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EDGE-TRANSITIVE MAPS

Doctoral thesis

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POVEZAVNO TRANZITIVNI ZEMLJEVIDI

Doktorska disertacija

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Abstract

In the thesis edge-transitive maps are studied through the theory of F-maps and from a computational point of view.

In the first part of the thesis the theory of F-maps is introduced. F-maps are a generalization of rooted counterparts of several objects in algebraic combinatorics (reflexible maps, orientably regular maps, holey maps, hypermaps, abstract polytopes, ...). The theory represents an unified approach to all these mathematical objects. It includes a study and analysis of quotients of F-maps and of the underlying algebraic lattice structure with emphasis on the operation parallel product (the join operation in the lattice). Projections and lifts of F-map automorphisms on F-maps in the lattice are studied. The main result is the decomposition theorem, which characterizes parallel-product decomposability of an F-map through its monodromy group. Regular F-maps are decomposable if and only if their monodromy groups have at least two minimal normal subgroups. The consequence of the decomposition theorem for regular F-maps is the algorithm for calculation of small quotients of finitely presented groups, which is described in the second part of the thesis. The last part is a study of edge-transitive maps using the theory and the algorithm from the first two parts. Degenerate cases of reflexible maps (i.e. edge-transitive maps with the highest symmetry) are classified and their decomposability characterized. All non-degenerate reflexible parallel-product indecomposable maps up to 100 edges are listed. Types of edge-transitive maps are analyzed through theory of G-orbit transitive F-maps. It is shown how edge-transitive maps can be viewed as certain kinds of regular F-maps. At the end, tables with complete listings of non-degenerate edge-transitive maps on surfaces of small genera are presented. The computational results of the thesis are also presented on the web page [53].

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Key words: maps on surfaces, edge-transitive maps, F-maps, parallel product, quotients of finitely presented groups.

Povzetek

V disertaciji se ukvarjamo s študijem povezavno tranzitivnih zemljevidov z uporabo teorije F-zemljevidov, prav tako pa povezavno tranzitivne zemljevide preučujemo iz računskega (računalniškega) vidika.

V prvem delu je predstavljena teorija F-zemljevidov. Ti so posplošitev zakoreninjenih inačic različnih objektov iz algebraične kombinatorike, kot so regularni zemljevidi, splošni zemljevidi, orientabilno regularni zemljevidi, hiper zemljevidi, abstraktni politopi, itd. Teorija predstavlja poenoten pristop k obravnavi vseh omenjenih matematičnih objektov. Vključuje analizo in preučevanje kvocientov F-zemljevidov, algebrajske mreže F-zemljevidov ter obnašanje avtomorfizmov F-zemljevidov pri kvocientih ter pri mrežnih operacijah. Poseben poudarek je na preučevanju operacije paralelnega produkta, ki je dejansko operacija kupa ($\vee$) v mreži. Glavni rezultat disertacije je izrek o razcepu, ki karakterizira razcepnost F-zemljevidov na paralelni produkt glede na monodromijsko grupo. Za regularne F-zemljevide velja, da so razcepni natanko tedaj, ko monodromijska grupa vsebuje vsaj dve različni minimalni edinki. Posledica tega je algoritem za izračun majhnih kvocientov končno prezentiranih grup, ki je opisan v drugem delu disertacije.

Zadnji del disertacije se ukvarja s študijem povezavno tranzitivnih zemljevidov preko teorije F-zemljevidov in omenjenega algoritma. Najprej klasificiramo vse degenerirane regularne zemljevide (t.j. povezavno tranzitivne zemljevide z najvišjo stopnjo simetrije) in karakteriziramo njihovo razcepnost. Podamo seznam nedegeneriranih regularnih zemljevidov do 100 povezav. Tipe povezavno tranzitivnih zemljevidov analiziramo s pomočjo teorije G-orbitno tranzitivnih F-zemljevidov. Prikazano je, kako lahko povezavno tranzitivne zemljevide obravnavamo kot ustrezne regularne F-zemljevide. Na koncu so podane tabele s popolno klasifikacijo vseh nedegeneriranih zemljevidov na nekaterih kompaktnih sklenjenih ploskvah majhnega roda. Računski rezultati disertacije so predstavljeni tudi na spletni strani [53].

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Ključne besede: zemljevidi na ploskvah, povezavno tranzitivni zemljevidi, F-zemljevidi, paralelni produkt, kvocienti končno prezentiranih grup.

Contents

Chapter 1

Introduction

1.1 Introduction

The history of edge-transitive maps, which also include regular (reflexible) and orientably regular maps, starts with ancient Greeks (the Platonic solids, also some of the Archimedian solids). In the 17th century, Kepler [29] worked on stellated polyhedra where some non-planar regular maps occurred. In the 19th century, Heffter [25] considered orientably regular embeddings of complete graphs, while Klein [30] and Dyck [16] constructed some cubic regular maps on the surface of orientable genus 3, in the context of automorphic functions. In the beginning of the 20th century, regular maps were first used as geometrical representations of groups (Burnside [11]). More systematic study of regular maps continued with Brahana [6] and Coxeter and Moser [15], where regular maps were treated as geometrical, combinatorial and group theoretical objects. The basis for the modern treatment of general maps was set by Jones and Singerman [26] for orientable surfaces and by Bryant and Singerman [9] for non-orientable ones. The classic reference for maps became the book by Gross and Tucker [21]. In the last decade, research on maps of high symmetry has mainly focused on regular (and orientably regular) maps and Cayley maps. The recent paper by Richter, Širáň, Jajcay, Tucker and Watkins [42] provides a nice survey for Cayley maps.

For edge-transitive maps, Graver and Watkins [20] give the fundamental classification into 14 types according to the possession of certain types of automorphisms. The existence of all the types on infinitely many orientable surfaces was shown in the important work by Širáň, Tucker and Watkins [44].

The central problems of edge-transitive maps are their construction and classification. The most common constructions of edge-transitive maps arise either from constructions of finite groups admitting one of 14 types of presentations [44] or as covers of smaller maps. Three natural approaches are used in the classification of edge-transitive maps, namely by the number of edges [53], by the underlying surface [12] and by the underlying graph [51].

It is known that all compact closed surfaces, other than the sphere, torus, projective plane and Klein bottle, necessarily contain a finite number of edge-transitive maps, where the upper bound of a size of a map's automorphism group depends on the surface and it is easily obtained from Euler's formula or the Riemann-Hurwitz equation. All edge-transitive maps on the torus were classified by Širáň, Tucker and Watkins [44], the classification for the sphere was done by Grünbaum and Shephard [22], while a part of the classification for the Klein bottle was done by Potočnik and Wilson [41].

Before the age of fast computers, many authors (Brahana [6], Coxeter and Moser [15], Sherk [43], Garbe [19], Bergau and Garbe [2]) worked on the classification of regular and orientably regular maps and managed to classify all regular and orientably regular maps on surfaces of orientable genus up to 7 and non-orientable genus up to 8. In the 1970s, Wilson in his Ph.D. thesis [49] calculated most reflexible and chiral maps up to 100 edges [53] using a computer and running his *Riemann surface algorithm* [52]. The recent breakthrough in this field is due to Conder and Dobcsányi [13], who calculated all orientably regular maps on surfaces from genera 2 up to 15 and all the non-orientable reflexible maps on surfaces from non-orientable genera 3 up to 30 (Conder&Dobcsányi's census [12]). In the time this thesis was nearing to its final form, a new version of extremely efficient `LowIndexNormalSubgroups` algorithm in MAGMA[10] was implemented. Using this algorithm Conder [14] extended the above mentioned result to orientable genera up to 100 and non-orientable genera up to 200.

Since Wilson's and Conder&Dobcsányi's censuses present different information, a census was needed that would contain the information from both of them. While Wilson tries to calculate all reflexible and chiral maps up to some size (currently 100 edges), Conder and Dobcsányi calculated only the maps for the above mentioned ranges of genera. Some maps in Wilson's census may exceed the genera in Conder&Dobcsányi's census, while in the latter there are also much larger maps (the largest on 546 edges). During the process of calculation Conder and Dobcsányi actually calculated all reflexible and chiral maps up to 140 edges. For higher number of edges they calculated only some special maps [13]. The above mentioned most recent result by Conder [14] uses a similar approach, but on much higher genera. Since chiral maps are not closed under the Petrie dual, the natural extension of the censuses, as observed by T. Pisanski, seemed to be edge-transitive maps.

One of the results of this thesis is the calculation of all edge-transitive maps up to at least 100 edges. A complete census of reflexible maps up to 500 edges and chiral maps up to 1000 edges is presented.

In this thesis the author presents alternative constructions of edge-transitive maps which are used to extend the known censuses. Since the censuses are large, the author sought a shorter description of maps in terms of some kind of "primitive" maps from which all other maps can be obtained using some set of operations. The algorithms for performing the operations needed to be of a relatively low time complexity so computations of "non-primitive" maps remain simple. It turned out that the appropriate operation is the parallel product introduced by Wilson [48].

The thesis is organized as follows. In Chapter 2, F-maps and the theory on F-maps is presented. F-maps are rooted generalizations of several objects that appear in algebraic combinatorics. Quotients of F-maps are characterized. Projections of automorphisms along quotients are studied. The parallel product is introduced and its properties studied. The parallel product is actually the operation join on the lattice of isomorphism classes of F-maps. The focus of the study is set on characterizations of decompositions of F-maps with prescribed symmetry. The main result of the theory is the decomposition theorem, Theorem 17. Isomorphism classes of finite regular F-maps form a sublattice of the lattice of all F-maps and several objects in algebraic combinatorics can be modeled by using them: reflexible maps, orientably regular maps, regular hypermaps, etc. According to the decomposition theorem, a finite regular F-map can be factored into a product of regular F-maps if and only if the F-map's monodromy group (which is a permutation group) has more than one minimal normal subgroup. At the end of Chapter 1 an application of the theory of F-maps on reflexible maps is presented.

Chapter 3 deals with the main consequence of the decomposition theorem. This is an algorithm for determining small quotients of finitely presented groups using a database of small groups. In this chapter the algorithm is described, analyzed and the implementation issues are discussed.

Chapter 4 deals with finite edge-transitive maps, the main topic of the thesis. The most important type of edge-transitive maps is the most symmetric one, namely reflexible maps. All degenerate reflexible maps are classified and for each one it is shown, whether it is decomposable. Some general theory on G-orbit transitive F-maps is introduced and using the theory, the analysis of the 14 classes of edge-transitive maps is studied. The theory of finite regular F-maps is applied on edge-transitive maps. It is shown how triality can help us in analysis of maps of certain types. The problem of a map being of the exact type is studied through F-map operations on edge-transitive maps viewed as F_T-maps, where F_T is a finitely presented group depending on the type of a map.

By using the algorithm from Chapter 2 all edge-transitive maps on surfaces of small genera are calculated. Some of the results of the thesis are different censuses of edge-transitive maps which are also published on Internet [53].

Chapter 2

F-maps

2.1 Definitions

A right *action* of a group G on a set Z is an operation $\cdot : Z \times G \to Z$, for which $z \cdot 1 = z$ and $z \cdot (gh) = (z \cdot g) \cdot h$, for every $z \in Z$ and $g, h \in G$. We denote the action by a pair (Z, G). Denote by $\mathrm{Sym}_{\mathrm{R}}(Z)$ the symmetric group on the set Z, where the bijections (permutations) are composed from the left to the right and naturally act on Z from the right. For $g \in G$, a mapping $\pi_g : Z \to Z$, $\pi_g : x \mapsto x \cdot g$ is a bijection on Z and therefore an element in $\mathrm{Sym}_{\mathrm{R}}(Z)$. The mapping $\chi : G \to \mathrm{Sym}_{\mathrm{R}}(Z)$, $\chi : g \mapsto \pi_g$ is a group homomorphism and is called the *action homomorphism*, The image $\chi(G) \leq \mathrm{Sym}_{\mathrm{R}}(Z)$ is called the *image* of the action and $\ker \chi$ is called the *kernel* of the action. The *stabilizer* of an element $z \in Z$ is the group $G_z = \{g \in G \mid z \cdot g = z\}$. The kernel of the action is exactly the intersection of all the stabilizers. The action is *semi-regular* if all G_z are trivial and *faithful* if the kernel is trivial. Any two $z, z' \in Z$, for which there exists $g \in G$ such that $z \cdot g = z'$, are in the same *orbit*. Orbits form a partition of Z. An action is *transitive* if there is only one orbit. Denote by $\mathrm{Core}_G(K) = \bigcap_{g \in G} K^g$, the core of a subgroup K in G, which is the intersection of all the conjugates of K and is also the maximal normal subgroup in G contained in K. All stabilizers of a transitive action (Z, G) are conjugate and the kernel equals to $\mathrm{Core}_G(G_z)$, for any $z \in Z$. A transitive semi-regular action is *regular*.

In the thesis we will denote the index of a subgroup K of a group G by $(G : K)$.

A *system of blocks of imprimitivity* for an action (Z, G) is a partition $\mathcal{B} = \{B_i\}_{i=1}^{k}$ of Z, such that for any $B \in \mathcal{B}$ and any $g \in G$, either $Bg = B$ or $B \cap Bg =$

$\emptyset$. Therefore, G naturally acts on any block system.

An *action epimorphism* of two right actions (Z, G) and (W, H) is a pair (ϕ, ψ), where $\phi : Z \to W$ is an onto mapping, $\psi : G \to H$ is a group epimorphism and for every $z \in Z$ and $g \in G$ it follows $\phi(z \cdot g) = \phi(z) \cdot \psi(g)$. If both ϕ and ψ are one-to-one, then (ϕ, ψ) is an *action isomorphism* .

In a similar manner, but by interchanging the sides, a left action is defined and denoted by (G, Z). Here a permutation group $\mathrm{Sym}_{\mathrm{L}}(Z)$ is used in place of $\mathrm{Sym}_{\mathrm{R}}(Z)$. In $\mathrm{Sym}_{\mathrm{L}}(Z)$ the bijections are composed as functions and $\mathrm{Sym}_{\mathrm{L}}(Z)$ naturally acts on Z from the left. The left action will be denoted by a pair (G, Z). In a case of a group G acting on itself, the notation (G, G) is confusing, therefore the nature of the action (left or right) should be explained in the context. Mostly, right actions will be used in the thesis. A use of left actions will be denoted explicitly.

A rooted transitive action (RTA) is a triple $(Z, G, \underline{\mathrm{id}})$, where (Z, G) is a transitive action and $\underline{\mathrm{id}} \in Z$ is the distinguished element called the *root*. An *RTA morphism* is an action epimorphism which maps a root to a root.

Let $F = \langle a_1, ..., a_k \mid R_1 = \ldots = R_n = 1\rangle$ be a finitely presented group with generators $\{a_i\}_{i=1}^{k}$ and relations $\{R_j\}_{j=1}^{n}$. A F*-group* is a triple (F, f, G), where $f : F \to G$ is an epimorphism. A F*-group morphism* of two F-groups $A_i = (F, f_i, G_i)$, $i = 1, 2$, is a group epimorphism $\psi : G_1 \to G_2$, such that $\psi \circ f_1 = f_2$. If ψ is an isomorphism, we denote this by $A_1 \simeq A_2$. An F-group can be considered as an abstract group with labeled generators respecting the relations in F.

A (finite) F*-map* is a 5-tuple $M = (F, f, G, Z, \underline{\mathrm{id}}) = (F, f_M, G_M, Z_M, \underline{\mathrm{id}}_M)$, where Z is a (finite) set of *flags*, $(Z, F, \underline{\mathrm{id}})$ is an RTA, $(Z, G, \underline{\mathrm{id}})$ is a faithful RTA, $(\mathrm{Id}, f) : (Z, F, \underline{\mathrm{id}}) \to (Z, G, \underline{\mathrm{id}})$ is an RTA morphism (here Id denotes the identity mapping) and (F, f, G) is an F-group. Let χ_M denote the action homomorphism $\chi_M : G_M \to \mathrm{Sym}_{\mathrm{R}}(Z_M)$. Then $\mathrm{Mon}(M) = (F, \chi_M \circ f_M, \chi_M(G_M))$ is an F-group called the *monodromy group*. $\mathrm{Mon}(M)$ is considered as a permutation group with labeled generators and as such a particularly convenient representation of an F-map when doing computer calculations. The *order* (or the *size*) of the F-map M is equal to $|Z|$ and denoted by $|M|$. Define $\mathrm{S}_F(M) = F_{\underline{\mathrm{id}}} = f^{-1}(G_{\underline{\mathrm{id}}})$, the stabilizer of $\underline{\mathrm{id}}$ in F. In the thesis we will work with finite F-maps only (the set Z will be finite). Nevertheless, in the claims where finiteness is crucial, we may emphasize it explicitly.

Examples of F-maps are maps on surfaces. At this point the reader is directed to the beginning of Section 2.4 for examples.

Let M and N be F-maps. An *F-map morphism* is a pair (ϕ, ψ), that is an RTA morphism $(\phi, \psi) : (Z_M, G_M, \underline{\mathrm{id}}_M) \to (Z_N, G_N, \underline{\mathrm{id}}_N)$ and $\psi : (F, f_M, G_M) \to (F, f_N, G_N)$ is an F-group morphism. The existence of such a morphism is denoted by $N \leq M$, and if it exists, it is unique (for $z \in Z_M$, there exists $w \in F$, such that $z = \underline{\mathrm{id}}_M \cdot f_M(w) = \underline{\mathrm{id}}_M \cdot w$ and therefore $\phi(z) = \phi(\underline{\mathrm{id}}_M) \cdot w = \underline{\mathrm{id}}_N \cdot w$). An F-map morphism is called *strict* if and only if $\ker \psi$ is non-trivial. If ϕ, ψ are one-to-one, we have an *F-map isomorphism* and denote it by $M \simeq N$. Note that $\mathrm{Mon}(M) \simeq \mathrm{Mon}(N)$ if and only $(F, f_M, Z_M) \simeq (F, f_N, Z_N)$. While $M \simeq N$ implies $\mathrm{Mon}(M) \simeq \mathrm{Mon}(N)$, the converse is far from being true.

An *automorphism* of an F-map M is an action isomorphism $(\phi, \mathrm{Id}) : (Z_M, G_M) \to (Z_M, G_M)$, where Id denotes the identity mapping. Note that contrary to an F-map morphism, here the condition $\phi(\underline{\mathrm{id}}_M) = \underline{\mathrm{id}}_N$ is omitted and therefore in general, an F-map automorphism is not an F-map morphism in the categorical sense. The group of all automorphisms is denoted by $\mathrm{Aut}(M)$. Since an automorphism is already determined by an image of a single flag (for the root: if $x = \underline{\mathrm{id}} \cdot w$, $w \in F$, then $\phi(x) = \phi(\underline{\mathrm{id}}) \cdot w$), the left action of $\mathrm{Aut}(M)$ on Z is semi-regular. The symbol α_w will denote an automorphism in $\mathrm{Aut}(M)$ that takes $\underline{\mathrm{id}}_M$ to $\underline{\mathrm{id}}_M \cdot w$, where $w \in F$. If $p \in \mathrm{Aut}(M)$ is such an automorphism, we will denote this by $p \equiv \alpha_w \in \mathrm{Aut}(M)$ and say that M *contains* α_w.

The *flag graph* of an F-map $M = (F, f, G, Z, \underline{\mathrm{id}})$ is the directed multi graph with labeled edges, where the set of the vertices is Z and for each $z \in Z$ and each $a \in \{a_i^{\pm 1}\}_{i=1}^k$ there is a directed edge from z to $z \cdot f(a) = z \cdot a$ with the label a. Note that in the situation where $z \cdot a = z$, we have a loop if the order of a in F is greater than 2, or a semi-edge otherwise.

Denote by F^+ a subgroup of even length words in F. Depending on the relations in F, F^+ can be either equal to F or be a subgroup of index 2. In the latter case, we can introduce the concept of orientability. An F-map M is *orientable* if and only if F^+ has exactly two orbits on Z_M.

If a root flag $\underline{\mathrm{id}}$ of an F-map M is changed to the flag $\underline{\mathrm{id}} \cdot w$, $w \in F$, a *re-rooted* F-map $R_w(M)$ is obtained. Note that $\mathrm{S}_F(R_w(M)) = w^{-1}\mathrm{S}_F(M)w$.

Let $d \in \mathrm{Aut}(F)$ and $M = (F, f, G, Z, \underline{\mathrm{id}})$. Then d induces an F-map operation $O_d(M) := (F, f \circ d, G, Z, \underline{\mathrm{id}})$. Also $\mathrm{S}_F(O_d(M)) = d^{-1}(\mathrm{S}_F(M))$. If d is an inner automorphism, say a conjugation by w, then it is quite easy to see that $O_d(M) \simeq R_w(M)$.

For a subgroup $G \leq F$ and an F-map M, the orbits of G acting on Z_M are called *G-orbits* and are blocks of imprimitivity for the left action of the automorphism group.

Let $p : M \to N$ be a morphism of F-maps and $f \in \mathrm{Aut}(M)$. If there exists $f' \in \mathrm{Aut}(N)$, such that $p \circ f = f' \circ p$, then we say that f *projects* (along p). On the other hand, if there is $f' \in \mathrm{Aut}(N)$ and there exists $f \in \mathrm{Aut}(M)$, such that $p \circ f = f' \circ p$, we say that f *lifts* (with p). Note that for $w \in F$, if $\alpha_w \in \mathrm{Aut}(M)$ projects, it projects to $\alpha_w \in \mathrm{Aut}(N)$.

A *partially ordered set* or a *poset* $(P, \leq)$ is a set P with a partial ordering relation $\leq$. Let f be a bijective mapping $f : (P, \leq) \to (P', \leq')$. If for any $a, b \in P$, $a \leq b$, it follows $f(a) \leq' f(b)$, f is a *poset isomorphism*; if it follows $f(b) \leq' f(a)$, it is a *poset anti-isomorphism*. For a finite subset $S \subseteq P$, any $u \in P$, such that $x \leq u$ for all $x \in S$, is called an *upper bound* of S. The exact upper bound of S, denoted by $\sup(S)$ and called a *supremum*, is any $v \in P$, such that v is an upper bound for S and for any upper bound u of S, it follows $v \leq u$. Similarly, lower bounds and an exact lower bound (infimum – $\inf(S)$) are defined. Instead of $\sup(\{x_1, \dots, x_n\})$ we often use the notation $\sup(x_1, \dots, x_n)$ and similarly for infimums.

For elements $a, b \in P$, $a \leq b$, the set $[a, b] := \{x \in P, a \leq x \leq b\}$ is called an *interval*.

A *lattice* $(L, \vee, \wedge)$ is an algebraic structure on a set L with operations $\vee$ and $\wedge$ admitting the following laws:

1. (Associativity): $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
2. (Commutativity): $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.
3. (Idempotence): $x \vee x = x$ and $x \wedge x = x$.
4. (Absorption): $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.

The operation $\vee$ is called the *join* and the operation $\wedge$ is called the *meet*. A lattice $(L, \vee, \wedge)$ is a poset for the relation $x \leq y \overset{\text{def}}{\Leftrightarrow} x \vee y = y$, or equivalently $x \leq y \overset{\text{def}}{\Leftrightarrow} x \wedge y = x$. On the other hand, if a poset $(P, \leq)$ has defined $\sup(a, b)$ and $\inf(a, b)$ for any two elements $a, b \in P$, then $(P, \sup, \inf)$ is a lattice.

A *lattice isomorphism* (anti-isomorphism) is a bijection preserving (interchanging) the operations $\vee$ and $\wedge$. A poset isomorphism (or anti-isomorphism) of a lattice L to a poset P implies that $(P, \sup, \inf)$ (or $(P, \inf, \sup)$) is a lattice.

For a lattice $(L, \vee, \wedge)$, a sublattice is any subset of L closed for the operations $\vee$ and $\wedge$.

A lattice $(L, \vee, \wedge)$ is *modular* if for every $x, y, z \in L$ it satisfies the identity

$$x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z). \tag{2.1}$$

Note that from the lattice laws it follows that this is true if and only if the identity with $\wedge$ and $\vee$ interchanged holds.

2.2 Quotients of F-maps

This section extends and generalizes the results by Wilson [48] and by Breda and Nedela [8]. Wilson was the first who introduced the parallel product, while Breda and Nedela discovered the connection between regular hypermaps and the lattice of normal subgroups of group $\mathcal{H} = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1\rangle$. Their idea is used to show the connection between F-maps and the lattice of (finite index) subgroups of F which is then used to prove some interesting results.

If (Z, G) is a transitive action with the kernel H, then the induced action $(Z, G/H)$ is faithful (note that the action is defined by $z \cdot Hg = z \cdot g$, for any $z \in Z$ and $g \in G$). For example, if $K \leq G$ are groups, then $(G/K, G, K)$ is an RTA and if $H = \mathrm{Core}_G(K)$, then $(G/K, G/H, K)$ is a faithful RTA explicitly defined by $Kw \cdot Hv = Kwv$, for $w, v \in G$.

For $K \leq F$, define $\mathrm{M}_F(K) = (F, q, F/\mathrm{Core}_F(K), F/K, K)$, where $q : F \to F/\mathrm{Core}_F(K)$ is the natural epimorphism. Obviously, $\mathrm{M}_F(K)$ is an F-map. Note that for $K \leq F$, $\mathrm{S}_F(\mathrm{M}_F(K)) = K$.

Proposition 1. *Let M be an F-map. Then $M \simeq \mathrm{M}_F(\mathrm{S}_F(M))$.*

Proof. Let $M = (F, f, G, Z, \underline{\mathrm{id}})$, $K = \mathrm{S}_F(M)$, $H = \mathrm{Core}_F(K)$, $q : F \to F/H$ be the natural epimorphism and $\mathrm{M}_F(\mathrm{S}_F(M)) = (F, q, F/H, F/K, K)$. Let $w \in F$, $z \in Z$, such that $z = \underline{\mathrm{id}} \cdot w$ and define $\phi(z) = Kw$. Since $\underline{\mathrm{id}} \cdot w = \underline{\mathrm{id}} \cdot v$ if and only if $wv^{-1} \in K$, ϕ is well defined and one-to-one. Obviously, ϕ is onto and $\phi(\underline{\mathrm{id}}) = K$. As (Z, G) is faithful, $\ker f = H$. Since f is an epimorphism with the kernel H and the image G, there exists an isomorphism $\psi : G \to F/H$, such that $\psi \circ f = q$.

Let $z \in Z$, $g \in G$ and $w, v \in F$, any such that $f(w) = g$ and $\underline{\mathrm{id}} \cdot v = z$. Then $\phi(z \cdot g) = \phi(z \cdot f(w)) = \phi((\underline{\mathrm{id}} \cdot v) \cdot w) = \phi(\underline{\mathrm{id}} \cdot (vw)) = Kvw$, while $\phi(z) \cdot \psi(g) = \phi(z) \cdot \psi(f(w)) = Kv \cdot q(w) = Kv \cdot Hw = Kvw$. Hence (ϕ, ψ) is the isomorphism. $\square$

Proposition 2. *Let $K_1, K_2 \leq F$ be subgroups. Then $K_1 \leq K_2$ if and only if there exists an F-map morphism $(\phi, \psi) : \mathrm{M}_F(K_1) \to \mathrm{M}_F(K_2)$.*

Proof. For $i = 1, 2$, let $H_i = \mathrm{Core}_F(K_i)$, $q_i : F \to F/H_i$ be the natural epimorphisms and $\mathrm{M}_F(K_i) = (F, q_i, F/H_i, F/K_i, K_i)$.

If $K_1 \le K_2$, then $H_1 \le H_2$ and the mappings $\psi : F/H_1 \to F/H_2$, $\psi : H_1w \mapsto H_2w$ and $\phi : F/K_1 \to F/K_2$, $\phi : K_1w \mapsto K_2w$, for any $w \in F$, are well defined. Also, ψ is an epimorphism and $\psi \circ q_1 = q_2$. For every $w, v \in F$, it follows $\phi(K_1w \cdot H_1v) = \phi(K_1wv) = K_2wv$ and $\phi(K_1w) \cdot \psi(H_1v) = K_2w \cdot H_2v = K_2wv$. Since $\phi(K_1) = K_2$ and ϕ and ψ are onto, $(\phi, \psi) : \mathrm{M}_F(K_1) \to \mathrm{M}_F(K_2)$ is an F-map morphism.

On the other hand, let $(\phi, \psi) : \mathrm{M}_F(K_1) \to \mathrm{M}_F(K_2)$ be an F-map morphism. Then $q_2 = \psi \circ q_1$ and $\phi(K_1) = K_2$. Let $x \in K_1$. Then $q_1(x) \in K_1/H_1 = (F/H_1)_{K_1}$. Since (ϕ, ψ) is an F-map morphism, it is true that $\psi((F/H_1)_{K_1}) \le (F/H_2)_{K_2} = K_2/H_2$. Therefore $(\psi \circ q_1)(x) = q_2(x) \in K_2/H_2$, implying that $x \in K_2$. Hence, $K_1 \le K_2$. $\square$

From the last two propositions the next corollary immediately follows.

Corollary 3. *Let M and N be F-maps. Then there exists an F-map morphism $(\phi, \psi) : M \to N$ if and only if $\mathrm{S}_F(N) \le \mathrm{S}_F(M)$. Therefore, $M \simeq N$ if and only if $\mathrm{S}_F(M) = \mathrm{S}_F(N)$.* $\square$

Recall the well known fourth isomorphism theorem for groups (the correspondence theorem).

Theorem 4. *Let G, G' be groups and $f : G \to G'$ epimorphism. Let $\mathcal{A} = \{K : \ker f \le K \le G\}$ and $\mathcal{B} = \{K' : K' \le G'\}$. Then a mapping $F : \mathcal{A} \to \mathcal{B}$ defined by $F : K \mapsto f(K)$ is a bijection. Under this bijection normal subgroups correspond to normal subgroups. For any two groups $K, H \in \mathcal{A}$, it follows $F(K \cap H) = F(K) \cap F(H)$, $F(\langle K, H \rangle) = \langle F(K), F(H) \rangle$ and if $K \le H$, then $F(K) \le F(H)$.* $\square$

Note that the sets of groups $\mathcal{A}$ and $\mathcal{B}$ are actually lattice intervals in the lattices of subgroups of G and G', respectively. The theorem is sometimes called the lattice theorem for groups, since it basically says that f induces a lattice isomorphism between the two lattice intervals with a special property of mapping normal subgroups to normal subgroups.

For an F-map $M = (F, f, G, Z, \underline{\mathrm{id}})$ and a subgroup $K \le G$, where $G_{\underline{\mathrm{id}}} \le K$, define $M/K = (F, q \circ f, G/H, G/K, K)$, where $H = \mathrm{Core}_G(K)$ and $q : G \to G/H$ is the natural epimorphism. The right action of G/H on G/K is faithful and

M/K is an F-map called a K-*quotient* of the F-map M. A K-quotient of M is *strict* if $\mathrm{Core}_G(K)$ is not trivial.

Theorem 5. *Let M and N be F-maps, such that there exists an F-map morphism $(\phi, \psi) : M \to N$. Let $K = \psi^{-1}((G_N)_{\underline{\mathrm{id}}_N}) \leq G_M$. Then $(G_M)_{\underline{\mathrm{id}}_M} \leq K$ and $M/K \simeq N$.*

For any two $N, N' \leq M$, where $N \simeq M/K$, $N' \simeq M/K'$ for some $K, K' \leq G_M$, it follows $N \leq N'$ if and only if $K' \leq K$. Also, $N \simeq N'$ if and only if $K = K'$.

Proof. Since $(\mathrm{Id}, f_M) : (Z_M, F, \underline{\mathrm{id}}_M) \to (Z_M, G_M, \underline{\mathrm{id}}_M)$ is an RTA morphism and Id is a bijection, it follows $f_M(F_{\underline{\mathrm{id}}}) = (G_M)_{\underline{\mathrm{id}}}$ and $f_M^{-1}((G_M)_{\underline{\mathrm{id}}_M}) = F_{\underline{\mathrm{id}}}$. As $N \leq M$, it follows $X = \mathrm{S}_F(N) \geq \mathrm{S}_F(M) = F_{\underline{\mathrm{id}}_M}$, by Corollary 3. Let $K = f_M(X)$. Since f_M is an epimorphism and $\ker f_M \leq F_{\underline{\mathrm{id}}_M} \leq X$, it is true that $f_M^{-1}(K) = X$ and $(G_M)_{\underline{\mathrm{id}}_M} \leq K$, by Theorem 4. But for $M/K = (F, q \circ f_M, G_M/H, G_M/K, K)$, where $H = \mathrm{Core}_{G_M}(K)$ and $q : G_M \to G_M/H$ is the natural epimorphism, it follows $(G_M/H)_K = K/H$, and thus $\mathrm{S}_F(M/K) = (q \circ f_M)^{-1}(K/H) = f_M^{-1}(K) = X$. By Corollary 3, $M/K \simeq N$. As f_M is an epimorphism, $K = f_M(X) = f_M(f_N^{-1}((G_N)_{\underline{\mathrm{id}}_N})) = f_M(f_M^{-1}(\psi^{-1}((G_N)_{\underline{\mathrm{id}}_N}))) = \psi^{-1}((G_N)_{\underline{\mathrm{id}}_N})$.

Let $N, N' \leq M$. Then $N \leq N'$ if and only if $\mathrm{S}_F(N) \geq \mathrm{S}_F(N')$, if and only if $K = f_M(\mathrm{S}_F(N)) \geq f_M(\mathrm{S}_F(N')) = K'$, by Theorem 4 and Corollary 3. Therefore, $K = K'$ if and only if $\mathrm{S}_F(N) = \mathrm{S}_F(N')$ if and only if $N \simeq N'$. □

The role of Theorem 5 on F-maps is similar to the role of the first isomorphism theorem on groups. From a computational point of view, it enables us to calculate all quotients of an F-map from the monodromy group.

Recall that the *normalizer* of $K \leq F$ is $N_F(K) = \{w \in F \mid w^{-1}Kw = K\} \leq F$.

Theorem 6. *Let $M = (F, f, G, Z, \underline{\mathrm{id}})$ and $\mathrm{S}_F(M) = K$. For $w \in F$, $\alpha_w \in \mathrm{Aut}(M)$ if and only if $w \in N_F(K)$. Furthermore, $\mathrm{Aut}(M) \simeq N_F(K)/K$.*

Proof. Let $H = \mathrm{Core}_F(K)$ and $L = \mathrm{M}_F(\mathrm{S}_F(M)) = (F, q, F/H, F/K, K)$. Obviously, $\alpha_w \in \mathrm{Aut}(L)$ if and only if $\alpha_w \in \mathrm{Aut}(M)$.

Let $\alpha_w \in \mathrm{Aut}(L)$ and $x \in w^{-1}Kw$. Then $x = w^{-1}kw$, for some $k \in K$, and $\alpha_w(Kw^{-1}) = \alpha_w(K) \cdot w^{-1} = Kww^{-1} = K$. Also $Kx = \alpha_w(Kw^{-1}) \cdot x = \alpha_w(Kw^{-1}wkw^{-1}) = \alpha(Kw^{-1}) = K$ and $x \in K$. Hence, $wKw^{-1} = K$ and $w \in N_F(K)$.

If $w \in N_F(K)$, then $wKw^{-1} = K$ and $wK = Kw$. Let $\phi : F/K \to F/K$ be defined by $\phi : Kx \mapsto wKx = Kwx$, for any $x \in F$. Obviously, ϕ is well defined and a bijection. Then for any $v \in F$, $\phi(Kx) \cdot v = Kwxv$ and $\phi(Kx \cdot v) = \phi(Kxv) = Kwxv$. Since $\phi(K) = Kw$, $(\phi, \mathrm{Id}) \equiv \alpha_w \in \mathrm{Aut}(L)$.

Define a mapping $\Phi : \mathrm{Aut}(L) \to N_F(K)/K$, where $\Phi : \alpha_w \mapsto Kw$. Since α_w and α_v represent the same automorphism in $\mathrm{Aut}(L)$ if and only if $wv^{-1} \in K$, Φ is well defined and one-to-one. Also, $\Phi(\alpha_{wv}) = Kwv = KwKv = \Phi(\alpha_w)\Phi(\alpha_v)$, since $K \lhd N_F(K)$. According to the discussion above, Φ is onto. □

Note that Theorem 6 appears in similar forms in several papers which deal with different types of F-maps (Cayley maps [42], hypermaps [8], abstract polytopes [24]).

Two corollaries immediately follow.

Corollary 7. *An F-map M is regular if and only if $\mathrm{S}_F(M) \lhd F$. M is regular if and only if $\mathrm{Aut}(M)$ and $\mathrm{Mon}(M)$ are isomorphic as abstract groups.* □

Proof. The first part follows directly from Theorem 6. If M is regular, let $H = \mathrm{S}_F(M) \lhd F$ and $q : F \to F/H$ the natural epimorphism. Then $\mathrm{M}_F(\mathrm{S}_F(M)) = (F, q, F/H, F/H, K)$, $\mathrm{Mon}(M) \simeq (F, q, F/H)$ (as F-groups) and $\mathrm{Aut}(M) \simeq F/H$, by Theorem 6. If M is not regular, then $|\mathrm{Aut}(M)| < |Z_M| \leq |\mathrm{Mon}(M)|$, since $\mathrm{Mon}(M)$ is transitive. □

Note that the last sentence is not true if F-map M is not finite (see [28]).

Corollary 8. *Let M, N be F-maps and $p : M \to N$ be an F-map morphism. Then $\mathrm{Aut}(M)$ projects if and only if $N_F(\mathrm{S}_F(M)) \leq N_F(\mathrm{S}_F(N))$.*

Proof. Note that $\alpha_w \in \mathrm{Aut}(M)$ projects if and only if it projects to $\alpha_w \in \mathrm{Aut}(N)$. □

Let $M = (F, f, G, Z, \underline{\mathrm{id}})$ and let $H \lhd G$. Consider the RTA morphism $(\mathrm{Id}, f) : (Z, F, \underline{\mathrm{id}}) \to (Z, G, \underline{\mathrm{id}})$ of M. By Theorem 4, $\mathrm{S}_F(M/G_{\underline{\mathrm{id}}}H) = f^{-1}(G_{\underline{\mathrm{id}}}H) = f^{-1}(G_{\underline{\mathrm{id}}}) \cdot f^{-1}(H) = \mathrm{S}_F(M)f^{-1}(H)$. Since $f^{-1}(H) \lhd F$, it follows $N_F(\mathrm{S}_F(M)) \leq N_F(\mathrm{S}_F(M/G_{\underline{\mathrm{id}}}H))$. Therefore $\mathrm{Aut}(M)$ projects, by Corollary 8. A K-quotient, where $K = G_{\underline{\mathrm{id}}}H$, $H \lhd G$ is called a *normal quotient* and is denoted by $M \triangle H$. The following proposition summarizes the discussion.

Proposition 9. *Let M be an F-map and $p : M \to M \triangle H$ be the F-map morphism onto the normal quotient. Then $\mathrm{Aut}(M)$ projects along p.* □

A normal quotient can be obtained also in a different way.

Proposition 10. *Let $M = (F, f, G, Z, \underline{\mathrm{id}})$ and $H \lhd G$ be a normal subgroup. Let $N = \mathrm{Core}_F(G_{\underline{\mathrm{id}}}H)$ and $q : G \to G/N$ be the natural epimorphism. Let Z/H denote the set of orbits of the action (Z, H) and let $p : Z \to Z/H$ be defined by $p : z \mapsto [z]$, where $[z]$ denotes the orbit containing the element $z \in Z$. Then $N = (F, q \circ f, G/\mathrm{Core}_G(G_{\underline{\mathrm{id}}}H), Z/H, p(\underline{\mathrm{id}}))$ is an F-map isomorphic to the $M/G_{\underline{\mathrm{id}}}H$.*

Proof. Since H is normal, the orbits in Z/H are blocks of imprimitivity for the action (Z, G). Obviously, the action is transitive. The action is explicitly defined by $[x] \cdot g = [x \cdot g]$, for $g \in G$ and $x \in Z$. Let $g \in G_{[\underline{\mathrm{id}}]}$. Since $[\underline{\mathrm{id}}] \cdot g = [\underline{\mathrm{id}} \cdot g] = [\underline{\mathrm{id}}]$, there exists some $h \in H$, such that $\underline{\mathrm{id}} \cdot g = \underline{\mathrm{id}} \cdot h$. Therefore, $gh^{-1} \in G_{\underline{\mathrm{id}}}$ and $g \in G_{\underline{\mathrm{id}}}H$.

For $g \in G_{\underline{\mathrm{id}}}H$, it follows $g = sh$, for some $s \in G_{\underline{\mathrm{id}}}$ and $h \in H$. Then $[\underline{\mathrm{id}}] \cdot g = [\underline{\mathrm{id}} \cdot sh] = [\underline{\mathrm{id}}]$. Hence, the stabilizer of $[\underline{\mathrm{id}}]$ is exactly $G_{\underline{\mathrm{id}}}H$. The action $(Z/H, G/\mathrm{Core}_G(G_{\underline{\mathrm{id}}}H))$ is faithful, therefore N is a map and it follows that $N \simeq M/G_{\underline{\mathrm{id}}}H$, by Theorem 5. □

Projecting of the whole automorphism group along a normal quotient makes normal quotients special. An interesting observation made by Tucker [46] is that any F-map morphism $(\phi, \psi) : M \to N$ factors through a normal quotient $M \to M \triangle \ker \psi \to N$, since $\mathrm{S}_F(M) \le \mathrm{S}_F(M) \cdot \mathrm{Core}_F(\mathrm{S}_F(N)) \le \mathrm{S}_F(N)$ (by Theorem 4, $\mathrm{Core}_F(\mathrm{S}_F(N)) \lhd F$ corresponds to $\ker \psi \lhd G_M$).

Consider a subgroup $K \le \mathrm{Aut}(M)$ and its orbits. Let $[x]$ denote the orbit containing an element $x \in Z_M$. If $y \in [x]$ then there exists $\gamma \in K$, such that $\gamma(x) = y$. Since for any $w \in F$ it follows

$$[y \cdot w] = [\gamma(x) \cdot w] = [\gamma(x \cdot w)] = [x \cdot w],$$

the orbits make a system of blocks of imprimitivity $\mathcal{B}$ for actions (Z_M, G_M) and (Z_M, F). Let $H \le G_M$ be the setwise stabilizer of the block $B_{\underline{\mathrm{id}}_M}$ containing the flag $\underline{\mathrm{id}}_M$. The action $(\mathcal{B}, G_M/\mathrm{Core}_{G_M}(H), B_{\underline{\mathrm{id}}_M})$ is a faithful RTA and therefore

$$N = (F, q \circ f_M, G_M/\mathrm{Core}_{G_M}(H), \mathcal{B}, B_{\underline{\mathrm{id}}})$$

is an F-map ($q : G_M \to G_M/\mathrm{Core}_{G_M}(H)$ is the natural epimorphism) and $N \le M$, since

$$\begin{aligned} \mathrm{S}_F(N) &= (q \circ f_M)^{-1}((G_M/\mathrm{Core}_{G_M}(H))_{B_{\underline{\mathrm{id}}_M}}) \\ &= f_M^{-1}((G_M)_{B_{\underline{\mathrm{id}}_M}}) \ge f_M^{-1}((G_M)_{\underline{\mathrm{id}}_M}) = \mathrm{S}_F(M). \end{aligned}$$

The inequality is due to the fact that fixing a point in a block causes fixing the whole block setwise. The F-map N is called a *K-automorphism quotient*.

Proposition 11. *Let M be F-maps, $K \leq \mathrm{Aut}(M)$ and $G = \{w \in F \mid \alpha_w \in K\}$. Then the K-automorphism quotient is isomorphic to $M/f_M(G)$.*

Proof. Let $[\underline{\mathrm{id}}_M]$ be the orbit of K containing $\underline{\mathrm{id}}_M$. Let $w \in F$ such that w stabilizes $[\underline{\mathrm{id}}_M]$. Therefore, $\underline{\mathrm{id}}_M \cdot w \in [\underline{\mathrm{id}}_M]$, implying that $\alpha_w \in \mathrm{Aut}(M)$ and $w \in G$.

Let $w \in G$. Then $\alpha_w \in K$ and $\underline{\mathrm{id}}_M \cdot w \in [\underline{\mathrm{id}}_M]$ meaning that w stabilizes $[\underline{\mathrm{id}}_M]$. Hence, $F_{[\underline{\mathrm{id}}_M]} = G$ and the K-automorphism quotient is isomorphic to $M/f_M(G)$. □

Corollary 12. *Let M be an F-map and $G = f_M(N_F(\mathrm{S}_F(M)))$. Then K-automorphism quotients are in one-to-one correspondence with K'-quotients, where $(G_M)_{\underline{\mathrm{id}}_M} \leq K' \leq G \leq G_M$.*

Proof. Each subgroup of $\mathrm{Aut}(M)$ is in one-to-one correspondence with subgroups of $N_F(\mathrm{S}_F(M))/\mathrm{S}_F(M)$, by Theorem 6, which are in one-to-one correspondence with subgroups of $N_F(\mathrm{S}_F(M))$ containing group $\mathrm{S}_F(M)$, by Theorem 4. Again by Theorem 4, these groups are in one-to-one correspondence with groups K', where $(G_M)_{\underline{\mathrm{id}}_M} \leq K' \leq G$. But this correspondence is induced exactly by the correspondence in Proposition 11. □

The consequence of the corollary is that for an F-map M, all K-quotients are H-automorphism quotients for some $H \leq \mathrm{Aut}(M)$ if and only if M is regular. In [33] they classified all quotients of orientably regular maps exactly through K-automorphism quotients.

Another very simple conclusion can be made.

Proposition 13. *Let M and N be F-maps and $g \in \mathrm{Aut}(F)$. Then $N \leq M$ if and only if $O_g(N) \leq O_g(M)$.*

Proof. $N \leq M$ if and only if $\mathrm{S}_F(M) \leq \mathrm{S}_F(N)$ and this is true if and only if

$$\mathrm{S}_F(O_g(M)) = g^{-1}(\mathrm{S}_F(M)) \leq g^{-1}(\mathrm{S}_F(N)) = \mathrm{S}_F(O_g(N)).$$

2.3 Parallel product and decomposition

The *parallel product* of two F-maps M and N is defined by

$$M \parallel N = (F, (f_M, f_N), G, Z, (\underline{\mathrm{id}}_M, \underline{\mathrm{id}}_N)),$$

where $G = (f_M, f_N)(F) \leq G_M \times G_N$ and Z is an orbit of the induced action of G on $Z_1 \times Z_2$ containing $(\underline{\mathrm{id}}_M, \underline{\mathrm{id}}_N)$. Since $(z_1, z_2) \cdot (g_1, g_2) = (z_1, z_2)$ if and only if $g_1 \in (G_M)_{\underline{\mathrm{id}}_M}$ and $g_2 \in (G_N)_{\underline{\mathrm{id}}_N}$, the kernel of the action is the direct product of the kernels of (Z_M, G_M) and (Z_N, G_N) and therefore trivial. The action (Z, G) is faithful and transitive on Z, $M \parallel N$ is an F-map and $\mathrm{S}_F(M \parallel N) = \mathrm{S}_F(M) \cap \mathrm{S}_F(N)$. From the definition it follows that the parallel product is an associative and commutative operation (see also [8, 48]).

The definition enables us to construct $\mathrm{Mon}(M \parallel N)$ from monodromy groups $\mathrm{Mon}(M)$ and $\mathrm{Mon}(N)$, which is useful for computational purposes. Let $\{a_i\}_{i=1}^k$ be the generators of F. Then $\{f_M(a_i)\}_{i=1}^k$ are the generators of $\mathrm{Mon}(M)$ and $\{f_N(a_i)\}_{i=1}^k$ of $\mathrm{Mon}(N)$, while $\{(f_M(a_i), f_N(a_i))\}_{i=1}^k$ are generators of $\mathrm{Mon}(M \parallel N)$. The latter group naturally acts on $Z_M \times Z_N$, but for a parallel product we consider only the action of $\mathrm{Mon}(M \parallel N)$ on the orbit containing pair $(\underline{\mathrm{id}}_M, \underline{\mathrm{id}}_N)$. This construction is due to Wilson [48].

The following proposition describes lifts of automorphisms in parallel products.

Proposition 14. *Let M and N be F-maps that both contain α_w. Then $M \parallel N$ contains α_w.*

Proof. $N_F(\mathrm{S}_F(M)) \cap N_F(\mathrm{S}_F(N))$ is a subgroup of $N_F(\mathrm{S}_F(M) \cap \mathrm{S}_F(N))$. □

The next proposition describes a relation between automorphisms, re-rootings and parallel products of re-rootings of a F-map M through monodromy groups.

Proposition 15. *Let M be an F-map.*

1. *For every $w \in F$, $\mathrm{Mon}(M) \simeq \mathrm{Mon}(R_w(M)) \simeq \mathrm{Mon}(M \parallel R_w(M))$.*
2. *$M \simeq R_w(M)$ if and only if $\alpha_w \in \mathrm{Aut}(M)$.*
3. *If $w^2 = 1$ then $\alpha_w \in \mathrm{Aut}(M \parallel R_w(M))$.*
4. *Let M^M denote the parallel product of all re-rootings of M. Then M^M is regular and for any regular F-map M', such that $M \leq M'$, it follows $M^M \leq M'$.*

Proof. Let $K = \mathrm{S}_F(M)$, $H = \mathrm{Core}_F(K)$ and $q : F \to F/H$ is the natural epimorphism. Since for any $w \in F$, $\mathrm{S}_F(R_w(M)) = w^{-1}Kw$ and $\mathrm{Core}_F(w^{-1}Kw) = \mathrm{Core}_F(K \cap w^{-1}Kw) = H$, all of the three monodromy groups in (1) are isomorphic to $(F, q, F/H)$ as F-groups. Since $\mathrm{S}_F(R_w(M)) = w^{-1}Kw = K$ if and only if $w \in N_F(K)$, (2) follows. Since $w^{-1}(K \cap w^{-1}Kw)w = w^{-1}Kw \cap w^{-2}Kw^2 = w^{-1}Kw \cap K$ it follows $w \in N_F(K \cap w^{-1}Kw)$ and (3) follows.

As $\mathrm{S}_F(M^M) = H$, M^M is regular. For any $N \lhd F$, $N \leq w^{-1}Kw$, for all $w \in F$, it follows $N \leq H$. Hence, for any regular F-map $M' \geq R_w(M)$, for all $w \in F$, it follows $M' \geq M^M$, yielding (4). □

The consequence of the proposition is that in a regular F-map, a root can be omitted, since all re-rootings are isomorphic.

Proposition 16. *Let M and N be F-maps and $f \in \mathrm{Aut}(F)$. Then $O_f(M \parallel N) = O_f(M) \parallel O_f(N)$.*

Proof. Note that $\mathrm{S}_F(O_f(M)) = f^{-1}(\mathrm{S}_F(M))$. The claim follows from the fact that $f^{-1}(\mathrm{S}_F(M) \cap \mathrm{S}_F(N)) = f^{-1}(\mathrm{S}_F(M)) \cap f^{-1}(\mathrm{S}_F(N))$ as f^{-1} is an isomorphism.□

A *trivial* F-map is an F-map N, such that $\mathrm{S}_F(N) = F$. A *decomposition pair* for an F-map M is any pair of F-maps (N_1, N_2), such that $M \simeq N_1 \parallel N_2$ and none of N_1, N_2 is isomorphic to M or to a trivial F-map. This is equivalent to $\mathrm{S}_F(N_1) \cap \mathrm{S}_F(N_2) = \mathrm{S}_F(M)$ and $\mathrm{S}_F(M) \lneq \mathrm{S}_F(N_i) \lneq F$, $i = 1, 2$. An F-map M is *parallel-product decomposable* if there exists a decomposition pair for M. If there exists a decomposition pair consisting of normal quotients, then M is *normally parallel-product decomposable*. If there exists a decomposition pair of strict K-quotients, then M is *strictly parallel-product decomposable*.

