Spectral Resolution of Symmetric Operators

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Outline

Basic Definitions

Definition

As we will be working over the field of real numbers, $\mathbb{R}$, we will use the term **symmetric**, instead of self-adjoint for operators $M \in \mathcal{L}(H)$ with the property $M = M^*$.

An operator $P : H \to H$ is called an **orthogonal projection**, if $P^2 = P$ and P is symmetric.

Recall we assign the antilinearity of a sesquilinear form to the second argument. A sesquilinear form $b : H \times H \to \mathbb{C}$ is called **hermitian**, if $b(x, y) = \overline{b(y, x)}$ for all $x, y \in H$.

The **weak operator topology** on $\mathcal{L}(H)$ is the initial topology with respect to $\mathcal{F} := \{f_{a,b} \in \mathcal{L}(H)' \mid f_{a,b}(M) := (Ma, b),\ a, b \in H\}$.

What is a Spectral Resolution?

Let H be a separable Hilbert Space, $(e_n) \subseteq H$ be an orthonormal basis (ONB) of eigenvectors of a given compact, symmetric operator $M : H \to H$, and $(\lambda_n) \in c_0$ the corresponding eigenvalues (cf. [B2] Thm. 6.16). An expression of the form

$$x = \sum_{n \in \mathbb{N}} (x, e_n) e_n \qquad Mx = \sum_{n \in \mathbb{N}} \lambda_n (x, e_n) e_n \tag{1}$$

is called a **spectral resolution** of M.

Our Goal: Spectral Resolution for Symmetric Operators

We shall use the projections E_n onto the eigenspace $\mathcal{E}_n$ of the eigenvalue λ_n, i.e. $E_n : H \to \mathcal{E}_n$ to get

$$x = \sum_{n \in \mathbb{N}} E_n x \qquad Mx = \sum_{n \in \mathbb{N}} \lambda_n E_n x$$

Finally, defining a projection-valued measure by

$$E(S) := \sum_{\lambda_n \in S} E_n \tag{2}$$

we rewrite (1) in the form of a spectral resolution of M,

$$x = \int dE(\lambda) x \qquad Mx = \int \lambda dE(\lambda) x \tag{3}$$

Functional Analysis

Section's Goal: Recall some results from the functional analysis lecture [B2].

Tools from Functional Analysis

Theorem 6.7 Let H be a Hilbert space and $M \in \mathcal{L}(H)$ a normal operator. Then, $r(M) = ||M||$.
Lemma 6.8 Let H be a Hilbert space and $M \in \mathcal{L}(H)$. Then, $\sigma(M^*) = \overline{\sigma(M)}$.
Assignment 40b Let H be a Hilbert space and b be a sesquilinear form. Then, b is bounded $\iff$ there is some $M \in \mathcal{L}(H)$ with the property $b(x, y) = (Mx, y)$, for all $x, y \in H$.
Proof's Idea: The forward direction is a consequence of Frechet-Riesz and the converse direction is simple.

Tools from Functional Analysis

Theorem 2.10 (Neumann Series) Let X be a Banach space and $M \in \mathcal{L}(X)$. If $\sum_{n\in\mathbb{N}} ||T^n|| < \infty$, then $\mathrm{id} - T$ is invertible in $\mathcal{L}(X)$.
Corollary 6.4 Let X be a Banach space and $M \in \mathcal{L}(X)$. Then, $\sigma(M) \subseteq \mathbb{C}$ is a compact set, which is bounded by $||M||$.
Corollary 4.17 Let $(X, ||\cdot||)$ and $(X, ||\cdot||_\star)$ be Banach spaces such that $\exists M > 0 : ||x|| \leq M||x||_\star$ for all $x \in X$. Then, both norms are equivalent.

