Examiners' commentaries 2019

MT1186 Mathematical methods

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2018–19. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the online subject guide. You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

General remarks

Learning outcomes

At the end of this half course and having completed the Essential reading and activities you should have:

- used the concepts, terminology, methods and conventions covered in the course to solve mathematical problems in this subject
- the ability to solve unseen mathematical problems involving the understanding of these concepts and application of these methods
- seen how calculus can be used to solve problems in economics and related subjects
- demonstrated knowledge and understanding of the underlying principles of calculus.

Showing your working

We start by emphasising that you should **always** include your working. This means two things. First, you should not simply write down the answer in the examination script, but you should explain the method by which it is obtained. Second, you should include rough working (even if it is messy!). The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined. We also stress that if you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if you have written down a wrong answer and nothing else, no marks can be awarded. So it is certainly in your interests to include all your workings.

Covering the syllabus and choosing questions

You should ensure that you have covered the syllabus in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: you should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, **any** topic could appear in the examination questions.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the candidate does indeed know the methods, understands them, and is able to use them, and not merely whether they vaguely remember them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different, from previous years' questions. You should **expect** to be surprised by some of the questions. Of course, you will only be examined on material in the syllabus, so all questions can be answered using the material of the course. There will be enough, routine, familiar content in the examination so that a candidate who has achieved competence in the course will pass, but, of course, for a high mark, more is expected: you will have to demonstrate an ability to solve new and unfamiliar problems.

Answer the question

Please do read the questions carefully. You might be asked to use specific methods, even when others could be used. The purpose of the examination is to test that you know certain methods, so the examiners might occasionally ask you to use a specific technique. In such circumstances, only limited partial credit can be given if you do not use the specified technique. It is also worth reading the question carefully so that you do not do more than is required (because it is unlikely that you would get extra marks for doing so). For instance, if a question asked you only to find the critical points of a function, but not their natures, then you should not determine their natures. Be careful to read all questions carefully because, although they may look like previous examination questions on first glance, there can be subtle differences.

Calculators

You are reminded that calculators are **not** permitted in the examination for this course, under any circumstances. The examiners know this, and so they set questions that do not require a calculator. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this course.

Examination revision strategy

Many candidates are disappointed to find that their examination performance is poorer than they expected. This may be due to a number of reasons, but one particular failing is '**question spotting**', that is, confining your examination preparation to a few questions and/or topics which have come up in past papers for the course. This can have serious consequences.

We recognise that candidates might not cover all topics in the syllabus in the same depth, but you need to be aware that examiners are free to set questions on **any aspect** of the syllabus. This means that you need to study enough of the syllabus to enable you to answer the required number of examination questions.

The syllabus can be found in the Course information sheet available on the VLE. You should read the syllabus carefully and ensure that you cover sufficient material in preparation for the examination. Examiners will vary the topics and questions from year to year and may well set questions that have not appeared in past papers. Examination papers may legitimately include questions on any topic in the syllabus. So, although past papers can be helpful during your revision, you cannot assume that topics or specific questions that have come up in past examinations will occur again.

If you rely on a question-spotting strategy, it is likely you will find yourself in difficulties when you sit the examination. We strongly advise you not to adopt this strategy.

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Comments on specific questions – Zone A

Candidates should answer all **FIVE** questions. All questions carry equal marks (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Question 1

(a) **The supply and demand functions for a market are:**

$$q^S(p) = \frac{p}{4} - 1 \quad \text{and} \quad q^D(p) = 11 - \frac{3}{4}p.$$

Find the equilibrium price and quantity.

The government imposes an excise (or per unit) tax of T on the market. Find the new equilibrium price and quantity. What percentage of the tax per unit has been passed onto the consumer?

Write down an expression for the government's tax revenue. What value of T will maximise this revenue?

Reading for this question

Block 1: Applications of functions in the subject guide.

Approaching the question

To find the equilibrium price, p^*, we solve the equation $q^S(p^*) = q^D(p^*)$ to see that:

$$\frac{p^*}{4} - 1 = 11 - \frac{3}{4}p^* \quad \Rightarrow \quad p^* = 12.$$

Using the demand function, say, we can then see that the equilibrium quantity is $q^* = 2$.

In the presence of an excise tax of T, the supply function becomes:

$$q_T^S(p) = q^S(p-T) = \frac{1}{4}(p-T) - 1$$

whereas the demand function is unchanged, i.e. $q_T^D(p) = q^D(p)$. (Notice that when a tax is introduced, only the 'supply-side' changes. This reflects the fact that, as far as the suppliers are concerned, the effective price is $p - T$ instead of the p seen by consumers.)

This means that the equilibrium price in the presence of the tax, p_T^*, is given by:

$$q_T^S(p_T^*) = q_T^D(p_T^*) \quad \Rightarrow \quad \frac{1}{4}(p_T^* - T) - 1 = 11 - \frac{3}{4}p_T^* \quad \Rightarrow \quad p_T^* = 12 + \frac{T}{4}$$

and, using $q_T^D(p)$ say, we see that the new equilibrium quantity is:

$$q_T^* = 11 - \frac{3}{4}\left(12 + \frac{T}{4}\right) = 2 - \frac{3}{16}T.$$

A common error at this point is to use the original supply equation, i.e. $q = q^S(p)$, instead of the modified one, i.e. $q = q_T^S(p)$, to find the new equilibrium quantity. To avoid this mistake, it is usually a good idea to use the demand equation, as we have just done, since this does not change!

With the introduction of the tax, the price paid by the consumers has increased from 12 to $12 + T/4$ which is an increase of $T/4$ per unit. However, the actual tax to be paid is T per unit and so only a quarter, i.e. 25%, of the tax per unit has been passed onto the consumer.

The tax revenue, R_T, is given by:

$$R_T = Tq_T^* = T\left(2 - \frac{3}{16}T\right) = 2T - \frac{3}{16}T^2$$

and this has a stationary point when $R_T' = 0$, i.e. when:

$$2 - \frac{3}{8}T = 0 \quad \Rightarrow \quad T = \frac{16}{3}$$

and, as $R_T'' = -3/8 < 0$ at $T = 16/3$, this is a local maximum of R_T. However, we need to guarantee that this is actually the global maximum of R_T and the easiest way to do this is to note that, as $R_T'' = -3/8 < 0$ for all T, the function R_T is concave and so the local maximum at $T = 16/3$ is a global maximum.

(b) **The elasticity of demand for a good is given by:**

$$\boldsymbol{\varepsilon(p) = \frac{3p}{p^2 + 3p + 2}.}$$

Find the demand function, $q^D(p)$, for this good if $q^D(1) = 27$.

Reading for this question

Block 8: First-order ODEs and Applications of ODEs in the subject guide.

Approaching the question

The elasticity of demand is given by:

$$\varepsilon = -\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p}$$

where $q = q^D(p)$ is the demand function. In this case we are given that:

$$\varepsilon = \frac{3p}{p^2 + 3p + 2}$$

and so we need to solve the differential equation:

$$-\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{3p}{p^2 + 3p + 2}$$

to find $q^D(p)$ given that we know that $q^D(1) = 27$. Indeed, this differential equation is separable and so using the standard method, we rewrite it as:

$$-\int \frac{\mathrm{d}q}{q} = \int \frac{3}{p^2 + 3p + 2}\,\mathrm{d}p$$

and determine the integrals. So, noting that partial fractions gives us:

$$\frac{3}{p^2 + 3p + 2} = \frac{3}{(p+1)(p+2)} = \frac{3}{p+1} - \frac{3}{p+2}$$

and that prices and quantities are always positive, we get:

$$-\ln q = 3\ln(p+1) - 3\ln(p+2) + c$$

where c is an arbitrary constant. So, to proceed, we could set $k = \mathrm{e}^c$ so that $c = \ln k$, and this then gives us:

$$\ln\left(\frac{1}{q}\right) = \ln(p+1)^3 - \ln(p+2)^3 + \ln k \quad \Rightarrow \quad \ln\left(\frac{1}{q}\right) = \ln\left[k\left(\frac{p+1}{p+2}\right)^3\right] \quad \Rightarrow \quad \frac{1}{q} = k\left(\frac{p+1}{p+2}\right)^3$$

which means that the general solution to our differential equation is:

$$q = \frac{1}{k}\left(\frac{p+2}{p+1}\right)^3.$$

Now, as we know that $q^D(1) = 27$, we get:

$$27 = \frac{1}{k}\left(\frac{1+2}{1+1}\right)^3 \quad \Rightarrow \quad 27 = \frac{27}{8k} \quad \Rightarrow \quad k = \frac{1}{8}$$

and so the particular solution to our differential equation is:

$$q = 8\left(\frac{p+2}{p+1}\right)^3 \quad \text{which means that} \quad q^D(p) = 8\left(\frac{p+2}{p+1}\right)^3$$

is the required demand function as we have $q = q^D(p)$.

