ASE 389P-7
Problem set 3

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1 Problem 1

1.1 Instruction

In this problem you will explore the difference between low- and high-side mixing on carrier phase tracking. Consider the process of converting a bandpass signal $2a(t)\cos 2\pi f_c$, centered at f_c, to an intermediate frequency via mixing with a signal $\cos(2\pi f_l t)$:

$$xu(t) = 2a(t)\cos 2\pi f_c t cos(2\pi f_l t) = a(t)\cos\left[2\pi(f_c - f_l)t\right] + HFT \tag{1}$$

Filtering at the intermediate frequency removes the high-frequency term (HFT), leaving only

$$x(t) = a(t)\cos 2\pi(f_c - f_l)t. \tag{2}$$

By definition, the frequency of a physical signal (the number of cycles per second) is a positive quantity. Therefore, if $f_c < f_l$, we express $x(t)$ as

$$x(t) = a(t)\cos\left[2\pi(-f_c + f_l)t\right] \tag{3}$$

by invoking the identity $\cos(y) = \cos(-y)$. The original bandpass signal's center frequency fc can be decomposed as $f_c = f_{c,nom} + f_{c,D}$ where $f_{c,nom}$ is the nominal value (e.g., 1575.42 MHz for GPS L1), and $f_{c,D}$ is a Doppler value due to satellite-to-receiver relative motion and satellite clock frequency offset.

Likewise,

$$f_l = f_{l,nom} + f_{l,D} \tag{4}$$

where $f_{l,nom}$ is the nominal local oscillator value and $f_{l,D}$ is a Doppler value due to receiver clock frequency offset. When $f_{c,nom} < f_{l,nom}$, the mixing operation is referred to as high-side mixing (HS mixing), whereas when $f_{l,nom} < f_{c,nom}$, it is low-side mixing (LS mixing). The intermediate frequency is defined as

$$fIF = |fc, nom - fl, nom| \tag{5}$$

Note that f_{IF} is always chosen to be large enough that $f_c < f_l$ for HS mixing and $f_l < f_c$ for LS mixing over the range of expected $f_{c,D}$ and $f_{l,D}$. The downmixed signal $x(t)$ is modeled as

$$x(t) = a(t)\cos 2\pi f_{IF} t + \theta(t) \tag{6}$$

Explain the different effect of HS and LS mixing on $\theta(t)$. What implications might this different effect have for signal acquisition and tracking?

1.2 Solution

The model of the received and downmixed signal is

$$x(t) = a(t)\cos 2\pi f_{IF} t + \theta(t) \tag{7}$$

where

$$\theta(t) = \begin{cases} f_{c,nom} - f_{l,nom} & (HSmixing) \\ f_{l,nom} - f_{c,nom} & (LSmixing) \end{cases} \tag{8}$$

Thus, we have that

$$\theta(t) = \begin{cases} \theta_{HS} = \int_0^t f_{l,D}(\tau) - f_{c,D}(\tau) d\tau + \theta(0) & (HSmixing) \\ \theta_{LS} = \int_0^t f_{c,D}(\tau) - f_{l,D}(\tau) d\tau + \theta(0) & (LSmixing) \end{cases} \tag{9}$$

Note that $\dot{\theta}_H S(t) = -\dot{\theta}_L S(t)$. In addition, if a high-side mixed signal ends completely on the other side of the zero frequency line, its phase gets reversed. Interestingly, observe that a small increment in f_c actually decreases the intermediate frequency f_{IF}.

