



CONVERGENCE TO EQUILIBRIUM

Entropy Methods in Partial Differential
Equations and Stochastic Analysis

Alejandro Morera Alvarez

MA8111: Projekt mit Kolloquium (Bachelor)

CONTENTS AND MAIN QUESTIONS

Entropy: Motivation and Problem Statement

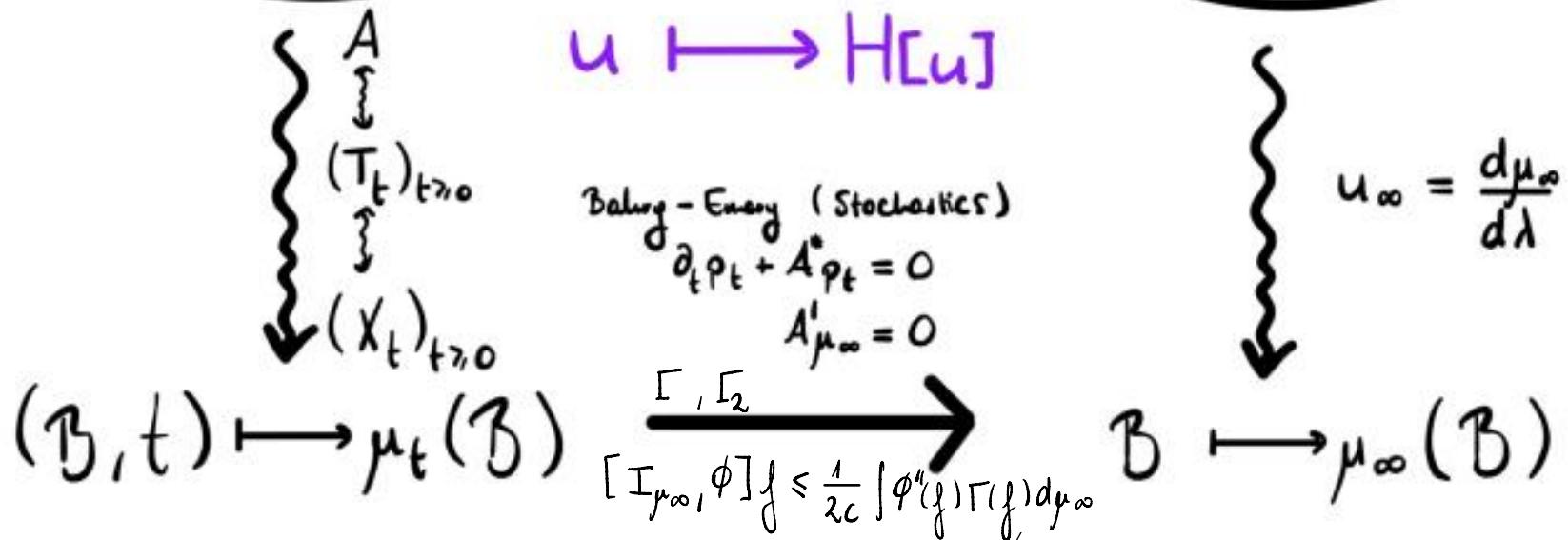
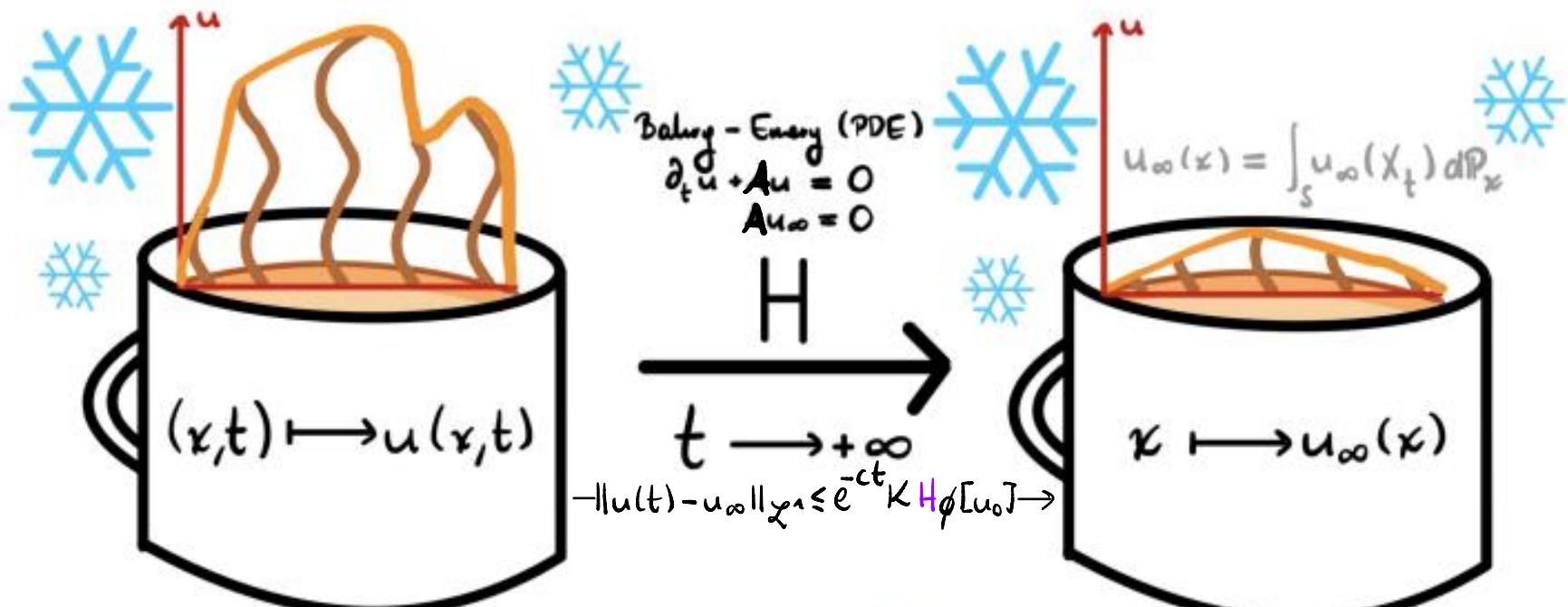
- **What is entropy? What is an entropy functional?**

