Examiners' commentaries 2022

MT1186 Mathematical methods: Preliminary examination

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2021–22. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the online subject guide. You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

General remarks

Learning outcomes

At the end of this half course and having completed the Essential reading and activities you should have:

- used the concepts, terminology, methods and conventions covered in the course to solve mathematical problems in this subject
- the ability to solve unseen mathematical problems involving the understanding of these concepts and application of these methods
- seen how calculus can be used to solve problems in economics and related subjects
- demonstrated knowledge and understanding of the underlying principles of calculus.

Showing your working

We start by emphasising that you should **always** include your working. This means two things. First, you should not simply write down the answer in the examination script, but you should explain the method by which it is obtained. Second, you should include rough working (even if it is messy!). The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined. We also stress that if you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if you have written down a wrong answer and nothing else, no marks can be awarded. So it is certainly in your interests to include all your workings.

Covering the syllabus and choosing questions

You should ensure that you have covered the syllabus in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: you should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, **any** topic could appear in the examination questions.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the candidate does indeed know the methods, understands them, and is able to use them, and not merely whether they vaguely remember them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different, from previous years' questions. You should **expect** to be surprised by some of the questions. Of course, you will only be examined on material in the syllabus, so all questions can be answered using the material of the course. There will be enough, routine, familiar content in the examination so that a candidate who has achieved competence in the course will pass, but, of course, for a high mark, more is expected: you will have to demonstrate an ability to solve new and unfamiliar problems.

Answer the question

Please do read the questions carefully. You might be asked to use specific methods, even when others could be used. The purpose of the examination is to test that you know certain methods, so the examiners might occasionally ask you to use a specific technique. In such circumstances, only limited partial credit can be given if you do not use the specified technique. It is also worth reading the question carefully so that you do not do more than is required (because it is unlikely that you would get extra marks for doing so). For instance, if a question asked you only to find the critical points of a function, but not their natures, then you should not determine their natures. Be careful to read all questions carefully because, although they may look like previous examination questions on first glance, there can be subtle differences.

Calculators

You are reminded that calculators are **not** permitted in the examination for this course, under any circumstances. The examiners know this, and so they set questions that do not require a calculator. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this course.

Examination revision strategy

Many candidates are disappointed to find that their examination performance is poorer than they expected. This may be due to a number of reasons, but one particular failing is '**question spotting**', that is, confining your examination preparation to a few questions and/or topics which have come up in past papers for the course. This can have serious consequences.

We recognise that candidates might not cover all topics in the syllabus in the same depth, but you need to be aware that examiners are free to set questions on **any aspect** of the syllabus. This means that you need to study enough of the syllabus to enable you to answer the required number of examination questions.

The syllabus can be found in the Course information sheet available on the VLE. You should read the syllabus carefully and ensure that you cover sufficient material in preparation for the examination. Examiners will vary the topics and questions from year to year and may well set questions that have not appeared in past papers. Examination papers may legitimately include questions on any topic in the syllabus. So, although past papers can be helpful during your revision, you cannot assume that topics or specific questions that have come up in past examinations will occur again.

If you rely on a question-spotting strategy, it is likely you will find yourself in difficulties when you sit the examination. We strongly advise you not to adopt this strategy.

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Comments on specific questions

Candidates should answer all **FIVE** questions. All questions carry equal marks.

Question 1

(a) **A market has supply and demand functions given by**

$$q^S(p) = 4p - 1 \quad \text{and} \quad q^D(p) = 5 - 2p$$

respectively. Find the equilibrium price and quantity.

An excise (or per unit) tax of T is imposed on this market. What is the new equilibrium price and quantity?

Find the value of T that maximises the tax revenue.

(b) **The function f is given by**

$$f(x, y) = x^3 - y^3 - 2xy.$$

Find the stationary points of f and determine their nature.

