Laplace's Method

Seminar: Applied Asymptotic Analysis

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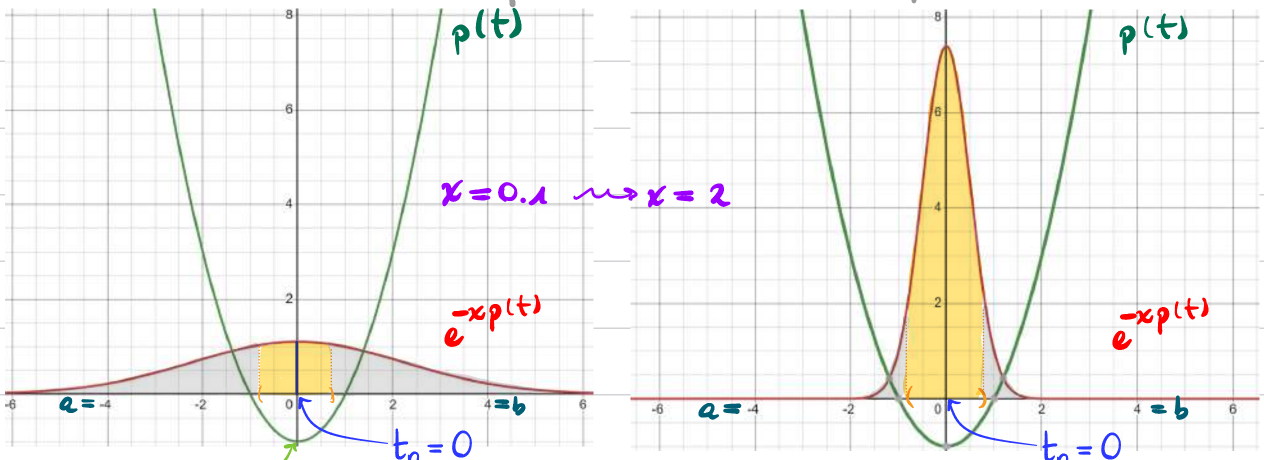
Laplace's Method

$$I(x) = \int_a^b e^{-x p(t)} q(t)\, dt$$

Idea: Expand p and q in ascending powers of $t - t_0$ (min of p) and replace $[a, b]$ with $(-\infty, +\infty)$.

Common Case: Simple minimum over (a,b)



minimum for p $\Rightarrow$ maximum for $\exp(-p)$

x large $\Rightarrow$ most of the area concentrated around t_0

Exa 1) $t_0 = a$ (In leading order), $p'(a) > 0$, $q(a) \neq 0$

$$\Rightarrow I(x) \doteq \int_a^b \exp(-x(p(a) + (t-a)p'(a))) \cdot \underbrace{q(a)}_{\neq 0}\, dt$$

$$\doteq q(a)\exp(-x(p(a))) \cdot \int_a^{\infty} \exp(-x(t-a)p'(a))\, dt = q(a)\exp(-xp(a)) \cdot \frac{1}{xp'(a)}$$

2) $t_0 \in (a,b)$ simple minimum of p and $q(t_0) \neq 0$.

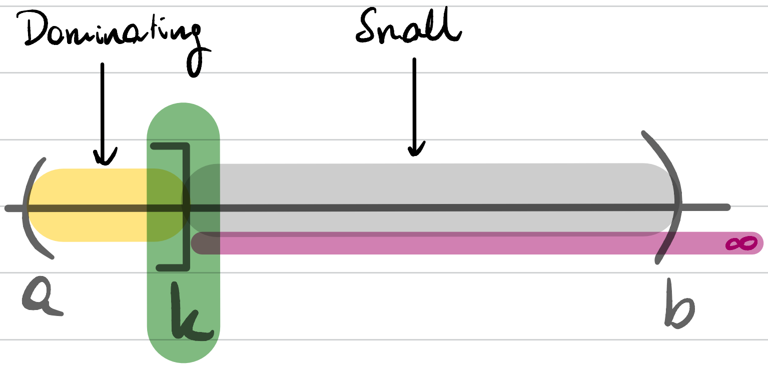
$$\Rightarrow I(x) \doteq \int_a^b \exp\left(-x\left(p(t_0) + (t-t_0)\underbrace{\cancel{p'(t_0)}}_{0} + \frac{1}{2}(t-t_0)^2 p''(t_0)\right)\right) \underbrace{q(t_0)}_{\neq 0}\, dt$$