Tools from Functional Analysis

The following may be known to the audience (or not):
Stone-Weierstrass Theorem Let X be a compact Hausdorff space and A an subalgebra of $\mathcal{C}(X;\mathbb{R})$ which contains a non-zero constant function. Then, A is dense in $\mathcal{C}(X;\mathbb{R})$ if and only if A separates points (i.e. for two different $x, y \in \mathbb{R}$, there exists some $f \in A$ with $f(x) \neq f(y)$).
Unique dense extension Let X be a normed space, $A \subseteq X$ be a dense subset, and Y be a Banach space. Assume $\gamma : A \to Y$ is an operator. Then, there exists a unique operator $\Gamma : X \to Y$ with the properties $\Gamma\big|_A = \gamma$ and $||\Gamma|| = ||\gamma||$. ([M], Thm. 1.9.1).
L^p**-Approximations** Let (Ω, Σ) be a measurable space. For any measurable set $S \in \Sigma$, we have that the simple functions are dense in $L^\infty(S)$. ([R], Sec. 7.4, Prop. 7.9).

Further Tools

Definition (Total Variation)

The total variation of a measure $\mu \in \mathcal{M}(\Omega)$ over a σ-field (Ω, Σ) is defined as

$$||\mu|| := \sup_{\Omega = \coprod_{i=1}^{n} E_i,\, E_i \in \Sigma} \sum_{j=1}^{n} |\mu(E_j)|$$

It is a norm in the linear space of measures (cf. [B2] Exa. 1.17).

The Riesz representation theorem says that for a compact metric space K, we have $\mathcal{C}(K)' \cong \mathcal{M}(K)$ with respect to the total variation as a norm on the linear space of measures.

Spectrum of Symmetric Operators

Section's Goal: Analyze the spectrum of symmetric operators.

Spectrum of Symmetric Operators is Real

Theorem

Symmetric operators have a real spectrum.

Proof: From ([B], L. 6.8b), we have that $\sigma(M^*) = \overline{\sigma(M)}$. So, using $M = M^*$, we get that $\sigma(M) \subseteq \mathbb{R}$.

Spectral Radii and Norms

Theorem (Spectral Radius of Symmetric Operators)

Symmetric operators have a spectral radius equal to its norm.

$$|\sigma(M)| = ||M|| \tag{4}$$

Proof: Symmetric operators are normal. Thus, using ([B], Thm. 6.7), we have $|\sigma(M)| = ||M||$.

Inverse commutes with Spectrum

Lemma

Let M be an invertible operator over a Hilbert space H. Then,

$$\sigma(M^{-1}) = (\sigma(M))^{-1}$$

Proof: First, note that since M is invertible, $0 \notin \sigma(M)$, since it would otherwise yield a contradiction to the invertibility of M. Now, let $\mu \in \mathbb{C}^{\times}$. Then,

$$M^{-1} - \mu \cdot \mathrm{id} = M^{-1}(\mathrm{id} - \mu M) = -\mu M^{-1}(M - \mu^{-1} \cdot \mathrm{id})$$

Whence, we see from the first and last expressions that

$$\mu \in \sigma(M^{-1}) \iff \mu \in (\sigma(M))^{-1}$$

.

Polynomials commute with Spectrum

Lemma (SP=PS)

Given a polynomial $p \in \mathbb{R}[x]$ and an operator M, it holds

$$\sigma(p(M)) = p(\sigma(M)) \tag{5}$$

Polynomials commute with Spectrum (Proof)

Proof: Let $p = \sum_{n=0}^{N} a_n z^n$ and assume it is not the zero polynomial – otherwise the result is trivially true. By the Fundamental Theorem of Algebra, we can write $\lambda - p$ in factored form, $\alpha \prod_{n=1}^{N}(z - b_n)$, where the $b_n \in \mathbb{C}$ are the complex roots of $\lambda - p$ and $\alpha \in \mathbb{C}^{\times}$. Inserting M instead of z yields

$$\lambda \cdot \mathrm{id} - p(M) = \alpha \prod_{n=1}^{N}(M - b_n \cdot \mathrm{id})$$

which is a product of commuting operators. Hence, $\lambda \cdot \mathrm{id} - p(M)$ is invertible if and only if all the factors are invertible, which is precisely the case whenever all the roots b_n are not in the spectrum of M. This, in turn, is equivalent to $p(z) \neq \lambda$, $\forall z \in \sigma(M)$. Hence, $\lambda \notin \sigma(p(M))$ if and only if $\lambda \notin p(\sigma(M))$.