Entropy Methods: Bakry-Emery

- **How can we use entropy functionals in order to determine the rate of convergence?**

Stationarity: Function-Measure Duality

- **Why are stationary solutions of interest in both PDEs and stochastic processes?**



Convergence to Equilibrium

pdedek04
@gmail.com

MOTIVATION FOR ENTROPY

Physics \Rightarrow Stochastics:

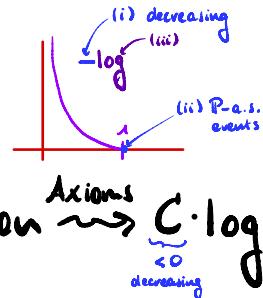
$$\mathcal{S} := E_{\mathbb{P}} I(X) := \int_{\Omega} I(f(X(\omega))) d\mathbb{P}(\omega)$$



PDEs

where $f: S \rightarrow [0,1]$
 $x \mapsto \mathbb{P}(X=x)$

$I: [0,1] \mapsto [0, \infty)$ \rightsquigarrow Information Function $\rightsquigarrow C \cdot \log$



$$\partial_t u(x, t) + Au(x, t) = 0 \text{ for } x \in \mathbb{R}^d, t > 0, \quad u(x, 0) = u_0(x) \text{ for } x \in \mathbb{R}^d$$