Reading for this question

For relevant reading, see Sections 1.6.2 and 3.5.3 of the subject guide for part (a), and see Section 6.2 of the subject guide for part (b).

Approaching the question

(a) Given the supply and demand functions:

$$q^S(p) = 4p - 1 \quad \text{and} \quad q^D(p) = 5 - 2p$$

respectively, we can easily find the equilibrium price, p^*, by solving the equation:

$$q^S(p^*) = q^D(p^*) \quad \Rightarrow \quad 4p^* - 1 = 5 - 2p^* \quad \Rightarrow \quad 6p^* = 6 \quad \Rightarrow \quad p^* = 1$$

and the equilibrium quantity, q^*, can then be found by using, say, $q^* = q^D(p^*) = 5 - 2 = 3$. If an excise tax of T is imposed on the market, then the supply function:

$$q^S(p) = 4p - 1 \quad \text{becomes} \quad q^S_T(p) = q^S(p - T) = 4(p - T) - 1$$

and the demand function stays unchanged, i.e. we have:

$$q^D_T(p) = q^D(p) = 5 - 2p.$$

The new equilibrium price, p^*_T, can then be found by solving the equation $q^S_T(p^*_T) = q^D_T(p^*_T)$, i.e. we have:

$$4(p^*_T - T) - 1 = 5 - 2p^*_T \quad \Rightarrow \quad 4p^*_T - 4T - 1 = 5 - 2p^*_T \quad \Rightarrow \quad 6p^*_T = 6 + 4T$$

which means that:

$$p^*_T = 1 + \frac{2}{3}T.$$

The new equilibrium quantity, q^*_T, can then be found using the demand function, i.e. we have:

$$q^*_T = q^D_T(p^*_T) = 5 - 2\left(1 + \frac{2}{3}T\right) = 3 - \frac{4}{3}T.$$

(A common error at this point is to use the original supply equation, i.e. $q = q^S(p)$, instead of the modified one, i.e. $q = q^S_T(p)$, to find the new equilibrium quantity. To avoid this mistake, it is usually a good idea to use the demand equation, as we do here, since this doesn't change!)

The tax revenue, R_T, is given by:

$$R_T = Tq^*_T = T\left(3 - \frac{4}{3}T\right) = 3T - \frac{4}{3}T^2$$

and this has a stationary point when $R'_T = 0$, i.e. when:

$$3 - \frac{8}{3}T = 0 \quad \Rightarrow \quad T = \frac{9}{8}$$

and, as $R''_T = -8/3 < 0$ at $T = 9/8$, this is a local maximum of R_T. However, we need to guarantee that this is actually the global maximum of R_T and the easiest way to do this is to note that, as $R''_T = -8/3 < 0$ for all T, the function R_T is concave and so the local maximum at $T = 9/8$ is a global maximum.

(b) To find the stationary points of the function:

$$f(x, y) = x^3 - y^3 - 2xy$$

we find its first-order partial derivatives, i.e. we have:

$$f_x(x, y) = 3x^2 - 2y \quad \text{and} \quad f_y(x, y) = -3y^2 - 2x$$

and solve the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously. This gives us the equations:

$$3x^2 - 2y = 0 \quad \text{and} \quad -3y^2 - 2x = 0$$

and, looking at the first equation, we must have $y = 3x^2/2$. Then, substituting this into the second equation, we then get:

$$-3\left(\frac{3}{2}x^2\right)^2 - 2x = 0 \quad \Rightarrow \quad \frac{27}{4}x^4 + 2x = 0 \quad \Rightarrow \quad x(27x^3+8) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = -\frac{2}{3}.$$

Thus, using $y = 3x^2/2$ again, the stationary points of the function are $(0,0)$ and $(-2/3, 2/3)$. To classify these stationary points, we note that:

$$f_{xx}(x,y) = 6x, \quad f_{xy}(x,y) = -2 = f_{yx}(x,y) \quad \text{and} \quad f_{yy}(x,y) = -6y.$$

so that the Hessian is given by:

$$H(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(-6y) - (-2)^2 = -4(9xy+1).$$

Thus, using this, we can see that:

- at $(0,0)$, we have $H(0,0) = -4 < 0$, i.e. it is a saddle point
- at $(-2/3, 2/3)$, we have $H(-2/3, 2/3) = 12 > 0$ and $f_{xx}(-2/3, 2/3) = -4 < 0$, i.e. it is a local maximum.