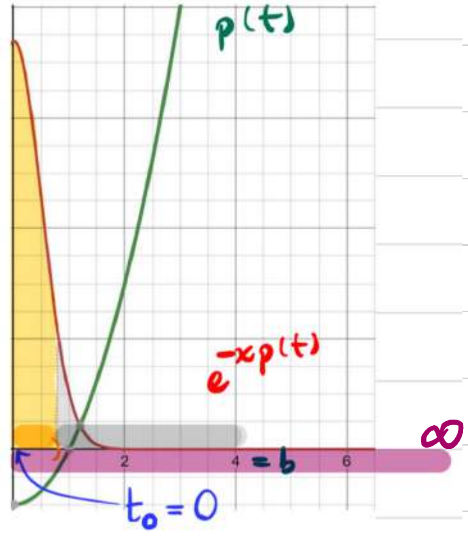
$$\doteq \int_{-\infty}^{+\infty} \exp\left(-x\left(p(t_0) + \frac{1}{2}(t-t_0)^2 p''(t_0)\right)\right) q(t_0)\, dt$$

$$= q(t_0)\exp(-xp(t_0)) \int_{-\infty}^{+\infty} \exp\left(-\frac{x}{2}(t-t_0)^2 p''(t_0)\right) dt \overset{\text{Normalize to } \mathcal{N}\left(t_0, \frac{1}{p''(t_0)}\right)}{=} q(t_0)\exp(-xp(t_0)) \left(\frac{2\pi}{xp''(t_0)}\right)^{1/2}$$

Formalizing : Proof Strategy

1) Split integral over two regions :





Don Knuth "Trading Tails"

2) Dominating : Replace integrand by leading order and $(a,k) \rightsquigarrow (a,\infty)$
Subdivide into three terms

I) What you want in the end,
II) Residual term by incomplete Gamma fct. behaviour (ε_1),
III) Residual term by assumed asymptotics (ε_2).

3) Small : Take advantage of exponential decay to show

I) $[k, b)$ - integral is small,
II) (k,∞) - integral added is small.

! Theorem 1 (Laplace, 1820)

Wlog $t_0 \overset{*}{=} a \in \mathbb{R}$ (cf. Exa. 1) $a < b \in (-\infty, \infty]$,

$\underbrace{t \mapsto p(t) \in \mathbb{R}\ ;\ t \mapsto q(t) \in \mathbb{C}}_{\text{independent of } x}$. Moreover,

* Else, subdivide $[a,b]$ at the minima and maxima of p and reverse signs where necessary.

i) $\forall t \in (a,b) : p(t) > p(a)$ and $\forall c \in (a,b) : \inf_{t \in [c,b)} \{p(t) - p(a)\} > 0$.

ii) p' and q are cts. in a neighborhood of a, except possibly at a.

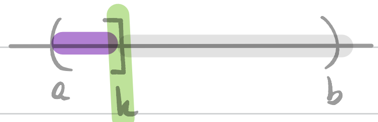
iii) $\underbrace{p - p(a) \sim P(\overbrace{t-a}^{>0})^{\mu}}_{\text{sp. differentiable}}$, $q \sim Q(t-a)^{\lambda-1}$ as $\underbrace{t \downarrow a}_{\text{i.e. as } t \text{ approaches } a \text{ from the right}}$, where $P, \mu, \lambda > 0$ and $Q \in \mathbb{C}^{\times}$.

iv) For x large $I(x)$ converges absolutely.

Then: $I(x) := \int_a^b e^{-xp(t)} q(t)\, dt \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-xp(a)}}{(Px)^{\lambda/\mu}}$ $(*)$

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt = (z-1)\Gamma(z-1)$

Pf: ⇒ $\exists k : \underbrace{p' > 0}$ cts. and q cts. over $(a,k]$.