Question 2

(a) Use a matrix method to solve the system of equations:

$$\begin{aligned} \boldsymbol{x + 2y - 3z} &= \mathbf{-4} \\ \boldsymbol{x - 6y + 7z} &= \mathbf{14} \\ \boldsymbol{x - 2y + 2z} &= \mathbf{5.} \end{aligned}$$

Express any solutions you find in vector form.

Reading for this question

Block 7: Systems of linear equations in the subject guide.

Approaching the question

We use row operations to deal with the given system of equations. Indeed, from the equations we get the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 1 & -6 & 7 & 14 \\ 1 & -2 & 2 & 5 \end{array}\right)$$

and, performing row operations we can then get:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 1 & -6 & 7 & 14 \\ 1 & -2 & 2 & 5 \end{array}\right) \xrightarrow[R_3 \to R_1 - R_3]{R_2 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 0 & 8 & -10 & -18 \\ 0 & 4 & -5 & -9 \end{array}\right) \xrightarrow{R_3 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 0 & 8 & -10 & -18 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This leaves us with two equations, namely:

$$x + 2y - 3z = -4 \quad \text{and} \quad 8y - 10z = -18$$

to solve by back-substitution. To do this, we let z be any real number, say t, so that the second equation gives us:

$$y = -\frac{9}{4} + \frac{5}{4}t$$

and then the first equation gives us:

$$x = -4 - 2\left(-\frac{9}{4} + \frac{5}{4}t\right) + 3t = \frac{1}{2} + \frac{t}{2}.$$

So, writing these in vector form we find that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2+2t \\ -9+5t \\ 4t \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ -9 \\ 0 \end{pmatrix} + \frac{t}{4}\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

for $t \in \mathbb{R}$, i.e. these equations have an infinite number of solutions.

Note: The positions of the 'leading ones' (in the first and second rows of the echelon form) mean that, when we do the back-substitution, we can easily find y in terms of z and then x in terms of z. This means that, in this case, we *should* pick z to be the parameter t which can be any real number. This holds more generally: Any variable associated with a 'leading one' in the echelon form *should* be determined (via back-substitution) in terms of the variables which are not associated with a 'leading one' and it is these latter variables which should be assigned parameters which can be any real number.

(b) State the compound angle formula for $\cos(A + B)$ and use it to show that:

$$\boldsymbol{\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta).}$$

Evaluate the definite integrals:

$$\textbf{(i)} \quad \int_0^{\pi/6} \cos^3(\theta)\,d\theta \quad \textbf{and} \quad \textbf{(ii)} \quad \int_0^{\pi/6} \cos(3\theta)\sec(\theta)\,d\theta.$$

Reading for this question

Block 1: Identities and Block 4 in the subject guide.

Approaching the question

The compound angle formula for $\cos(A + B)$ is:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

and, using this, we can see that:

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta+\theta) && \text{As } 3\theta = 2\theta+\theta \\
&= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta && \text{Compound angle formula} \\
&= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta && \text{Double angle formulae} \\
&= \cos^3\theta - 3\sin^2\theta\cos\theta && \text{Simplifying} \\
&= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta && \text{Pythagorean identity} \\
&= 4\cos^3\theta - 3\cos\theta && \text{Simplifying}
\end{aligned}$$

as required.

For (i), we can use this trigonometric identity to see that:

$$\begin{aligned}
\int_0^{\pi/6} \cos^3\theta \, \mathrm{d}\theta &= \frac{1}{4}\int_0^{\pi/6} \Big[\cos(3\theta) + 3\cos\theta\Big] \, \mathrm{d}\theta \\
&= \frac{1}{4}\left[\frac{\sin(3\theta)}{3} + 3\sin\theta\right]_0^{\pi/6} \\
&= \frac{1}{4}\left[\frac{1}{3}\sin\frac{\pi}{2} + 3\sin\frac{\pi}{6}\right] - \frac{1}{4}\left[\frac{1}{3}\sin 0 + 3\sin 0\right] \\
&= \frac{1}{4}\left[\frac{1}{3} + \frac{3}{2}\right] - \frac{1}{4}[0+0] \\
&= \frac{11}{24}
\end{aligned}$$

is the final answer.

For (ii), using this trigonometric identity again, we can see that:

$$\int_0^{\pi/6} \cos(3\theta)\sec\theta \, \mathrm{d}\theta = \int_0^{\pi/6} \frac{4\cos^3\theta - 3\cos\theta}{\cos\theta} \, \mathrm{d}\theta = \int_0^{\pi/6} \Big[4\cos^2\theta - 3\Big] \, \mathrm{d}\theta.$$

To deal with this, we can use the double angle formula $\cos(2\theta) = 2\cos^2\theta - 1$ to get:

$$\int_0^{\pi/6} \Big[4\cos^2\theta - 3\Big] \, \mathrm{d}\theta = \int_0^{\pi/6} \Big[2\cos(2\theta) - 1\Big] \, \mathrm{d}\theta = \Big[\sin(2\theta) - \theta\Big]_0^{\pi/6} = \left[\sin\frac{\pi}{3} - \frac{\pi}{6}\right] - \Big[\sin 0 - 0\Big]$$

and so we find that:

$$\int_0^{\pi/6} \cos(3\theta)\sec\theta \, \mathrm{d}\theta = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

is the final answer.

Question 3

(a) **Find the eigenvalues of the matrix:**

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-2} \end{pmatrix}$$

and find an eigenvector corresponding to each eigenvalue. Hence find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

Reading for this question

Block 7: Diagonalisation in the subject guide.

Approaching the question

To find the eigenvalues of this matrix, we solve the equation:

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)(-2-\lambda) + 2 = 0$$

which, multiplying out the brackets, gives us the quadratic equation:

$$(\lambda^2 + \lambda - 2) + 2 = 0 \quad \Rightarrow \quad \lambda(\lambda + 1) = 0$$

and so the eigenvalues are -1 and 0. To find the corresponding eigenvectors we seek a non-zero vector, $\mathbf{x}$, which is a solution to the equation $(A - \lambda I)\mathbf{x} = 0$, i.e. we have:

- for $\lambda = -1$, we solve:

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Rightarrow \quad 2x + y = 0 \quad \Rightarrow \quad y = -2x \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

  (or any non-zero multiple of this) is an eigenvector

- for $\lambda = 0$, we solve:

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Rightarrow \quad x + y = 0 \quad \Rightarrow \quad y = -x \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

  (or any non-zero multiple of this) is an eigenvector.

Consequently, if we take:

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

we have an invertible matrix, P, and a diagonal matrix, D, such that $P^{-1}AP = D$.

Of course, this is only one of the many possible pairs of matrices which we could choose for P and D: others are possible depending on which eigenvectors we choose when we form the columns of P and the order in which we choose to place them in P. For instance, choosing the other order for the eigenvectors we found above, we can see that:

$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

is another possible answer here.

(b) Use your result to find the functions $f(t)$ and $g(t)$ that satisfy the system of differential equations:

$$f'(t) = f(t) + g(t)$$
$$g'(t) = -2f(t) - 2g(t)$$

and the initial conditions $f(0) = 2$ and $g(0) = -3$.

Describe how these functions behave as t increases.

Reading for this question

Block 8: Systems of first-order ODEs in the subject guide.

Approaching the question

We can now *use this result* to solve the given coupled first-order differential equations. (It would, of course, be unwise to use any other method given the phrasing of the question and all the work that we have already done!) We let:

$$\mathbf{f}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \quad \text{so that the given differential equations can be written as} \quad \mathbf{f}'(t) = A\mathbf{f}(t)$$

and, using $P^{-1}AP = D$, we have $A = PDP^{-1}$. Putting this together, we then have:

$$\mathbf{f}'(t) = PDP^{-1}\mathbf{f}(t) \quad \Rightarrow \quad P^{-1}\mathbf{f}'(t) = DP^{-1}\mathbf{f}(t) \quad \Rightarrow \quad \mathbf{z}'(t) = D\mathbf{z}(t)$$

where $\mathbf{z}(t) = P^{-1}\mathbf{f}(t)$. Using this, we have (say):

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \Rightarrow \quad u'(t) = -u(t) \quad \text{and} \quad v'(t) = 0$$

and this pair of differential equations can easily be solved to yield:

$$u(t) = A\mathrm{e}^{-t} \quad \text{and} \quad v(t) = B$$

for arbitrary constants A and B. This means that, using $\mathbf{z}(t) = P^{-1}\mathbf{f}(t)$, we have $\mathbf{f}(t) = P\mathbf{z}(t)$ and so:

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} A\mathrm{e}^{-t} \\ B \end{pmatrix} = \begin{pmatrix} A\mathrm{e}^{-t} + B \\ -2A\mathrm{e}^{-t} - B \end{pmatrix}$$

is the general solution to our coupled system of differential equations. Indeed, using the initial conditions $f(0) = 2$ and $g(0) = -3$, we can solve the equations:

$$2 = A + B \quad \text{and} \quad -3 = -2A - B$$

by adding them to get $A = 1$ and, in turn, this gives us $B = 1$. Consequently, we find that:

$$f(t) = \mathrm{e}^{-t} + 1 \quad \text{and} \quad g(t) = -2\mathrm{e}^{-t} - 1$$

are the required solutions to our coupled system of differential equations. Indeed, as t increases, we can see that $f(t)$ decreases to 1 and $g(t)$ increases to -1.