Question 2

(a) Suppose that the equation:

$$\boldsymbol{xy^3 + x^2y^2 + x^3y = 3}$$

implicitly defines y as a function of x around $x = 1$. Find the values of $y(1)$ and $y'(1)$.

(b) For $x > 0$, the function f is given by

$$\boldsymbol{f(x) = a\ln x - b(\ln x)^2}$$

where a and b are constants.

For which values of a and b does this function have one stationary point?

When it exists, determine the coordinates and nature of this stationary point.

Reading for this question

For relevant reading, see Section 5.3.3 of the subject guide for part (a), and see Chapter 3 of the subject guide for part (b).

Approaching the question

(a) If the equation:

$$xy^3 + x^2y^2 + x^3y = 3$$

implicitly defines y as a function of x around $x = 1$, then $y(1)$ is a solution to the equation:

$$y^3 + y^2 + y = 3 \quad \Rightarrow \quad y^3 + y^2 + y - 3 = 0.$$

We can easily see that $y = 1$ is a solution because $1^3 + 1^2 + 1 - 3 = 0$ and then, using any appropriate method, we can easily factorise this equation to get:

$$(y-1)(y^2+2y+3) = 0.$$

Now, using the discriminant (say) we can see that the quadratic factor gives us no further solutions as $2^2 - 4(1)(3) = -8 < 0$ and so, using the solution we did find, we have $y(1) = 1$.

To find $y'(1)$, we can define the function $g(x, y)$ where:

$$g(x, y) = y^3 + y^2 + y = 3$$

and then find $y'(x)$ by using the formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial g/\partial x}{\partial g/\partial y} = -\frac{y^3 + 2xy^2 + 3x^2y}{3xy^2 + 2x^2y + x^3}$$

as long as $3xy^2 + 2x^2y + x^3 \neq 0$. In this case, we have $y(1) = 1$ and so:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = -\frac{1+2+3}{3+2+1} = -1$$

i.e. $y'(1) = -1$.

(b) For $x > 0$, the function:

$$f(x) = a \ln x - b(\ln x)^2$$

where a and b are constants has stationary points when $f'(x) = 0$, i.e. when:

$$\frac{a}{x} - \frac{2b}{x} \ln x = 0 \quad \Rightarrow \quad 2b \ln x = a$$

as $x > 0$. Now, if $b \neq 0$, this then gives us:

$$\ln x = \frac{a}{2b} \quad \Rightarrow \quad x = \mathrm{e}^{a/2b}.$$

That is, for any a and for $b \neq 0$ this function has exactly one stationary point with coordinates:

$$x = \mathrm{e}^{a/2b} \quad \text{and} \quad y = a\left(\frac{a}{2b}\right) - b\left(\frac{a}{2b}\right)^2 = \frac{2a^2 - a^2}{4b} = \frac{a^2}{2b}.$$

To determine the nature of this stationary point, we note that:

$$f''(x) = -\frac{a}{x^2} - 2b\left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right) = -\frac{1}{x^2}(a + 2b - 2b \ln x)$$

so that, at the stationary point, we have:

$$f''(\mathrm{e}^{a/2b}) = -\mathrm{e}^{-a/b}\left(a + 2b - 2b\,\frac{a}{2b}\right) = -2b\mathrm{e}^{-a/b}.$$

So, with $b \neq 0$, we see that:

- when $b > 0$, we have $f''(\mathrm{e}^{a/2b}) < 0$ giving us a local maximum
- when $b < 0$, we have $f''(\mathrm{e}^{a/2b}) > 0$ giving us a local minimum.