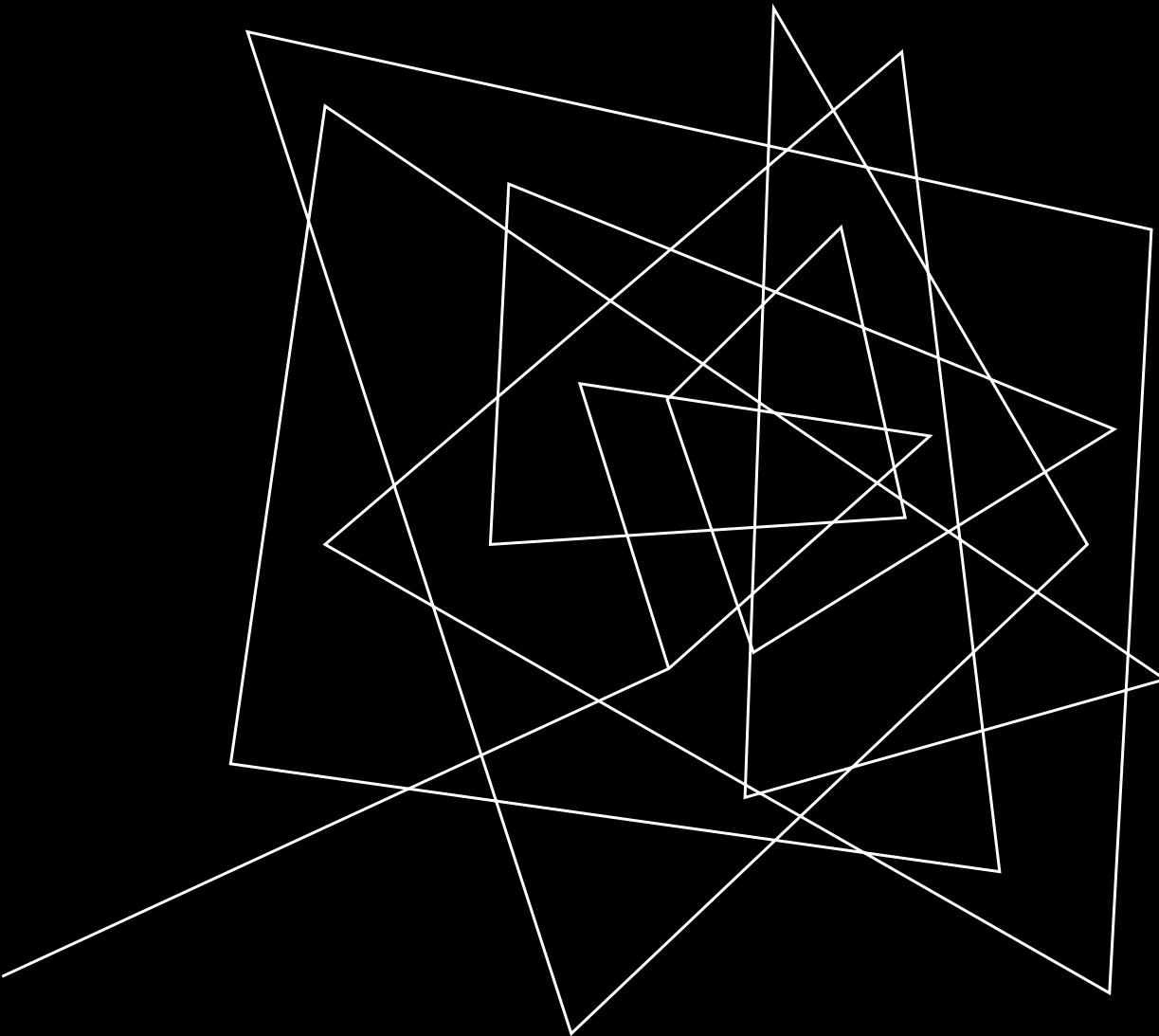
$A: D(A) \rightarrow X'$ where $D(A) \subseteq X$ Banach space.

Lyapunov functional - $H: D(A) \rightarrow \mathbb{R}$

Entropy functional - Convex Lyapunov functional.

$$H[f] = \int_{\mathbb{R}^d \times \mathbb{R}^d} f \log f d\lambda d\nu$$

$$H_1[u] := \int_{\mathbb{T}^d} u(\log(u) - 1) d\lambda$$



ENTROPY METHODS

Convergence to equilibrium with various distance notions and nonlinear evolution equations

- **How can we use entropy functionals in order to determine the rate of convergence?**

GENERAL ENTROPY METHODS

1. Compute the entropy production

$$-\frac{d}{dt}H[u(t)] = \underbrace{\langle A(u(t)),}_{\text{Recall } A \text{ maps into } X^*} \underbrace{H'[u(t)] \rangle}_{\text{where the braces } \langle \cdot, \cdot \rangle \text{ denote some kind of Banach space product.}}$$

Recall A maps into X^* .
E.g. $\langle f, g \rangle = \int fg d\lambda$ for
 $p \geq 1$, $f \in L^p$, $g \in L^q$, $\bar{p} + \bar{q} = 1$

2. Derive an inequality of the form

$$cH[u] \leq \langle A(u(t)), H'[u(t)] \rangle$$

($H'[u(t)]$ denotes the Fréchet derivative for general normed spaces of H).

exist. This implies $dH/dt \leq -cH$.

3. Use Gronwall's lemma to conclude that

$$H[u(t)] \leq H[u_0] \exp(-ct).$$

This structure will arise naturally by using the evolution equation itself.



Bakry-Emery: One step by
bounding second derivative of
entropy functional.

EXAMPLE: LINEAR FOKKER-PLANCK

$$\partial_t u = \nabla \cdot (\nabla u + u \nabla V) =: A^* u, \quad t > 0, \quad u(0) = u_0 \text{ in } \mathbb{R}^d \quad (2.1)$$