Question 4

(a) Find and classify the stationary points of the function:

$$\boldsymbol{f(x,y) = x^3 + 3x^2y + 2y^2 - y + 5.}$$

Does this function have a global maximum or global minimum? Justify your answer.

Reading for this question

Block 6: Unconstrained optimisation in the subject guide.

Approaching the question

To find the stationary points of the function, we note that:

$$f_x(x,y) = 3x^2 + 6xy \quad \text{and} \quad f_y(x,y) = 3x^2 + 4y - 1$$

so that we can solve the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$. The second of these equations gives us:

$$3x^2 + 6xy = 0 \quad \Rightarrow \quad x(x+2y) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = -2y.$$

Now, substituting these into the first equation, we find that:

- with $x = 0$, the first equation gives us $y = 1/4$ and so $(0, 1/4)$ is a stationary point
- with $x = -2y$, the first equation gives us:

$$12y^2 + 4y - 1 = 0 \quad \Rightarrow \quad (6y-1)(2y+1) = 0 \quad \Rightarrow \quad y = \frac{1}{6} \text{ or } y = -\frac{1}{2}.$$

  Therefore, using $x = -2y$ again, we see that $(-1/3, 1/6)$ and $(1, -1/2)$ are also stationary points.

Hence the stationary points of the given function are $(0, 1/4)$, $(-1/3, 1/6)$ and $(1, -1/2)$.

To classify these stationary points, we note that:

$$f_{xx}(x,y) = 6x + 6y, \quad f_{xy}(x,y) = 6x = f_{yx}(x,y) \quad \text{and} \quad f_{yy}(x,y) = 4$$

so that the Hessian is:

$$H(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x+6y)(4) - (6x)^2 = 12(2x + 2y - 3x^2).$$

Therefore, evaluating this at each of the stationary points we find that:

- at $(0, 1/4)$, we have $H(0, 1/4) = 12(1/2) = 6 > 0$ and $f_{xx}(1/4, 0) = 6(1/4) > 0$
- at $(-1/3, 1/6)$, we have $H(-1/3, 1/6) = 12(-2/3) < 0$
- at $(1, -1/2)$, we have $H(1, -1/2) = 12(-2) < 0$.

Hence $(-1/3, 1/6)$ and $(1, -1/2)$ are saddle points and $(0, 1/4)$ is a local minimum.

To see whether this function has a global maximum or a global minimum, we can set $y = 0$ and see how it is changing as we move along the x-axis. That is, if we consider the function:

$$f(x, 0) = x^3 + 5$$

we can see that:

- as $x \to \infty$, $f(x, 0) \to \infty$ and so $f(x, y)$ has no global maximum
- as $x \to -\infty$, $f(x, 0) \to -\infty$ and so $f(x, y)$ has no global minimum.

(b) When a consumer has quantities, x_1 and x_2, of apples and oranges respectively, his utility is given by:

$$u(x_1, x_2) = x_1^{1/3} x_2^{2/3}.$$

Given that each apple costs p_1 dollars and each orange costs p_2 dollars, use the method of Lagrange multipliers to find the bundle of goods, (x_1, x_2), that will maximise his utility given that he spends M dollars on apples and oranges.

[You are not required to justify the use of the method of Lagrange multipliers here.]

Also find his maximum utility, $U(M)$, and verify that $U'(M)$ is equal to the value of the Lagrange multiplier.

Reading for this question

Block 6: Constrained optimisation in the subject guide.

Approaching the question

We are being asked to maximise $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$ subject to the constraint $p_1 x_1 + p_2 x_2 = M$ using the method of Lagrange multipliers. To do this, we construct the Lagrangean:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{1/3} x_2^{2/3} - \lambda(p_1 x_1 + p_2 x_2 - M)$$

whose first-order derivatives are given by:

$$\mathcal{L}_{x_1}(x_1, x_2, \lambda) = \frac{1}{3} x_1^{-2/3} x_2^{2/3} - \lambda p_1$$

$$\mathcal{L}_{x_2}(x_1, x_2, \lambda) = \frac{2}{3} x_1^{1/3} x_2^{-1/3} - \lambda p_2$$

$$\mathcal{L}_{\lambda}(x_1, x_2, \lambda) = -(p_1 x_1 + p_2 x_2 - M)$$

and solve the equations $\mathcal{L}_{x_1}(x_1, x_2, \lambda) = 0$, $\mathcal{L}_{x_2}(x_1, x_2, \lambda) = 0$ and $\mathcal{L}_{\lambda}(x_1, x_2, \lambda) = 0$, i.e. we have:

$$\frac{1}{3} x_1^{-2/3} x_2^{2/3} - \lambda p_1 = 0, \quad \frac{2}{3} x_1^{1/3} x_2^{-1/3} - \lambda p_2 \quad \text{and} \quad p_1 x_1 + p_2 x_2 - M = 0$$

to find the required point. We start by eliminating λ from the first two equations to see that:

$$\lambda = \frac{x_2^{2/3}}{3p_1 x_1^{2/3}} = \frac{2x_1^{1/3}}{3p_2 x_2^{1/3}} \quad \Rightarrow \quad x_2 = \frac{2p_1}{p_2} x_1$$

and, solving this simultaneously with the third equation, we get:

$$p_1 x_1 + p_2 x_2 = M \quad \Rightarrow \quad p_1 x_1 + p_2 \left(\frac{2p_1}{p_2} x_1 \right) = M \quad \Rightarrow \quad x_1 = \frac{M}{3p_1}$$

so that using:

$$x_2 = \frac{2p_1}{p_2} x_1 \quad \text{again, we get} \quad x_2 = \frac{2p_1}{p_2} \left(\frac{M}{3p_1} \right) = \frac{2M}{3p_2}.$$

Therefore, the consumer's optimal bundle contains $x_1 = M/3p_1$ apples and $x_2 = 2M/3p_2$ oranges.

The consumer's maximum utility is given by:

$$U(M) = u\left(\frac{M}{3p_1}, \frac{2M}{3p_2} \right) = \left(\frac{M}{3p_1} \right)^{1/3} \left(\frac{2M}{3p_2} \right)^{2/3} = \frac{2^{2/3} M}{3p_1^{1/3} p_2^{2/3}}$$

and, using one of our expressions for λ from above, the value of the Lagrange multiplier is:

$$\lambda = \frac{1}{3p_1} \left(\frac{2M/3p_2}{M/3p_1} \right)^{2/3} = \frac{1}{3p_1} \left(\frac{2p_1}{p_2} \right)^{2/3} = \frac{2^{2/3}}{3p_1^{1/3} p_2^{2/3}}.$$

Hence we can see that $U'(M) = \lambda$, as required.

Question 5

(a) **The price, $p(t)$, of a commodity varies continuously with time according to the differential equation:**

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 2p(t) + \mathrm{e}^{-t}$$

and the initial price is $p(0) = 2/3$. By solving this differential equation, find $p(t)$.

Hence find the approximate change in $p(t)$ when t changes from 0 to 1/2.

Reading for this question

Block 2: Using derivatives and Block 8: First-Order ODEs in the subject guide.

Approaching the question

The given differential equation is linear and so, following the standard method, we solve:

$$\frac{\mathrm{d}p}{\mathrm{d}t} - 2p(t) = \mathrm{e}^{-t} \quad \text{by comparing it to} \quad \frac{\mathrm{d}p}{\mathrm{d}t} + P(t)p(t) = Q(t)$$

to get $P(t) = -2$ and $Q(t) = \mathrm{e}^{-t}$. This means that the integrating factor is given by:

$$\mu(t) = \mathrm{e}^{\int P(t)\,\mathrm{d}t} = \mathrm{e}^{\int -2\,\mathrm{d}t} = \mathrm{e}^{-2t}$$

and so, using the formula:

$$\mu(t)p(t) = \int \mu(t)Q(t)\,\mathrm{d}t$$

we get:

$$\mathrm{e}^{-2t}p(t) = \int \mathrm{e}^{-t}\mathrm{e}^{-2t}\,\mathrm{d}t = \int \mathrm{e}^{-3t}\,\mathrm{d}t = \frac{\mathrm{e}^{-3t}}{-3} + c$$

where c is an arbitrary constant. Consequently, we find that:

$$p(t) = -\frac{e^{-t}}{3} + ce^{2t}$$

is the general solution. However, $p(0) = 2/3$ and so we have:

$$\frac{2}{3} = -\frac{e^0}{3} + ce^0 \quad \Rightarrow \quad c = 1$$

which means that:

$$p(t) = -\frac{e^{-3t}}{3} + e^{2t}$$

is the particular solution we seek.

