Appendix B
Solutions to the sample examination paper

Question 1.

(a) The elasticity of demand is given by the formula

$$\varepsilon(p) = -\frac{p}{q} \cdot \frac{\mathrm{d}q}{\mathrm{d}p},$$

where $q = q^D(p)$ is the demand function. In this question, we are told that

$$\varepsilon = \frac{p}{26-p},$$

and so putting this into the formula above we get

$$-\frac{p}{q} \cdot \frac{\mathrm{d}q}{\mathrm{d}p} = \frac{p}{26-p} \quad \Longrightarrow \quad \frac{1}{q} \cdot \frac{\mathrm{d}q}{\mathrm{d}p} = \frac{1}{p-26},$$

which is a separable differential equation. As such, we solve this by 'separating' the variables and integrating both sides to get

$$\int \frac{1}{q}\,\mathrm{d}q = \int \frac{1}{p-26}\,\mathrm{d}p \quad \Longrightarrow \quad \ln|q| = \ln|p-26| + c \quad \Longrightarrow \quad q = A(p-26),$$

for some arbitrary constant, A. Then, using the fact that the equilibrium price is 14 and the equilibrium quantity is 6, we can see that A must satisfy the equation

$$6 = A(14-26) \quad \Longrightarrow \quad A = -\frac{6}{12} = -\frac{1}{2}.$$

Putting this all together, we then see that we have $q = q^D(p)$ where

$$q^D(p) = 13 - \frac{p}{2},$$

is the sought after demand function.

(b) The producer surplus is given by

$$\mathrm{PS} = p^* q^* - \int_0^{q^*} p^S(q)\,\mathrm{d}q,$$

where p^* and q^* are the equilibrium price and quantity, and $p^S(q)$ is the inverse supply function. So, using the information given in the question, we have

$$36 = (14)(6) - \int_0^6 (aq+b)\,\mathrm{d}q \quad \Longrightarrow \quad 48 = \left[a\frac{q^2}{2} + bq\right]_0^6 \quad \Longrightarrow \quad 48 = 18a + 6b,$$

or, indeed, $8 = 3a + b$ as our first equation for a and b. Another equation that needs to be satisfied is

$$14 = 6a + b,$$

as the equilibrium quantity must give the equilibrium price when we use the inverse supply function. We can easily solve these equations for the constants a and b by subtracting one from the other to get $a = 2$ and then, using the first equation again, we get $b = 2$. Consequently, we have

$$p^S(q) = 2q + 2 \quad \text{so that} \quad q^S(p) = \frac{p}{2} - 1,$$

is the supply function for this market.[1]

(c) In the presence of an excise tax of T, the supply function becomes

$$q_T^S(p) = q^S(p - T) = \frac{1}{2}(p - T) - 1,$$

whereas the demand function is unchanged, i.e. $q_T^D(p) = q^D(p)$.

This means that, in the presence of the excise tax of T, the new equilibrium price is given by

$$q_T^S(p) = q_T^D(p) \quad \Longrightarrow \quad \frac{1}{2}(p - T) - 1 = 13 - \frac{p}{2} \quad \Longrightarrow \quad p = 14 + \frac{T}{2},$$

and, using $q_T^D(p)$ say, we see that the new equilibrium quantity is

$$q = 13 - \frac{1}{2}\left(14 + \frac{T}{2}\right) = 6 - \frac{T}{4}.$$

We can now find the tax revenue, $R(T)$, which is the tax per unit, T, multiplied by q, the amount being sold in the presence of the tax, i.e. we have

$$R(T) = Tq = T\left(6 - \frac{T}{4}\right) = 6T - \frac{T^2}{4}.$$

To see where this is maximised, we start by noting that $R(T)$ has a stationary point when $R'(T) = 0$, i.e. when

$$6 - \frac{T}{2} = 0 \quad \Longrightarrow \quad T = 12,$$

[1] Of course, an alternative method here would be to observe that the supply function is a straight line and so the producer surplus is the area of a triangular region whose height is $p^* - b$ and whose width is q^*. This means that, if we find the area of this triangle, we have

$$36 = \tfrac{1}{2}(14 - b)(6) \quad \Longrightarrow \quad 14 - b = 12 \quad \Longrightarrow \quad b = 2.$$

Then, again using the fact that equilibrium quantity must give the equilibrium price when we use the inverse supply function, we use $b = 2$ to see that

$$14 = 6a + b \quad \Longrightarrow \quad a = \frac{14 - 2}{6} = 2,$$

so that, once again, we find that

$$p^S(q) = 2q + 2 \quad \text{so that} \quad q^S(p) = \frac{p}{2} - 1,$$

is the supply function for this market.

and since $R''(T) = -1/2 < 0$ this turning point is indeed a maximum. Thus, the tax revenue is maximised when $T = 12$.