2 Problem 3

2.1 Instruction

In lecture we considered an analog signal $x_a(t)$ sampled by impulses:

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT) \tag{10}$$

We showed that the Fourier transform of the impulse-sampled signal $x_\delta(t)$ is related to $X_a(f)$, the Fourier transform of $x_a(t)$, by

$$X_\delta(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a(f - \frac{n}{T}) \tag{11}$$

We can derive a similar relationship between $X_a(f)$ and the Fourier transform of the discrete-time signal

$$x(n) = x_a(nT), -\infty < n < \infty \tag{12}$$

To make this easier, we'll define the frequency variable $\tilde{f} = fT = \frac{f}{f_s}$. This variable, which has units of cycles per sample and is often called the normalized frequency, is used as the frequency variable for discrete-time signals. For example, a discrete-time sinusoid can be represented as

$2x(n) = cos(2\pi \tilde{f} n)$. For discrete-time signals, only frequencies in the range $-1/2 \leq \tilde{f} \leq 1/2$ are unique; all frequencies $|\tilde{f}| > 1/2$ are aliases.

The Fourier transform of a discrete-time signal $x(n)$ is defined by

$$X(\tilde{f}) = \sum_{n=-\infty}^{\infty} x(n) \exp\left(-j2\pi \tilde{f} n\right) \tag{13}$$

and the inverse transform is defined by

$$x(n) = \int_{-1/2}^{1/2} X(\tilde{f}) \exp\left(j2\pi \tilde{f} n\right) d\tilde{f} \tag{14}$$

Derive the relationship between $X(\tilde{f})$ and $X_a(f)$.

Hint: This is a standard relationship whose derivation can be found in many texts that treat digital signal processing. You're free to use such a text as a guide or you may perform the derivation yourself following these steps:

- Express $x(n) = x_a(nT)$ in terms of $X_a(f)$.
- Equate this expression with the inverse transform definition given above.
- Express the integral that goes from $-\infty to \infty$ as an infinite sum of integrals of width f_s.
- Make a change of variable $f = \tilde{f} fs$ in this infinite sum of integrals expression to eliminate f in favor of $\tilde{f}$.
- Make some deductions to arrive at the desired relationship between $X(\tilde{f})$ and $X_a(f = \tilde{f} fs)$.

2.2 Solution

It is possible to consider equation 10 to calculate the Fourier Transform of $x_\delta(t)$.

$$X_\delta(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_a(f - \frac{n}{T}) = \sum_{n=-\infty}^{\infty} x_a(nT) \exp\left(-j2\pi f nT\right) \tag{15}$$

Then, it is possible to see that formula for $X_\delta(f)$ can be equated to 13 as follows

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x_a(nT) \exp\left(-j2\pi f nT\right) = \sum_{n=-\infty}^{\infty} x(n) \exp\left(-j2\pi \tilde{f} n\right) = X(\tilde{f}) \tag{16}$$

This leads to the following relationship

$$X(\tilde{f}) = X_\delta(\frac{\tilde{f}}{T}) \tag{17}$$

Now, using equation 11 it is possible to reach

$$X(\tilde{f}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_a(\frac{\tilde{f} - n}{T}) \tag{18}$$

