A Generalized Constructive Proof for Brouwer Fixed-Point Theorem on D^2 and D^3

Ruohan Yang*
Fudan University , China

Abstract

This article presents a constructive proof by analyzing decompositions of continuous vector field. The original proof of Brouwer's theorem relies on a contradiction argument, which, while effective, does not offer a constructive method for locating the fixed point. Through projecting arbitrary vector field on the basis of the vector field, it can be proved there exists zero points on both of the basis. The article will also generalize the proof from 2D to 3D, 4D dimensions. The method is valid under contraction map or non-contraction map.

1 Introduction

This article provides a direct proof of Brouwer's Fixed-Point Theorem by decomposing the vector field into n components in D^n (for dimensions $n = 2$, $n = 3$ and $n = 4$).It explains why a fixed point must exist by using the theory of intersections of surfaces and lines. The proof is valid for any continuous map that maps a compact convex set to itself. Additionally, the article includes a series of code experiments that verify the proof through numerical results.

2 Background

The fixed-point theorem is a fundamental result in mathematics that asserts the existence of points that remain invariant under a given function. One of the most well-known fixed-point theorems is Brouwer's Fixed-Point Theorem[2], which states that any continuous function mapping a compact convex set to itself has at least one fixed point. This theorem is particularly significant in topology and has applications across various fields, including economics, game theory, and differential equations.

Theorem 2.1 (Brouwer's Fixed-Point Theorem)**.** *Let D be a compact convex subset of $\mathbf{R}^n$. If $f : D \to D$ is a continuous function, then there exists at least one point $x \in D$ such that $f(x) = x$.*

Original Proof of Brouwer's Fixed-Point Theorem: Brouwer's original proof employs topological arguments to demonstrate the existence of a fixed point for continuous mappings. The essence of the proof relies on the assumption that if a continuous mapping f does not have a fixed point, then a retraction r can be constructed to project the entire disk onto its boundary. However, it is known that there exists a homotopy f_0 in D^2 that can be continuously deformed to a constant loop. If we compose this homotopy with the retraction r, it implies the existence of a homotopy from S^1 (the boundary of the disk) to a constant loop, which is a contradiction in topology. Therefore, the assumption that f has no fixed point must be false, confirming that at least one fixed point exists within the disk.

3 Continuous Vector fields representation

For any continuous map f mapping $D^n \to D^n$, X is a set of points in D^n, f can be identified by a continuous vector field $\vec{F}(X)$. The vector field has the following relations:

$$\vec{F}(X) = f(X) - X$$

*Email: lyf@mail.ustc.edu.cn

,for a single point positions p in X,

$$\vec{F}(p) = f(p) - p$$

,$\vec{F}(p)$ represent the vector pointing from the original position p to where $f(p)$ has send p.

4 Basis of Projection

This section examines how a continuous vector field on a disk D^n can be expressed and analyzed to determine conditions for fixed points, i.e., points where the vector field vanishes.

Let $p \in D^n$ and consider the vector field $\vec{F}(p)$, which can be expressed as:

$$\vec{F}(p) = \sum_{i=1}^{n} a^i(p)\hat{e}_i,$$

where $a^i(p)$ represents the i-th component of $\vec{F}(p)$ along the standard basis $\hat{e}_i$. In two dimensions, this reduces to:

$$\vec{F}(p) = a^1(p)\hat{e}_1 + a^2(p)\hat{e}_2.$$

The projection function P_i extracts the i-th component of $\vec{F}(p)$:

$$P_i\left(\vec{F}(p)\right) = a^i(p).$$

To find a fixed point where $\vec{F}(p) = 0$, all components of the vector field must vanish:

$$a^i(p) = 0 \quad \text{for all} \quad i = 1, \dots, n.$$

In two dimensions ($n = 2$), this condition becomes:

$$a^1(p) = 0 \quad \text{and} \quad a^2(p) = 0.$$

In n-dimensions, the same principle applies: a fixed point exists if and only if all n components $a^1(p), a^2(p), \dots, a^n(p)$ are zero.

5 Analyze the zero components on basis

This chapter will start discussing the situations of using 2D disks first,then in next chapter it can be generalized to higher dimension.