Theorem 17. *(Decomposition theorem) An F-map $M = (F, f, G, Z, \underline{\mathrm{id}})$ is parallel-product decomposable if and only if there exist two different subgroups K_1, $K_2 \leq G$, such that $G_{\underline{\mathrm{id}}} \lneq K_i \lneq G$, $i = 1, 2$, and $G_{\underline{\mathrm{id}}} = K_1 \cap K_2$. Furthermore, M is normally parallel-product decomposable if and only if there exist two different non-trivial normal subgroups $H_1, H_2 \lhd G$ acting non-transitively on Z and $G_{\underline{\mathrm{id}}}H_1 \cap G_{\underline{\mathrm{id}}}H_2 = G_{\underline{\mathrm{id}}}$. Also, M is normally parallel-product decomposable if and only if it is strictly parallel-product decomposable.*

Proof. Consider the RTA morphism $(\mathrm{Id}, f) : (Z, F, \underline{\mathrm{id}}) \to (Z, G, \underline{\mathrm{id}})$ in the F-map M. Then $\mathrm{S}_F(M) = f^{-1}(G_{\underline{\mathrm{id}}}) = F_{\underline{\mathrm{id}}} \geq \ker f$. By Theorem 4, such K_1, K_2

exist if and only if there exist L_i, $i = 1, 2$, where $F_{\underline{\mathrm{id}}} \lneq L_i \lneq F$, $f(L_i) = K_i$, $L_i = f^{-1}(K_i)$, and $L_1 \cap L_2 = F_{\underline{\mathrm{id}}}$. By Theorem 5, this is true if and only if M is parallel product decomposable and one of decomposition pairs is $(M/K_1, M/K_2)$.

Since (Z, G) is faithful and transitive, $\mathrm{Core}_G(G_{\underline{\mathrm{id}}}) = \{1\}$ and non-triviality and non-transitivity of H_1 and H_2 is equivalent to $G_{\underline{\mathrm{id}}} \lneq G_{\underline{\mathrm{id}}} H_i \lneq G$. Together with the condition $G_{\underline{\mathrm{id}}} H_1 \cap G_{\underline{\mathrm{id}}} H_2 = G_{\underline{\mathrm{id}}}$ this is equivalent to normal parallel-product decomposability, where one of decomposition pairs is $(M/G_{\underline{\mathrm{id}}} H_1, M/G_{\underline{\mathrm{id}}} H_2)$.

An F-map is strictly parallel-product decomposable if and only if there is a decomposition pair $(M/K_1, M/K_2)$, where the cores $N_i = \mathrm{Core}(K_i)$, $i = 1, 2$, are non-trivial. But since $\mathrm{Core}_G(G_{\underline{\mathrm{id}}})$ is trivial, it follows $G_{\underline{\mathrm{id}}} \lneq G_{\underline{\mathrm{id}}} N_i \leq K_i$, $i = 1, 2$, and obviously $G_{\underline{\mathrm{id}}} N_1 \cap G_{\underline{\mathrm{id}}} N_2 = G_{\underline{\mathrm{id}}}$ (as $K_1 \cap K_2 = G_{\underline{\mathrm{id}}}$ and $G_{\underline{\mathrm{id}}} \leq G_{\underline{\mathrm{id}}} N_1 \cap G_{\underline{\mathrm{id}}} N_2$). Obviously, all (nontrivial) normal quotients are strict. □

When computing with F-maps, we mostly operate with (permutation) monodromy groups. The theorem tells us exactly how to determine decomposability of an F-map and how to decompose it, if possible. Often, we would like to decompose a monodromy group into a parallel product of monodromy groups of strictly smaller order, i.e., we want strict parallel-product decomposability. The theorem says that if we are able to achieve this for an F-map M, we can do this in a way where both factors preserve the symmetry of M.

For an F-map $M = (F, f, G, Z, \underline{\mathrm{id}})$, some normal non-trivial and non-transitive subgroups $H_1, H_2 \lhd G$, such that $G_{\underline{\mathrm{id}}} H_1 \cap G_{\underline{\mathrm{id}}} H_2 = G_{\underline{\mathrm{id}}}$, exist, if and only if there exist minimal normal non-trivial and non-transitive subgroups $N_1, N_2 \lhd G$, $N_i \leq H_i$, $i = 1, 2$, and still $G_{\underline{\mathrm{id}}} N_1 \cap G_{\underline{\mathrm{id}}} N_2 = G_{\underline{\mathrm{id}}}$. Note that here we use the fact that M is finite. Therefore, it is sufficient to check minimal normal subgroups of G to determine normal (or strict) parallel-product decomposability. Together with the fact that in a finite regular F-map the stabilizers are trivial, the theorem follows.

Theorem 18. *A regular F-map M is normally parallel-product decomposable if and only if $\mathrm{Mon}(M)$ (and therefore also $\mathrm{Aut}(M)$) contains at least two non-trivial minimal normal subgroups. In this case both of factors are regular F-maps.* □

The groups with a unique minimal normal subgroup, including the simple ones, will be called *monolithic groups*.

Consider a finite regular F-map M and the F-map $\mathrm{M}_F(\mathrm{S}_F(M)) = (F, q, F/H, F/H, H)$, for $H = \mathrm{S}_F(M) \lhd F$ and $q : F \to F/H$, the natural epimorphism. Recall that $F = \langle a_1, \ldots, a_k \mid R_1 = \ldots = R_n \rangle$. Since F/H is finite, H can be expressed as a normal closure of a finite set of words $\{R_{n+j}\}_{j=1}^{m}$ and $F/H = \langle a_1, \ldots, a_k \mid R_1 = \ldots = R_{n+m}\rangle$. Note that the presentation of F/H completely encodes all the information about M up to isomorphism. From now on, a finite regular F-map will be represented in such a presentation, which will be called a *F-map group*.

Consider a finite regular F-map M represented by an F-map group $M = \langle a_1, \ldots, a_k \mid W_1^{e_1} = \ldots W_s^{e_s} = 1\rangle$, where e_i is the exact order of the word W_i in F, $i = 1, \ldots, s$. A sequence of words $(W_i)_{i=1}^{s}$ is called the *context*. In the given context, M can be encoded by the vector $(e_i)_{i=1}^{s}$ and we will write $M = (e_i)_{i=1}^{s}$. Let some other finite regular F-map M' be presented in a different context C'. The *common context* C'' is any context which contains exactly all the words from C and C'. The F-map M (and similarly M') can be represented in C'' by calculating the orders of the words in C'' and adding additional (redundant) relations. For a given context C and a finite regular F-map M, we will say that C is *sufficient for* M, if M has a presentation in C.

Let M and N be regular F-maps and $w \in F$. Let a denote the exact order of $f_M(w)$ and b denote the exact order of $f_N(w)$. Then the exact order of $(f_M, f_N)(w)$ is obviously $\mathrm{lcm}(a, b)$. The following lemma is straightforward.

Lemma 19. *Let $M = (a_i)_{i=1}^{s}$, $N = (b_i)_{i=1}^{s}$ be two regular F-maps represented in a common context $(W_i)_{i=1}^{s}$. Suppose that the common context is sufficient for the F-map $M \parallel N$. Then $M \parallel N = (\mathrm{lcm}(a_i, b_i))_{i=1}^{s}$.* □

It is well known that all subgroups of a particular fixed group form a lattice. Recall that for $K, H \leq F$, the operations $\vee$ and $\wedge$ are defined by $K \vee H = \langle K, H\rangle$ and $K \wedge H = K \cap H$. Let K and H be of finite index in F. Then it is obvious that $K \vee H = \langle K, H\rangle$ is of finite index. The index $(K : K \cap H)$ is smaller or equal to the index $(F : H)$ and therefore $(F, K \cap H)$ is finite. Therefore, finite index subgroups of F make a sublattice denoted by Sub_F.

Let $\mathcal{M}_F$ denote the set of all isomorphism classes of F-maps and let Let $[M]$, $[N] \in \mathcal{M}_F$. Define a relation:

$$[N] \leq [M] \overset{\text{def}}{\Longleftrightarrow} N \leq M. \tag{2.2}$$

According to Corollary 3, the relation is well defined, it is a partial ordering relation and the mapping $\Theta : \mathcal{M}_F \to \mathrm{Sub}_F$ defined by $\Theta : [M] \mapsto \mathrm{S}_F(M)$ is a poset anti-isomorphism. Since Sub_F is a lattice, it follows that $\mathcal{M}_F$ is also a lattice.

Note that the inverse of the mapping Θ is the mapping $\Theta^{-1} : \mathrm{Sub}_F \to \mathcal{M}_F$ defined by $\Theta^{-1} : K \mapsto [\mathrm{M}_F(K)]$. The operations join and meet in $\mathcal{M}_F$ are defined by $[M] \vee [N] = \Theta^{-1}(\Theta([M]) \wedge \Theta([N]))$ and $[M] \wedge [N] = \Theta^{-1}(\Theta([M]) \vee \Theta([N]))$.

Let M, N be F-maps. For any $M' \simeq M$ and $N' \simeq N$ it follows $\mathrm{S}_F(M' \parallel N') = \mathrm{S}_F(M') \cap \mathrm{S}_F(N') = \mathrm{S}_F(M) \cap \mathrm{S}_F(N) = \mathrm{S}_F(M \parallel N)$, implying $[M \parallel N] = [M' \parallel N']$. The parallel product can be also defined on $\mathcal{M}_F$ by:

$$[M] \parallel [N] = [M \parallel N].$$

The discussion above tells us that the parallel product in $\mathcal{M}_F$ is exactly the operation $\vee$ in the lattice $\mathcal{M}_F$.

Note that the lattice Sub_F has the unique maximal element, namely F. Therefore the lattice $\mathcal{M}_F$ has the unique minimal element, that is the equivalence class of trivial maps.

From now on, we will consider F-maps as isomorphism classes rather then as particular representations although we will often primarily operate with some special class representatives. If we say "M is an F-map" we will actually mean $[M] \in \mathcal{M}_F$. Therefore $M = N$ would mean $[M] = [N]$ (and $M \simeq N$).

Beside the common properties of a lattice let us consider some additional properties of $\mathcal{M}_F$. Most of them were already noticed in [48] or [8] in some form.

For a subgroup $G \leq \mathrm{Aut}(F)$, an F-map is called G-symmetric if for every $f \in G$ it follows $O_f(M) \simeq M$. The following proposition is immediate consequence of Proposition 16 and its trivial extension to meets of F-maps.

Proposition 20. *For $G \leq \mathrm{Aut}(F)$, The parallel product and meet of two G-symmetric F-maps are again G-symmetric.* □

All G-symmetric F-maps form a sublattice in $\mathcal{M}_F$. In particular, regular F-maps are $\mathrm{Inn}(F)$-symmetric and thus form a sublattice in $\mathcal{M}_F$ denoted by $\mathcal{R}_F$.

For finite regular F-maps, there is a particularly nice numerical connection between the sizes of two F-maps and the sizes of their parallel product and meet. The connection was discovered by Breda and Nedela in [8] on regular hypermaps. Here it is rephrased in the language of regular F-maps.

Proposition 21. *Let M and N be regular F-maps. Then*

$$|M \wedge N| \cdot |M \parallel N| = |M| \cdot |N|. \tag{2.3}$$

Proof. Let $H = \mathrm{S}_F^{-1}(M)$ and $K = \mathrm{S}_F^{-1}(N)$ and since the F-maps are regular $H, K \lhd F$. Note that $|M| = |F/H|$, $|N| = |F/K|$, $|M \parallel N| = |F/(H \cap K)|$ and $|M \wedge N| = |F/HK|$.

According to the second and the third isomorphism theorems for groups, it follows:

$$\begin{aligned} (F/H)/(HK/H) = F/HK, &\quad \text{and thus} \quad & \frac{|F/H|}{|F/HK|} = |HK/H|, \\ (F/H \cap K)/(K/H \cap K) = F/K, &\quad \text{and thus} \quad & \frac{|F/H \cap K|}{|F/K|} = |K/H \cap K|, \\ KH/H = K/H \cap K, &\quad \text{and thus} \quad & |KH/H| = |K/H \cap K|. \end{aligned}$$

Combining the equations the result follows. □

As already noted in [8], the equation (2.3) is very similar to the equation $ab = \gcd(a, b) \cdot \mathrm{lcm}(a, b)$, which might give us a false hope that $|\mathrm{Mon}(M \parallel N)| = \mathrm{lcm}(|\mathrm{Mon}(M)|, |\mathrm{Mon}(N)|)$. Combining Proposition 21 with the fact that for any finite F-maps $N \leq M$ it follows $|N|$ divides $|M|$, we can claim only that

$$\begin{aligned} \mathrm{lcm}(|\mathrm{Mon}(M)|, |\mathrm{Mon}(N)|) \quad & \text{divides} \quad |\mathrm{Mon}(M \parallel N)| \\ |\mathrm{Mon}(M \wedge N)| \quad & \text{divides} \quad \gcd(|\mathrm{Mon}(M)|, |\mathrm{Mon}(N)|). \end{aligned}$$

For a regular F-map M let a *direct regular quotient* be a regular F-map $N \leq M$ and $N \neq M$, such that for any F-map X, where $N \leq X \leq M$, it follows $X = N$ or $X = M$. Applying the anti-isomorphism Θ, the definition is equivalent to saying that the interval $[\mathrm{S}_F(M), \mathrm{S}_F(N)]$ in the lattice of normal subgroups contains exactly two elements, namely $\mathrm{S}_F(M)$ and $\mathrm{S}_F(N)$. Therefore, for $H = f_M(\mathrm{S}_F(N))$, H must be a minimal normal subgroup in G_M and $N \simeq M \triangle H$.

It is well known that the lattice of normal subgroups of any group is modular. It is also quite easy to see that the sublattice of finite index normal subgroups is also modular. Therefore the lattice $\mathcal{R}_F$ is modular.

2.4 Examples of F-maps

2.4.1 Holey maps

The following important example shows the application of the theory on rooted holey maps. A special class of rooted holey maps are edge-transitive maps, which are the main topic of the thesis.

A finite map on a closed compact surface S is an embedding of a finite graph G on S, where $S \setminus G$ consists of connected parts homeomorphic to disks (faces). According to [17, 32, 47], such a map can be combinatorially represented by a finite set of *flags* Z and three fixed-point-free involutions $T, L, R \in \mathrm{Sym}_{\mathrm{R}}(Z)$, where T and L commute and TL is also fixed-point-free. The involutions act on the set of *flags* and generate the *monodromy group*. Imagine the flags as triangles with the sides labeled by T, L and R and glue two triangles a and b along the side labeled by T, if $a\,T = b$ and similarly for the labels L and R. The conditions on the involutions imply that the surface obtained by gluing is a compact closed surface and the sides labeled by T define an embedding of a graph. If we do not insist on the involutions being fixed-point free, we obtain algebraic objects called *holey maps* as defined in [1]. If we additionally root them, we get F-maps for $F = \langle T, L, R \mid T^2 = L^2 = R^2 = (TL)^2 = 1\rangle$, where the involutions correspond to labeled generators of monodromy groups. The combinatorial edges, vertices, faces and Petrie circuits are the $\langle T, L\rangle$-, $\langle T, R\rangle$-, $\langle L, R\rangle$- and $\langle LT, R\rangle$-orbits, respectively. Consider the automorphisms $\mathbf{d}, \mathbf{p} \in \mathrm{Aut}(F)$, defined by the assignments $\mathbf{d} : T \mapsto L$, $L \mapsto T$, $R \mapsto R$ and $\mathbf{p} : T \mapsto T$, $L \mapsto TL$, $R \mapsto R$. They induce the well known map operations, the dual $\mathrm{D}(M) = O_{\mathbf{d}}(M)$ and the Petrie dual $\mathrm{P}(M) = O_{\mathbf{p}}(M)$. Regular maps on surfaces are called *reflexible maps*.

From now on in this section, the word "map" will mean a holey map. For a reflexible map M, $F(M)$ will denote the corresponding F-map group. The following example shows an application of the theory of F-maps on reflexible maps.

Example 1. Figures 2.1 and 2.2 demonstrate an application of Theorem 18 and the difference between a normal and a general map quotient, both on a map M, a 4-cycle on the sphere. $F(M) \simeq \mathbb{Z}_2 \times \mathrm{D}_4$ and has exactly 3 minimal normal subgroups which induce three normal quotients. In Figure 2.1, M and the three normal quotients are represented by flag graphs. By Theorem 18, M is isomorphic to a parallel product of any two of the quotients.

In Figure 2.2, a non-normal quotient N is presented. Still, $N \parallel R_L(N) \simeq M$. Both N and $R_L(N)$ contain α_T and α_R which lift to $N \parallel R_L(N)$, by Proposition 14. Also, by Proposition 15, claim 3, α_L lifts. Since T, L and R generate F, $N \parallel R_L(N)$ is reflexible.

A morphism of rooted maps $(p, q) : M \to N$ is in the case of rooted holey maps also called a *covering projection*. In this case the holey map M is called a *cover* of the map N. Note that the notion of covering projection corresponds

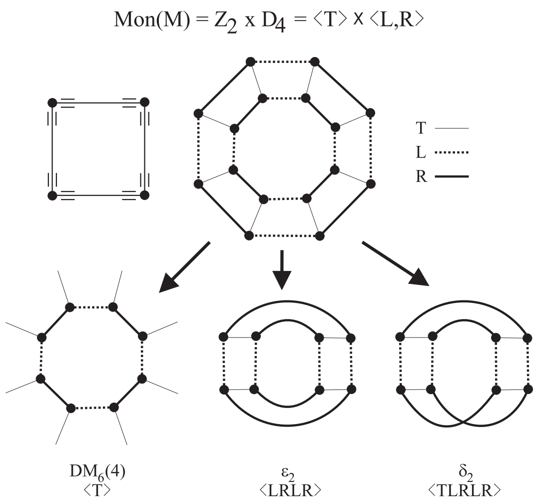


Figure 2.1: Normal quotients of a 4-cycle on the sphere that yield a non-trivial normal parallel-product decomposition.

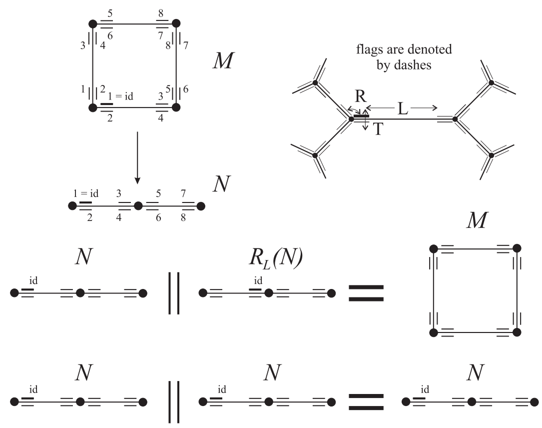


Figure 2.2: A non-normal quotient of C_4.

to the notion of covering projection of flag graphs described in [34]. Since such a projection can take an edge to a semi-edge, this kind of a projection is not a covering projection in the sense of topology (if we consider maps topologically), namely a local homeomorphism, but is more like the projection associated with an orbifold.

It should be noted that given a map M, both $\mathrm{D}(M)$ and $\mathrm{P}(M)$ have the same edges as M (namely the same $\langle T, L\rangle$-orbits), but $\mathrm{D}(M)$ interchanges faces and vertices leaving Petrie circuits the same, while $\mathrm{P}(M)$ interchanges faces and Petrie circuits leaving vertices the same. Since $\langle \mathbf{d}, \mathbf{p}\rangle \simeq \mathrm{Out}(F) \simeq S_3$ (see [27]), at most 6 non-isomorphic maps can be produced from one map applying these two operations. Since all the maps obtained using the operations D and P have the same automorphism group (only the roles of automorphisms are changed), we will often analyze only one representative of the class. The symmetry provided by $\langle \mathbf{d}, \mathbf{p}\rangle$ will be called a *triality* and a class of maps obtained from a single map by applying the operations will be called a *triality class*.

2.4.2 Orientable maps

Consider an embedding of a graph into an orientable surface. Divide each edge $e = uv$ on two halfs and denote the halves by pairs (u, e) and (v, e). The pairs will define the set of flags. Let L be an involution, such that $(v, e) \cdot L = (u, e)$, $(u, e) \cdot L = (v, e)$ (and similarly for all other vertices and edges). Since the surface is orientable, a (chosen) positive orientation determines circular orderings of edges emanating around vertices. For a vertex v, let $e_0, e_2, \ldots, e_{\deg(v)-1}$ be the edges emanating from v numbered according to the circular ordering. Let R be a permutation, such that $(v, e_i) \cdot R = (v, e_{(i+1) \bmod \deg(v)})$ (and similarly for all other vertices and edges). An orientable map can be encoded by the permutations R and L, which generate monodromy group. The faces are determined by the orbits of $\langle RL\rangle$. On the other hand, if we allow certain degeneracies, any two permutations R and L on a set Z, where L is an involution, such that $\langle R, L\rangle$ acts transitively on Z determine an orientable map (possibly degenerate). If we add rooting, we get F-maps for $F = \langle R, L \mid L^2 = 1\rangle$. Regular F-maps correspond to orientably regular maps.

Note that here the concept of orientability in topological sense does not match to the concept of orientability in the sense of F-maps.

Chapter 3

Computation of small finite quotients of finitely presented groups

3.1 The small quotients algorithm

We will present an algorithm which for a finitely presented group $F = \langle a_1, ..., a_k \mid R_1 = \ldots = R_m = 1\rangle$ determines all quotients up to order $N_{\max}$. These quotients are in one-to-one correspondence with isomorphism classes of finite regular F-maps up to order $N_{\max}$. In this section an F-map is said to be (in-)decomposable if it is normally parallel-product (in-)decomposable.

The algorithm can be implemented within any computational algebra programming environment (CAPE) containing a database of all small groups up to some order. An isomorphism class represented by a group G has a unique *id pair* denoted by $\mathrm{idPair}(G) := (n, o)$. Here $n = |G|$. Let $m(n)$ be the number of isomorphism classes of groups of order n. Then $o \in \{1, \ldots, m(n)\}$, depending on the isomorphism class of G. For a group G, where $|G| \leq \left\lfloor \frac{N_{\max}}{2} \right\rfloor$, CAPE should be able to quickly and efficiently determine $\mathrm{idPair}(G)$. Within CAPE we have to build a database of id pairs of monolithic groups.

Nowadays, such CAPEs are MAGMA[10] and GAP [18]. Currently (2006), both of them use the same database of small groups which contains all the groups up to order 2000, except the groups of order 1024. They can efficiently calculate $\mathrm{idPair}(G)$, if $|G| \leq 1000$ and $|G| \neq 512$. The identification of groups of order 512 would be needed only in the part of the algorithm when we want to identify direct quotients of monolithic groups of order 1024. Since we do not have those groups, the identification at order 512 is not needed. With the current database available, the algorithm can efficiently calculate all F-maps up to order $N_{\max} = 2000$, except the ones on monolithic groups of order 1024. With the database updated in future, capabilities of the algorithm extend.

An alternative efficient algorithm for calculation of small quotients of a finitely presented group F is the algorithm LOWX by Conder and Dobcsányi [12]. This algorithm does not use any additional pre-calculated information about the known small groups and it is therefore usually slower for this purpose. But it has some other very useful properties. By the time this thesis was nearing to its final form a new algorithm was implemented as a function `LowIndexNormalSubgroups` in MAGMA[10]. The algorithm is highly efficient and even much faster that the algorithm described in this section.

Basically, there are two kinds of algorithms for determining small quotients of a finitely presented group F. The first kind is based on coset enumeration (LOWX), while the second kind uses the database of small groups and for each group tries to determine generating sets that admit the relations in F. Our algorithm is of the second kind. The virtue of the algorithm is that it performs the checking for 'good' generating sets only on a very small number of groups, namely monolithic ones. Note that monolithic groups make only 0.1% of all groups up to order 1000.

The new algorithm is based on a simple idea. At first we calculate all F-maps on monolithic groups by searching for sets of generators that admit the relations of F. Then we calculate all other regular F-maps (or quotients) using parallel products. This is basically all what the algorithm does, of course in a much more efficient way.

3.1.1 Pre-calculation of a database of monolithic groups

At the beginning, a database of monolithic groups up to some order has to be built. The ratio of monolithic groups to all groups up to some order seems to be very small. For instance, for groups up to order 1000 it is 12860/11758814 and if we exclude p-groups it is 1153/1205037. The author conjectures that the ratio actually converges to 0 when the maximal order goes towards infinity.

Since for some orders there are large numbers isomorphism classes of groups, even a determination of monolithic ones among them can be a computationally difficult task. For the majority of orders we can do a filtering by simply calculating and counting minimal normal subgroups. The real challenge represent the orders

p^n and $p^n q$, for p and q distinct primes, since most of groups have these orders. Especially tough are the cases when $p = 2$ and $q = 3$. In the current situation, this means that the problematic orders are 512, 768, 1024, 1280, 1536, 1792. For each of these orders there are more than a million of groups [4]. At order 1024 there are over $49 \cdot 10^9$ groups and a database of those is not available yet (mainly due to its enormous size). A larger numbers of groups appear also at orders 1152 and 1920.

Let us first introduce some properties of p-groups. For proofs and details the reader is referred to [45]. A *p-group* is a group of order p^n, $n \geq 1$ and p is a prime. The centre $Z(G)$ of such a group is nontrivial and contains an element of order p. Each nontrivial normal subgroup intersects the centre nontrivially. All minimal normal subgroups are cyclic of order p. A p-group has a unique minimal normal subgroup if and only if the centre is cyclic. Each minimal set of generators of a p-group has the same size and this number is called the *Frattini rank*.

By checking whether the centre is cyclic, representatives for monolithic groups of order 512 are extracted among representatives of about 10^7 isomorphism classes at order 512. This takes about 8 hours on an average home computer in 2006.

A finite group is *nilpotent* if and only if it is a direct product of its Sylow subgroups. Therefore, if G is not a p-group and is nilpotent, it cannot be monolithic.

The groups of all the problematic orders have been stored into the small groups database in MAGMA (or GAP) in special orderings that are consequences of particular group generation algorithms. The groups of order $2^n q$, where q is an odd prime, especially the ones of orders 768, 1280, 1536 and 1972, have been generated by algorithms by Besche and Eick [3] and are stored in the following order: nilpotent groups, groups with a normal q-Sylow subgroup, groups with a normal 2-Sylow subgroup and others. It turns out that the vast majority (more than 99%) of the groups of those orders are the ones with a normal q-Sylow subgroup.

The following proposition reduces our calculation significantly.

Proposition 22. *Let G be a finite group and $N \lhd G$ a normal subgroup, such that $(G : N) = p^n$, where $n \geq 2$, p is a prime, which does not divide $|N|$, and $|\mathrm{Aut}(N)| \neq p^n$. Then G has at least two minimal normal subgroups.*

Proof. It suffices to find a normal subgroup $S \lhd G$, such that $S \cap N = \{1\}$. Let $Q \leq G$ be any p-Sylow subgroup and $|Q| = p^n$. Since p does not divide $|N|$, it must be $N \cap Q = \{1\}$ and $NQ = G$. Therefore, $G = N \rtimes Q$. Each such semidirect product arises from a homomorphism $\theta : Q \to \mathrm{Aut}(N)$, such that G is isomorphic to $(N \times Q, \cdot)$ where $(n, q) \cdot (n', q') = (n \cdot \theta(q)(n'), qq')$. Since $p^n = |Q| \neq |\mathrm{Aut}(N)|$, the kernel of θ is nontrivial. Each nontrivial normal subgroup of a p-group intersects the centre $Z(Q)$ nontrivialy. Hence, there exists an element $c \in \ker\theta \cap Z(Q)$ of order p. Let $(n, q) \in (N \times Q, \cdot)$. Then its inverse is (n', q^{-1}), where $\theta(q)(n') = n^{-1}$. It follows $(n, q) \cdot (1, c^k) \cdot (n', q^{-1}) = (n, qc^k) \cdot (n', q^{-1}) = (n\theta(qc^k)(n'), c^k) = (n\theta(q)(n'), c^k) = (1, c^k)$, for $k \in \mathbb{N}$. Therefore, the group $S = \langle(1, c)\rangle \cong \mathbb{Z}_p$ is a minimal normal subgroup in G. Obviously, $N \cap S = \{1\}$. □

The immediate corollary of the proposition above is the following.

Corollary 23. *Let G be a group of order $p^n q$, where p and q are distinct primes, $q - 1 \neq p^n$ and G has a normal q-Sylow subgroup. Then G has at least two minimal normal subgroups.* □

The corollary enables us to omit the majority of groups of orders $768 = 2^8 3$, $1280 = 2^8 5$, $1536 = 2^9 3$, $1792 = 2^8 7$, leaving us to check only 6763, 153, 114465 and 81 groups, respectively.

In the case of order 1152 we similarly use the information about nilpotency and normality of a 3-Sylow subgroup. Here we have to check only 4565 groups.

In the case of order 1920 we have to check only 4660 groups. Here we use the information about nilpotency and normality of $(3, 5)$-Hall subgroups.

3.1.2 The algorithm

Unfortunately, decomposable regular F-maps do not decompose in a unique way like numbers are factorized into primes. An F-map can occur as a parallel product of different sets of (indecomposable) F-maps. By Theorem 18, for a regular F-map M, any pair of normal subgroups $H, K \leq G_M$, $H \cap K = \{1\}$, yields two quotients whose parallel product is M. From a computational point of view this means a lot of isomorphism checks when computing a set of non-isomorphic F-maps from all possible parallel products. Also, we would like to omit all F-maps of the order greater than $N_{\max}$ without calculating the parallel products and save the time and a computer memory. An algorithm that explicitly calculates parallel products for only one representative per an isomorphism class of F-maps up to order $N_{\max}$ is presented. The algorithm efficiently omits all sets of F-maps whose parallel products would yield an F-map from an isomorphism class we already have. Additionally, F-maps on monolithic groups are calculated after all the information about all direct quotients is already known which also helps us in reducing calculations significantly.

Namely, if for some monolithic group G there is no F-map on the direct quotient, then there is also no F-map on the group G.

Let $\mathrm{dir}(M) = \{N \in \mathcal{M}_F \mid N \leq M, N \text{ is a direct quotient}\}$ be the set of direct quotients of an F-map M. Unless M is indecomposable, $\mathrm{dir}(M)$ completely determines M, since M is a parallel product of any two elements in $\mathrm{dir}(M)$. Since the lattice of regular F-maps ($\mathcal{R}_F$) is modular, the *Dedekind's transposition principle* applies:

If $a, b \in \mathcal{R}_F$ then the function $\rho : [a \wedge b, a] \to [b, a \vee b]$ defined by $\rho : x \mapsto x \vee b$ is a lattice isomorphism.

The direct consequence of the Dedekind's transposition principle is the following proposition.

Proposition 24. *Regular F-maps A and B, $A \neq B$, are direct quotients of a regular F-map M if and only if $M = A \parallel B$ and $A \wedge B$ is a direct quotient of both A and B.* □

The algorithm follows.

Algorithm 1. Computes all regular F-maps (or equivalently, all quotients of F) up to order $N_{\max}$.
`Input:`

- $N_{\max}$ ... the maximal order of F-maps to be determined.
- F ... the finitely presented group whose quotients we are determining.
- $f(G, p, G/H, E)$ ... the function that calculates and returns a list of all representatives of isomorphism classes of regular F-maps with the F-map group isomorphic to the monolithic group G. This is done by searching for generating sets admitting the relations of F. The search is performed by (custom) backtracking style algorithm which can use the information on already calculated F-maps stored in the *calculation environment* E (see below). In particular, it can use the information about F-maps on the unique direct quotient G/H (where H is the unique minimal normal subgroup and $p : G \mapsto G/H$ the natural epimorphism).
- Invars ... a finite sequence $(w_i)_{i=1}^n$ of words in F. A *signature* $\mathrm{sig}(M)$ *of an F-map M by* Invars is a $(|\mathrm{Invars}|+1)$-tuple in $\mathbb{N}^{|\mathrm{Invars}|+1}$. The first component

equals to the order $|G_M|$. The i-th component, for $i > 1$, equals to the order of $f_M(w_{i-1})$.

We use the lexicographic ordering on the set of signatures and denote it by $\leq_{\text{sig}}$. Note that for any two F-maps N and M, it follows $\text{sig}(N) \leq_{\text{sig}} \text{sig}(M) \implies |N| \leq |M|$. As we will see, signatures are relatively easy to calculate. Since two isomorphic F-maps have the same signatures, signatures are used to reduce isomorphism checks. The efficiency of the reduction depends on a particular choice of words in Invars.

`Temporary variables:`

- E ... the *calculation environment* which contains:
  - Maps ... a set of representatives of isomorphism classes of F-maps currently calculated.
  - $\text{dir} : \text{Maps} \to \mathcal{P}(\text{Maps})$ ... a mapping that for an F-map $M \in \text{Maps}$ returns the set of its direct quotients.
  - $\text{sig} : \text{Maps} \to \mathbb{N}^{|\text{Invars}|+1}$ ... a mapping that for an F-map $M \in \text{Maps}$ returns its signature.
  - $\text{groupId} : \text{Maps} \to \mathbb{N}^2$ ... a mapping that for an F-map $M \in \text{Maps}$ returns $\text{idPair}(\text{Mon}(M))$.
  - $\text{signatureToMaps} : \mathbb{N}^{|\text{Invars}|+1} \to \mathcal{P}(\text{Maps})$ ... a mapping that for an F-map signature s by Invars returns the set of all F-maps in Maps with the signature s.
  - $\text{groupIdToMaps} : \mathbb{N}^2 \to \mathcal{P}(\text{Maps})$ ... a mapping that for a group id pair returns the set of all F-maps on the group with the id pair.

Note that in the actual implementation of the algorithm each F-map in Maps is represented by its unique id number (or a pointer). Therefore, the set Maps is actually a set of id numbers. The set $\mathcal{P}(\text{Maps})$ represents the subsets of id numbers. In an implementation we need a mapping from id numbers to presentations of F-maps for parts of the algorithm where we actually need the presentations.

Updating the calculation environment E. For a new F-map M with the signature s_M, the environment E is updated in the following way. Add M to the set Maps. Update the mapping sig by the entry $M \mapsto s_M$. Calculate $ip :=$

$\mathrm{idPair}(\mathrm{Mon}(M))$, update the mapping $\mathrm{groupId}$ by the entry $M \mapsto ip$ and update the mapping $\mathrm{groupIdToMaps}$ by the entry $ip \mapsto \mathrm{groupIdToMaps}(ip) \cup M$. Calculate the minimal normal subgroups of $\mathrm{Mon}(M)$ and the set of direct quotients Q_{dir}. For each $D \in Q_{\mathrm{dir}}$ calculate the signature $s_D := \mathrm{sig}(D)$ by Invars and get $id_D := \mathrm{idPair}(\mathrm{Mon}(D))$. Among the F-maps in the set $\mathrm{signatureToMaps}(s_D) \cap \mathrm{groupIdToMaps}(ip_D)$ determine the F-map representative $X_D \in \mathrm{Maps}$ isomorphic to D. Note that this is the only part in the actual implementation of the algorithm that uses isomorphism checks. Update the mapping dir by the entry $M \mapsto \{X_D | D \in Q_{\mathrm{dir}}\}$.

- Q ... a priority queue which contains the records of the form $(s, (A, B))$ representing the F-map with a signature s, that is obtained either by a parallel product of two non-isomorphic F-maps ($A \neq B$), which are direct quotients, or it is an indecomposable F-map ($A = B$ and A is the indecomposable F-map). The priority queue Q respects the ordering $\leq_{\mathrm{sig}}$ on signatures of records (the smaller element, the higher priority).

- P ... a set of $\|$-*processed* F-maps.

  For an F-map $M \notin P$, but $M \in E$, the $\|$-*processing of the F-map M* will denote the procedure that for each $X \in P$ having a common direct quotient with M, calculates the signature $s := \mathrm{sig}(M \parallel X)$. If the order of the F-map $M \parallel X$ (i.e. the first component of the signature) does not exceed $N_{\max}$, the record $(s, (M, X))$ is inserted into the priority queue Q.

  The signature s is determined as follows. The F-maps M and X have the common direct quotient $J = M \wedge X$. By Proposition 24 and Proposition 21, we know that $|M \parallel X| = \frac{|M|\cdot|X|}{|J|}$. The rest of the signature is calculated from $\mathrm{sig}(M)$ and $\mathrm{sig}(X)$ using Lemma 19.

  Note that $\|$-processing does not include an actual calculation of any parallel product and does not use any actual presentations of F-maps. After the above procedure is completed, the F-map M is considered as $\|$-*processed* and included into the set P. F-maps of order greater then $\lfloor \frac{N_{\max}}{2} \rfloor$ do not need $\|$-processing since they cannot be factors in parallel products yielding F-maps of orders up to $N_{\max}$. All other F-maps that occur in the algorithm need $\|$-processing.

`Output:`

The algorithm returns the calculation environment E, containing all necessary information on all F-maps of order less or equal to $N_{\max}$.

Notes. If $M \in \mathrm{Maps}$, we will denote this by $M \in E$. Also, if G is a group, such that $\mathrm{idPair}(G)$ is in the domain of $\mathrm{groupId}$, we will denote this by $G \in E$.

Invariants. To make the definitions of E and P sensible, the following invariants must hold throughout the algorithm.

- *P-invariant:* The set P consists of exactly all the F-maps $X \in E$ which are $\|$-processed and every $\|$-processed F-map is in E.
- *E-invariant:* If i is the current order of F-maps being processed, then E contains all F-maps of order strictly smaller than i. All F-maps in E of order strictly smaller then $i-1$ are either $\|$-processed or they do not need $\|$-processing.
- *Q-invariant:* If $i \leq N_{\max}$ is the current order of F-maps being processed and M is any F-map of order i, then either $M \in E$, M is indecomposable or there is an F-map record representing M in Q.
- *PQ-invariant:* If $i \leq N_{\max}$ is the current order of F-maps being processed, then all the F-maps of order less or equal to $\min\left\{\left\lfloor\frac{N_{\max}}{2}\right\rfloor, i-2\right\}$ are $\|$-processed.

For the purpose of easier understanding of the algorithm some of optimizations will be omitted in the pseudo-code. They will be discussed later. In the comments below in the code F-maps will be simply called 'maps'.

```
$i \leftarrow 1$
initialize $Q$, $E$ and $P$ with a trivial map.
**while** $i \leq N_{\max}$ **do**
    **while** not empty(Q) **do**                    {*process map records* }
        $R = (s, (A, B)) \leftarrow \mathrm{top}(Q)$              {*the current map record*}
        **if** $s[1] > i$ **then**        $\{s[1]$ *is the order of the map represented by* $R\}$
            break the inner **while** loop   {*go to indecompos. maps of order* $i$}
        removeTop(Q)
        **if** $A \neq B$ **then**                    {*a map record for a parallel product*}
            **if** exists $N \in E$, such that $\{A, B\} \subseteq \mathrm{dir}(N)$ **then**
                continue at the beginning of the inner **while** loop
                                                      {*already have this map*}
            $M \leftarrow A \parallel B$  {*otherwise a new map is obtained and calculated*}
            Update the environment $E$ with the information on $M$
        **else** $M \leftarrow A$          { *the record represents an indecomposable map*}
        **if** $i \leq \left\lfloor \frac{N_{\max}}{2} \right\rfloor + 1$ **then**          $\{M$ *can be a parallel product factor*$\}$
            **forall** $X \in P$, where $\mathrm{dir}(X) \cap \mathrm{dir}(M) \neq \emptyset$ **do**
                                                      { $\parallel$-*process the map* $M$}
                $\{J\} \leftarrow \mathrm{dir}(X) \cap \mathrm{dir}(M)$
                $o_{\mathrm{temp}} \leftarrow \frac{|X| \cdot |M|}{|J|}$                    {*the order of* $M \parallel X$}
                **if** $o_{\mathrm{temp}} \leq N_{\max}$ **then**
                    $s_{\mathrm{temp}} \leftarrow$ calculate $\mathrm{sig}(M \parallel X)$
                    insert $(s_{\mathrm{temp}}, (M, X))$ into $Q$
                **end if**
            **end for**
            $P \leftarrow P \cup \{M\}$                         { $M$ *is now* $\parallel$-*processed* }
        **end if**
    **end while**                    {*end processing map records for order* $i$}
    $L \leftarrow$ all monolithic groups of order $i$
    **for all** $G \in L$ **do**
        let $H \lhd G$ minimal normal and $p : G \rightarrow G/H$ the natural epimorph.
        id $\leftarrow$ id pair for a direct quotient $G/H$
        **if** id $\notin E$ **then** continue at the beginning of the **forall** loop
                                                      {*no maps on* $G$}
        $Z \leftarrow f(G, p, G/H, E)$                    {calculate all maps on $G$}
        Update the environment $E$ with the information on the maps in $Z$
```

37 **if** $i \leq \lfloor \frac{N_{\max}}{2} \rfloor$ **then** include all the map records for maps in Z into Q
38 **end for**
39 $i \leftarrow i + 1$ $\{$*proceed with the next order*$\}$
40 **end while**

Comments on the algorithm and notes on the implementation.

Basically, the algorithm consists of two essential parts that interchange in the main **while** loop (lines 03–40), the first being the calculation and $\|$-processing of F-maps obtained from F-map records in Q (lines 04–28), and the second being the part where indecomposable F-maps are calculated (lines 30–38).

In the first part the records are being removed from Q and processed, if they do not exceed the current order i (lines 05–08). Each such a record stores information on an F-map obtained by a parallel product of direct quotients (lines 09–14) or on an indecomposable F-map (line 15). In the first case, several records can yield an isomorphic F-map. The lines 09–14 discard the F-map records of already obtained F-maps. Since for every already obtained map we have a list of its direct quotients, an F-map record $(s, (A, B))$ represents an already obtained map M if and only $\{A, B\} \subseteq \mathrm{dir}(M)$. This is a fast isomorphism check on F-map records. For each record that is not discarded, a new F-map M is calculated (line 13) and the environment E is updated with the information on M (line 14). The procedure just described is skipped in the case when an F-map record represents an indecomposable F-map (only the line 15 is executed).

After the line 15, we have an F-map $M \in E$, $M \notin P$, ready to be $\|$-processed in the lines 16–27. The $\|$-processing is not needed if the order of M exceeds $\lfloor \frac{N_{\max}}{2} \rfloor$ (line 16). After the $\|$-processing, the F-map M is included into P and the P-invariant is preserved.

The second part of the algorithm (lines 29–38) calculates all the F-maps on monolithic groups of order i. Here for each such group G the unique direct quotient G/H for the unique minimal subgroup $H \lhd G$ is calculated together with the natural epimorphism $p : G \to G/H$. If no F-map in the environment E is an F-map with the monodromy group isomorphic to G/H, the group G is discarded and the calculation continues with the next group (lines 32–33). Checking whether $G \in E$ can be done by verifying whether $\mathrm{idPair}(G)$ is a member of the domain of the mapping $\mathrm{groupIdToMaps}$. If the group G is not discarded, F-maps with monodromy groups isomorphic to G are calculated using the (custom) algorithm (line 35). The environment E is then updated with those F-maps (line 36). If the current order i does not exceed $\left\lfloor \frac{N_{\max}}{2} \right\rfloor$ the obtained F-maps need to be $\|$-processed and are therefore inserted into the priority queue Q.

After all F-maps on monolithic groups of order i are obtained, the current order increases by 1 (line 39). The algorithm continues in the first part (lines 04–28) processing the records of the indecomposable F-maps just obtained in the last execution of the second part. Those F-maps have the highest priority in Q. After that, the records representing the F-maps of the new order i are processed and the algorithm repeated.

Proposition 25. *Algorithm 1 terminates and when it does, the calculation environment E contains exactly all non-isomorphic F-maps up to order $N_{\max} \geq 1$. During the run of the algorithm, each F-map is explicitly calculated exactly once.*

Proof. It suffices to prove that the algorithm preserves the P, Q, E and PQ invariants, that algorithm terminates and that $i > N_{\max}$ upon the termination of the algorithm. The fact that $i > N_{\max}$ and the E-invariant at the end of the algorithm imply the claim of the theorem.

Obviously, immediately after the line 02, all the invariants are true, since Q contains only a trivial F-map, which is automatically $\|$-processed. Also, the P-invariant is always true during the run of the program, unless immediately after the line 25. But since the algorithm always proceeds from there to the line 26, the P-invariant immediately becomes true.

Due to the fact that there are finitely many groups up to some order, there are also finitely many F-maps up to some order. The variable i determines the current order of F-maps being processed. Let b_i be the number of all F-maps up to order i and let c_i be the maximal number of minimal normal subgroups of monodromy groups of F-maps up to order i. Let m_i be the maximal number of monolithic groups on orders up to i and let n_i be the maximal number of F-maps on a single monolithic group up to order i. Then the **for all** loop in the lines 17–25 terminates in at most b_i steps (a candidate for X can be any already calculated map, but usually this upper bound is far too high). For each decomposable F-map there might be up to $\binom{c_i}{2}$ F-map records representing it. Therefore the **while** loop in the lines 04–28 surely terminates in $b_i\binom{c_i}{2}$ steps (again, usually a way to high upper bound). The **for all** loop in the lines 30–38 terminates in at most $m_i n_i$ steps. Since i increases after every run of the **while** loop in the lines 03–40, the algorithm terminates after exactly $N_{\max}$ runs of that loop.

We need to verify that if the invariants are true in the line 03, at the point when the condition of the **while** loop is verified (03-point), then they will still be true when 03-point is reached the next time. This can only happen if exactly one run through the **while** loop is executed.

For the P-invariant, this is true (according to the discussion in comments above).

According to the Q-invariant, all F-maps up to order i are either in E either there are records representing them in Q or they are indecomposable. The **while** loop in the lines 4–28 checks and processes all those records in Q and transfers all the parallel-product decomposable F-maps of order i into E. Also it $\|$-processes all indecomposable F-maps of order $i-1$ and all decomposable F-maps of order i in Q, if $\|$-processing is needed. The **for all** loop in the lines 30–38 includes all the F-maps of order i with monolithic groups into E. Just before the line 39, the E-invariant is true for $i+1$. In the line 39, i is increased and after that, 03-point is reached with the E-invariant being true.

Now, let $i \leq N_{\max} - 1$. If $i \leq 3$ there are no parallel-product decomposable F-maps, therefore the Q-invariant is automatically true for $i \leq 3$. If $i \geq 3$, then the PQ-invariant implies that all the F-maps of order smaller and equal to $\left\lfloor \frac{i}{2} \right\rfloor$ have already been $\|$-processed. The **while** loop in the lines 04–28 processes all possibly remaining F-maps of order $i-1$. This is true since an F-map record for $i-1$ cannot enter Q after i was increased to the current value (in the line 39). Since $\left\lfloor \frac{i+1}{2} \right\rfloor \geq i-1$ for $i \geq 3$, all possible direct parallel-product factors for F-maps of order $i+1$, have been surely $\|$-processed, therefore, after the line 28, the Q and PQ invariants are true for $i+1$. Since the lines 29–38 do not change Q and do not perform any $\|$-processing, the Q and PQ invariants remain true until the 03-point is reached.

Each F-map is either calculated in the line 35 (indecomposable) or in the line 13 (decomposable). The lines 11–12 discard all F-map records that yield a decomposable F-map, which has already been calculated. $\square$

3.1.3 Optimizations

The implementation of the algorithm (the author used MAGMA) exposes a bottleneck, namely an efficient implementation of the algorithm for function f, which determines F-maps on monolithic groups. The solution depends on how efficiently one can use the relations in F to reduce the calculation.

Some information can be calculated in advance and reused. Consider a situation where for some monolithic group G, $\mathrm{Aut}(G)$ is exactly the lift of all automorphisms in $\mathrm{Aut}(G/H)$ and let the minimal normal subgroup H be small. Since there is the unique epimorphism from G to G/H, it is not hard to see that we may assume that the generators of F-maps projecting to M are contained in the preimages of the generators of M (recall that we consider regular F-maps as F-map groups here).

Since the situation with a complete lift of automorphisms is not very likely to happen, the author used the following approach. For each monolithic group G precalculate its subgroup lattice and construct the poset consisting of the subgroups (inheriting the lattice partial ordering) and the elements of the group, where the partial ordering between the elements and the subgroups is the inclusion relation. For each subset S of elements of this poset (these can be both subgroups or group elements), a supremum is uniquely defined and it is exactly the group which is generated by elements from S. A matrix where (i, j)-th element represents the supremum of elements i and j can be precalculated and stored. It can be used to efficiently determine whether some set of group elements generates the whole group. Unfortunately, a monolithic group can have many subgroups and the matrix can be extremely large. Instead of a matrix some other representation of a poset with supremums can be used to trade-off the space complexity with the time complexity. The elements of the poset can be distinct numbers. We only need to store the group G in some presentation, the list of the elements of G and the mapping between the list and the corresponding element numbers in the poset. We can also store generators of automorphism group G in terms of permutations on the numbers representing the group elements in the poset.

Let $A = \mathrm{Aut}(G)$. Here is a sketch of a backtracking style algorithm for determining F-maps on G. First we calculate orbits of A. All elements from the same orbit are equivalent in a choice of the first generator since in any F-map group we can map generators by any automorphism and still obtain an F-map group representing the same (isomorphic) F-map. For each orbit and a choice of its representative x we do the following. First we check whether there are relations that forbid the choice of x. If there are, the choice of x is not good and we proceed to the next orbit. Otherwise x becomes a candidate for the first generator. Now we calculate the stabilizer of x in A and denote it by S. Each choice of the second generator from an orbit of S is equivalent. For each orbit of S we choose a representative y and verify whether the relations of F allow the choice. If the choice is not allowed, we choose the next y form the next orbit. When y is successfully chosen we proceed recursively to the choice of the third generator and so on. When all the orbits for the choice of i-th generator are checked, we backtrack to the choice of a representative of the next orbit for the choice of the $(i-1)$-th generator. In the case when we successfully choose the last generator and verify that all the chosen generators generate the whole group (using the poset described above) we have an F-map and we store it. At the end of the algorithm all stored F-maps (i.e. sequences of generators) are returned.

When the algorithm is finished, we obtain F-map groups representing non-isomorphic regular F-maps. Consider any F-map group (or F-map) M. The first generator is contained in some orbit of A. There is an automorphism taking the first generator to the representative x of the orbit, where x is the choice made during the run of the algorithm. Then the (mapped) second generator is in some orbit of the stabilizer of x and therefore there is an automorphism in the stabilizer taking the second generator to the representative y of the corresponding orbit, where y is the choice made during the run of the algorithm. We can use a similar argument for the third generator and so on and gradually find a sequence of automorphisms that maps M to some F-map group obtained during the run of the algorithm. Therefore the algorithm determines all F-maps on the group G. Similarly, it is not hard to see that the sets of generators calculated during the run of the algorithm determine non-isomorphic F-maps (or their F-map groups).

It is also easy to see that the condition in the line 16 can be optimized:

$$\left(i \leq \left\lfloor \frac{N_{\max}}{2} \right\rfloor \wedge A \neq B\right) \vee \left(i \leq \left\lfloor \frac{N_{\max}}{2} \right\rfloor + 1 \wedge A = B\right).$$

This is due to the fact that the records of maps on monolithic groups of order j are $\|$-processed when $i = j + 1$.

Chapter 4

Edge-transitive maps

4.1 Reflexible maps

4.1.1 Degeneracy of reflexible maps

In this section reflexible maps are classified into three families according to their degeneracy. For a given reflexible map M, let $e_1, \ldots, e_7$ be the exact orders of the words T, L, R, TL, TR, LR, TLR, respectively. A map M is *slightly-degenerate* if it follows $e_i \geq 2$, for all $i = 1, \ldots, 7$, and at least one of e_5, e_6, e_7 equals to 2. It is *degenerate* if at least one of e_i, $i = 1, \ldots, 7$, equals to 1. If a map is not degenerate or slightly-degenerate then it is *non-degenerate*. In this case $e_i \geq 3$, $i = 5, 6, 7$.