Question 3

(a) Evaluate the definite integrals

$$\textbf{(i)} \quad \int_{-1}^{1} \sqrt{1-x}\,\mathrm{d}x \qquad \textbf{and} \qquad \textbf{(ii)} \quad \int_{-1}^{1} x\sqrt{1-x}\,\mathrm{d}x.$$

(b) State the compound angle formula for $\cos(A + B)$ and use it to show that

$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

Hence, by differentiating both sides of this result, deduce a formula for $\sin(2\theta)$.

(c) Evaluate the definite integrals

$$\textbf{(i)} \quad \int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x \qquad \textbf{and} \qquad \textbf{(ii)} \quad \int_{-1}^{1} x^2\sqrt{1-x^2}\,\mathrm{d}x.$$

Reading for this question

For relevant reading, see Chapter 4 of the subject guide.

Approaching the question

(a) i. We make the substitution $u = 1 - x$ so that $\mathrm{d}u = -\mathrm{d}x$ to get:

$$\int_{-1}^{1} \sqrt{1-x}\,\mathrm{d}x = \int_{2}^{0} \sqrt{u}(-\mathrm{d}u) = \int_{0}^{2} u^{1/2}\,\mathrm{d}u = \left[\frac{u^{3/2}}{3/2}\right]_0^2 = \frac{2}{3}\left[2^{3/2} - 0\right] = \frac{4}{3}\sqrt{2}.$$

ii. We can use the same substitution to get:

$$\begin{aligned}
\int_{-1}^{1} x\sqrt{1-x}\,\mathrm{d}x &= \int_{2}^{0} (1-u)u^{1/2}(-\mathrm{d}u) \\
&= \int_{0}^{2} (u^{1/2} - u^{3/2})\,\mathrm{d}u \\
&= \left[\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_0^2 \\
&= \frac{2}{3}\left[2^{3/2} - 0\right] + \frac{2}{5}\left[2^{5/2} - 0\right] \\
&= -\frac{4}{15}\sqrt{2}.
\end{aligned}$$

(b) As in Section 1.5.2 of the subject guide, the compound angle formula for $\cos(A+B)$ is:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

and, using this, we can see that:

$$\begin{aligned}
\cos(2\theta) &= \cos(\theta + \theta) && \text{(As } 2\theta = \theta + \theta\text{)} \\
&= \cos\theta\cos\theta - \sin\theta\sin\theta && \text{(Compound angle formula)} \\
&= \cos^2\theta - \sin^2\theta && \text{(Simplifying)} \\
&= \cos^2\theta - (1 - \cos^2\theta) && \text{(Pythagorean identity)} \\
&= 2\cos^2\theta - 1 && \text{(Simplifying)}
\end{aligned}$$

as required. Indeed, if we differentiate both sides of this identity, we get:

$$-2\sin(2\theta) = -4\cos\theta\sin\theta \quad \Rightarrow \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

which is the sought-after formula for $\sin(2\theta)$.

(c) i. We make the substitution $x = \sin\theta$ so that $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ to get:

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \cos\theta(\cos\theta\,\mathrm{d}\theta) = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta.$$

We can now use the first double angle formula from (b) to get:

$$\int_{-\pi/2}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos(2\theta))\,\mathrm{d}\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}.$$

Thus, the final answer is:

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

ii. We can use the same substitution to get:

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \sin^2\theta\cos\theta(\cos\theta\,\mathrm{d}\theta) = \int_{-\pi/2}^{\pi/2} (\sin\theta\cos\theta)^2\,\mathrm{d}\theta.$$