Distance between Spectra

Lemma (Distance)

For two symmetric operators M, N, we have that

$$dist(\sigma(M), \sigma(N)) \leq ||M - N||$$

where the distance of two closed point-sets is defined as

$$\max\left\{\max_{\nu \in \sigma(N)} \min_{\mu \in \sigma(M)} |\nu - \mu|, \max_{\mu \in \sigma(M)} \min_{\nu \in \sigma(N)} |\nu - \mu|\right\}$$

Q: Can someone explain the intuition on how to get the distance?

Distance between Spectra

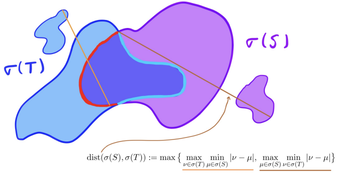

Figure: Distance between two spectra on the complex plane.

In order to calculate $\max_{\nu \in \sigma(T)} \min_{\mu \in \sigma(S)} |\nu - \mu|$, we first fix some $\nu \in \sigma(T)$ and run through all $\mu \in \sigma(S)$ with the goal of *minimizing* the amount $|\nu - \mu|$. We denote the set of $\mu \in \sigma(S)$ such that the minimum is achieved by $Q_\nu \subseteq \sigma(S)$. We then go through all $\nu \in \sigma(T)$ with the aim of *maximizing* the distance from it to its corresponding set Q_ν. Note that both steps are well-defined, since the spectrum of any operator is compact and the absolute value is continuous.

Distance between Spectra (Proof)

Assume $\delta := \mathrm{dist}(\sigma(M), \sigma(N)) = \max_{\nu\in\sigma(N)} \min_{\mu\in\sigma(M)} |\nu - \mu|$ and now, for contradiction that $d := ||M - N|| < \delta$. Then, there exists some $\nu \in \sigma(N)$ with the property

$$\min_{\mu\in\sigma(M)} |\mu - \nu| > d \tag{6}$$

Note that $\nu \in \mathbb{R} \setminus \sigma(M)$, since we would otherwise have by definition of spectral distance $0 = \delta > d \geq 0$; a contradiction. Hence, $M - \nu \cdot \mathrm{id}$ is invertible. Using (6), we have in particular, $|\sigma(M - \nu \cdot \mathrm{id})| > d$. Applying the commuting property of the inverse and spectra together with the previous inequality, we get

$$|\sigma((M - \nu \cdot \mathrm{id})^{-1})| = |\sigma(M - \nu \cdot \mathrm{id})|^{-1} < d^{-1}$$

Distance between Spectra (Proof, ctd.)

By [B], Cor. 6.4, the spectral radius is bounded by the norm of the corresponding operator, i.e. $|\sigma(M)| \leq ||M||$ and hence, taking the inverse, we have $||(M - \nu \cdot \mathrm{id})^{-1}|| \leq |\sigma(M - \nu \cdot \mathrm{id})|^{-1} < d^{-1}$. We get the bound

$$||(M - \nu \cdot \mathrm{id})^{-1}|| < d^{-1} \tag{7}$$

Let us rewrite,

$$N - \nu \cdot \mathrm{id} = N - \nu \cdot \mathrm{id} + M - M = (M - \nu \cdot \mathrm{id})(\mathrm{id} + (M - \nu \cdot \mathrm{id})^{-1}(N - M))$$

and define $K := (M - \nu \cdot \mathrm{id})^{-1}(N - M)$. Using submultiplicativity of norms, Eqn. 7 and $d = ||N - M||$, we get,

$$||K|| \leq ||(M - \nu \cdot \mathrm{id})^{-1}|| \, ||N - M|| < d^{-1} d = 1$$

which by the Neumann Series Theorem implies that $\mathrm{id} + K$ is invertible. So, since both factors of $N - \nu \cdot \mathrm{id}$ are invertible, the operator itself is invertible as well. But this contradicts $\nu \in \sigma(N)$.

