1. Functional Analysis

Theorem 6.7 Let H be a Hilbert space and $M \in \mathcal{L}(H)$ a normal operator. Then, $r(M) = ||M||$.
Lemma 6.8 Let H be a Hilbert space and $M \in \mathcal{L}(H)$. Then, $\sigma(M^*) = \overline{\sigma(M)}$.
Lax-Milgram (A. 40b) Let H be a Hilbert space and b be a sesquilinear form. Then, b is bounded $\iff$ there is some $M \in \mathcal{L}(H)$ with the property $b(x, y) = (Mx, y)$, for all $x, y \in H$.
Theorem 2.10 (Neumann Series) Let X be a Banach space and $M \in \mathcal{L}(X)$. If $\sum_{n\in\mathbb{N}} ||T^n|| < \infty$, then $\mathrm{id} - T$ is invertible in $\mathcal{L}(X)$.
Corollary 6.4 Let X be a Banach space and $M \in \mathcal{L}(X)$. Then, $\sigma(M) \subseteq \mathbb{C}$ is a compact set, which is bounded by $||M||$.
Stone-Weierstrass Theorem Let X be a compact Hausdorff space and A an subalgebra of $\mathcal{C}(X; \mathbb{R})$ which contains a non-zero constant function. Then, A is dense in $\mathcal{C}(X; \mathbb{R})$ if and only if A separates points (i.e. for two different $x, y \in \mathbb{R}$, there exists some $f \in A$ with $f(x) \neq f(y)$).
Unique dense extension Let X be a normed space, $A \subseteq X$ be a dense subset, and Y be a Banach space. Assume $\gamma : A \to Y$ is an operator. Then, there exists a unique operator $\Gamma : X \to Y$ with the properties $\Gamma\big|_A = \gamma$ and $||\Gamma|| = ||\gamma||$. ([M], Thm. 1.9.1).
L^p-Approximations Let (Ω, Σ) be a measurable space. For any measurable set $S \in \Sigma$, we have that the simple functions are dense in $L^\infty(S)$. ([R], Sec. 7.4, Prop. 7.9).

2. The Heart of an Operator: The Spectrum

Theorem (Distance) For two symmetric operators M, N, we have that

$$\mathrm{dist}(\sigma(M), \sigma(N)) \leq ||M - N||$$

where the distance of two closed point-sets is defined as

$$\max\big\{ \max_{\nu\in\sigma(N)} \min_{\mu\in\sigma(M)} |\nu - \mu|, \max_{\mu\in\sigma(M)} \min_{\nu\in\sigma(N)} |\nu - \mu| \big\}$$

3. Functional Calculus

$\star$ **Theorem** (PFC) Let $M \in \mathcal{L}(H)$ be a symmetric operator. Then, the map

$$\Psi : \mathbb{R}[x]\big|_{\sigma(M)} \to \mathcal{L}(H),\ p \mapsto p(M)$$

is a unital, isometric algebra homomorphism with symmetric images which commute with M, i.e. $[M, p(M)] = 0$.

Theorem (CFC) Let $M \in \mathcal{L}(H)$ be a symmetric operator and $\Psi : p \mapsto p(M)$ as in PFC. Then, there exists a unique, unital, isometric algebra homomorphism $\hat{\Psi}$ with symmetric images such that $[M, f(M)] = 0$ and $\hat{\Psi}$ makes the following diagram commute:

$$\begin{tikzcd} \mathbb{K}[x]\big|_{\sigma(M)} \arrow[r, "\gamma"] \arrow[d, hook] & \mathcal{L}(H) \\ \mathcal{C}(\sigma(M)) \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

i.e. $\hat{\Psi}$ extends Ψ to continuous functions over the spectrum of M and is the unique extension preserving the properties from the PFC.
Moreover, for all $f \in \mathcal{C}(\sigma(M))$

$$\sigma(f(M)) = f(\sigma(M))$$

Corollary (Existence of Positive Square Roots) Every positive-semidefinite, symmetric operator has a positive square root.

4. Spectral Resolution

$$\ell_{x,y} : \mathcal{C}(\sigma(M)) \to \mathbb{R},\ f \mapsto \ell_{x,y}(f) := (f(M)x, y)$$

$$(f(M)x, y) = \int f(\lambda) d\mu_{x,y}$$

$$(x, y) \mapsto \mu_{x,y}(S) = (E(S)x, y)$$

Theorem (Projection-Valued Integral) Let H be a Hilbert Space and $M \in \mathcal{L}(H)$ be a symmetric operator. Then, there exists a unique orthogonal projection-valued measure E on the spectrum of M such that

$$E(S \cap T) = E(S)E(T)$$

and for all continuous functions f on $\sigma(M)$

$$f(M) = \int_{\sigma(M)} f(\lambda) dE$$

Corollary (Spectral Resolution) The spectral resolution of a symmetric operator M is given by

$$\mathrm{id} = \int_{\sigma(M)} dE(\lambda), \qquad M = \int_{\sigma(M)} \lambda dE(\lambda)$$