⇒ p strictly increasing over (a,k) ⇝ $v := p - p(a)$ variable sub. over (a,k)

$e^{xp(a)} \int_a^k e^{-xp(t)} q(t)\, dt \overset{\text{Sub.}}{=} \int_{\underset{p(a)-p(a)}{0}}^{\kappa} e^{-xv} \underbrace{q(t) \overbrace{\frac{dt}{dv}}^{=1/p'}}\, dv = \cdots$, $\kappa := p(k) - p(a)$ finite and positive.

$=: f(v)$ cts. over $(0, \kappa]$ because v and t are cts functions of each other

$v \sim P(t-a)^{\mu} \Rightarrow \frac{v}{P} \sim (t-a)^{\mu} \Rightarrow t - a \sim \left(\frac{v}{P}\right)^{1/\mu}$ as $v \downarrow 0$ (compare this to Thm. 2) $\Rightarrow f = \frac{q}{p'_{t=v'}} \sim \frac{Q v^{\lambda/\mu - 1}}{\mu P^{\lambda/\mu}}$ as $v \downarrow 0$.

$$\dots = \underbrace{\frac{Q}{\mu p^{\lambda/\mu}} \int_0^\infty e^{-xv} v^{\lambda/\mu - 1} dv}_{} - \frac{Q}{\mu p^{\lambda/\mu}} \underbrace{\mathcal{E}_1(x) + \mathcal{E}_2(x)}_{\text{Rest}} \; ; \quad ③ \; \mathcal{E}_1(x) = \frac{1}{x^{\lambda/\mu}} \underbrace{\Gamma\left(\frac{\lambda}{\mu}, \kappa x\right)}_{\text{i.e. Gamma function, but integrating over } (\kappa x, \infty)} \overset{\text{Follows from elementary methods (IbP)}}{=} O\left(\frac{e^{-\kappa x}}{x}\right)$$

① $\sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-xp(a)}}{(px)^{\lambda/\mu}}$ by Euler's integral $\leftarrow$ cf. $(*)$

$\overset{\kappa>0}{\Rightarrow} \exists\, x$ large s.t. $\mathcal{E}_1$ bdd. by $\varepsilon x^{-\lambda/\mu}$

$$\mathcal{E}_2(x) = \int_0^\kappa e^{-\kappa v} \Big(\underbrace{f(v) - \frac{Q v^{\lambda/\mu - 1}}{\mu p^{\lambda/\mu}}}_{} \Big) dv$$

② $\forall \varepsilon > 0 \; \exists \kappa$ s.t. $|\cdot| < \varepsilon \underbrace{\frac{|Q| v^{\lambda/\mu - 1}}{\mu p^{\lambda/\mu}}}_{\text{Integrand for an Euler integral}}$ for $v \in (0, \kappa]$

$\Rightarrow |\mathcal{E}_2(x)| < \varepsilon \frac{|Q|}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{1}{(Px)^{\lambda/\mu}}$

Small : $[k,b)$-integral

Let X be s.t. $I(X)$ converges absolutely[(iv)] and $\eta := \inf_{t \in [k,b)} \{p(t) - p(a)\} \overset{(i)}{>} 0$. (Recall from before)

$\forall x > X$:

$$xp(t) - xp(a) = \overbrace{(x - X)(p(t) - p(a)) + X(p(t) - p(a))}^{\text{additive zero}} \overset{\inf}{\geq} (x - X)\eta + Xp(t) - Xp(a)$$

$$\Rightarrow \left| e^{xp(a)} \int_k^b e^{-xp(t)} q(t)\, dt \right| \overset{\Delta + \text{monotonicity}}{\leq} e^{\overbrace{-(x-X)}^{<0}\underbrace{\eta}_{>0} + Xp(a)} \int_k^b e^{-Xp(t)} |q(t)|\, dt \;\; \text{bdd. by } \varepsilon x^{-\lambda/\mu} \text{ for } x > X \text{ large.}$$