Isometry

Lemma (Isometry)

Given a polynomial $p \in \mathbb{R}[x]\big|_{\sigma(M)}$ *and a symmetric operator* M*, it holds*

$$||p||_\infty = ||p(M)||$$

where $||p||_\infty = \max_{\lambda \in \sigma(M)} |p(M)|$

Proof: Assume the polynomial p has real coefficients. Since $M = M^*$ it follows that $p(M)$ is also a symmetric operator – and these in turn are normal. We can thus apply ([B2], Thm. 6.7) in order to get $r(p(M)) = ||p(M)||$. By Gelfand's Theorem ([B2], Thm. 6.6) and the commuting property of polynomials and spectra,

$$||p(M)|| = \max_{\lambda \in \sigma(M)} |p(\lambda)| = ||p||_\infty$$

Functional Calculus

Section's Goal: Define a map $f \mapsto f(M)$ for a symmetric operator $M \in \mathcal{L}(H)$ on the set of real-valued continuous functions over $\sigma(M)$.

Polynomial Functional Calculus

Theorem (PFC)

Let $M \in \mathcal{L}(H)$ be a symmetric operator. Then, the map

$$\gamma : \mathbb{R}[x]\big|_{\sigma(M)} \to \mathcal{L}(H),\ p \mapsto p(M)$$

is a unital, isometric algebra homomorphism with symmetric images which commute with M, i.e. $[M, p(M)] = 0$.

Polynomial Functional Calculus (Proof)

Proof: The isometry property follows from the Isometry Lemma, and the commuting property $[M, p(M)] = 0$ is obvious. Furthermore, the symmetry of the images follows from $\overline{p} = p$, $M^* = M$, and the (anti)linearity of taking adjoints.
The homomorphism properties follow straight-forwardly from the definition of polynomial addition and multiplication respectively:

$$\gamma(p+q) = (p+q)(M) = p(M) + q(M) = \gamma(p) + \gamma(q)$$

$$\gamma(pq) = (pq)(M) = p(M)q(M) = \gamma(p)\gamma(q)$$

Finally, we obviously have $\gamma(1) = \mathrm{id}$.

Continuous Functional Calculus

Theorem (CFC)

Let $M \in \mathcal{L}(H)$ *be a symmetric operator and* $\gamma : p \mapsto p(M)$ *as in PFC. Then, there exists a unique, unital, isometric algebra homomorphism* Γ *with symmetric images such that* $[M, f(M)] = 0$ *and*

$$\begin{tikzcd} \mathbb{K}[x]\big|_{\sigma(M)} \arrow[r, "\gamma"] \arrow[d, hook] & \mathcal{L}(H) \\ \mathcal{C}(\sigma(M)) \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

i.e. Γ *extends* γ *to continuous functions over the spectrum of* M *and is the unique extension preserving the properties from the PFC.*
Moreover, for all $f \in \mathcal{C}(\sigma(M))$

$$\sigma(f(M)) = f(\sigma(M)) \tag{8}$$

Continuous Functional Calculus (Proof)

Step 1: First, note that since $\sigma(M)$ is compact, we get by the Stone-Weierstrass theorem that the space $\mathbb{K}[x]\big|_{\sigma(M)}$ is dense in $\mathcal{C}(\sigma(M))$. Using the unique extension theorem, we get that γ extends uniquely to an operator

$$\Gamma : \mathcal{C}(\sigma(M)) \to \mathcal{L}(H),\ f \mapsto \Gamma(f)$$

with $f(M) := \Gamma(f) = \lim_{n\to\infty} \gamma(p_n)$, where $(p_n) \subseteq \mathbb{K}[x]\big|_{\sigma(M)}$ is some sequence of polynomials which converges uniformly to f, and the limit in the assignment is with respect to the operator norm. Moreover, the isometry property follows directly from the continuity of norms and the fact that γ is itself an isometry:

$$||\Gamma(f)|| = \lim_{n\to\infty} ||\gamma(p_n)|| = \lim_{n\to\infty} ||p_n|| = ||f||$$

Similarly, the unital, algebra $\star$-homomorphism properties of γ are passed on to Γ.

Continuous Functional Calculus (Proof, ctd.)

Step 2: Show the commuting property (8). Using the inequality from the distance lemma, we get

$$\mathrm{dist}(\sigma(f(M)) - \sigma(p_n(M))) \leq ||f(M) - p_n(M)|| \xrightarrow{n\to\infty} 0$$

and hence,

$$\sigma(f(M)) = \lim_{n\to\infty} \sigma(p_n(M))$$

Taking the polynomial p_n out of the spectrum using the SP=PS lemma and subsequently taking the limit yields the desired result.