Lemma 5.1. *Let $\vec{F}$ be a continuous vector field on a disk D^n, mapping $D^n \to D^n$, and let $P_i(\vec{F}(p))$ denote the projection of the vector field $\vec{F}(p)$ onto the i-th axis. For each $i \in \{1, \dots, n\}$, there exists at least one point where $P_i(\vec{F}(p)) = 0$ along a path within D^n. This path connects the two intersection points of D^n with the i-th axis and the zero occurs either between the boundary points or exactly at one of the boundary points.*

Proof. To begin, consider the two-dimensional case, namely, a disk $D \subset \mathbf{R}^2$, where the vector field $\vec{F}$ is continuous.

The basis chosen for the two-dimensional disk is simply in Cartesian coordinates, $\hat{e_x}$, $\hat{e_y}$. Let $p \in \partial D$ be a point on the boundary of the disk. The point p is represented in polar coordinates as $p(\theta)$, where $\theta \in [0, 2\pi]$ parameterizes the boundary ∂D.

Consider the projection of $\vec{F}(p)$ onto the x-axis, $P_x(\vec{F}(p))$. By the boundary condition and continuity of the map f, it is evident that at the rightmost point of the boundary, denoted as x_0 ($\theta = 0$), the vector field $\vec{F}(p)$ must point inward or remain stable, implying

$$P_x(\vec{F}(x_0)) \leq 0$$

. Similarly, at the leftmost point x_0', ($\theta = \pi$)

$$P_x(\vec{F}(x_0')) \geq 0$$

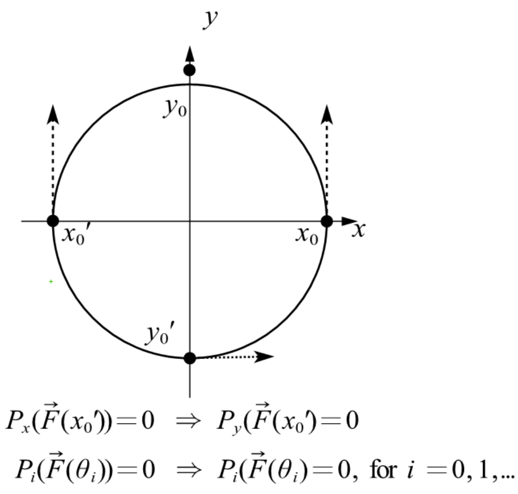

Figure 1: Trivial cases

Remark. Before proceeding with the proof, it is necessary to first address certain special conditions.Specifically, when $P_x(\vec{F}(x_0')) = 0$ at the leftmost point x_0'—it follows directly that $P_y(\vec{F}(x_0')) = 0$, as the vector field $\vec{F}$ is constrained to lie entirely within the disk. In this case, the existence of a fixed point is straightforward.This holds for any i.*(Remark end)*

The following proof focuses exclusively on non-trivial cases where these special conditions do not apply.

Consider when $\vec{F}(p)$ always point inward. Since $P_x(\vec{F}(p))$ is a continuous function on any path $\mathcal{C}$ start from the point x_0 and end with x_0', and the sign of the function changes between them, by the Intermediate Value Theorem (IVT) [1], there exists a point $x \in \mathcal{C}$ where $P_x(\vec{F}(x)) = 0$.

Next, consider the projection onto the y-axis, $P_y(\vec{F}(p))$. At the top of the disk $\theta = \frac{\pi}{2}$, the vector field $\vec{F}(p)$ must again point inward , implying $P_y(\vec{F}(\frac{\pi}{2})) < 0$, and at the bottom $\theta = \frac{3\pi}{2}$, we have $P_y(\vec{F}(\frac{3\pi}{2})) > 0$.

By the IVT, there exists a point $y \in \mathcal{C}$ where $P_y(\vec{F}(y)) = 0$ for every path $\mathcal{C}$ start from $\theta = \frac{\pi}{2}$ to $\theta = \frac{3\pi}{2}$.