Question 2.

(a) To find the eigenvalues of this matrix, we solve the equation

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 4-\lambda & 1 \\ 2 & 5-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (4-\lambda)(5-\lambda) - 2 = 0,$$

which, multiplying out the brackets, gives us the quadratic equation

$$\lambda^2 - 9\lambda + 18 = 0 \quad \Longrightarrow \quad (\lambda - 3)(\lambda - 6) = 0,$$

and so the eigenvalues are 3 and 6. To find the corresponding eigenvectors we seek a non-zero vector, $\boldsymbol{x}$, which is a solution to the equation $(A - \lambda I)\boldsymbol{x} =$, i.e.

- For $\lambda = 3$, we solve

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \quad \Longrightarrow \quad x + y = 0 \quad \Longrightarrow \quad y = -x \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

  (or any non-zero multiple of this) is an eigenvector.

- For $\lambda = 6$, we solve

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \quad \Longrightarrow \quad -2x + y = 0 \quad \Longrightarrow \quad y = 2x \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

  (or any non-zero multiple of this) is an eigenvector.

Consequently, if we take

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix},$$

we have an invertible matrix, P, and a diagonal matrix, D, such that $P^{-1}AP = D$.[2]

The given system of difference equations can be written as $\boldsymbol{x}_t = A\boldsymbol{x}_{t-1}$ if we let

$$\boldsymbol{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}. \quad \text{We then define the vector} \quad \boldsymbol{u}_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

[2] Of course, this is only one of the many possible pairs of matrices that we could choose for P and D: others are possible depending on which eigenvectors we choose when we form the columns of P and the order in which we choose to place them in P. For instance, choosing the other order for the eigenvectors we found above, we can see that

$$P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix},$$

is another possible answer here.

where $\boldsymbol{x}_t$ and $\boldsymbol{u}_t$ are related by $\boldsymbol{x}_t = P\boldsymbol{u}_t$. This means that $\boldsymbol{x}_{t-1} = P\boldsymbol{u}_{t-1}$ and, substituting this into $\boldsymbol{x}_t = A\boldsymbol{x}_{t-1}$, we get

$$P\boldsymbol{u}_t = AP\boldsymbol{u}_{t-1} \quad \Longrightarrow \quad \boldsymbol{u}_t = P^{-1}AP\boldsymbol{u}_{t-1} \quad \Longrightarrow \quad \boldsymbol{u}_t = D\boldsymbol{u}_{t-1},$$

as $P^{-1}AP = D$. Using this, we have

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} u_{t-1} \\ v_{t-1} \end{pmatrix} \quad \Longrightarrow \quad u_t = 3u_{t-1} \quad \text{and} \quad v_t = 6v_{t-1},$$

and this pair of difference equations can easily be solved to get

$$u_t = A(3^t) \qquad \text{and} \qquad v_t = B(6^t),$$

for arbitrary constants A and B. This means that, using $\boldsymbol{x}_t = P\boldsymbol{u}_t$, we have

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A(3^t) \\ B(6^t) \end{pmatrix} = \begin{pmatrix} A(3^t) + B(6^t) \\ -A(3^t) + 2B(6^t) \end{pmatrix},$$

which means that

$$x_t = A(3^t) + B(6^t) \quad \text{and} \quad y_t = -A(3^t) + 2B(6^t),$$

is the general solution to our coupled system of difference equations. Then with the initial conditions $x_0 = 1$ and $y_0 = 1$, we get the equations

$$1 = A + B \qquad \text{and} \qquad 1 = -A + 2B,$$

which are easily solved to get $A = 1/3$ and $B = 2/3$. Consequently, we find that

$$x_t = \frac{3^t + 2(6^t)}{3} \qquad \text{and} \qquad y_t = \frac{-(3^t) + 4(6^t)}{3},$$

is the required particular solution to our coupled system of difference equations.