To find the *approximate* change, Δp, in $p(t)$ when t changes from 0 to 1/2, we use:

$$\Delta p \simeq p'(0)\Delta t$$

where $\Delta t = 1/2 - 0 = 1/2$. So, differentiating our expression for $p(t)$ with respect to t, we get:

$$p'(t) = \frac{1}{3}e^{-t} + 2e^{2t} \quad \Rightarrow \quad p'(0) = \frac{1}{3} + 2 = \frac{7}{3}$$

and so:

$$\Delta p \simeq \left(\frac{7}{3}\right)\left(\frac{1}{2}\right) = \frac{7}{6}$$

is the required *approximate* change in $p(t)$.

A common error here was to say that Δp is given by:

$$p\left(\frac{1}{2}\right) - p(0)$$

but this is the *exact* change in $p(t)$ when t changes from 0 to 1/2 and this is not what the question was asking for.

Another common error was to say that Δp is given by the definite integral:

$$\int_0^{1/2} p(t)\,dt$$

but this is just wrong. Indeed, this error probably arose because people were thinking of the definite integral:

$$\int_0^{1/2} p'(t)\,dt$$

which is, incidentally, another way of finding the *exact* change in $p(t)$ since we have:

$$\int_0^{1/2} p'(t)\,dt = \Big[p(t)\Big]_0^{1/2} = p\left(\frac{1}{2}\right) - p(0).$$

(b) Find the sequence, x_t, that satisfies the difference equation:

$$x_t + 2x_{t-1} + x_{t-2} = 4 \quad \text{for } t \geq 2$$

and the initial conditions $x_0 = 2$ and $x_1 = 0$.

Describe how this sequence behaves as t increases.

Reading for this question

Block 9: Second-order difference equations in the subject guide.

Approaching the question

To solve the second-order difference equation:

$$x_{t+2} + 2x_{t+1} + x_t = 4$$

we start by solving the auxiliary equation which is:

$$m^2 + 2m + 1 = 0 \quad \Rightarrow \quad (m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1.$$

Therefore, as we have a repeated solution to the auxiliary equation, the complementary sequence is:

$$x_t = (A + Bt)(-1)^t$$

for some arbitrary constants A and B. To find a particular sequence, as the right-hand side is a constant, we try something of the form $x_t = \alpha$ where α is a constant to be determined. So, as $x_{t+1} = \alpha$ and $x_{t+2} = \alpha$, we can substitute these in to the given difference equation, to get:

$$\alpha + 2\alpha + \alpha = 4 \quad \Rightarrow \quad 4\alpha = 4 \quad \Rightarrow \quad \alpha = 1.$$

Therefore, the particular sequence is $x_t = 1$ and, adding this to the complementary sequence, we get:

$$x_t = (A + Bt)(-1)^t + 1$$

as the general solution of our difference equation. Indeed, since $x_0 = 2$, we have:

$$2 = A + 1 \quad \Rightarrow \quad A = 1$$

and, since $x_1 = 0$, we find that:

$$0 = (A + B)(-1) + 1 \quad \Rightarrow \quad A + B = 1 \quad \Rightarrow \quad B = 0.$$

Therefore, the sought-after particular solution is:

$$x_t = (-1)^t + 1.$$

Indeed, depending on whether t is even or odd, we see that x_t will be 2 or 0 and so, as t increases, x_t oscillates constantly between 2 and 0.

Examiners' commentaries 2019

MT1186 Mathematical methods

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2018–19. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the online subject guide. You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions – Zone B

Candidates should answer all **FIVE** questions. All questions carry equal marks (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Question 1

(a) **The supply and demand functions for a market are:**

$$q^S(p) = \frac{p}{2} - 1 \quad \text{and} \quad q^D(p) = 13 - \frac{p}{2}.$$

Find the equilibrium price and quantity.

The government imposes an excise (or per unit) tax of T on the market. Find the new equilibrium price and quantity. What percentage of the tax per unit has been passed onto the consumer?

Write down an expression for the government's tax revenue. What value of T will maximise this revenue?

Reading for this question

Block 1: Applications of functions in the subject guide.

Approaching the question

To find the equilibrium price, p^*, we solve the equation $q^S(p^*) = q^D(p^*)$ to see that:

$$\frac{p^*}{2} - 1 = 13 - \frac{p^*}{2} \quad \Rightarrow \quad p^* = 14.$$

Using the demand function, say, we can then see that the equilibrium quantity is $q^* = 6$.

In the presence of an excise tax of T, the supply function becomes:

$$q_T^S(p) = q^S(p - T) = \frac{1}{2}(p - T) - 1$$

whereas the demand function is unchanged, i.e. $q_T^D(p) = q^D(p)$. (Notice that when a tax is introduced, only the 'supply-side' changes. This reflects the fact that, as far as the suppliers are concerned, the effective price is $p - T$ instead of the p seen by consumers.)

This means that the equilibrium price in the presence of the tax, p_T^*, is given by:

$$q_T^S(p_T^*) = q_T^D(p_T^*) \quad \Rightarrow \quad \frac{1}{2}(p_T^* - T) - 1 = 13 - \frac{p_T^*}{2} \quad \Rightarrow \quad p_T^* = 14 + \frac{T}{2}$$

and, using $q_T^D(p)$ say, we see that the new equilibrium quantity is:

$$q_T^* = 13 - \frac{1}{2}\left(14 + \frac{T}{2}\right) = 6 - \frac{T}{4}.$$

A common error at this point is to use the original supply equation, i.e. $q = q^S(p)$, instead of the modified one, i.e. $q = q_T^S(p)$, to find the new equilibrium quantity. To avoid this mistake, it is usually a good idea to use the demand equation, as we have just done, since this does not change!

With the introduction of the tax, the price paid by the consumers has increased from 14 to $14 + T/2$ which is an increase of $T/2$ per unit. However, the actual tax to be paid is T per unit and so only a half, i.e. 50%, of the tax per unit has been passed onto the consumer.

The tax revenue, R_T, is given by:

$$R_T = Tq_T^* = T\left(6 - \frac{T}{4}\right) = 6T - \frac{T^2}{4}$$

and this has a stationary point when $R_T' = 0$, i.e. when:

$$6 - \frac{T}{2} = 0 \quad \Rightarrow \quad T = 12$$

and, as $R_T'' = -1/2 < 0$ at $T = 12$, this is a local maximum of R_T. However, we need to guarantee that this is actually the global maximum of R_T and the easiest way to do this is to note that, as $R_T'' = -1/2 < 0$ for all T, the function R_T is concave and so the local maximum at $T = 12$ is a global maximum.

(b) The elasticity of demand for a good is given by:

$$\boldsymbol{\varepsilon(p) = \frac{2p}{p^2 + 5p + 6}.}$$

Find the demand function, $\boldsymbol{q^D(p)}$, for this good if $\boldsymbol{q^D(1) = 16}$.

Reading for this question

Block 8: First-order ODEs and Applications of ODEs in the subject guide.

Approaching the question

The elasticity of demand is given by:

$$\varepsilon = -\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p}$$

where $q = q^D(p)$ is the demand function. In this case we are given that:

$$\varepsilon = \frac{2p}{p^2 + 5p + 6}$$

and so we need to solve the differential equation:

$$-\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{2p}{p^2 + 5p + 6}$$

to find $q^D(p)$ given that we know that $q^D(1) = 16$. Indeed, this differential equation is separable and so using the standard method, we rewrite it as:

$$-\int \frac{\mathrm{d}q}{q} = \int \frac{2}{p^2 + 5p + 6}\,\mathrm{d}p$$

and determine the integrals. So, noting that partial fractions gives us:

$$\frac{2}{p^2 + 5p + 6} = \frac{2}{(p+2)(p+3)} = \frac{2}{p+2} - \frac{2}{p+3}$$

and that prices and quantities are always positive, we get:

$$-\ln q = 2\ln(p+2) - 2\ln(p+3) + c$$

where c is an arbitrary constant. So, to proceed, we could set $k = \mathrm{e}^c$ so that $c = \ln k$, and this then gives us:

$$\ln\left(\frac{1}{q}\right) = \ln(p+2)^2 - \ln(p+3)^2 + \ln k \quad\Rightarrow\quad \ln\left(\frac{1}{q}\right) = \ln\left[k\left(\frac{p+2}{p+3}\right)^2\right] \quad\Rightarrow\quad \frac{1}{q} = k\left(\frac{p+2}{p+3}\right)^2$$

which means that the general solution to our differential equation is:

$$q = \frac{1}{k}\left(\frac{p+3}{p+2}\right)^2.$$

Now, as we know that $q^D(1) = 16$, we get:

$$16 = \frac{1}{k}\left(\frac{1+3}{1+2}\right)^2 \quad\Rightarrow\quad 16 = \frac{16}{9k} \quad\Rightarrow\quad k = \frac{1}{9}$$

and so the particular solution to our differential equation is:

$$q = 9\left(\frac{p+3}{p+2}\right)^2 \quad \text{which means that} \quad q^D(p) = 9\left(\frac{p+3}{p+2}\right)^2$$

is the required demand function as we have $q = q^D(p)$.

