)^2\, d\mu \overset{DCT+}{=} \int \frac{d}{dt}(T_t f)^2\, d\mu = 2\int T_t f\, \mathcal{L} T_t f\, d\mu.$$

$$\text{II)} \quad \mathrm{Var}_\mu f \overset{T_0 = id}{=} \mathrm{Var}_\mu T_0 f - \underbrace{\mathrm{Var}_\mu\Big(\int f\, d\mu\Big)}_{\substack{=0 \\ \text{"additive zero"}}} \overset{\text{(Ergodicity)}+DCT}{=} \dots \lim_{t\to\infty}\mathrm{Var}_\mu(T_t f) \quad \overset{FTC}{=} -\int_0^\infty \frac{d}{dt}\mathrm{Var}_\mu T_t f\, d\lambda(t) \overset{\text{I)}}{=} 2\int_0^\infty \mathcal{E}(T_t f, T_t f)\, d\lambda(t)$$

(I) $\frac{d}{dt}\mathrm{Var}_\mu T_t f = -2\mathcal{E}(T_t f, T_t f)$.

(II) $\mathrm{Var}_\mu f = 2\int_0^\infty \mathcal{E}(T_t f, T_t f)\, d\lambda(t)$.

1) $\mathrm{Var}_\mu f \leq c\,\mathcal{E}(f,f)$.

2) $\mathrm{Var}_\mu T_t f \leq e^{-2t/c}\,\mathrm{Var}_\mu f$

3) $\mathcal{E}(T_t f, T_t f) \leq e^{-2t/c}\,\mathcal{E}(f,f)$.

See later (Reversibility)

Poincaré 2) $\Leftrightarrow$ 1) $\Leftarrow$ 3)

Pf:

"1) $\Rightarrow$ 2)"
$$\frac{d}{dt}\mathrm{Var}_\mu T_t f \overset{\mathrm{I)}}{=} -2\,\mathcal{E}(T_t f, T_t f) \overset{1)}{\leq} -\frac{2}{c}\mathrm{Var}_\mu T_t f$$

$$\overset{\text{Grönwall}}{\Longrightarrow} \mathrm{Var}_\mu T_t f \leq e^{-2t/c}\,\mathrm{Var}_\mu f.$$

"2) $\Rightarrow$ 1)"
$$2\,\mathcal{E}(f,f) \overset{\mathrm{I)}}{=} -\frac{d}{dt}\mathrm{Var}_\mu(T_t f)\Big|_{t=0} = \lim_{t\downarrow 0}\frac{\mathrm{Var}_\mu f - \overbrace{\mathrm{Var}_\mu T_t f}^{\overset{2)}{\leq} e^{-2t/c}\underbrace{\mathrm{Var}_\mu f}_{\text{independent of } t}}}{t}$$

$$\overset{2)}{\geq} \lim_{t\downarrow 0}\frac{1-e^{-2t/c}}{t}\mathrm{Var}_\mu f \overset{-d/dt|_{t=0}}{=} \frac{2}{c}\mathrm{Var}_\mu f.$$

"3) $\Rightarrow$ 1)"
$$\mathrm{Var}_\mu f \overset{\mathrm{II)}}{=} 2\int_0^\infty \mathcal{E}(T_t f, T_t f)\, d\lambda(t) \overset{3)}{\leq} 2\int_0^\infty \underbrace{\mathcal{E}(f,f)}_{\text{independent of } t} e^{-2t/c}\, d\lambda(t)$$

$$= c\,\mathcal{E}(f,f).$$

□

1) $\mathrm{Var}_\mu f \le c\,\mathcal{E}(f,f)$.

2) $\mathrm{Var}_\mu T_t f \le e^{-2t/c}\,\mathrm{Var}_\mu f$

3) $\mathcal{E}(T_t f, T_t f) \le e^{-2t/c}\,\mathcal{E}(f,f)$.

For "2) $\Rightarrow$ 3)", we require reversibility.

Lemma III (T_t) reversible $\Rightarrow t \overset{\Phi}{\longmapsto} \log \mathrm{Var}_\mu T_t f$ convex.

$\Rightarrow$ Use $\frac{d}{dt}\Phi$ nondecreasing.

Hint: $f \in \mathcal{C}^2(\mathbb{R})$ convex $\Leftrightarrow f'' \geq 0$

and prove $\mathcal{E}(\cdot,\cdot)$ satisfies a CS-like inequality $\mathcal{E}(f,g)^2 \le \mathcal{E}(f,f)\,\mathcal{E}(g,g)$.