$(\eta > 0$ and $\kappa = p(k) - p(a) > 0)$

$\square$ 🥵

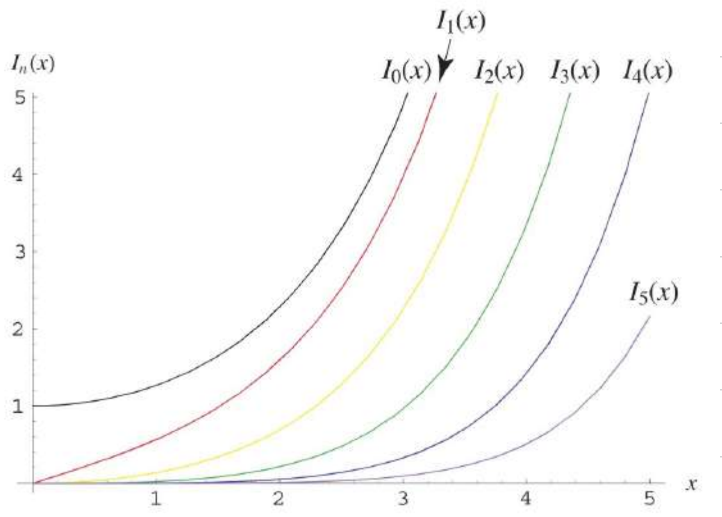
Exa (Modified Bessel Function)

$$I_n(x) := \frac{1}{\pi}\int_0^{\pi} e^{-x(-\cos(t))}\cos(nt)\,dt\,, \quad n\in\mathbb{N}.$$

Increasing for $t\in(0,\pi)$

$$p(t) = \underbrace{-1}_{p(a)=} + \underbrace{\frac{1}{2}}_{=P}\, t^{\overset{=\mu}{2}} + O(t^4)$$

$$q(t) = \underbrace{\frac{1}{\pi}}_{=Q} + O(t^2) \qquad \lambda = 1 \qquad (\text{"}1\text{-}1\text{"})$$



Wolfram MathWorld

Thm. 1 : $$I_n(x) \sim \frac{Q}{\mu}\,\Gamma\left(\frac{\lambda}{\mu}\right)\frac{\exp(-x\,p(a))}{(Px)^{\lambda/\mu}} \overset{\Gamma(\frac{1}{2})=\sqrt{\pi}}{=} (2\pi x)^{-1/2}e^{x} \quad \text{as } x\to\infty$$

Exa

1) Find equation for the peak value t of the whole integrand.

2) Solve the above (possibly trascendental) equation asymptotically for large x $\rightsquigarrow t = \xi(x)$. cf. 2nd talk

3) New integration variable $\tau := t / \xi(x)$ to make the approx. location of the peak independent of x.

$$I(x) = \int_0^\infty e^{\underbrace{xt-(t-1)\log(t)}} \, dt$$

$\frac{d}{dt} \cdots = x - 1 - \log(t) + \frac{1}{t} \overset{!}{=} 0$ for candidate extrema of whole integrand.

Why not simply $p(t) = -t$? Because p has no min. on $\underbrace{(0,\infty)}_{\text{integration range}}$.

i.e. analogous to prev. example by $e^{xt} = e^{-x(-t)}$ and the rest using the functional eqn. of exp. $\rightsquigarrow q(t)$

cf. assumptions Thm. 1

x large : $x - 1 - \log(t) \sim 0 \Rightarrow t \sim e^{x-1} =: \xi(x)$

$$\tau := t/\xi \Rightarrow \frac{d\tau}{dt} = \frac{1}{\xi} \Rightarrow I(x) = \xi^2 \int_0^\infty e^{-\xi p(\tau)} q(\tau)\, d\tau \; ; \; p := \tau(\log(\tau) - 1), \; q := \tau$$

$\rightsquigarrow$ Apply Thm. 1 for $[1,\infty)$ and $[0,1]$ (split at minimum).