For higher dimensions, consider a disk D^n, where a continuous map f and its corresponding vector field $\vec{F}$ can be decomposed into components along the standard basis vectors. Denoting the points where the i-th axis intersects the boundary of the disk as i_0 and i_0', with i_0 representing the point on the positive side of the i-th axis and i_0' on the negative side, the following relations hold:

On the loop passing through i_0 and i_0' of D^n, we have:

$$P_i\left(\vec{F}(i_0)\right) \leq 0 \quad \text{and} \quad P_i\left(\vec{F}(i_0')\right) \geq 0,$$

If $\vec{F}(p)$ consistently points inward, the Intermediate Value Theorem guarantees the existence of at least one point i_1 on the curve $\gamma(\theta)$, which connects i_0 and i_0', such that $P_i\left(\vec{F}(\theta)\right) = 0$.

Since this argument holds for each axis, there are at least two distinct points on ∂D^n where the projection function $P_i\left(\vec{F}(p)\right)$ vanishes, i.e., $P_i\left(\vec{F}(p)\right) = 0$.

□

Claim 5.2. *In two dimensions, the set of points where $P_i\left(\vec{F}(p)\right) = 0$ forms a curve that connects two distinct points on the boundary of the disk.*

Proof. From **Lemma 5.1**, it follows that for the vector field $\vec{F}$, there exist at least two points on the boundary ∂D^2, denoted as p_i and p_i', where $P_i(\vec{F}(p)) = 0$. Additionally, the rightmost points on the disk, x_0 and x_0', correspond to $\theta = 0$ and $\theta = \pi$ in polar coordinates.

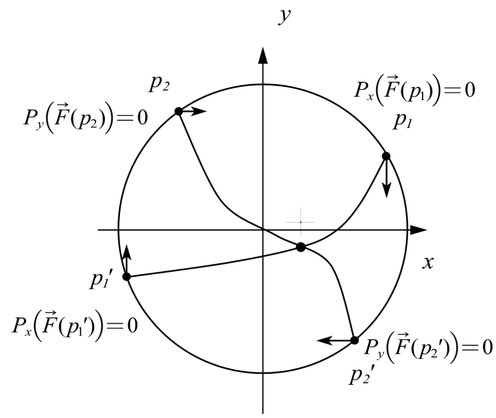

Figure 2: intersection of two lines

To show that the set of points where $P_i(\vec{F}(p)) = 0$ forms a continuous curve, consider the case where i corresponds to the x-axis. Construct a path $\mathcal{C}$ on D^2 connecting x_0 to x_0'. Using a homotopy h_t, $t \in [0,1]$, this path $\mathcal{C}$ can be continuously deformed between two boundary paths $\mathcal{A}$ and $\mathcal{B}$ without self-intersections.

Since $P_i(\vec{F}(p))$ is continuous and $P_i(\vec{F}(x_0))$ and $P_i(\vec{F}(x_0'))$ have opposite signs, the Intermediate Value Theorem ensures that there exists at least one point on each $\mathcal{C}$ where $P_i(\vec{F}(p)) = 0$. These points trace out a continuous curve under the homotopy h_t, connecting p_i to p_i'. A similar argument applies to the y-axis, completing the proof.

□

Claim 5.3. *In two dimensions, the two points on the boundary ∂D^2 where $P_y\left(\vec{F}(p)\right) = 0$ must lie within the different section of ∂D^2 that has points where $P_x\left(\vec{F}(p)\right) = 0$ as its boundary.*

Proof. Let p_1 and p_1' be the points on the boundary ∂D^2 where $P_x\left(\vec{F}(p)\right) = 0$, as guaranteed by **Lemma 5.1**. These points lie on opposite sides of the x-axis in non-trivial cases: p_1 on the upper semicircle and p_1' on the lower semicircle, since P_x changes sign between them.

At p_1, since $P_x\left(\vec{F}(p)\right) = 0$ and $\vec{F}(p_1)$ must point inward, the y-component satisfies $P_y\left(\vec{F}(p_1)\right) < 0$. Similarly, at p_1', $P_y\left(\vec{F}(p_1')\right) > 0$.

Now, consider the continuous function $P_y\left(\vec{F}(p)\right)$ on ∂D^2. By the Intermediate Value Theorem (IVT), there must exist at least one point on the boundary between p_1 and p_1' where $P_y\left(\vec{F}(p)\right) = 0$.

Thus, the points where $P_y\left(\vec{F}(p)\right) = 0$ either coincide with p_1 and p_1' or lie strictly between them on the boundary arc connecting p_1 and p_1'. This completes the proof.