Note that the set of the chosen words represents exactly the generators whose orders determine the map's properties, such as the degrees of the vertices, the co-degrees of the faces and the sizes of the Petrie circuits.

Let $C = (W_i)_{i=1}^7 = (T, L, R, TL, TR, LR, TLR)$ be the context and $(e_i)_{i=1}^7$ be a vector denoting a map for which C is sufficient. In analysis we use triality. Note that the operations D (dual) and P (Petrie dual) permute the triple (e_1, e_2, e_4) by the same permutation as the triple (e_5, e_6, e_7). To describe the action of D and P on the indices $i = 1, \ldots, 7$ of e_i, we can represent D by the permutation $(1, 2)(5, 6)$ and P by $(2, 4)(6, 7)$.

Proposition 26. *All degenerate reflexible maps are shown in Table 4.1.*

Name	$(T,$	$L,$	$R,$	$TL,$	$TR,$	$LR,$	$TLR)$	$\lvert F(M)\rvert$
DM_1	(1,	1,	1,	1,	1,	1,	1)	1
DM_2	(1,	1,	2,	1,	2,	2,	2)	2
DM_4	(2,	1,	1,	2,	2,	1,	2)	2
DM_3	(1,	2,	1,	2,	1,	2,	2)	2
DM_8	(2,	2,	1,	1,	2,	2,	1)	2
$\mathrm{DM}_5(k)$, $k > 0$	(2,	1,	2,	2,	k,	2,	k)	$2k$
$\mathrm{DM}_6(k)$, $k > 0$	(1,	2,	2,	2,	2,	k,	k)	$2k$
$\mathrm{EM}_3(k)$, $k > 0$	(2,	2,	2,	1,	k,	k,	2)	$2k$
DM_7	(2,	2,	1,	2,	2,	2,	2)	4
EM_2	(2,	2,	2,	2,	1,	2,	2)	4
ε_1	(2,	2,	2,	2,	2,	1,	2)	4
δ_1	(2,	2,	2,	2,	2,	2,	1)	4

Table 4.1: Degenerate reflexible maps.

Proof. First we prove that all the map groups in Table 4.1 are uniquely determined by the context C. For all the maps in the table except $\mathrm{DM}_5(k)$, $\mathrm{DM}_6(k)$ and $\mathrm{EM}_3(k)$, this is pretty obvious. By triality it is enough to check the group of $\mathrm{DM}_5(k)$. The relations here determine a dihedral group D_{2k} generated by $a = T$ and $b = TR$ and $D_{2k} = \langle a, b \mid a^2 = b^k = (ab)^2 = 1\rangle$. One can easily see that any quotient of D_{2k} strictly decreases the orders of at least one of the (projected) generators.

Now we will make an analysis of what kind of degenerate maps can occur. Let $e_1 = e_2 = 1$. Then $e_4 = 1$. If $e_3 = 1$ we get DM_1. If $e_3 = 2$ then it must be $e_5 = e_6 = e_7 = 2$ (DM_2). Now, let $e_1 = 1$ and $e_2 = 2$. Since $e_4 = 1$ implies $e_2 = e_1$, it must be $e_4 = 2$. If $e_3 = 1$ then it must be $e_5 = 1$, $e_6 = e_7 = 2$ (DM_3 and by triality DM_4 and DM_8). If $e_3 = 2$ then $e_5 = 2$ and $e_6 = e_7 = k \geq 1$ ($\mathrm{DM}_6(k)$ and by triality $\mathrm{DM}_5(k)$ and $\mathrm{EM}_3(k)$). By triality, all the possibilities where one of e_1, e_2, e_4 is 1 are exhausted. Assume $e_1 = e_2 = e_4 = 2$. If $e_3 = 1$ then $e_5 = e_6 = e_7 = 2$ (DM_7). Let now $e_3 = 2$. Since a map has to be degenerate, one of e_5, e_6, e_7 must be equal to 1. By triality we can assume $e_5 = 1$. Then it must be $e_6 = e_7 = 2$, otherwise the orders e_1, e_2 collapse (EM_2, ε_1, δ_1). This exhausts all the possibilities for degenerate maps. □

A similar analysis of degenerate maps was done in [31], but their definition of degeneracy is different from ours and uses an automorphism group. According to [31], a reflexible map M is degenerate if one of the generators $x = \alpha_L$, $y = \alpha_T$,

$z = \alpha_R \in \mathrm{Aut}(M)$ equals to the identity. It is easy to see that their degeneracy is equivalent to saying that one of e_1, e_2 or e_3 is equal to 1. Unfortunately, in [31] they forgot to include the map DM_8.

In Figure 4.1 all the flag graphs for degenerate maps are shown.

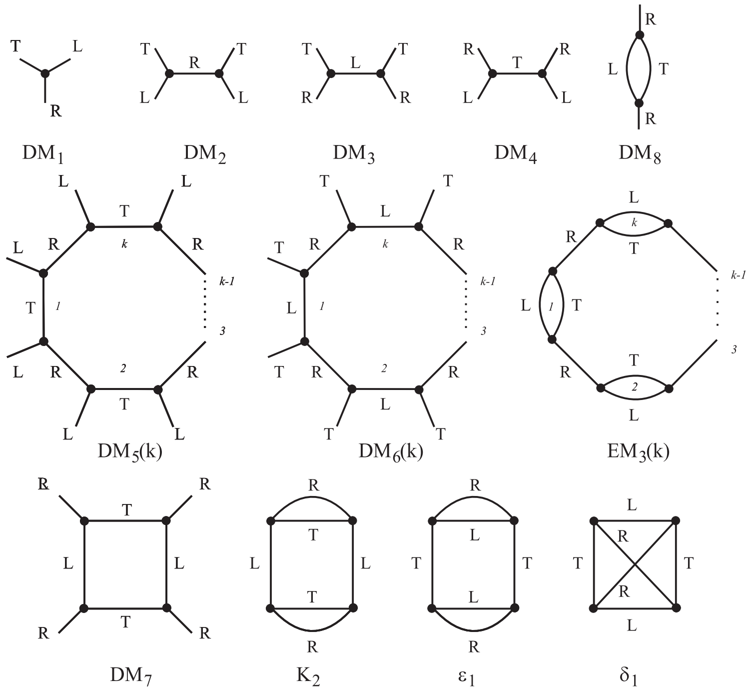


Figure 4.1: Flag graphs of degenerate reflexible maps.

If a reflexible map is not degenerate then all the involutions T, L, R, TL are fixed-point-free. Such a map corresponds to a reflexible 2-cell embedding of some graph into a compact closed surface. Slightly-degenerate maps can be constructed using the operations D and P from a reflexible embedding of a cycle in some compact closed surface. The only possible such 2-cell embeddings are the embeddings of k-cycle in the sphere, denoted by ε_k, and in the projective plane with the k-cycle embedded as a non-contractible curve, denoted by δ_k. Here the names are adopted from [50].

The map group presentations of maps ε_k and δ_k are shown in Table 4.2.

Name	Additional relations	Order
ε_k, $k > 0$ even	$(LR)^k, (TLR)^k$	$4k$
ε_k, $k > 1$ odd	$(LR)^k, (TLR)^{2k}$	$4k$
δ_k, $k > 0$ even	$T(LR)^k$, $T(TLR)^k$	$4k$
δ_k, $k > 1$ odd	$(LR)^{2k}, (TLR)^k$	$4k$

Table 4.2: A map group of each map in this table is obtained as $\langle T, L, R \mid T^2 = L^2 = R^2 = (TL)^2 = (RT)^2 = \ldots = 1\rangle$, where instead of "..." one should put the additional relations. All slightly-degenerate reflexible maps can be constructed from the maps in this table by using the operations D and P. Note that ε_1 and δ_1 are degenerate and so not included in this table.

4.1.2 Normal parallel-product decomposition of reflexible maps

Proposition 27. *The map* $\mathrm{DM}_5(k)$ *(*$\mathrm{DM}_6(k)$, $\mathrm{EM}_3(k)$*),* $k > 2$ *is normally parallel-product decomposable if and only if* k *is not a prime power.*

Proof. Number k is not a prime power if and only if there exist $a, b > 1$, such that $\gcd(a, b) = 1$ and $k = ab$. Using Lemma 19 and Table 4.1 it is easy to see that for any $a, b > 1$, $\mathrm{DM}_5(a) \parallel \mathrm{DM}_5(b) \simeq \mathrm{DM}_5(\mathrm{lcm}(a, b))$. Nontrivial factors of $\mathrm{DM}_5(k)$ can be only degenerate maps with $L = 1$, so only: $\mathrm{DM}_5(l)$, $l \geq 1$, DM_2 and DM_4. Since DM_2 and DM_4 are quotients of any $\mathrm{DM}_5(l)$, $l > 2$, a parallel product with $\mathrm{DM}_5(l)$ absorbs them. Also $\mathrm{DM}_2 \parallel \mathrm{DM}_4 \simeq \mathrm{DM}_2 \parallel \mathrm{DM}_5(1) \simeq \mathrm{DM}_4 \parallel \mathrm{DM}_5(1) \simeq \mathrm{DM}_5(2)$. So if $k > 2$ and $\mathrm{DM}_5(k)$ is normally parallel-product decomposable, then it must be a product of two factors of the form $\mathrm{DM}_5(l)$. By Table 4.1 and Lemma 19 this is possible only when the conditions of the lemma are fulfilled. Using triality, the proofs for $\mathrm{DM}_6(k)$ and $\mathrm{EM}_3(k)$ immediately follow. $\square$

The monodromy groups of the maps $\mathrm{DM}_7, \mathrm{EM}_2, \varepsilon_1$ and δ_1, are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and thus by Theorem 18 the maps are normally parallel-product decomposable. The monodromy groups of DM_1, DM_2,DM_3,DM_4 and DM_8 are either trivial or isomorphic to $\mathbb{Z}_2$, implying that those maps are normally parallel-product indecomposable.

The following corollary immediately follows.

Corollary 28. *All degenerate reflexible maps are normally parallel-product indecomposable except:*

1. $\mathrm{EM}_3(k), \mathrm{DM}_5(k)$ *and* $\mathrm{DM}_6(k)$, *for* $k = 2$ *and any* $k > 2$ *which is not a power of a prime,*

2. DM_7, EM_2, ε_1 *and* δ_1. □

Proposition 29. *The only normally parallel-product indecomposable slightly-degenerate maps are the maps* δ_k, *where* $k = 2^n$, $n \geq 1$.

Proof. Since $\mathrm{P}(\varepsilon_k) \simeq \delta_k$, for k odd, we have to consider only normal parallel-product decompositions of maps ε_k for all $k > 1$ and δ_k, for $k > 1$ even.

By Lemma 19, it is easily seen that $\varepsilon_k \simeq \mathrm{DM}_6(k) \parallel \mathrm{DM}_4$, for any $k > 1$.

Now, let $k > 0$ and let $l \geq 1$ be any odd number. The following monodromy groups are defined by relations:

$$\mathrm{Mon}(\delta_{2^k}) : T^2 = L^2 = R^2 = (TL)^2 = (RT)^2 = 1, (RL)^{2^k} = (TLR)^{2^k} = T,$$
$$\mathrm{Mon}(\mathrm{DM}_6(2^k l)) : T = L^2 = R^2 = (TL)^2 = (RT)^2 = (RL)^{2^k l} = (TLR)^{2^k l} = 1.$$

A pretty straightforward relation chasing helps us to see that $\delta_{2^k l} \simeq \mathrm{DM}_6(2^k l) \parallel \delta_{2^k}$. This means that for any even u not equal to a power of 2, δ_u is normally parallel-product decomposable.

For a given map M, denote by $e_5(M)$, $e_6(M)$ and $e_7(M)$ the exponents of the words RT, RL, TLR, respectively. For δ_{2^n} it follows $e_5 = 2$, $e_6 = e_7 = 2^{n+1}$. Since these values are powers of 2 and $\mathrm{lcm}(2^x, 2^y) = \max(2^x, 2^y)$, at least one of e_5, e_6, e_7 must be reached with the corresponding values e'_5, e'_6, e'_7 and e''_5, e''_6, e''_7 in two possible factors. Since the factors must be either degenerate or slightly degenerate maps, one of them must be one of $\mathrm{DM}_6(2^{n+1})$, δ_{2^n} or $\varepsilon_{2^{n+1}}$. A map δ_{2^n} is not admissible factor in a non-trivial decomposition, while a map $\varepsilon_{2^{n+1}}$ in a product would yield orientable map. Thus one of the factors must be $\mathrm{DM}_6(2^{n+1})$. Since the context C is not sufficient to obtain the map δ_{2^n}, one of the maps must be δ_l, for some $l = 2^u$, $u < n$. But since $\mathrm{DM}_6(2^{n+1}) \parallel \delta_l \simeq \varepsilon_{2^{n+1}}$ this is not possible. Thus δ_{2^n}, $n \geq 1$ is normally parallel-product indecomposable. □

Using computer programs LOWX [13] and MAGMA [10] all non-degenerate reflexible maps were calculated up to 100 edges. The results of the calculation match with *Wilson's census of rotary maps* [53]. Among them, the ones with the monolithic monodromy group were selected and they are shown in Table 4.3.

Name	**Wilson cen.**	e_5	e_6	e_7	**Additional relations**	**Monolith**
MN_1	6,1-3,1	3	3	4		$\mathbb{Z}_2^2$
MN_2	8,2-4,2	4	8	8	$(RTRL)^2, (LRT)^2(LR)^2$	$\mathbb{Z}_2$
MN_3	15,1-3,2	3	5	5		$\{1\} \leq A_5$
MN_4	16,1-1,1	4	4	4		$\mathbb{Z}_2$
MN_5	16,5-7,5	4	8	8	$((LR)^2T)^2$	$\mathbb{Z}_2$
MN_6	16,2-4,2	4	16	16	$(RTRL)^2, TLRT(LR)^7$	$\mathbb{Z}_2$
MN_7	18,1-3,1	4	4	6	$(RTRL)^3$	$\mathbb{Z}_3^2$
MN_8	24,19-24,21	3	8	12	$(LRTLR)^2T(LR)^2T$	$\mathbb{Z}_2$
MN_9	24,28-33,30	6	8	12	$TL(RT)^2(LR)^3$	$\mathbb{Z}_2$
MN_{10}	27,1-3,3	3	6	6		$\mathbb{Z}_3$
MN_{11}	30,28-33,30	4	5	6	$L(RTRL)^2(RT)^2$	$A_5 \leq S_5$
MN_{12}	30,37-37,37	6	6	6	$L(RT)^2RL(RT)^3, T(LR)^3TR(LR)^2$	$A_5 \leq S_5$
MN_{13}	32,4-6,4	4	4	8	$(RTRL)^4$	$\mathbb{Z}_2$
MN_{14}	32,19-21,19	4	16	16	$(LRT)^2(RL)^2(RT)^2, (LR)^2T(LR)^6T$	$\mathbb{Z}_2$
MN_{15}	32,1-3,1	4	32	32	$(RTRL)^2, (LRT)^2(LR)^{14}$	$\mathbb{Z}_2$
MN_{16}	32,26-26,26	8	8	8	$(LRT)^2(RL)^2(RT)^2, (LRT)^2(LR)^2(TR)^2$	$\mathbb{Z}_2$
MN_{17}	32,13-18,16	8	16	16	$(RTRL)^2, (LRT)^4(LR)^4$	$\mathbb{Z}_2$
MN_{18}	40,19-21,21	4	5	5		$\mathbb{Z}_2^4$
MN_{19}	48,19-24,21	3	6	8		$\mathbb{Z}_2^2$
MN_{20}	48,70-72,72	4	6	6	$(T(LR)^2)^3$	$\mathbb{Z}_2^2$
MN_{21}	48,76-78,78	6	6	8	$LRTRLRTLRL(RT)^2$	$\mathbb{Z}_2$
MN_{22}	48,73-75,75	8	12	12	$(RT)^2(LR)^2TRLRTL, (LR)^3TL(RL)^2RT$	$\mathbb{Z}_2$
MN_{23}	48,61-63,63	8	24	24	$(RT(RL)^2)^2, (TLR)^3(LR)^3$	$\mathbb{Z}_2$
MN_{24}	48,64-66,66	8	24	24	$(RT(RL)^2)^2, T(LR)^2T(LR)^2LTRLR$	$\mathbb{Z}_2$
MN_{25}	50,1-3,1	4	4	10	$(RTRL)^5$	$\mathbb{Z}_5^2$
MN_{26}	54,10-15,13	4	6	12	$T(LRTR)^3$	$\mathbb{Z}_3$
MN_{27}	54,19-21,21	6	12	12	$L(RT)^2RL(RT)^3, (TLRLR)^3$	$\mathbb{Z}_3$
MN_{28}	64,4-6,4	4	4	8		$\mathbb{Z}_2$
MN_{29}	64,49-54,51	4	8	8	$(LRTR)^2(LR)^2LTRLRT$	$\mathbb{Z}_2$
MN_{30}	64,40-42,42	4	16	16	$(RTRL)^4, (RTRL(RL)^2)^2, (LRT)^4(LR)^4$	$\mathbb{Z}_2$
MN_{31}	64,25-27,25	4	32	32	$(LRT)^2(RL)^2(RT)^2, (LR)^2T(LR)^{14}T$	$\mathbb{Z}_2$
MN_{32}	64,1-3,1	4	64	64	$(RTRL)^2, TLRT(LR)^{31}$	$\mathbb{Z}_2$
MN_{33}	64,58-60,59	8	8	8	$(LRT)^2(LR)^2(TR)^2, T(RTRL)T(RTRL)^3$	$\mathbb{Z}_2$
MN_{34}	64,34-36,34	8	16	16	$(LRTRLRT)^2, (RT)^2RL(RT)^2(RL)^3$	$\mathbb{Z}_2$
MN_{35}	64, 43-45,45	8	16	16	$(LRT)^2(RL)^2(RT)^2, ((RT)^3RL)^2,$ $(LRT)^2(LR)^2T(LR)^3LTR$	$\mathbb{Z}_2$
MN_{36}	64, 7-9,7	8	16	16	$((LR)^2T)^2$	$\mathbb{Z}_2$
MN_{37}	64, 19-24,24	8	32	32	$(RTRL)^2, (LRT)^4(LR)^{12}$	$\mathbb{Z}_2$
MN_{38}	75,7-12,9	3	6	10		$\mathbb{Z}_5^2$
MN_{39}	80,37-39,39	5	5	8	$(LRTR)^2T(LR)^2TRLRT$	$\mathbb{Z}_2$
MN_{40}	80,40-45,42	5	8	10	$(RT(RL)^3)^2, (TLR)^3TR(LR)^2TR$	$\mathbb{Z}_2$
MN_{41}	80,46-48,46,	8	10	10	$(RT)^3(LR)^4TL, (TLR)^3LR(TR)^2LR$	$\mathbb{Z}_2$
MN_{42}	81,1-6,3	3	6	18	$((LR)^2T)^6$	$\mathbb{Z}_3$
MN_{43}	81,28-33,31	6	6	9	$(LRTLR)^2T(LR)^2T, T(LR(TR)^2)^3$	$\mathbb{Z}_3$
MN_{44}	81,22-27,27	6	9	18	$(RT(RL)^2)^2, (LRT)^4RL(RT)^2$	$\mathbb{Z}_3$
MN_{45}	84,28-33,30	3	7	8		PSL(2,7)
MN_{46}	84,49-51,49	3	8	8	$(TLR)^2(LRT)^2(LR)^3LT(RL)^2R$	PSL(2,7)
MN_{47}	84,43-48,44	4	6	8	$T(RTRL)^4, (RT(RL)^2)^3$	PSL(2,7)
MN_{48}	84,37-42,39	4	7	8	$(RTRL)^3$	PSL(2,7)
MN_{49}	84,53-55,55	6	6	8	$(L(TR)^2)^3, (T(LR)^2)^3$	PSL(2,7)
MN_{50}	84,34-36,34	6	7	7	$RTL(RT)^2RL(RT)^2$	PSL(2,7)
MN_{51}	84,52-52,52	8	8	8	$(RTRL)^3, TL(RT)^2LRTRL(TR)^2,$ $(T(LR)^2)^3$	PSL(2,7)
MN_{52}	96,82-87,85	4	6	24	$(LRT)^3(RL)^2TRL(RT)^2$	$\mathbb{Z}_2$
MN_{53}	96,73-78,76	4	12	24	$(LRTLR)^2LTRLRT$	$\mathbb{Z}_2$
MN_{54}	96,184-186,186	6	6	8	$(RTRL)^3, L(RT)^2(LR)^2L(TR)^2T(LR)^2T$	$\mathbb{Z}_2$
MN_{55}	96,187-189,189	6	6	8	$(LRT)^3(RTRL)^2R$	$\mathbb{Z}_2$
MN_{56}	96,178-180,180	8	12	12	$(T(LR)^2)^3, ((RT)^3RL)^2,$ $(RTRL)^4, L(RT)^3(LR)^5T$	$\mathbb{Z}_2$
MN_{57}	96,181-183,183	8	12	12	$((RT)^3RL)^2, (RTRL)^4,$ $T(LR)^2T(RL)^3RTRLR, L(RT)^3(LR)^5T$	$\mathbb{Z}_2$
MN_{58}	96,97-99,99	8	24	24	$L(RT)^2(LR)^2TRLRT,$ $((LR)^3T)^2(LR)^6$	$\mathbb{Z}_2$
MN_{59}	96,64-69,68	8	48	48	$(RT(RL)^2)^2, (LRT)^2RTLRLT(RT)^2,$ $(TLR)^3(LR)^9$	$\mathbb{Z}_2$
MN_{60}	98,1-3,1	4	4	14	$(RTRL)^7$	$\mathbb{Z}_7^2$

Table 4.3: Normally parallel-product indecomposable non-degenerate reflexible maps up to triality and up to 100 edges. The second column is a reference to Wilson census [53]. A triple $e, s-f, n$, denotes that the corresponding map has the code (e, n) in Wilson census, where e denotes the number of edges. The map represents the triality class on maps with codes $(e, s), (e, s+1), \ldots, (e, f)$. A presentation of any of the maps in the table can be obtained by using a presentation $\langle T, L, R \mid T^2 = L^2 = R^2 = (TL)^2 = (RT)^{e_5} = (RL)^{e_6} = (TLR)^{e_7} = \ldots = 1\rangle$, where the corresponding additional relations should be put instead of "…".

Theorem 30. *Up to triality, all normally parallel-product indecomposable non-degenerate reflexible maps up to 100 edges are presented in Table 4.3.* □

Name	Genus symbol	Hex. n.	Underlying graphs
MN_1	$[0, -1, -1]$	3	$K_4, K_4, C_3(2)$
MN_2	$[2, 2, 3]$	3	$C_4(2), K_2(8), K_2(8)$
MN_3	$[-1, -1, -5]$	3	Petersen, K_6, K_6
MN_4	$[1, 1, 1]$	1	$K_{4,4}, K_{4,4}, K_{4,4}$
MN_5	$[3, 3, 5]$	3	$K_{4,4}, C_4(4), C_4(4)$
MN_6	$[4, 4, 7]$	3	$C_8(2), K_2(16), K_2(16)$
MN_7	$[1, -5, -5]$	3	$DK_{3,3,3}, DK_{3,3,3}, K_{3,3}(2)$
MN_8	$[2, 3, -16]$	6	Gen. Petersen $G(8, 3), K_{2,2,2}(2), K_4(4)$
MN_9	$[6, 7, -16]$	6	$Q_3(2), K_{2,2,2}(2), K_4(4)$
MN_{10}	$[1, 1, -11]$	3	Pappus, $K_{3,3,3}, K_{3,3,3}$

Table 4.4: The normally parallel-product indecomposable non-degenerate reflexible maps MN_1 to MN_{10} in detail. A *genus symbol* contains genera of the maps M, $\mathrm{P}(M)$ and $\mathrm{P}(\mathrm{D}(\mathrm{P}(M)))$. Note that the operation dual preserves the genus of a map while the operation Petrie dual preserves its underlying graph. An entry $x \geq 0$ in a genus symbol denotes orientable genus x, while $x < 0$ denotes nonorientable genus $-x$. The *hexagonal number* is the number of non-isomorphic maps in the triality class. Underlying graphs of maps M, $\mathrm{D}(M)$ and $\mathrm{P}(\mathrm{D}(\mathrm{P}(M)))$ are described in the last column. An edge multiplicity $k > 1$ of the underlying graph G is denoted by $G(k)$.

There are exactly 2424 reflexible maps up to 100 edges. Among them, there are 1223 non-degenerate and they are presented in [53]; 229 of non-degenerate are normally parallel-product indecomposable and are obtained from Table 4.3 (calculating whole triality classes). There are 1201 degenerate and slightly degenerate maps, among which 203 are normally parallel-product indecomposable and are obtained from the classification above.

4.1.3 Calculation of reflexible maps up to 500 edges

In the case of reflexible maps the following proposition resolves the problem for order 1024.

Proposition 31. *Every non-degenerate reflexible map M with a monodromy (automorphism) group of order 2^n, $n \geq 2$ is orientable.*

Proof. Non-degenerate maps cannot be degenerate or redundant in the sense of [31]. Therefore it follows that the set of three involutions generating the monodromy (automorphism) group of such reflexible map is a minimal generating set. This means that the Frattini rank of G equals 3. The subgroup $\langle RT, LR\rangle \leq \mathrm{Mon}(M)$ is generated by two generators and cannot generate the whole group. Therefore the index of $\langle RT, LR\rangle$ in $\mathrm{Mon}(M)$ is 2 and M is orientable. $\square$

Each reflexible orientable map is also orientably regular and therefore already determined by the subgroup of orientation preserving automorphisms. All degenerate and slightly degenerate maps, even those of order 1024, are already classified. To obtain all reflexible orientable maps with automorphism group of order 1024 we need to calculate all orientably regular maps with automorphism group of order 512 and filter out chiral maps.

Normally decomposable reflexible maps on groups of order 2^n can be decomposed only into parallel products of maps on groups of order 2^m for $m < n$. Note that the nature of our algorithm for calculating small quotients of finitely presented groups allows us to run the algorithm on 2-group (or p-groups) only.

4.2 G-orbit transitive F-maps

Consider a finite subgroup $G \leq F$ and an F-map M. Note that G-orbits form a system of blocks of imprimitivity for the action of $\mathrm{Aut}(M)$ on Z_M. As we know, $\mathrm{Aut}(M)$ acts semi-regularly on Z_M. Our question is, when does $\mathrm{Aut}(M)$ act transitively on the G-orbits.

Proposition 32. *An F-map M is G-orbit transitive if and only if $N_F(\mathrm{S}_F(M)) \cdot G = F$.*

Proof. M is G-orbit transitive if and only if the G-orbit containing $\underline{\mathrm{id}}$ can be mapped by some automorphism to any other G-orbit. This is true if and only if for any $v \in F$, there exist $g \in G$ and $w \in N_F(\mathrm{S}_F(M))$, such that $\alpha_w(\underline{\mathrm{id}})g = \underline{\mathrm{id}} \cdot v$. But this is equivalent to saying that $\underline{\mathrm{id}} \cdot wg = \underline{\mathrm{id}} \cdot v$.

From here it follows $wgv^{-1} \in \mathrm{S}_F(M) \leq N_F(\mathrm{S}_F(M))$ and $v \in N_F(\mathrm{S}_F(M))g \subseteq N_F(\mathrm{S}_F(M))G$, for every $v \in F$. On the other hand, if $N_F(\mathrm{S}_F(M))G = F$, every

$v \in F$ can be expressed as $v = wg$, for some $w \in N_F(\mathrm{S}_F(M))$ and $g \in G$, implying that $\underline{\mathrm{id}} \cdot wg = \underline{\mathrm{id}} \cdot v$. □

Note that for two subgroups $H, K \leq F$ the product HK is not a group in general. The condition from the proposition implies that every element form F can be written as a product of an element from the normalizer and an element from G.

Let

$$\mathcal{T}_G = \{K : K \leq F, KG = F, K \text{ is a normalizer for some } H \leq F \text{ of a finite index}\}.$$

For $T \in \mathcal{T}_G$, an F-map M is *T-admissible* if and only if $\mathrm{S}_F(M) \lhd T$. Note that this means that $N_F(\mathrm{S}_F(M)) \geq T$. A map is of *type* T if and only if $N_F(\mathrm{S}_F(M)) = T$. For a T-admissible map M, the group $\mathrm{Aut}^T(M) = \{\alpha_w \mid w \in T\} \leq \mathrm{Aut}(M)$ is called the *T-admissible automorphism group* of the map M.

For $T \in \mathcal{T}_G$, $TG = F$ and every element $v \in F$ can be expressed as a pair $v = tg$, $t \in T$, $g \in G$. A conjugation by $w \in G$ is an automorphism of F and maps v to $(w^{-1}tw)(w^{-1}gw) \in (w^{-1}Tw)G$. Therefore, $(w^{-1}Tw)G = F$, for $w \in G$.

Let $w \in F$ and $w = tg$, for some $t \in T$ and $g \in G$. Therefore $w^{-1}Tw = g^{-1}t^{-1}Ttg = g^{-1}Tg$ and thus $(w^{-1}Tw)G = F$, for every $w \in F$. Since $T \in \mathcal{T}_G$, $T = N_F(\mathrm{S}_F(M))$ for some map M (according to the definition of $\mathcal{T}_G$), it follows $w^{-1}Tw = N_F(\mathrm{S}_F(R_w(M)))$. Therefore, if $T \in \mathcal{T}_G$, all conjugates of T are also in $\mathcal{T}_G$. For a group $T \in \mathcal{T}_G$, let $[T]$ denote its conjugacy class. A map is *$[T]$-admissible* if and only if it is T'-admissible for some $T' \in [T]$. Similarly, a map is of *type* $[T]$, if it is of type T', where $T' \in [T]$.

Proposition 33. *Let $G \leq F$ be finite and $T \in \mathcal{T}_G$ and let F_T be any finite presentation of T. Then T-admissible maps are in one-to-one correspondence with regular F_T-maps.*

Proof. T-admissible F-maps are in one-to-one correspondence with normal subgroups of finite index of T and therefore in one-to-one correspondence with regular F_T-maps. □

4.3 Edge-transitive maps

In this section we will discuss edge-transitive maps. These are rooted holey maps or F-maps for $F = \langle T, L, R \mid T^2 = L^2 = R^2 = (TL)^2 = 1\rangle$, which are Q-orbit transitive, for $Q = \langle T, L\rangle$. Note that $|Q| = 4$, since $Q \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

First we have to determine $\mathcal{T}_Q$. Fortunately, this was already done by Graver and Watkins [20] through the study of local automorphisms. Let

$$A = \{T, L, R, TL, RT, RL, LTRT, TRLT, LTR, TRL, LRL, TRT, LTRTL\}$$

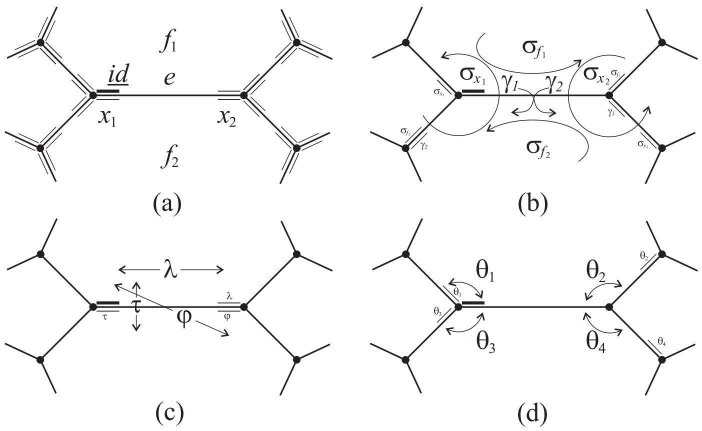
be a set of words in F representing walks from the root flag to some flags in the close neighbourhood of the root. Graver and Watkins [20] give the fundamental classification of edge-transitive maps into 14 types according to possesion of automorphisms α_w, $w \in A$. The types match to exactly 14 conjugacy classes of subgroups of F of index up to 4 admitting the condition of Proposition 32. They mainly work with maps that are not rooted, but for the purpose of analysis they choose a representative rooting, which we will call the *canonical rooting* and which matches to the chosen representative of the conjugacy class from $\mathcal{T}_Q$. In this section, we will operate mainly with canonically rooted maps that will be typed by the chosen representatives. The types in [20] are named by special names and we will use the same names for the corresponding representatives of the conjugacy classes. Each type is determined by a set A_T, which is a subset of A and represents all the elements from A that are in the representative $T \in \mathcal{T}_Q$ and also generate it. Also, α_w, $w \in A_T$ are the automorphisms that generate T-admissible automorphism groups for T-admissible maps.

In Figure 4.2, the automorphisms $\alpha_w, w \in A$, are presented. Graver and Watkins use special names for them and we will continue the tradition. Subsets for the types are defined in Table 4.5. Combining the table in Figure 4.2 and Table 4.5, the sets A_T can be defined for each of the types.

Applying a conjugation by elements from Q on the elements of the sets A_T, one can easily see that for all types except 4, 4^* and 4^P the conjugacy classes contain only one group, while for these three types there are two groups in each of the conjugacy classes. There is a partial ordering relation $\preceq$ on the set of the types defined by $T \preceq T' \Leftrightarrow A_T \subseteq A_{T'}$. The Hasse diagram for this ordering is shown in Figure 4.3. Note that Figure 4.3 actually represents a part of a lattice of subgroups in F, where all groups but 4, 4^* and 4^P are normal.

An interesting observation is the following. The classification by Graver and Watkins [20] can be also easily verified using the computer and MAGMA and a function `LowIndexSubgroups` to calculate all conjugacy classes of subgroups of F of index up to 4 (33 conjugacy classes) and establish that exactly 14 admit the condition of Proposition 32.

An edge-transitive map can have at most two orbits of vertices, faces and Petrie-circuits. The degrees of the vertices in each of the orbits are denoted by a_1, a_2, the sizes of the faces by b_1, b_2, and the sizes of the Petrie circuits by c_1, c_2. Here a_1 is the degree of the vertex adjacent to the root flag, while b_1 and c_1 are the sizes of the face and the Petrie circuit containing the root flag. By $\langle a_1, a_2; b_1, b_2; c_1, c_2\rangle$ we denote the *map symbol*. If a map is vertex-transitive then $a_1 = a_2 = a$ and we reduce the symbol to $\langle a; b_1, b_2; c_1, c_2\rangle$. If $a_1 = a_2$, but the map is not vertex-transitive the symbol is not reduced. A similar rule extends to faces and Petrie circuits.



Definition
$\sigma_{x_1} = \alpha_{RT}$
$\sigma_{x_2} = \alpha_{LTRL}$
$\sigma_{f_1} = \alpha_{LR}$
$\sigma_{f_2} = \alpha_{TRLT}$
$\gamma_1 = \alpha_{LTR}$
$\gamma_2 = \alpha_{TRL}$
$\theta_1 = \alpha_R$
$\theta_2 = \alpha_{LRL}$
$\theta_3 = \alpha_{TRT}$
$\theta_4 = \alpha_{LTRTL}$
$\tau = \alpha_T$
$\lambda = \alpha_L$
$\varphi = \alpha_{LT}$

Figure 4.2: Automorphisms "around" the edge e with the root flag $\underline{\mathrm{id}}$.

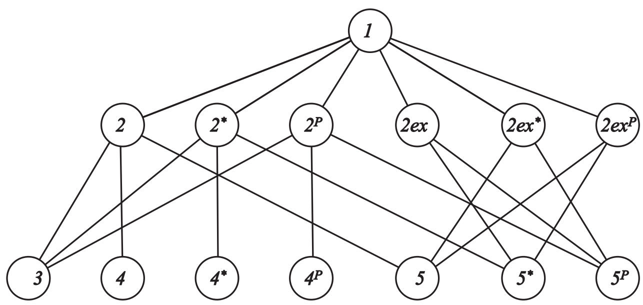


Figure 4.3: The partial ordering of the types of edge-transitive maps.

Consider the impact of the operations D and P on edge-transitive maps. For the purpose of an easier consideration, we denote $1 = 1^* = 1^P$ and $3 = 3^* = 3^P$.

$\mathrm{Type}(M)$	A_M	**Map symbol**	**Comments**
1	τ, λ, φ, σ_{x_1}, σ_{x_2}, σ_{f_1}, σ_{f_2}, γ_1, γ_2, θ_1, θ_2, θ_3, θ_4	$\langle a; b; c\rangle$	
2	τ, σ_{x_1}, σ_{x_2}, θ_1, θ_2, θ_3, θ_4	$\langle a_1, a_2; b; c\rangle$	$2\|b$, $2\|c$
2^*	λ, σ_{f_1}, σ_{f_2}, θ_1, θ_2, θ_3, θ_4	$\langle a; b_1, b_2; c\rangle$	$2\|a$, $2\|c$
2^P	φ, γ_1, γ_2, θ_1, θ_2, θ_3, θ_4	$\langle a; b; c_1, c_2\rangle$	$2\|a$, $2\|b$
2ex	τ, σ_{f_1}, σ_{f_2}, γ_1, γ_2	$\langle a; b; c\rangle$	$a\|2$
2^*ex	λ, σ_{x_1}, σ_{x_2}, γ_1, γ_2,	$\langle a; b; c\rangle$	$b\|2$
2^Pex	φ, σ_{x_1}, σ_{x_2}, σ_{f_1}, σ_{f_2}	$\langle a; b; c\rangle$	$c\|2$
3	θ_1, θ_2, θ_3, θ_4	$\langle a_1, a_2; b_1, b_2; c_1, c_2\rangle$	all even
4	σ_{x_1}, θ_2, θ_4	$\langle a_1, a_2; b; c\rangle$	$2\|a_1$, $2\|a_2$, $4\|b$, $4\|c$
4^*	σ_{f_1}, θ_3, θ_4	$\langle a; b_1, b_2; c\rangle$	$4\|a$, $2\|b_1$, $2\|b_2$, $4\|c$
4^P	γ_1, θ_2, θ_3	$\langle a; b; c_1, c_2\rangle$	$4\|a$, $4\|b$, $2\|c_1$, $2\|c_2$
5	σ_{x_1}, σ_{x_2}	$\langle a_1, a_2; b; c\rangle$	$2\|b$, $2\|c$
5^*	σ_{f_1}, σ_{f_2}	$\langle a; b_1, b_2; c\rangle$	$2\|a$, $2\|c$
5^P	γ_1, γ_2	$\langle a; b; c_1, c_2\rangle$	$2\|a$, $2\|b$

Table 4.5: A classification of edge-transitive maps on 14 types according to the possession of automorphisms around the root flag $\underline{\mathrm{id}}$.

Proposition 34. *Let M be an edge-transitive map of type T. Then $\mathrm{D}(M)$ and $\mathrm{P}(M)$ are also edge-transitive maps. Furthermore, if $T \in \{1, 2, 2\mathrm{ex}, 3, 4, 5\}$ then (by abusing the notation), the types convert as follows:*

$$\begin{array}{lll} \mathrm{D}(T) = T^*, & \mathrm{D}(T^*) = T, & \mathrm{D}(T^P) = T^P, \\ \mathrm{P}(T) = T, & \mathrm{P}(T^*) = T^P, & \mathrm{P}(T^P) = T^*. \end{array}$$

Proof. For a map M, $\mathrm{S}_F(\mathrm{D}(M)) = \mathbf{d}^{-1}(\mathrm{S}_F(M)) = \mathbf{d}(\mathrm{S}_F(M))$, where $\mathbf{d} \in \mathrm{Aut}(F)$ and $\mathrm{D} \equiv O_{\mathbf{d}}$. Note that $\mathbf{d}(Q) = Q$. Similar is true for $\mathbf{p} \in \mathrm{Aut}(M)$, where $\mathrm{P} \equiv O_{\mathbf{p}}$. Therefore, if $T \in \mathcal{T}_Q$, then $\mathbf{d}(T), \mathbf{p}(T) \in \mathcal{T}_Q$. Now we will use the part of subgroup lattice in Figure 4.3. All subgroups but 4, 4^* and 4^P are normal. Since $\mathbf{d}$ and $\mathbf{p}$ are isomorphisms and permute conjugacy classes in $\mathcal{T}_Q$, they must respect the partial ordering in the subgroup lattice, or in our case, in the Hasse diagram in the figure. It is easy to see from the diagram, that if we prove the rules claimed in the proposition on types 2, 2^* and 2^P, all the rules for all other transformations on all types follow. This is due to the properties of the diagram. Recall that the automorphisms $\mathbf{d}$ and $\mathbf{p}$ are defined on the set of generators of F by

$\mathbf{d} : T \mapsto L, L \mapsto T, R \mapsto R$ and $\mathbf{p} : T \mapsto T, L \mapsto TL, R \mapsto R$. But since T is only in 2, L is only in 2^* and TL is only in 2^P and according to diagram $\mathbf{d}$ and $\mathbf{p}$ can only permute the three types, the rules claimed in the proposition follow. $\square$

The corollary immediately follows.

Corollary 35. *Each edge-transitive map can be obtained from some map of type 1, 2, 2ex, 3, 4 or 5 by one of 6 possible compositions of the operations* D *and* P. $\square$

Therefore we can focus on these 6 types only.

From the classification in [20, 44] the partial presentations of automorphism groups for edge-transitive maps can be extracted. They are shown in Table 4.6.

Type	A partial presentation for a given map symbol.
1	$\langle \tau, \lambda, \theta_1 \mid \underline{\tau^2, \lambda^2, {\theta_1}^2, (\tau\lambda)^2}, (\theta_1\tau)^a, (\lambda\theta_1)^b, (\tau\lambda\theta)^c, \ldots \rangle$
2	$\langle \tau, \theta_1, \theta_2 \mid \underline{\tau^2, \theta_1^2, \theta_2^2}, (\theta_1\tau)^{a_1}, (\tau\theta_2)^{a_2}, (\theta_2\theta_1)^{\frac{b}{2}}, (\tau\theta_2\tau\theta_1)^{\frac{c}{2}}, \ldots \rangle$
2ex	$\langle \tau, \sigma_{f_1} \mid \underline{\tau^2}, (\sigma_{f_1}^{-1}\tau\sigma_{f_1}\tau)^{\frac{a}{2}}, \sigma_{f_1}^b, (\tau\sigma_{f_1})^c, \ldots \rangle$
3	$\langle \theta_1, \theta_2, \theta_3, \theta_4 \mid \underline{\theta_1^2, \theta_2^2, \theta_3^2, \theta_4^2}, (\theta_1\theta_3)^{\frac{a_1}{2}}, (\theta_4\theta_2)^{\frac{a_2}{2}}, (\theta_2\theta_1)^{\frac{b_1}{2}},$ $(\theta_3\theta_4)^{\frac{b_2}{2}}, (\theta_4\theta_1)^{\frac{c_1}{2}}, (\theta_3\theta_2)^{\frac{c_2}{2}}, \ldots \rangle$
4	$\langle \sigma_{x_1}, \theta_2, \theta_4 \mid \underline{\theta_2^2, \theta_4^2}, \sigma_{x_1}^{a_1}, (\theta_4\theta_2)^{\frac{a_2}{2}}, (\sigma_{x_1}\theta_4\sigma_{x_1}^{-1}\theta_2)^{\frac{b}{4}}, (\sigma_{x_1}^{-1}\theta_4\sigma_{x_1}\theta_2)^{\frac{c}{4}}, \ldots \rangle$
5	$\langle \sigma_{x_1}, \sigma_{x_2} \mid \sigma_{x_1}^{a_1}, \sigma_{x_2}^{a_2}, (\sigma_{x_1}\sigma_{x_2})^{\frac{b}{2}}, (\sigma_{x_1}\sigma_{x_2}^{-1})^{\frac{c}{2}}, \ldots \rangle$

Table 4.6: Partial presentations for automorphism groups of types 1, 2, 2ex, 3, 4, 5.

Note that the values of map symbols are used in presentations to denote partial presentations of maps having the prescribed map symbol. The relations, that are independent of a specific map symbol and therefore are present in any partial presentation of the corresponding type, are underlined in Table 4.6. The generators and those relations alone determine the *universal automorphism group* for the corresponding type T, denoted by F_T.

Not surprisingly, F_T is just a partial presentation for the group $T \in \mathcal{T}_G$. Namely, $F_T = \langle \alpha_w : w \in B_T \rangle$, where $B_T \subseteq A_T$ is the chosen generating set for T according to Table 4.6.

Let $s : F_T \to T$ be the isomorphism mapping $\alpha_w \mapsto w$, $w \in F$. For a T-admissible map M, consider the isomorphism $\Phi : \mathrm{Aut}(M) \to N_F(\mathrm{S}_F(M))/\mathrm{S}_F(M)$

from Theorem 6. This isomorphism induces a group epimorphism $\Lambda : N_F(\mathrm{S}_F(M)) \to \mathrm{Aut}(M)$ defined by $\Lambda : w \mapsto \alpha_w$, with the kernel $\mathrm{S}_F(M)$ and the image of T is exactly $\mathrm{Aut}^T(M)$. Let $f = \Lambda \circ s$, $f : F_T \to \mathrm{Aut}^T(M)$. Then $F_T/\ker f \simeq \mathrm{Aut}^T(M)$ where the isomorphism is given by the rule $w \mapsto \alpha_w$, $w \in F_T/\ker f$ and $F_T/\ker f$ is viewed as a finitely presented quotient of F_T. This is exactly the correspondence from Proposition 33.

To determine all T-admissible maps up to some order, we can first determine all regular F_T-maps, where F_T is some finite presentation of T and use the correspondence above to actually construct T-admissible F-maps.

Beside the construction described above there is also a direct construction of the monodromy group of an F-map that corresponds to a given regular F_T-map.

Proposition 36. *For an F_T-map group G of a regular F_T-map, Table 4.7 gives a definition of a new set of flags and three involutions defining the monodromy group of a T-admissible map corresponding to G by Proposition 33.*

Proof. Consider a T-admissible F-map M, where G is its T-admissible subgroup. Since G is edge-transitive, it can have at most 4 orbits on the flags and a subgroup Q acts on the set of the orbits transitively.

Since Q is a small group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, one can by analyzing all transitive actions of Q verify that there is always a subgroup $S \leq Q$, such that S acts regularly on the set of the orbits. Each flag x can be uniquely labeled by a pair (α, w), $\alpha \in G$ and $w \in S$, such that $x = \alpha(\underline{\mathrm{id}}) \cdot w$. To see that, let $x = \alpha_1(\underline{\mathrm{id}}) \cdot w_1 = \alpha_2(\underline{\mathrm{id}}) \cdot w_2$ for some $\alpha_1, \alpha_2 \in G$ and $w_1, w_2 \in S$. This would imply $\alpha_2^{-1}(\alpha_1(\underline{\mathrm{id}})) \cdot w_1 w_2^{-1} = \underline{\mathrm{id}}$. Since $\underline{\mathrm{id}}$ is in the same orbit as $\alpha_2^{-1}(\alpha_1(\underline{\mathrm{id}}))$ and S acts regularly on the orbits, it first follows $w_1 = w_2$ and then by semi-regularity of G it follows $\alpha_1 = \alpha_2$. Thus the labelling is unique and any edge-transitive map corresponds to the unique labelling $G \times S$.

The unique labelling alone already determines the map, since a label (α_w, v) corresponds to the flag $\underline{\mathrm{id}} \cdot wv$. From this information it is straightforward to calculate the actions of the involutions T, L and R on the flags with the labels of the form (Id, w), $w \in S$. Since for $W \in \{T, L, R\}$, $x = (\alpha, w)$, $\alpha \in \mathrm{Aut}(M)$, it follows $x \cdot W = \alpha(\alpha^{-1}(x) \cdot W)$, the map is uniquely determined by the labelling.

Since the existence of the T-admissible map corresponding to the F_T-map group G is assured by Proposition 33, all we need is to find the labelling and define the local action of the involutions on the labels like in the paragraph above.

Type	**Flags**	T	L	R
1	G	$g \cdot T = g\tau$	$g \cdot L = g\lambda$	$g \cdot R = g\theta_1$
2	$G \times \mathbb{Z}_2$	$(g, j) \cdot T =$ $(g\tau, j)$	$(g, j) \cdot L =$ $(g, j + 1)$	$(g, 0) \cdot R = (g\theta_1, 0)$ $(g, 1) \cdot R = (g\theta_2, 1)$
2ex	$G \times \mathbb{Z}_2$	$(g, j) \cdot T =$ $(g\tau, j)$	$(g, j) \cdot L =$ $(g, j + 1)$	$(g, 0) \cdot R = (g\sigma_{f_1}^{-1}, 1)$ $(g, 1) \cdot R = (g\sigma_{f_1}, 0)$
3	$G \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$(g, j, k) \cdot T =$ $(g, j + 1, k)$	$(g, j, k) \cdot L =$ $(g, j, k + 1)$	$(g, 0, 0) \cdot R = (g\theta_1, 0, 0)$ $(g, 0, 1) \cdot R = (g\theta_2, 0, 1)$ $(g, 1, 0) \cdot R = (g\theta_3, 1, 0)$ $(g, 1, 1) \cdot R = (g\theta_4, 1, 1)$
4	$G \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$(g, j, k) \cdot T =$ $(g, j + 1, k)$	$(g, j, k) \cdot L =$ $(g, j, k + 1)$	$(g, 0, 0) \cdot R = (g\sigma_{x_1}, 1, 0)$ $(g, 0, 1) \cdot R = (g\theta_2, 0, 1)$ $(g, 1, 0) \cdot R = (g\sigma_{x_1}^{-1}, 0, 1)$ $(g, 1, 1) \cdot R = (g\theta_4, 1, 1)$
5	$G \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$(g, j, k) \cdot T =$ $(g, j + 1, k)$	$(g, j, k) \cdot L =$ $(g, j, k + 1)$	$(g, 0, 0) \cdot R = (g\sigma_{x_1}, 1, 0)$ $(g, 0, 1) \cdot R = (g\sigma_{x_2}^{-1}, 1, 1)$ $(g, 1, 0) \cdot R = (g\sigma_{x_1}^{-1}, 0, 0)$ $(g, 1, 1) \cdot R = (g\sigma_{x_2}, 0, 1)$

Table 4.7: A construction of the corresponding T-admissible map from a partially presented group G of type T.

Note that S is determined by the type of the map. If the type is 1,2, 2ex, 3, 4, 5, then, according to [20, 44], the corresponding groups S are: $\{1\}$, $\langle L\rangle$, $\langle L\rangle$, Q, Q and Q, respectively.

For a given finitely presented F_T-map group G representing some regular F_T-map, the unique labelling $G \times S$ is therefore easily obtained. In Table 4.7 is modelled by a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

For $\alpha, \beta \in G$, $w \in S$, $x = (\alpha, w)$. Define the right action $\beta(x) = \beta((\alpha, w)) = (\beta \circ \alpha, w)$. It is easy to see that this is exactly the action of G as a T-admissible automorphism group on the flags (labels). $\square$

4.4 Map operations on edge-transitive maps and type

In this section, we will investigate some of operations on F_T-maps and characterize through them when a regular F_T-map represents a T-admissible edge-transitive F-map of exact type T.

Using Proposition 15 and definitions of the types, the obvious proposition follows.

Proposition 37. *1. A 2-admissible or 2ex^P-admissible F-map M is 1-admissible, if and only if $R_L(M) \simeq M$.*

2. A 3-admissible F-map is

(a) 2-admissible if and only if $R_T(M) \simeq M$.

(b) 2^*-admissible if and only if $R_L(M) \simeq M$.

(c) 2^P-admissible if and only if $R_{TL}(M) \simeq M$.

3. A 4-admissible map is 2-admissible if and only if $R_T(M) \simeq M$.

4. A 5-admissible map is

(a) 2-admissible if and only if $R_T(M) \simeq M$.

(b) 2^*ex-admissible if and only if $R_L(M) \simeq M$.

(c) 2^Pex-admissible if and only if $R_{TL}(M) \simeq M$.