"2) $\Rightarrow$ 3)" $t \overset{\frac{d}{dt}\Phi}{\longmapsto} \frac{d}{dt}\log \mathrm{Var}_\mu T_t f \overset{\text{Calculate}}{=} -\frac{2\,\mathcal{E}(T_t f, T_t f)}{\mathrm{Var}_\mu T_t f} \overset{\substack{T_0 = id \\ + \\ \text{III})}}{\geq} -\frac{2\,\mathcal{E}(f,f)}{\mathrm{Var}_\mu f}$.

$$\overset{\text{Rearrange}}{\Longrightarrow} \frac{\mathcal{E}(T_t f, T_t f)}{\mathcal{E}(f,f)} \le \frac{\mathrm{Var}_\mu T_t f}{\mathrm{Var}_\mu f} \overset{2)}{\le} e^{-2t/c}.$$

$\square$

1) Spectral Gap

μ-adjoint

$$S = \{1,\dots,d\} \Rightarrow \mathbb{R}^{d\times d} \ni \mathcal{L} \overset{\text{Reversible}}{=} \mathcal{L}^{*_\mu} \overset{\text{Spectral Theorem}}{\Longrightarrow} \sigma(\mathcal{L}) \subseteq \mathbb{R} \text{ with ONB.}$$

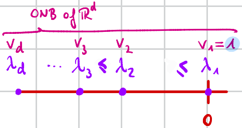
$$(T_t f)(x) := e^{t\mathcal{L}} f_x \quad \xleftrightarrow{x \in S = \{1,\dots,d\}} \quad \mathcal{L} f_x$$

"Maximum Principle"

$$0 \overset{(G2)}{\le} \mathcal{E}(f,f) = -\langle f, \mathcal{L} f \rangle_\mu \Rightarrow \sigma(\mathcal{L}) \le 0.$$

Probability Generator

$$\mathcal{L}\mathbb{1} \overset{(G1)}{=} 0 = 0 \cdot \mathbb{1} \Rightarrow \lambda_1 := 0 \in \sigma(\mathcal{L}).$$



$$f \perp \mathbb{1} \Rightarrow \mathcal{E}(f,f) = -\langle f, \mathcal{L} f \rangle_\mu \ge -\lambda_2 \|f\|^2_{\mathcal{L}^2(\mu)} = (\underbrace{\lambda_1 - \lambda_2}_{\text{Spectral Gap} = c^{-1}}) \mathrm{Var}_\mu(f)$$

$f = v_2 \Rightarrow$ becomes equality $\rightsquigarrow$ is tight.

$\mathcal{L} v_2 = \lambda_2 v_2$

$f = \sum_{i=1}^d a_i v_i$ + bound is tight

Controls precisely exponential cvg. rate of semigroup

$$\mathrm{Var}_\mu f \le \underbrace{(\lambda_1 - \lambda_2)^{-1}}_{= c} \mathcal{E}(f,f) \Rightarrow 1) \overset{\text{Poincaré}}{\iff} 2) \iff \sup_{f = \sum_{i=1}^d a_i v_i} \frac{\sum_{i=2}^d e^{2\lambda_i t} a_i^2}{\sum_{i=2}^d a_i^2} = e^{-2(\lambda_1 - \lambda_2)t}.$$

2) Sample Paths $t \longmapsto X_t^{(\sigma)}(\omega)$ for $\omega \in \Omega$ and $dX_t^{(\sigma)} = -X_t\,dt + \sigma\,dB_t$, $\sigma \in \{1, \dots, 6\}$

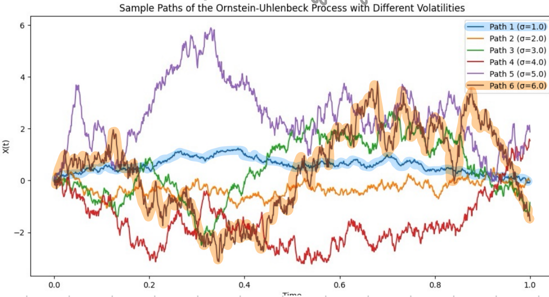
$X_0^{(\sigma)} = 0$

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters for the Ornstein-Uhlenbeck process
a = 1          # mean reversion rate
theta = 0      # long-term mean
x0 = 0         # initial value
T = 1          # total time
dt = 0.001     # time step
n = int(T / dt)  # number of steps
num_paths = 6    # number of sample paths

# Define different volatilities for each path
volatilities = [1.0, 2.0, 3.0, 4.0, 5.0, 6.0]  # Adjust each value for each path

# Generate the paths
time = np.linspace(0, T, n)
paths = np.zeros((num_paths, n))
for i in range(num_paths):
    sigma = volatilities[i]  # Set sigma for each path
    x = x0
    for j in range(1, n):
        dx = a * (theta - x) * dt + sigma * np.sqrt(dt) * np.random.randn()
        x += dx
        paths[i, j] = x

# Plot the sample paths
plt.figure(figsize=(12, 6))
for i in range(num_paths):
    plt.plot(time, paths[i], label=f'Path {i + 1} (σ={volatilities[i]})')
plt.xlabel("Time")
plt.ylabel("X(t)")
plt.title("Sample Paths of the Ornstein-Uhlenbeck Process with Different Volatilities")
plt.legend()
plt.show()
```

Scaling property of BM

$= 0$ here

(One) Motivation: Physics — velocity of a particle undergoing BM (recall $t \mapsto B_t(\omega)$ is not diff'ble for a.e. $\omega \in \Omega$). [3]



$dX_t = -a X_t\,dt + \sigma\,dB_t$

$X_0 = 0$

The End.

Questions?

References: 1) Van Handel, Probability in High Dimensions.
2) Liggett, Continuous Time Markov Processes.
3) Yeo, Eventually Almost Everywhere, Ornstein-Uhlenbeck Process.