$\tau = 1$ unique minimum for p. Expand in powers of $\tau - 1$:

$$p = \underbrace{-1}_{p(a)} + \underbrace{\tfrac{1}{2}}_{P}(\tau-1)^{\overset{\mu}{2}} - \frac{1}{6}(\tau-1)^3 + \frac{1}{12}(\tau-1)^4 \mp \cdots$$

$$q = \underbrace{1}_{Q} + (\tau - 1)^{\lambda = 1}$$

$$\overset{\text{Thm. 1}}{\Longrightarrow} \int_1^\infty e^{-\xi p(\tau)} q(\tau)\, d\tau \sim \left(\frac{\pi}{2\xi}\right)^{1/2} e^{\xi}$$

Expansion point (recall wlog assp. in Thm. 1)

How about integral over $[0,1]$?

Replace τ with $2-\tau$ $\rightsquigarrow$ same asymptotic. Plug in to and substitute x back into equation.

Overall:

$$\boxed{I(x) \sim (2\pi)^{1/2} \exp\left(3(x-1)/2 + e^{x-1}\right) \quad \text{as} \quad x \to \infty}$$

Rule: Series Reversion (General Case, for reference)

Two methods: Algebraic (this slide), fixed point (next slide) — Relevant for today

Sp. $f = c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ Observe we assume $c_0 = 0$ so that $g \circ f$ is wd. ($\Leftarrow$ coefficients of $g \circ f$ depend only on finitely many c_i) See below

Ansatz: $g = b_1 y + b_2 y^2 + b_3 y^3 + \cdots$

$b_0 = 0$ for $f \circ g$ wd. (see above)

Plug in and arrange powers: $\underbrace{f \circ g}_{\stackrel{!}{=} id} = c_1 b_1 y + (c_2 b_1^2 + c_1 b_2) y^2 + (c_3 b_1^3 + 2 c_2 b_1 b_2 + c_1 b_3) y^3 + \cdots$

$\stackrel{!}{=} id = 1 \cdot y + 0 \cdot y^2 + 0 y^3 + \cdots$

Equating coefficients: $b_1 = c_1^{-1}$ $\longleftarrow$ Note $\underbrace{c_1 \neq 0}$ $\Leftrightarrow$ f is invertible (wrt $\circ$) in the ring of formal power series

Through a change of variables, one can achieve this, cf. next slide.

$b_2 = -c_1^{-3} c_2$

$b_3 = c_1^{-5} (2c_2^2 - c_1 c_3)$

$\vdots$

Notation suggests the coefficients can be elements of rings.

Exa (Algebraic)

$p = x + x^2 + x^3 + \cdots$

Ansatz: $p^{-1} = c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ — Or simply use the above formula

$\Rightarrow p^{-1} = x - x^2 + x^3 \mp \cdots$

Alternatively:

Exa ("Fixed Point")

$$\underbrace{t - \log(1+t)}_{\text{will be used later...}} = \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4 + \mathcal{O}(t^5) =: v \qquad t \in (-1, +1)$$

Solve for t^2

$$\Rightarrow t^2 = 2v + \frac{2}{3}t^3 - \frac{1}{4}t^4 + \mathcal{O}(t^5)$$

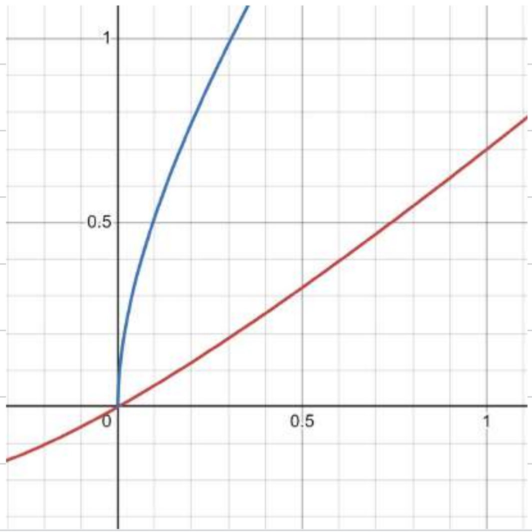
Insert $t^3 = (t^2)^{3/2}$, $t^4 = (t^2)^2$, ... using $t^2 = t^2(v)$

$$\Rightarrow t^2 = 2v + \frac{2}{3}\left(2v + \frac{2}{3}t^3 - \frac{1}{4}t^4 + \mathcal{O}(t^5)\right)^{3/2}$$
$$- \frac{1}{4}\left(2v + \frac{2}{3}t^3 - \frac{1}{4}t^4 + \mathcal{O}(t^5)\right)^2 + \mathcal{O}(t^5)$$