□

Theorem 5.4. *In a two dimension disk, the lines $P_y\left(\vec{F}(p)\right) = 0$ and $P_x\left(\vec{F}(p)\right) = 0$ must have at least one intersection corresponds to the point where $\vec{F}(p) = 0$.*

Proof. Let p_1 and p_1' be the points on the boundary ∂D^2 where $P_x\left(\vec{F}(p)\right) = 0$. By the same reasoning as in Claim 5.2, at p_1, $P_y\left(\vec{F}(p_1)\right) < 0$, and at p_1', $P_y\left(\vec{F}(p_1')\right) > 0$. By the Intermediate Value Theorem, there must exist at least one point on the boundary between p_1 and p_1' where $P_y\left(\vec{F}(p)\right) = 0$.
□

6 Fixed point Theorem on D^3

This chapter proves the fixed point theorem in three dimensions, specifically in the 3-ball D^3. The proof is similar to the two-dimensional case (D^2) but involves more complexity due to the additional dimension.

Lemma 6.1. *In three dimensions, the set of points where $P_i\left(\vec{F}(p)\right) = 0$ forms a surface within D^3. On the boundary ∂D^3, this surface intersects the boundary to form a closed loop.*

Proof. Consider the case where the i-th axis corresponds to the y-axis. Once the proof is established for the y-axis, it generalizes to the other axes.

Let y_0 and y_0' denote the rightmost and leftmost points on the boundary ∂D^3. In spherical coordinates, these correspond to $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$, respectively. By **Lemma 5.1**, every path through these boundary points must have at least one zero point for the function $P_y(\vec{F}(p))$.

Since $P_i(\vec{F}(p))$ is continuous, we can apply a homotopy h_t that deforms paths continuously across the interior of the sphere. As these paths are deformed, the zero points of $P_i(\vec{F}(p))$ also move continuously. Hence, the set of zero points traces out a continuous surface between the boundary points y_0 and y_0'.

Therefore, the set of points where $P_i(\vec{F}(p)) = 0$ forms a surface in D^3, and on the boundary ∂D^3, it forms a loop. □

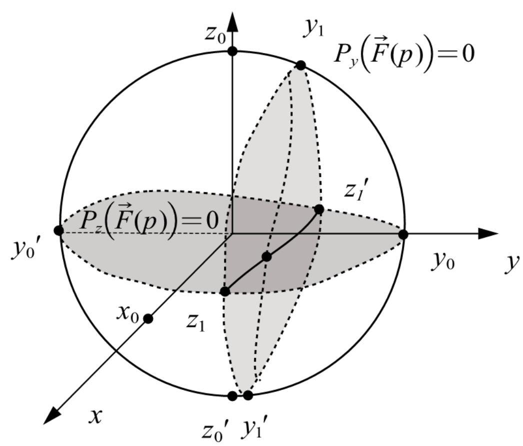

Figure 3: intersection of three surface on D^3

Theorem 6.2. *In three dimensions,there at least one exist intersection point of $P_i\left(\vec{F}(p)\right) = 0$ for each $i = x, y, z$.*

Proof. By **Lemma 5.1**, for any path starting from z_0 to z_0' on ∂D^3, there exists at least one point where $P_z\left(\vec{F}(p)\right) = 0$. Similarly, for any path starting from y_0 to y_0', there is a point where $P_y\left(\vec{F}(p)\right) = 0$.

Consider a path lying in the section between y_0 and y_0', which corresponds in spherical coordinates to the arc $\phi \in [0, \frac{\pi}{2}]$ with $\theta = 0$ and π. By **Lemma 5.1**, there exists a point y_1 on this arc where $P_y\left(\vec{F}(p)\right) = 0$, and this point lies on ∂D^3.

On this path, y_1 always lies on the upper half of the sphere, we can construct y_1' in the lower half of the sphere use the same way.

Remark. Since y_1 is the outermost point with the largest norm of components on the y-axis, and satisfies $P_y\left(\vec{F}(p)\right) = 0$, its mapping vector must lie on the local z-x plane. This implies that either $P_z\left(\vec{F}(p)\right) = 0$, or the vector always points inside the surface with $P_z\left(\vec{F}(p)\right) < 0$.

In the former case, where both $P_y\left(\vec{F}(p)\right)=0$ and $P_z\left(\vec{F}(p)\right)=0$, the point cannot move elsewhere without leaving the disk, thus it is a fixed point.