Now, let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a smooth, strictly convex function with $\phi(1) = \phi'(1) = 0$ and $1/\phi''$ concave. Define an entropy functional by

$$H_\phi[u] := \int \phi(u/u_\infty) d\lambda.$$

Theorem 2.2.3 (Exponential Decay of Fokker-Planck, [J], Thm. 2.1) Let $\phi \in C^4([0, \infty))$ be as above, assume $H_\phi[u_0] < \infty$ and let V be a potential satisfying the Bakry-Emery condition: there exists some $\lambda > 0$ with

$$\boxed{\nabla^2 V - \lambda \cdot \text{id} \geq 0.} \quad (2.2)$$

Uniformly
convex
potential

Then, any normalized, smooth solution to (2.1) converges exponentially fast to the steady state in L^1 , i.e. for all $t > 0$, we have

$$\|u(t) - u_\infty\|_{L^1} \leq \exp(-t\lambda) K_\phi H_\phi[u_0]^{1/2}$$

$$\int \phi\left(\frac{u}{u_\infty}\right) u_\infty d\lambda \leq \frac{1}{2\lambda} \int \phi''\left(\frac{u}{u_\infty}\right) \left\| \nabla \frac{u}{u_\infty} \right\|^2 u_\infty d\lambda$$

KEY STEP: UPPER BOUND SECOND DERIVATIVE

1. Compute the entropy production

$$\begin{aligned} -\frac{d}{dt} H[u(t)] &= -\frac{d}{dt} \int \phi(\tilde{u}/u_\infty) d\lambda \\ &\stackrel{\text{PDE and IBP}}{=} \int \phi''(g) \|\nabla g\|^2 u_\infty d\lambda \end{aligned}$$