□

Let $T \in \mathcal{T}_Q$ be a representative of a conjugation class determining the type T. When the conjugation by $w \in Q$ preserves T, it defines an automorphism of T which is not necessarily inner, if $w \notin T$. Table 4.8 defines the operations on F_T-maps that are induced by the automorphisms of F_T which arise from conjugations in F with elements $w \in Q$. Note that the conjugations of T by Q correspond to re-rootings when a map of type T is viewed as an F-map (or as a holey map).

The following example shows an application of Proposition 37 and operations in Table 4.8.

Example 2. Consider an F_T-map M and let N be its corresponding edge-transitive map (F-map). We would like to determine whether N is of exact type T or only T-admissible. From Proposition 37 it follows that N is of type T if and only if it

Type	**Gens. of** F_T	R_T	R_L	R_{TL}
2	$(\tau, \theta_1, \theta_2)$	–	$(\tau, \theta_2, \theta_1)$	–
2ex	(τ, σ_{f_1})	–	$(\tau, \sigma_{f_1}^{-1})$	–
3	$(\theta_1, \theta_2, \theta_3, \theta_4)$	$(\theta_3, \theta_4, \theta_1, \theta_2)$	$(\theta_2, \theta_1, \theta_4, \theta_3)$	$(\theta_4, \theta_3, \theta_2, \theta_1)$
4	$(\sigma_{x_1}, \theta_2, \theta_4)$	$(\sigma_{x_1}^{-1}, \theta_4, \theta_2)$	–	–
5	$(\sigma_{x_1}, \sigma_{x_2})$	$(\sigma_{x_1}^{-1}, \sigma_{x_2}^{-1})$	$(\sigma_{x_2}^{-1}, \sigma_{x_1}^{-1})$	$(\sigma_{x_2}, \sigma_{x_1})$

Table 4.8: The table defines the automorphisms on types $T \in \mathcal{T}_Q$ induced by conjugations of T by elements of Q, for the cases we are interested in. The automorphisms defining the operations are given in terms of images of generators of F_T (see Table 4.6). Note that conjugations by T and TL do define automorphisms on types 2 and 2ex, but the definitions are omitted.

is not isomorphic to $R_W(N)$, where W is one of T, L, TL, depending on type. In Table 4.8 find the operation O on F_T-maps that corresponds to the operation R_W on edge-transitive maps (F-maps). Check whether $O(M) \simeq M$. If this is the case, then N is only T-admissible, otherwise it is of type T. ■

4.5 Non-degenerate edge-transitive maps

The purpose of this section is to present all non-degenerate edge-transitive maps on orientable and non-orientable surfaces of small genus. An edge-transitive map is non-degenerate if the map symbol contains all the values greater or equal to 3. We will show that automorphism groups of edge-transitive maps are bounded for surfaces of orientable genus $g > 1$ and non-orientable genus $g > 2$ and therefore only a finite number of edge-transitive maps can occur on those surfaces.

All edge-transitive maps on the torus were classified by Širáň, Tucker and Watkins [44], the classification for the sphere was done by Grünbaum and Shephard [22], while a part of the classification for the Klein bottle was done by Potočnik and Wilson [41].

There are 9 non-degenerate edge-transitive maps on the sphere, namely, cuboctahedron, icosidodecahedron, their duals and 5 Platonic solids.

Up to author's knowledge, edge-transitive maps on the projective plane are not classified yet.

For a non-degenerate map M the *Euler characteristic* of the underlying surface

$S(M)$ of a (non-)orientable genus g is

$$\chi(S(M)) = |V| - |E| + |F| = \begin{cases} 2-2g, & M \text{ is orientable,} \\ 2-g, & M \text{ is not orientable.} \end{cases}$$

In the case of edge-transitive maps, $\mathrm{Aut}(M) = k|E|$, where $k \in \{1,2,4\}$. If an edge-transitive map is vertex-transitive, then all the vertices have the same degree a and $|V| = \frac{2|E|}{a}$. Similarly, if there is only one orbit on faces of size b, $|F| = \frac{2|E|}{b}$. If there are two orbits on vertices with degrees a_1 and a_2 having v_1 and v_2 vertices, respecitvely, then $|V| = v_1 + v_2 = \frac{|E|}{a_1} + \frac{|E|}{a_2}$. Similarly, if a map is not not face-transitive, we have $|F| = \frac{|E|}{b_1} + \frac{|E|}{b_2}$, where b_1 and b_2 are the sizes of faces in each orbit. Combining the Euler formula and the facts from this paragraph we obtain

$$|\mathrm{Aut}(M)| = \frac{k(2-sg)}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{b_1} + \frac{1}{b_2} - 1}, \quad k \in \{1,2,4\}, s \in \{1,2\}, \tag{4.1}$$

where k depends on type of M and s on its orientability. Note that for vertex-transitive maps we use $a = a_1 = a_2$, while for face-transitive maps $b = b_1 = b_2$. In an orientable case, for $g > 1$, and in non-orientable case, for $g > 2$, the nominator of the right side of the equation is negative and therefore the denominator must be also negative. The size of the automorphism group is maximal possible for given k and s, if the value of the denominator is maximal possible but still negative.

The following lemma tells us what are the maximal values of the denominator at given conditions.

Lemma 38. *Let* $X(a_1, a_2, b_1, b_2) = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{b_1} + \frac{1}{b_2} - 1$.

1. *Let* $a = a_1 = a_2$, $b = b_1 = b_2$, *let* $a, b \geq 3$ *and* $X < 0$. *Subject to these conditions, the maximum of* X *is* $-\frac{1}{21}$ *at* $(a,b) \in \{(3,7),(7,3)\}$.

2. *Let* $a = a_1 = a_2$, $b = b_1 = b_2$, *let* $a, b \geq 4$, *both even and* $X < 0$. *Subject to these conditions, the maximum of* X *is* $-\frac{1}{6}$ *at* $(a,b) \in \{(4,6),(6,4)\}$.

3. *Let* $a = a_1 = a_2$, $b = b_1 = b_2$, *let* $a \geq 3$ *and* $b \geq 4$, *where* $2|b$ *and* $X < 0$. *Subject to these conditions, the maximum of* X *is* $-\frac{1}{12}$ *at* $(a,b) = (3,8)$.

4. *Let* $a = a_1 = a_2$, $b = b_1 = b_2$, *let* $a, b \geq 4$, *both divisible by 4 and* $X < 0$. *Subject to these conditions, the maximum of* X *is* $-\frac{1}{4}$ *at* $(a,b) \in \{(4,8),(8,4)\}$.

5. *Let* $a_1, a_2 \geq 3$ *and* $b = b_1 = b_2$*, where* $b \geq 4$, $2|b$ *and* $X < 0$*. Subject to these conditions the maximum of* X *is* $-\frac{1}{42}$ *for* $(a_1, a_2, b) \in \{(3,7,4),(7,3,4)\}$.

6. *Let* $a = a_1 = a_2$*, where* $4|a$*; let* $b_1, b_2 \geq 4$ *and* $2|b_1$, $2|b_2$ *and* $X < 0$*. Subject to these conditions, the maximum of* X *is* $-\frac{1}{12}$ *for*

$$(a, b_1, b_2) \in \{(4,4,6),(4,6,4)\}.$$

7. *Let all* $a_1, a_2, b_1, b_2 \geq 4$ *and even. Let* $X < 0$*. Subject to these conditions, the maximum of* X *is* $-\frac{1}{12}$ *for*

$$(a_1, a_2, b_1, b_2) \in \{(6,4,4,4),(4,6,4,4),(4,4,6,4),(4,4,4,6)\}.$$

Proof. 1. We may assume that $a \leq b$. If both $a, b \geq 5$, then we cannot get more than $-\frac{1}{5}$ at $(a,b) = (5,5)$. If $a = 3$, the best possible choice is $b = 7$, yielding $-\frac{1}{21}$, while if $a = 4$, the best possible choice is $b = 5$, yielding $-\frac{1}{10}$.

2. We may assume that $a \leq b$. Not both a and b can be 4. If $a, b \geq 6$, we cannot get more than $-\frac{1}{3}$ at $(a,b) = (6,6)$. For $a = 4$, the optimal value is $b = 6$, yielding $-\frac{1}{6}$.

3. If $a \geq 5$ and $b \geq 6$ we cannot get more than $-\frac{4}{15}$ at $(a,b) = (5,6)$. If $a < 5$, then the best choices for (a,b) are $(3,8)$, $(4,6)$, yielding $-\frac{1}{12}$ and $-\frac{1}{6}$, respectively. If $b < 6$, then $b = 4$ and the best option is $a = 5$, yielding $-\frac{1}{10}$.

4. We may assume that $a \leq b$. Not both a, b can equal 4. Therefore the best possibility is $b = 8$.

5. We may assume that $a_1 \leq a_2$. If $a_1, a_2 \geq 5$, then we cannot get more than $-\frac{1}{10}$. If $a_1 = 3$ and $a_2 \geq 7$ we cannot get more better than $-\frac{1}{42}$ for $(a_1, a_2, b) = (3,7,4)$. For the remaining possibilities for (a_1, a_2): $(3,3)$, $(3,4)$, $(3,5)$ and $(3,6)$ the optimal choices of b are 8, 6, 6, 6, yielding $-\frac{1}{12}$, $-\frac{1}{12}$, $-\frac{2}{15}$, $-\frac{1}{6}$, respectively. If $a_1 = 4$ and $a_2 \geq 6$, we cannot get more than $-\frac{1}{12}$ in the case $a_2 = 6$ and $b = 4$. In the remaining cases, where (a_1, a_2): $(4,4)$ and $(4,5)$, the optimal choices for b are 6 and 4 yielding $-\frac{1}{20}$ and $-\frac{1}{6}$, respectively.

6. We may assume that $b_1 \leq b_2$. If $a \geq 8$, then we cannot get more than $-\frac{1}{4}$ at $(a, b_1, b_2) = (8,4,4)$. If $a = 4$, then the optimal value occurs at $(4,4,6)$, namely $-\frac{1}{12}$.

7. Since the combination $(a_1, a_2, b_1, b_2) = (4, 4, 4, 4)$ is the only one making X non-negative, the best choice is to have 3 parameters equal to 4 and the remaining equal to 6. □

Recall that an F-map M is orientable if and only if the action of F^+ on cosets $F/\mathrm{S}_F(M)$ has 2 orbits. Since the index of F^+ in F is 2, M is non-orientable if and only if $\mathrm{S}_F(M) \leq F^+$. Note that F^+ is exactly $2\mathrm{ex}^P \in \mathcal{T}_Q$. Therefore, according to the Hasse diagram in Figure 4.3, the proposition follows.

Proposition 39. *Edge-transitive maps of types $2\mathrm{ex}^P$, 5 and 5^* are necessarily orientable. Let M be an orientable edge-transitive F-map.*

1. *If M is 1-admissible then it is also $2\mathrm{ex}^P$-admissible.*
2. *If M is 2-admissible then it is also 5-admissible.*
3. *If M is 2^*-admissible then it is also 5^*-admissible.*

□

From this point of view, the types are just an algebraic generalization of the concept of orientability on a map.

Combining the information about values of a map symbol in Table 4.5, Lemma 38, and the equation (4.1) the proposition follows.

Proposition 40. *Let M be a non-degenerate edge-transitive map of type T embedded on a compact closed surface of orientable genus $g > 1$ or non-orientable genus $g > 2$. Then the maximal possible order of $\mathrm{Aut}(M)$ is bounded from above and the bounds are shown in Table 4.9.* □

Using the algorithm for calculation of small quotients we calculated all small quotients for groups F_T, where T is 1, 2, 2ex, 3, 4 and 5 up to orders 2000, 767, 1942, 100, 200, and 255, respectively. All those F_T-maps are available on [53]. Using Table 4.9 we can determine all edge-transitive maps on surfaces up to some genus. The tables with the results are shown below. Note that the maps of genera (orientable or non-orientable) larger than 30 are omitted from tables for types with large numbers of maps. Maps of types T^* are not shown in tables, since they are obtained as duals of maps of types T.

Type	Orientable $\max \lvert \mathrm{Aut}(M) \rvert$	Non-orientable $\max \lvert \mathrm{Aut}(M) \rvert$
1	$168(g-1)$	$84(g-2)$
2	$168(g-1)$	$84(g-2)$
2^*	$168(g-1)$	$84(g-2)$
2^P	$24(g-1)$	$12(g-2)$
2ex	$48(g-1)$	$24(g-2)$
2ex*	$48(g-1)$	$24(g-2)$
2ex^P	$84(g-1)$	–
3	$24(g-1)$	$12(g-2)$
4	$24(g-1)$	$12(g-2)$
4^*	$24(g-1)$	$12(g-2)$
4^P	$8(g-1)$	$4(g-2)$
5	$84(g-1)$	–
5^*	$84(g-1)$	–
5^P	$12(g-1)$	$6(g-2)$

Table 4.9: The upper bounds on orders of automorphism groups of non-degenerate edge-transitive maps on compact closed surfaces of orientable genus $g > 1$ and non-orientable genus $g > 2$.

Unfortunately, we were not able to calculate all reflexible (type 1) non-orientable maps on any of surfaces of genera higher then 30. A reader interested in those maps can see [13].

For orientable reflexible maps we were able to improve the bound in [13] and calculate all those maps on surfaces up to orientable genus 24. We were also able to improve the bound for chiral maps (type 2ex^P) and calculate all chiral maps on surfaces of genus up to 24. But as already mentioned in the introduction of the theses, M. Conder [14] recently improved the result significantly.

The id numbers of maps refer to the libraries of maps which can be obtained from [53]. In those libraries only one representative from each triality class of edge-transitive maps is stored.

Tables for types 1 and 3 are organized as follows. For a chosen line (i.e. map M), the first column contains the *id number* of the representative of the triality class in which the map is. The second column describes what composition of the operations D and P constructs the map from the representative. The third column denotes the genus of the map and the fourth contains the number of the edges. The column *Trial.* contains the information about the number of maps in the triality class. Let R be the representative of the triality class of the map M. In this column D means that R is self-dual, P means that R is self-Petrie, PD means that $\mathrm{P}(\mathrm{D}(R)) \simeq R$ and PDP means $\mathrm{P}(\mathrm{D}(\mathrm{P}(R))) \simeq R$. If both D and P appear in the column, then the representative of the triality class is self-dual and self-Petrie and the triality class contains only one map. The last column denotes the map symbol of the map M.

Example 3. For instance, consider the first type 3 map of non-orientable genus 8. The id number of the representative R of the triality class is 13. One can use the library of type 3 maps from [53] to obtain the presentation of the representative in MAGMA. The map M which is represented by that line is $M = \mathrm{P}(\mathrm{D}(R))$, since in the entry in the second column equals PD. The genus of the map is 8 (3rd column), the number of the edges is 16 (4th column) and the representative R is self-Petrie (5th column). Therefore the triality class of M contains 3 maps and the pairs $(R, \mathrm{P}(R))$, $(\mathrm{D}(R), \mathrm{D}(\mathrm{P}(R)))$ and $(M = \mathrm{P}(\mathrm{D}(R)), \mathrm{P}(\mathrm{D}(\mathrm{P}(R))))$ are isomorphic pairs and M is self-Petrie. The last column denotes the map symbol of M, which is $\langle 4, 16; 4, 16; 4, 4\rangle$. ■

Tables for types 2, 2ex, 4 and 5 and the types obtained from those by triality are organized in a different way. For a type T (one of the types above), there are up to 4 tables, namely for type T orientable maps, type T non-orientable, type T^P orientable and type T^P non-orientable. Not all combinations are possible; for instance there are no non-orientable maps of type $2\mathrm{ex}^P$. The maps of type T^* at a given genus are exactly the duals of the maps T at the genus and therefore no tables for types T^* are needed. All the columns in the tables have the same meaning that in tables for types 1 and 3 – the only exception is the 5th column. For type T this column tells us whether the map is self-Petrie. If it is, then the triality class contains 3 maps; otherwise it contains 6 maps. The 5th column for types T^P denotes whether the maps is self-dual. Note that if such a map is self dual, its dual is also of type T^P and has the same genus. In this case only one of maps is present in tables.

Example 4. For example, let us answer the question: *What are the sizes of the faces of the largest map of type 2^* of orientable genus 4?*

Since maps of type 2^* are obtained from the maps of type 2, we have to check the table for orientable maps of type 2. The largest such map of genus 4 has 120 edges, it is obtained as the map with id number 4469 from the library for type 2 [53], it is not self-Petrie (the triality class has 6 maps) and its map symbol is $\langle 4, 5; 4; 12\rangle$. The dual of this map is the map we are looking for. The map symbol of the dual is $\langle 4; 4, 5; 12\rangle$. Therefore the sizes of the faces are 4 and 5. ■

4.5.1 Type 1 (reflexible) orientable maps

The reader is referred to read previous two pages to understand the organization of the following tables.

Type 1 - orientable genus 2 to 24

The reader may compare this list with the census in [13] and establish that some of maps are missing here. These are exaclty the degenerate ones – the ones with Petrie circuits of size 2. Non-orientable maps of type 1 are not listed here (see [13, 14]).

Id	Oper.	Genus	$\lvert E\rvert$	Trial.	Map symb.
3	P	2	8	D	$\langle 8; 4; 8\rangle$
5	P	2	12		$\langle 6; 4; 12\rangle$
25	P	2	24		$\langle 3; 8; 12\rangle$
3	–	3	8	D	$\langle 8; 8; 4\rangle$
5	PD	3	12		$\langle 12; 4; 6\rangle$
7	–	3	12	D	$\langle 6; 6; 4\rangle$
9	–	3	16	P	$\langle 4; 8; 8\rangle$
11	P	3	16	D	$\langle 8; 4; 8\rangle$
22	–	3	24	P	$\langle 4; 6; 6\rangle$
25	–	3	24		$\langle 3; 12; 8\rangle$
77	–	3	48		$\langle 3; 8; 6\rangle$
194	–	3	84		$\langle 3; 7; 8\rangle$
5	–	4	12		$\langle 6; 12; 4\rangle$
12	P	4	16	D	$\langle 16; 4; 16\rangle$
13	–	4	18	D	$\langle 6; 6; 6\rangle$
13	P	4	18	D	$\langle 6; 6; 6\rangle$
16	–	4	20		$\langle 4; 10; 20\rangle$
34	–	4	30	D	$\langle 5; 5; 6\rangle$
50	P	4	36	D	$\langle 4; 6; 4\rangle$
52	–	4	36		$\langle 3; 12; 6\rangle$
112	P	4	60		$\langle 5; 4; 6\rangle$
8	PD	5	15		$\langle 15; 6; 10\rangle$
9	PD	5	16	P	$\langle 8; 8; 4\rangle$
11	–	5	16	D	$\langle 8; 8; 4\rangle$
16	P	5	20		$\langle 4; 20; 10\rangle$
21	–	5	24	P	$\langle 4; 12; 12\rangle$
22	PD	5	24	P	$\langle 6; 6; 4\rangle$
37	P	5	32	D	$\langle 8; 4; 8\rangle$
41	–	5	32	PDP	$\langle 4; 8; 4\rangle$
58	–	5	40	D	$\langle 5; 5; 4\rangle$
72	P	5	48		$\langle 4; 6; 12\rangle$
107	–	5	60	P	$\langle 3; 10; 10\rangle$
170	P	5	80		$\langle 5; 4; 10\rangle$
237	–	5	96		$\langle 3; 8; 12\rangle$
8	–	6	15		$\langle 10; 15; 6\rangle$
14	–	6	18	D	$\langle 9; 9; 4\rangle$
18	P	6	24		$\langle 6; 8; 24\rangle$
20	–	6	24	P	$\langle 4; 24; 24\rangle$
26	P	6	24		$\langle 6; 8; 12\rangle$

Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Trial.	Map symb.
27	–	6	25	P	$\langle 5; 10; 10\rangle$
30	–	6	28		$\langle 4; 14; 28\rangle$
48	P	6	36		$\langle 9; 4; 18\rangle$
113	–	6	60		$\langle 4; 6; 10\rangle$
163	–	6	75		$\langle 3; 10; 6\rangle$
12	–	7	16	D	$\langle 16; 16; 4\rangle$
17	–	7	21		$\langle 6; 21; 14\rangle$
26	–	7	24		$\langle 6; 12; 8\rangle$
28	–	7	27		$\langle 6; 9; 18\rangle$
30	P	7	28		$\langle 4; 28; 14\rangle$
36	–	7	32	P	$\langle 4; 16; 16\rangle$
39	–	7	32	P	$\langle 4; 16; 16\rangle$
156	–	7	72		$\langle 3; 12; 24\rangle$
1047	–	7	252		$\langle 3; 7; 18\rangle$
16	PD	8	20		$\langle 10; 20; 4\rangle$
18	–	8	24		$\langle 6; 24; 8\rangle$
19	P	8	24		$\langle 12; 8; 24\rangle$
31	P	8	30		$\langle 10; 6; 30\rangle$
43	–	8	32	P	$\langle 4; 32; 32\rangle$
51	P	8	36		$\langle 4; 18; 36\rangle$
567	P	8	168		$\langle 3; 8; 14\rangle$
575	–	8	168	P	$\langle 3; 8; 8\rangle$
17	PD	9	21		$\langle 21; 14; 6\rangle$
18	PD	9	24		$\langle 24; 8; 6\rangle$
19	PD	9	24		$\langle 24; 8; 12\rangle$
21	PD	9	24	P	$\langle 12; 12; 4\rangle$
23	–	9	24	D	$\langle 12; 12; 4\rangle$
24	–	9	24	D	$\langle 12; 12; 4\rangle$
35	–	9	32	P	$\langle 8; 8; 8\rangle$
35	PD	9	32	P	$\langle 8; 8; 8\rangle$
37	–	9	32	D	$\langle 8; 8; 4\rangle$
38	–	9	32	D, P	$\langle 8; 8; 8\rangle$
42	–	9	32	D, P	$\langle 8; 8; 8\rangle$
51	–	9	36		$\langle 4; 36; 18\rangle$
57	–	9	40	P	$\langle 4; 20; 20\rangle$
66	–	9	48	P	$\langle 4; 12; 12\rangle$
68	P	9	48	D	$\langle 12; 4; 12\rangle$
72	–	9	48		$\langle 4; 12; 6\rangle$
78	–	9	48	D	$\langle 6; 6; 4\rangle$
81	P	9	48	PDP	$\langle 6; 6; 8\rangle$
108	P	9	60		$\langle 5; 6; 10\rangle$
112	–	9	60		$\langle 5; 6; 4\rangle$
122	P	9	64	D	$\langle 8; 4; 8\rangle$
131	P	9	64	D	$\langle 4; 8; 4\rangle$
132	P	9	64		$\langle 8; 4; 8\rangle$
132	PD	9	64		$\langle 8; 4; 8\rangle$
178	–	9	80	D	$\langle 5; 5; 8\rangle$
237	P	9	96		$\langle 3; 12; 8\rangle$
239	–	9	96	P	$\langle 4; 6; 6\rangle$
256	P	9	96		$\langle 6; 4; 24\rangle$
521	–	9	160		$\langle 4; 5; 20\rangle$
19	–	10	24		$\langle 12; 24; 8\rangle$
28	PD	10	27		$\langle 9; 18; 6\rangle$
31	PD	10	30		$\langle 30; 6; 10\rangle$
46	P	10	36	D	$\langle 12; 6; 12\rangle$
47	P	10	36		$\langle 12; 6; 12\rangle$
47	PD	10	36		$\langle 12; 6; 12\rangle$
56	–	10	40	P	$\langle 4; 40; 40\rangle$
61	P	10	44		$\langle 4; 22; 44\rangle$
91	–	10	54	D, P	$\langle 6; 6; 6\rangle$
92	–	10	54	P	$\langle 6; 6; 6\rangle$
92	PD	10	54	P	$\langle 6; 6; 6\rangle$
93	–	10	54		$\langle 4; 12; 6\rangle$
156	P	10	72		$\langle 3; 24; 12\rangle$
186	–	10	81		$\langle 3; 18; 6\rangle$
195	–	10	84		$\langle 4; 7; 8\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Trial.	Map symb.
207	–	10	90		$\langle 3; 15; 10\rangle$
282	P	10	108		$\langle 6; 4; 12\rangle$
286	P	10	108		$\langle 4; 6; 12\rangle$
290	P	10	108		$\langle 3; 12; 6\rangle$
541	–	10	162		$\langle 3; 9; 12\rangle$
625	–	10	180		$\langle 4; 5; 8\rangle$
20	PD	11	24	P	$\langle 24; 24; 4\rangle$
40	–	11	32		$\langle 8; 16; 16\rangle$
40	P	11	32		$\langle 8; 16; 16\rangle$
44	PD	11	33		$\langle 33; 6; 22\rangle$
61	–	11	44		$\langle 4; 44; 22\rangle$
65	P	11	48	D	$\langle 24; 4; 24\rangle$
67	–	11	48		$\langle 6; 8; 12\rangle$
70	P	11	48	D	$\langle 24; 4; 24\rangle$
81	–	11	48	PDP	$\langle 6; 8; 6\rangle$
114	–	11	60	D	$\langle 6; 6; 6\rangle$
333	P	11	120		$\langle 4; 6; 10\rangle$
30	PD	12	28		$\langle 14; 28; 4\rangle$
31	–	12	30		$\langle 10; 30; 6\rangle$
32	–	12	30	D	$\langle 15; 15; 4\rangle$
54	P	12	40		$\langle 8; 10; 40\rangle$
59	P	12	42		$\langle 14; 6; 42\rangle$
73	P	12	48	D	$\langle 48; 4; 48\rangle$
87	P	12	52		$\langle 4; 26; 52\rangle$
106	P	12	60		$\langle 15; 4; 30\rangle$
36	PD	13	32	P	$\langle 16; 16; 4\rangle$
39	PD	13	32	P	$\langle 16; 16; 4\rangle$
40	PD	13	32		$\langle 16; 16; 8\rangle$
46	–	13	36	D	$\langle 12; 12; 6\rangle$
47	–	13	36		$\langle 12; 12; 6\rangle$
48	–	13	36		$\langle 9; 18; 4\rangle$
53	PD	13	39		$\langle 39; 6; 26\rangle$
63	P	13	45		$\langle 15; 6; 30\rangle$
67	P	13	48		$\langle 6; 12; 8\rangle$
72	PD	13	48		$\langle 12; 6; 4\rangle$
87	–	13	52		$\langle 4; 52; 26\rangle$
97	P	13	56	D	$\langle 28; 4; 28\rangle$
108	–	13	60		$\langle 5; 10; 6\rangle$
118	P	13	64	D	$\langle 16; 4; 16\rangle$
125	P	13	64	D	$\langle 16; 4; 16\rangle$
153	P	13	72	D	$\langle 4; 12; 4\rangle$
155	P	13	72		$\langle 6; 6; 12\rangle$
463	–	13	144	P	$\langle 3; 12; 12\rangle$
611	P	13	180		$\langle 3; 10; 30\rangle$
45	PD	14	35		$\langle 35; 10; 14\rangle$
55	P	14	40		$\langle 8; 20; 40\rangle$
59	PD	14	42		$\langle 42; 6; 14\rangle$
69	P	14	48		$\langle 6; 16; 48\rangle$
99	P	14	56	D	$\langle 56; 4; 56\rangle$
111	P	14	60		$\langle 30; 4; 60\rangle$
3401	–	14	546		$\langle 3; 7; 26\rangle$
3402	–	14	546		$\langle 3; 7; 14\rangle$
3403	–	14	546		$\langle 3; 7; 12\rangle$
43	PD	15	32	P	$\langle 32; 32; 4\rangle$
44	–	15	33		$\langle 22; 33; 6\rangle$
45	–	15	35		$\langle 14; 35; 10\rangle$
49	–	15	36	D	$\langle 18; 18; 4\rangle$
54	–	15	40		$\langle 8; 40; 10\rangle$
55	–	15	40		$\langle 8; 40; 20\rangle$
64	P	15	48		$\langle 12; 8; 24\rangle$
67	PD	15	48		$\langle 8; 12; 6\rangle$
71	–	15	48		$\langle 8; 12; 24\rangle$
82	–	15	48	P	$\langle 8; 12; 12\rangle$
83	–	15	49	P	$\langle 7; 14; 14\rangle$
109	–	15	60		$\langle 6; 10; 10\rangle$
111	PD	15	60		$\langle 60; 4; 30\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Trial.	Map symb.
124	–	15	64	P	$\langle 4; 32; 32\rangle$
129	–	15	64	P	$\langle 4; 32; 32\rangle$
148	P	15	72	D	$\langle 18; 4; 18\rangle$
341	–	15	120	P	$\langle 3; 20; 20\rangle$
469	–	15	147		$\langle 3; 14; 6\rangle$
576	–	15	168		$\langle 4; 6; 8\rangle$
1046	–	15	252		$\langle 3; 9; 14\rangle$
51	PD	16	36		$\langle 36; 18; 4\rangle$
54	PD	16	40		$\langle 40; 10; 8\rangle$
69	–	16	48		$\langle 6; 48; 16\rangle$
84	–	16	50	D	$\langle 10; 10; 10\rangle$
84	P	16	50	D	$\langle 10; 10; 10\rangle$
89	–	16	54		$\langle 6; 18; 18\rangle$
89	P	16	54		$\langle 6; 18; 18\rangle$
134	P	16	64	D	$\langle 64; 4; 64\rangle$
138	–	16	68		$\langle 4; 34; 68\rangle$
152	–	16	72	P	$\langle 6; 8; 8\rangle$
262	–	16	100	PDP	$\langle 4; 10; 4\rangle$
476	–	16	150	D	$\langle 5; 5; 6\rangle$
1404	P	16	300		$\langle 5; 4; 30\rangle$
1829	P	16	360		$\langle 3; 8; 10\rangle$
57	PD	17	40	P	$\langle 20; 20; 4\rangle$
64	PD	17	48		$\langle 24; 8; 12\rangle$
66	PD	17	48	P	$\langle 12; 12; 4\rangle$
68	–	17	48	D	$\langle 12; 12; 4\rangle$
71	P	17	48		$\langle 8; 24; 12\rangle$
82	PD	17	48	P	$\langle 12; 12; 8\rangle$
86	PD	17	51		$\langle 51; 6; 34\rangle$
102	–	17	60		$\langle 6; 15; 20\rangle$
119	–	17	64	D, P	$\langle 8; 8; 8\rangle$
122	–	17	64	D	$\langle 8; 8; 4\rangle$
123	–	17	64	PDP	$\langle 8; 8; 8\rangle$
123	P	17	64	PDP	$\langle 8; 8; 8\rangle$
132	–	17	64		$\langle 8; 8; 4\rangle$
133	–	17	64	P	$\langle 8; 8; 8\rangle$
133	PD	17	64	P	$\langle 8; 8; 8\rangle$
138	P	17	68		$\langle 4; 68; 34\rangle$
149	–	17	72	P	$\langle 4; 36; 36\rangle$
170	–	17	80		$\langle 5; 10; 4\rangle$
172	P	17	80	D	$\langle 20; 4; 20\rangle$
179	–	17	80		$\langle 5; 10; 8\rangle$
223	P	17	96	D	$\langle 12; 4; 12\rangle$
229	–	17	96		$\langle 4; 12; 24\rangle$
236	–	17	96		$\langle 6; 6; 8\rangle$
239	PD	17	96	P	$\langle 6; 6; 4\rangle$
250	–	17	96	D	$\langle 6; 6; 8\rangle$
251	–	17	96	D	$\langle 6; 6; 8\rangle$
255	P	17	96		$\langle 12; 4; 24\rangle$
339	P	17	120		$\langle 5; 6; 8\rangle$
371	P	17	128		$\langle 8; 4; 8\rangle$
371	PD	17	128		$\langle 8; 4; 8\rangle$
380	P	17	128	D	$\langle 8; 4; 8\rangle$
393	P	17	128		$\langle 8; 4; 16\rangle$
567	–	17	168		$\langle 3; 14; 8\rangle$
701	–	17	192		$\langle 4; 6; 12\rangle$
741	–	17	192		$\langle 4; 6; 12\rangle$
743	–	17	192	P	$\langle 4; 6; 6\rangle$
2134	P	17	384		$\langle 3; 8; 12\rangle$
53	–	18	39		$\langle 26; 39; 6\rangle$
55	PD	18	40		$\langle 40; 20; 8\rangle$
59	–	18	42		$\langle 14; 42; 6\rangle$
60	–	18	42	D	$\langle 21; 21; 4\rangle$
62	PD	18	45		$\langle 45; 10; 18\rangle$
74	P	18	48		$\langle 16; 12; 48\rangle$
98	–	18	56		$\langle 8; 14; 56\rangle$
104	P	18	60		$\langle 20; 6; 60\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Trial.	Map symb.
160	–	18	72	P	$\langle 4;72;72\rangle$
164	P	18	76		$\langle 4;38;76\rangle$
192	–	18	84		$\langle 4;21;42\rangle$
56	PD	19	40	P	$\langle 40;40;4\rangle$
63	–	19	45		$\langle 15;30;6\rangle$
64	–	19	48		$\langle 12;24;8\rangle$
71	PD	19	48		$\langle 12;24;8\rangle$
90	–	19	54	D	$\langle 12;12;6\rangle$
94	–	19	54	D	$\langle 12;12;6\rangle$
100	PD	19	57		$\langle 57;6;38\rangle$
110	–	19	60	D	$\langle 10;10;6\rangle$
115	–	19	63		$\langle 6;21;42\rangle$
152	PD	19	72	P	$\langle 8;8;6\rangle$
154	P	19	72	PDP	$\langle 8;8;12\rangle$
155	–	19	72		$\langle 6;12;6\rangle$
155	PD	19	72		$\langle 12;6;6\rangle$
158	–	19	72		$\langle 6;12;24\rangle$
164	–	19	76		$\langle 4;76;38\rangle$
168	P	19	80	D	$\langle 40;4;40\rangle$
173	P	19	80	D	$\langle 40;4;40\rangle$
183	–	19	81		$\langle 6;9;18\rangle$
184	–	19	81		$\langle 6;9;6\rangle$
184	PD	19	81		$\langle 9;6;6\rangle$
185	P	19	81		$\langle 9;6;18\rangle$
196	–	19	84	D	$\langle 7;7;6\rangle$
282	PD	19	108		$\langle 12;4;6\rangle$
286	–	19	108		$\langle 4;12;6\rangle$
333	–	19	120		$\langle 4;10;6\rangle$
444	P	19	144		$\langle 8;4;24\rangle$
447	–	19	144		$\langle 4;8;24\rangle$
450	P	19	144		$\langle 3;24;6\rangle$
571	P	19	168		$\langle 7;4;8\rangle$
626	–	19	180	D	$\langle 5;5;10\rangle$
853	P	19	216		$\langle 3;12;24\rangle$
1823	P	19	360		$\langle 5;4;8\rangle$
61	PD	20	44		$\langle 44;22;4\rangle$
62	–	20	45		$\langle 18;45;10\rangle$
74	PD	20	48		$\langle 48;12;16\rangle$
96	P	20	56		$\langle 8;28;56\rangle$
104	PD	20	60		$\langle 60;6;20\rangle$
105	P	20	60		$\langle 10;12;60\rangle$
137	P	20	66		$\langle 6;22;66\rangle$
157	–	20	72		$\langle 8;9;36\rangle$
176	P	20	80	D	$\langle 80;4;80\rangle$
189	P	20	84		$\langle 4;42;84\rangle$
65	–	21	48	D	$\langle 24;24;4\rangle$
69	PD	21	48		$\langle 48;16;6\rangle$
70	–	21	48	D	$\langle 24;24;4\rangle$
74	–	21	48		$\langle 16;48;12\rangle$
75	–	21	48	D	$\langle 24;24;4\rangle$
76	–	21	48	D	$\langle 24;24;4\rangle$
79	–	21	48	D	$\langle 24;24;8\rangle$
80	–	21	48	D	$\langle 24;24;8\rangle$
96	–	21	56		$\langle 8;56;28\rangle$
98	P	21	56		$\langle 8;56;14\rangle$
117	–	21	64		$\langle 8;16;16\rangle$
117	P	21	64		$\langle 8;16;16\rangle$
120	–	21	64		$\langle 8;16;16\rangle$
120	P	21	64		$\langle 8;16;16\rangle$
121	–	21	64	P	$\langle 8;16;16\rangle$
126	–	21	64	P	$\langle 8;16;16\rangle$
127	P	21	64	D	$\langle 16;8;16\rangle$
128	P	21	64	D	$\langle 16;8;16\rangle$
189	–	21	84		$\langle 4;84;42\rangle$
199	P	21	88	D	$\langle 44;4;44\rangle$
214	P	21	96		$\langle 8;6;24\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Trial.	Map symb.
219	–	21	96	P	$\langle 4;24;24\rangle$
229	P	21	96		$\langle 4;24;12\rangle$
231	–	21	96	P	$\langle 4;24;24\rangle$
234	P	21	96		$\langle 6;8;12\rangle$
236	P	21	96		$\langle 6;8;6\rangle$
236	PD	21	96		$\langle 6;8;6\rangle$
238	–	21	96		$\langle 6;8;12\rangle$
248	–	21	96	P	$\langle 4;24;24\rangle$
255	PD	21	96		$\langle 24;4;12\rangle$
256	PD	21	96		$\langle 24;4;6\rangle$
334	–	21	120		$\langle 4;12;10\rangle$
335	–	21	120		$\langle 4;12;10\rangle$
336	–	21	120	D, P	$\langle 6;6;6\rangle$
728	–	21	192		$\langle 3;16;6\rangle$
729	–	21	192		$\langle 3;16;24\rangle$
970	–	21	240		$\langle 4;6;20\rangle$
89	PD	22	54		$\langle 18;18;6\rangle$
95	–	22	55		$\langle 10;55;22\rangle$
137	–	22	66		$\langle 6;66;22\rangle$
145	–	22	72		$\langle 6;24;24\rangle$
145	P	22	72		$\langle 6;24;24\rangle$
147	–	22	72	P	$\langle 6;24;24\rangle$
154	–	22	72	PDP	$\langle 8;12;8\rangle$
158	P	22	72		$\langle 6;24;12\rangle$
202	P	22	88	D	$\langle 88;4;88\rangle$
211	–	22	92		$\langle 4;46;92\rangle$
339	–	22	120		$\langle 5;8;6\rangle$
570	–	22	168		$\langle 4;8;14\rangle$
576	P	22	168		$\langle 4;8;6\rangle$
2980	–	22	504		$\langle 3;8;42\rangle$
2988	P	22	504		$\langle 3;8;24\rangle$
73	–	23	48	D	$\langle 48;48;4\rangle$
103	P	23	60		$\langle 20;12;30\rangle$
130	–	23	64		$\langle 8;32;32\rangle$
130	P	23	64		$\langle 8;32;32\rangle$
139	PD	23	69		$\langle 69;6;46\rangle$
211	P	23	92		$\langle 4;92;46\rangle$
221	–	23	96	P	$\langle 4;48;48\rangle$
224	–	23	96	P	$\langle 4;48;48\rangle$
86	–	24	51		$\langle 34;51;6\rangle$
87	PD	24	52		$\langle 52;26;4\rangle$
88	–	24	54	D	$\langle 27;27;4\rangle$
98	PD	24	56		$\langle 14;56;8\rangle$
102	PD	24	60		$\langle 15;20;6\rangle$
103	PD	24	60		$\langle 30;12;20\rangle$
105	–	24	60		$\langle 10;60;12\rangle$
140	P	24	70		$\langle 14;10;70\rangle$
142	P	24	72		$\langle 18;8;72\rangle$
159	P	24	72		$\langle 18;8;36\rangle$
166	P	24	78		$\langle 26;6;78\rangle$
242	–	24	96	P	$\langle 4;96;96\rangle$
265	P	24	100		$\langle 4;50;100\rangle$
277	–	24	108		$\langle 4;27;54\rangle$

4.5.2 Type 2, 2^* and 2^P maps

Type 2 - orientable genus 2 to 5

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
1	–	2	5	yes	$\langle 5,5;10;10\rangle$
2	P	2	6	no	$\langle 3,6;12;4\rangle$
9	–	2	10	no	$\langle 5,10;4;20\rangle$
18	–	2	12	yes	$\langle 3,4;8;8\rangle$
40	–	2	16	yes	$\langle 4,8;4;4\rangle$
91	–	2	24	yes	$\langle 4,6;4;4\rangle$
118	–	2	24	no	$\langle 3,4;6;12\rangle$
611	–	2	48	no	$\langle 3,8;4;16\rangle$
3	–	3	7	no	$\langle 7,7;14;14\rangle$
3	P	3	7	no	$\langle 7,7;14;14\rangle$
4	–	3	8	no	$\langle 4,8;16;16\rangle$
4	P	3	8	no	$\langle 4,8;16;16\rangle$
5	–	3	9	no	$\langle 9,9;6;18\rangle$
6	–	3	9	no	$\langle 3,9;18;18\rangle$
6	P	3	9	no	$\langle 3,9;18;18\rangle$
12	–	3	12	no	$\langle 4,12;6;12\rangle$
15	–	3	12	yes	$\langle 4,6;8;8\rangle$
16	–	3	12	no	$\langle 3,12;8;24\rangle$
17	–	3	12	yes	$\langle 3,4;24;24\rangle$
26	–	3	14	no	$\langle 7,14;4;28\rangle$
32	–	3	16	yes	$\langle 4,4;8;8\rangle$
92	–	3	24	yes	$\langle 4,12;4;4\rangle$
113	P	3	24	no	$\langle 3,4;8;4\rangle$
116	–	3	24	no	$\langle 3,6;6;8\rangle$
202	–	3	32	yes	$\langle 4,8;4;4\rangle$
222	–	3	32	no	$\langle 4,8;4;8\rangle$
519	–	3	48	no	$\langle 4,6;4;8\rangle$
601	–	3	48	yes	$\langle 3,4;6;6\rangle$
612	–	3	48	no	$\langle 3,12;4;24\rangle$
2826	–	3	96	no	$\langle 3,8;4;16\rangle$
9329	–	3	168	no	$\langle 3,7;4;14\rangle$
5	P	4	9	no	$\langle 9,9;18;6\rangle$
7	–	4	10	no	$\langle 5,10;20;20\rangle$
7	P	4	10	no	$\langle 5,10;20;20\rangle$
8	–	4	10	yes	$\langle 10,10;10;10\rangle$
9	P	4	10	no	$\langle 5,10;20;4\rangle$
12	P	4	12	no	$\langle 4,12;12;6\rangle$
13	–	4	12	no	$\langle 6,12;8;24\rangle$
14	–	4	12	yes	$\langle 4,6;24;24\rangle$
16	P	4	12	no	$\langle 3,12;24;8\rangle$
27	–	4	15	no	$\langle 3,15;10;30\rangle$
28	–	4	15	yes	$\langle 3,5;30;30\rangle$
31	–	4	15	no	$\langle 5,15;6;30\rangle$
38	–	4	16	yes	$\langle 4,8;8;8\rangle$
45	P	4	18	no	$\langle 3,6;12;4\rangle$
46	–	4	18	yes	$\langle 3,6;12;12\rangle$
47	–	4	18	yes	$\langle 3,6;12;12\rangle$
54	–	4	18	no	$\langle 9,18;4;36\rangle$
60	–	4	20	yes	$\langle 4,5;8;8\rangle$
89	–	4	24	no	$\langle 6,12;4;12\rangle$
116	P	4	24	no	$\langle 3,6;8;6\rangle$
117	–	4	24	no	$\langle 4,6;6;12\rangle$
118	P	4	24	no	$\langle 3,4;12;6\rangle$
230	–	4	32	yes	$\langle 4,16;4;4\rangle$
267	–	4	36	no	$\langle 6,6;4;12\rangle$
299	–	4	36	yes	$\langle 3,6;6;6\rangle$
305	–	4	36	yes	$\langle 3,4;8;8\rangle$
342	–	4	40	yes	$\langle 4,10;4;4\rangle$
1447	–	4	72	no	$\langle 3,12;4;24\rangle$
1516	–	4	72	no	$\langle 4,6;4;12\rangle$
4469	–	4	120	no	$\langle 4,5;4;12\rangle$
10	–	5	11	no	$\langle 11,11;22;22\rangle$
10	P	5	11	no	$\langle 11,11;22;22\rangle$
11	–	5	11	no	$\langle 11,11;22;22\rangle$
11	P	5	11	no	$\langle 11,11;22;22\rangle$
13	P	5	12	no	$\langle 6,12;24;8\rangle$
27	P	5	15	no	$\langle 3,15;30;10\rangle$
33	–	5	16	yes	$\langle 4,8;16;16\rangle$
39	–	5	16	yes	$\langle 4,8;16;16\rangle$
59	–	5	20	yes	$\langle 4,10;8;8\rangle$
77	–	5	22	no	$\langle 11,22;4;44\rangle$
96	–	5	24	yes	$\langle 4,6;8;8\rangle$
115	P	5	24	no	$\langle 3,6;12;4\rangle$
165	–	5	30	no	$\langle 6,15;4;12\rangle$
225	–	5	32	yes	$\langle 4,4;8;8\rangle$
341	–	5	40	yes	$\langle 4,20;4;4\rangle$
507	–	5	48	yes	$\langle 4,12;4;4\rangle$
522	–	5	48	no	$\langle 3,4;8;8\rangle$
522	P	5	48	no	$\langle 3,4;8;8\rangle$
529	–	5	48	yes	$\langle 3,4;8;8\rangle$
530	–	5	48	no	$\langle 4,4;6;12\rangle$
947	–	5	60	no	$\langle 3,5;6;10\rangle$
1070	–	5	64	no	$\langle 4,8;4;8\rangle$
1137	–	5	64	no	$\langle 4,8;4;8\rangle$
2489	–	5	96	no	$\langle 4,6;4;8\rangle$
2797	–	5	96	no	$\langle 3,4;6;12\rangle$
4416	–	5	120	no	$\langle 3,10;4;20\rangle$
8330	–	5	160	no	$\langle 4,5;4;8\rangle$
12738	–	5	192	no	$\langle 3,8;4;16\rangle$

Type 2 - nonorientable genus 3 to 11

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
19	P	4	12	no	$\langle 3,4;8;4\rangle$
112	–	4	24	no	$\langle 4,6;4;8\rangle$
114	–	4	24	no	$\langle 4,6;4;8\rangle$
303	–	5	36	no	$\langle 4,6;4;12\rangle$
969	–	5	60	no	$\langle 4,5;4;12\rangle$
190	–	6	30	no	$\langle 3,5;6;10\rangle$
939	–	6	60	no	$\langle 3,10;4;20\rangle$
945	–	6	60	no	$\langle 3,10;4;20\rangle$
1848	–	6	80	no	$\langle 4,5;4;8\rangle$
298	–	7	36	no	$\langle 4,9;4;8\rangle$
968	–	7	60	no	$\langle 4,6;4;6\rangle$
21694	–	8	252	no	$\langle 3,7;4;14\rangle$
167	–	9	30	yes	$\langle 6,10;4;4\rangle$
2032	–	9	84	no	$\langle 3,4;6;8\rangle$
9309	–	9	168	no	$\langle 3,8;4;16\rangle$
9333	–	9	168	no	$\langle 3,8;4;16\rangle$
112	P	10	24	no	$\langle 4,6;8;4\rangle$
114	P	10	24	no	$\langle 4,6;8;4\rangle$
189	P	10	30	no	$\langle 3,5;10;4\rangle$
190	P	10	30	no	$\langle 3,5;10;6\rangle$
521	–	10	48	no	$\langle 4,12;4;8\rangle$
523	–	10	48	no	$\langle 4,12;4;8\rangle$
527	–	10	48	yes	$\langle 3,4;8;8\rangle$
941	–	10	60	no	$\langle 5,6;4;20\rangle$
944	–	10	60	no	$\langle 5,6;4;20\rangle$
959	–	10	60	no	$\langle 5,6;4;8\rangle$
2823	–	10	96	no	$\langle 4,6;4;8\rangle$
3523	–	11	108	no	$\langle 4,6;4;12\rangle$