Positivity Characterization for Symmetric Operators

Theorem (Positivity Characterization)

A symmetric operator is positive semi-definite if and only if its spectrum contains only nonnegative numbers, i.e.

$$(Mx, x) \geq 0 \iff \sigma(M) \geq 0$$

Proof's Idea: For the forward direction, construct an appropriate lower bound on the spectrum and show it is nonnegative. We only show the converse direction, since it is the one required later. For this, we will use our functional calculus on $f(\lambda) = \sqrt{\lambda}$.

Positivity Characterization for Symmetric Operators (Proof)

Assume $\sigma(M) \geq 0$. The function $x \mapsto \sqrt{x}$ is continuous over the nonnegative reals – which by assumption contain the spectrum $\sigma(M)$. We thus define the symmetric operator $\sqrt{M}$ using our CFC. It fulfills the property $\sqrt{M}^2 = M$, hence,

$$(Mx, x) = (\sqrt{M}^2 x) = (\sqrt{M}x, \sqrt{M}x) \geq 0$$

Existence of Positive Square Roots

Corollary

Every positive, symmetric operator has a positive square root.

Spectral Resolution for Symmetric Operators

Section's Goal: Put all the results we have gotten so far together in order to state and prove the Spectral Resolution for Symmetric Operators.

μ's Blues

For any symmetric operator M and $x, y \in H$, we can define the functional

$$\ell_{x,y} : \mathcal{C}(\sigma(M)) \to \mathbb{R},\ f \mapsto \ell_{x,y}(f) := (f(M)x, y) \tag{9}$$

Note $|\ell_{x,y}(f)| = |(f(M)x, y)| \leq ||f(M)||\,||x||\,||y|| = ||f||_\infty\,||x||\,||y|| < \infty$ $(\star\star)$ where in the last equality, we used the isometry property in CFC. This says $||\ell_{x,y}|| \leq ||x||\,||y||$. By Riesz' Representation Theorem, $\mathcal{C}(\sigma(M))' \cong \mathcal{M}(\sigma(M))$, there exists a unique measure $\mu_{x,y}$ such that its total variation is bounded by $||x||\,||y||$ and

$$(f(M)x, y) = \int f(\lambda) d\mu_{x,y} \tag{10}$$

Theorem (Measure)

The measures $\mu_{x,y}$ are sesquilinear, hermitian, and the measures $\mu_{x,x}$ are real and nonnegative.

Measure (Proof)

Proof:
i) For each $x, y \in H$, the functional $\ell_{x,y}$ is sesquilinear because of the scalar product. Since $\mu_{(\cdot,\cdot)}$ is uniquely determined by $\ell_{(\cdot,\cdot)}$, the measure $\mu_{x,y}$ is sesquilinear as well.
ii) Since $f(M)$ is symmetric and scalar products are hermitian, the integral $\int f(\lambda) d\mu_{x,y} = (f(M)x, y)$ is hermitian in x and y. Again, using the uniqueness of the representing measure, we get that $\mu_{x,y}$ is hermitian.
iii) By the CFC, $\sigma(f(M)) = f(\sigma(M))$ and $f(M)$ is symmetric. Assume $f > 0$. Then, $f(\sigma(M)) > 0$, which by the positivity characterization, it implies $f(M)$ is a positive operator, i.e. $0 \leq (f(M)x, x) = \ell_{x,x}(f)$ and hence, so are the measures $0 \leq \mu_{x,x}$.

Remark: The latter theorem states that for any Borel subset S of the spectrum of M, the measure $\mu_{(\cdot,\cdot)}(S)$ is a symmetric, sesquilinear form. Using Lax-Milgram's Theorem, we conclude that for each S there is a symmetric operator $E(S) \in \mathcal{L}(H)$ with the property

$$\mu_{x,y}(S) = (E(S)x, y) \tag{11}$$

We shall analyze the collection $(E(S))_{S \in \mathcal{B}(\sigma(M))}$ in the following theorem.