Question 2

(a) Use a matrix method to solve the system of equations:

$$\begin{aligned} \boldsymbol{x + 3y - 2z} &= \boldsymbol{-2} \\ \boldsymbol{x + 5y + 6z} &= \boldsymbol{4} \\ \boldsymbol{x + 4y + 2z} &= \boldsymbol{1.} \end{aligned}$$

Express any solutions you find in vector form.

Reading for this question

Block 7: Systems of linear equations in the subject guide.

Approaching the question

We use row operations to deal with the given system of equations. Indeed, from the equations we get the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -2 \\ 1 & 5 & 6 & 4 \\ 1 & 4 & 2 & 1 \end{array}\right)$$

and, performing row operations we can then get:

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -2 \\ 1 & 5 & 6 & 4 \\ 1 & 4 & 2 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -2 \\ 0 & 2 & 8 & 6 \\ 0 & 1 & 4 & 3 \end{array}\right) \xrightarrow[R_3 \to R_2 - 2R_3]{R_2 - R_2/2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This leaves us with two equations, namely:

$$x + 3y - 2z = -2 \quad \text{and} \quad y + 4z = 3$$

to solve by back-substitution. To do this, we let z be any real number, say t, so that the second equation gives us:

$$y = 3 - 4t$$

and then the first equation gives us:

$$x = -2 - 3(3 - 4t) + 2t = -11 + 14t.$$

So, writing these in vector form we find that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 + 14t \\ 3 - 4t \\ t \end{pmatrix} = \begin{pmatrix} -11 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 14 \\ -4 \\ 1 \end{pmatrix}$$

for $t \in \mathbb{R}$, i.e. these equations have an infinite number of solutions.

Note: The positions of the 'leading ones' (in the first and second rows of the echelon form) mean that, when we do the back-substitution, we can easily find y in terms of z and then x in terms of z. This means that, in this case, we *should* pick z to be the parameter t which can be any real number. This holds more generally: Any variable associated with a 'leading one' in the echelon form *should* be determined (via back-substitution) in terms of the variables which are not associated with a 'leading one' and it is these latter variables which should be assigned parameters which can be any real number.

(b) State the compound angle formula for $\sin(A + B)$ and use it to show that:

$$\boldsymbol{\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta).}$$

Evaluate the definite integrals:

$$\textbf{(i)} \quad \int_0^{\pi/6} \sin^3(\theta)\,d\theta \quad \textbf{and} \quad \textbf{(ii)} \quad \int_0^{\pi/6} \sin(3\theta)\operatorname{cosec}(\theta)\,d\theta.$$

Reading for this question

Block 1: Identities and Block 4 in the subject guide.

Approaching the question

The compound angle formula for $\sin(A + B)$ is:

$$\sin(A + B) = \sin A \cos B - \cos A \sin B$$

and, using this, we can see that:

$$\begin{aligned}
\sin(3\theta) &= \sin(2\theta+\theta) && \text{As } 3\theta = 2\theta+\theta \\
&= \sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta && \text{Compound angle formula} \\
&= (2\sin\theta\cos\theta)\cos\theta - (\cos^2\theta - \sin^2\theta)\sin\theta && \text{Double angle formulae} \\
&= 3\sin\theta\cos^2\theta - \sin^3\theta && \text{Simplifying} \\
&= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta && \text{Pythagorean identity} \\
&= 3\sin\theta - 4\sin^3\theta && \text{Simplifying}
\end{aligned}$$

as required.

For (i), we can use this trigonometric identity to see that:

$$\begin{aligned}
\int_0^{\pi/6} \sin^3\theta \, \mathrm{d}\theta &= \frac{1}{4}\int_0^{\pi/6} \Big[3\sin\theta - \sin(3\theta)\Big] \, \mathrm{d}\theta \\
&= \frac{1}{4}\left[-3\cos\theta + \frac{\cos(3\theta)}{3}\right]_0^{\pi/6} \\
&= \frac{1}{4}\left[-3\cos\frac{\pi}{6} + \frac{1}{3}\cos\frac{\pi}{2}\right] - \frac{1}{4}\left[-3\cos 0 + \frac{1}{3}\cos 0\right] \\
&= \frac{1}{4}\left[-3\frac{\sqrt{3}}{2} + 0\right] - \frac{1}{4}\left[-3 + \frac{1}{3}\right] \\
&= \frac{2}{3} - \frac{3\sqrt{3}}{8}
\end{aligned}$$

is the final answer.

For (ii), using this trigonometric identity again, we can see that:

$$\int_0^{\pi/6} \sin(3\theta)\mathrm{cosec}\theta \, \mathrm{d}\theta = \int_0^{\pi/6} \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} \, \mathrm{d}\theta = \int_0^{\pi/6} \Big[3 - 4\sin^2\theta\Big] \, \mathrm{d}\theta.$$

To deal with this, we can use the double angle formula $\cos(2\theta) = 1 - 2\sin^2\theta$ to get:

$$\int_0^{\pi/6} \Big[3-4\sin^2\theta\Big] \, \mathrm{d}\theta = \int_0^{\pi/6} \Big[1+2\cos(2\theta)\Big] \, \mathrm{d}\theta = \Big[\theta + \sin(2\theta)\Big]_0^{\pi/6} = \left[\frac{\pi}{6} + \sin\frac{\pi}{3}\right] - \Big[0 + \sin 0\Big]$$

and so we find that:

$$\int_0^{\pi/6} \sin(3\theta)\mathrm{cosec}\theta \, \mathrm{d}\theta = \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

is the final answer.

Question 3

(a) **Find the eigenvalues of the matrix:**

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{-3} & \mathbf{-4} \\ \mathbf{2} & \mathbf{3} \end{pmatrix}$$

and find an eigenvector corresponding to each eigenvalue. Hence find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

Reading for this question

Block 7: Diagonalisation in the subject guide.

Approaching the question

To find the eigenvalues of this matrix, we solve the equation:

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} -3-\lambda & -4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (-3-\lambda)(3-\lambda) + 8 = 0$$

which, multiplying out the brackets, gives us the quadratic equation:

$$(\lambda^2 - 9) + 8 = 0 \quad \Rightarrow \quad \lambda^2 = 1$$

and so the eigenvalues are -1 and 1. To find the corresponding eigenvectors we seek a non-zero vector, $\mathbf{x}$, which is a solution to the equation $(A - \lambda I)\mathbf{x} = 0$, i.e. we have:

- for $\lambda = -1$, we solve:

$$\begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Rightarrow \quad x + 2y = 0 \quad \Rightarrow \quad x = -2y \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

  (or any non-zero multiple of this) is an eigenvector

- for $\lambda = 1$, we solve:

$$\begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Rightarrow \quad x + y = 0 \quad \Rightarrow \quad x = -y \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

  (or any non-zero multiple of this) is an eigenvector.

Consequently, if we take:

$$P = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

we have an invertible matrix, P, and a diagonal matrix, D, such that $P^{-1}AP = D$.

Of course, this is only one of the many possible pairs of matrices which we could choose for P and D: others are possible depending on which eigenvectors we choose when we form the columns of P and the order in which we choose to place them in P. For instance, choosing the other order for the eigenvectors we found above, we can see that:

$$P = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is another possible answer here.

(b) Use your result to find the sequences x_t and y_t that satisfy the system of difference equations:

$$\begin{aligned} x_t &= -3x_{t-1} - 4y_{t-1} \\ y_t &= 2x_{t-1} + 3y_{t-1} \end{aligned}$$

for $t \geq 1$ and the initial conditions $x_0 = 2$ and $y_0 = 1$.

Describe how these sequences behave as t increases.

Reading for this question

Block 9: Systems of first-order difference equations in the subject guide.

