Alejandro Morera Alvarez

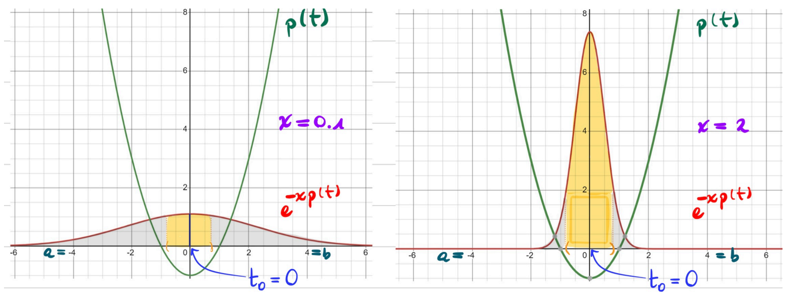
# Laplace's Method

**Goal:** Analyze the asymptotics of the expression

$$\boxed{I(x) = \int_{[a,b]} \exp\left(-xp(t)\right) q(t) \, \mathrm{d}t}$$

for large $x$.

**Idea:** Expand $p$ and $q$ in ascending powers of $t - t_0$ and replace $[a, b]$ with $\mathbb{R}$.




**Theorem 1 (Laplace)**

1. $\forall t \in (a, b) : p(t) > p(a)$ and $\forall c \in (a, b) : \inf_{t \in [c,b)}\{p(t) - p(a)\} > 0$.
2. $p'$ and $q$ are continuous in a neighborhood of $a$ except possibly at $a$.
3. $p - p(a) \sim P(t - a)^{\mu}$ and $q - q(a) \sim Q(t - a)^{\lambda - 1}$ as $t \searrow a$, where $P, \mu, \lambda > 0$ and $Q \in \mathbb{C}^{\times}$.
4. For large $x$, the expression $I(x)$ converges absolutely.

Then:

$$I(x) \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{\exp\left(-xp(a)\right)}{(Px)^{\lambda/\mu}}$$

**Procedure**

1. Find equation for the peak value of the whole integrand.
2. Solve the (possibly transcendental) equation from the previous step (asymptotically for large $x$). This yields $t = \xi(x)$.
3. Introduce $\tau := t/\xi(x)$ so as to make the approximate location of the peak independent of $x$.

**Examples:**

$$I_n(x) = \frac{1}{\pi} \int_{[a,b]} \exp(x \cos(nt)) \, \mathrm{d}t \qquad I(x) = \int_{[0,\infty)} \exp(xt - (t-1)\log(t)) \, \mathrm{d}t$$

**Theorem 2** Assume 1, 2 and 4 from Theorem 1. Moreover, suppose

$$p - p(a) \sim \sum_{s=0}^{\infty} p_s (t-a)^{s+\mu} \qquad q \sim \sum_{s=0}^{\infty} q_s (t-a)^{s+\lambda-1} \qquad p' \sim \sum_{s=0}^{\infty} (s+\mu) p_s (t-a)^{s+\mu-1} \quad \text{as } t \searrow a$$

Then, there exists some sequence $(a_s)_{s\in\mathbb{N}_0}$ with:

$$I(x) \sim \exp(-xp(a)) \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}}. \tag{1}$$

**Example:** Stirling's Formula

$$\Gamma(x) \sim \exp(-x)x^x \left(\frac{2\pi}{x}\right)^{1/2} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots\right).$$

### Error Bounds in Watson's Lemma

The $n$-th truncation error of the expansion in (1) (i.e. the difference between both sides of (1)) can be expressed as

$$\begin{aligned}
&\int_{[0,\infty)} \exp(-xt)q(t)\,\mathrm{d}t - \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \\
&= \underbrace{-\exp(-xp(a))\epsilon_{n,1}(x)}_{|\cdot| \le \exp(-xp(k))/(\kappa x - \alpha_n) \sum_{s=0}^{n-1} |a_s| \kappa^{(s+\lambda)/\mu}} + \underbrace{\exp(-xp(a))\epsilon_{n,2}(x)}_{(\star)} + \underbrace{\int_{[k,b]} \exp(-xp(t))q(t)\,\mathrm{d}t}_{\text{cf. Theorem 1}}
\end{aligned}$$

Example on how to bound $(\star)$: Assume $p, q$ have Taylor expansions for all $t \in (a, b)$, $p$ has a simple minimum at $t_0 = 0 \in (a, b)$, and $q$ does not vanish at 0. Apply strategies from Thm. 1 and, finally, assume a bound of the type

$$|F(v)| \le M|a_n| v^{\frac{n+\lambda-\mu}{\mu}} \exp(\hat{\sigma}_n v)$$

for some finite $\hat{\sigma}_n$, $M > 1$ and $t \in (0, \infty)$. Then, conclude with ideas from the proof of [O], Thm. 3.1 (cf. Eqn. (3.06)) that

$$\left| \int_{[0,\infty)} \exp(-xv)F(v)\,\mathrm{d}v \le \Gamma\left(\frac{n+\lambda}{\mu}\right) \frac{M|a_n|}{(x-\hat{\sigma}_n)^{(n+\lambda)/\mu}}\right.$$

for $x > \hat{\sigma}_n \vee 0$.

**References:**


- [O] Olver, F. W. J., *Asymptotics and Special Functions*, reprint, AK Peters, 1997.