Type 2^P - orientable genus 2 to 30

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
8	PD	4	10	yes	$\langle 10;10;10,10\rangle$
12	PD	4	12	no	$\langle 6;12;4,12\rangle$
112	PD	4	24	no	$\langle 4;8;4,6\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
32	PD	5	16	yes	$\langle 8; 8; 4, 4\rangle$
38	PD	5	16	yes	$\langle 8; 8; 4, 8\rangle$
89	PD	5	24	no	$\langle 4; 12; 6, 12\rangle$
222	PD	5	32	no	$\langle 4; 8; 4, 8\rangle$
25	PD	6	14	no	$\langle 14; 14; 14, 14\rangle$
34	PD	6	16	no	$\langle 8; 16; 16, 16\rangle$
50	PD	6	18	no	$\langle 6; 18; 18, 18\rangle$
968	PD	6	60	no	$\langle 4; 6; 4, 6\rangle$
33	PD	7	16	yes	$\langle 16; 16; 4, 8\rangle$
39	PD	7	16	yes	$\langle 16; 16; 4, 8\rangle$
96	PD	7	24	yes	$\langle 8; 8; 4, 6\rangle$
111	PD	7	24	yes	$\langle 8; 8; 4, 12\rangle$
267	PD	7	36	no	$\langle 4; 12; 6, 6\rangle$
505	PD	7	48	no	$\langle 4; 8; 8, 24\rangle$
519	PD	7	48	no	$\langle 4; 8; 4, 6\rangle$
573	PD	7	48	no	$\langle 4; 8; 8, 24\rangle$
53	PD	8	18	no	$\langle 18; 18; 6, 18\rangle$
61	PD	8	20	no	$\langle 10; 20; 4, 20\rangle$
65	PD	8	20	no	$\langle 10; 20; 20, 20\rangle$
109	PD	8	24	no	$\langle 6; 24; 8, 24\rangle$
64	PD	9	20	yes	$\langle 20; 20; 10, 10\rangle$
90	PD	9	24	yes	$\langle 12; 12; 4, 6\rangle$
93	PD	9	24	yes	$\langle 12; 12; 4, 12\rangle$
95	PD	9	24	no	$\langle 8; 24; 6, 12\rangle$
203	PD	9	32	yes	$\langle 8; 8; 4, 8\rangle$
207	PD	9	32	yes	$\langle 8; 8; 4, 8\rangle$
208	PD	9	32	yes	$\langle 8; 8; 4, 8\rangle$
209	PD	9	32	yes	$\langle 8; 8; 4, 8\rangle$
221	PD	9	32	yes	$\langle 8; 8; 4, 8\rangle$
225	PD	9	32	yes	$\langle 8; 8; 4, 4\rangle$
228	PD	9	32	yes	$\langle 8; 8; 4, 16\rangle$
346	PD	9	40	no	$\langle 4; 20; 10, 20\rangle$
503	PD	9	48	no	$\langle 4; 12; 6, 24\rangle$
590	PD	9	48	no	$\langle 4; 12; 12, 24\rangle$
1035	PD	9	64	no	$\langle 4; 8; 8, 8\rangle$
1070	PD	9	64	no	$\langle 4; 8; 4, 8\rangle$
1118	PD	9	64	no	$\langle 4; 8; 4, 16\rangle$
1129	PD	9	64	no	$\langle 4; 8; 8, 16\rangle$
1137	PD	9	64	no	$\langle 4; 8; 4, 8\rangle$
80	PD	10	22	no	$\langle 22; 22; 22, 22\rangle$
83	PD	10	22	no	$\langle 22; 22; 22, 22\rangle$
105	PD	10	24	no	$\langle 12; 24; 8, 24\rangle$
180	PD	10	30	no	$\langle 6; 30; 10, 30\rangle$
291	PD	10	36	no	$\langle 6; 12; 4, 12\rangle$
304	PD	10	36	yes	$\langle 8; 8; 6, 6\rangle$
607	PD	10	48	no	$\langle 4; 16; 8, 12\rangle$
613	PD	10	48	no	$\langle 4; 16; 8, 12\rangle$
942	PD	10	60	no	$\langle 4; 10; 6, 10\rangle$
1450	PD	10	72	no	$\langle 4; 8; 6, 12\rangle$
1492	PD	10	72	no	$\langle 4; 8; 4, 18\rangle$
3528	PD	10	108	no	$\langle 4; 6; 4, 12\rangle$
94	PD	11	24	yes	$\langle 24; 24; 4, 6\rangle$
110	PD	11	24	yes	$\langle 24; 24; 4, 12\rangle$
352	PD	11	40	yes	$\langle 8; 8; 4, 10\rangle$
372	PD	11	40	yes	$\langle 8; 8; 4, 20\rangle$
867	PD	11	60	no	$\langle 4; 12; 6, 30\rangle$
1734	PD	11	80	no	$\langle 4; 8; 8, 40\rangle$
1817	PD	11	80	no	$\langle 4; 8; 8, 40\rangle$
4466	PD	11	120	no	$\langle 4; 6; 4, 6\rangle$
128	PD	12	26	no	$\langle 26; 26; 26, 26\rangle$
130	PD	12	26	yes	$\langle 26; 26; 26, 26\rangle$
133	PD	12	26	no	$\langle 26; 26; 26, 26\rangle$
144	PD	12	28	no	$\langle 14; 28; 4, 28\rangle$
150	PD	12	28	no	$\langle 14; 28; 28, 28\rangle$
152	PD	12	28	no	$\langle 14; 28; 28, 28\rangle$
187	PD	12	30	no	$\langle 10; 30; 6, 30\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
188	PD	12	30	no	$\langle 10; 30; 30, 30\rangle$
216	PD	12	32	no	$\langle 8; 32; 32, 32\rangle$
276	PD	12	36	no	$\langle 6; 36; 36, 36\rangle$
149	PD	13	28	no	$\langle 28; 28; 14, 14\rangle$
198	PD	13	32	yes	$\langle 16; 16; 4, 8\rangle$
199	PD	13	32	yes	$\langle 16; 16; 4, 8\rangle$
200	PD	13	32	yes	$\langle 16; 16; 4, 8\rangle$
201	PD	13	32	yes	$\langle 16; 16; 4, 8\rangle$
206	PD	13	32	yes	$\langle 16; 16; 4, 4\rangle$
210	PD	13	32	yes	$\langle 16; 16; 4, 8\rangle$
223	PD	13	32	yes	$\langle 16; 16; 4, 4\rangle$
224	PD	13	32	yes	$\langle 16; 16; 4, 8\rangle$
226	PD	13	32	yes	$\langle 16; 16; 8, 8\rangle$
227	PD	13	32	yes	$\langle 16; 16; 8, 8\rangle$
266	PD	13	36	yes	$\langle 12; 12; 6, 6\rangle$
508	PD	13	48	yes	$\langle 8; 8; 4, 12\rangle$
517	PD	13	48	yes	$\langle 8; 8; 4, 6\rangle$
518	PD	13	48	yes	$\langle 8; 8; 4, 12\rangle$
524	PD	13	48	no	$\langle 6; 12; 4, 12\rangle$
530	PD	13	48	no	$\langle 6; 12; 4, 4\rangle$
533	PD	13	48	yes	$\langle 8; 8; 4, 12\rangle$
535	PD	13	48	yes	$\langle 8; 8; 4, 12\rangle$
570	PD	13	48	yes	$\langle 8; 8; 4, 24\rangle$
571	PD	13	48	yes	$\langle 8; 8; 8, 12\rangle$
576	PD	13	48	yes	$\langle 8; 8; 6, 8\rangle$
599	PD	13	48	no	$\langle 6; 12; 4, 6\rangle$
603	PD	13	48	no	$\langle 6; 12; 4, 8\rangle$
776	PD	13	56	no	$\langle 4; 28; 14, 28\rangle$
871	PD	13	60	no	$\langle 4; 20; 10, 30\rangle$
1127	PD	13	64	no	$\langle 4; 16; 16, 16\rangle$
1428	PD	13	72	no	$\langle 4; 12; 6, 12\rangle$
1434	PD	13	72	no	$\langle 4; 12; 12, 12\rangle$
1454	PD	13	72	yes	$\langle 6; 6; 6, 6\rangle$
1458	PD	13	72	no	$\langle 6; 6; 6, 12\rangle$
1459	PD	13	72	yes	$\langle 6; 6; 4, 6\rangle$
1463	PD	13	72	yes	$\langle 6; 6; 4, 12\rangle$
1470	PD	13	72	no	$\langle 4; 12; 6, 12\rangle$
1516	PD	13	72	no	$\langle 4; 12; 4, 6\rangle$
2464	PD	13	96	no	$\langle 4; 8; 8, 24\rangle$
2489	PD	13	96	no	$\langle 4; 8; 4, 6\rangle$
2490	PD	13	96	no	$\langle 4; 8; 4, 12\rangle$
2494	PD	13	96	no	$\langle 4; 8; 4, 12\rangle$
2705	PD	13	96	no	$\langle 4; 8; 4, 24\rangle$
2706	PD	13	96	no	$\langle 4; 8; 8, 12\rangle$
2739	PD	13	96	no	$\langle 4; 8; 4, 12\rangle$
169	PD	14	30	yes	$\langle 30; 30; 10, 10\rangle$
178	PD	14	30	no	$\langle 30; 30; 10, 30\rangle$
179	PD	14	30	yes	$\langle 30; 30; 6, 10\rangle$
215	PD	14	32	no	$\langle 16; 32; 32, 32\rangle$
217	PD	14	32	no	$\langle 16; 32; 32, 32\rangle$
278	PD	14	36	no	$\langle 12; 18; 36, 36\rangle$
413	PD	14	42	no	$\langle 6; 42; 14, 42\rangle$
204	PD	15	32	no	$\langle 32; 32; 8, 16\rangle$
205	PD	15	32	yes	$\langle 32; 32; 4, 16\rangle$
229	PD	15	32	no	$\langle 32; 32; 8, 16\rangle$
231	PD	15	32	yes	$\langle 32; 32; 4, 16\rangle$
275	PD	15	36	no	$\langle 12; 36; 18, 18\rangle$
351	PD	15	40	no	$\langle 8; 40; 10, 20\rangle$
781	PD	15	56	yes	$\langle 8; 8; 4, 14\rangle$
810	PD	15	56	yes	$\langle 8; 8; 4, 28\rangle$
949	PD	15	60	no	$\langle 6; 10; 6, 10\rangle$
1940	PD	15	84	no	$\langle 4; 12; 6, 42\rangle$
3677	PD	15	112	no	$\langle 4; 8; 8, 56\rangle$
3779	PD	15	112	no	$\langle 4; 8; 8, 56\rangle$
245	PD	16	34	no	$\langle 34; 34; 34, 34\rangle$
247	PD	16	34	yes	$\langle 34; 34; 34, 34\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
250	PD	16	34	no	$\langle 34; 34; 34, 34\rangle$
253	PD	16	34	no	$\langle 34; 34; 34, 34\rangle$
272	PD	16	36	no	$\langle 18; 36; 4, 36\rangle$
283	PD	16	36	no	$\langle 18; 36; 12, 36\rangle$
285	PD	16	36	no	$\langle 18; 36; 12, 36\rangle$
370	PD	16	40	no	$\langle 10; 40; 8, 40\rangle$
371	PD	16	40	no	$\langle 10; 40; 40, 40\rangle$
563	PD	16	48	no	$\langle 6; 48; 16, 48\rangle$
630	PD	16	50	yes	$\langle 10; 10; 10, 10\rangle$
716	PD	16	54	no	$\langle 6; 18; 18, 18\rangle$
4173	PD	16	120	no	$\langle 4; 8; 6, 20\rangle$
4181	PD	16	120	no	$\langle 4; 8; 4, 30\rangle$
4437	PD	16	120	no	$\langle 4; 8; 6, 10\rangle$
282	PD	17	36	no	$\langle 36; 36; 6, 18\rangle$
343	PD	17	40	yes	$\langle 20; 20; 4, 10\rangle$
344	PD	17	40	no	$\langle 20; 20; 10, 20\rangle$
347	PD	17	40	yes	$\langle 20; 20; 4, 20\rangle$
348	PD	17	40	yes	$\langle 20; 20; 20, 20\rangle$
502	PD	17	48	yes	$\langle 12; 12; 6, 8\rangle$
509	PD	17	48	yes	$\langle 12; 12; 4, 12\rangle$
513	PD	17	48	no	$\langle 8; 24; 6, 12\rangle$
514	PD	17	48	no	$\langle 8; 24; 12, 12\rangle$
567	PD	17	48	no	$\langle 8; 24; 12, 24\rangle$
574	PD	17	48	no	$\langle 8; 24; 6, 24\rangle$
589	PD	17	48	yes	$\langle 12; 12; 8, 12\rangle$
591	PD	17	48	yes	$\langle 12; 12; 4, 24\rangle$
592	PD	17	48	yes	$\langle 12; 12; 4, 8\rangle$
1038	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1047	PD	17	64	yes	$\langle 8; 8; 4, 16\rangle$
1052	PD	17	64	yes	$\langle 8; 8; 8, 8\rangle$
1053	PD	17	64	no	$\langle 8; 8; 8, 8\rangle$
1065	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1066	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1067	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1068	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1069	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1071	PD	17	64	no	$\langle 8; 8; 4, 8\rangle$
1081	PD	17	64	yes	$\langle 8; 8; 8, 8\rangle$
1084	PD	17	64	yes	$\langle 8; 8; 4, 4\rangle$
1086	PD	17	64	yes	$\langle 8; 8; 4, 16\rangle$
1087	PD	17	64	yes	$\langle 8; 8; 4, 16\rangle$
1090	PD	17	64	yes	$\langle 8; 8; 4, 16\rangle$
1114	PD	17	64	yes	$\langle 8; 8; 8, 8\rangle$
1117	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1120	PD	17	64	yes	$\langle 8; 8; 4, 16\rangle$
1135	PD	17	64	yes	$\langle 8; 8; 8, 16\rangle$
1138	PD	17	64	yes	$\langle 8; 8; 4, 4\rangle$
1139	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1140	PD	17	64	yes	$\langle 8; 8; 4, 4\rangle$
1141	PD	17	64	yes	$\langle 8; 8; 4, 4\rangle$
1142	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1143	PD	17	64	yes	$\langle 8; 8; 4, 8\rangle$
1144	PD	17	64	no	$\langle 8; 8; 4, 8\rangle$
1155	PD	17	64	yes	$\langle 8; 8; 4, 32\rangle$
1422	PD	17	72	no	$\langle 4; 36; 18, 36\rangle$
1733	PD	17	80	no	$\langle 4; 20; 10, 40\rangle$
1843	PD	17	80	no	$\langle 4; 20; 20, 40\rangle$
2472	PD	17	96	no	$\langle 4; 12; 12, 24\rangle$
2525	PD	17	96	no	$\langle 4; 12; 6, 12\rangle$
2555	PD	17	96	no	$\langle 4; 12; 6, 48\rangle$
2772	PD	17	96	no	$\langle 4; 12; 12, 48\rangle$
2796	PD	17	96	yes	$\langle 6; 6; 4, 6\rangle$
2821	PD	17	96	yes	$\langle 6; 6; 4, 6\rangle$
2831	PD	17	96	yes	$\langle 6; 6; 4, 8\rangle$
4869	PD	17	128	no	$\langle 4; 8; 8, 16\rangle$
4918	PD	17	128	no	$\langle 4; 8; 4, 16\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
5019	PD	17	128	no	$\langle 4; 8; 8, 16\rangle$
5020	PD	17	128	no	$\langle 4; 8; 8, 16\rangle$
5071	PD	17	128	no	$\langle 4; 8; 8, 16\rangle$
5086	PD	17	128	no	$\langle 4; 8; 8, 8\rangle$
5087	PD	17	128	no	$\langle 4; 8; 4, 8\rangle$
5197	PD	17	128	no	$\langle 4; 8; 4, 16\rangle$
5206	PD	17	128	no	$\langle 4; 8; 8, 16\rangle$
5268	PD	17	128	no	$\langle 4; 8; 4, 32\rangle$
5278	PD	17	128	no	$\langle 4; 8; 8, 32\rangle$
317	PD	18	38	no	$\langle 38; 38; 38, 38\rangle$
320	PD	18	38	no	$\langle 38; 38; 38, 38\rangle$
323	PD	18	38	no	$\langle 38; 38; 38, 38\rangle$
327	PD	18	38	no	$\langle 38; 38; 38, 38\rangle$
364	PD	18	40	no	$\langle 20; 40; 8, 40\rangle$
365	PD	18	40	no	$\langle 20; 40; 40, 40\rangle$
391	PD	18	42	no	$\langle 14; 42; 6, 42\rangle$
403	PD	18	42	no	$\langle 14; 42; 42, 42\rangle$
406	PD	18	42	no	$\langle 14; 42; 42, 42\rangle$
537	PD	18	48	no	$\langle 8; 48; 48, 48\rangle$
747	PD	18	54	no	$\langle 6; 54; 54, 54\rangle$
345	PD	19	40	yes	$\langle 40; 40; 10, 10\rangle$
349	PD	19	40	no	$\langle 40; 40; 10, 20\rangle$
350	PD	19	40	yes	$\langle 40; 40; 4, 10\rangle$
373	PD	19	40	yes	$\langle 40; 40; 4, 20\rangle$
374	PD	19	40	yes	$\langle 40; 40; 20, 20\rangle$
495	PD	19	48	yes	$\langle 16; 16; 6, 8\rangle$
497	PD	19	48	yes	$\langle 16; 16; 4, 24\rangle$
501	PD	19	48	yes	$\langle 16; 16; 8, 12\rangle$
506	PD	19	48	no	$\langle 12; 24; 8, 24\rangle$
565	PD	19	48	yes	$\langle 16; 16; 4, 12\rangle$
572	PD	19	48	no	$\langle 12; 24; 8, 24\rangle$
581	PD	19	48	yes	$\langle 16; 16; 8, 12\rangle$
582	PD	19	48	yes	$\langle 16; 16; 4, 24\rangle$
585	PD	19	48	yes	$\langle 16; 16; 4, 6\rangle$
588	PD	19	48	yes	$\langle 16; 16; 6, 8\rangle$
751	PD	19	54	no	$\langle 12; 12; 6, 6\rangle$
1369	PD	19	72	yes	$\langle 8; 8; 4, 18\rangle$
1373	PD	19	72	yes	$\langle 8; 8; 4, 36\rangle$
1444	PD	19	72	yes	$\langle 8; 8; 6, 12\rangle$
1457	PD	19	72	yes	$\langle 8; 8; 6, 6\rangle$
1517	PD	19	72	yes	$\langle 8; 8; 6, 6\rangle$
1518	PD	19	72	yes	$\langle 8; 8; 4, 6\rangle$
1946	PD	19	84	no	$\langle 4; 28; 14, 42\rangle$
2537	PD	19	96	no	$\langle 4; 16; 16, 48\rangle$
2781	PD	19	96	no	$\langle 4; 16; 16, 48\rangle$
2814	PD	19	96	no	$\langle 4; 16; 8, 12\rangle$
2816	PD	19	96	no	$\langle 4; 16; 6, 8\rangle$
2827	PD	19	96	no	$\langle 4; 16; 6, 8\rangle$
2861	PD	19	96	no	$\langle 4; 16; 8, 12\rangle$
2870	PD	19	96	no	$\langle 4; 16; 6, 8\rangle$
3370	PD	19	108	no	$\langle 4; 12; 6, 18\rangle$
3515	PD	19	108	no	$\langle 4; 12; 6, 6\rangle$
4420	PD	19	120	no	$\langle 4; 10; 6, 10\rangle$
6302	PD	19	144	no	$\langle 4; 8; 8, 72\rangle$
6336	PD	19	144	no	$\langle 4; 8; 6, 12\rangle$
6532	PD	19	144	no	$\langle 4; 8; 8, 72\rangle$
6737	PD	19	144	no	$\langle 4; 8; 4, 18\rangle$
388	PD	20	42	yes	$\langle 42; 42; 6, 14\rangle$
395	PD	20	42	no	$\langle 42; 42; 14, 14\rangle$
400	PD	20	42	no	$\langle 42; 42; 14, 42\rangle$
409	PD	20	42	no	$\langle 42; 42; 14, 42\rangle$
431	PD	20	44	no	$\langle 22; 44; 44, 44\rangle$
433	PD	20	44	no	$\langle 22; 44; 44, 44\rangle$
437	PD	20	44	no	$\langle 22; 44; 4, 44\rangle$
441	PD	20	44	no	$\langle 22; 44; 44, 44\rangle$
443	PD	20	44	no	$\langle 22; 44; 44, 44\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
538	PD	20	48	no	$\langle 16; 24; 48, 48\rangle$
541	PD	20	48	no	$\langle 12; 48; 16, 48\rangle$
634	PD	20	50	no	$\langle 10; 50; 50, 50\rangle$
641	PD	20	50	no	$\langle 10; 50; 50, 50\rangle$
882	PD	20	60	no	$\langle 6; 60; 20, 60\rangle$
430	PD	21	44	no	$\langle 44; 44; 22, 22\rangle$
440	PD	21	44	no	$\langle 44; 44; 22, 22\rangle$
493	PD	21	48	no	$\langle 16; 48; 6, 24\rangle$
499	PD	21	48	no	$\langle 16; 48; 12, 24\rangle$
510	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
511	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
512	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
515	PD	21	48	yes	$\langle 24; 24; 4, 6\rangle$
516	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
534	PD	21	48	yes	$\langle 24; 24; 4, 4\rangle$
566	PD	21	48	yes	$\langle 24; 24; 8, 12\rangle$
568	PD	21	48	yes	$\langle 24; 24; 4, 24\rangle$
569	PD	21	48	yes	$\langle 24; 24; 4, 8\rangle$
575	PD	21	48	yes	$\langle 24; 24; 6, 8\rangle$
578	PD	21	48	no	$\langle 16; 48; 12, 24\rangle$
584	PD	21	48	no	$\langle 16; 48; 6, 12\rangle$
586	PD	21	48	no	$\langle 16; 48; 6, 24\rangle$
782	PD	21	56	no	$\langle 8; 56; 14, 28\rangle$
866	PD	21	60	yes	$\langle 12; 12; 6, 10\rangle$
1123	PD	21	64	no	$\langle 8; 16; 16, 16\rangle$
1753	PD	21	80	yes	$\langle 8; 8; 4, 10\rangle$
1754	PD	21	80	yes	$\langle 8; 8; 4, 20\rangle$
1757	PD	21	80	yes	$\langle 8; 8; 4, 20\rangle$
1760	PD	21	80	yes	$\langle 8; 8; 4, 20\rangle$
1766	PD	21	80	yes	$\langle 8; 8; 4, 20\rangle$
1810	PD	21	80	yes	$\langle 8; 8; 8, 20\rangle$
1816	PD	21	80	yes	$\langle 8; 8; 4, 40\rangle$
1823	PD	21	80	yes	$\langle 8; 8; 8, 10\rangle$
2127	PD	21	88	no	$\langle 4; 44; 22, 44\rangle$
2701	PD	21	96	no	$\langle 4; 24; 12, 24\rangle$
2817	PD	21	96	no	$\langle 4; 24; 6, 12\rangle$
2994	PD	21	100	no	$\langle 4; 20; 10, 10\rangle$
4139	PD	21	120	no	$\langle 4; 12; 12, 60\rangle$
4316	PD	21	120	no	$\langle 4; 12; 6, 60\rangle$
4332	PD	21	120	no	$\langle 4; 12; 12, 30\rangle$
4465	PD	21	120	no	$\langle 4; 12; 4, 6\rangle$
4467	PD	21	120	no	$\langle 4; 12; 4, 6\rangle$
4470	PD	21	120	no	$\langle 4; 12; 4, 10\rangle$
8017	PD	21	160	no	$\langle 4; 8; 8, 40\rangle$
8206	PD	21	160	no	$\langle 4; 8; 8, 20\rangle$
8214	PD	21	160	no	$\langle 4; 8; 4, 40\rangle$
8254	PD	21	160	no	$\langle 4; 8; 4, 20\rangle$
8332	PD	21	160	no	$\langle 4; 8; 4, 10\rangle$
19982	PD	21	240	no	$\langle 4; 6; 4, 12\rangle$
468	PD	22	46	no	$\langle 46; 46; 46, 46\rangle$
471	PD	22	46	no	$\langle 46; 46; 46, 46\rangle$
474	PD	22	46	no	$\langle 46; 46; 46, 46\rangle$
477	PD	22	46	no	$\langle 46; 46; 46, 46\rangle$
480	PD	22	46	no	$\langle 46; 46; 46, 46\rangle$
536	PD	22	48	no	$\langle 24; 48; 16, 48\rangle$
539	PD	22	48	no	$\langle 24; 48; 16, 48\rangle$
540	PD	22	48	no	$\langle 24; 48; 16, 16\rangle$
715	PD	22	54	yes	$\langle 18; 18; 6, 18\rangle$
720	PD	22	54	no	$\langle 18; 18; 6, 18\rangle$
1189	PD	22	66	no	$\langle 6; 66; 22, 66\rangle$
1468	PD	22	72	no	$\langle 6; 24; 8, 24\rangle$
8942	PD	22	168	no	$\langle 4; 8; 6, 28\rangle$
8989	PD	22	168	no	$\langle 4; 8; 4, 42\rangle$
494	PD	23	48	yes	$\langle 48; 48; 6, 8\rangle$
496	PD	23	48	yes	$\langle 48; 48; 4, 8\rangle$
498	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
500	PD	23	48	yes	$\langle 48; 48; 8, 12\rangle$
564	PD	23	48	yes	$\langle 48; 48; 4, 12\rangle$
577	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
579	PD	23	48	yes	$\langle 48; 48; 8, 12\rangle$
580	PD	23	48	yes	$\langle 48; 48; 4, 8\rangle$
583	PD	23	48	yes	$\langle 48; 48; 4, 6\rangle$
587	PD	23	48	yes	$\langle 48; 48; 6, 8\rangle$
2122	PD	23	88	yes	$\langle 8; 8; 4, 22\rangle$
2126	PD	23	88	yes	$\langle 8; 8; 4, 44\rangle$
5508	PD	23	132	no	$\langle 4; 12; 6, 66\rangle$
9849	PD	23	176	no	$\langle 4; 8; 8, 88\rangle$
9998	PD	23	176	no	$\langle 4; 8; 8, 88\rangle$
638	PD	24	50	no	$\langle 50; 50; 50, 50\rangle$
643	PD	24	50	yes	$\langle 50; 50; 50, 50\rangle$
646	PD	24	50	no	$\langle 50; 50; 10, 50\rangle$
649	PD	24	50	no	$\langle 50; 50; 50, 50\rangle$
652	PD	24	50	no	$\langle 50; 50; 10, 50\rangle$
671	PD	24	52	no	$\langle 26; 52; 52, 52\rangle$
673	PD	24	52	no	$\langle 26; 52; 52, 52\rangle$
678	PD	24	52	no	$\langle 26; 52; 52, 52\rangle$
681	PD	24	52	no	$\langle 26; 52; 4, 52\rangle$
685	PD	24	52	no	$\langle 26; 52; 52, 52\rangle$
687	PD	24	52	no	$\langle 26; 52; 52, 52\rangle$
738	PD	24	54	no	$\langle 18; 54; 54, 54\rangle$
741	PD	24	54	no	$\langle 18; 54; 54, 54\rangle$
744	PD	24	54	no	$\langle 18; 54; 54, 54\rangle$
806	PD	24	56	no	$\langle 14; 56; 8, 56\rangle$
807	PD	24	56	no	$\langle 14; 56; 56, 56\rangle$
808	PD	24	56	no	$\langle 14; 56; 56, 56\rangle$
883	PD	24	60	no	$\langle 12; 30; 20, 60\rangle$
887	PD	24	60	no	$\langle 10; 60; 12, 60\rangle$
888	PD	24	60	no	$\langle 10; 60; 60, 60\rangle$
1094	PD	24	64	no	$\langle 8; 64; 64, 64\rangle$
1396	PD	24	72	no	$\langle 6; 72; 72, 72\rangle$
670	PD	25	52	no	$\langle 52; 52; 26, 26\rangle$
677	PD	25	52	yes	$\langle 52; 52; 26, 26\rangle$
684	PD	25	52	no	$\langle 52; 52; 26, 26\rangle$
770	PD	25	56	yes	$\langle 28; 28; 4, 14\rangle$
773	PD	25	56	no	$\langle 28; 28; 14, 28\rangle$
775	PD	25	56	no	$\langle 28; 28; 14, 28\rangle$
777	PD	25	56	no	$\langle 28; 28; 28, 28\rangle$
778	PD	25	56	yes	$\langle 28; 28; 4, 28\rangle$
869	PD	25	60	yes	$\langle 20; 20; 6, 10\rangle$
870	PD	25	60	no	$\langle 20; 20; 10, 30\rangle$
877	PD	25	60	no	$\langle 12; 60; 10, 30\rangle$
1032	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
1033	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
1034	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1036	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1037	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1048	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1049	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1050	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1051	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1054	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1055	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1056	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1057	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
1058	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
1059	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
1072	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1073	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1074	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1075	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1076	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1077	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
					Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
1078	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1079	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1080	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1082	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1083	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1085	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
1115	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1116	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1128	PD	25	64	yes	$\langle 16; 16; 4, 16\rangle$
1134	PD	25	64	yes	$\langle 16; 16; 4, 16\rangle$
1136	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1145	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1146	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
1147	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1148	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1149	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
1153	PD	25	64	yes	$\langle 16; 16; 8, 16\rangle$
1154	PD	25	64	yes	$\langle 16; 16; 8, 16\rangle$
1362	PD	25	72	yes	$\langle 12; 12; 6, 12\rangle$
1429	PD	25	72	yes	$\langle 12; 12; 6, 12\rangle$
1431	PD	25	72	yes	$\langle 12; 12; 12, 12\rangle$
1432	PD	25	72	yes	$\langle 12; 12; 4, 12\rangle$
1433	PD	25	72	yes	$\langle 12; 12; 4, 12\rangle$
1435	PD	25	72	yes	$\langle 12; 12; 4, 12\rangle$
1438	PD	25	72	no	$\langle 8; 24; 6, 12\rangle$
1439	PD	25	72	no	$\langle 8; 24; 6, 12\rangle$
1460	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
1464	PD	25	72	yes	$\langle 12; 12; 4, 12\rangle$
1465	PD	25	72	no	$\langle 12; 12; 6, 12\rangle$
1466	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
1471	PD	25	72	no	$\langle 8; 24; 6, 6\rangle$
1476	PD	25	72	no	$\langle 8; 24; 12, 12\rangle$
1477	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
1482	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
1484	PD	25	72	yes	$\langle 12; 12; 6, 12\rangle$
1515	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
1849	PD	25	80	yes	$\langle 10; 10; 4, 4\rangle$
2484	PD	25	96	yes	$\langle 8; 8; 4, 24\rangle$
2486	PD	25	96	yes	$\langle 8; 8; 6, 8\rangle$
2487	PD	25	96	yes	$\langle 8; 8; 8, 24\rangle$
2491	PD	25	96	no	$\langle 8; 8; 4, 6\rangle$
2492	PD	25	96	no	$\langle 8; 8; 4, 12\rangle$
2493	PD	25	96	no	$\langle 8; 8; 4, 6\rangle$
2501	PD	25	96	no	$\langle 8; 8; 4, 12\rangle$
2512	PD	25	96	yes	$\langle 8; 8; 4, 12\rangle$
2513	PD	25	96	yes	$\langle 8; 8; 4, 6\rangle$
2514	PD	25	96	yes	$\langle 8; 8; 4, 6\rangle$
2518	PD	25	96	yes	$\langle 8; 8; 4, 12\rangle$
2534	PD	25	96	yes	$\langle 8; 8; 4, 12\rangle$
2563	PD	25	96	yes	$\langle 8; 8; 8, 12\rangle$
2572	PD	25	96	yes	$\langle 8; 8; 8, 12\rangle$
2574	PD	25	96	yes	$\langle 8; 8; 8, 24\rangle$
2578	PD	25	96	yes	$\langle 8; 8; 4, 24\rangle$
2605	PD	25	96	yes	$\langle 8; 8; 4, 24\rangle$
2611	PD	25	96	no	$\langle 8; 8; 8, 24\rangle$
2612	PD	25	96	yes	$\langle 8; 8; 4, 24\rangle$
2615	PD	25	96	yes	$\langle 8; 8; 8, 12\rangle$
2616	PD	25	96	yes	$\langle 8; 8; 8, 12\rangle$
2617	PD	25	96	yes	$\langle 8; 8; 4, 6\rangle$
2627	PD	25	96	no	$\langle 6; 12; 4, 12\rangle$
2695	PD	25	96	yes	$\langle 8; 8; 4, 24\rangle$
2696	PD	25	96	yes	$\langle 8; 8; 8, 12\rangle$
2699	PD	25	96	yes	$\langle 8; 8; 6, 8\rangle$
2722	PD	25	96	yes	$\langle 8; 8; 8, 24\rangle$
2731	PD	25	96	yes	$\langle 8; 8; 4, 6\rangle$
2736	PD	25	96	yes	$\langle 8; 8; 4, 12\rangle$

Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
2737	PD	25	96	yes	$\langle 8; 8; 4, 12\rangle$
2758	PD	25	96	yes	$\langle 8; 8; 4, 48\rangle$
2759	PD	25	96	yes	$\langle 8; 8; 12, 16\rangle$
2793	PD	25	96	yes	$\langle 8; 8; 6, 16\rangle$
2805	PD	25	96	no	$\langle 6; 12; 4, 8\rangle$
2874	PD	25	96	no	$\langle 6; 12; 4, 8\rangle$
2875	PD	25	96	no	$\langle 6; 12; 4, 8\rangle$
3244	PD	25	104	no	$\langle 4; 52; 26, 52\rangle$
3380	PD	25	108	no	$\langle 4; 36; 18, 18\rangle$
3675	PD	25	112	no	$\langle 4; 28; 14, 56\rangle$
3810	PD	25	112	no	$\langle 4; 28; 28, 56\rangle$
4130	PD	25	120	no	$\langle 4; 20; 10, 60\rangle$
4326	PD	25	120	no	$\langle 4; 20; 20, 60\rangle$
4342	PD	25	120	no	$\langle 4; 20; 20, 30\rangle$
4419	PD	25	120	no	$\langle 4; 20; 6, 10\rangle$
4421	PD	25	120	no	$\langle 4; 20; 6, 10\rangle$
4854	PD	25	128	no	$\langle 4; 16; 16, 16\rangle$
5023	PD	25	128	no	$\langle 4; 16; 16, 16\rangle$
5225	PD	25	128	no	$\langle 4; 16; 8, 16\rangle$
5232	PD	25	128	no	$\langle 4; 16; 8, 16\rangle$
5294	PD	25	128	no	$\langle 4; 16; 4, 8\rangle$
5302	PD	25	128	no	$\langle 4; 16; 4, 8\rangle$
5303	PD	25	128	no	$\langle 4; 16; 4, 8\rangle$
5307	PD	25	128	no	$\langle 4; 16; 8, 8\rangle$
5328	PD	25	128	no	$\langle 4; 16; 8, 8\rangle$
6313	PD	25	144	no	$\langle 4; 12; 12, 12\rangle$
6373	PD	25	144	no	$\langle 4; 12; 6, 6\rangle$
6619	PD	25	144	no	$\langle 4; 12; 6, 24\rangle$
6621	PD	25	144	no	$\langle 4; 12; 6, 24\rangle$
6650	PD	25	144	no	$\langle 4; 12; 12, 24\rangle$
6651	PD	25	144	no	$\langle 4; 12; 12, 24\rangle$
6812	PD	25	144	no	$\langle 4; 12; 4, 12\rangle$
6815	PD	25	144	no	$\langle 4; 12; 6, 8\rangle$
6818	PD	25	144	no	$\langle 4; 12; 8, 12\rangle$
6847	PD	25	144	yes	$\langle 6; 6; 8, 24\rangle$
6909	PD	25	144	no	$\langle 6; 6; 6, 24\rangle$
6910	PD	25	144	yes	$\langle 6; 6; 6, 8\rangle$
6917	PD	25	144	yes	$\langle 6; 6; 6, 12\rangle$
11430	PD	25	192	no	$\langle 4; 8; 6, 8\rangle$
11431	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
11447	PD	25	192	no	$\langle 4; 8; 4, 12\rangle$
11485	PD	25	192	no	$\langle 4; 8; 4, 24\rangle$
11581	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
11617	PD	25	192	no	$\langle 4; 8; 4, 24\rangle$
11619	PD	25	192	no	$\langle 4; 8; 8, 12\rangle$
11724	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
11844	PD	25	192	no	$\langle 4; 8; 4, 24\rangle$
11981	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
11987	PD	25	192	no	$\langle 4; 8; 8, 12\rangle$
11993	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
12174	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
12351	PD	25	192	no	$\langle 4; 8; 4, 48\rangle$
12352	PD	25	192	no	$\langle 4; 8; 12, 16\rangle$
12448	PD	25	192	no	$\langle 4; 8; 16, 24\rangle$
12449	PD	25	192	no	$\langle 4; 8; 8, 48\rangle$
12503	PD	25	192	no	$\langle 4; 8; 8, 12\rangle$
12504	PD	25	192	no	$\langle 4; 8; 4, 24\rangle$
12527	PD	25	192	no	$\langle 4; 8; 4, 12\rangle$
12577	PD	25	192	no	$\langle 4; 8; 8, 24\rangle$
12723	PD	25	192	no	$\langle 4; 8; 4, 6\rangle$
726	PD	26	54	no	$\langle 54; 54; 18, 54\rangle$
729	PD	26	54	no	$\langle 54; 54; 18, 54\rangle$
732	PD	26	54	no	$\langle 54; 54; 18, 54\rangle$
735	PD	26	54	no	$\langle 54; 54; 6, 54\rangle$
798	PD	26	56	no	$\langle 28; 56; 8, 56\rangle$
799	PD	26	56	no	$\langle 28; 56; 56, 56\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
800	PD	26	56	no	$\langle 28; 56; 56, 56\rangle$
885	PD	26	60	no	$\langle 20; 30; 12, 60\rangle$
886	PD	26	60	no	$\langle 20; 30; 60, 60\rangle$
1703	PD	26	78	no	$\langle 6; 78; 26, 78\rangle$
3061	PD	26	100	no	$\langle 8; 8; 10, 10\rangle$
7329	PD	26	150	yes	$\langle 6; 6; 6, 10\rangle$
7338	PD	26	150	yes	$\langle 6; 6; 10, 10\rangle$
774	PD	27	56	no	$\langle 56; 56; 14, 14\rangle$
779	PD	27	56	no	$\langle 56; 56; 14, 28\rangle$
780	PD	27	56	no	$\langle 56; 56; 14, 28\rangle$
783	PD	27	56	yes	$\langle 56; 56; 4, 14\rangle$
809	PD	27	56	yes	$\langle 56; 56; 4, 28\rangle$
811	PD	27	56	no	$\langle 56; 56; 28, 28\rangle$
873	PD	27	60	no	$\langle 20; 60; 6, 30\rangle$
876	PD	27	60	no	$\langle 20; 60; 30, 30\rangle$
1063	PD	27	64	no	$\langle 16; 32; 32, 32\rangle$
1158	PD	27	64	no	$\langle 16; 32; 32, 32\rangle$
1423	PD	27	72	no	$\langle 8; 72; 18, 36\rangle$
1495	PD	27	72	no	$\langle 12; 18; 18, 18\rangle$
3252	PD	27	104	yes	$\langle 8; 8; 4, 26\rangle$
3256	PD	27	104	yes	$\langle 8; 8; 4, 52\rangle$
7811	PD	27	156	no	$\langle 4; 12; 6, 78\rangle$
14518	PD	27	208	no	$\langle 4; 8; 8, 104\rangle$
14687	PD	27	208	no	$\langle 4; 8; 8, 104\rangle$
834	PD	28	58	no	$\langle 58; 58; 58, 58\rangle$
837	PD	28	58	no	$\langle 58; 58; 58, 58\rangle$
840	PD	28	58	no	$\langle 58; 58; 58, 58\rangle$
843	PD	28	58	no	$\langle 58; 58; 58, 58\rangle$
846	PD	28	58	no	$\langle 58; 58; 58, 58\rangle$
849	PD	28	58	no	$\langle 58; 58; 58, 58\rangle$
851	PD	28	58	yes	$\langle 58; 58; 58, 58\rangle$
878	PD	28	60	no	$\langle 30; 60; 4, 60\rangle$
879	PD	28	60	no	$\langle 30; 60; 20, 60\rangle$
880	PD	28	60	no	$\langle 30; 60; 20, 60\rangle$
881	PD	28	60	no	$\langle 30; 60; 12, 20\rangle$
884	PD	28	60	no	$\langle 30; 60; 4, 12\rangle$
889	PD	28	60	no	$\langle 30; 60; 4, 20\rangle$
890	PD	28	60	no	$\langle 30; 60; 20, 20\rangle$
1093	PD	28	64	no	$\langle 16; 64; 64, 64\rangle$
1096	PD	28	64	no	$\langle 16; 64; 64, 64\rangle$
1321	PD	28	70	no	$\langle 10; 70; 14, 70\rangle$
1333	PD	28	70	no	$\langle 10; 70; 70, 70\rangle$
1449	PD	28	72	no	$\langle 12; 24; 4, 6\rangle$
1467	PD	28	72	no	$\langle 12; 24; 8, 24\rangle$
1959	PD	28	84	no	$\langle 6; 84; 28, 84\rangle$
2292	PD	28	90	no	$\langle 6; 30; 10, 30\rangle$
3504	PD	28	108	no	$\langle 6; 12; 4, 12\rangle$
3514	PD	28	108	no	$\langle 6; 12; 4, 12\rangle$
3520	PD	28	108	no	$\langle 6; 12; 12, 12\rangle$
3538	PD	28	108	no	$\langle 6; 12; 4, 12\rangle$
6869	PD	28	144	no	$\langle 4; 16; 12, 24\rangle$
6870	PD	28	144	no	$\langle 4; 16; 8, 36\rangle$
6921	PD	28	144	no	$\langle 4; 16; 12, 24\rangle$
6922	PD	28	144	no	$\langle 4; 16; 8, 36\rangle$
8474	PD	28	162	yes	$\langle 6; 6; 6, 6\rangle$
8477	PD	28	162	no	$\langle 4; 12; 6, 18\rangle$
8588	PD	28	162	yes	$\langle 6; 6; 6, 18\rangle$
8605	PD	28	162	no	$\langle 4; 12; 6, 6\rangle$
8618	PD	28	162	no	$\langle 4; 12; 6, 18\rangle$
8627	PD	28	162	no	$\langle 6; 6; 18, 18\rangle$
8630	PD	28	162	no	$\langle 6; 6; 6, 18\rangle$
8635	PD	28	162	no	$\langle 4; 12; 6, 18\rangle$
15416	PD	28	216	no	$\langle 4; 8; 6, 36\rangle$
15917	PD	28	216	no	$\langle 4; 8; 4, 54\rangle$
15937	PD	28	216	no	$\langle 4; 8; 12, 18\rangle$
868	PD	29	60	yes	$\langle 60; 60; 10, 10\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
874	PD	29	60	yes	$\langle 60; 60; 6, 10\rangle$
875	PD	29	60	no	$\langle 60; 60; 10, 30\rangle$
1039	PD	29	64	no	$\langle 32; 32; 8, 16\rangle$
1040	PD	29	64	yes	$\langle 32; 32; 8, 16\rangle$
1041	PD	29	64	no	$\langle 32; 32; 8, 16\rangle$
1042	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1043	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1044	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1045	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1088	PD	29	64	yes	$\langle 32; 32; 4, 4\rangle$
1089	PD	29	64	yes	$\langle 32; 32; 8, 16\rangle$
1091	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1119	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1121	PD	29	64	yes	$\langle 32; 32; 4, 4\rangle$
1122	PD	29	64	no	$\langle 32; 32; 8, 16\rangle$
1124	PD	29	64	no	$\langle 32; 32; 8, 16\rangle$
1125	PD	29	64	no	$\langle 32; 32; 8, 16\rangle$
1126	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1131	PD	29	64	yes	$\langle 32; 32; 4, 16\rangle$
1132	PD	29	64	yes	$\langle 32; 32; 4, 8\rangle$
1133	PD	29	64	yes	$\langle 32; 32; 4, 8\rangle$
1150	PD	29	64	no	$\langle 32; 32; 4, 16\rangle$
1151	PD	29	64	yes	$\langle 32; 32; 8, 16\rangle$
1152	PD	29	64	yes	$\langle 32; 32; 8, 16\rangle$
1384	PD	29	72	no	$\langle 12; 36; 18, 36\rangle$
1386	PD	29	72	no	$\langle 12; 36; 18, 36\rangle$
1387	PD	29	72	no	$\langle 12; 36; 36, 36\rangle$
1498	PD	29	72	no	$\langle 18; 18; 6, 18\rangle$
1752	PD	29	80	no	$\langle 8; 40; 10, 20\rangle$
1764	PD	29	80	no	$\langle 8; 40; 20, 20\rangle$
1814	PD	29	80	no	$\langle 8; 40; 20, 40\rangle$
1822	PD	29	80	no	$\langle 8; 40; 10, 40\rangle$
1941	PD	29	84	yes	$\langle 12; 12; 6, 14\rangle$
2509	PD	29	96	no	$\langle 6; 24; 8, 24\rangle$
2523	PD	29	96	no	$\langle 6; 24; 8, 8\rangle$
2702	PD	29	96	no	$\langle 8; 12; 12, 24\rangle$
2842	PD	29	96	no	$\langle 6; 24; 8, 24\rangle$
2857	PD	29	96	no	$\langle 6; 24; 4, 8\rangle$
2868	PD	29	96	no	$\langle 8; 12; 6, 12\rangle$
3699	PD	29	112	yes	$\langle 8; 8; 4, 14\rangle$
3702	PD	29	112	yes	$\langle 8; 8; 4, 28\rangle$
3705	PD	29	112	yes	$\langle 8; 8; 4, 28\rangle$
3708	PD	29	112	yes	$\langle 8; 8; 4, 28\rangle$
3711	PD	29	112	yes	$\langle 8; 8; 4, 28\rangle$
3771	PD	29	112	yes	$\langle 8; 8; 8, 28\rangle$
3774	PD	29	112	yes	$\langle 8; 8; 4, 56\rangle$
3785	PD	29	112	yes	$\langle 8; 8; 8, 14\rangle$
4309	PD	29	120	no	$\langle 4; 60; 30, 60\rangle$
4427	PD	29	120	no	$\langle 6; 10; 6, 10\rangle$
4452	PD	29	120	no	$\langle 6; 10; 4, 6\rangle$
5276	PD	29	128	no	$\langle 4; 32; 32, 32\rangle$
6049	PD	29	140	no	$\langle 4; 20; 10, 70\rangle$
8932	PD	29	168	no	$\langle 4; 12; 12, 84\rangle$
9162	PD	29	168	no	$\langle 4; 12; 6, 84\rangle$
9163	PD	29	168	no	$\langle 4; 12; 12, 42\rangle$
16846	PD	29	224	no	$\langle 4; 8; 8, 56\rangle$
17066	PD	29	224	no	$\langle 4; 8; 8, 28\rangle$
17069	PD	29	224	no	$\langle 4; 8; 4, 56\rangle$
17122	PD	29	224	no	$\langle 4; 8; 4, 28\rangle$
41154	PD	29	336	no	$\langle 4; 6; 6, 8\rangle$
41233	PD	29	336	no	$\langle 4; 6; 4, 14\rangle$
987	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
990	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
993	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
996	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
999	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
					Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
1002	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
1006	PD	30	62	no	$\langle 62; 62; 62, 62\rangle$
1095	PD	30	64	no	$\langle 32; 64; 64, 64\rangle$
1101	PD	30	64	no	$\langle 32; 64; 64, 64\rangle$
1102	PD	30	64	no	$\langle 32; 64; 64, 64\rangle$
1104	PD	30	64	no	$\langle 32; 64; 64, 64\rangle$
1200	PD	30	66	no	$\langle 22; 66; 66, 66\rangle$
1203	PD	30	66	no	$\langle 22; 66; 66, 66\rangle$
1210	PD	30	66	no	$\langle 22; 66; 6, 66\rangle$
1222	PD	30	66	no	$\langle 22; 66; 66, 66\rangle$
1225	PD	30	66	no	$\langle 22; 66; 66, 66\rangle$
1342	PD	30	70	no	$\langle 14; 70; 10, 70\rangle$
1343	PD	30	70	no	$\langle 14; 70; 70, 70\rangle$
1344	PD	30	70	no	$\langle 14; 70; 70, 70\rangle$
1393	PD	30	72	no	$\langle 12; 72; 72, 72\rangle$
1395	PD	30	72	no	$\langle 18; 24; 72, 72\rangle$
1772	PD	30	80	no	$\langle 8; 80; 80, 80\rangle$
2272	PD	30	90	no	$\langle 6; 90; 90, 90\rangle$