$(1-t)^{1/2} = 1 - t/2 - t^2/8 + \mathcal{O}(t^3)$

Taylor root, solve for t, then Taylor again $\sqrt{}$ and $\overbrace{\text{iterate}}^{\text{Fixed Point}}$ (... solve for lowest power of t ...)

$$\Rightarrow t = 2^{1/2} v^{1/2} + \frac{2}{3}v + \frac{2^{1/2}}{18}v^{3/2} - \frac{2}{135}v^2 \pm \cdots$$



Extension of Thm. 1

Theorem 2 (Asymptotic Expansion in Descending Orders)

Wlog $t_0 = a \in \mathbb{R}$ with $a < b \in (-\infty, \infty]$,

$\underbrace{t \mapsto p(t) \in \mathbb{R} \;;\; t \mapsto q(t) \in \mathbb{C}}_{\text{independent of } x}$. Moreover,

i) $\forall t \in (a,b)$: $p(t) > p(a)$ and $\forall c \in (a,b)$: $\inf_{t \in [c,b)} \{p(t) - p(a)\} > 0$.

ii) p' and q are cts. in a neighborhood of a, except possibly at a.

iii) $p \sim p(a) + \sum_{s=0}^{\infty} p_s (t-a)^{s+\mu}$ $\qquad q \sim \sum_{s=0}^{\infty} q_s (t-a)^{s+\lambda-1}$ as $t \downarrow a$; $\mu, \lambda > 0$

wlog $p_0 \neq 0$, $q_0 \neq 0$. Since $p(a)$ is min. $\rightsquigarrow$ $p_0 > 0$.

$p' \sim \sum_{s=0}^{\infty} (s+\mu) p_s (t-a)^{s+\mu-1}$ as $t \downarrow a$.

Rmk: The expansions need not be convergent and the powers $s + \begin{cases} \mu \\ \lambda - 1 \end{cases}$ need not be integers.

iv) For x large $I(x)$ converges absolutely.

Then:

$$I(x) := \int_a^b e^{-xp(t)} q(t)\, dt \sim e^{-xp(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}}$$

for some coefficients $(a_s)_{s \in \mathbb{N}_0}$.

Pf:

cf. following example: Stirling

Main difference to Pf. Thm. 1

Set $v := p - p(a)$ and revert to get asymptotic for $t - a \sim \sum_{s=1}^{\infty} c_s v^{s/\mu}$ as $v \downarrow 0$

Coefficient Comparison / Fixed Point Method

$c_1 = 1/p_0^{1/\mu}$, $c_2 = -p_1/\mu p_0^{1+2/\mu}$, ...

Use $f := \frac{q}{p'}$ and similar to the previous proof to get $f \sim \sum_{s=0}^{\infty} a_s v^{s+\lambda-1}$ as $v \downarrow 0$, where the a_s are expressible in terms of p_s, q_s.

$a_0 = q_0 / \mu p_0^{\lambda/\mu}$, $a_1 = \left(\frac{q_1}{\mu} - \frac{(\lambda+1)}{\mu^2 p_0^2} p_1 q_0\right) \frac{1}{p_0^{(\lambda+1)/\mu}}$ ...

cf. prev. proof

$\Rightarrow \exists k : p' > 0$ cts. and q cts. over $(a, k]$.

$$e^{xp(a)} \int_a^k e^{-xp(t)} q(t)\, dt \overset{\text{Sub.}}{=} \int_0^K e^{-xv} \underbrace{q(t) \frac{dt}{dv}}_{=: f(v) \text{ cts. because } v \text{ and } t \text{ are cts functions of each other over } (0,K]} dv = \dots$$

, $K := p(k) - p(a)$ finite and positive.