Approaching the question

We can now *use this result* to solve the given coupled first-order difference equations. (It would, of course, be unwise to use any other method given the phrasing of the question and all the work that we have already done!) We let:

$$\mathbf{y}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad \text{so that the given difference equations can be written as} \quad \mathbf{y}_t = A\mathbf{y}_{t-1}$$

and, using $P^{-1}AP = D$, we have $A = PDP^{-1}$. Putting this together, we then have:

$$\mathbf{y}_t = PDP^{-1}\mathbf{y}_{t-1} \quad \Rightarrow \quad P^{-1}\mathbf{y}_t = DP^{-1}\mathbf{y}_{t-1} \quad \Rightarrow \quad \mathbf{z}_t = D\mathbf{z}_{t-1}$$

where $\mathbf{z}_t = P^{-1}\mathbf{y}_t$. Using this, we have (say):

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{t-1} \\ v_{t-1} \end{pmatrix} \quad \Rightarrow \quad u_t = -u_{t-1} \quad \text{and} \quad v_t = v_{t-1}$$

and this pair of difference equations can easily be solved to yield:

$$u_t = A(-1)^t \quad \text{and} \quad v_t = B(1)^t = B$$

for arbitrary constants A and B. This means that, using $\mathbf{z}_t = P^{-1}\mathbf{y}_t$, we have $\mathbf{y}_t = P\mathbf{z}_t$ and so:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A(-1)^t \\ B \end{pmatrix} = \begin{pmatrix} -2A(-1)^t - B \\ A(-1)^t + B \end{pmatrix}$$

is the general solution to our coupled system of difference equations. Indeed, using the initial conditions $x_0 = 2$ and $y_0 = 1$, we can solve the equations:

$$2 = -2A - B \quad \text{and} \quad 1 = A + B$$

by adding them to get $A = -3$ and, in turn, this gives us $B = 4$. Consequently, we find that:

$$x_t = 6(-1)^t - 4 \quad \text{and} \quad y_t = -3(-1)^t + 4$$

are the required solutions to our coupled system of difference equations. Indeed, depending on whether t is even or odd, we see that x_t will be 2 or -10 and y_t will be 1 or 7. Therefore, as t increases, x_t oscillates constantly between 2 and -10 and y_t oscillates constantly between 1 and 7.

Question 4

(a) **Find and classify the stationary points of the function:**

$$\boldsymbol{f(x, y) = 2x^2 - x + y^3 + 3xy^2 - 7.}$$

Does this function have a global maximum or global minimum? Justify your answer.

Reading for this question

Block 6: Unconstrained optimisation in the subject guide.

Approaching the question

To find the stationary points of the function, we note that:

$$f_x(x, y) = 4x - 1 + 3y^2 \quad \text{and} \quad f_y(x, y) = 3y^2 + 6xy$$

so that we can solve the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$. The second of these equations gives us:

$$3y^2 + 6xy = 0 \quad \Rightarrow \quad y(y + 2x) = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = -2x.$$

Now, substituting these into the first equation, we find that:

- with $y = 0$, the first equation gives us $x = 1/4$ and so $(1/4, 0)$ is a stationary point
- with $y = -2x$, the first equation gives us:

$$12x^2 + 4x - 1 = 0 \quad \Rightarrow \quad (6x - 1)(2x + 1) = 0 \quad \Rightarrow \quad x = \frac{1}{6} \text{ or } x = -\frac{1}{2}.$$

  Therefore, using $y = -2x$ again, we see that $(1/6, -1/3)$ and $(-1/2, 1)$ are also stationary points.

Hence the stationary points of the given function are $(1/4, 0)$, $(1/6, -1/3)$ and $(-1/2, 1)$.

To classify these stationary points, we note that:

$$f_{xx}(x,y) = 4, \quad f_{xy}(x,y) = 6y = f_{yx}(x,y) \quad \text{and} \quad f_{yy}(x,y) = 6y + 6x$$

so that the Hessian is:

$$H(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (4)(6y+6x) - (6y)^2 = 12(2y + 2x - 3y^2).$$

Therefore, evaluating this at each of the stationary points we find that:

- at $(1/4, 0)$, we have $H(1/4, 0) = 12(1/2) = 6 > 0$ and $f_{xx}(1/4, 0) = 4 > 0$
- at $(1/6, -1/3)$, we have $H(1/6, -1/3) = 12(-2/3) < 0$
- at $(-1/2, 1)$, we have $H(-1/2, 1) = 12(-2) < 0$.

Hence $(1/6, -1/3)$ and $(-1/2, 1)$ are saddle points and $(1/4, 0)$ is a local minimum.

To see whether this function has a global maximum or a global minimum, we can set $x = 0$ and see how it is changing as we move along the y-axis. That is, if we consider the function:

$$f(0, y) = y^3 - 7$$

we can see that:

- as $y \to \infty$, $f(0, y) \to \infty$ and so $f(x, y)$ has no global maximum
- as $y \to -\infty$, $f(0, y) \to -\infty$ and so $f(x, y)$ has no global minimum.

(b) When a consumer has quantities, x_1 and x_2, of apples and oranges respectively, his utility is given by:

$$u(x_1, x_2) = x_1^{2/3} x_2^{1/3}.$$

Given that each apple costs p_1 dollars and each orange costs p_2 dollars, use the method of Lagrange multipliers to find the bundle of goods, (x_1, x_2), that will maximise his utility given that he spends M dollars on apples and oranges.

[You are not required to justify the use of the method of Lagrange multipliers here.]

Also find his maximum utility, $U(M)$, and verify that $U'(M)$ is equal to the value of the Lagrange multiplier.

Reading for this question

Block 6: Constrained optimisation in the subject guide.

Approaching the question

We are being asked to maximise $u(x_1, x_2) = x_1^{2/3} x_2^{1/3}$ subject to the constraint $p_1 x_1 + p_2 x_2 = M$ using the method of Lagrange multipliers. To do this, we construct the Lagrangean:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{2/3} x_2^{1/3} - \lambda(p_1 x_1 + p_2 x_2 - M)$$

whose first-order derivatives are given by:

$$\mathcal{L}_{x_1}(x_1, x_2, \lambda) = \frac{2}{3} x_1^{-1/3} x_2^{1/3} - \lambda p_1$$

$$\mathcal{L}_{x_2}(x_1, x_2, \lambda) = \frac{1}{3} x_1^{2/3} x_2^{-2/3} - \lambda p_2$$

$$\mathcal{L}_{\lambda}(x_1, x_2, \lambda) = -(p_1 x_1 + p_2 x_2 - M)$$

and solve the equations $\mathcal{L}_{x_1}(x_1, x_2, \lambda) = 0$, $\mathcal{L}_{x_2}(x_1, x_2, \lambda) = 0$ and $\mathcal{L}_{\lambda}(x_1, x_2, \lambda) = 0$, i.e. we have:

$$\frac{2}{3} x_1^{-1/3} x_2^{1/3} - \lambda p_1 = 0, \quad \frac{1}{3} x_1^{2/3} x_2^{-2/3} - \lambda p_2 \quad \text{and} \quad p_1 x_1 + p_2 x_2 - M = 0$$

to find the required point. We start by eliminating λ from the first two equations to see that:

$$\lambda = \frac{2x_2^{1/3}}{3p_1x_1^{1/3}} = \frac{x_1^{2/3}}{3p_2x_2^{2/3}} \quad \Rightarrow \quad x_2 = \frac{p_1}{2p_2}x_1$$

and, solving this simultaneously with the third equation, we get:

$$p_1x_1 + p_2x_2 = M \quad \Rightarrow \quad p_1x_1 + p_2\left(\frac{p_1}{2p_2}x_1\right) = M \quad \Rightarrow \quad x_1 = \frac{2M}{3p_1}$$

so that using:

$$x_2 = \frac{p_1}{2p_2}x_1 \quad \text{again, we get} \quad x_2 = \frac{p_1}{2p_2}\left(\frac{2M}{3p_1}\right) = \frac{M}{3p_2}.$$

Therefore, the consumer's optimal bundle contains $x_1 = 2M/3p_1$ apples and $x_2 = M/3p_2$ oranges.

The consumer's maximum utility is given by:

$$U(M) = u\left(\frac{2M}{3p_1}, \frac{M}{3p_2}\right) = \left(\frac{2M}{3p_1}\right)^{2/3}\left(\frac{M}{3p_2}\right)^{1/3} = \frac{2^{2/3}M}{3p_1^{2/3}p_2^{1/3}}$$

and, using one of our expressions for λ from above, the value of the Lagrange multiplier is:

$$\lambda = \frac{2}{3p_1}\left(\frac{M/3p_2}{2M/3p_1}\right)^{1/3} = \frac{2}{3p_1}\left(\frac{p_1}{2p_2}\right)^{1/3} = \frac{2^{2/3}}{3p_1^{2/3}p_2^{1/3}}.$$

Hence we can see that $U'(M) = \lambda$, as required.

Question 5

(a) **The value, $v(t)$, of an asset varies continuously with time according to the differential equation:**

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(t) + \mathrm{e}^{-2t}$$

and the initial value is $v(0) = 2/3$. By solving this differential equation, find $v(t)$.

Hence find the approximate change in $v(t)$ when t changes from 0 to 1/2.

Reading for this question

Block 2: Using derivatives and Block 8: First-Order ODEs in the subject guide.