3 Problem 5

3.1 Code

```
function [IVec,QVec] = if2iq(xVec,T,fIF)
% IF2IQ : Convert intermediate frequency samples to baseband I and Q samples.
%
% Let x(n) = I(n*T)*cos(2*pi*fIF*n*T) - Q(n*T)*sin(2*pi*fIF*n*T) be a
% discrete-time bandpass signal centered at the user-specified intermediate
% frequency fIF, where T is the bandpass sampling interval. Then this
% function converts the bandpass samples to quadrature samples from a complex
% discrete-time baseband representation of the form xl(m*Tl) = I(m*Tl) +
% j*Q(m*Tl), where Tl = 2*T.
%
%
% INPUTS
%
% xVec ------------ N-by-1 vector of intermediate frequency samples with
%                   sampling interval T.
%
% T --------------- Sampling interval of intermediate frequency samples, in
%                   seconds.
%
% fIF ------------- Intermediate frequency of the bandpass signal, in Hz.
%
%
% OUTPUTS
%
% IVec ------------ N/2-by-1 vector of in-phase baseband samples.
%
% QVec ------------ N/2-by-1 vector of quadrature baseband samples.
%
%
%+------------------------------------------------------------------------------+
% References:
%
%
%+==============================================================================+
n = 0:1:length(xVec)-1; n=n';
IVec = 2*cos(2*pi*fIF*n*T).*xVec;
QVec = -2*sin(2*pi*fIF*n*T).*xVec; % figure out the sign

IVec = decimate(IVec,2)/sqrt(2);
QVec = decimate(QVec,2)/sqrt(2);

function [xVec] = iq2if(IVec,QVec,Tl,fIF)
```

```
% IQ2IF : Convert baseband I and Q samples to intermediate frequency samples.
%
% Let xl(m*Tl) = I(m*Tl) + j*Q(m*Tl) be a discrete-time baseband
% representation of a bandpass signal. This function converts xl(n) to a
% discrete-time bandpass signal x(n) = I(n*T)*cos(2*pi*fIF*n*T) -
% Q(n*T)*sin(2*pi*fIF*n*T) centered at the user-specified intermediate
% frequency fIF, where T = Tl/2.
%
%
% INPUTS
%
% IVec ---------- N-by-1 vector of in-phase baseband samples.
%
% QVec ---------- N-by-1 vector of quadrature baseband samples.
%
% Tl ------------ Sampling interval of baseband samples (complex sampling
%                 interval), in seconds.
%
% fIF ----------- Intermediate frequency to which the baseband samples will
%                 be up-converted, in Hz.
%
%
% OUTPUTS
%
% xVec ---------- 2*N-by-1 vector of intermediate frequency samples with
%                 sampling interval T = Tl/2.
%
%
%+------------------------------------------------------------------------------+
% References:
%
%
%+==============================================================================+
IVec_resampled = interp(IVec,2);
QVec_resampled = interp(QVec,2);

T = Tl/2;
n = 0:1:length(IVec_resampled)-1; n=n';
xVec = IVec_resampled.*cos(2*pi*fIF*n*T) ...
       - QVec_resampled.*sin(2*pi*fIF*n*T);

xVec = sqrt(2)*xVec;
```

3.2 Results

The power spectral density of the original signal (baseband representation), the bandpass representation and the reconstructed baseband representation can be observed in Figures 1, 2 and 3, respectively. Note that the bandpass representation had to be centered at a $f_{IF} = 5MHz$ to be able to reconstruct the signal without distortion since the bandwidth of the original signal is approximately 5MHz.

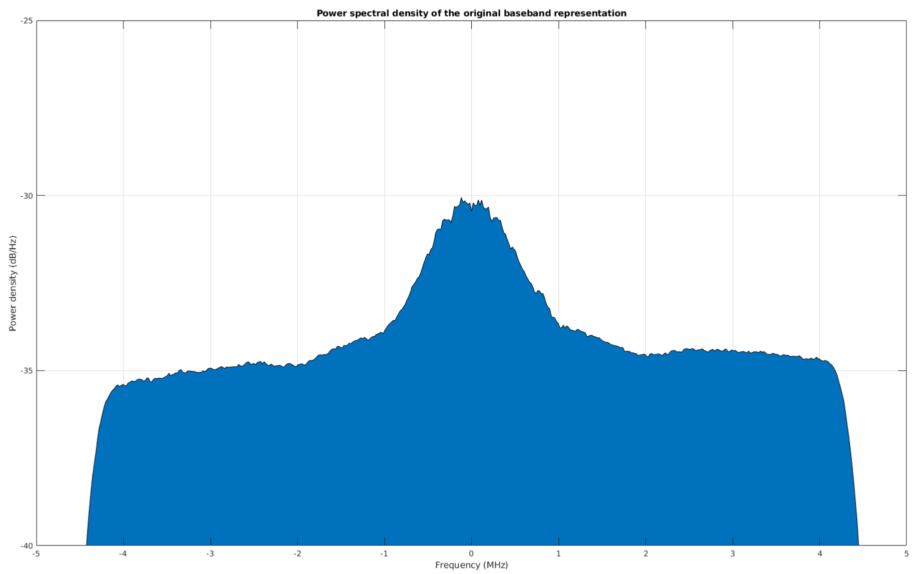

Figure 1: Power spectral density of the original signal in its baseband representation.