(b) The first-order partial derivatives of $f(x, y)$ are

$$f_x(x, y) = 2x - 2 \qquad \text{and} \qquad f_y(x, y) = -3y^2 + 2y + 1.$$

At a stationary point, both of these first-order partial derivatives are zero, i.e. we must have $f_x(x, y) = 0$ and $f_y(x, y) = 0$. Thus, to find the stationary points we have to solve the simultaneous equations

$$2x - 2 = 0 \qquad \text{and} \qquad -3y^2 + 2y + 1 = 0.$$

But, the first equation gives us $x = 1$ and the second equation gives us

$$3y^2 - 2y - 1 = 0 \quad \Longrightarrow \quad (3y + 1)(y - 1) = 0 \quad \Longrightarrow \quad y = -\frac{1}{3} \text{ or } 1.$$

Consequently, the points $(1, -1/3)$ and $(1, 1)$ are the stationary points of this function.

The second-order partial derivatives of this function are

$$f_{xx}(x, y) = 2, \qquad f_{xy}(x, y) = 0 = f_{yx}(x, y) \qquad \text{and} \qquad f_{yy}(x, y) = -6y + 2,$$

and, as such, the Hessian is given by

$$H(x, y) = (2)(-6y + 2) - (0)^2 = 4(1 - 3y).$$

Evaluating this at each of the stationary points we then find that:

- At $(1, -1/3)$, the Hessian is
$$H(1, -1/3) = 4(2) > 0 \qquad \text{and} \qquad f_{xx}(1, -1/3) = 2 > 0,$$
so this is a local minimum.
- At $(1, 1)$, the Hessian is
$$H(1, 1) = 4(-2) < 0,$$
and so this is a saddle point.

Thus, the stationary points $(1, -1/3)$ and $(1, 1)$ are a local minimum and a saddle point respectively.

Question 3.

(a) We use integration by parts to see that, differentiating the t and integrating the $\cos t$, we get
$$\int t \cos t \, \mathrm{d}t = t \sin t - \int \sin t \, \mathrm{d}t = t \sin t + \cos t + c,$$
where c is an arbitrary constant.

(b) The augmented matrix for the given matrix equation is
$$\begin{pmatrix} 1 & -3 & 1 & | & 1 \\ 1 & -1 & -1 & | & -1 \\ 1 & -5 & a & | & b \end{pmatrix},$$
where a and b are any numbers. We can reduce this to its row-echelon form by performing the following row operations.
$$\begin{pmatrix} 1 & -3 & 1 & | & 1 \\ 1 & -1 & -1 & | & -1 \\ 1 & -5 & a & | & b \end{pmatrix} \xrightarrow[R_3 \to R_1 - R_3]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -3 & 1 & | & 1 \\ 0 & 2 & -2 & | & -2 \\ 0 & 2 & 1-a & | & 1-b \end{pmatrix}$$
$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{pmatrix} 1 & -3 & 1 & | & 1 \\ 0 & 1 & -1 & | & -1 \\ 0 & 2 & 1-a & | & 1-b \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & -3 & 1 & | & 1 \\ 0 & 1 & -1 & | & -1 \\ 0 & 0 & 3-a & | & 3-b \end{pmatrix}.$$
From the last line, we see that **if** $a \neq 3$, we get a **unique solution** given by
$$z = \frac{3-b}{3-a},$$
so that
$$y = -1 + z = -1 + \frac{3-b}{3-a} = \frac{-3 + a + (3-b)}{3-a} = \frac{a-b}{3-a},$$
and
$$x = 1 + 3y - z = 1 + 3\frac{a-b}{3-a} - \frac{3-b}{3-a} = \frac{3 - a + (3a - 3b) - (3-b)}{3-a} = \frac{2(a-b)}{3-a}.$$

and these can easily be written in vector form. However, **if** $a = 3$ we have the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 3-b \end{array}\right),$$

so that, if in addition to $a = 3$, we have

- $b \neq 3$, we have no solutions, whereas
- $b = 3$, we have an infinite number of solutions.

For the latter solutions, we use the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + 3R_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

giving us solutions

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2+2z \\ -1+z \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

for any $z \in \mathbb{R}$.

Question 4.