Approaching the question

The given differential equation is linear and so, following the standard method, we solve:

$$\frac{\mathrm{d}v}{\mathrm{d}t} - v(t) = \mathrm{e}^{-2t} \quad \text{by comparing it to} \quad \frac{\mathrm{d}v}{\mathrm{d}t} + p(t)v(t) = q(t)$$

to get $p(t) = -1$ and $q(t) = \mathrm{e}^{-2t}$. This means that the integrating factor is given by:

$$\mu(t) = \mathrm{e}^{\int p(t)\,\mathrm{d}t} = \mathrm{e}^{\int -1\,\mathrm{d}t} = \mathrm{e}^{-t}$$

and so, using the formula:

$$\mu(t)v(t) = \int \mu(t)q(t)\,\mathrm{d}t$$

we get:

$$\mathrm{e}^{-t}v(t) = \int \mathrm{e}^{-t}\mathrm{e}^{-2t}\,\mathrm{d}t = \int \mathrm{e}^{-3t}\,\mathrm{d}t = \frac{\mathrm{e}^{-3t}}{-3} + c$$

where c is an arbitrary constant. Consequently, we find that:

$$v(t) = -\frac{e^{-2t}}{3} + ce^t$$

is the general solution. However, $v(0) = 2/3$, and so we have:

$$\frac{2}{3} = -\frac{e^0}{3} + ce^0 \quad \Rightarrow \quad c = 1$$

which means that:

$$v(t) = -\frac{e^{-2t}}{3} + e^t$$

is the particular solution we seek.

To find the *approximate* change, Δv, in $v(t)$ when t changes from 0 to 1/2 we use:

$$\Delta v \simeq v'(0)\Delta t$$

where $\Delta t = 1/2 - 0 = 1/2$. So, differentiating our expression for $v(t)$ with respect to t, we get:

$$v'(t) = \frac{2}{3}e^{-t} + e^t \quad \Rightarrow \quad v'(0) = \frac{2}{3} + 1 = \frac{5}{3}$$

and so:

$$\Delta v \simeq \left(\frac{5}{3}\right)\left(\frac{1}{2}\right) = \frac{5}{6}$$

is the required *approximate* change in $v(t)$.

A common error here was to say that Δv is given by:

$$v\left(\frac{1}{2}\right) - v(0)$$

but this is the *exact* change in $v(t)$ when t changes from 0 to 1/2 and this is not what the question was asking for.

Another common error was to say that Δv is given by the definite integral:

$$\int_0^{1/2} v(t)\,dt$$

but this is just wrong. Indeed, this error probably arose because people were thinking of the definite integral:

$$\int_0^{1/2} v'(t)\,dt$$

which is, incidentally, another way of finding the *exact* change in $v(t)$ since we have:

$$\int_0^{1/2} v'(t)\,dt = \Big[v(t)\Big]_0^{1/2} = v\left(\frac{1}{2}\right) - v(0).$$

(b) Find the function, $f(t)$, that satisfies the differential equation:

$$f''(t) + 6f'(t) + 9f(t) = 18$$

and the initial conditions $f(0) = 1$ and $f'(0) = 3$.

Describe how this function behaves as t increases.

Reading for this question

Block 8: Second-order ODEs in the subject guide.

Approaching the question

To solve the second-order differential equation:

$$f''(t) + 6f'(t) + 9f(t) = 18$$

we start by solving the auxiliary equation which is:

$$m^2 + 6m + 9 = 0 \quad \Rightarrow \quad (m+3)^2 = 0 \quad \Rightarrow \quad m = -3, -3.$$

Therefore, as we have a repeated solution to the auxiliary equation, the complementary function is:

$$f(t) = (A + Bt)\mathrm{e}^{-3t}$$

for some arbitrary constants A and B. To find a particular integral, as the right-hand side is a constant, we try something of the form $f(t) = \alpha$ where α is a constant to be determined. So, as $f'(t) = 0$ and $f''(t) = 0$, we can substitute these in to the given differential equation, to get:

$$0 + 6(0) + 9\alpha = 18 \quad \Rightarrow \quad \alpha = 2.$$

Therefore, the particular integral is $f(t) = 2$ and, adding this to the complementary function, we get:

$$f(t) = (A + Bt)\mathrm{e}^{-3t} + 2$$

as the general solution of our differential equation. Indeed, since $f(0) = 1$, we have:

$$1 = A + 2 \quad \Rightarrow \quad A = -1$$

and, since $f'(0) = 3$, we find that:

$$f'(t) = B\mathrm{e}^{-3t} - 3(A + Bt)\mathrm{e}^{-3t} \quad \Rightarrow \quad 3 = B - 3A \quad \Rightarrow \quad B = 3 + 3(-1) = 0.$$

Therefore, the sought-after particular solution is:

$$f(t) = -\mathrm{e}^{-3t} + 2.$$

Indeed, we see that as t increases, this function increases to 2.

Examiners' commentaries 2020

MT1186 Mathematical methods

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2019–20. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the online subject guide. You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

General remarks

Learning outcomes

At the end of this half course and having completed the Essential reading and activities you should have:

- used the concepts, terminology, methods and conventions covered in the course to solve mathematical problems in this subject
- the ability to solve unseen mathematical problems involving the understanding of these concepts and application of these methods
- seen how calculus can be used to solve problems in economics and related subjects
- demonstrated knowledge and understanding of the underlying principles of calculus.

Showing your working

We start by emphasising that you should **always** include your working. This means two things. First, you should not simply write down the answer in the examination script, but you should explain the method by which it is obtained. Second, you should include rough working (even if it is messy!). The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined. We also stress that if you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if you have written down a wrong answer and nothing else, no marks can be awarded. So it is certainly in your interests to include all your workings.

Covering the syllabus and choosing questions

You should ensure that you have covered the syllabus in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: you should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, **any** topic could appear in the examination questions.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the candidate does indeed know the methods, understands them, and is able to use them, and not merely whether they vaguely remember them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different, from previous years' questions. You should **expect** to be surprised by some of the questions. Of course, you will only be examined on material in the syllabus, so all questions can be answered using the material of the course. There will be enough, routine, familiar content in the examination so that a candidate who has achieved competence in the course will pass, but, of course, for a high mark, more is expected: you will have to demonstrate an ability to solve new and unfamiliar problems.

Answer the question

Please do read the questions carefully. You might be asked to use specific methods, even when others could be used. The purpose of the examination is to test that you know certain methods, so the examiners might occasionally ask you to use a specific technique. In such circumstances, only limited partial credit can be given if you do not use the specified technique. It is also worth reading the question carefully so that you do not do more than is required (because it is unlikely that you would get extra marks for doing so). For instance, if a question asked you only to find the critical points of a function, but not their natures, then you should not determine their natures. Be careful to read all questions carefully because, although they may look like previous examination questions on first glance, there can be subtle differences.

Calculators

You are reminded that calculators are **not** permitted in the examination for this course, under any circumstances. The examiners know this, and so they set questions that do not require a calculator. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this course.

Examination revision strategy

Many candidates are disappointed to find that their examination performance is poorer than they expected. This may be due to a number of reasons, but one particular failing is '**question spotting**', that is, confining your examination preparation to a few questions and/or topics which have come up in past papers for the course. This can have serious consequences.

We recognise that candidates might not cover all topics in the syllabus in the same depth, but you need to be aware that examiners are free to set questions on **any aspect** of the syllabus. This means that you need to study enough of the syllabus to enable you to answer the required number of examination questions.

The syllabus can be found in the Course information sheet available on the VLE. You should read the syllabus carefully and ensure that you cover sufficient material in preparation for the examination. Examiners will vary the topics and questions from year to year and may well set questions that have not appeared in past papers. Examination papers may legitimately include questions on any topic in the syllabus. So, although past papers can be helpful during your revision, you cannot assume that topics or specific questions that have come up in past examinations will occur again.

If you rely on a question-spotting strategy, it is likely you will find yourself in difficulties when you sit the examination. We strongly advise you not to adopt this strategy.

Examiners' commentaries 2020

MT1186 Mathematical methods

Important note

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Comments on specific questions

Candidates should answer all of the following **FIVE** questions. All questions carry equal marks.

Question 1

(a) **The demand equation for a market is given by:**

$$q(4 + p^2) = 10.$$

Find the elasticity of demand and show that it never exceeds two.

(b) **Using the demand equation in part (a), find the revenue function, $R(p)$, for this market.**

For what values of p is $R(p)$ a decreasing function?

If p changes from 1 to 1.1, what is the approximate change in revenue?

(c) **Suppose that the inverse supply function for this market has the form $p^S(q) = \alpha q + \beta$ where α and β are positive constants.**

Given that the equilibrium point for this market is $(q, p) = (2, 4)$ and the producer surplus is two, find the values of α and β.

Reading for this question

For (a), Block 2: Applications of derivatives. For (b), Block 2: Tangent lines and linear approximations and Block 3: Increasing and decreasing functions. For (c), Block 4: Consumer and producer surpluses.

Approaching the question

(a) Writing the demand equation as:

$$q(p) = \frac{10}{4+p^2} \quad \Rightarrow \quad \frac{\mathrm{d}q}{\mathrm{d}p} = -\frac{20p}{(4+p^2)^2}$$

and so the elasticity of demand is given by:

$$\varepsilon = -\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p} = -p\left(\frac{4+p^2}{10}\right)\left(-\frac{20p}{(4+p^2)^2}\right) = \frac{2p^2}{4+p^2}.$$

Then, as we can see that:

$$\frac{p^2}{4+p^2} < 1 \quad \text{for all } p \in \mathbb{R}$$

we can conclude that $\varepsilon < 2$, i.e. this elasticity of demand never exceeds two.