Write $f = \sum_{s=0}^{n-1} a_s v^{(s+\lambda-\mu)/\mu} + \overbrace{v^{(n+\lambda-\mu)/\mu} f_n(v)}^{\text{Tail of the Series}}$ for $v > 0$, where $f_n(0) = a_n$, $n \in \mathbb{N}_0$.

analogous to Thm. 1

$$\Rightarrow \dots = \sum_{s=0}^{n-1} \left(\Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} - \varepsilon_{n,1}(x) + \varepsilon_{n,2}(x)\right)$$

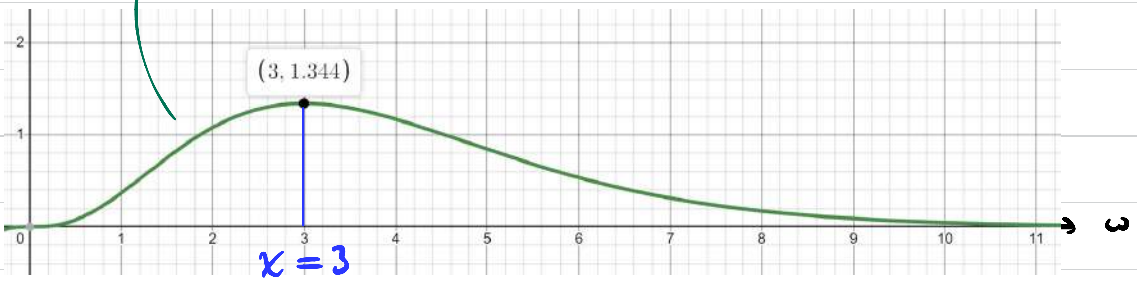
$$\varepsilon_{n,2}(x) = \int_0^K e^{-xv} v^{(n+\lambda-\mu)/\mu} f_n(v)\, dv \underset{\text{cf. prev. proof}}{=} \mathcal{O}\left(\frac{1}{x^{(n+\lambda)/\mu}}\right)$$

$$\varepsilon_{n,1}(x) = \sum_{s=0}^{n-1} \Gamma\left(\frac{s+\lambda}{\mu}, Kx\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \underset{\text{cf. prev. proof}}{=} \mathcal{O}(e^{-Kx}/x)$$

Further details and rest of the arguments are analogous to the previous proof. $\square$

! Exa (Stirling) $\Gamma(x) \sim e^{-x} x^{x} \left(\frac{2\pi}{x}\right)^{1/2} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right)$

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-w} w^{x}\, dw = \cdots$$



Substitution

Independence of max. of x by taking t as $w = x(1+t)$ ⇝ $\frac{dw}{dt} = x$

Recall Reversion Example

$$\rightsquigarrow \cdots = e^{-x} x^{x} \int_{-1}^{\infty} e^{-xt}(1+t)^{x}\, dt = e^{-x} x^{x} \int_{-1}^{\infty} e^{-x(\underbrace{t - \log(1+t)}_{=p})} \cdot \underbrace{1}_{=q}\, dt$$

Split at min.

$$\rightsquigarrow e^{x} x^{-x} \Gamma(x) \overset{t \mapsto -t}{=} \underbrace{\int_0^\infty e^{-xp(t)}\, dt}_{=:\alpha(x)} + \underbrace{\int_0^1 e^{-xp(-t)}\, dt}_{=:\beta(x)}$$

(Split done to subdivide at min. (cf. Rmk. Thm. 1 (*)))
at min: $t_0 = 0$

See example before

Note $p = \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4 \mp \cdots$ $t \in (-1, +1)$ ⇒ Thm 2 holds.

$p' = \frac{t}{1+t}$ ⇝ $v = p(t) - 0$ and reversion of is: $t = 2^{1/2} v^{1/2} + \frac{2}{3} v + \frac{2^{1/2}}{18} v^{3/2} - \frac{2}{135} v^2 + \cdots$

$\frac{dt}{dv} = \underbrace{\frac{2^{1/2}}{2}}_{=a_0} v^{-1/2} + \underbrace{\frac{2}{3}}_{=a_1} + \underbrace{\frac{2^{1/2}}{12}}_{=a_2} v^{1/2} \mp \cdots$ Convergent for small v

From the "$-t$" we got from substitution ... carries over.