Here the cost function is

$$C(k, l) = vk + wl,$$

and we want to minimise this subject to the constraint $q(k, l) = Q$ where $k, l > 0$. So, writing the constraint in the form $q(k, l) - Q = 0$, we get the Lagrangean

$$L(k, l, \lambda) = vk + wl - \lambda(q(k, l) - Q) = vk + wl - \lambda(k^\alpha l^\alpha - Q).$$

and we seek the points which simultaneously satisfy the equations $L_k(k, l, \lambda) = 0$, $L_l(k, l, \lambda) = 0$ and $L_\lambda(k, l, \lambda) = 0$. As such, we find the first-order partial derivatives of $L(k, l, \lambda)$, i.e.

$$L_k(k, l, \lambda) = v - \lambda\alpha k^{\alpha-1} l^\alpha, \; L_l(k, l, \lambda) = w - \lambda\alpha k^\alpha l^{\alpha-1} \text{ and } L_\lambda(k, l, \lambda) = -\left(k^\alpha l^\alpha - Q\right),$$

and set these equal to zero to yield the equations

$$v - \lambda\alpha k^{\alpha-1} l^\alpha = 0, \qquad w - \lambda\alpha k^\alpha l^{\alpha-1} = 0 \qquad \text{and} \qquad k^\alpha l^\alpha - Q = 0.$$

We now solve these by eliminating λ from the first two equations, i.e. we get

$$v - \lambda\alpha k^{\alpha-1} l^\alpha = 0, \qquad \Longrightarrow \qquad \lambda = \frac{v}{\alpha k^{\alpha-1} l^\alpha} = \frac{vk}{\alpha k^\alpha l^\alpha},$$

from the first equation, and

$$w - \lambda\alpha k^\alpha l^{\alpha-1} = 0 \qquad \Longrightarrow \qquad \lambda = \frac{w}{\alpha k^\alpha l^{\alpha-1}} = \frac{wl}{\alpha k^\alpha l^\alpha},$$

from the second equation. As such, we can equate these expressions for λ to get

$$\frac{vk}{\alpha k^\alpha l^\alpha} = \frac{wl}{\alpha k^\alpha l^\alpha} \qquad \Longrightarrow \qquad l = \frac{v}{w}k.$$

We then use this new relationship between k and l in the third equation, which is just the constraint $k^\alpha l^\alpha = Q$, to get

$$Q = k^\alpha \left(\frac{v}{w}k\right)^\alpha \quad \Longrightarrow \quad Q = \left(\frac{v}{w}\right)^\alpha k^{2\alpha} \quad \Longrightarrow \quad k^{2\alpha} = \left(\frac{w}{v}\right)^\alpha Q \quad \Longrightarrow \quad k = \sqrt{\frac{w}{v}}Q^{\frac{1}{2\alpha}},$$

and then, using this in the equation $l = vk/w$, we get

$$l = \frac{v}{w}\left(\sqrt{\frac{w}{v}}Q^{\frac{1}{2\alpha}}\right) = \sqrt{\frac{v}{w}}Q^{\frac{1}{2\alpha}}.$$

Thus, these values of k and l minimise the cost of producing Q units. The minimum cost is then given by

$$\hat{C}(Q) = C\left(\sqrt{\frac{w}{v}}Q^{\frac{1}{2\alpha}}, \sqrt{\frac{v}{w}}Q^{\frac{1}{2\alpha}}\right) = v\sqrt{\frac{w}{v}}Q^{\frac{1}{2\alpha}} + w\sqrt{\frac{v}{w}}Q^{\frac{1}{2\alpha}} = 2\sqrt{vw}\,Q^{\frac{1}{2\alpha}},$$

as required.

To justify this, we note that the constraint $k^\alpha l^\alpha = Q$ looks a bit like a rectangular hyperbola and, for $k, l > 0$, this is illustrated in Figure B.1(a). The objective function, $C(k,l) = vk + wl$ has contours $C(k,l) = c$, where c is a constant, that are straight lines as illustrated in Figure B.1(b). The direction in which $C(k,l)$ is decreasing is indicated in this figure along with the point we found above using the Lagrange multiplier method — i.e. a point where we have a contour of $C(k,l)$ which is both tangential to the constraint and touching the constraint. Having seen this, it should be clear that this point will minimise C subject to the constraint.