Orthogonal Projection-Valued Measure

Theorem (OPVM)

The family of operators $(E(S))_{S\in\mathcal{B}(\sigma(M))}$ *has the following properties:*

1. $E^*(S) = E(S)$
2. $||E(S)|| \leq 1$
3. $E(\emptyset) = 0 \qquad E(\sigma(M)) = id$
4. $S \cap T = \emptyset \implies E(S \cup T) = E(S) + E(T)$
5. $\forall S \in \mathcal{B}(\sigma(M)) : [E(S), M] = 0$
6. $E(S \cap T) = E(S)E(T)$
7. *Each of the* $E(S)$ *is an orthogonal projection.*
8. $S \cap T = \emptyset \implies (E(S))(H) \perp (E(T))(H)$
9. $[E(S), E(T)] = 0$

Orthogonal Projection-Valued Measure (Proof)

Proof: The integral representation $(f(M)x, y) = \int f(\lambda)d\mu_{x,y}$ is crucial for this proof.
1) This follows from Lax-Milgram.
2) This follows from $(\star\star)$, since the bound derived here represents the bound of the total variation of $\mu_{x,y}$.
3) We have $0 = \mu_{x,y}(\emptyset) = (E(\emptyset)x, y) \implies E(\emptyset) = 0$ and with $f = 1$, it follows $f(M) = \mathrm{id}$ and hence,
$(\mathrm{id}x, y) = (x, y) = \int 1\, d\mu_{x,y} = (E(\sigma(M))x, y)$. Thus, $E(\sigma(M)) = \mathrm{id}$.
4) This is a consequence of the additivity of measures for pairwise disjoint sets.

Orthogonal Projection-Valued Measure (Proof, ctd.)

5) Since $[M, f(M)] = 0$ and M is symmetric, we have $(f(M)Mx, y) = (Mf(M)x, y) = (f(M)x, My)$. Note that the first term in the equality chain is represented by the measure $\mu_{Mx,y}$; and the last term is represented by $\mu_{x,My}$. It follows $\mu_{Mx,y} = \mu_{x,My}$. Hence, by the latter and the symmetry of M,

$$(E(S)Mx, y) = \mu_{Mx,y} = \mu_{x,My} = (E(S)x, My) = (ME(S)x, y)$$

from the first and last term we read $[E(S), M] = 0$.
6) See next slide.
7), 8) By (6), we get $E(S) = E(S)^2$ and by (1) $E(S)$ is symmetric. Hence, we get an orthogonal projection. Moreover, if $S \cap T = \emptyset$, then, using (3) and (6), $0 = E(\emptyset) = E(S \cap T) = E(S)E(T)$ and thus their ranges must be orthogonal.
9) Follows from interchanging S and T in (6).

Orthogonal Projection-Valued Measure (Proof, ctd. 2)

6) *Proof's Idea:* ([L], pp. 365-366) Let J_x be the set of elements of the form $z = f(M)x$ for a continuous function f. We say $z \in J_x$ is represented by f for $\lambda \in \sigma(M)$.
Exercise: We have $MJ_x \subseteq J_x$ and Mz is represented by λf. Moreover, $(f(M)x, y) = \int_{\sigma(M)} f(\lambda) d\mu_{x,y}$.
Let $K_x := \overline{J_x}$. The above isometry induces for every $z \in K_x$ an isometric representation by some function h; and conversly, for every h there exists some $z \in K_x$. We call this a spectral representation of M acting on K_x.
Through some further steps (involving Zorn's Lemma), get a spectral representation of M on H, i.e. a family of pairwise orthogonal, closed subspaces (K_j) spanning H, such that for all $j \in J$, we have $MK_j \subseteq K_j$ and K_j is spectrally represented as $L^2(m_j)$.
This yields $E(S)$ as a multiplication by an indicator function $\mathbb{1}_S$. Using $\mathbb{1}_{S \cap T} = \mathbb{1}_S \mathbb{1}_T$, we get $E(S \cap T) = E(S)E(T)$

Symmetric Operators and Projection-Valued Measures

Theorem (Projection-Valued Integral)

Let H be a Hilbert Space and $M \in \mathcal{L}(H)$ be a symmetric operator. Then, there exists a unique orthogonal projection-valued measure E on the spectrum of M such that

$$E(S \cap T) = E(S)E(T) \tag{12}$$

and for all continuous functions f on $\sigma(M)$

$$f(M) = \int_{\sigma(M)} f(\lambda) dE \tag{13}$$

Symmetric Operators and Projection-Valued Measures (Proof)