In the latter case, where $P_z\left(\vec{F}(y_1)\right)$ and $P_z\left(\vec{F}(y_1')\right)$ have different signs, the mapping still indicates a transition across the surface. (*Remark end*)

A path can be constructed like follows: Start from z_0, then connected with y_1 ,then extend it within the surface $P_y\left(\vec{F}(p)\right)=0$ to y_1', finally go to z_0'. The path $\mathcal{C}$

$$\mathcal{C}=z_0\to y_1\to y_1'\to z_0'$$

A homotopy again can be constructed from y_1 to y_1' within the surface $P_y\left(\vec{F}(p)\right)=0$.

As the path deforms and sweeps across the surface, by **Lemma 5.1** and the remark again, there must be points where $P_z\left(\vec{F}(p)\right)=0$ between y_1 and y_1'. This proves that these two surfaces must has intersections points. And due to continuity,intersections points of $P_z\left(\vec{F}(p)\right)=0$ form a curve $\mathcal{L}$. Also, because path $\mathcal{C}$ also lies on the surface $P_y\left(\vec{F}(p)\right)=0$, we have this $\mathcal{L}$ with z, y components of its vector field all zero.

Let $\mathcal{A}$ denote the portion of the path on the hemisphere with positive x-coordinates, and $\mathcal{B}$ the portion on the hemisphere with negative x-coordinates. To prove that there is always an intersection on $\mathcal{L}$ where $P_x\left(\vec{F}(p)\right)=0$, denote the points where $P_z\left(\vec{F}(p)\right)=0$ as $z_1\in\mathcal{A}$ and $z_1'\in\mathcal{B}$. Since z_1 has positive x-coordinates and z_1' has negative x-coordinates and $P_y\left(\vec{F}(p)\right)=0$,$P_z\left(\vec{F}(p)\right)=0$, a path $\mathcal{C}'$ can be constructed.

$$\mathcal{C}'=x_0\to z_1\to\mathcal{L}\to z_1'\to x_0'$$

By Lemma 5.1 and remarks, there must exist a point on this path where $P_x\left(\vec{F}(p)\right)=0$. Therefore, an intersection point can be found on $\mathcal{L}$ where $P_x\left(\vec{F}(p)\right)=0$, and at this point, the vector field satisfies $P_x\left(\vec{F}(p)\right)=P_y\left(\vec{F}(p)\right)=P_z\left(\vec{F}(p)\right)=0$, which proves the existence of a fixed point. □

7 Fixed point Theorem on D^4 and higher dimensions D^n

In four dimension,the coordinates are respectively w,z,y,x. As the same, we discussed the non-trivialized cases.

Lemma 7.1. *For any i , in the set $\{p|P_i\left(\vec{F}(p)\right)=0\}$, then for $k\neq i$, it can always be found i_1, i_1' that in the different partial hyperspheres of ∂D^n separated by plane $k=0$. Denote the intersection points of k- th axis and the disk as k_0, k_0', then a path can be construct: $k_0\to i_1\to i_1'\to k_0'$ with $P_k\left(\vec{F}(i_1)\right)<0$,$P_k\left(\vec{F}(i_1')\right)>0$.*

Proof. Consider the case when $n=4$. The set

$$\mathcal{A}=\{p\mid P_w\left(\vec{F}(p)\right)=0\}$$

can be visualized in three dimensions if the other three axes (x,y,z) are "compressed" into a two-dimensional surface, as shown in the following figure.

For every curve that passes through the polar points w_0 and w_0' of D^4, there exists at least one zero point of $P_w\left(\vec{F}(p)\right)=0$ on the curve. These zero points collectively form a surface, as established in **Lemma 6.1**.