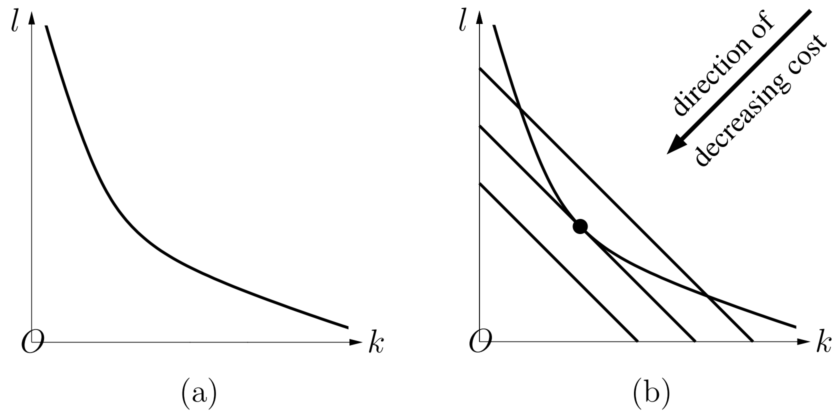

Figure B.1: (a) The constraint $q(k,l) = Q$. (b) Adding three contours, $C(k,l) = c$, where the direction in which $C(k,l)$ is decreasing is as indicated. Clearly, we are interested in the point which is indicated in the figure.

Using the given information, we can see that if Q is produced then the revenue generated will be $\mathrm{R}(Q) = pQ$ and the costs incurred will be

$$\mathrm{C}(Q) = cQ + \hat{C}(Q) + \mathrm{FC} = cQ + 2\sqrt{vw}\,Q^{\frac{1}{2\alpha}} + \mathrm{FC},$$

which is the cost of the raw materials plus the costs of capital and labour plus any fixed costs the firm may have. As such, the profit function for the firm is

$$\pi(Q) = \mathrm{R}(Q) - \mathrm{C}(Q) = pQ - cQ - 2\sqrt{vw}\, Q^{\frac{1}{2\alpha}} - \mathrm{FC},$$

and we want to find the value of Q that maximises this. As such, we find that

$$\pi'(Q) = p - c - 2\sqrt{vw}\left(\frac{1}{2\alpha} Q^{\frac{1}{2\alpha}-1}\right) = p - c - \frac{\sqrt{vw}}{\alpha} Q^{\frac{1-2\alpha}{2\alpha}},$$

as the fixed costs, FC, are a constant and, setting this equal to zero, we find that

$$\pi'(Q) = 0 \quad \Longrightarrow \quad Q^{\frac{1-2\alpha}{2\alpha}} = \alpha \frac{p-c}{\sqrt{vw}} \quad \Longrightarrow \quad Q = \left(\alpha \frac{p-c}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}},$$

is the only stationary point. Indeed, notice that this value of Q is positive as $p > c$ and $\alpha > 0$. Furthermore, we have

$$\pi''(Q) = -\frac{\sqrt{vw}}{\alpha}\left(\frac{2\alpha}{1-2\alpha} Q^{\frac{2\alpha}{1-2\alpha}-1}\right),$$

and as this is negative at the stationary point (since $0 < \alpha < 1/2$ implies that $\alpha > 0$ and $1 - 2\alpha > 0$) we see that our stationary point is a local maximum. Thus,

$$Q = \left(\alpha \frac{p-c}{\sqrt{vw}}\right)^{\frac{2\alpha}{1-2\alpha}},$$

is the value of Q that maximises the firm's profit.

Question 5.

(a) We make the substitution $g = \sin x$ so that

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \cos x \qquad \Longrightarrow \qquad \cos x \mathrm{d}x = \mathrm{d}g,$$

and so we have

$$\int \frac{\cos x}{(1-\sin x)(2+\sin x)}\,\mathrm{d}x = \int \frac{1}{(1-g)(2+g)}\,\mathrm{d}g.$$

Thus, using partial fractions, we have

$$\frac{1}{(1-g)(2+g)} = \frac{A}{1-g} + \frac{B}{2+g} \qquad \Longrightarrow \qquad 1 = A(2+g) + B(1-g),$$

so that, setting $g = 1$ we get $A = 1/3$ and setting $g = -2$ we get $B = 1/3$. Consequently, we have

$$\begin{aligned}
\int \frac{1}{(1-g)(2+g)}\,\mathrm{d}g &= \int \left(\frac{1/3}{1-g} + \frac{1/3}{2+g}\right)\mathrm{d}g \\
&= \frac{1}{3}\Big(-\ln|1-g| + \ln|2+g|\Big) + c \\
&= \frac{1}{3}\ln\left|\frac{2+\sin x}{1-\sin x}\right| + c,
\end{aligned}$$

as the answer.