By $(f(M)x, y) = \int f(\lambda) d\mu_{x,y}$ and $\mu_{x,y}(S) = (E(S)x, y)$, we have that the claim holds in the weak operator topology (recall from before). Hence, it is enough to show that the integral converges in the strong topology.
Let $(I_j) \subseteq \sigma(M)$ be a finite, measurable partition of $\sigma(M)$. Then, using that the ranges $E(I_j)$ are pairwise orthogonal and that orthogonal projections are isometries, we get

$$\left\| \sum_j a_j E(I_j) \right\| \leq \max_j |a_j|$$

Symmetric Operators and Projection-Valued Measures (Proof, ctd.)

Since $\mathcal{C}(\sigma(M)) \subseteq L^\infty(\sigma(M))$, we have that for any continuous function $f \in \mathcal{C}(\sigma(M))$, there exists a sequence of simple functions (s_n) which converge uniformly to f. Thus, using the nonexpansiveness from the previous slide,

$$\left|\left| \int_{\sigma(M)} f(\lambda) dE \right|\right| \leq \int_{\sigma(M)} \lim_{n\to\infty} |s_n(\lambda)| dE \leq ||f||_\infty E(\sigma(M)) = ||f||_\infty < \infty$$

The uniqueness of the measures in $\int f(\lambda) d\mu_{x,y}$ imply the uniqueness of $E(S)$.

Spectral Resolution of Symmetric Operators

Corollary

The spectral resolution of a symmetric operator M is given by

$$id = \int_{\sigma(M)} dE(\lambda), \qquad M = \int_{\sigma(M)} \lambda dE(\lambda) \tag{14}$$

Proof: This follows from setting $f = 1$ and $f = \mathrm{id}$ in the previous theorem.

But wait, there's more!

You may be wondering: Can we construct such an integral in a different way? Moreover, can we enlarge the domain in which our integral is defined?
The answer is yes. In this bonus section, we sketch how we would construct an integral for the set of bounded, Borel-measurable functions.

Sketch

Let us sketch my Bachelor thesis in two slides:
We start in the same way we did here: define a PFC, and extend it to a CFC.

$$
\begin{tikzcd}
 & \mathbb{K}[x]\big|_{\sigma(M)} \arrow[d, hook] \arrow[dr, "\gamma"] & & \\
 & \mathcal{C}(\sigma(M)) \arrow[r, "\Gamma"] \arrow[d, hook] & \mathcal{L}(H) \arrow[d, hook] & \\
\mathcal{B}(\sigma(M)) \arrow[r, "\Phi"] \arrow[rrr, bend right, "\hat{\Gamma}"'] & \mathcal{C}(\sigma(M))'' \arrow[r, "\Gamma''"] & \mathcal{L}(H)'' \arrow[r, two heads, "P_{\mathcal{L}(H)}"] & \mathcal{L}(H)
\end{tikzcd}
$$

PFC

CFC

Riesz Representation Theorem

Double adjoint makes diagram commute

$\mathcal{L}(H)$ is a dual space

Unique Extension with special continuity properties (Goldstine's Theorem and Unique Extension Theorem)

Figure: Construction sketch of the BFC.

Sketch (ctd.)

BFC

$$\mathbb{1} : \mathcal{B}(\sigma(M)) \to \mathcal{L}(H),\ S \mapsto \mathbb{1}_S(M) := \hat{\Gamma}(\mathbb{1}_S)$$

Induces a spectral measure
(special lattice homomorphism)

$$\Theta : (\sigma(M), \mathcal{B}(\sigma(M))) \to (H, \Gamma(H)),\ S \mapsto \Theta(S) := \operatorname{ran}(\mathbb{1}_S(M))$$

Construct integral

Show the BST

$$f(M) = \int_{\sigma(M)} f\, dP = \mathcal{U} M_f \mathcal{U}^*$$

$$\Downarrow$$

$$M = \mathcal{U} M_{\mathrm{id}} \mathcal{U}^*$$

Figure: Proof sketch of the Bounded Spectral Theorem (BST)

References

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Thank You

Thank you for your time and attention. If you have further questions, I will be glad to discuss them.