We can now use the second double angle formula from (b) to get:

$$\int_{-\pi/2}^{\pi/2} (\sin\theta\cos\theta)^2\,\mathrm{d}\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \sin^2(2\theta)\,\mathrm{d}\theta = \frac{1}{8}\int_{-\pi/2}^{\pi/2} (1-\cos(4\theta))\,\mathrm{d}\theta$$

if we use the $\cos(2\theta) = 1 - 2\sin^2\theta$ variant of the first double angle formula from (b). Then:

$$\int_{-\pi/2}^{\pi/2} (1-\cos(4\theta))\,\mathrm{d}\theta = \left[\theta - \frac{\sin(4\theta)}{4}\right]_{-\pi/2}^{\pi/2} = \pi$$

and so:

$$\int_{-1}^{1} x^2\sqrt{1-x}\,\mathrm{d}x = \frac{\pi}{8}$$

is the final answer.

Question 4

(a) Use a matrix method to solve the system of equations

$$\mathbf{2x - 3y + z = -1}$$

$$\mathbf{x - y - z = 0}$$

$$\mathbf{2x - 5y + 7z = -3.}$$

Express any solutions you find in vector form.

(b) With quantities k and l of capital and labour, respectively, a firm can produce a quantity $q(k, l)$ given by

$$q(k,l) = \left(\frac{\alpha}{k} + \frac{\beta}{l}\right)^{-1}$$

where α and β are positive constants. Capital and labour can be purchased at a cost of v and w dollars per unit, respectively.

Use the method of Lagrange multipliers to find the firm's maximum level of production if the budget for expenditure on capital and labour is M dollars. You should simplify your answer as far as possible.

[You are not required to justify the use of the method of Lagrange multipliers here.]

Reading for this question

For relevant reading, see Section 7.2 of the subject guide for part (a), and see Section 6.3 of the subject guide for part (b).

Approaching the question

(a) We use row operations to deal with the given system of equations. Indeed, from the equations we get the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & -1 & -1 & 0 \\ 2 & -5 & 7 & -3 \end{array}\right)$$

and, after swapping rows one and two (i.e. the row operation $R_1 \leftrightharpoons R_2$), we then get:

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & -3 & 1 & -1 \\ 2 & -5 & 7 & -3 \end{array}\right) \xrightarrow[R_3 \to 2R_1 - R_3]{R_2 \to 2R_1 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 3 & -9 & 3 \end{array}\right) \xrightarrow{R_3 \to R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This leaves us with two equations, namely:

$$x - y - z = 0 \quad \text{and} \quad y - 3z = 1$$

to solve by back-substitution. To do this, we set $z = t \in \mathbb{R}$ so that the second equation gives us:

$$y = 1 + 3t$$

and then the first equation gives us:

$$x = (1 + 3t) + t = 1 + 4t.$$

So, writing these in vector form we find that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+4t \\ 1+3t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

for $t \in \mathbb{R}$, i.e. these equations have an infinte number of solutions.

Note: The positions of the 'leading ones' (in the first and second rows of the echelon form) mean that, when we do the back-substitution, we can easily find y in terms of z and then x in terms of z. This means that, in this case, we should pick z to be the parameter t which can be any real number. This holds more generally: Any variable associated with a 'leading one' in the echelon form should be determined (via back-substitution) in terms of the variables that are not associated with a 'leading one' and it is these latter variables that should be assigned parameters which can be any real number.

(b) Given the information in the question, it should be clear that we want to:

$$\text{maximise} \quad \left(\frac{\alpha}{k} + \frac{\beta}{l}\right)^{-1} \quad \text{subject to the constraint } vk + wl = M$$

where $k, l > 0$, and we can do this by finding the stationary points of the Lagrangean:

$$\mathcal{L}(x, y, \lambda) = \left(\frac{\alpha}{k} + \frac{\beta}{l}\right)^{-1} - \lambda(vk + wl - M).$$