Type 2^P - nonorientable genus 3 to 30

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
2	PD	4	6	no	$\langle 4; 12; 3, 6\rangle$
1	PD	5	5	yes	$\langle 10; 10; 5, 5\rangle$
19	PD	5	12	no	$\langle 4; 8; 3, 4\rangle$
9	PD	6	10	no	$\langle 4; 20; 5, 10\rangle$
3	PD	7	7	no	$\langle 14; 14; 7, 7\rangle$
5	PD	7	9	no	$\langle 6; 18; 9, 9\rangle$
4	PD	8	8	no	$\langle 16; 16; 4, 8\rangle$
15	PD	8	12	yes	$\langle 8; 8; 4, 6\rangle$
18	PD	8	12	yes	$\langle 8; 8; 3, 4\rangle$
26	PD	8	14	no	$\langle 4; 28; 7, 14\rangle$
45	PD	8	18	no	$\langle 4; 12; 3, 6\rangle$
113	PD	8	24	no	$\langle 4; 8; 3, 4\rangle$
114	PD	8	24	no	$\langle 4; 8; 4, 6\rangle$
6	PD	9	9	no	$\langle 18; 18; 3, 9\rangle$
7	PD	10	10	no	$\langle 20; 20; 5, 10\rangle$
13	PD	10	12	no	$\langle 8; 24; 6, 12\rangle$
16	PD	10	12	no	$\langle 8; 24; 3, 12\rangle$
54	PD	10	18	no	$\langle 4; 36; 9, 18\rangle$
115	PD	10	24	no	$\langle 4; 12; 3, 6\rangle$
10	PD	11	11	no	$\langle 22; 22; 11, 11\rangle$
11	PD	11	11	no	$\langle 22; 22; 11, 11\rangle$
31	PD	11	15	no	$\langle 6; 30; 5, 15\rangle$
189	PD	11	30	no	$\langle 4; 10; 3, 5\rangle$
298	PD	11	36	no	$\langle 4; 8; 4, 9\rangle$
14	PD	12	12	yes	$\langle 24; 24; 4, 6\rangle$
17	PD	12	12	yes	$\langle 24; 24; 3, 4\rangle$
59	PD	12	20	yes	$\langle 8; 8; 4, 10\rangle$
60	PD	12	20	yes	$\langle 8; 8; 4, 5\rangle$
77	PD	12	22	no	$\langle 4; 44; 11, 22\rangle$
116	PD	12	24	no	$\langle 6; 8; 3, 6\rangle$
165	PD	12	30	no	$\langle 4; 12; 6, 15\rangle$
20	PD	13	13	no	$\langle 26; 26; 13, 13\rangle$
21	PD	13	13	yes	$\langle 26; 26; 13, 13\rangle$
22	PD	13	13	no	$\langle 26; 26; 13, 13\rangle$
27	PD	13	15	no	$\langle 10; 30; 3, 15\rangle$
30	PD	13	15	no	$\langle 10; 30; 15, 15\rangle$
23	PD	14	14	no	$\langle 28; 28; 7, 14\rangle$
24	PD	14	14	no	$\langle 28; 28; 7, 14\rangle$
46	PD	14	18	yes	$\langle 12; 12; 3, 6\rangle$
47	PD	14	18	yes	$\langle 12; 12; 3, 6\rangle$
117	PD	14	24	no	$\langle 6; 12; 4, 6\rangle$
118	PD	14	24	no	$\langle 6; 12; 3, 4\rangle$
134	PD	14	26	no	$\langle 4; 52; 13, 26\rangle$
171	PD	14	30	no	$\langle 4; 20; 10, 15\rangle$
					Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
299	PD	14	36	yes	$\langle 6; 6; 3, 6\rangle$
303	PD	14	36	no	$\langle 4; 12; 4, 6\rangle$
521	PD	14	48	no	$\langle 4; 8; 4, 12\rangle$
523	PD	14	48	no	$\langle 4; 8; 4, 12\rangle$
28	PD	15	15	yes	$\langle 30; 30; 3, 5\rangle$
29	PD	15	15	no	$\langle 30; 30; 5, 15\rangle$
76	PD	15	21	no	$\langle 6; 42; 7, 21\rangle$
35	PD	16	16	no	$\langle 32; 32; 8, 16\rangle$
36	PD	16	16	no	$\langle 32; 32; 4, 16\rangle$
37	PD	16	16	no	$\langle 32; 32; 8, 16\rangle$
48	PD	16	18	no	$\langle 12; 36; 9, 18\rangle$
49	PD	16	18	no	$\langle 12; 36; 9, 18\rangle$
68	PD	16	20	no	$\langle 8; 40; 10, 20\rangle$
69	PD	16	20	no	$\langle 8; 40; 5, 20\rangle$
147	PD	16	28	yes	$\langle 8; 8; 4, 14\rangle$
148	PD	16	28	yes	$\langle 8; 8; 4, 7\rangle$
186	PD	16	30	no	$\langle 4; 60; 15, 30\rangle$
190	PD	16	30	no	$\langle 6; 10; 3, 5\rangle$
392	PD	16	42	no	$\langle 4; 12; 6, 21\rangle$
41	PD	17	17	no	$\langle 34; 34; 17, 17\rangle$
42	PD	17	17	yes	$\langle 34; 34; 17, 17\rangle$
43	PD	17	17	no	$\langle 34; 34; 17, 17\rangle$
44	PD	17	17	no	$\langle 34; 34; 17, 17\rangle$
933	PD	17	60	no	$\langle 4; 8; 4, 15\rangle$
959	PD	17	60	no	$\langle 4; 8; 5, 6\rangle$
51	PD	18	18	no	$\langle 36; 36; 6, 9\rangle$
52	PD	18	18	no	$\langle 36; 36; 3, 18\rangle$
254	PD	18	34	no	$\langle 4; 68; 17, 34\rangle$
601	PD	18	48	yes	$\langle 6; 6; 3, 4\rangle$
55	PD	19	19	no	$\langle 38; 38; 19, 19\rangle$
56	PD	19	19	no	$\langle 38; 38; 19, 19\rangle$
57	PD	19	19	no	$\langle 38; 38; 19, 19\rangle$
58	PD	19	19	no	$\langle 38; 38; 19, 19\rangle$
71	PD	19	21	no	$\langle 14; 42; 3, 21\rangle$
73	PD	19	21	no	$\langle 14; 42; 21, 21\rangle$
74	PD	19	21	no	$\langle 14; 42; 21, 21\rangle$
143	PD	19	27	no	$\langle 6; 54; 27, 27\rangle$
62	PD	20	20	yes	$\langle 40; 40; 4, 10\rangle$
63	PD	20	20	yes	$\langle 40; 40; 4, 5\rangle$
66	PD	20	20	no	$\langle 40; 40; 10, 20\rangle$
67	PD	20	20	no	$\langle 40; 40; 5, 20\rangle$
99	PD	20	24	yes	$\langle 16; 16; 3, 8\rangle$
104	PD	20	24	no	$\langle 16; 16; 8, 12\rangle$
108	PD	20	24	yes	$\langle 16; 16; 6, 8\rangle$
270	PD	20	36	yes	$\langle 8; 8; 4, 18\rangle$
271	PD	20	36	yes	$\langle 8; 8; 4, 9\rangle$
305	PD	20	36	yes	$\langle 8; 8; 3, 4\rangle$
324	PD	20	38	no	$\langle 4; 76; 19, 38\rangle$
410	PD	20	42	no	$\langle 4; 28; 14, 21\rangle$
609	PD	20	48	no	$\langle 4; 16; 6, 8\rangle$
611	PD	20	48	no	$\langle 4; 16; 3, 8\rangle$
709	PD	20	54	no	$\langle 4; 12; 6, 9\rangle$
752	PD	20	54	no	$\langle 4; 12; 3, 6\rangle$
940	PD	20	60	no	$\langle 4; 10; 3, 5\rangle$
943	PD	20	60	no	$\langle 4; 10; 6, 10\rangle$
1490	PD	20	72	no	$\langle 4; 8; 4, 9\rangle$
1491	PD	20	72	no	$\langle 4; 8; 4, 18\rangle$
70	PD	21	21	yes	$\langle 42; 42; 3, 7\rangle$
72	PD	21	21	no	$\langle 42; 42; 7, 21\rangle$
75	PD	21	21	no	$\langle 42; 42; 7, 21\rangle$
119	PD	21	25	no	$\langle 10; 50; 25, 25\rangle$
121	PD	21	25	no	$\langle 10; 50; 25, 25\rangle$
78	PD	22	22	no	$\langle 44; 44; 11, 22\rangle$
79	PD	22	22	no	$\langle 44; 44; 11, 22\rangle$
81	PD	22	22	no	$\langle 44; 44; 11, 22\rangle$
82	PD	22	22	no	$\langle 44; 44; 11, 22\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
97	PD	22	24	no	$\langle 16; 48; 3, 24\rangle$
101	PD	22	24	no	$\langle 16; 48; 12, 24\rangle$
102	PD	22	24	no	$\langle 16; 48; 12, 24\rangle$
106	PD	22	24	no	$\langle 16; 48; 6, 24\rangle$
156	PD	22	28	no	$\langle 8; 56; 14, 28\rangle$
157	PD	22	28	no	$\langle 8; 56; 7, 28\rangle$
166	PD	22	30	yes	$\langle 12; 12; 5, 6\rangle$
414	PD	22	42	no	$\langle 4; 84; 21, 42\rangle$
610	PD	22	48	no	$\langle 4; 24; 6, 12\rangle$
612	PD	22	48	no	$\langle 4; 24; 3, 12\rangle$
629	PD	22	50	no	$\langle 4; 20; 5, 10\rangle$
953	PD	22	60	no	$\langle 4; 12; 5, 10\rangle$
954	PD	22	60	no	$\langle 4; 12; 5, 10\rangle$
969	PD	22	60	no	$\langle 4; 12; 4, 5\rangle$
1848	PD	22	80	no	$\langle 4; 8; 4, 5\rangle$
84	PD	23	23	no	$\langle 46; 46; 23, 23\rangle$
85	PD	23	23	no	$\langle 46; 46; 23, 23\rangle$
86	PD	23	23	no	$\langle 46; 46; 23, 23\rangle$
87	PD	23	23	no	$\langle 46; 46; 23, 23\rangle$
88	PD	23	23	no	$\langle 46; 46; 23, 23\rangle$
135	PD	23	27	yes	$\langle 18; 18; 3, 9\rangle$
232	PD	23	33	no	$\langle 6; 66; 11, 33\rangle$
2024	PD	23	84	no	$\langle 4; 8; 4, 21\rangle$
98	PD	24	24	yes	$\langle 48; 48; 3, 8\rangle$
100	PD	24	24	no	$\langle 48; 48; 4, 24\rangle$
103	PD	24	24	no	$\langle 48; 48; 8, 12\rangle$
107	PD	24	24	yes	$\langle 48; 48; 6, 8\rangle$
185	PD	24	30	no	$\langle 12; 20; 15, 30\rangle$
426	PD	24	44	yes	$\langle 8; 8; 4, 22\rangle$
427	PD	24	44	yes	$\langle 8; 8; 4, 11\rangle$
481	PD	24	46	no	$\langle 4; 92; 23, 46\rangle$
1183	PD	24	66	no	$\langle 4; 12; 6, 33\rangle$
120	PD	25	25	no	$\langle 50; 50; 25, 25\rangle$
122	PD	25	25	yes	$\langle 50; 50; 25, 25\rangle$
123	PD	25	25	no	$\langle 50; 50; 5, 25\rangle$
124	PD	25	25	no	$\langle 50; 50; 25, 25\rangle$
125	PD	25	25	no	$\langle 50; 50; 5, 25\rangle$
140	PD	25	27	no	$\langle 18; 54; 27, 27\rangle$
141	PD	25	27	no	$\langle 18; 54; 27, 27\rangle$
142	PD	25	27	no	$\langle 18; 54; 27, 27\rangle$
126	PD	26	26	no	$\langle 52; 52; 13, 26\rangle$
127	PD	26	26	no	$\langle 52; 52; 13, 26\rangle$
129	PD	26	26	no	$\langle 52; 52; 13, 26\rangle$
131	PD	26	26	no	$\langle 52; 52; 13, 26\rangle$
132	PD	26	26	no	$\langle 52; 52; 13, 26\rangle$
168	PD	26	30	yes	$\langle 20; 20; 3, 10\rangle$
170	PD	26	30	no	$\langle 20; 20; 10, 15\rangle$
174	PD	26	30	no	$\langle 12; 60; 10, 15\rangle$
177	PD	26	30	no	$\langle 12; 60; 5, 30\rangle$
293	PD	26	36	no	$\langle 8; 24; 6, 12\rangle$
296	PD	26	36	no	$\langle 8; 24; 3, 12\rangle$
302	PD	26	36	yes	$\langle 12; 12; 4, 6\rangle$
520	PD	26	48	no	$\langle 8; 8; 4, 6\rangle$
522	PD	26	48	no	$\langle 8; 8; 3, 4\rangle$
527	PD	26	48	yes	$\langle 8; 8; 3, 4\rangle$
528	PD	26	48	yes	$\langle 8; 8; 4, 6\rangle$
529	PD	26	48	yes	$\langle 8; 8; 3, 4\rangle$
635	PD	26	50	no	$\langle 4; 100; 25, 50\rangle$
723	PD	26	54	no	$\langle 4; 36; 9, 18\rangle$
939	PD	26	60	no	$\langle 4; 20; 3, 10\rangle$
941	PD	26	60	no	$\langle 4; 20; 5, 6\rangle$
944	PD	26	60	no	$\langle 4; 20; 5, 6\rangle$
945	PD	26	60	no	$\langle 4; 20; 3, 10\rangle$
1461	PD	26	72	no	$\langle 4; 12; 3, 6\rangle$
2497	PD	26	96	no	$\langle 4; 8; 4, 24\rangle$
2499	PD	26	96	no	$\langle 4; 8; 4, 24\rangle$
2823	PD	26	96	no	$\langle 4; 8; 4, 6\rangle$
136	PD	27	27	no	$\langle 54; 54; 9, 27\rangle$
137	PD	27	27	no	$\langle 54; 54; 9, 27\rangle$
138	PD	27	27	no	$\langle 54; 54; 9, 27\rangle$
139	PD	27	27	no	$\langle 54; 54; 3, 27\rangle$
340	PD	27	39	no	$\langle 6; 78; 13, 39\rangle$
1596	PD	27	75	yes	$\langle 6; 6; 3, 5\rangle$
145	PD	28	28	yes	$\langle 56; 56; 4, 14\rangle$
146	PD	28	28	yes	$\langle 56; 56; 4, 7\rangle$
151	PD	28	28	no	$\langle 56; 56; 14, 28\rangle$
153	PD	28	28	no	$\langle 56; 56; 14, 28\rangle$
154	PD	28	28	no	$\langle 56; 56; 7, 28\rangle$
155	PD	28	28	no	$\langle 56; 56; 7, 28\rangle$
181	PD	28	30	no	$\langle 20; 60; 6, 15\rangle$
182	PD	28	30	no	$\langle 20; 60; 15, 30\rangle$
183	PD	28	30	no	$\langle 20; 60; 15, 30\rangle$
184	PD	28	30	no	$\langle 20; 60; 3, 30\rangle$
289	PD	28	36	no	$\langle 8; 72; 18, 36\rangle$
290	PD	28	36	no	$\langle 8; 72; 9, 36\rangle$
301	PD	28	36	no	$\langle 12; 18; 9, 9\rangle$
693	PD	28	52	yes	$\langle 8; 8; 4, 26\rangle$
694	PD	28	52	yes	$\langle 8; 8; 4, 13\rangle$
748	PD	28	54	no	$\langle 4; 108; 27, 54\rangle$
1707	PD	28	78	no	$\langle 4; 12; 6, 39\rangle$
158	PD	29	29	no	$\langle 58; 58; 29, 29\rangle$
159	PD	29	29	no	$\langle 58; 58; 29, 29\rangle$
160	PD	29	29	no	$\langle 58; 58; 29, 29\rangle$
161	PD	29	29	no	$\langle 58; 58; 29, 29\rangle$
162	PD	29	29	no	$\langle 58; 58; 29, 29\rangle$
163	PD	29	29	no	$\langle 58; 58; 29, 29\rangle$
164	PD	29	29	yes	$\langle 58; 58; 29, 29\rangle$
264	PD	29	35	no	$\langle 10; 70; 7, 35\rangle$
265	PD	29	35	no	$\langle 10; 70; 35, 35\rangle$
1886	PD	29	81	yes	$\langle 6; 6; 3, 9\rangle$
3495	PD	29	108	no	$\langle 4; 8; 4, 27\rangle$
172	PD	30	30	no	$\langle 60; 60; 10, 15\rangle$
173	PD	30	30	yes	$\langle 60; 60; 3, 10\rangle$
175	PD	30	30	no	$\langle 60; 60; 5, 30\rangle$
176	PD	30	30	yes	$\langle 60; 60; 5, 6\rangle$
300	PD	30	36	no	$\langle 18; 18; 6, 9\rangle$
393	PD	30	42	yes	$\langle 12; 12; 6, 7\rangle$
595	PD	30	48	no	$\langle 8; 12; 3, 6\rangle$
596	PD	30	48	no	$\langle 8; 12; 6, 6\rangle$
831	PD	30	58	no	$\langle 4; 116; 29, 58\rangle$
947	PD	30	60	no	$\langle 6; 10; 3, 5\rangle$
950	PD	30	60	no	$\langle 6; 10; 6, 10\rangle$
964	PD	30	60	no	$\langle 6; 10; 4, 6\rangle$
1303	PD	30	70	no	$\langle 4; 20; 10, 35\rangle$
9302	PD	30	168	no	$\langle 4; 6; 6, 8\rangle$
9327	PD	30	168	no	$\langle 4; 6; 4, 7\rangle$

Continued ...

4.5.3 Type 2ex, 2ex* and 2ex^P maps

Type 2ex - orientable genus 2 to 41

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
3	–	3	21	no	$\langle 14; 3; 6\rangle$
2	–	4	20	yes	$\langle 10; 4; 4\rangle$
10	–	6	39	no	$\langle 26; 3; 6\rangle$
21	–	7	56	no	$\langle 4; 7; 14\rangle$
13	–	9	40	yes	$\langle 20; 4; 4\rangle$
23	–	9	57	no	$\langle 38; 3; 6\rangle$
5	P	10	27	no	$\langle 18; 9; 6\rangle$
27	–	10	63	no	$\langle 42; 3; 6\rangle$
34	–	10	72	no	$\langle 8; 4; 8\rangle$
12	–	12	40	yes	$\langle 10; 8; 8\rangle$
14	–	12	42	no	$\langle 14; 6; 6\rangle$
14	P	12	42	no	$\langle 14; 6; 6\rangle$
17	–	12	52	yes	$\langle 26; 4; 4\rangle$
53	–	12	84	no	$\langle 28; 3; 6\rangle$
11	–	14	40	yes	$\langle 20; 8; 8\rangle$
26	–	14	60	no	$\langle 30; 4; 12\rangle$
19	–	15	55	no	$\langle 22; 5; 10\rangle$
20	–	15	55	no	$\langle 22; 5; 10\rangle$
56	–	15	93	no	$\langle 62; 3; 6\rangle$
18	–	16	54	no	$\langle 18; 6; 18\rangle$
30	–	16	68	yes	$\langle 34; 4; 4\rangle$
33	–	16	72	yes	$\langle 6; 8; 8\rangle$
59	–	16	100	no	$\langle 10; 4; 20\rangle$
42	–	17	80	yes	$\langle 20; 4; 4\rangle$
103	–	17	128	no	$\langle 8; 4; 16\rangle$
79	–	18	111	no	$\langle 74; 3; 6\rangle$
34	P	19	72	no	$\langle 8; 8; 4\rangle$
41	–	19	80	no	$\langle 40; 4; 8\rangle$
44	–	19	80	no	$\langle 40; 4; 8\rangle$
85	–	19	117	no	$\langle 78; 3; 6\rangle$
121	–	19	144	no	$\langle 8; 4; 8\rangle$
282	–	19	216	no	$\langle 12; 3; 8\rangle$
25	–	20	60	yes	$\langle 10; 12; 12\rangle$
28	–	21	63	no	$\langle 14; 9; 18\rangle$
29	–	21	64	yes	$\langle 8; 16; 16\rangle$
57	–	21	96	yes	$\langle 8; 6; 6\rangle$
80	–	21	112	yes	$\langle 4; 14; 14\rangle$
105	–	21	129	no	$\langle 86; 3; 6\rangle$
18	P	22	54	no	$\langle 18; 18; 6\rangle$
26	P	24	60	no	$\langle 30; 12; 4\rangle$
37	–	24	78	no	$\langle 26; 6; 6\rangle$
37	P	24	78	no	$\langle 26; 6; 6\rangle$
60	–	24	100	yes	$\langle 50; 4; 4\rangle$
128	–	24	147	no	$\langle 98; 3; 6\rangle$
141	–	24	156	no	$\langle 52; 3; 6\rangle$
64	–	25	104	yes	$\langle 52; 4; 4\rangle$
51	–	26	84	no	$\langle 28; 6; 12\rangle$
52	–	26	84	no	$\langle 28; 6; 12\rangle$
176	–	26	168	no	$\langle 56; 3; 12\rangle$
237	–	26	200	yes	$\langle 4; 8; 8\rangle$
500	–	26	300	no	$\langle 6; 4; 12\rangle$
1714	–	26	600	no	$\langle 8; 3; 12\rangle$
39	–	27	80	yes	$\langle 20; 8; 8\rangle$
43	–	27	80	yes	$\langle 20; 8; 8\rangle$
40	–	28	80	yes	$\langle 10; 16; 16\rangle$
46	–	28	81	no	$\langle 18; 9; 18\rangle$
84	–	28	116	yes	$\langle 58; 4; 4\rangle$
187	–	28	171	no	$\langle 114; 3; 6\rangle$
205	–	28	189	no	$\langle 42; 3; 6\rangle$
351	–	28	243	no	$\langle 18; 3; 18\rangle$
41	P	29	80	no	$\langle 40; 8; 4\rangle$
44	P	29	80	no	$\langle 40; 8; 4\rangle$
90	–	29	120	no	$\langle 60; 4; 12\rangle$
50	–	30	84	no	$\langle 14; 12; 12\rangle$
50	P	30	84	no	$\langle 14; 12; 12\rangle$
201	–	30	183	no	$\langle 122; 3; 6\rangle$
45	–	32	80	yes	$\langle 20; 16; 16\rangle$
51	P	33	84	no	$\langle 28; 12; 6\rangle$
52	P	33	84	no	$\langle 28; 12; 6\rangle$
104	–	33	128	no	$\langle 8; 8; 16\rangle$
114	–	33	136	yes	$\langle 68; 4; 4\rangle$
150	–	33	160	no	$\langle 20; 4; 8\rangle$
221	–	33	192	no	$\langle 12; 4; 6\rangle$
238	–	33	201	no	$\langle 134; 3; 6\rangle$
71	–	34	108	no	$\langle 36; 6; 36\rangle$
119	–	34	140	no	$\langle 70; 4; 28\rangle$
170	–	35	168	no	$\langle 4; 21; 42\rangle$
58	–	36	100	yes	$\langle 10; 20; 20\rangle$
59	P	36	100	no	$\langle 10; 20; 4\rangle$
63	–	36	104	yes	$\langle 26; 8; 8\rangle$
83	–	36	114	no	$\langle 38; 6; 6\rangle$
83	P	36	114	no	$\langle 38; 6; 6\rangle$
129	–	36	148	yes	$\langle 74; 4; 4\rangle$
285	–	36	219	no	$\langle 146; 3; 6\rangle$
305	–	36	228	no	$\langle 76; 3; 6\rangle$
47	P	37	81	no	$\langle 54; 27; 6\rangle$
74	–	37	108	no	$\langle 18; 9; 12\rangle$
121	P	37	144	no	$\langle 8; 8; 4\rangle$
146	–	37	160	no	$\langle 40; 4; 8\rangle$
151	–	37	160	yes	$\langle 40; 4; 4\rangle$
378	–	37	252	no	$\langle 42; 3; 12\rangle$
456	–	37	288	no	$\langle 8; 4; 8\rangle$
65	–	38	104	yes	$\langle 52; 8; 8\rangle$
137	–	38	156	no	$\langle 78; 4; 12\rangle$
66	–	39	105	no	$\langle 14; 15; 30\rangle$
154	–	39	160	no	$\langle 80; 4; 16\rangle$
158	–	39	160	no	$\langle 80; 4; 16\rangle$
318	–	39	237	no	$\langle 158; 3; 6\rangle$
70	–	40	108	no	$\langle 18; 12; 36\rangle$
73	P	40	108	no	$\langle 36; 9; 6\rangle$
77	–	40	110	no	$\langle 22; 10; 10\rangle$
77	P	40	110	no	$\langle 22; 10; 10\rangle$
78	–	40	110	no	$\langle 22; 10; 10\rangle$
78	P	40	110	no	$\langle 22; 10; 10\rangle$
98	–	40	126	no	$\langle 42; 6; 6\rangle$
98	P	40	126	no	$\langle 42; 6; 6\rangle$
100	–	40	126	no	$\langle 42; 6; 6\rangle$
100	P	40	126	no	$\langle 42; 6; 6\rangle$
123	–	40	144	yes	$\langle 6; 16; 16\rangle$
164	–	40	164	yes	$\langle 82; 4; 4\rangle$
198	–	40	180	yes	$\langle 30; 4; 4\rangle$
375	–	40	252	no	$\langle 84; 3; 6\rangle$
103	P	41	128	no	$\langle 8; 16; 4\rangle$
104	P	41	128	no	$\langle 8; 16; 8\rangle$
215	–	41	192	no	$\langle 8; 6; 12\rangle$
216	–	41	192	no	$\langle 8; 6; 12\rangle$
218	–	41	192	yes	$\langle 8; 6; 6\rangle$
230	–	41	200	no	$\langle 20; 4; 20\rangle$
300	–	41	224	no	$\langle 4; 14; 28\rangle$
795	–	41	384	no	$\langle 16; 3; 12\rangle$

Continued ...

Type 2ex - nonorientable genus 3 to 82

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
1	–	5	10	yes	$\langle 10; 4; 4\rangle$
6	–	8	28	yes	$\langle 4; 7; 7\rangle$
3	P	13	21	no	$\langle 14; 6; 3\rangle$
4	–	13	26	yes	$\langle 26; 4; 4\rangle$

Continued on next page

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
5	–	17	27	no	$\langle 18; 6; 9\rangle$
8	–	17	34	yes	$\langle 34; 4; 4\rangle$
9	–	17	36	yes	$\langle 6; 8; 8\rangle$
7	–	21	30	yes	$\langle 10; 12; 12\rangle$
21	P	22	56	no	$\langle 4; 14; 7\rangle$
22	–	22	56	yes	$\langle 4; 14; 14\rangle$
10	P	25	39	no	$\langle 26; 6; 3\rangle$
16	–	25	50	yes	$\langle 50; 4; 4\rangle$
24	–	29	58	yes	$\langle 58; 4; 4\rangle$
49	–	36	84	yes	$\langle 4; 21; 21\rangle$
15	–	37	50	yes	$\langle 10; 20; 20\rangle$
23	P	37	57	no	$\langle 38; 6; 3\rangle$
35	–	37	74	yes	$\langle 74; 4; 4\rangle$
19	P	41	55	no	$\langle 22; 10; 5\rangle$
20	P	41	55	no	$\langle 22; 10; 5\rangle$
27	P	41	63	no	$\langle 42; 6; 3\rangle$
48	–	41	82	yes	$\langle 82; 4; 4\rangle$
54	–	41	90	yes	$\langle 30; 4; 4\rangle$
94	–	46	120	yes	$\langle 4; 15; 15\rangle$
75	–	47	108	yes	$\langle 6; 8; 8\rangle$
28	P	49	63	no	$\langle 14; 18; 9\rangle$
31	–	49	68	no	$\langle 34; 8; 8\rangle$
31	P	49	68	no	$\langle 34; 8; 8\rangle$
81	–	50	112	yes	$\langle 4; 28; 28\rangle$
82	–	50	112	yes	$\langle 4; 28; 28\rangle$
159	–	50	160	no	$\langle 10; 4; 8\rangle$
53	P	52	84	no	$\langle 28; 6; 3\rangle$
32	–	53	70	yes	$\langle 10; 28; 28\rangle$
47	–	53	81	no	$\langle 54; 6; 27\rangle$
68	–	53	106	yes	$\langle 106; 4; 4\rangle$
61	–	57	100	yes	$\langle 10; 8; 8\rangle$
36	–	61	78	yes	$\langle 26; 12; 12\rangle$
38	–	61	78	no	$\langle 26; 12; 12\rangle$
38	P	61	78	no	$\langle 26; 12; 12\rangle$
56	P	61	93	no	$\langle 62; 6; 3\rangle$
95	–	61	122	yes	$\langle 122; 4; 4\rangle$
120	–	64	140	yes	$\langle 4; 35; 35\rangle$
46	P	65	81	no	$\langle 18; 18; 9\rangle$
69	–	65	108	yes	$\langle 6; 24; 24\rangle$
108	–	65	130	yes	$\langle 130; 4; 4\rangle$
109	–	65	130	yes	$\langle 130; 4; 4\rangle$
73	–	68	108	no	$\langle 36; 6; 9\rangle$
55	–	69	90	yes	$\langle 10; 36; 36\rangle$
67	–	69	105	no	$\langle 70; 6; 15\rangle$
79	P	73	111	no	$\langle 74; 6; 3\rangle$
125	–	73	146	yes	$\langle 146; 4; 4\rangle$
85	P	77	117	no	$\langle 78; 6; 3\rangle$
170	P	78	168	no	$\langle 4; 42; 21\rangle$
171	–	78	168	yes	$\langle 4; 42; 42\rangle$
127	–	79	147	no	$\langle 14; 6; 21\rangle$
74	P	80	108	no	$\langle 18; 12; 9\rangle$
62	–	81	102	yes	$\langle 34; 12; 12\rangle$
219	–	82	192	yes	$\langle 8; 6; 6\rangle$

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
34	PD	10	72	no	$\langle 4; 8; 8\rangle$
282	PD	10	216	no	$\langle 3; 8; 12\rangle$
7	PD	11	30	yes	$\langle 12; 12; 10\rangle$
11	PD	11	40	yes	$\langle 8; 8; 20\rangle$
12	PD	11	40	yes	$\langle 8; 8; 10\rangle$
26	PD	11	60	no	$\langle 4; 12; 30\rangle$
41	PD	11	80	no	$\langle 4; 8; 40\rangle$
44	PD	11	80	no	$\langle 4; 8; 40\rangle$
19	PD	12	55	no	$\langle 5; 10; 22\rangle$
20	PD	12	55	no	$\langle 5; 10; 22\rangle$
37	PD	14	78	no	$\langle 6; 6; 26\rangle$
176	PD	15	168	no	$\langle 3; 12; 56\rangle$
18	PD	16	54	no	$\langle 6; 18; 18\rangle$
21	PD	17	56	no	$\langle 7; 14; 4\rangle$
57	PD	17	96	yes	$\langle 6; 6; 8\rangle$
221	PD	17	192	no	$\langle 4; 6; 12\rangle$
2206	PD	17	672	no	$\langle 3; 7; 16\rangle$
31	PD	18	68	no	$\langle 8; 8; 34\rangle$
33	PD	19	72	yes	$\langle 8; 8; 6\rangle$
121	PD	19	144	no	$\langle 4; 8; 8\rangle$
83	PD	20	114	no	$\langle 6; 6; 38\rangle$
15	PD	21	50	yes	$\langle 20; 20; 10\rangle$
22	PD	21	56	yes	$\langle 14; 14; 4\rangle$
25	PD	21	60	yes	$\langle 12; 12; 10\rangle$
39	PD	21	80	yes	$\langle 8; 8; 20\rangle$
43	PD	21	80	yes	$\langle 8; 8; 20\rangle$
59	PD	21	100	no	$\langle 4; 20; 10\rangle$
90	PD	21	120	no	$\langle 4; 12; 60\rangle$
146	PD	21	160	no	$\langle 4; 8; 40\rangle$
150	PD	21	160	no	$\langle 4; 8; 20\rangle$
159	PD	21	160	no	$\langle 4; 8; 10\rangle$
28	PD	22	63	no	$\langle 9; 18; 14\rangle$
51	PD	22	84	no	$\langle 6; 12; 28\rangle$
52	PD	22	84	no	$\langle 6; 12; 28\rangle$
98	PD	22	126	no	$\langle 6; 6; 42\rangle$
100	PD	22	126	no	$\langle 6; 6; 42\rangle$
378	PD	22	252	no	$\langle 3; 12; 42\rangle$

Type 2ex^P - orientable genus 2 to 24

Map of type 2ex^P are also called chiral maps.

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
5	PD	7	27	no	$\langle 6; 9; 18\rangle$
6	PD	7	28	yes	$\langle 7; 7; 4\rangle$
14	PD	8	42	no	$\langle 6; 6; 14\rangle$
9	PD	10	36	yes	$\langle 8; 8; 6\rangle$
					Continued ...

4.5.4 Type 3 maps

Type 3 - orientable genus 2 to 5

Id	Oper.	Genus	$\|E\|$	Trial.	Map symb.
4	–	2	12	D, P	$\langle 4,6;4,6;4,6\rangle$
11	–	2	16	P	$\langle 4,4;4,8;8,4\rangle$
33	P	2	24		$\langle 4,4;6,4;4,12\rangle$
6	–	3	12	P	$\langle 4,6;12,6;12,6\rangle$
11	PD	3	16	P	$\langle 4,8;8,4;4,4\rangle$
15	–	3	16	P	$\langle 8,4;8,4;8,4\rangle$
15	PD	3	16	P	$\langle 8,4;8,4;8,4\rangle$
33	–	3	24		$\langle 4,4;4,12;6,4\rangle$
99	–	3	32	P	$\langle 4,4;8,4;8,4\rangle$
371	–	3	48	P	$\langle 4,4;4,6;6,4\rangle$
372	–	3	48		$\langle 4,4;6,4;8,4\rangle$
6	PD	4	12	P	$\langle 12,6;12,6;4,6\rangle$
19	–	4	18	D, P	$\langle 6,6;6,6;6,6\rangle$
23	–	4	20	P	$\langle 4,10;4,10;4,10\rangle$
23	PD	4	20	P	$\langle 4,10;4,10;4,10\rangle$
33	PD	4	24		$\langle 4,12;6,4;4,4\rangle$
40	P	4	24	D	$\langle 4,12;6,4;4,12\rangle$
41	–	4	24	PDP	$\langle 4,12;6,4;12,4\rangle$
50	–	4	24	P	$\langle 4,6;6,6;6,6\rangle$
103	–	4	32	P	$\langle 4,4;4,16;16,4\rangle$
178	–	4	36	D, P	$\langle 4,6;6,4;6,4\rangle$
189	–	4	36		$\langle 6,4;6,4;12,6\rangle$
190	–	4	36	D	$\langle 6,4;6,4;12,6\rangle$
244	P	4	40		$\langle 4,4;10,4;4,20\rangle$
372	P	4	48		$\langle 4,4;8,4;6,4\rangle$
1761	P	4	72		$\langle 4,4;6,4;4,12\rangle$
18	–	5	16		$\langle 4,16;8,16;16,8\rangle$
18	P	5	16		$\langle 4,16;16,8;8,16\rangle$
40	–	5	24	D	$\langle 4,12;4,12;6,4\rangle$
41	P	5	24	PDP	$\langle 4,12;12,4;6,4\rangle$
46	–	5	24	D, P	$\langle 4,12;4,12;12,4\rangle$
50	PD	5	24	P	$\langle 6,6;6,6;4,6\rangle$
54	–	5	24		$\langle 8,4;8,6;8,12\rangle$
99	PD	5	32	P	$\langle 8,4;8,4;4,4\rangle$
109	–	5	32	D	$\langle 4,8;4,8;8,4\rangle$
109	P	5	32	D	$\langle 4,8;8,4;4,8\rangle$
111	–	5	32	P	$\langle 8,4;8,4;8,4\rangle$
111	PD	5	32	P	$\langle 8,4;8,4;8,4\rangle$
113	–	5	32	D, P	$\langle 4,8;4,8;4,8\rangle$
114	–	5	32	D	$\langle 8,4;8,4;8,4\rangle$
114	P	5	32	D	$\langle 8,4;8,4;8,4\rangle$
117	–	5	32	D, P	$\langle 8,4;8,4;8,4\rangle$
119	–	5	32	D	$\langle 8,4;8,4;16,4\rangle$
136	–	5	32	P	$\langle 4,4;8,8;8,8\rangle$
137	–	5	32	D	$\langle 4,8;4,8;8,8\rangle$
244	–	5	40		$\langle 4,4;4,20;10,4\rangle$
371	PD	5	48	P	$\langle 4,6;6,4;4,4\rangle$
379	–	5	48	P	$\langle 4,4;4,12;12,4\rangle$
387	–	5	48	D, P	$\langle 4,6;4,6;4,6\rangle$
388	P	5	48	PDP	$\langle 4,6;6,4;4,8\rangle$
390	–	5	48		$\langle 4,6;4,6;12,4\rangle$
400	–	5	48		$\langle 4,4;4,12;8,6\rangle$
428	–	5	48		$\langle 4,4;4,12;12,8\rangle$
1103	–	5	64	P	$\langle 4,4;8,4;8,4\rangle$
1108	–	5	64		$\langle 4,4;4,8;4,16\rangle$
1127	–	5	64		$\langle 4,4;4,8;8,8\rangle$
1139	–	5	64		$\langle 4,4;4,8;8,16\rangle$
3772	–	5	96		$\langle 4,4;4,6;8,4\rangle$
3773	P	5	96		$\langle 4,4;6,4;4,12\rangle$

Type 3 - nonorientable genus 3 to 10

Id	Oper.	Genus	$\|E\|$	Trial.	Map symb.
2	–	3	8	P	$\langle 4,4;4,8;4,8\rangle$
3	–	3	12		$\langle 4,4;4,6;4,12\rangle$
2	PD	4	8	P	$\langle 4,8;4,8;4,4\rangle$
3	P	4	12		$\langle 4,4;4,12;4,6\rangle$
8	P	4	16	D	$\langle 4,4;8,4;4,4\rangle$
9	–	4	16	P	$\langle 4,4;4,8;8,4\rangle$
10	–	4	16	P	$\langle 4,4;4,8;4,8\rangle$
12	–	4	16	P	$\langle 4,4;4,8;8,4\rangle$
3	PD	5	12		$\langle 4,6;4,12;4,4\rangle$
5	–	5	12	PDP	$\langle 4,12;6,4;12,4\rangle$
13	–	5	16	P	$\langle 4,4;4,16;4,16\rangle$
22	–	5	20		$\langle 4,4;10,4;20,4\rangle$
5	P	6	12	PDP	$\langle 4,12;12,4;6,4\rangle$
9	PD	6	16	P	$\langle 4,8;8,4;4,4\rangle$
10	PD	6	16	P	$\langle 4,8;4,8;4,4\rangle$
12	PD	6	16	P	$\langle 4,8;8,4;4,4\rangle$
14	–	6	16	D, P	$\langle 4,8;4,8;4,8\rangle$
22	P	6	20		$\langle 4,4;20,4;10,4\rangle$
32	P	6	24	D	$\langle 4,4;12,4;4,4\rangle$
34	–	6	24	P	$\langle 4,4;4,12;4,12\rangle$
35	–	6	24		$\langle 4,4;12,4;12,4\rangle$
35	P	6	24		$\langle 4,4;12,4;12,4\rangle$
37	–	6	24	D	$\langle 4,6;4,6;8,4\rangle$
98	–	6	32		$\langle 4,4;8,4;8,4\rangle$
98	P	6	32		$\langle 4,4;8,4;8,4\rangle$
100	–	6	32	P	$\langle 4,4;4,8;8,4\rangle$
101	P	6	32		$\langle 4,4;8,4;4,16\rangle$
373	–	6	48		$\langle 4,4;4,6;8,4\rangle$
374	–	6	48		$\langle 4,4;4,6;12,4\rangle$
392	–	6	48		$\langle 4,4;4,6;8,8\rangle$
16	–	7	16	PDP	$\langle 4,16;8,4;16,4\rangle$
17	–	7	16	P	$\langle 4,8;16,4;16,4\rangle$
36	–	7	24	P	$\langle 4,4;4,24;24,4\rangle$
37	P	7	24	D	$\langle 4,6;8,4;4,6\rangle$
38	–	7	24	P	$\langle 4,6;8,4;8,4\rangle$
39	–	7	24		$\langle 4,6;4,8;4,24\rangle$
48	–	7	24	P	$\langle 4,4;6,8;6,8\rangle$
49	–	7	24		$\langle 4,6;4,8;8,8\rangle$
71	P	7	28		$\langle 4,4;14,4;4,28\rangle$
13	PD	8	16	P	$\langle 4,16;4,16;4,4\rangle$
16	P	8	16	PDP	$\langle 4,16;16,4;8,4\rangle$
17	PD	8	16	P	$\langle 16,4;16,4;4,8\rangle$
38	PD	8	24	P	$\langle 8,4;8,4;4,6\rangle$
43	–	8	24	D	$\langle 4,8;8,4;12,4\rangle$
71	–	8	28		$\langle 4,4;4,28;14,4\rangle$
97	P	8	32	D	$\langle 4,4;4,16;4,4\rangle$
101	–	8	32		$\langle 4,4;4,16;8,4\rangle$
102	–	8	32	P	$\langle 4,4;4,16;16,4\rangle$
104	–	8	32	P	$\langle 4,4;4,16;16,4\rangle$
105	–	8	32	P	$\langle 4,4;4,16;16,4\rangle$
106	–	8	32	P	$\langle 4,4;4,16;16,4\rangle$
180	P	8	36		$\langle 4,4;12,4;6,12\rangle$
373	P	8	48		$\langle 4,4;8,4;4,6\rangle$
375	P	8	48		$\langle 4,4;8,4;4,12\rangle$
376	–	8	48		$\langle 4,4;8,4;12,4\rangle$
377	P	8	48		$\langle 4,4;8,4;4,12\rangle$
378	–	8	48		$\langle 4,4;8,4;24,4\rangle$
414	–	8	48		$\langle 4,4;8,4;8,12\rangle$
415	–	8	48		$\langle 4,4;8,4;8,12\rangle$
22	PD	9	20		$\langle 10,4;20,4;4,4\rangle$
24	–	9	20	PDP	$\langle 4,20;10,4;20,4\rangle$
25	–	9	20		$\langle 4,10;4,20;4,20\rangle$
25	P	9	20		$\langle 4,10;4,20;4,20\rangle$

Continued ...

Id	Oper.	Genus	$\vert E\vert$	Trial.	Map symb.
39	P	9	24		$\langle 4, 6; 4, 24; 4, 8\rangle$
42	–	9	24	P	$\langle 4, 6; 24, 4; 24, 4\rangle$
43	P	9	24	D	$\langle 4, 8; 12, 4; 8, 4\rangle$
44	–	9	24		$\langle 4, 8; 12, 4; 24, 4\rangle$
51	–	9	24	P	$\langle 4, 6; 6, 8; 6, 8\rangle$
52	–	9	24	P	$\langle 4, 6; 6, 8; 6, 8\rangle$
107	–	9	32	P	$\langle 4, 4; 4, 32; 32, 4\rangle$
177	–	9	36		$\langle 4, 4; 18, 4; 36, 4\rangle$
24	P	10	20	PDP	$\langle 4, 20; 20, 4; 10, 4\rangle$
25	PD	10	20		$\langle 4, 20; 4, 20; 4, 10\rangle$
34	PD	10	24	P	$\langle 4, 12; 4, 12; 4, 4\rangle$
35	PD	10	24		$\langle 12, 4; 12, 4; 4, 4\rangle$
39	PD	10	24		$\langle 4, 8; 4, 24; 4, 6\rangle$
44	P	10	24		$\langle 4, 8; 24, 4; 12, 4\rangle$
45	–	10	24	D	$\langle 4, 12; 4, 12; 4, 12\rangle$
45	P	10	24	D	$\langle 4, 12; 4, 12; 4, 12\rangle$
49	P	10	24		$\langle 4, 6; 8, 8; 4, 8\rangle$
53	–	10	24	P	$\langle 8, 4; 8, 6; 8, 6\rangle$
98	PD	10	32		$\langle 8, 4; 8, 4; 4, 4\rangle$
100	PD	10	32	P	$\langle 4, 8; 8, 4; 4, 4\rangle$
108	–	10	32	P	$\langle 4, 8; 8, 4; 8, 4\rangle$
108	PD	10	32	P	$\langle 8, 4; 8, 4; 4, 8\rangle$
110	–	10	32	P	$\langle 4, 8; 8, 4; 8, 4\rangle$
110	PD	10	32	P	$\langle 8, 4; 8, 4; 4, 8\rangle$
112	–	10	32	P	$\langle 4, 8; 8, 4; 8, 4\rangle$
112	PD	10	32	P	$\langle 8, 4; 8, 4; 4, 8\rangle$
115	–	10	32	P	$\langle 4, 8; 8, 4; 8, 4\rangle$
115	PD	10	32	P	$\langle 8, 4; 8, 4; 4, 8\rangle$
116	–	10	32	D, P	$\langle 4, 8; 4, 8; 8, 4\rangle$
118	–	10	32	D	$\langle 4, 8; 4, 8; 8, 4\rangle$
118	P	10	32	D	$\langle 4, 8; 8, 4; 4, 8\rangle$
120	–	10	32	D	$\langle 4, 8; 8, 4; 16, 4\rangle$
177	P	10	36		$\langle 4, 4; 36, 4; 18, 4\rangle$
243	P	10	40	D	$\langle 4, 4; 20, 4; 4, 4\rangle$
245	–	10	40		$\langle 4, 4; 4, 20; 4, 20\rangle$
245	P	10	40		$\langle 4, 4; 4, 20; 4, 20\rangle$
246	–	10	40	P	$\langle 4, 4; 4, 20; 4, 20\rangle$
374	P	10	48		$\langle 4, 4; 12, 4; 4, 6\rangle$
375	–	10	48		$\langle 4, 4; 4, 12; 8, 4\rangle$
376	P	10	48		$\langle 4, 4; 12, 4; 8, 4\rangle$
377	–	10	48		$\langle 4, 4; 4, 12; 8, 4\rangle$
380	–	10	48	P	$\langle 4, 4; 4, 12; 4, 12\rangle$
381	P	10	48		$\langle 4, 4; 12, 4; 4, 24\rangle$
389	–	10	48	D	$\langle 4, 6; 6, 4; 8, 4\rangle$
391	PD	10	48	P	$\langle 6, 4; 6, 4; 4, 12\rangle$
411	P	10	48		$\langle 4, 4; 12, 4; 8, 8\rangle$
1102	–	10	64	P	$\langle 4, 4; 8, 4; 8, 4\rangle$
1104	–	10	64		$\langle 4, 4; 4, 8; 16, 4\rangle$
1105	–	10	64		$\langle 4, 4; 8, 4; 16, 4\rangle$
1106	P	10	64		$\langle 4, 4; 8, 4; 4, 16\rangle$
1107	–	10	64		$\langle 4, 4; 4, 8; 4, 16\rangle$
1109	–	10	64		$\langle 4, 4; 4, 8; 4, 32\rangle$
1128	–	10	64		$\langle 4, 4; 4, 8; 8, 8\rangle$
1134	–	10	64		$\langle 4, 4; 4, 8; 8, 8\rangle$
3774	–	10	96		$\langle 4, 4; 4, 6; 16, 4\rangle$
3775	–	10	96		$\langle 4, 4; 4, 6; 24, 4\rangle$

4.5.5 Type 4, 4^* and 4^P maps

Type 4 - orientable genus 2 to 9

Id	Oper.	Genus	$\vert E\vert$	Self Pe.	Map symb.
5	–	2	12	yes	$\langle 3, 4; 8; 8\rangle$
7	–	3	16	yes	$\langle 4, 4; 8; 8\rangle$
44	–	3	24	yes	$\langle 3, 4; 8; 8\rangle$
192	–	3	42	no	$\langle 3, 14; 4; 28\rangle$
19	–	4	20	yes	$\langle 5, 4; 8; 8\rangle$
53	–	4	24	yes	$\langle 3, 4; 12; 12\rangle$
55	–	4	24	no	$\langle 3, 6; 8; 12\rangle$
135	–	4	36	yes	$\langle 3, 4; 8; 8\rangle$
178	–	4	40	no	$\langle 4, 10; 4; 20\rangle$
15	–	5	16	yes	$\langle 4, 8; 16; 16\rangle$
25	–	5	24	yes	$\langle 6, 4; 8; 8\rangle$
55	P	5	24	no	$\langle 3, 6; 12; 8\rangle$
67	–	5	32	yes	$\langle 4, 4; 8; 8\rangle$
70	–	5	32	yes	$\langle 4, 4; 8; 8\rangle$
87	–	5	32	yes	$\langle 4, 4; 8; 8\rangle$
115	–	5	32	yes	$\langle 4, 4; 8; 8\rangle$
248	–	5	48	yes	$\langle 3, 4; 8; 8\rangle$
339	–	5	48	yes	$\langle 3, 4; 8; 8\rangle$
27	–	6	24	yes	$\langle 4, 6; 12; 12\rangle$
29	–	6	24	yes	$\langle 3, 8; 16; 16\rangle$
63	–	6	28	yes	$\langle 7, 4; 8; 8\rangle$
186	–	6	40	no	$\langle 4, 4; 8; 8\rangle$
186	P	6	40	no	$\langle 4, 4; 8; 8\rangle$
1115	–	6	78	no	$\langle 3, 26; 4; 52\rangle$
30	–	7	24	yes	$\langle 4, 6; 24; 24\rangle$
33	–	7	24	no	$\langle 4, 8; 16; 16\rangle$
33	P	7	24	no	$\langle 4, 8; 16; 16\rangle$
37	–	7	24	yes	$\langle 4, 12; 12; 12\rangle$
79	–	7	32	yes	$\langle 8, 4; 8; 8\rangle$
81	–	7	32	yes	$\langle 8, 4; 8; 8\rangle$
88	–	7	32	yes	$\langle 8, 4; 8; 8\rangle$
120	–	7	32	yes	$\langle 8, 4; 8; 8\rangle$
124	–	7	32	yes	$\langle 8, 4; 8; 8\rangle$
128	–	7	36	yes	$\langle 6, 4; 8; 8\rangle$
136	–	7	36	no	$\langle 3, 12; 8; 24\rangle$
146	–	7	36	yes	$\langle 3, 4; 24; 24\rangle$
204	–	7	48	yes	$\langle 4, 4; 8; 8\rangle$
216	–	7	48	yes	$\langle 4, 4; 8; 8\rangle$
221	–	7	48	yes	$\langle 4, 4; 8; 8\rangle$
237	–	7	48	yes	$\langle 4, 4; 8; 8\rangle$
355	–	7	48	yes	$\langle 4, 4; 8; 8\rangle$
3150	–	7	112	no	$\langle 7, 4; 4; 8\rangle$
28	–	8	24	yes	$\langle 4, 12; 24; 24\rangle$
42	–	8	24	yes	$\langle 6, 8; 16; 16\rangle$
150	–	8	36	yes	$\langle 9, 4; 8; 8\rangle$
342	–	8	48	no	$\langle 3, 8; 8; 16\rangle$
345	–	8	48	yes	$\langle 3, 4; 16; 16\rangle$
348	–	8	48	no	$\langle 3, 8; 8; 16\rangle$
351	–	8	48	yes	$\langle 3, 4; 16; 16\rangle$
1376	–	8	84	no	$\langle 3, 4; 8; 8\rangle$
1376	P	8	84	no	$\langle 3, 4; 8; 8\rangle$
69	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
71	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
72	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
73	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
76	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
77	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
89	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
90	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$
110	–	9	32	yes	$\langle 4, 8; 16; 16\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
117	–	9	32	yes	$\langle 8,4;16;16\rangle$
123	–	9	32	yes	$\langle 8,8;8;8\rangle$
125	–	9	32	yes	$\langle 4,8;16;16\rangle$
169	–	9	40	yes	$\langle 10,4;8;8\rangle$
229	–	9	48	yes	$\langle 6,4;8;8\rangle$
231	–	9	48	yes	$\langle 6,4;8;8\rangle$
233	–	9	48	yes	$\langle 6,4;8;8\rangle$
240	–	9	48	yes	$\langle 6,4;8;8\rangle$
246	–	9	48	yes	$\langle 6,4;8;8\rangle$
247	–	9	48	yes	$\langle 6,4;8;8\rangle$
331	–	9	48	yes	$\langle 3,4;24;24\rangle$
337	–	9	48	no	$\langle 3,12;8;24\rangle$
541	–	9	64	yes	$\langle 4,4;8;8\rangle$
542	–	9	64	yes	$\langle 4,4;8;8\rangle$
578	–	9	64	yes	$\langle 4,4;8;8\rangle$
581	–	9	64	yes	$\langle 4,4;8;8\rangle$
599	–	9	64	yes	$\langle 4,4;8;8\rangle$
606	–	9	64	yes	$\langle 4,4;8;8\rangle$
619	–	9	64	yes	$\langle 4,4;8;8\rangle$
638	–	9	64	yes	$\langle 4,4;8;8\rangle$
672	–	9	64	yes	$\langle 4,4;8;8\rangle$
766	–	9	64	yes	$\langle 4,4;8;8\rangle$
840	–	9	64	yes	$\langle 4,4;8;8\rangle$
1263	–	9	80	no	$\langle 4,20;4;20\rangle$
2372	–	9	96	yes	$\langle 3,4;8;8\rangle$
2407	–	9	96	yes	$\langle 3,4;8;8\rangle$
3187	–	9	114	no	$\langle 3,38;4;76\rangle$

Type 4 - nonorientable genus 3 to 18

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
1	–	4	8	yes	$\langle 4,4;8;8\rangle$
2	–	4	8	yes	$\langle 4,4;8;8\rangle$
20	–	5	20	no	$\langle 4,10;4;20\rangle$
3	–	6	12	yes	$\langle 6,4;8;8\rangle$
4	–	6	12	yes	$\langle 6,4;8;8\rangle$
8	–	6	16	yes	$\langle 4,4;8;8\rangle$
9	–	6	16	yes	$\langle 4,4;8;8\rangle$
10	–	6	16	yes	$\langle 4,4;8;8\rangle$
13	–	6	16	yes	$\langle 4,4;8;8\rangle$
46	–	6	24	yes	$\langle 3,4;8;8\rangle$
48	–	6	24	yes	$\langle 3,4;8;8\rangle$
50	–	6	24	no	$\langle 3,4;8;8\rangle$
50	P	6	24	no	$\langle 3,4;8;8\rangle$
6	–	8	16	yes	$\langle 8,4;8;8\rangle$
11	–	8	16	yes	$\langle 8,4;8;8\rangle$
12	–	8	16	yes	$\langle 8,4;8;8\rangle$
14	–	8	16	yes	$\langle 8,4;8;8\rangle$
24	–	8	24	yes	$\langle 4,4;8;8\rangle$
35	–	8	24	yes	$\langle 4,4;8;8\rangle$
450	–	8	56	no	$\langle 7,4;4;8\rangle$
56	–	9	24	no	$\langle 3,8;8;16\rangle$
57	–	9	24	yes	$\langle 3,4;16;16\rangle$
58	–	9	24	no	$\langle 3,8;8;16\rangle$
59	–	9	24	yes	$\langle 3,4;16;16\rangle$
17	–	10	20	yes	$\langle 10,4;8;8\rangle$
18	–	10	20	yes	$\langle 10,4;8;8\rangle$
40	–	10	24	yes	$\langle 6,4;8;8\rangle$
41	–	10	24	yes	$\langle 6,4;8;8\rangle$
43	–	10	24	yes	$\langle 6,4;8;8\rangle$
45	–	10	24	yes	$\langle 6,4;8;8\rangle$
47	–	10	24	yes	$\langle 6,4;8;8\rangle$
49	–	10	24	no	$\langle 6,4;8;8\rangle$
49	P	10	24	no	$\langle 6,4;8;8\rangle$
80	–	10	32	yes	$\langle 4,4;8;8\rangle$
83	–	10	32	yes	$\langle 4,4;8;8\rangle$
84	–	10	32	yes	$\langle 4,4;8;8\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
93	–	10	32	no	$\langle 4,4;8;8\rangle$
93	P	10	32	no	$\langle 4,4;8;8\rangle$
95	–	10	32	yes	$\langle 4,4;8;8\rangle$
98	–	10	32	yes	$\langle 4,4;8;8\rangle$
105	–	10	32	yes	$\langle 4,4;8;8\rangle$
377	–	10	48	yes	$\langle 3,4;8;8\rangle$
381	–	10	48	yes	$\langle 3,4;8;8\rangle$
16	–	11	18	yes	$\langle 6,6;12;12\rangle$
51	–	11	24	yes	$\langle 4,4;16;16\rangle$
52	–	11	24	yes	$\langle 4,8;8;8\rangle$
23	–	12	24	yes	$\langle 12,4;8;8\rangle$
26	–	12	24	yes	$\langle 12,4;8;8\rangle$
34	–	12	24	yes	$\langle 12,4;8;8\rangle$
36	–	12	24	yes	$\langle 12,4;8;8\rangle$
54	–	12	24	no	$\langle 4,6;12;12\rangle$
54	P	12	24	no	$\langle 4,6;12;12\rangle$
56	P	12	24	no	$\langle 3,8;16;8\rangle$
58	P	12	24	no	$\langle 3,8;16;8\rangle$
158	–	12	40	yes	$\langle 4,4;8;8\rangle$
160	–	12	40	yes	$\langle 4,4;8;8\rangle$
187	–	12	40	no	$\langle 4,4;8;8\rangle$
187	P	12	40	no	$\langle 4,4;8;8\rangle$
20	P	13	20	no	$\langle 4,10;20;4\rangle$
21	–	13	20	no	$\langle 4,10;20;20\rangle$
21	P	13	20	no	$\langle 4,10;20;20\rangle$
22	–	13	20	yes	$\langle 4,10;20;20\rangle$
195	–	13	42	no	$\langle 6,14;4;28\rangle$
391	–	13	52	no	$\langle 4,26;4;52\rangle$
60	–	14	24	yes	$\langle 4,8;16;16\rangle$
61	–	14	28	yes	$\langle 14,4;8;8\rangle$
62	–	14	28	yes	$\langle 14,4;8;8\rangle$
68	–	14	32	yes	$\langle 8,4;8;8\rangle$
74	–	14	32	yes	$\langle 8,4;8;8\rangle$
75	–	14	32	yes	$\langle 8,4;8;8\rangle$
82	–	14	32	yes	$\langle 8,4;8;8\rangle$
85	–	14	32	yes	$\langle 8,4;8;8\rangle$
86	–	14	32	yes	$\langle 8,4;8;8\rangle$
96	–	14	32	yes	$\langle 8,4;8;8\rangle$
97	–	14	32	no	$\langle 8,4;8;8\rangle$
97	P	14	32	no	$\langle 8,4;8;8\rangle$
102	–	14	32	yes	$\langle 8,4;8;8\rangle$
127	–	14	36	yes	$\langle 6,4;8;8\rangle$
131	–	14	36	yes	$\langle 6,4;8;8\rangle$
134	–	14	36	yes	$\langle 6,4;8;8\rangle$
258	–	14	48	yes	$\langle 4,4;8;8\rangle$
262	–	14	48	yes	$\langle 4,4;8;8\rangle$
264	–	14	48	yes	$\langle 4,4;8;8\rangle$
296	–	14	48	yes	$\langle 4,4;8;8\rangle$
354	–	14	48	yes	$\langle 4,4;8;8\rangle$
905	–	14	72	yes	$\langle 3,4;8;8\rangle$
912	–	14	72	yes	$\langle 3,4;8;8\rangle$
915	–	14	72	no	$\langle 3,4;8;8\rangle$
915	P	14	72	no	$\langle 3,4;8;8\rangle$
31	–	16	24	yes	$\langle 6,8;16;16\rangle$
32	–	16	24	yes	$\langle 6,8;16;16\rangle$
38	–	16	24	yes	$\langle 6,8;16;16\rangle$
39	–	16	24	yes	$\langle 6,8;16;16\rangle$
78	–	16	32	yes	$\langle 16,4;8;8\rangle$
103	–	16	32	yes	$\langle 16,4;8;8\rangle$
104	–	16	32	yes	$\langle 16,4;8;8\rangle$
106	–	16	32	yes	$\langle 16,4;8;8\rangle$
446	–	16	56	yes	$\langle 4,4;8;8\rangle$
448	–	16	56	yes	$\langle 4,4;8;8\rangle$
151	–	17	36	yes	$\langle 4,6;12;12\rangle$
152	–	17	36	no	$\langle 4,6;12;12\rangle$
152	P	17	36	no	$\langle 4,6;12;12\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
408	–	17	54	no	$\langle 6,18;4;12\rangle$
866	–	17	68	no	$\langle 4,34;4;68\rangle$
1069	–	17	72	no	$\langle 8,6;4;12\rangle$
91	–	18	32	no	$\langle 4,8;16;16\rangle$
91	P	18	32	no	$\langle 4,8;16;16\rangle$
92	–	18	32	yes	$\langle 4,8;16;16\rangle$
94	–	18	32	yes	$\langle 4,8;16;16\rangle$
99	–	18	32	yes	$\langle 4,8;16;16\rangle$
101	–	18	32	yes	$\langle 4,8;16;16\rangle$
148	–	18	36	yes	$\langle 18,4;8;8\rangle$
149	–	18	36	yes	$\langle 18,4;8;8\rangle$
157	–	18	40	yes	$\langle 10,4;8;8\rangle$
175	–	18	40	yes	$\langle 10,4;8;8\rangle$
244	–	18	48	yes	$\langle 6,4;8;8\rangle$
245	–	18	48	yes	$\langle 6,4;8;8\rangle$
249	–	18	48	yes	$\langle 6,4;8;8\rangle$
252	–	18	48	yes	$\langle 6,4;8;8\rangle$
253	–	18	48	yes	$\langle 6,4;8;8\rangle$
255	–	18	48	no	$\langle 6,4;8;8\rangle$
255	P	18	48	no	$\langle 6,4;8;8\rangle$
256	–	18	48	no	$\langle 6,4;8;8\rangle$
256	P	18	48	no	$\langle 6,4;8;8\rangle$
275	–	18	48	yes	$\langle 6,4;8;8\rangle$
276	–	18	48	yes	$\langle 6,4;8;8\rangle$
338	–	18	48	yes	$\langle 6,4;8;8\rangle$
378	–	18	48	yes	$\langle 6,4;8;8\rangle$
382	–	18	48	yes	$\langle 6,4;8;8\rangle$
601	–	18	64	yes	$\langle 4,4;8;8\rangle$
615	–	18	64	yes	$\langle 4,4;8;8\rangle$
616	–	18	64	yes	$\langle 4,4;8;8\rangle$
627	–	18	64	yes	$\langle 4,4;8;8\rangle$
628	–	18	64	yes	$\langle 4,4;8;8\rangle$
629	–	18	64	yes	$\langle 4,4;8;8\rangle$
701	–	18	64	no	$\langle 4,4;8;8\rangle$
701	P	18	64	no	$\langle 4,4;8;8\rangle$
713	–	18	64	yes	$\langle 4,4;8;8\rangle$
714	–	18	64	yes	$\langle 4,4;8;8\rangle$
715	–	18	64	no	$\langle 4,4;8;8\rangle$
715	P	18	64	no	$\langle 4,4;8;8\rangle$
716	–	18	64	yes	$\langle 4,4;8;8\rangle$
722	–	18	64	yes	$\langle 4,4;8;8\rangle$
723	–	18	64	yes	$\langle 4,4;8;8\rangle$
725	–	18	64	yes	$\langle 4,4;8;8\rangle$
741	–	18	64	yes	$\langle 4,4;8;8\rangle$
757	–	18	64	yes	$\langle 4,4;8;8\rangle$
1763	–	18	96	yes	$\langle 3,4;8;8\rangle$
2445	–	18	96	no	$\langle 3,4;8;8\rangle$
2445	P	18	96	no	$\langle 3,4;8;8\rangle$
2452	–	18	96	yes	$\langle 3,4;8;8\rangle$