Lebesgue

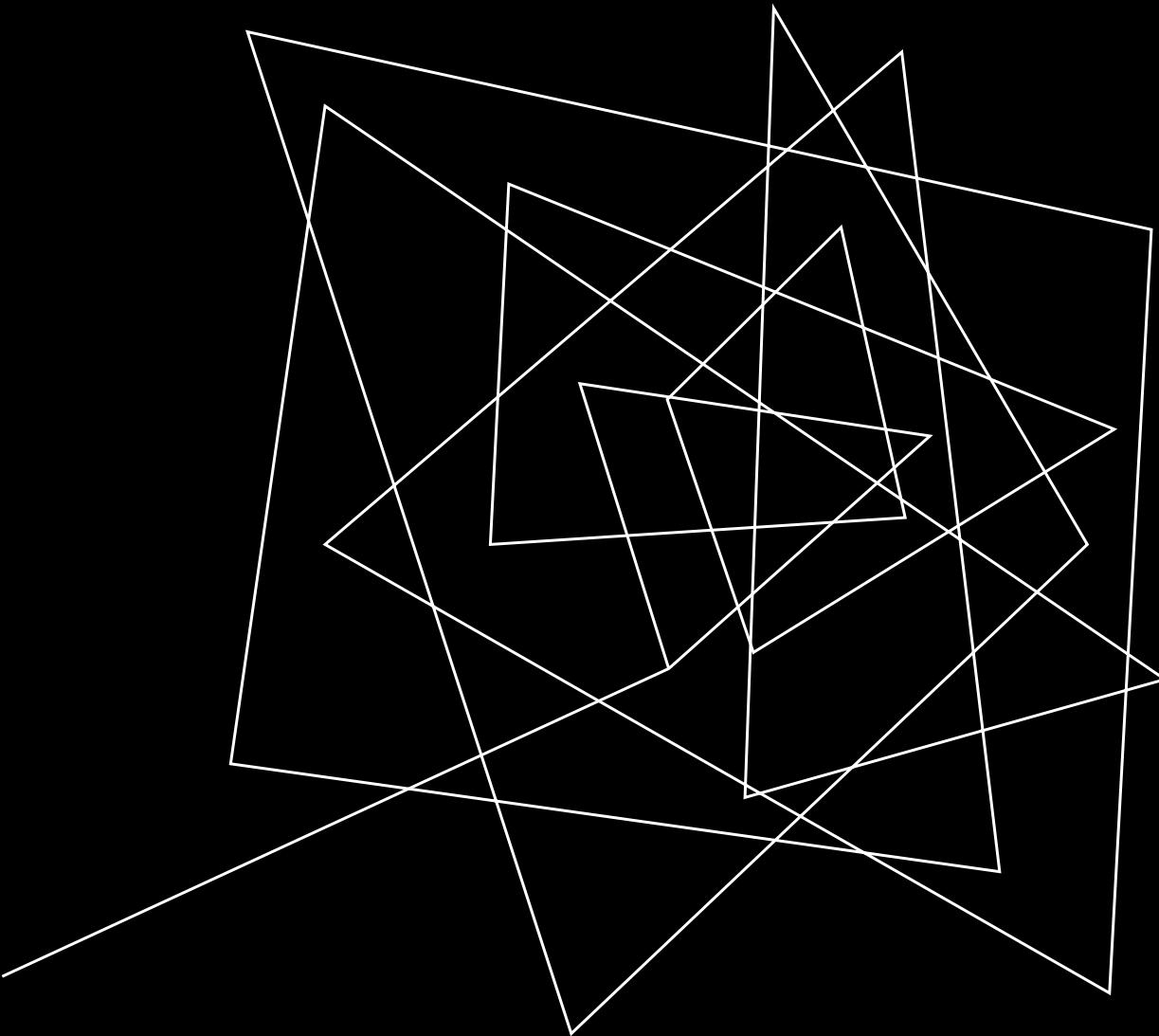
* Assumptions on ϕ

Key Step

$$\begin{aligned} 2. \text{ Bound second time-derivative: } \frac{d^2}{dt^2} H_\phi[u] &= - \int \phi'''(g) \partial_t u \|\nabla g\|^2 + 2\phi''(g) \nabla g^\top \partial_t \nabla g u_\infty d\lambda \\ &= - \int \phi''(g) (\nabla \cdot (\nabla^2 g \nabla g) - \|\nabla^2 g\|^2 - \langle \nabla g, \nabla V \rangle_{\nabla^2 g} - \langle \nabla g, \nabla g \rangle_{\nabla^2 V} u_\infty d\lambda \\ &\quad + \int (\phi''''(g) \|\nabla g\|^4 + 2\phi'''(g) \langle \nabla g, \nabla g \rangle_{\nabla^2 g}) u_\infty d\lambda) \\ &\stackrel{\substack{\star \\ C^0[0, \infty)}}{\leq} -\lambda \|\nabla g\|^2 \\ &\stackrel{\substack{\star \\ \text{IBP and convexity}}}{{\geq} 2 \int \phi''(g) \|\nabla^2 g + \phi'''(g)/\phi''(g) \nabla g \otimes \nabla g\|^2 u_\infty d\lambda} + \int (\phi''''(g) - 2\phi'''(g)^2/\phi''(g)) \|\nabla g\|^4 u_\infty d\lambda \\ &\quad - 2\lambda \frac{d}{dt} H[u] \Rightarrow \boxed{\frac{d^2}{dt^2} H_\phi[u] \geq -2\lambda H_\phi[u]} \end{aligned}$$

Assumption
 ≤ 0
 ≥ 0

3. Conclude with Cronwall and a special inequality (Csiszar-Kullback-Pinsker).



STATIONARITY

Time-independent solutions to evolution equations

**Why are stationary solutions of interest in both
PDEs and stochastic processes?**

WHY STATIONARITY?

1. Convergence to time-independent solutions (cf. Linear Fokker-Planck).

$$\|u(t) - u_\infty\|_{L^1} \leq \exp(-t\lambda) K_\phi H_\phi [u_0]^{1/2}$$

2. Dynkin's Equation: Mean Value Representation of stationary solutions.

$$u_\infty(x) = \int_S u_\infty(X_t) d\mathbb{P}_x$$

3. Function-Measure Correspondence.

$$\begin{array}{ccc} A & \xleftrightarrow{\text{Hille-Yosida}} & (X_t)_{t \geq 0} \\ \downarrow & & \downarrow \\ u_\infty & \xleftrightarrow{u_\infty = \frac{d\mu_\infty}{d\lambda}} & \mu_\infty \end{array}$$

HILLE-YOSIDA THEOREM: MOTIVATION

$$X_t : \overbrace{\{1, \dots, d\}}^{=: \mathcal{S}} \rightarrow \mathbb{R}, j \mapsto (x_0)_j = v_0(j)$$

Can we generalize this
for stochastic
processes?

$$T_t x_0 := \exp(tA)x_0 \quad \dot{x}(t) = Ax(t), \quad x(0) = x_0$$

HILLE-YOSIDA THEOREM: INGREDIENTS

Feller Process $(X_t)_{t \geq 0}$ $(\mathcal{B}_t)_{t \geq 0}$

- Markov Process
- (IC) $\forall x \in S : \mathbb{P}_x(X_0 = x) = 1.$
 - (MP) $\forall s, t \geq 0 : E_x(f(X_{s+t}) | \mathcal{F}_t) = E_{X_s}f(X_s).$
 - (FP) $\forall f \in \mathcal{C}_0(S), t \geq 0 : x \mapsto E_x f(X_t)$ is in $\mathcal{C}_0(S).$

Locally Compact Space

$$\mathcal{C}_0(S) := \{ f \in \mathcal{C}(S) \mid \forall \varepsilon > 0 \exists K \subset S : |f(x)| < \varepsilon, \forall x \in S \setminus K \}$$

Probability Semigroup $(T_t)_{t \geq 0}$ $T_t : \mathcal{C}_0(S) \xrightarrow{\cong} \mathcal{C}_0(S)$