Thus, we need to solve the equations:

$$\mathcal{L}_k(x, y, \lambda) = 0 \quad \Rightarrow \quad \frac{\alpha}{k^2}\left(\frac{\alpha}{k} + \frac{\beta}{l}\right)^{-2} - \lambda v = 0$$

$$\mathcal{L}_l(x, y, \lambda) = 0 \quad \Rightarrow \quad \frac{\beta}{l^2}\left(\frac{\alpha}{k} + \frac{\beta}{l}\right)^{-2} - \lambda w = 0$$

$$\mathcal{L}_\lambda(x, y, \lambda) = 0 \quad \Rightarrow \quad -(vk + wl - M) = 0$$

and we do this by following the standard method. Firstly, we use the first two equations to eliminate λ by noting that:

$$\lambda\left(\frac{\alpha}{k}+\frac{\beta}{l}\right)^2=\frac{\alpha}{vk^2}\quad\text{and}\quad\lambda\left(\frac{\alpha}{k}+\frac{\beta}{l}\right)^2=\frac{\beta}{wl^2}$$

so that we get the tangency condition:

$$\frac{\alpha}{vk^2}=\frac{\beta}{wl^2}\quad\Rightarrow\quad k^2=\frac{\alpha w}{\beta v}\,l^2\quad\Rightarrow\quad k=\sqrt{\frac{\alpha w}{\beta v}}\,l$$

as $k, l > 0$. We can then substitute this into the third equation, the constraint, to see that:

$$v\sqrt{\frac{\alpha w}{\beta v}}\,l+wl=M\quad\Rightarrow\quad\left(\sqrt{\alpha vw}+\sqrt{\beta}w\right)l=\sqrt{\beta}M\quad\Rightarrow\quad l=\frac{\sqrt{\beta}M}{\sqrt{w}(\sqrt{\alpha v}+\sqrt{\beta w})}$$

is the value of l and, using the tangency condition again, we see that:

$$k=\sqrt{\frac{\alpha w}{\beta v}}\,\frac{\sqrt{\beta}M}{\sqrt{w}(\sqrt{\alpha v}+\sqrt{\beta w})}=\frac{\sqrt{\alpha}M}{\sqrt{v}(\sqrt{\alpha v}+\sqrt{\beta w})}$$

is the value of k. Consequently, using the original production function, we can see that the sought after maximum level of production is:

$$\left(\alpha\,\frac{\sqrt{v}(\sqrt{\alpha v}+\sqrt{\beta w})}{\sqrt{\alpha}M}+\beta\,\frac{\sqrt{w}(\sqrt{\alpha v}+\sqrt{\beta w})}{\sqrt{\beta}M}\right)^{-1}=\left(\frac{(\sqrt{\alpha v}+\sqrt{\beta w})^2}{M}\right)^{-1}=\frac{M}{(\sqrt{\alpha v}+\sqrt{\beta w})^2}$$

if we simplify it as far as possible.

Question 5

(a) **Given the differential equation**

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2xy}{x^2+1}=x^3$$

and the initial condition $y(0)=1$, find $y(x)$.

(b) **Consumers and suppliers anticipate market trends according to the demand equation**

$$q=-2p+6-\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{\mathrm{d}^2p}{\mathrm{d}t^2}$$

and the supply equation

$$q=3p-4+\frac{\mathrm{d}p}{\mathrm{d}t}+2\,\frac{\mathrm{d}^2p}{\mathrm{d}t^2}$$

respectively. Write down the differential equation that must hold at equilibrium.

Hence find $p(t)$ if $p(0)=1$ and $p'(0)=1$.

Describe the long-term behaviour of $p(t)$.

Reading for this question

For relevant reading, see Sections 8.2.2, 8.3 and 8.5.3 of the subject guide.