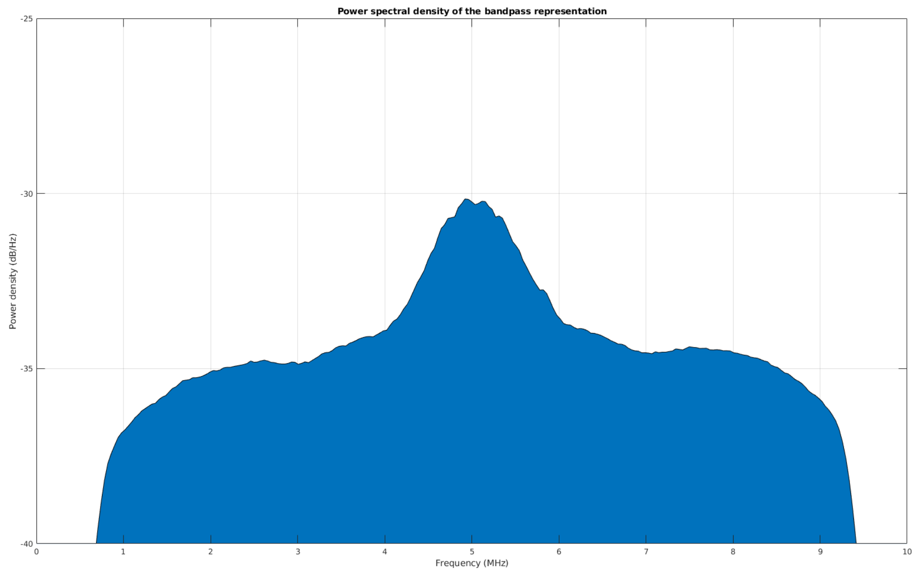

Figure 2: Power spectral density of the bandpass representation of the original signal.

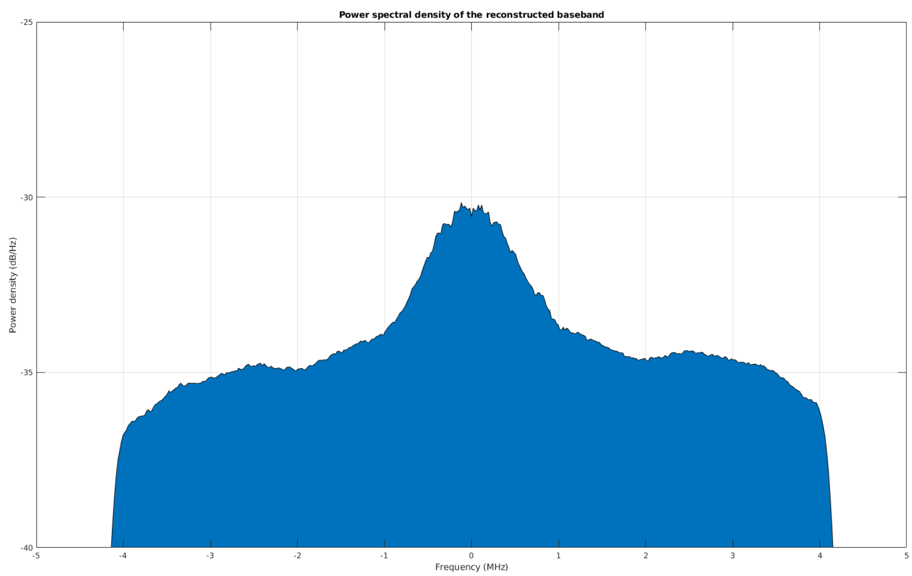

Figure 3: Power spectral density of the reconstructed signal in its baseband representation.