- Exponential Properties
- (S1) $T_0 = \text{id}.$
 - (S2) $\forall f \in \mathcal{C}_0(S) : T_t f \xrightarrow{t \searrow 0} f$ with respect to $\|\cdot\|_\infty.$
 - (S3) $T_{t+s} = T_t T_s.$
 - (S4) $f \geq 0 \implies T_t f \geq 0.$

This is partly why we require (FP)

$$T_t f := f * g_t, \quad g_t := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\|\cdot\|^2}{2t}\right)$$

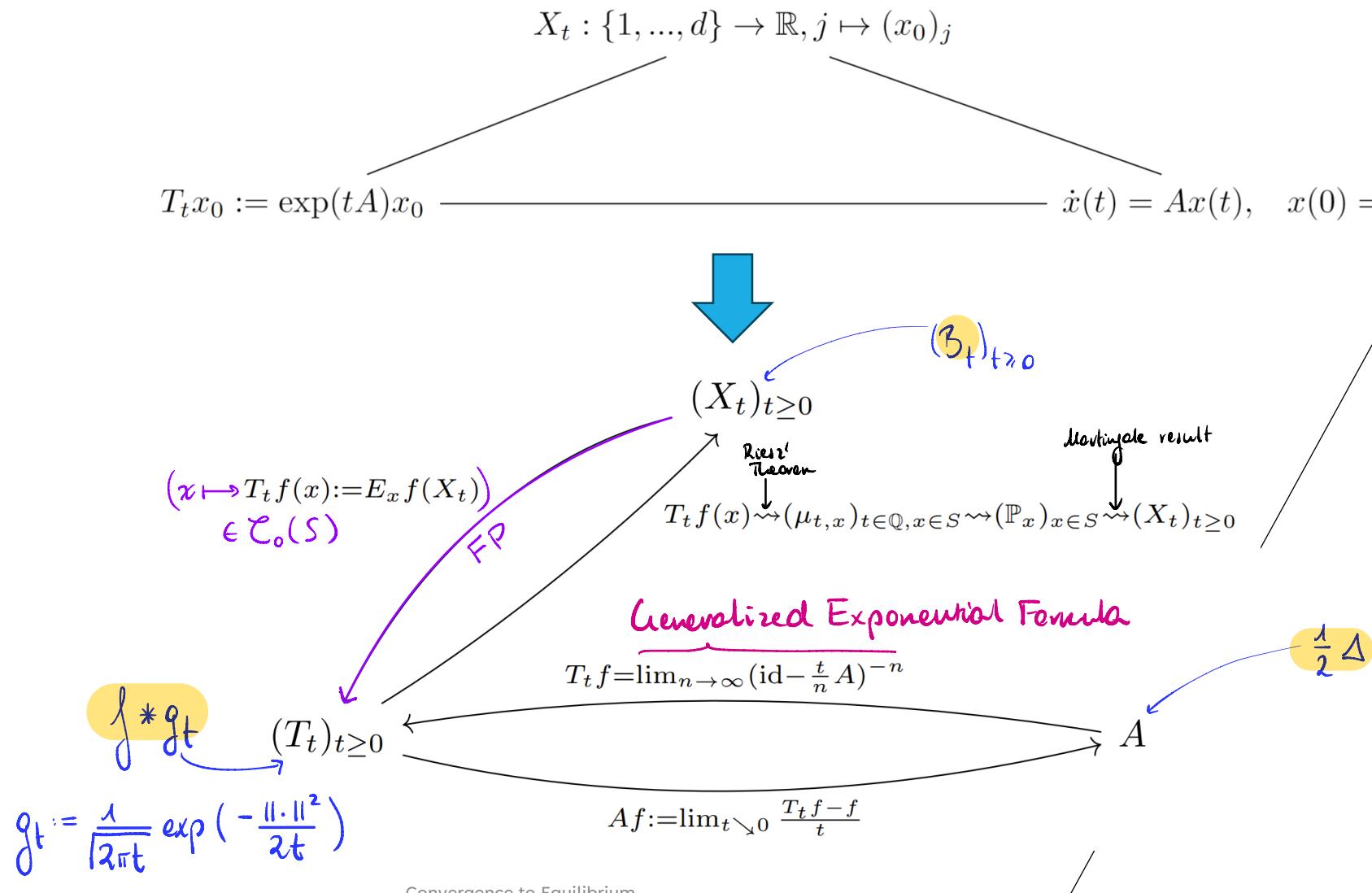
- Stochastic Property
- (S5) There exists a sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}_0(S)$ which is bounded in norm, i.e. $\sup \|f_n\| < \infty$, such that for all $x \in S$ we have $f_n(x) \rightarrow 1$ and $T_t f_n(x) \rightarrow 1$ as $n \rightarrow \infty$. ("Sum of rows = 1" of A)