(b) The differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} - \frac{S(t)}{10} = 1,$$

is linear and so we start by finding the integrating factor, i.e. $\mathrm{e}^{\int -\frac{\mathrm{d}t}{10}} = \mathrm{e}^{-\frac{t}{10}}$ and this gives us

$$\mathrm{e}^{-\frac{t}{10}} S(t) = \int \mathrm{e}^{-\frac{t}{10}}\, \mathrm{d}t = -10\,\mathrm{e}^{-\frac{t}{10}} + c,$$

Thus, we find that

$$S(t) = c\,\mathrm{e}^{\frac{t}{10}} - 10,$$

as the general solution. Now, given that $S(0) = P$, we have

$$P = c - 10 \implies c = P + 10,$$

and so the particular solution in this case is

$$S(t) = (P + 10)\,\mathrm{e}^{\frac{t}{10}} - 10.$$

(c) The corresponding homogeneous second-order ODE is

$$y_1'' - 2y_1' + y_1 = 0,$$

and so the auxiliary equation is

$$k^2 - 2k + 1 = 0 \implies (k-1)^2 = 0,$$

which has one real solution given by $k = 1$ (twice). Consequently, the complementary function, $y_c(t)$, is

$$y_1(t) = (At + B)\,\mathrm{e}^t,$$

for arbitrary constants A and B.

The right-hand-side of the given ODE is e^t and our first reaction in this case would be to take $y_p(t) = \alpha\,\mathrm{e}^t$ where α is a constant that has to be determined. But, this won't work as, taking $A = 0$ and $B = \alpha$, we see that this is 'part' of the complementary function. As such, we 'multiply by t' and try $y_p(t) = \alpha t\,\mathrm{e}^t$ which won't work either because, taking $A = \alpha$ and $B = 0$, we see that this is 'part' of the complementary function as well. Consequently, we multiply by t again and try $y_p(t) = \alpha t^2\,\mathrm{e}^t$ which, thankfully, will work because it is not 'part' of the complementary function. So, differentiating this using the product rule, we have

$$y_p'(t) = (2\alpha t)\,\mathrm{e}^t + (\alpha t^2)\,\mathrm{e}^t = \alpha(2t + t^2)\,\mathrm{e}^t,$$

and

$$y_p''(t) = \alpha(2 + 2t)\,\mathrm{e}^t + \alpha(2t + t^2)\,\mathrm{e}^t = \alpha(2 + 4t + t^2)\,\mathrm{e}^t,$$

which means that, substituting these into our ODE, we get

$$\alpha(2 + 4t + t^2)\,\mathrm{e}^t - 2\alpha(2t + t^2)\,\mathrm{e}^t + \alpha t^2\,\mathrm{e}^t = \mathrm{e}^t \implies 2\alpha\,\mathrm{e}^t = \mathrm{e}^t \implies \alpha = \frac{1}{2}.$$

B

Consequently, we see that

$$y_p(t) = \frac{t^2}{2}\,\mathrm{e}^t,$$

is the particular integral we seek

The general solution to our ODE is then given by the sum of its complementary function and its particular integral, i.e. we have

$$y(t) = (At + B)\,\mathrm{e}^t + \frac{t^2}{2}\,\mathrm{e}^t,$$

where A and B are arbitrary constants.

Then given the conditions $y(0) = 1$ and $y(1) = 0$, we have the equations

$$1 = B\,\mathrm{e}^0 \qquad \text{and} \qquad 0 = (A + B)\,\mathrm{e}^1 + \frac{\mathrm{e}^1}{2},$$

respectively. The first of these gives $B = 1$ and then the second gives

$$0 = A + B + \frac{1}{2} \qquad \Longrightarrow \qquad 0 = A + 1 + \frac{1}{2} \qquad \Longrightarrow \qquad A = -\frac{3}{2}.$$

Thus, we find that

$$y(t) = \left(-\frac{3}{2}t + 1\right)\mathrm{e}^t + \frac{t^2}{2}\,\mathrm{e}^t = \frac{t^2 - 3t + 2}{2}\,\mathrm{e}^t,$$

is the particular solution we seek.