Recall $f = \frac{q}{p'} = \frac{dt}{dv} = a_0 v^{-1/2} + a_1 + a_2 v^{1/2} + \cdots$ Thm 2: $\alpha(x) \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{x^{(s+1)/2}}$, $\beta(x) \sim \sum_{s=0}^{\infty} (-1)^s \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{x^{(s+1)/2}}$

Thus:

Odd terms cancel
Even terms duplicate

$$e^x x^{-x}\Gamma(x) \sim 2\sum_{s=0}^{\infty}\Gamma\left(\frac{2s+1}{2}\right)\frac{a_{2s}}{x^{(2s+1)/2}}$$

$$= 2\sqrt{\pi}\,\frac{a_0}{x^{1/2}} + 2\sum_{s=1}^{\infty}\frac{(2s-1)!!}{2^{s-1}}a_{2s}\sqrt{\pi}\,\frac{1}{x^{(2s+1)/2}}$$

$$= \sqrt{\frac{2\pi}{x}}\Bigg(\underbrace{\underbrace{a_0}_{=\sqrt{2}/2}\sqrt{2}}_{=1} + \sum_{s=1}^{\infty}\frac{(2s-1)!!}{2^{s-1}}\sqrt{2}\,a_{2s}\,\frac{1}{x^s}\Bigg)$$

$$\Gamma\left(\frac{2s+1}{2}\right) = \begin{cases}\sqrt{\pi}, & s=0\\[1ex] \dfrac{2s-1}{2}\cdot\Gamma\left(\dfrac{2s-1}{2}\right) = \dfrac{(2s-1)(2s-3)\cdots 3}{2^{s-1}}\underbrace{\Gamma(1/2)}_{=\sqrt{\pi}} \\[2ex] \qquad = \dfrac{(2s-1)!!}{2^{s-1}}\sqrt{\pi} & s\geq 1\end{cases}$$

Rmk: The coefficients a_s from above satisfy

$$\sum_{\ell=0}^{s}\frac{1}{\ell+1}a_\ell a_{s-\ell} = \frac{1}{s}a_{s-1} \qquad s\geq 1$$

Exa : $(a_0, a_1, a_2, \dots) = \left(\frac{1}{\sqrt{2}}, \frac{2}{3}, \frac{\sqrt{2}}{12}, \dots\right)$

$\uparrow a_0$ $\uparrow a_2$

Pf:

For $f = \sum_{s=0}^{\infty} a_s v^{\frac{s-1}{2}}$ we have

Formal integral

$$\int f \, dv = \sum_{s=0}^{\infty} \frac{2a_s}{s+1} v^{(s+1)/2}$$

$$\| $$

$$\int \frac{dt}{dv} dv = \int dt = t$$

which gives a representation of the inversion t in terms of the coefficients of f.

Moreover: $$tf = t\frac{q}{p'} = t\,\frac{1}{t}(1+t) = 1+t = 1 + \sum_{s=0}^{\infty} \frac{2a_s}{s+1} v^{(s+1)/2} = 1 + \sum_{s=1}^{\infty} \boxed{\frac{2a_{s-1}}{s}} v^{s/2}$$

$$\|$$

$$\left(\sum_{s=0}^{\infty} \frac{2a_s}{s+1} v^{(s+1)/2}\right)\left(\sum_{s=0}^{\infty} a_s v^{\frac{s-1}{2}}\right) \overset{CP}{=} \sum_{s=0}^{\infty}\left(\sum_{l=0}^{s} \frac{2a_l}{l+1} a_{s-l} \underbrace{v^{\frac{l+1}{2}} v^{\frac{s-l-1}{2}}}_{= v^{s/2}}\right)$$

$$= \sum_{s=0}^{\infty}\left(\sum_{l=0}^{s} \frac{2}{l+1} a_l a_{s-l}\right) v^{s/2} = 1 + \sum_{s=1}^{\infty}\left(\boxed{\sum_{l=0}^{s} \frac{2}{l+1} a_l a_{s-l}}\right) v^{s/2}$$

$\square$

Thank You!

Questions?