It can also be proved that the outermost points in this set are separately distributed in different parts of the hemisphere for each axis. For example, along the z-axis, the two points w_1 and w_1' always lie on opposite sides of the origin, separated by the plane $z=0$. Similarly, for x,y, there are two points on ∂D can always be found. The proof for $P_z\left(\vec{F}(w_1)\right)<0$ and $P_z\left(\vec{F}(w_1)\right)>0$ is simple like

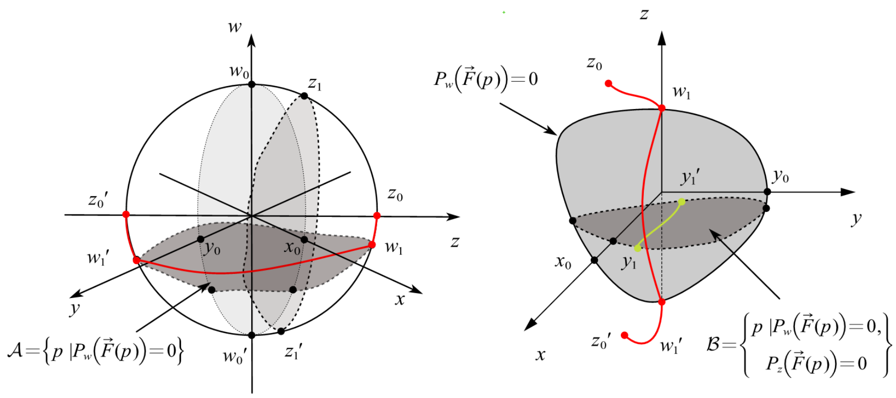

Figure 4: Compress the other three axes in a two dimension surface

the situations in the remark of **Theorem 6.2** . Because the points w_1 and w_1' always point inward, it can only have different signs.

Therefore, on this surface, through the outermost points w_1 and w_1' (the rightmost and leftmost points along the z-axis of the set $\mathcal{A}$), a path $\mathcal{L}$ can be constructed as $z_0 \to w_1 \to w_1' \to z_0'$. The proofs for other axes are the same.

□

Corollary 7.2. *In four dimensions,there at least one exist intersection point of* $P_i\left(\vec{F}(p)\right) = 0$ *for each* $i = x, y, z, w$.

Proof. Once the set $\mathcal{A} = \{p \mid P_w\left(\vec{F}(p)\right) = 0\}$ is obtained,the homotopy class of the path $z_0 \to w_1 \to w_1' \to z_0'$ generates a surface $\mathcal{B} = \{p \mid P_z\left(\vec{F}(p)\right) = 0\}$, which is precisely the surface directly generated by the path $z_0 \to z_0'$.

By "uncompressing" the x-, y-, and z-axes, the surface can be visualized as a volume. Hence, the set $\mathcal{A}$ forms a three-dimensional volume that covers the origin shows by Figure 4 (left) .

With a similar approach, a path can be construct through in terms of y-axis, intersect the the surface $\{p|P_z\left(\vec{F}(p)\right) = 0\}$ in a curve from $y_1 \to y_1'$, as the figure depicts.Finally, using the same approach again, a fixed point lie on this curve can be find. □

8 Experiment

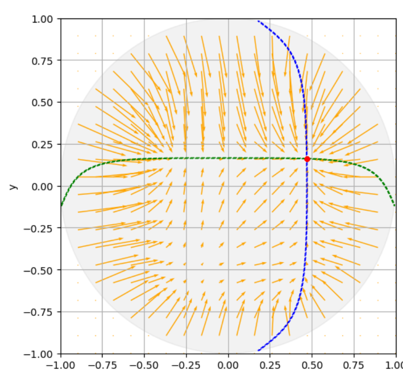

Figure 5: sinusoidal field $U, V =< sin(5x) + 0.4(x - 0.3), cos(5y) - 0.4(y + 0.2) >$

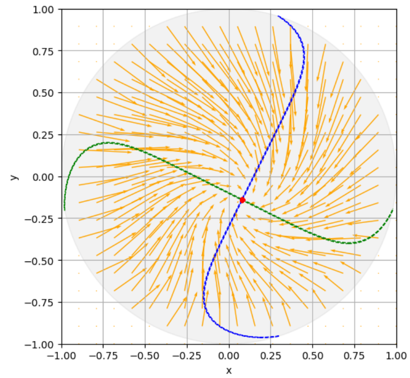

Figure 6: Rotation field add shrink field

References

[1] R. G. Bartle and D. R. Sherbert. *Introduction to Real Analysis*. 4th. John Wiley & Sons, 2011.

[2] L. E. J. Brouwer. "Die fixed-points der Funktionaloperationen". In: *Proceedings of the Amsterdam Academy of Sciences* 14 (1911), pp. 39–49.