Type 4^P - orientable genus 2 to 26

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
3	PD	4	12	yes	$\langle 8;8;6,4\rangle$
4	PD	4	12	yes	$\langle 8;8;6,4\rangle$
7	PD	5	16	yes	$\langle 8;8;4,4\rangle$
17	PD	6	20	yes	$\langle 8;8;10,4\rangle$
18	PD	6	20	yes	$\langle 8;8;10,4\rangle$
15	PD	7	16	yes	$\langle 16;16;4,8\rangle$
16	PD	7	18	yes	$\langle 12;12;6,6\rangle$
25	PD	7	24	yes	$\langle 8;8;6,4\rangle$
43	PD	7	24	yes	$\langle 8;8;6,4\rangle$
61	PD	8	28	yes	$\langle 8;8;14,4\rangle$
62	PD	8	28	yes	$\langle 8;8;14,4\rangle$
27	PD	9	24	yes	$\langle 12;12;4,6\rangle$
37	PD	9	24	yes	$\langle 12;12;4,12\rangle$
54	PD	9	24	no	$\langle 12;12;4,6\rangle$
67	PD	9	32	yes	$\langle 8;8;4,4\rangle$
70	PD	9	32	yes	$\langle 8;8;4,4\rangle$
79	PD	9	32	yes	$\langle 8;8;8,4\rangle$
81	PD	9	32	yes	$\langle 8;8;8,4\rangle$
87	PD	9	32	yes	$\langle 8;8;4,4\rangle$
88	PD	9	32	yes	$\langle 8;8;8,4\rangle$
115	PD	9	32	yes	$\langle 8;8;4,4\rangle$
120	PD	9	32	yes	$\langle 8;8;8,4\rangle$
123	PD	9	32	yes	$\langle 8;8;8,8\rangle$
124	PD	9	32	yes	$\langle 8;8;8,4\rangle$
178	PD	9	40	no	$\langle 4;20;4,10\rangle$
31	PD	10	24	yes	$\langle 16;16;6,8\rangle$
32	PD	10	24	yes	$\langle 16;16;6,8\rangle$
38	PD	10	24	yes	$\langle 16;16;6,8\rangle$
39	PD	10	24	yes	$\langle 16;16;6,8\rangle$
127	PD	10	36	yes	$\langle 8;8;6,4\rangle$
130	PD	10	36	yes	$\langle 8;8;6,12\rangle$
131	PD	10	36	yes	$\langle 8;8;6,4\rangle$
133	PD	10	36	yes	$\langle 8;8;6,12\rangle$
134	PD	10	36	yes	$\langle 8;8;6,4\rangle$
148	PD	10	36	yes	$\langle 8;8;18,4\rangle$
149	PD	10	36	yes	$\langle 8;8;18,4\rangle$
195	PD	10	42	no	$\langle 4;28;6,14\rangle$
408	PD	10	54	no	$\langle 4;12;6,18\rangle$
28	PD	11	24	yes	$\langle 24;24;4,12\rangle$
30	PD	11	24	yes	$\langle 24;24;4,6\rangle$
64	PD	11	30	yes	$\langle 12;12;10,6\rangle$
169	PD	11	40	yes	$\langle 8;8;10,4\rangle$
186	PD	11	40	no	$\langle 8;8;4,4\rangle$
200	PD	12	44	yes	$\langle 8;8;22,4\rangle$
201	PD	12	44	yes	$\langle 8;8;22,4\rangle$
65	PD	13	30	yes	$\langle 20;20;6,10\rangle$
66	PD	13	30	yes	$\langle 20;20;6,10\rangle$
69	PD	13	32	yes	$\langle 16;16;4,8\rangle$
71	PD	13	32	yes	$\langle 16;16;4,8\rangle$
72	PD	13	32	yes	$\langle 16;16;4,8\rangle$
73	PD	13	32	yes	$\langle 16;16;4,8\rangle$
76	PD	13	32	yes	$\langle 16;16;4,8\rangle$
77	PD	13	32	yes	$\langle 16;16;4,8\rangle$
89	PD	13	32	yes	$\langle 16;16;4,8\rangle$
90	PD	13	32	yes	$\langle 16;16;4,8\rangle$
108	PD	13	32	yes	$\langle 16;16;4,16\rangle$
109	PD	13	32	yes	$\langle 16;16;8,8\rangle$
110	PD	13	32	yes	$\langle 16;16;4,8\rangle$
112	PD	13	32	yes	$\langle 16;16;8,8\rangle$
114	PD	13	32	yes	$\langle 16;16;8,8\rangle$
117	PD	13	32	yes	$\langle 16;16;8,4\rangle$
121	PD	13	32	yes	$\langle 16;16;8,8\rangle$
125	PD	13	32	yes	$\langle 16;16;4,8\rangle$
126	PD	13	32	yes	$\langle 16;16;8,8\rangle$
142	PD	13	36	yes	$\langle 12;12;6,6\rangle$
143	PD	13	36	no	$\langle 12;12;6,6\rangle$
147	PD	13	36	yes	$\langle 12;12;6,6\rangle$
203	PD	13	48	yes	$\langle 8;8;12,4\rangle$
204	PD	13	48	yes	$\langle 8;8;4,4\rangle$
211	PD	13	48	yes	$\langle 8;8;12,4\rangle$
214	PD	13	48	yes	$\langle 8;8;4,12\rangle$
215	PD	13	48	yes	$\langle 8;8;12,4\rangle$
216	PD	13	48	yes	$\langle 8;8;4,4\rangle$
221	PD	13	48	yes	$\langle 8;8;4,4\rangle$
222	PD	13	48	yes	$\langle 8;8;12,4\rangle$
229	PD	13	48	yes	$\langle 8;8;6,4\rangle$
231	PD	13	48	yes	$\langle 8;8;6,4\rangle$
233	PD	13	48	yes	$\langle 8;8;6,4\rangle$
237	PD	13	48	yes	$\langle 8;8;4,4\rangle$
240	PD	13	48	yes	$\langle 8;8;6,4\rangle$
241	PD	13	48	yes	$\langle 8;8;12,4\rangle$
244	PD	13	48	yes	$\langle 8;8;6,4\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
245	PD	13	48	yes	$\langle 8;8;6,4\rangle$
247	PD	13	48	yes	$\langle 8;8;6,4\rangle$
252	PD	13	48	yes	$\langle 8;8;6,4\rangle$
255	PD	13	48	no	$\langle 8;8;6,4\rangle$
327	PD	13	48	yes	$\langle 8;8;4,8\rangle$
338	PD	13	48	yes	$\langle 8;8;6,4\rangle$
353	PD	13	48	no	$\langle 8;8;4,12\rangle$
354	PD	13	48	yes	$\langle 8;8;4,4\rangle$
387	PD	14	52	yes	$\langle 8;8;26,4\rangle$
388	PD	14	52	yes	$\langle 8;8;26,4\rangle$
107	PD	15	32	yes	$\langle 32;32;4,8\rangle$
111	PD	15	32	yes	$\langle 32;32;4,16\rangle$
113	PD	15	32	yes	$\langle 32;32;4,16\rangle$
116	PD	15	32	no	$\langle 32;32;8,16\rangle$
118	PD	15	32	yes	$\langle 32;32;4,16\rangle$
119	PD	15	32	yes	$\langle 32;32;4,16\rangle$
122	PD	15	32	yes	$\langle 32;32;4,16\rangle$
188	PD	15	42	yes	$\langle 12;12;14,6\rangle$
336	PD	15	48	no	$\langle 8;12;6,6\rangle$
426	PD	15	56	yes	$\langle 8;8;14,4\rangle$
3148	PD	15	112	no	$\langle 4;8;14,4\rangle$
137	PD	16	36	yes	$\langle 24;24;6,4\rangle$
138	PD	16	36	yes	$\langle 24;24;6,4\rangle$
139	PD	16	36	yes	$\langle 24;24;6,12\rangle$
140	PD	16	36	yes	$\langle 24;24;6,12\rangle$
141	PD	16	36	yes	$\langle 24;24;6,12\rangle$
144	PD	16	36	yes	$\langle 24;24;6,12\rangle$
154	PD	16	40	yes	$\langle 16;16;10,8\rangle$
156	PD	16	40	yes	$\langle 16;16;10,8\rangle$
172	PD	16	40	yes	$\langle 16;16;10,8\rangle$
173	PD	16	40	yes	$\langle 16;16;10,8\rangle$
341	PD	16	48	no	$\langle 8;16;6,8\rangle$
347	PD	16	48	no	$\langle 8;16;6,8\rangle$
364	PD	16	48	no	$\langle 8;16;8,6\rangle$
380	PD	16	48	no	$\langle 8;16;12,8\rangle$
452	PD	16	60	yes	$\langle 8;8;30,4\rangle$
453	PD	16	60	yes	$\langle 8;8;10,12\rangle$
454	PD	16	60	yes	$\langle 8;8;30,4\rangle$
459	PD	16	60	yes	$\langle 8;8;10,4\rangle$
466	PD	16	60	yes	$\langle 8;8;6,4\rangle$
468	PD	16	60	yes	$\langle 8;8;6,4\rangle$
487	PD	16	60	yes	$\langle 8;8;30,4\rangle$
490	PD	16	60	yes	$\langle 8;8;10,4\rangle$
492	PD	16	60	yes	$\langle 8;8;10,12\rangle$
498	PD	16	60	yes	$\langle 8;8;6,20\rangle$
499	PD	16	60	yes	$\langle 8;8;6,20\rangle$
506	PD	16	60	yes	$\langle 8;8;30,4\rangle$
161	PD	17	40	yes	$\langle 20;20;4,10\rangle$
164	PD	17	40	yes	$\langle 20;20;4,20\rangle$
165	PD	17	40	yes	$\langle 20;20;4,10\rangle$
168	PD	17	40	yes	$\langle 20;20;4,20\rangle$
182	PD	17	40	no	$\langle 20;20;4,10\rangle$
185	PD	17	40	yes	$\langle 20;20;4,10\rangle$
210	PD	17	48	yes	$\langle 12;12;8,6\rangle$
212	PD	17	48	no	$\langle 8;24;12,12\rangle$
230	PD	17	48	no	$\langle 8;24;6,12\rangle$
266	PD	17	48	yes	$\langle 12;12;4,12\rangle$
271	PD	17	48	yes	$\langle 12;12;8,12\rangle$
302	PD	17	48	yes	$\langle 12;12;4,24\rangle$
330	PD	17	48	yes	$\langle 12;12;6,4\rangle$
334	PD	17	48	no	$\langle 12;12;4,6\rangle$
363	PD	17	48	yes	$\langle 12;12;8,8\rangle$
368	PD	17	48	yes	$\langle 12;12;8,4\rangle$
528	PD	17	64	yes	$\langle 8;8;8,4\rangle$
537	PD	17	64	yes	$\langle 8;8;8,4\rangle$
541	PD	17	64	yes	$\langle 8;8;4,4\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
542	PD	17	64	yes	$\langle 8;8;4,4\rangle$
545	PD	17	64	yes	$\langle 8;8;8,4\rangle$
547	PD	17	64	yes	$\langle 8;8;8,4\rangle$
549	PD	17	64	yes	$\langle 8;8;8,4\rangle$
553	PD	17	64	yes	$\langle 8;8;8,8\rangle$
575	PD	17	64	yes	$\langle 8;8;8,8\rangle$
578	PD	17	64	yes	$\langle 8;8;4,4\rangle$
581	PD	17	64	yes	$\langle 8;8;4,4\rangle$
585	PD	17	64	yes	$\langle 8;8;8,4\rangle$
587	PD	17	64	yes	$\langle 8;8;8,4\rangle$
599	PD	17	64	yes	$\langle 8;8;4,4\rangle$
606	PD	17	64	yes	$\langle 8;8;4,4\rangle$
617	PD	17	64	yes	$\langle 8;8;8,4\rangle$
618	PD	17	64	yes	$\langle 8;8;8,4\rangle$
619	PD	17	64	yes	$\langle 8;8;4,4\rangle$
625	PD	17	64	yes	$\langle 8;8;8,4\rangle$
626	PD	17	64	yes	$\langle 8;8;16,4\rangle$
630	PD	17	64	yes	$\langle 8;8;16,4\rangle$
638	PD	17	64	yes	$\langle 8;8;4,4\rangle$
639	PD	17	64	yes	$\langle 8;8;16,4\rangle$
672	PD	17	64	yes	$\langle 8;8;4,4\rangle$
688	PD	17	64	yes	$\langle 8;8;8,4\rangle$
748	PD	17	64	yes	$\langle 8;8;8,16\rangle$
763	PD	17	64	yes	$\langle 8;8;8,4\rangle$
766	PD	17	64	yes	$\langle 8;8;4,4\rangle$
773	PD	17	64	yes	$\langle 8;8;8,8\rangle$
775	PD	17	64	yes	$\langle 8;8;4,8\rangle$
776	PD	17	64	yes	$\langle 8;8;8,8\rangle$
786	PD	17	64	yes	$\langle 8;8;8,4\rangle$
796	PD	17	64	yes	$\langle 8;8;4,8\rangle$
797	PD	17	64	yes	$\langle 8;8;8,8\rangle$
798	PD	17	64	yes	$\langle 8;8;4,8\rangle$
802	PD	17	64	yes	$\langle 8;8;8,4\rangle$
803	PD	17	64	yes	$\langle 8;8;8,8\rangle$
807	PD	17	64	yes	$\langle 8;8;4,8\rangle$
811	PD	17	64	yes	$\langle 8;8;4,8\rangle$
840	PD	17	64	yes	$\langle 8;8;4,4\rangle$
848	PD	17	64	yes	$\langle 8;8;16,4\rangle$
849	PD	17	64	yes	$\langle 8;8;16,4\rangle$
1258	PD	17	80	no	$\langle 4;20;8,10\rangle$
1263	PD	17	80	no	$\langle 4;20;4,20\rangle$
1265	PD	17	80	no	$\langle 4;20;8,20\rangle$
860	PD	18	68	yes	$\langle 8;8;34,4\rangle$
861	PD	18	68	yes	$\langle 8;8;34,4\rangle$
162	PD	19	40	yes	$\langle 40;40;4,20\rangle$
163	PD	19	40	yes	$\langle 40;40;4,10\rangle$
166	PD	19	40	yes	$\langle 40;40;4,20\rangle$
167	PD	19	40	yes	$\langle 40;40;4,10\rangle$
189	PD	19	42	yes	$\langle 28;28;6,14\rangle$
190	PD	19	42	yes	$\langle 28;28;6,14\rangle$
191	PD	19	42	yes	$\langle 28;28;6,14\rangle$
193	PD	19	42	yes	$\langle 28;28;6,14\rangle$
196	PD	19	42	no	$\langle 28;28;6,14\rangle$
199	PD	19	42	no	$\langle 28;28;6,14\rangle$
223	PD	19	48	yes	$\langle 16;16;6,8\rangle$
225	PD	19	48	yes	$\langle 16;16;6,8\rangle$
227	PD	19	48	yes	$\langle 16;16;12,8\rangle$
238	PD	19	48	yes	$\langle 16;16;4,8\rangle$
239	PD	19	48	yes	$\langle 16;16;6,8\rangle$
242	PD	19	48	yes	$\langle 16;16;6,8\rangle$
274	PD	19	48	no	$\langle 12;24;8,24\rangle$
294	PD	19	48	no	$\langle 12;24;8,24\rangle$
307	PD	19	48	yes	$\langle 16;16;4,8\rangle$
308	PD	19	48	yes	$\langle 16;16;12,8\rangle$
310	PD	19	48	yes	$\langle 16;16;6,8\rangle$
311	PD	19	48	yes	$\langle 16;16;12,8\rangle$
					Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
313	PD	19	48	yes	$\langle 16; 16; 4, 8\rangle$
315	PD	19	48	yes	$\langle 16; 16; 4, 24\rangle$
316	PD	19	48	yes	$\langle 16; 16; 12, 8\rangle$
317	PD	19	48	yes	$\langle 16; 16; 4, 8\rangle$
321	PD	19	48	yes	$\langle 16; 16; 12, 8\rangle$
324	PD	19	48	yes	$\langle 16; 16; 4, 4\rangle$
344	PD	19	48	yes	$\langle 16; 16; 6, 4\rangle$
350	PD	19	48	yes	$\langle 16; 16; 6, 4\rangle$
356	PD	19	48	yes	$\langle 16; 16; 4, 8\rangle$
360	PD	19	48	no	$\langle 12; 24; 8, 12\rangle$
362	PD	19	48	no	$\langle 12; 24; 8, 12\rangle$
367	PD	19	48	no	$\langle 12; 24; 8, 6\rangle$
370	PD	19	48	no	$\langle 12; 24; 8, 6\rangle$
379	PD	19	48	yes	$\langle 16; 16; 12, 4\rangle$
397	PD	19	54	yes	$\langle 12; 12; 18, 6\rangle$
398	PD	19	54	yes	$\langle 12; 12; 6, 18\rangle$
403	PD	19	54	yes	$\langle 12; 12; 6, 6\rangle$
404	PD	19	54	no	$\langle 12; 12; 6, 6\rangle$
415	PD	19	54	yes	$\langle 12; 12; 6, 18\rangle$
882	PD	19	72	yes	$\langle 8; 8; 6, 4\rangle$
883	PD	19	72	yes	$\langle 8; 8; 6, 4\rangle$
892	PD	19	72	yes	$\langle 8; 8; 6, 4\rangle$
894	PD	19	72	yes	$\langle 8; 8; 6, 4\rangle$
896	PD	19	72	yes	$\langle 8; 8; 6, 12\rangle$
904	PD	19	72	yes	$\langle 8; 8; 6, 12\rangle$
943	PD	19	72	yes	$\langle 8; 8; 18, 4\rangle$
963	PD	19	72	yes	$\langle 8; 8; 18, 4\rangle$
1064	PD	19	72	yes	$\langle 8; 8; 4, 4\rangle$
1095	PD	19	72	yes	$\langle 8; 8; 4, 6\rangle$
1119	PD	19	78	no	$\langle 4; 52; 6, 26\rangle$
1373	PD	19	84	no	$\langle 4; 28; 6, 14\rangle$
1384	PD	19	84	no	$\langle 4; 28; 6, 14\rangle$
2861	PD	19	108	no	$\langle 4; 12; 6, 18\rangle$
7496	PD	19	144	no	$\langle 4; 8; 8, 8\rangle$
1096	PD	20	76	yes	$\langle 8; 8; 38, 4\rangle$
1097	PD	20	76	yes	$\langle 8; 8; 38, 4\rangle$
205	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
209	PD	21	48	yes	$\langle 24; 24; 8, 12\rangle$
213	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
217	PD	21	48	yes	$\langle 24; 24; 4, 4\rangle$
218	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
219	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
220	PD	21	48	yes	$\langle 24; 24; 12, 4\rangle$
224	PD	21	48	no	$\langle 16; 48; 6, 24\rangle$
232	PD	21	48	yes	$\langle 24; 24; 6, 4\rangle$
243	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
260	PD	21	48	yes	$\langle 24; 24; 4, 6\rangle$
261	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
267	PD	21	48	yes	$\langle 24; 24; 4, 12\rangle$
272	PD	21	48	yes	$\langle 24; 24; 8, 6\rangle$
304	PD	21	48	yes	$\langle 24; 24; 4, 24\rangle$
312	PD	21	48	no	$\langle 16; 48; 12, 24\rangle$
332	PD	21	48	no	$\langle 24; 24; 4, 12\rangle$
333	PD	21	48	no	$\langle 24; 24; 4, 12\rangle$
352	PD	21	48	yes	$\langle 24; 24; 4, 4\rangle$
359	PD	21	48	yes	$\langle 24; 24; 4, 4\rangle$
385	PD	21	50	yes	$\langle 20; 20; 10, 10\rangle$
386	PD	21	50	yes	$\langle 20; 20; 10, 10\rangle$
461	PD	21	60	yes	$\langle 12; 12; 10, 6\rangle$
844	PD	21	64	no	$\langle 8; 16; 16, 16\rangle$
1121	PD	21	80	yes	$\langle 8; 8; 20, 4\rangle$
1122	PD	21	80	yes	$\langle 8; 8; 10, 4\rangle$
1127	PD	21	80	yes	$\langle 8; 8; 4, 4\rangle$
1129	PD	21	80	yes	$\langle 8; 8; 4, 4\rangle$
1130	PD	21	80	yes	$\langle 8; 8; 4, 4\rangle$
1146	PD	21	80	yes	$\langle 8; 8; 20, 4\rangle$

Continued ...

Id	Oper.	Genus	$\lvert E\rvert$	Self Du.	Map symb.
1148	PD	21	80	yes	$\langle 8; 8; 20, 4\rangle$
1149	PD	21	80	yes	$\langle 8; 8; 20, 4\rangle$
1151	PD	21	80	yes	$\langle 8; 8; 10, 4\rangle$
1152	PD	21	80	yes	$\langle 8; 8; 10, 4\rangle$
1154	PD	21	80	yes	$\langle 8; 8; 4, 20\rangle$
1160	PD	21	80	yes	$\langle 8; 8; 4, 4\rangle$
1161	PD	21	80	yes	$\langle 8; 8; 20, 4\rangle$
1180	PD	21	80	yes	$\langle 8; 8; 10, 4\rangle$
1293	PD	21	80	no	$\langle 8; 8; 4, 4\rangle$
1295	PD	21	80	no	$\langle 8; 8; 4, 4\rangle$
1298	PD	21	80	no	$\langle 8; 8; 4, 4\rangle$
284	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
285	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
286	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
287	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
290	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
291	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
292	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
293	PD	22	48	yes	$\langle 32; 32; 6, 16\rangle$
422	PD	22	56	yes	$\langle 16; 16; 14, 8\rangle$
424	PD	22	56	yes	$\langle 16; 16; 14, 8\rangle$
430	PD	22	56	yes	$\langle 16; 16; 14, 8\rangle$
432	PD	22	56	yes	$\langle 16; 16; 14, 8\rangle$
1304	PD	22	84	yes	$\langle 8; 8; 42, 4\rangle$
1305	PD	22	84	yes	$\langle 8; 8; 14, 4\rangle$
1307	PD	22	84	yes	$\langle 8; 8; 14, 12\rangle$
1308	PD	22	84	yes	$\langle 8; 8; 42, 4\rangle$
1314	PD	22	84	yes	$\langle 8; 8; 42, 4\rangle$
1318	PD	22	84	yes	$\langle 8; 8; 14, 12\rangle$
1320	PD	22	84	yes	$\langle 8; 8; 14, 4\rangle$
1325	PD	22	84	yes	$\langle 8; 8; 42, 4\rangle$
1358	PD	22	84	yes	$\langle 8; 8; 6, 4\rangle$
1360	PD	22	84	yes	$\langle 8; 8; 6, 4\rangle$
1367	PD	22	84	yes	$\langle 8; 8; 6, 28\rangle$
1368	PD	22	84	yes	$\langle 8; 8; 6, 28\rangle$
1375	PD	22	84	no	$\langle 8; 8; 6, 4\rangle$
1398	PD	22	84	no	$\langle 8; 8; 6, 4\rangle$
226	PD	23	48	yes	$\langle 48; 48; 6, 8\rangle$
228	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
283	PD	23	48	yes	$\langle 48; 48; 8, 6\rangle$
289	PD	23	48	yes	$\langle 48; 48; 8, 12\rangle$
298	PD	23	48	yes	$\langle 48; 48; 8, 6\rangle$
303	PD	23	48	yes	$\langle 48; 48; 4, 6\rangle$
305	PD	23	48	yes	$\langle 48; 48; 4, 12\rangle$
306	PD	23	48	yes	$\langle 48; 48; 8, 12\rangle$
309	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
314	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
318	PD	23	48	yes	$\langle 48; 48; 4, 8\rangle$
319	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
320	PD	23	48	yes	$\langle 48; 48; 4, 24\rangle$
322	PD	23	48	yes	$\langle 48; 48; 12, 8\rangle$
854	PD	23	66	yes	$\langle 12; 12; 22, 6\rangle$
1405	PD	23	88	yes	$\langle 8; 8; 22, 4\rangle$
1454	PD	24	92	yes	$\langle 8; 8; 46, 4\rangle$
1455	PD	24	92	yes	$\langle 8; 8; 46, 4\rangle$
399	PD	25	54	yes	$\langle 36; 36; 6, 6\rangle$
400	PD	25	54	yes	$\langle 36; 36; 6, 18\rangle$
401	PD	25	54	yes	$\langle 36; 36; 6, 18\rangle$
402	PD	25	54	yes	$\langle 36; 36; 6, 18\rangle$
414	PD	25	54	no	$\langle 36; 36; 6, 18\rangle$
417	PD	25	54	no	$\langle 36; 36; 6, 18\rangle$
418	PD	25	54	no	$\langle 36; 36; 6, 6\rangle$
419	PD	25	54	no	$\langle 36; 36; 6, 18\rangle$
434	PD	25	56	yes	$\langle 28; 28; 4, 14\rangle$
437	PD	25	56	yes	$\langle 28; 28; 4, 28\rangle$
438	PD	25	56	yes	$\langle 28; 28; 4, 14\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
441	PD	25	56	yes	$\langle 28; 28; 4, 28\rangle$
442	PD	25	56	yes	$\langle 28; 28; 4, 14\rangle$
445	PD	25	56	yes	$\langle 28; 28; 4, 28\rangle$
464	PD	25	60	no	$\langle 12; 60; 10, 30\rangle$
475	PD	25	60	yes	$\langle 20; 20; 6, 10\rangle$
482	PD	25	60	yes	$\langle 20; 20; 6, 10\rangle$
527	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
529	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
530	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
534	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
535	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
536	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
538	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
539	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
543	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
546	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
548	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
550	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
554	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
565	PD	25	64	yes	$\langle 16; 16; 8, 4\rangle$
566	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
576	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
577	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
579	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
580	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
582	PD	25	64	yes	$\langle 16; 16; 8, 4\rangle$
583	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
584	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
586	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
588	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
589	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
590	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
591	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
592	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
593	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
594	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
596	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
603	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
622	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
623	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
636	PD	25	64	yes	$\langle 16; 16; 4, 16\rangle$
637	PD	25	64	yes	$\langle 16; 16; 4, 16\rangle$
640	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
641	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
642	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
643	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
644	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
645	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
646	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
653	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
654	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
655	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
656	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
657	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
658	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
659	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
666	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
667	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
668	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
669	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
673	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
674	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
681	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
682	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
684	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
685	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
689	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
690	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
691	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
692	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
693	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
733	PD	25	64	yes	$\langle 16; 16; 8, 16\rangle$
737	PD	25	64	yes	$\langle 16; 16; 4, 16\rangle$
746	PD	25	64	yes	$\langle 16; 16; 8, 16\rangle$
754	PD	25	64	yes	$\langle 16; 16; 4, 32\rangle$
761	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
762	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
764	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
765	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
774	PD	25	64	no	$\langle 16; 16; 8, 8\rangle$
777	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
778	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
781	PD	25	64	yes	$\langle 16; 16; 8, 4\rangle$
782	PD	25	64	yes	$\langle 16; 16; 8, 4\rangle$
783	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
785	PD	25	64	no	$\langle 16; 16; 8, 8\rangle$
789	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
795	PD	25	64	yes	$\langle 16; 16; 8, 4\rangle$
799	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
800	PD	25	64	yes	$\langle 16; 16; 8, 4\rangle$
801	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
804	PD	25	64	yes	$\langle 16; 16; 8, 8\rangle$
805	PD	25	64	no	$\langle 16; 16; 4, 8\rangle$
806	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
808	PD	25	64	no	$\langle 16; 16; 4, 8\rangle$
809	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
810	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
812	PD	25	64	yes	$\langle 16; 16; 4, 4\rangle$
829	PD	25	64	yes	$\langle 16; 16; 16, 8\rangle$
831	PD	25	64	yes	$\langle 16; 16; 16, 8\rangle$
833	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
834	PD	25	64	yes	$\langle 16; 16; 16, 8\rangle$
839	PD	25	64	yes	$\langle 16; 16; 16, 8\rangle$
852	PD	25	64	yes	$\langle 16; 16; 4, 8\rangle$
853	PD	25	64	yes	$\langle 16; 16; 16, 8\rangle$
885	PD	25	72	yes	$\langle 12; 12; 12, 6\rangle$
889	PD	25	72	no	$\langle 8; 24; 6, 12\rangle$
891	PD	25	72	no	$\langle 8; 24; 6, 12\rangle$
934	PD	25	72	yes	$\langle 12; 12; 12, 12\rangle$
937	PD	25	72	yes	$\langle 12; 12; 6, 12\rangle$
948	PD	25	72	yes	$\langle 12; 12; 4, 18\rangle$
951	PD	25	72	yes	$\langle 12; 12; 4, 36\rangle$
979	PD	25	72	no	$\langle 12; 12; 12, 6\rangle$
982	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
984	PD	25	72	yes	$\langle 12; 12; 12, 6\rangle$
1002	PD	25	72	no	$\langle 12; 12; 12, 12\rangle$
1003	PD	25	72	yes	$\langle 12; 12; 4, 12\rangle$
1004	PD	25	72	yes	$\langle 12; 12; 12, 12\rangle$
1013	PD	25	72	no	$\langle 12; 12; 6, 12\rangle$
1014	PD	25	72	yes	$\langle 12; 12; 6, 12\rangle$
1031	PD	25	72	no	$\langle 12; 12; 12, 6\rangle$
1032	PD	25	72	no	$\langle 12; 12; 4, 6\rangle$
1036	PD	25	72	yes	$\langle 12; 12; 6, 6\rangle$
1058	PD	25	72	yes	$\langle 12; 12; 4, 6\rangle$
1062	PD	25	72	no	$\langle 12; 12; 4, 6\rangle$
1081	PD	25	72	yes	$\langle 12; 12; 4, 4\rangle$
1458	PD	25	96	yes	$\langle 8; 8; 8, 4\rangle$
1459	PD	25	96	yes	$\langle 8; 8; 24, 4\rangle$
1461	PD	25	96	yes	$\langle 8; 8; 6, 4\rangle$
1462	PD	25	96	yes	$\langle 8; 8; 12, 4\rangle$
1465	PD	25	96	yes	$\langle 8; 8; 12, 4\rangle$
1470	PD	25	96	yes	$\langle 8; 8; 12, 4\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
1475	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1476	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1479	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1504	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1507	PD	25	96	yes	$\langle 8;8;4,12\rangle$
1508	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1509	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1514	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1515	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1516	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1517	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1524	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1527	PD	25	96	yes	$\langle 8;8;8,12\rangle$
1528	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1529	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1534	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1535	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1536	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1537	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1539	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1541	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1544	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1545	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1548	PD	25	96	yes	$\langle 8;8;8,12\rangle$
1549	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1550	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1556	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1557	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1561	PD	25	96	yes	$\langle 8;8;8,8\rangle$
1562	PD	25	96	yes	$\langle 8;8;24,8\rangle$
1580	PD	25	96	yes	$\langle 8;8;6,8\rangle$
1587	PD	25	96	yes	$\langle 8;8;8,8\rangle$
1588	PD	25	96	yes	$\langle 8;8;24,8\rangle$
1590	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1593	PD	25	96	yes	$\langle 8;8;4,12\rangle$
1594	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1595	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1600	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1601	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1605	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1608	PD	25	96	yes	$\langle 8;8;8,12\rangle$
1609	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1610	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1615	PD	25	96	yes	$\langle 8;8;8,4\rangle$
1616	PD	25	96	yes	$\langle 8;8;24,4\rangle$
1617	PD	25	96	yes	$\langle 8;8;4,8\rangle$
1618	PD	25	96	yes	$\langle 8;8;12,8\rangle$
1670	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1676	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1678	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1683	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1685	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1704	PD	25	96	yes	$\langle 8;8;4,12\rangle$
1750	PD	25	96	yes	$\langle 8;8;12,4\rangle$
1752	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1755	PD	25	96	yes	$\langle 8;8;4,4\rangle$
1791	PD	25	96	yes	$\langle 8;8;4,8\rangle$
1839	PD	25	96	yes	$\langle 8;8;6,4\rangle$
1906	PD	25	96	yes	$\langle 8;8;4,4\rangle$
2187	PD	25	96	yes	$\langle 8;8;12,4\rangle$
2189	PD	25	96	yes	$\langle 8;8;4,4\rangle$
2191	PD	25	96	yes	$\langle 8;8;4,12\rangle$
2192	PD	25	96	yes	$\langle 8;8;12,4\rangle$
2193	PD	25	96	yes	$\langle 8;8;4,4\rangle$
2197	PD	25	96	yes	$\langle 8;8;12,4\rangle$
2251	PD	25	96	yes	$\langle 8;8;6,4\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
2253	PD	25	96	yes	$\langle 8;8;6,4\rangle$
2254	PD	25	96	yes	$\langle 8;8;6,4\rangle$
2273	PD	25	96	yes	$\langle 8;8;24,4\rangle$
2275	PD	25	96	yes	$\langle 8;8;8,4\rangle$
2277	PD	25	96	yes	$\langle 8;8;8,12\rangle$
2278	PD	25	96	yes	$\langle 8;8;24,4\rangle$
2279	PD	25	96	yes	$\langle 8;8;8,4\rangle$
2283	PD	25	96	yes	$\langle 8;8;24,4\rangle$
2309	PD	25	96	yes	$\langle 8;8;24,8\rangle$
2311	PD	25	96	yes	$\langle 8;8;8,8\rangle$
2313	PD	25	96	yes	$\langle 8;8;8,24\rangle$
2314	PD	25	96	yes	$\langle 8;8;24,8\rangle$
2315	PD	25	96	yes	$\langle 8;8;8,8\rangle$
2319	PD	25	96	yes	$\langle 8;8;24,8\rangle$
2321	PD	25	96	yes	$\langle 8;8;24,4\rangle$
2323	PD	25	96	yes	$\langle 8;8;8,4\rangle$
2325	PD	25	96	yes	$\langle 8;8;8,12\rangle$
2326	PD	25	96	yes	$\langle 8;8;24,4\rangle$
2327	PD	25	96	yes	$\langle 8;8;8,4\rangle$
2331	PD	25	96	yes	$\langle 8;8;24,4\rangle$
2336	PD	25	96	yes	$\langle 8;8;4,8\rangle$
2337	PD	25	96	yes	$\langle 8;8;12,8\rangle$
2340	PD	25	96	yes	$\langle 8;8;6,8\rangle$
2368	PD	25	96	yes	$\langle 8;8;8,4\rangle$
2371	PD	25	96	yes	$\langle 8;8;6,4\rangle$
2380	PD	25	96	yes	$\langle 8;8;8,4\rangle$
2399	PD	25	96	yes	$\langle 8;8;6,4\rangle$
2406	PD	25	96	yes	$\langle 8;8;6,4\rangle$
2408	PD	25	96	yes	$\langle 8;8;6,4\rangle$
2410	PD	25	96	yes	$\langle 8;8;12,4\rangle$
2414	PD	25	96	yes	$\langle 8;8;12,4\rangle$
2533	PD	25	96	yes	$\langle 8;8;8,16\rangle$
2553	PD	25	96	yes	$\langle 8;8;8,16\rangle$
2567	PD	25	96	yes	$\langle 8;8;12,8\rangle$
2569	PD	25	96	yes	$\langle 8;8;12,8\rangle$
2702	PD	25	104	no	$\langle 4;52;4,26\rangle$
2865	PD	25	108	no	$\langle 4;36;18,18\rangle$
3301	PD	25	120	no	$\langle 4;20;12,10\rangle$
3306	PD	25	120	no	$\langle 4;20;4,30\rangle$
5086	PD	25	128	no	$\langle 4;16;16,8\rangle$
7350	PD	25	144	no	$\langle 4;12;8,6\rangle$
7471	PD	25	144	no	$\langle 4;12;4,8\rangle$
456	PD	26	60	yes	$\langle 24;24;10,4\rangle$
457	PD	26	60	yes	$\langle 24;24;10,12\rangle$
458	PD	26	60	yes	$\langle 24;24;10,12\rangle$
493	PD	26	60	yes	$\langle 24;24;10,4\rangle$
494	PD	26	60	yes	$\langle 24;24;10,12\rangle$
495	PD	26	60	yes	$\langle 24;24;10,12\rangle$
2587	PD	26	100	yes	$\langle 8;8;10,4\rangle$
2600	PD	26	100	yes	$\langle 8;8;10,4\rangle$
2605	PD	26	100	yes	$\langle 8;8;10,4\rangle$
2608	PD	26	100	yes	$\langle 8;8;10,20\rangle$
2609	PD	26	100	yes	$\langle 8;8;10,20\rangle$
2631	PD	26	100	yes	$\langle 8;8;50,4\rangle$
2632	PD	26	100	yes	$\langle 8;8;50,4\rangle$
2959	PD	26	110	no	$\langle 4;44;10,22\rangle$
2970	PD	26	110	no	$\langle 4;44;10,22\rangle$

Type 4^P - nonorientable genus 3 to 30

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
1	PD	6	8	yes	$\langle 8;8;4,4\rangle$
2	PD	6	8	yes	$\langle 8;8;4,4\rangle$
5	PD	8	12	yes	$\langle 8;8;3,4\rangle$
6	PD	10	16	yes	$\langle 8;8;8,4\rangle$
8	PD	10	16	yes	$\langle 8;8;4,4\rangle$
9	PD	10	16	yes	$\langle 8;8;4,4\rangle$
10	PD	10	16	yes	$\langle 8;8;4,4\rangle$
11	PD	10	16	yes	$\langle 8;8;8,4\rangle$
12	PD	10	16	yes	$\langle 8;8;8,4\rangle$
					Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
13	PD	10	16	yes	$\langle 8;8;4,4\rangle$
14	PD	10	16	yes	$\langle 8;8;8,4\rangle$
20	PD	10	20	no	$\langle 4;20;4,10\rangle$
19	PD	12	20	yes	$\langle 8;8;5,4\rangle$
23	PD	14	24	yes	$\langle 8;8;12,4\rangle$
24	PD	14	24	yes	$\langle 8;8;4,4\rangle$
26	PD	14	24	yes	$\langle 8;8;12,4\rangle$
34	PD	14	24	yes	$\langle 8;8;12,4\rangle$
35	PD	14	24	yes	$\langle 8;8;4,4\rangle$
36	PD	14	24	yes	$\langle 8;8;12,4\rangle$
40	PD	14	24	yes	$\langle 8;8;6,4\rangle$
41	PD	14	24	yes	$\langle 8;8;6,4\rangle$
44	PD	14	24	yes	$\langle 8;8;3,4\rangle$
45	PD	14	24	yes	$\langle 8;8;6,4\rangle$
46	PD	14	24	yes	$\langle 8;8;3,4\rangle$
47	PD	14	24	yes	$\langle 8;8;6,4\rangle$
48	PD	14	24	yes	$\langle 8;8;3,4\rangle$
49	PD	14	24	no	$\langle 8;8;6,4\rangle$
50	PD	14	24	no	$\langle 8;8;3,4\rangle$
52	PD	14	24	yes	$\langle 8;8;4,8\rangle$
55	PD	16	24	no	$\langle 8;12;3,6\rangle$
63	PD	16	28	yes	$\langle 8;8;7,4\rangle$
450	PD	16	56	no	$\langle 4;8;7,4\rangle$
56	PD	17	24	no	$\langle 8;16;3,8\rangle$
58	PD	17	24	no	$\langle 8;16;3,8\rangle$
21	PD	18	20	no	$\langle 20;20;4,10\rangle$
22	PD	18	20	yes	$\langle 20;20;4,10\rangle$
53	PD	18	24	yes	$\langle 12;12;3,4\rangle$
68	PD	18	32	yes	$\langle 8;8;8,4\rangle$
74	PD	18	32	yes	$\langle 8;8;8,4\rangle$
75	PD	18	32	yes	$\langle 8;8;8,4\rangle$
78	PD	18	32	yes	$\langle 8;8;16,4\rangle$
80	PD	18	32	yes	$\langle 8;8;4,4\rangle$
82	PD	18	32	yes	$\langle 8;8;8,4\rangle$
83	PD	18	32	yes	$\langle 8;8;4,4\rangle$
84	PD	18	32	yes	$\langle 8;8;4,4\rangle$
85	PD	18	32	yes	$\langle 8;8;8,4\rangle$
86	PD	18	32	yes	$\langle 8;8;8,4\rangle$
93	PD	18	32	no	$\langle 8;8;4,4\rangle$
95	PD	18	32	yes	$\langle 8;8;4,4\rangle$
96	PD	18	32	yes	$\langle 8;8;8,4\rangle$
97	PD	18	32	no	$\langle 8;8;8,4\rangle$
98	PD	18	32	yes	$\langle 8;8;4,4\rangle$
102	PD	18	32	yes	$\langle 8;8;8,4\rangle$
103	PD	18	32	yes	$\langle 8;8;16,4\rangle$
104	PD	18	32	yes	$\langle 8;8;16,4\rangle$
105	PD	18	32	yes	$\langle 8;8;4,4\rangle$
106	PD	18	32	yes	$\langle 8;8;16,4\rangle$
29	PD	20	24	yes	$\langle 16;16;3,8\rangle$
33	PD	20	24	no	$\langle 16;16;4,8\rangle$
42	PD	20	24	yes	$\langle 16;16;6,8\rangle$
51	PD	20	24	yes	$\langle 16;16;4,4\rangle$
57	PD	20	24	yes	$\langle 16;16;3,4\rangle$
59	PD	20	24	yes	$\langle 16;16;3,4\rangle$
60	PD	20	24	yes	$\langle 16;16;4,8\rangle$
128	PD	20	36	yes	$\langle 8;8;6,4\rangle$
135	PD	20	36	yes	$\langle 8;8;3,4\rangle$
150	PD	20	36	yes	$\langle 8;8;9,4\rangle$
192	PD	20	42	no	$\langle 4;28;3,14\rangle$
153	PD	22	40	yes	$\langle 8;8;20,4\rangle$
155	PD	22	40	yes	$\langle 8;8;20,4\rangle$
157	PD	22	40	yes	$\langle 8;8;10,4\rangle$
158	PD	22	40	yes	$\langle 8;8;4,4\rangle$
160	PD	22	40	yes	$\langle 8;8;4,4\rangle$
170	PD	22	40	yes	$\langle 8;8;20,4\rangle$
174	PD	22	40	yes	$\langle 8;8;20,4\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
175	PD	22	40	yes	$\langle 8;8;10,4\rangle$
187	PD	22	40	no	$\langle 8;8;4,4\rangle$
202	PD	24	44	yes	$\langle 8;8;11,4\rangle$
91	PD	26	32	no	$\langle 16;16;4,8\rangle$
92	PD	26	32	yes	$\langle 16;16;4,8\rangle$
94	PD	26	32	yes	$\langle 16;16;4,8\rangle$
99	PD	26	32	yes	$\langle 16;16;4,8\rangle$
100	PD	26	32	no	$\langle 16;16;8,8\rangle$
101	PD	26	32	yes	$\langle 16;16;4,8\rangle$
129	PD	26	36	no	$\langle 8;24;6,12\rangle$
136	PD	26	36	no	$\langle 8;24;3,12\rangle$
151	PD	26	36	yes	$\langle 12;12;4,6\rangle$
152	PD	26	36	no	$\langle 12;12;4,6\rangle$
206	PD	26	48	yes	$\langle 8;8;24,4\rangle$
207	PD	26	48	yes	$\langle 8;8;24,4\rangle$
208	PD	26	48	yes	$\langle 8;8;8,4\rangle$
234	PD	26	48	yes	$\langle 8;8;12,4\rangle$
235	PD	26	48	yes	$\langle 8;8;12,4\rangle$
236	PD	26	48	yes	$\langle 8;8;12,4\rangle$
246	PD	26	48	yes	$\langle 8;8;6,4\rangle$
248	PD	26	48	yes	$\langle 8;8;3,4\rangle$
249	PD	26	48	yes	$\langle 8;8;6,4\rangle$
250	PD	26	48	yes	$\langle 8;8;12,4\rangle$
251	PD	26	48	yes	$\langle 8;8;12,4\rangle$
253	PD	26	48	yes	$\langle 8;8;6,4\rangle$
254	PD	26	48	yes	$\langle 8;8;12,4\rangle$
256	PD	26	48	no	$\langle 8;8;6,4\rangle$
257	PD	26	48	no	$\langle 8;8;12,4\rangle$
258	PD	26	48	yes	$\langle 8;8;4,4\rangle$
259	PD	26	48	yes	$\langle 8;8;12,4\rangle$
262	PD	26	48	yes	$\langle 8;8;4,4\rangle$
263	PD	26	48	yes	$\langle 8;8;12,4\rangle$
264	PD	26	48	yes	$\langle 8;8;4,4\rangle$
265	PD	26	48	yes	$\langle 8;8;12,4\rangle$
268	PD	26	48	yes	$\langle 8;8;24,4\rangle$
269	PD	26	48	yes	$\langle 8;8;8,4\rangle$
270	PD	26	48	yes	$\langle 8;8;24,4\rangle$
275	PD	26	48	yes	$\langle 8;8;6,4\rangle$
276	PD	26	48	yes	$\langle 8;8;6,4\rangle$
278	PD	26	48	yes	$\langle 8;8;24,4\rangle$
279	PD	26	48	yes	$\langle 8;8;8,4\rangle$
280	PD	26	48	yes	$\langle 8;8;24,4\rangle$
295	PD	26	48	yes	$\langle 8;8;12,4\rangle$
296	PD	26	48	yes	$\langle 8;8;4,4\rangle$
297	PD	26	48	yes	$\langle 8;8;12,4\rangle$
299	PD	26	48	yes	$\langle 8;8;24,4\rangle$
300	PD	26	48	yes	$\langle 8;8;24,4\rangle$
301	PD	26	48	yes	$\langle 8;8;8,4\rangle$
326	PD	26	48	yes	$\langle 8;8;4,8\rangle$
328	PD	26	48	yes	$\langle 8;8;4,8\rangle$
339	PD	26	48	yes	$\langle 8;8;3,4\rangle$
355	PD	26	48	yes	$\langle 8;8;4,4\rangle$
361	PD	26	48	yes	$\langle 8;8;8,4\rangle$
365	PD	26	48	yes	$\langle 8;8;8,4\rangle$
377	PD	26	48	yes	$\langle 8;8;3,4\rangle$
378	PD	26	48	yes	$\langle 8;8;6,4\rangle$
381	PD	26	48	yes	$\langle 8;8;3,4\rangle$
382	PD	26	48	yes	$\langle 8;8;6,4\rangle$
383	PD	26	48	yes	$\langle 8;8;12,4\rangle$
384	PD	26	48	yes	$\langle 8;8;12,4\rangle$
391	PD	26	52	no	$\langle 4;52;4,26\rangle$
409	PD	26	54	no	$\langle 4;36;9,18\rangle$
508	PD	26	60	no	$\langle 4;20;12,10\rangle$
1069	PD	26	72	no	$\langle 4;12;8,6\rangle$
389	PD	28	52	yes	$\langle 8;8;13,4\rangle$
177	PD	30	40	no	$\langle 8;40;4,20\rangle$

Continued ...