Approaching the question

(a) The given differential equation is linear and so, following the method given in Section 8.2.2 of the subject guide, we solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy}{x^2+1} = x^3 \quad \text{by comparing it to} \quad \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y(x) = Q(x)$$

to get $P(x) = 2x/(x^2+1)$ and $Q(x) = x^3$. This means that the integrating factor is given by:

$$\mu(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{\int \frac{2x\,\mathrm{d}x}{x^2+1}} = \mathrm{e}^{\ln(x^2+1)} = x^2+1$$

and so, using the formula

$$\mu(x)y(x) = \int \mu(x)Q(x)\,\mathrm{d}x$$

we get:

$$(x^2+1)y(x) = \int x^3(x^2+1)\,\mathrm{d}x = \int (x^5+x^3)\,\mathrm{d}x = \frac{x^6}{6} + \frac{x^4}{4} + c$$

where c is an arbitrary constant. Consequently, we find that:

$$y(x) = \frac{2x^6+3x^4+12c}{12(x^2+1)}$$

is the general solution. However, $y(0) = 1$ and so we have:

$$1 = \frac{12c}{12} \quad \Rightarrow \quad c = 1$$

which means that:

$$y(x) = \frac{2x^6+3x^4+12}{12(x^2+1)}$$

is the particular solution we seek.

(b) At equilibrium, for any price p, the quantity demanded is equal to the quantity supplied, i.e. we have:

$$-2p + 6 - \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}^2p}{\mathrm{d}t^2} = 3p - 4 + \frac{\mathrm{d}p}{\mathrm{d}t} + 2\,\frac{\mathrm{d}^2p}{\mathrm{d}t^2}$$

and this gives us the second-order differential equation:

$$\frac{\mathrm{d}^2p}{\mathrm{d}t^2} + 2\,\frac{\mathrm{d}p}{\mathrm{d}t} + 5p = 10.$$

So, following the standard method, we start by solving the auxiliary equation which, in this case, is:

$$m^2 + 2m + 5 = 0 \quad \Rightarrow \quad m = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2\sqrt{-1}.$$

In this case, as there are no real solutions, we take $\gamma = -1$ and $\delta = 2$ so that:

$$p(t) = \mathrm{e}^{-t}\left(A\cos(2t) + B\sin(2t)\right)$$

is the complementary function for some arbitrary constants A and B. To find a particular integral, as the right-hand side is a constant, we try something of the form $p(t) = \alpha$, where α is a constant to be determined. Thus, $p'(t) = 0$ and $p''(t) = 0$ and, substituting these in to the given differential equation, we get:

$$0 - 0 + 5\alpha = 10 \quad \Rightarrow \quad 5\alpha = 10 \quad \Rightarrow \quad \alpha = 2.$$

Consequently, the particular integral is $p(t) = 2$ and, adding this to the complementary function, we get:

$$p(t) = \mathrm{e}^{-t}\left(A\cos(2t) + B\sin(2t)\right) + 2$$

as the general solution of our differential equation. Indeed, since $p(0) = 1$, we have:

$$1 = A + 2 \quad \Rightarrow \quad A = -1$$

and, since $p'(0) = 1$ as well, we find that:

$$p'(t) = -\mathrm{e}^{-t}\left(A\cos(2t) + B\sin(2t)\right) + \mathrm{e}^{-t}\left(-2A\sin(2t) + 2B\cos(2t)\right)$$

so that we get:

$$1 = -A + 2B \quad \Rightarrow \quad 2B = 1 + A = 0 \quad \Rightarrow \quad B = 0.$$

Thus, putting these values of A and B into our general solution, we then see that:

$$p(t) = 2 - \mathrm{e}^{-t}\cos(2t)$$

is the sought-after particular solution. We can now see that, as t increases, this function will oscillate decreasingly to 2.

(Observe, in particular, that the *behaviour* of $p(t)$ as t increases is that it is *oscillating decreasingly* to 2, its limit as $t \to \infty$. That is, it is always *oscillating about* 2 but the *amplitude* of the oscillations is *decreasing to zero* as t increases.)