"Infinitesimal Probability" Generator A

$$A = \frac{1}{2} \Delta$$

- "RHS derivative at zero"
- $A : D(A) \rightarrow \mathcal{C}_0(S), f \mapsto Af := \lim_{t \searrow 0} \frac{T_t f - f}{t}$

HILLE-YOSIDA THEOREM



HILLE-YOSIDA THEOREM

Theorem 3.1.9 (Hille-Yosida, [P], Thm. 3.1; [B], Thm. 2.14) A linear operator A is the generator of a probability semigroup $(T_t)_{t \geq 0}$ if and only if the following conditions are met:

(L1) A is a closed operator, i.e. $(f_n) \subseteq D(A)$ with $f_n \xrightarrow{\|\cdot\|_\infty} f \in D(A)$ and $Af_n \xrightarrow{\|\cdot\|_\infty} g$ implies $g = Af$.

(L2) $D(A)$ is dense in $\mathcal{C}_0(S)$.

(L3) $(0, \infty) \subseteq \rho(A)$. *Resolvent*

(L4) $\forall \lambda > 0 : \|(\lambda \cdot \text{id} - A)^{-1}\| \leq 1/\lambda$.

(L5) $\forall \lambda > 0 \exists (f_n)_{n \in \mathbb{N}} \subseteq D(A) : ((\text{id} - \lambda A)f_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}_0(S), \sup_{n \in \mathbb{N}} \|f_n\| < \infty$, and for all $x \in S$:

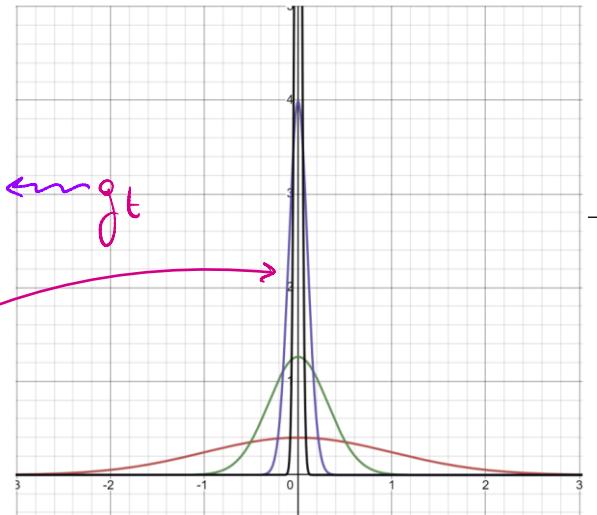
$$f_n(x) \rightarrow 1 \quad Af_n(x) \rightarrow 0. \quad \text{"Sum of rows = 0"}$$

GENERALIZED DYNKIN'S FORMULA

$$M_t := \overbrace{f(X_t)}^{\in L^1(\mathbb{P}_x)} - \int_{[0,t]} \underbrace{A f(X_s)}_{\text{Assumption: 'Behaves like } \frac{1}{2} \Delta\text{'}} d\lambda(s)$$

is a martingale with respect to the filtration induced by the Feller process $(X_t)_{t \geq 0}$.

$$T_t = f * g_t \leftarrow g_t$$



Corollary 3.2.5 (Martingale Mean Value Property) Consider the setting of Thm. 3.2.3, and let $u_\infty \in \ker A$ be a stationary solution to (1.3) such that $u_\infty(X_t) \in L^1(\mathbb{P}_x)$ for all $x \in S$ and $t \geq 0$. Then,

$$(u_\infty(X_t))_{t \geq 0}$$

is a martingale. In particular, stationary solutions can be written as the expected value of u_∞ evaluated at a process X_t started at $x \in S$:

$$u_\infty(x) = \int_S u_\infty(X_t) d\mathbb{P}_x$$

for any $t \geq 0$.

STATIONARY MEASURES

$$(\mathcal{B}, t) \mapsto \mu_t(B) := \int_S \underbrace{\mathbb{P}_x(X_t \in B)}_{\text{Transition probability}} d\mu(x)$$

From Feller process
where the process starts
Prob. Measure

Definition 3.3.1 ([B], Def. 7.1) A non-zero measure $\mu_\infty \in \mathcal{M}(S)$ is said to be **invariant** or **stationary** under the Feller process $(X_t)_{t \geq 0}$, if

$$\forall t \geq 0 : \mu_t = \mu_\infty.$$

Theorem 3.3.2 (Stationarity, [B] Thm. 7.2) The following statements are equivalent:

1. The measure $\mu \in \mathcal{M}(S)$ is invariant under $(X_t)_{t \geq 0}$.
2. For all $f \in \mathcal{C}_0(S)$ and $t \geq 0$:

$$\int T_t f d\mu = \int f d\mu.$$

3. For all $f \in D(A)$:

$$\int A f d\mu = 0.$$

STATIONARY MEASURES: EXAMPLE

Example 3.3.5 (Parts from [B], H.5.3) Consider a bounded generator $A \in \mathcal{L}(\ell^2)$ for the Hilbert space $\ell^2 := \ell^2(\mathbb{N})$. We can write the set of stationary probability measures as

$$\mathcal{I}_1 = \ker(A^*) \cap \partial B_1^{||\cdot||_{\ell^1}}(0)$$

constraining ourselves to sequences with non-negative entries (i.e. we do not deal with signed measures with mass 1).

$$A = \lambda \begin{pmatrix} -1 & 1 & 0 & 0 & \dots \\ 1/2 & -1 & 1/2 & 0 & \dots \\ 0 & 1/2 & -1 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\mathcal{I}_1 = \emptyset$$

Conservation of Mass:

$$A = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/3 & -1 & 2/3 \\ 1/2 & 1/2 & -1 \end{pmatrix}$$

Sum of rows is zero, only non-negative entries on the diagonal.

$$\mu_\infty = \frac{1}{27} (8, 9, 10)$$

$$\begin{aligned} \forall f \in \overline{\mathbb{R}^3}: \int_{\mathbb{N}_2, \mathbb{N}_3} Af d\mu_\infty &= (\mu_\infty, Af)_{\ell^2} \\ &= (A^* \mu_\infty, f)_{\ell^2} \Rightarrow A^* \mu_\infty = 0 \end{aligned}$$

Notice that mass is conserved (cf. PDE case).

$$\sum_{i=1}^3 (\mu_\infty)_i = 1$$

STATIONARITY: FUNCTION-MEASURE CORRESPONDENCE

$$0 = \int Af \ d\mu_\infty = \langle \mu_\infty, Af \rangle = \langle A'\mu_\infty, f \rangle$$

$$d\mu_\infty = u_\infty d\lambda$$

$$\begin{array}{ccc} A & \xleftrightarrow{\text{Hille-Yosida}} & (X_t)_{t \geq 0} \\ \downarrow & & \downarrow \\ u_\infty & \xleftarrow{u_\infty = \frac{d\mu_\infty}{d\lambda}} & \mu_\infty \end{array}$$

ORNSTEIN-UHLENBECK \leftrightarrow FOKKER-PLANCK

$$T_t f(x) = \int_S p_t(x, y) f(y) d\mu(y)$$

$$\int_S \partial_t p_t(x, y) f(y) d\mu(y) = \frac{d}{dt} T_t f(x) \stackrel{\text{Kolmogorov}}{=} A_x T_t f(x) = \int_S f(y) A_x p_t(x, y) d\mu(y)$$

Primal

One can also consider : $\partial_t p_t(x, y) = A_y^* p_t(x, y)$ Dual (where $\int f A g d\mu = \int g A^* f d\mu$)

Example 3.5.2 (Dual Operator: Ornstein-Uhlenbeck, [J], Exa. 2.4, p. 28) Consider the generator A defined by

$$Af = \Delta f - \langle \nabla V, \nabla f \rangle$$

$$0 = \int_{\mathbb{R}^d} Af d\nu \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} (A^* u_\infty) f d\lambda$$

\Downarrow L^2 dual through partial integration (assuming the functions are smooth enough and have vanishing support)

$$A^* u = \nabla \cdot (\nabla u + u \nabla V)$$

Rmk: $A := \frac{1}{2} \Delta$ is self-dual wrt $L^2(\lambda)$.

BARKY-EMERY: STOCHASTIC ANALYSIS

$$\Gamma(f, g) := \frac{1}{2}(A(fg) - fA(g) - gA(f))$$

$$\Gamma_2(f, g) := \frac{1}{2}(A\Gamma(f, g) - \Gamma(Af, g) - \Gamma(f, Ag))$$

Theorem 3.5.3 ([J], Thm. 2.2) Let $\phi \in \mathcal{C}^2([0, \infty))$ be convex with $1/\phi''$ concave. Assume there exists some $\lambda > 0$ such that for all suitable non-negative functions f

$$\Gamma_2(f) \geq \lambda \Gamma(f) \quad \nabla^2 \phi \geq \lambda \text{id} \quad (3.15)$$

holds. Then,

$$\int \phi(f) d\mu_\infty - \phi\left(\int f d\mu_\infty\right) \leq \frac{1}{2\lambda} \int \phi''(f) \Gamma(f) d\mu_\infty.$$

c.f. uniform convex potential from before



THANK YOU

Alejandro Morera Alvarez

alexmorera.a@gmail.com