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
179	PD	30	40	no	$\langle 8;40;4,20\rangle$
421	PD	30	56	yes	$\langle 8;8;28,4\rangle$
423	PD	30	56	yes	$\langle 8;8;28,4\rangle$
425	PD	30	56	yes	$\langle 8;8;14,4\rangle$
427	PD	30	56	yes	$\langle 8;8;28,4\rangle$
429	PD	30	56	yes	$\langle 8;8;28,4\rangle$
431	PD	30	56	yes	$\langle 8;8;14,4\rangle$
446	PD	30	56	yes	$\langle 8;8;4,4\rangle$
448	PD	30	56	yes	$\langle 8;8;4,4\rangle$
449	PD	30	56	no	$\langle 8;8;7,4\rangle$
451	PD	30	56	no	$\langle 8;8;7,4\rangle$
3150	PD	30	112	no	$\langle 4;8;7,4\rangle$
3151	PD	30	112	no	$\langle 4;8;14,4\rangle$

4.5.6 Type 5, 5* and 5^P maps

Type 5 - orientable genus 2 to 4

Id	Oper.	Genus	$\|E\|$	Self Pe.	Map symb.
2	–	3	21	yes	$\langle 3,7;6;6\rangle$
1	–	4	20	yes	$\langle 4,5;8;8\rangle$

Type 5^P - orientable genus 2 to 22

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
11	PD	8	42	yes	$\langle 6;6;14,6\rangle$
6	PD	11	40	yes	$\langle 8;8;4,10\rangle$
48	PD	14	78	yes	$\langle 6;6;26,6\rangle$
347	PD	14	156	no	$\langle 4;6;12,12\rangle$
83	PD	15	84	no	$\langle 4;12;6,6\rangle$
42	PD	17	64	yes	$\langle 8;8;16,8\rangle$
96	PD	17	96	yes	$\langle 6;6;6,4\rangle$
275	PD	18	136	no	$\langle 4;8;8,8\rangle$
133	PD	19	108	no	$\langle 4;12;18,6\rangle$
165	PD	20	114	yes	$\langle 6;6;38,6\rangle$
53	PD	21	80	yes	$\langle 8;8;4,4\rangle$
54	PD	21	80	yes	$\langle 8;8;4,4\rangle$
59	PD	21	80	yes	$\langle 8;8;20,4\rangle$
61	PD	21	80	yes	$\langle 8;8;10,4\rangle$
62	PD	21	80	yes	$\langle 8;8;4,20\rangle$
18	PD	22	54	yes	$\langle 18;18;6,18\rangle$
79	PD	22	84	no	$\langle 6;12;28,12\rangle$
86	PD	22	84	no	$\langle 6;12;12,4\rangle$
210	PD	22	126	yes	$\langle 6;6;42,6\rangle$
212	PD	22	126	yes	$\langle 6;6;6,14\rangle$
226	PD	22	126	yes	$\langle 6;6;6,6\rangle$

Type 5^P - nonorientable genus 3 to 44

Id	Oper.	Genus	$\|E\|$	Self Du.	Map symb.
2	PD	9	21	yes	$\langle 6;6;3,7\rangle$
1	PD	12	20	yes	$\langle 8;8;4,5\rangle$
4	PD	15	39	yes	$\langle 6;6;13,3\rangle$
13	PD	16	42	no	$\langle 4;12;6,3\rangle$
20	PD	20	54	no	$\langle 4;12;6,9\rangle$
26	PD	21	57	yes	$\langle 6;6;3,19\rangle$
3	PD	23	27	yes	$\langle 18;18;3,9\rangle$
36	PD	23	63	yes	$\langle 6;6;3,21\rangle$
15	PD	28	52	yes	$\langle 8;8;13,4\rangle$
51	PD	28	78	no	$\langle 4;12;6,3\rangle$
10	PD	30	42	yes	$\langle 12;12;14,3\rangle$
12	PD	30	42	yes	$\langle 12;12;7,6\rangle$
14	PD	30	42	yes	$\langle 12;12;6,7\rangle$
82	PD	30	84	yes	$\langle 6;6;14,3\rangle$
5	PD	32	40	no	$\langle 16;16;4,8\rangle$
7	PD	32	40	yes	$\langle 16;16;10,8\rangle$
8	PD	32	40	yes	$\langle 16;16;5,8\rangle$
29	PD	32	60	yes	$\langle 8;8;4,6\rangle$
33	PD	32	60	yes	$\langle 8;8;15,4\rangle$
92	PD	33	93	yes	$\langle 6;6;31,3\rangle$
9	PD	34	42	no	$\langle 12;28;6,3\rangle$
22	PD	35	55	yes	$\langle 10;10;5,11\rangle$
23	PD	35	55	no	$\langle 10;10;5,5\rangle$
24	PD	35	55	yes	$\langle 10;10;11,5\rangle$
44	PD	36	68	yes	$\langle 8;8;17,4\rangle$
21	PD	38	54	yes	$\langle 12;12;9,6\rangle$
158	PD	39	111	yes	$\langle 6;6;37,3\rangle$
167	PD	40	114	no	$\langle 4;12;3,6\rangle$
170	PD	41	117	yes	$\langle 6;6;39,3\rangle$
25	PD	42	56	no	$\langle 14;14;7,7\rangle$
28	PD	42	60	no	$\langle 8;24;12,6\rangle$
32	PD	42	60	no	$\langle 8;24;15,12\rangle$
16	PD	44	54	no	$\langle 12;36;18,9\rangle$
222	PD	44	126	no	$\langle 4;12;6,3\rangle$

4.6 Conclusion

The main results of the thesis are theory of F-maps which generalizes several objects in algebraic combinatorics, the application of this theory on edge-transitive maps, the decomposition theorem, the algorithm for determining small quotients of finitely presented groups and a classification of edge-transitive maps on surfaces of small genera.

From here the research can go to several directions. A classification of finite monolithic groups could be one, perhaps the most (or maybe too) ambitious. The theory of F-maps should continue to develop using the lattice theory. The algorithm can be surely improved – some new optimizations can be introduced. Applications of the parallel product and its limitations on (regular) abstract polytopes and Cayley maps should be studied.

For some families of highly symmetrical graphs several constructions of "optimal" embeddings determining their genus are known. But only a few of those embeddings are highly symmetrical as maps. Studying maps from this point of view would be also an interesting topic (see [5, 38, 40]).

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Index

Razširjeni povzetek

Uvod

Zgodovina in motivacija. Zgodovina povezavno tranzitivnih zemljevidov, med katere spadajo tudi regularni zemljevidi, se začne že s starimi Grki (platonska telesa, nekatera izmed arhimedskih teles). V 17. stoletju je Kepler [29] obravnaval nekonveksne poliedre in pri tem naletel na nekatere neravninske regularne zemljevide. V 19. stoletju je Heffter [25] preučeval orientabilne regularne vložitve polnih grafov, Dyck [16] in Klein [30] pa sta v kontekstu avtomorfnih funkcij konstruirala nekaj kubičnih regularnih zemljevidov na orientabilni kompaktni sklenjeni ploskvi roda 3. Na začetku 20. stoletja so bili regularni zemljevidi prvič uporabljeni kot geometrične predstavitve grup (Burnside [11]). Obdobje bolj sistematičnega preučevanja regularnih zemljevidov se je začelo z Brahano [6] ter Coxeterjem in Moserjem [15]. Pri tem so regularne zemljevide obravnavali kot geometrične, kombinatorične in grupno teoretične objekte. Osnove za moderno algebraično-kombinatorično obravnavo splošnih zemljevidov sta za orientabilne ploskve postavila Jones in Singerman [26], za neorientabilne pa Bryant in Singerman [9]. Osnovna referenca na tem področju je postala knjiga Grossa in Tuckerja [21]. V zadnjem desetletju so raziskave na področju zemljevidov z visoko simetrijo osredotočene predvsem na regularne in orientabilno regularne zemljevide ter na Cayleyeve zemljevide. Pregled teorije slednjih je podan v [42].

Za povezavno tranzitivne zemljevide sta Graver in Watkins [20] podala klasifikacijo, v kateri sta jih razdelila na 14 tipov glede na vsebovanost določenih vrst avtomorfizmov. Širáň, Tucker and Watkins [44] so klasificirali povezavno tranzitivne zemljevide na torusu. Pokazali so tudi, da za vsak tip T obstaja neskončno orientabilnih rodov, pri katerih obstaja zemljevid tipa T vložen na ploskev tega roda.

Osnovna problema pri povezavno tranzitivnih zemljevidih sta njihova konstrukcija in klasifikacija. Običajno take zemljevide konstruiramo s pomočjo kvocientov določenih končno prezentiranih grup ali pa s pomočjo krovnih konstrukcij nad manjšimi zemljevidi. V uporabi so trije pristopi za klasifikacijo povezavno tranzitivnih zemljevidov, in sicer glede na velikost (število povezav) [53], glede na rod ploskve [12] in glede na graf [51].

Znano je, da je na sklenjenih kompaktnih ploskvah, razen morda na sferi, torusu, projektivni ravini in Kleinovi steklenici le končno mnogo povezavno tranzitivnih zemljevidov, saj je velikost njihove grupe avtomorfizmov omejena. Klasifikacijo za sfero sta naredila Grünbaum in Shephard [22], klasifikacijo za torus Širáň, Tucker in Watkins [44], delno klasifikacijo za Kleinovo steklenico sta opravila Potočnik and Wilson [41], medtem ko klasifikacija za projektivno ravnino verjetno še ni bila opravljena.

Pred dobo računalnikov so mnogi avtorji (Brahana [6], Coxeter in Moser [15], Sherk [43], Garbe [19], Bergau in Garbe [2]) delali na klasifikaciji majhnih regularnih in orientabilno regularnih zemljevidov ter klasificirali vse regularne in orientabilno regularne zemljevide na ploskvah do orientabilnega roda 7 in neorientabilnega roda 8. V sedemdesetih letih je Wilson v svoji disertaciji [49] izračunal večino regularnih in orientabilno regularnih zemljevidov, ki vsebujejo do 100 povezav [53]. Uporabljal je računalnik in poseben algoritem [52]. Nedavno sta na tem področju naredila preboj Conder in Dobcsányi [13], ki sta izračunala vse orientabilno regularne zemljevide na ploskvah z rodom od 2 do 15 in neorientabilne regularne zemljevide na neorientabilnih ploskvah z rodom od 3 do 30 [12]. V času zaključevanja te disertacije je M. Conder s pomočjo novega algoritma v MAGMI[10] bistveno izboljšal omenjeni rezultat.

Namen te disertacije je združiti, razširiti in nadgraditi obstoječe cenzuse. Orientabilno regularni zemljevidi niso zaprti za operacijo Petrijev dual, zato je Pisanski predlagal naravno razširitev na povezavno tranzitivne zemljevide. Eden od rezultatov disertacije je cenzus vseh povezavno tranzitivnih zemljevidov do 100 povezav. Poleg tega pa je cenzus regularnih zemljevidov dopolnjen na vse zemljevide do 500 povezav.

V disertaciji so predstavljeni alternativni postopki za konstrukcijo povezavno tranzitivnih zemljevidov. Ker je povezavno tranzitivnih zemljevidov relativno veliko, je zaželjen čimbolj kompakten opis teh. Zanimiv bi bil opis v obliki nekakšnih primitivnih zemljevidov in osnovnih operacij, pri čemer bi lahko vse ostale zemljevide sestavili z operacijami na primitivnih zemljevidih, računska zahtevnost operacij pa bi bila čim manjša. Izkaže se, da je ena od pomembnejših operacij ravno paralelni produkt, ki ga je prvi predstavil Wilson [48].

F-zemljevidi

V 2. poglavju disertacije je predstavljena teorija F-zemljevidov. Teorija je posplošitev konceptov zakoreninjenih zemljevidov, hiper zemljevidov abstraktnih politopov in podobnih (zakoreninjenih) algebraično-kombinatoričnih objektov.

Delovanje grupe G na (končni) množici Z označimo s parom (Z, G) (*desno delovanje*). *Epimorfizem* dveh delovanj (Z, G) in (Z', G') je par (ϕ, ψ), surjekcije $\phi : Z \to Z'$ in epimorfizma $\psi : G \to G'$, pri čemer velja za vsak $z \in Z$ ter $g \in G$: $\phi(z \cdot g) = \phi(z) \cdot \psi(g)$. *Zakoreninjeno (desno) delovanje* je trojica $(Z, G, \underline{\mathrm{id}})$, kjer (Z, G) predstavlja (desno) tranzitivno delovanje in $\underline{\mathrm{id}} \in Z$ je izbran element imenovan *koren*. Morfizmi zakoreninjenih delovanj so epimorfizmi delovanj, pri katerih dodatno zahtevamo, da se koren slika v koren.

Naj bo $F = \langle a_1, ..., a_k \mid R_1 = \ldots = R_n = 1\rangle$ končno prezentirana grupa z generatorji $\{a_i\}_{i=1}^k$ in relacijami $\{R_j\}_{j=1}^n$. Kot *F-grupo* definiramo trojico (F, f, G), pri čemer je $f : F \to G$ epimorfizem grup. *Morfizem F-grup* $A_i = (F, f_i, G_i)$, $i = 1, 2$, je epimorfizem $\psi : G_1 \to G_2$ in $\psi \circ f_1 = f_2$. Če je ψ izomorfizem, to zapišemo kot $A_1 \simeq A_2$. F-grupo si lahko predstavljamo kot grupo s poimenovanimi generatorji, ki izpolnjujejo določene relacije (relacije v F).

F-zemljevid je petorka $M = (F, f, G, Z, \underline{\mathrm{id}}) = (F, f_M, G_M, Z_M, \underline{\mathrm{id}}_M)$, kjer je Z množica *praporjev*, $(Z, F, \underline{\mathrm{id}})$ ter $(Z, G, \underline{\mathrm{id}})$ sta zakoreninjeni delovanji, pri čemer je slednje zvesto, $(\mathrm{Id}, f) : (Z, F, \underline{\mathrm{id}}) \to (Z, G, \underline{\mathrm{id}})$ je epimorfizem zakoreninjenih delovanj (Id predstavlja identično preslikavo) in (F, f, G) je F-grupa. Naj $\chi : G \to \mathrm{Sym}_{\mathrm{R}}(Z)$ določa homomorfizem grupe G na permutacijsko reprezentacijo delovanja ($\mathrm{Sym}_{\mathrm{R}}(Z)$ je množica permutacij na Z, ki jih množimo od leve proti desni). Permutacijsko F-grupo $\mathrm{Mon}(M) = (F, \chi \circ f, \chi(G))$ imenujemo *monodromijska grupa*. Predstavljamo si jo lahko kot permutacijsko reprezentacijo grupe G na množici Z, kjer so izbrani generatorji $\mathrm{Mon}(M)$ poimenovani. Označimo z $\mathrm{S}_F(M) = F_{\underline{\mathrm{id}}} = f^{-1}(G_{\underline{\mathrm{id}}})$ stabilizator korena za delovanje grupe F na Z.

Morfizem F-zemljevidov M in N je par (ϕ, ψ), ki je morfizem zakoreninjenih delovanj $(\phi, \psi) : (Z_M, G_M, \underline{\mathrm{id}}_M) \to (Z_N, G_N, \underline{\mathrm{id}}_N)$ in $\psi : (F, f_M, G_M) \to (F, f_N, G_N)$ je morfizem F-grup. Obstoj takega morfizma naznačimo z zapisom $N \leq M$. Če tak morfizem obstaja, je enoličen. Izomorfizem dobimo, če sta obe preslikavi v paru še injektivni; izomorfizem naznačimo z $M \simeq N$. Avtomorfizem

F-zemljevida M je morfizem $(\phi, \mathrm{Id}) : M \to M$, pri katerem je ϕ bijekcija, opustimo pa pogoj, da se mora koren preslikati v koren. Grupa avtomorfizmov deluje na praporjih z leve strani in to delovanje je semi-regularno. To sledi iz dejstva, da je avtomorfizem povsem določen že s sliko korena. S simbolom α_w, $w \in F$ označimo avtomorfizem, ki slika $\underline{\mathrm{id}}$ v $\underline{\mathrm{id}} \cdot w$, če seveda za dani zemljevid M tak avtomorfizem obstaja (takrat rečemo, da M *vsebuje* α_w, oziroma zapišemo $\alpha_w \in \mathrm{Aut}(M)$).

Naj bo F^+ podgrupa F, ki je generirana z vsemi besedami v F, ki so sode dolžine v generatorjih. Odvisno od relacij v F je indeks F^+ v F bodisi 1 bodisi 2. V slednjem primeru lahko definiramo koncept orientabilnosti. Zemljevid je *orientabilen* natanko tedaj, ko ima delovanje F^+ na praporjih natanko dve orbiti.

Če koren premaknemo, smo zemljevid *prekoreninili*. Prekoreninjenje smatramo kot operacijo na F-zemljevidih in če pri F-zemljevidu M koren premaknemo iz $\underline{\mathrm{id}}$ v $\underline{\mathrm{id}} \cdot w$, $w \in F$, dobimo zemljevid $R_w(M)$. Velja $\mathrm{S}_F(R_w(M)) = w^{-1}\mathrm{S}_F(M)w$.

Naj bo $d \in \mathrm{Aut}(F)$ in $M = (F, f, G, Z, \underline{\mathrm{id}})$. Potem d inducira operacijo na F-zemljevidih definirano z $O_d(M) := (F, f \circ d, G, Z, \underline{\mathrm{id}})$. Velja $\mathrm{S}_F(O_d(M)) = d^{-1}(\mathrm{S}_F(M))$. Če je d notranji avtomorfizem (t.j. konjugacija z nekim elementom $w \in F$), potem velja $O_d(M) \simeq R_w(M)$.

V nadaljevanju disertacije se osredotočimo na kvociente F-zemljevidov. Pri tem se zgledujemo po teoriji Brede in Nedele [8], ki sta pokazala povezavo med mrežo edink v F (za ustrezno grupo F na regularnih hiper zemljevidih) in mrežo izomorfnostnih razredov regularnih hiper zemljevidov. Njihovo teorijo razširimo in prikažemo povezavo med mrežo podgrup F in delno urejeno množico izomorfnostnih razredov F-zemljevidov. Tako pokažemo, da je slednja dejansko mreža.

Za podgrupo $K \leq F$ velja, da $\mathrm{M}_F(K) = (F, q, F/\mathrm{Core}_F(K), F/K, K)$ predstavlja F-zemljevid, pri čemer $\mathrm{Core}_F(K)$ predstavlja presek vseh konjugirank grupe K v F, oziroma ekvivalentno, največjo edinko v F, ki je vsebovana v K. Preslikava $q : F \to F/\mathrm{Core}_F(K)$ je naravni epimorfizem. Seveda je $\mathrm{S}_F(\mathrm{M}_F(K)) = K$.

Veljata naslednji trditvi:

Trditev 1. *Naj bo M F-zemljevid. Potem je $M \simeq \mathrm{M}_F(\mathrm{S}_F(M))$.*

Trditev 2. *Naj bosta $K_1, K_2 \leq F$ podgrupi. Potem velja $K_1 \leq K_2$ natanko tedaj, ko obstaja morfizem F-zemljevidov $(\phi, \psi) : \mathrm{M}_F(K_1) \to \mathrm{M}_F(K_2)$.*

Posledica zgornjih trditev je naslednja pomembna trditev.

Posledica 3. *Naj bosta M in N F-zemljevida. Potem obstaja morfizem F-zemljevidov $(\phi, \psi) : M \to N$ natanko tedaj, ko $\mathrm{S}_F(N) \leq \mathrm{S}_F(M)$. Zato velja $M \simeq N$ natanko tedaj, ko $\mathrm{S}_F(M) = \mathrm{S}_F(N)$.*

Ključni izrek, ki ga bomo uporabili pri klasifikaciji kvocentov in pri izreku o razcepu, je znani izrek iz teorije grup – izrek o korespondenci. Zato je smiselno, da ga na tem mestu omenimo.

Izrek 4. *Naj bosta G, G' grupi in $f : G \to G'$ epimorfizem. Definirajmo množici $\mathcal{A} = \{K : \ker f \leq K \leq G\}$ in $\mathcal{B} = \{K' : K' \leq G'\}$. Potem je preslikava $F : \mathcal{A} \to \mathcal{B}$ definirana s pravilom $F : K \mapsto f(K)$ bijekcija. Preslikava preslika edinke v edinke. Za katerikoli podgrupi $K, H \in \mathcal{A}$ sledi: $F(K \cap H) = F(K) \cap F(H)$, $F(\langle K, H \rangle) = \langle F(K), F(H) \rangle$, in če velja $K \leq H$, potem $F(K) \leq F(H)$.*

Za F-zemljevid $M = (F, f, G, Z, \underline{\mathrm{id}})$ in podgrupo $K \leq G$, kjer $G_{\underline{\mathrm{id}}} \leq K$, definirajmo $M/K = (F, q \circ f, G/H, G/K, K)$, pri čemer je $H = \mathrm{Core}_G(K)$ in $q : G \to G/H$ naravni epimorfizem. Definicija predstavlja F-zemljevid, ki ga imenujemo *K-kvocient* F-zemljevida M. K-kvocient je *strog*, če je H netrivialna.

Naslednji izrek karakterizira morfizme danega F-zemljevida M preko K-kvocientov.

Izrek 5. *Naj bosta M in N F-zemljevida in naj bo $(\phi, \psi) : M \to N$ morfizem F-zemljevidov. Potem obstaja podgrupa $K \leq G_M$, da velja $(G_M)_{\underline{\mathrm{id}}_M} \leq K$ in $M/K \simeq N$. Velja tudi $K = \psi^{-1}((G_N)_{\underline{\mathrm{id}}_N})$.*

Za katerakoli F-zemljevida $N, N' \leq M$, pri čemer velja $N \simeq M/K$ in $N' \simeq M/K'$ za neke podgrupe $K, K' \leq G_M$, sledi $N \leq N'$ natanko tedaj, ko velja $K' \leq K$. Zato je $N \simeq N'$ natanko tedaj, ko je $K = K'$.

Vloga tega izreka pri F-zemljevidih je podobna vlogi prvega izreka o izomorfizmih pri grupah. Z računskega vidika nam omogoča izračun vseh možnih kvocientov zemljevida kar iz monodromijske grupe.

Kvocient N F-zemljevida M imenujemo *direktni kvocient*, če za vsak N', $N \leq N' \leq M$ sledi, bodisi $N' \simeq N$ ali $N' \simeq M$.

Grupa avtomorfizmov danega zemljevida M je odvisna od normalizatorja $N_F(\mathrm{S}_F(M))$.

Izrek 6. *Naj bo $M = (F, f, G, Z, \underline{\mathrm{id}})$ in $\mathrm{S}_F(M) = K$. Potem za $w \in F$, sledi $\alpha_w \in \mathrm{Aut}(M)$ natanko tedaj, ko je $w \in N_F(K)$. Velja tudi $\mathrm{Aut}(M) \simeq N_F(K)/K$.*

Direktni (uporabni) posledici tega izreka sledita.

Posledica 7. *F-zemljevid M je regularen natanko tedaj, ko je $\mathrm{S}_F(M) \lhd F$, in natanko tedaj, ko sta $\mathrm{Aut}(M)$ in $\mathrm{Mon}(M)$ izomorfni kot abstraktni grupi.*

Posledica 8. *Naj bosta M in N F-zemljevida in $p : M \to N$ morfizem F-*

zemljevidov. Potem se grupa $\mathrm{Aut}(M)$ *projicira natanko tedaj, ko* $N_F(\mathrm{S}_F(M)) \leq N_F(\mathrm{S}_F(N))$.

Naj bo $M = (F, f, G, Z, \underline{\mathrm{id}})$ F-zemljevid in $H \lhd G$. Potem je $G_{\underline{\mathrm{id}}}H$ podgrupa v G, ki vsebuje $G_{\underline{\mathrm{id}}}$, zato lahko izračunamo $G_{\underline{\mathrm{id}}}H$-kvocient. Zapišemo ga kot $M \triangle H$. Kvociente dobljene na ta način imenujemo *normalni kvocienti*. Zanje velja naslednja lepa lastnost.

Trditev 9. *Naj bo* M F*-zemljevid in naj bo* $p : M \to M \triangle H$ *morfizem na normalni kvocient. Potem se* $\mathrm{Aut}(M)$ *projicira.*

Normalne kvociente lahko dobimo tudi z naslednjo konstrukcijo.

Trditev 10. *Naj bo* $M = (F, f, G, Z, \underline{\mathrm{id}})$ F*-zemljevid in* $H \lhd G$. *Naj bo* $q : G \to G/H$ *naravni epimorfizem in* Z/H *množica orbit za delovanje edinke* H *na množici* Z. *Definirajmo preslikavo* $p : Z \to Z/H$, $p : z \mapsto [z]$, *kjer* $[z]$ *predstavlja orbito, v kateri je element* $z \in Z$. *Potem je*

$$N = (F, q \circ f, G/\mathrm{Core}_G(G_{\underline{\mathrm{id}}}H), Z/H, p(\underline{\mathrm{id}}))$$

F*-zemljevid izomorfen* $M/G_{\underline{\mathrm{id}}}H$.

Zanimivo je dejstvo, ki ga je opazil Tucker [46], da se vsak morfizem F-zemljevidov $(\phi, \psi) : M \to N$ razcepi preko morfizma na normalni kvocient $M \to M \triangle \ker \psi \to N$.

Namesto blokov neprimitivnosti, ki so orbite neke edinke v monodormijski grupi, lahko uporabimo orbite katerekoli podgrupe grupe avtomorfizmov (te orbite so namreč prav tako bloki neprimitivnosti za delovanje monodormijske grupe na praporjih). Naj bo M F-zemljevid, $\mathcal{B}$ sistem blokov neprimitivnosti, ki so orbite neke podgrupe grupe avtomorfizmov, in naj bo $K \leq G_M$ stabilizator orbite (kot množice), ki vsebuje koren $B_{\underline{\mathrm{id}}}$. Delovanje $(\mathcal{B}, G_M/\mathrm{Core}_G(K), B_{\underline{\mathrm{id}}_M})$ je zvesto zakoreninjeno delovanje in $N = (F, q \circ f_M, G_M/\mathrm{Core}_G(K), \mathcal{B}, B_{\underline{\mathrm{id}}})$ je F-zemljevid, za katerega velja $N \leq M$, saj

$$\mathrm{S}_F(N) = (q \circ f_M)^{-1}((G_M/\mathrm{Core}_G(K))_{B_{\underline{\mathrm{id}}_M}}) \geq f_M^{-1}(\underline{\mathrm{id}}_M) = \mathrm{S}_F(M).$$

Kvociente tega tipa imenujemo K*-avtomorfizemski kvocienti*.

Malnič, Nedela in Škoviera [33] so preučevali morfizme orientabilno regularnih zemljevidov in jih uspeli klasificirati preko K-avtomorfizemskih kvocientov. V splošnem pa to ne gre. Posplošitev njihovega rezultata na F-zemljevidih je naslednja trditev.

Trditev 11 *Naj bo* M F*-zemljevid,* $K \leq \mathrm{Aut}(M)$ *in* $G = \{w \in F \mid \alpha_w \in K\}$. *Potem je* K*-avtomorfizemski kvocient izomorfen* $M/f_M(G)$.

Posledica tega je:

Posledica 12. *Naj bo M F-zemljevid in $G = f_M(N_F(\mathrm{S}_F(M)))$. Potem so K-avtomorfizemski kvocienti v bijektivni korespondenci s K'-kvocienti, kjer je*

$$(G_M)_{\underline{\mathrm{id}}_M} \leq K' \leq G \leq G_M.$$

Za dan F-zemljevid M sledi, da so vsi K-kvocienti, $K \leq G_M$, tudi K'-avtomorfizemski kvocienti za nek $K' \leq \mathrm{Aut}(M)$, natanko tedaj, ko je M regularen F-zemljevid.

V nadaljevanju se osredotočimo na operacije na zemljevidih. Velja:

Trditev 13. *Naj bosta M in N F-zemljevida in naj bo $g \in \mathrm{Aut}(F)$. Potem velja: $N \leq M$ natanko tedaj, ko $O_g(N) \leq O_g(M)$.*

V nadaljevanju disertacije se osredotočimo na paralelni produkt. Ta je definiran kot

$$M \parallel N = (F, (f_M, f_N), G, Z, (\underline{\mathrm{id}}_M, \underline{\mathrm{id}}_N)),$$

kjer je $G = (f_M, f_N)(F) \leq G_M \times G_N$ in Z je orbita pri delovanju G na $Z_M \times Z_N$, ki vsebuje $(\underline{\mathrm{id}}_M, \underline{\mathrm{id}}_N)$. Izkaže se, da je paralelni produkt F-zemljevidov res F-zemljevid in da velja $\mathrm{S}_F(M \parallel N) = \mathrm{S}_F(M) \cap \mathrm{S}_F(N)$. Paralelni produkt je asociativna in komutativna binarna operacija. Definicija nam omogoča izračun $\mathrm{Mon}(M \parallel N)$ iz $\mathrm{Mon}(M)$ ter $\mathrm{Mon}(N)$, kar je posebej uporabno pri računanju.

Za paralelni produkt in simetrije v faktrojih velja naslednja lepa lastnost.

Trditev 14. *Naj bosta M in N F-zemljevida, ki oba vsebujeta avtomorfizem α_w. Potem tudi $M \parallel N$ vsebuje α_w.*

Naslednja trditev opisuje obnašanje paralelnega produkta na prekoreninjenih zemljevidih.

Trditev 15. *Naj bo M F-zemljevid.*

1. *Za vsak $w \in F$, velja $\mathrm{Mon}(M) \simeq \mathrm{Mon}(R_w(M)) \simeq \mathrm{Mon}(M \parallel R_w(M))$.*
2. *$M \simeq R_w(M)$ natanko tedaj, ko $\alpha_w \in \mathrm{Aut}(M)$.*
3. *Če je $w^2 = 1$, potem $\alpha_w \in \mathrm{Aut}(M \parallel R_w(M))$.*
4. *Naj M^M predstavlja paralelni produkt vseh prekoreninjenj danega F-zemljevida. Potem je M^M regularen in za vsak regularen F-zemljevid M', za katerega velja $M \leq M'$, sledi $M^M \leq M'$.*

Obnašanje operacij na zemljevidih glede na paralelni produkt opisuje naslednja trditev.

Trditev 16. *Naj bosta M in N F-zemljevida in naj bo $f \in \mathrm{Aut}(F)$. Potem je $O_f(M \parallel N) \simeq O_f(M) \parallel O_f(N)$.*

Zemljevid N je *trivialen*, če $\mathrm{S}_F(N) = F$. *Razcepni par* za F-zemljevid M je par F-zemljevidov (N_1, N_2), za katera velja $M \simeq N_1 \parallel N_2$ in nobeden od N_1, N_2 ni izomorfen M niti ni trivialen zemljevid.

F-zemljevid M je *razcepen*, če obstaja kak razcepni par za M. Če obstaja razcepni par sestavljen iz normalnih kvocientov, pravimo, da je M *normalno razcepen*. Če obstaja razcepni par iz strogih kvocientov, potem rečemo, da je M *strogo razcepen*.

Sledi glavni rezultat, izrek o razcepu.

Izrek (o razcepu) 17. *F-zemljevid $M = (F, f, G, Z, \underline{\mathrm{id}})$ je razcepen natanko tedaj, ko v G obstajata dve podgrupi $K_1, K_2 \leq G$, da velja $G_{\underline{\mathrm{id}}} \lneqq K_i \lneqq G$, $i = 1, 2$, in $G_{\underline{\mathrm{id}}} = K_1 \cap K_2$.*

F-zemljevid M je normalno razcepen natanko tedaj, ko obstajata dve različni netrivialni edinki $H_1, H_2 \lhd G$, ki delujeta na Z netranzitivno in $G_{\underline{\mathrm{id}}}H_1 \cap G_{\underline{\mathrm{id}}}H_2 = G_{\underline{\mathrm{id}}}$.

F-zemljevid M je normalno razcepen natanko tedaj, ko je strogo razcepen.

Pri regularnih F-zemljevidih ima izrek še posebno lepo obliko.

Izrek 18. *Naj bo M regularen F-zemljevid. M je normalno razcepen natanko tedaj, ko $\mathrm{Mon}(M)$ (oziroma $\mathrm{Aut}(M)$) vsebuje vsaj dve različni netrivialni minimalni edinki. V tem primeru sta faktorja regularna F-zemljevida.*

Grupe z eno samo minimalno edinko in enostavne grupe bomo imenovali *monolitske* grupe.

Regularne F-zemljevide lahko enačimo kar s končnimi kvocienti končno prezentirane grupe F. Taka prezentacija hrani celotno informacijo o regularnem F-zemljevidu, saj si za tako končno prezentirano grupo G lahko predstavljamo pripadajoči F-zemljevid kar kot $(F, q, G, G, 1)$, kjer je $q : F \to G$ naravni epimorfizem na kvocient G. Naj poudarimo, da si prezentacijo grupe G lahko predstavljamo kot prezentacijo F z dodatnimi relacijami.

Zaporedje besed v F, $(W_i)_{i=1}^{s}$, bomo imenovali *kontekst*, če obstaja kakšen kvocient G grupe F, katerega relacije so natanko potence besed iz zaporedja (torej $W_i^{e_i} = 1$, za vsak i in nek pripadajoči $e_i \geq 1$). Tako lahko pripadajočo prezentacijo za G za dan kontekst C zapišemo kot vektor $(e_i)_{i=1}^s$ oziroma pripadajoči regularni zemljevid zapišemo z $M = (e_i)_{i=1}^s$. Če sta dva kvocienta (regularna F-zemljevida) podana v dveh kontekstih, potem *skupen kontekst* vsebuje besede iz obeh in oba zemljevida lahko zapišemo v skupnem kontekstu (dodamo ustrezne redundantne relacije). Za nek kvocient G (regularen F-zemljevid) je kontekst C *zadosten*, če obstaja prezentacija za G v kontekstu C.

Naj bosta M in N regularna F-zemljevida in $w \in F$. Naj bo a natančen red elementa $f_M(w)$ in naj bo b natančen red elementa $f_N(w)$. Potem je natančen red elementa $(f_M, f_N)(w) \in (f_M, f_N)(F)$ enak $\mathrm{lcm}(a, b)$.

Lema 19. *Naj bosta $M = (a_i)_{i=1}^s$ in $N = (b_i)_{i=1}^s$ dva regularna F-zemljevida predstavljena v skupnem kontekstu $(W_i)_{i=1}^s$. Naj bo skupni kontekst zadosten za zemljevid $M \parallel N$. Potem velja $M \parallel N = (\mathrm{lcm}(a_i, b_i))_{i=1}^s$.*

Naj $\mathcal{M}_F$ predstavlja množico izomorfnostnih razredov zemljevidov. Za izomorfnostna razreda $[M]$, $[N] \in \mathcal{M}_F$ definiramo relacijo

$$[N] \leq [M] \overset{\text{def}}{\iff} N \leq M. \tag{4.2}$$

Relacija je delna urejenost in $\mathcal{M}_F$ je delno urejena množica. Izkaže se, da lahko paralelni produkt definiramo na izomorfnostnih razredih kar kot izomorfnostni razred paralelnega produkta dveh poljubnih predstavnikov faktorjev. Naj bo Sub_F mreža podgrup končnega indeksa v F. V tej mreži sta operaciji kupa ($\vee$) in kapa ($\wedge$) definirani kot: $K \vee H = \langle K, H \rangle$ in $K \wedge H = K \cap H$. Preslikava $\Theta : \mathcal{M}_F \to \mathrm{Sub}_F$ definirana z $\Theta : [M] \mapsto \mathrm{S}_F(M)$ je anti-izomorfizem dveh delno urejenih množic. Toda z njim se struktura mreže prenese iz Sub_F na $\mathcal{M}_F$. Pri tem paralelni produkt sovpada ravno z operacijo kupa ($\vee$) v $\mathcal{M}_F$. Preko anti-izomorfizma pa definiramo še operacijo kapa ($\wedge$).

Naslednja trditev, ki sledi iz [8], je pomembna pri izpeljavi algoritma v nadaljevanju.

Trditev 21. *Naj bosta M in N regularna F-zemljevida. Potem velja:*

$$|M \wedge N| \cdot |M \parallel N| = |M| \cdot |N|. \tag{4.3}$$

S tem se poglavje o teoriji F-zemljevidov zaključi.

Algoritem za izračun majhnih kvocientov končno prezentiranih grup

V naslednjem poglavju obravnavamo algoritem za izračun majhnih kvocientov končno prezentiranih grup. Kot smo že ugotovili, lahko končne kvociente končno prezentirane grupe F enačimo z regularnimi F-zemljevidi.

Predvideno je, da bo algoritem implementiran v okolju, ki podpira računanje z grupami, npr. MAGMA [10] ali GAP[18]. Ideja algoritma je naslednja: najprej izračunamo vse regularne F-zemljevide na monolitskih grupah, potem pa s paralelnimi produkti še ostale.

V obeh omenjenih okoljih obstajata (isti) podatkovni bazi majhnih grup do reda 2000 (razen reda 1024, ker je teh grup preveč). Na prvi problem naletimo, ko želimo iz te baze konstruirati bazo monolitskih grup. Za večino redov to ni težko, obstaja pa nekaj redov, kot so 512, 768, 1280, 1536, 1792, pri katerih je zelo veliko grup (to so redi oblike 2^n in $2^n \cdot 3$ za ustrezne n). Izločanje nemonolitskih grup izmed velikega števila vseh grup je sicer enkratno opravilo, a bi v teh primerih vseeno potekalo predolgo časa. S pomočjo teorije p-grup in teorije algoritmov za generiranje teh grup, si delo bistveno olajšamo.

Naslednja trditev in posledica nam pomagata izločiti veliko večino grup pri problematičnih redih.

Trditev 22. *Naj bo G končna grupa in naj bo $N \lhd G$, tako da velja, da je indeks $(G : N)$ enak p^n, za $n \geq 2$, p je praštevilo, ki ne deli $|N|$, ter $|\mathrm{Aut}(N)| \neq p^n$. Potem ima G vsaj dve različni minimalni edinki.*

Posledica 23. *Naj bo G grupa reda $p^n q$, kjer sta p in q različni praštevili, $q - 1 \neq p^n$ in G ima q-Sylowko, ki je edinka. Potem ima G vsaj dve različni minimalni edinki.*

Izračunana baza monolitskih grup bo temelj algoritma. V nadaljevanju bomo skicirali delovanje algoritma.

Osnovna ideja je, da s pomočjo zunanjega algoritma, ki temelji na strategiji sestopanja, poiščemo vse nekongruentne prezentacije kvocientov F na monolitskih grupah. Tako dobljene prezentacije gledamo kot regularne F-zemljevide in s pomočjo njih lahko s paralelnimi produkti izrazimo vse ostale regularne F-zemljevide (oziroma kvociente F). Zavedati se moramo, da lahko isti (do izomorfizma natančno) razcepen regularen F-zemljevid dobimo kot paralelni produkt različnih množic manjših zemljevidov. Ker operacijo paralelni produkt smatramo kot računsko drago, jo želimo za vsak zemljevid izvesti največ enkrat. Poleg tega pa bi radi izračunali vse regularne F-zemljevide le do nekega maksimalnega reda $N_{\max}$, če želimo, da se izračun konča. Zato bi paralelne produkte prevelikih zemljevidov radi zavrgli brez dejanskega računanja.

Računanje začnemo pri majhnih zemljevidih. Šele ko izračunamo zemljevide reda n, se lotimo izračuna regularnih F-zemljevidov reda $n+1$. Na trenutnem koraku (izračun regularnih F-zemljevidov reda n) predpostavimo, da imamo izračunane vse regularne zemljevide do reda $n-1$ ter za njih shranjene naslednje informacije.

1. Grupo, ki določa zemljevid (grupa + ustrezni generatorji); F-zemljevidu (natančneje: izomorfnostnem razredu, kateremu zemljevid pripada) je prirejena *identifikacijska številka F-zemljevida*.

2. *Identifikacijsko številko grupe* (vsak izomorfnostni razred grup v MAGMI ali GAPU ima svojo identifikacijsko številko grupe).

3. Množico direktnih kvocientov (v obliki množice identifikacijskih številk F-zemljevidov).

4. Morebitne dodatne informacije, ki olajšajo identifikacijo (pri določanju direktnih kvocientov).

Pri teh predpostavkah se lotimo izračuna regularnih F-zemljevidov reda n. Za vsako monolitsko grupo G reda n (iz baze monolitskih grup), za katero obstaja kak regularni F-zemljevid na G/H (H minimalna edinka – na tem mestu potrebujemo identifikacijsko številko grupe G/H), uporabimo zunanji algoritem za izračun regularnih F-zemljevidov na G. Ustrezne zapise za dobljene regularne F-zemljevide vstavimo v vrsto s prednostjo Q, katere vlogo bomo sedaj obrazložili. V vrsti Q so pari oblike $(k,(A,B))$, ki v primeru $A \neq B$ pomenijo, da obstaja regularen F-zemljevid reda k, ki je paralelni produkt zemljevidov A in B in sta to direktna kvocienta produkta (A in B sta v resnici identifikacijski številki F-zemljevidov), ali pa je $A = B$ in gre za zapis o zemljevidu A na monolitski grupi.

V vrsti imajo prednost pari z manjšo prvo komponento. Ko se računa regularne F-zemljevide reda n, so na začetku vrste pari s prvo komponento enako n. Algoritem nadaljujemo tako, da vsak tak par vzamemo iz vrste in ga obdelamo. Obdelava pomeni naslednje. Če za zapis velja $A = B$, označimo $M = A$ in preskočimo del obdelave v nadaljevanju tega odstavka. Naj za zapis velja $A \neq B$. Če že obstaja kak zemljevid reda n, ki ima A in B za direktna kvocienta, zapis predstavlja zemljevid, ki ga že imamo (sledi iz modularnosti mreže regularnih F-zemljevidov). V tem primeru zapis zavržemo. Sicer gre za nov razcepen zemljevid, ki ga izračunamo skupaj z ustreznimi zgoraj omenjenimi informacijami, ter vse skupaj shranimo. Novi F-zemljevid označimo z M.

Na tem mestu imamo zemljevid M in nadaljujemo obdelavo. Za vsak že izračunan zemljevid X, ki ima z M kak skupen direkten kvocient, je ta direktni kvocient enak $M \wedge X$. Zaradi modularnosti mreže regularnih F-zemljevidov sledi, da sta v tem primeru M in X direktna kvocienta paralelnega produkta $M \parallel X$. Po trditvi 21 določimo velikost $|M \parallel X|$, in če ta ne presega $N_{\max}$, vstavimo ustrezen zapis v vrsto s prednostjo Q. Na tak način vstavljamo v vrsto s prednostjo vse možne paralelne produkte zemljevida M z že izračunanimi zemljevidi X, kjer sta oba faktorja direktna kvocienta produkta.

Ko obdelamo vse zapise reda n v vrsti s prednostjo Q, se lotimo izračuna F-zemljevidov reda $n+1$. Del obdelave v prejšnejem odstavku lahko izpustimo, če je $n > \frac{N_{\max}}{2}$.

Povezavno tranzitivni zemljevidi

V tem poglavju uporabimo na povezavno tranzitivnih zemljevidih teorijo F-zemljevidov ter algoritem iz prejšnjih dveh poglavij. Najprej se lotimo najbolj simetričnih, to je regularnih zemljevidov oziroma zemljevidov tipa 1. Klasificiramo degenerirane in rahlo degenerirane zemljevide. To so regularni zemljevidi, pri katerih gre za vložitve na ploskve z robom ali pa vsebujejo neprava lica (dolžine 2 ali manj), ter za vse zemljevide dobljene iz teh s pomočjo operacij dual in Petrijev dual. Poleg klasifikacije teh zemljevidov je podana še karakterizacija tistih, ki so nerazcepni. Sledi prikaz vseh nerazcepnih nedegeneriranih regularnih zemljevidov do 100 povezav. Ti so bili že predstavljeni v Wilsonovem cenzusu [53] skupaj z vsemi drugimi regularnimi zemljevidi do 100 povezav.

V nadaljevanju je prikazan način, kako lahko s pomočjo algoritma iz prejšnjega poglavja izračunamo čisto vse regularne zemljevide do 500 povezav (moč grupe avtomorfizmov oziroma monodromijske grupe do 2000), kljub temu, da nimamo baze monolitskih grup na 1024 točkah. Pokažemo, da so nedegenerirani regularni zemljevidi z grupo avtomorfizmov moči 1024 orientabilni in jih izračunamo kot orientabilno regularne zemljevide. Te se namreč da predstaviti kot ustrezne regularne

F'-zemljevide, kjer so grupe za polovico manjše.

Sledi teorija G-orbitno tranzitivnih zemljevidov. Tu je najpomembnejša naslednja trditev.

Trditev: 32. *Naj bo M F-zemljevid. Potem je M G-orbitno tranzitiven natanko tedaj, ko velja $N_F(\mathrm{S}_F(M)) \cdot G = F$.*

Definirajmo:

$$\mathcal{T}_G = \{K : K \leq F, KG = F, K \text{ je normalizator za nek } H \leq F, \\ H \text{ končnega indeksa}\}.$$

Grupe v $\mathcal{T}_G$ se imenujejo *tipi*. F-zemljevid je *T-dopusten* ($T \in \mathcal{T}_G$), če velja $\mathrm{S}_F(M) \lhd T$. F-zemljevid je *tipa* T, če velja $N_F(\mathrm{S}_F(M)) = T$. Za T-dopusten zemljevid imenujemo grupo $\mathrm{Aut}^T(M) = \{\alpha_w \mid w \in T\} \leq \mathrm{Aut}(M)$ *T-dopustna grupa avtomorfizmov* F-zemljevida M. Ugotovimo lahko, da so za vsak $T \in \mathcal{T}_G$ tudi vse konjugiranke T v $\mathcal{T}_G$. Zato v nekaterih primerih za tipe proglasimo kar konjugiranostne razrede in za vsak konjugiranostni razred izberemo kanoničnega predstavnika.

Velja naslednja pomembna trditev:

Trditev 33. *Naj bo $G \leq F$ končna, $T \in \mathcal{T}_G$ ter naj bo F_T končna prezentacija grupe T. Potem so T-dopustni F-zemljevidi v bijektivni korespondenci z regularnimi F_T-zemljevidi.*

Sledi obravnava tipov povezavno tranzitivnih zemljevidov s pomočjo teorije G-orbitno tranzitivnih zemljevidov.

Povezavno tranzitivni zemljevidi so Q-orbitno tranzitivni F-zemljevidi za $F = \langle T, L, R \mid T^2 = L^2 = R^2 = (TL)^2 = 1\rangle$ in $Q = \langle T, L\rangle$. Klasifikacijo tipov sta opravila že Graver and Watkins [20] preko študija lokalnih avtomorfizmov. Naj bo

$$A = \{T, L, R, TL, RT, RL, LTRT, TRLT, LTR, TRL, LRL, TRT, LTRTL\}$$

množica predpisanih besed v F, ki predstavljajo premike od korena do določenih praporjev v okolici korena. Graver and Watkins [20] sta glede na vsebovanost avtomorfizmov α_w, $w \in A$, razdelila zemljevide na 14 tipov (glej sliko 4.2 in tabelo 4.5). Tipi sovpadajo s podgrupami F indeksa do 4 (moč grupe Q je 4), ki ustrezajo pogoju iz trditve 32. Vsak tip T je podan s podmnožico množice A, ki generira grupo $T \in \mathcal{T}_Q$. Tipe označujeta s svojimi oznakami, mi pa bomo te oznake privzeli. Za vse tipe T razen za tipe z oznakami 4, 4^* in 4^P so pripadajoče grupe v $\mathcal{T}_Q$ edinke, pri teh tipih pa imajo konjugiranostni razredi po dve grupi. V tem primeru je definicija teh treh tipov pri omenjenih avtorjih ekvivalentna izboru ustrezne grupe izmed dveh v dotičnem ekvivalenčnem razredu.

Povezavno tranzitivni zemljevidi imajo največ po dve orbiti na točkah, oziroma na povezavah, oziroma na Petrijevih licih. Stopnje teh zapišemo v simbolu zemljevida v obliki $\langle a_1, a_2; b_1, b_2; c_1, c_2 \rangle$ (a-ji za stopnje točk, b-ji za velikosti lic in c-ji za velikosti Petrijevih lic). V primeru, da gre za točkovno tranzitiven zemljevid (ena orbita na točkah), namesto para a_1, a_2 v simbolu pišemo le eno stopnjo a. Pravilo ustrezno razširimo za primera tranzitivnosti po licih oziroma Petrijevih licih.

V tabeli 4.5 so podane omejitve za vrednosti v simbolu zemljevida.

Sledi obravnava vpliva operacij dual in Petrijev dual na spremembo tipov. Velja naslednja trditev, pri kateri zaradi enostavnejšega zapisa predpostavimo, da za tipa 1 in 3 velja: $1 = 1^* = 1^P$ ter $3 = 3^* = 3^P$.

Trditev 34. *Naj bo M povezavno tranzitiven zemljevid tipa T. Potem sta $\mathrm{D}(M)$ in $\mathrm{P}(M)$ prav tako povezavno tranzitivna zemljevida in za $T \in \{1, 2, 2\mathrm{ex}, 3, 4, 5\}$ veljajo naslednja pravila transformiranja tipov:*

$$\begin{array}{lll} \mathrm{D}(T) = T^*, & \mathrm{D}(T^*) = T, & \mathrm{D}(T^P) = T^P, \\ \mathrm{P}(T) = T, & \mathrm{P}(T^*) = T^P, & \mathrm{P}(T^P) = T^*. \end{array}$$

Zato se lahko, vsaj pri izračunu in klasifikaciji, omejimo na tipe $1, 2, 2\mathrm{ex}, 3, 4, 5$. V tabeli 4.6 so podane delne prezentacije T-dopustnih grup avtomorfizmov, pri čemer podčrtane relacije določajo prezentacijo F_T grupe T. V nadaljevanju pokažemo, da lahko T-dopustne zemljevide obravnavamo kot regularne F_T-zemljevide glede na tabelo 4.6. Zato jih lahko s pomočjo algoritma iz prejšnjega poglavja izračunamo in rezultate predstavimo na koncu disertacije ter na [53].

V nadaljevanju si postavimo vprašanje, kdaj je T-dopusten zemljevid natanko tipa T. Odgovor sledi iz trditve 15. Prekoreninjenja povezavno tranzitivnih T-dopustnih zemljevidov, ki so predstavljeni kot F-zemljevidi, predstavljajo neke operacije na pripadajočih (po trditvi 33 in tabeli 4.6) regularnih F_T-zemljevidih. Odgovor na zgornje vprašanje je podan v smislu fiksnih točk določenih operacij (glej tabelo 4.8 v kombinaciji s trditvijo 37).

Na koncu s pomočjo Eulerjeve formule (v stilu [13]) določimo zgornje meje redov grup avtomorfizmov za nedegenerirane povezavno tranzitivne zemljevide vseh tipov vloženih na kompaktne sklenjene ploskve orientabilnega roda večjega od 1 in neorientabilnega roda večjega od 2. Meje so linearno odvisne od roda. Če izračunamo vse regularne F_T zemljevide za $T \in \{1, 2, 2\mathrm{ex}, 3, 4, 5\}$ do nekega reda $N_{\max}$, lahko glede na omenjene zgornje meje z gotovostjo podamo vse nedegenerirane zemljevide tipa T do nekega maksimalnega roda.

Rezultati so podani na koncu zadnjega poglavja (glej tudi [53]). Kot zanimivost, cenzus Conderja in Dobcsányi-ja [13] je za orientabilne regularne in kiralne zemljevide razširjen do roda 24. Kot smo že omenili, je Conder ravno v času dokončevanja te disertacije rezultat že izboljšal [14].

Izjava

Izjavljam, da je disertacija plod lastnega raziskovalnega dela.

Alen Orbanić