(b) Using the demand equation in part (a), the revenue function is given by:

$$R(p) = pq(p) = \frac{10p}{4+p^2}.$$

Using the quotient rule, we can see that:

$$R'(p) = \frac{10(4+p^2) - 10p(2p)}{(4+p^2)^2} = 10\frac{4-p^2}{(4+p^2)^2}$$

and so, as the revenue is decreasing when $R'(p) < 0$, we see that this occurs when $p^2 > 4$. That is, as $p \geq 0$ because it represents a price, we can conclude that the revenue is decreasing when $p > 2$.

If p changes from 1 to 1.1, an increase of a tenth, the approximate change in revenue is given by:

$$\Delta R = R'(1)\Delta p = 10\left(\frac{3}{25}\right)\left(\frac{1}{10}\right) = \frac{3}{25}$$

which is 0.12.

(Note that this question is asking for the *approximate* change in the revenue and not for $R(1.1) - R(1)$ which is the *exact* change.)

(c) The inverse supply function for the market is $p^S(q) = \alpha q + \beta$ for some constants $\alpha, \beta > 0$ and the equilibrium point is $(q, p) = (2, 4)$. In particular, this means that α and β must satisfy the equation:

$$4 = 2\alpha + \beta$$

as the equilibrium point must be on the supply curve. We are also told that the producer surplus is two and so, as the producer surplus is given by:

$$PS = p^*q^* - \int_0^{q^*} p^S(q)\,\mathrm{d}q$$

we have:

$$\int_0^2 (\alpha q + \beta)\,\mathrm{d}q = \left[\alpha\frac{q^2}{2} + \beta q\right]_0^2 = 2\alpha + 2\beta - (0+0) = 2\alpha + 2\beta$$

so that:

$$2 = (4)(2) - (2\alpha + 2\beta) \quad \Rightarrow \quad \alpha + \beta = 3$$

is another equation that α and β must satisfy. Consequently, solving these two equations simultaneously, we find that $\alpha = 1$ and $\beta = 2$.

(An alternative method here would be to observe that the supply function is a straight line and so the producer surplus is the area of a triangular region whose height is $4-\beta$ and whose base is 2. This means that, if we find the area of this triangle, we have:

$$2 = \frac{1}{2}(4-\beta)(2) \quad \Rightarrow \quad 4-\beta = 2 \quad \Rightarrow \quad \beta = 2.$$

Hence, using the equation $4 = 2\alpha + \beta$ that we found above, we use $\beta = 2$ to see that $\alpha = 1$. Of course, if you wished to tackle the question this way, the examiners would expect your answer to be justified by means of a sketch of the supply function and an identification of this clearly indicated triangular region with the producer surplus!)

Question 2

(a) Use row operations to determine the values of k for which the system of equations:

$$\boldsymbol{x + y + z = 2}$$

$$\boldsymbol{x + 2y + 2z = 3}$$

$$\boldsymbol{2x + 3y + kz = 5}$$

has a solution.

Hence, for each of these values of k, find the solution(s) to this system of equations.

(b) Evaluate the definite integral:

$$\int_0^1 \frac{3x^2 + 2x + 3}{(1+x)(1+x^2)}\, dx$$

simplifying your answer as far as possible.

Reading for this question

For (a), Block 7: Systems of linear equations. For (b), Block 4: Using partial fractions to simplify integrands.

Approaching the question

(a) We use row operations to deal with the given system of equations. Indeed, from the equations we get the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & k & 5 \end{array}\right)$$

and, performing row operations we can then get:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & k & 5 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & k-2 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & k-3 & 0 \end{array}\right).$$

From this row-echelon form, we can then see that the system of equations has solutions for all values of k. As such, solving these equations, we get the following.

- If $k \neq 3$, we have three equations, namely:

$$x + y + z = 2, \quad y + z = 1 \quad \text{and} \quad z = 0$$

  which are easily solved by back-substitution to give us the unique solution $x = 1$, $y = 1$ and $z = 0$.

- If $k = 3$, we have two equations, namely:

$$x + y + z = 2 \quad \text{and} \quad y + z = 1$$

  to solve by back-substitution. To do this, we let z be any real number, say t, so that the second equation gives us $y = 1 - t$ and then the first equation gives us:

$$x = 2 - (1 - t) - t = 1.$$

  Therefore, we have an infinite number of solutions given by $x = 1$, $y = 1 - t$ and $z = t$ for $t \in \mathbb{R}$.

Note: The positions of the 'leading ones' (in the first and second rows of the echelon form) mean that, when we do the back-substitution in the $k = 3$ case, we can easily find y in terms of z and then x in terms of z. This means that, in this case, we **should** pick z to be the parameter t which can be any real number. This holds more generally: Any variable associated with a 'leading one' in the echelon form **should** be determined (via back-substitution) in terms of the variables that are not associated with a 'leading one' and it is these latter variables that should be assigned parameters which can be any real number.

(b) Here the integrand is a rational function of two polynomials and the degree of the numerator is less than the degree of the denominator. As such, we can use the method of partial fractions and, looking at the denominator, we have to find the numbers A, B and C that make:

$$\frac{3x^2 + 2x + 3}{(1 + x)(1 + x^2)} = \frac{A}{1 + x} + \frac{Bx + C}{1 + x^2} \quad \Rightarrow \quad 3x^2 + 2x + 3 = A(1 + x^2) + (Bx + C)(1 + x).$$

So, looking at this and picking some useful values of x, we see that using $x = -1$ we get $A = 2$, using $x = 0$ we get $A + C = 3$ so that $C = 1$, and using $x = 1$, say, we get $4 = A + B + C$ so that $B = 1$. This means that we have:

$$\frac{3x^2 + 2x + 3}{(1 + x)(1 + x^2)} = \frac{2}{1 + x} + \frac{x + 1}{1 + x^2}$$

when it is expressed in partial fractions. With this, the given integral becomes:

$$\begin{aligned}
\int_0^1 \frac{3x^2 + 2x + 3}{(1 + x)(1 + x^2)}\,\mathrm{d}x &= \int_0^1 \left(\frac{2}{1 + x} + \frac{1}{2}\frac{2x}{1 + x^2} + \frac{1}{1 + x^2} \right) \mathrm{d}x \\
&= \left[2\ln|1 + x| + \frac{1}{2}\ln|1 + x^2| + \tan^{-1}(x) \right]_0^1 \\
&= \frac{5}{2}\ln(2) + \frac{\pi}{4}.
\end{aligned}$$

Question 3

(a) **Minouche invests \$1000 at an interest rate of 5% per annum at the start of 2019. She wants to withdraw a fixed amount \$$I$ at the end of each year for the next 12 years.**

i. Model this situation using a difference equation and find the value of her investment after n years have elapsed.

ii. Given that $(1.05)^{12}$ is approximately 1.8, what is the corresponding maximum value of I?

(b) Find and classify the stationary points of the function:

$$f(x, y) = 2x^3 - 3x^2 - 12x + y2^+6y - 7.$$

Reading for this question

For (a), Block 9: Applications of difference equations. For (b), Block 6: Unconstrained optimisation.

Approaching the question

(a) i. To model the situation given in the question it makes sense to let:

$$y_n = \text{the value (in dollars) of the investment after } n \text{ years have elapsed}$$

so that $y_0 = 1000$ and:

$$y_n = 1.05y_{n-1} - I$$

for $n \geq 1$. Comparing this difference equation with $y_n = ay_{n-1} + b$ we can see that $a = 1.05 \neq 1$ and so a time-independent solution, y^*, exists and is given by:

$$y^* = 1.05y^* - I \quad \Rightarrow \quad 0.05y^* = I \quad \Rightarrow \quad y^* = 20I.$$

The general solution is then given by:

$$y_n = y^* + (y_0 - y^*)a^n$$

and, as $y_0 = 1000$ here, we have:

$$y_n = 20I + (1000 - 20I)(1.05)^n$$

as the value of the investment (in dollars) after n years have elapsed.

ii. Given that Minouche wants to withdraw a fixed amount $\$I$ for the next 12 years, the value of I she chooses must satisfy $y_{12} \geq 0$. That is, we must have:

$$20I+(1000-20I)(1.05)^{12} \geq 0 \quad \Rightarrow \quad 50((1.05)^{12}) \geq I((1.05)^{12}-1) \quad \Rightarrow \quad I \leq 50\times\frac{(1.05)^{12}}{(1.05)^{12}-1}.$$

Therefore, the maximum value of I that is consistent with this is given by:

$$I_{\max} = 50 \times \frac{(1.05)^{12}}{(1.05)^{12} - 1} \simeq 50 \times \frac{1.8}{1.8 - 1} = 50 \times \frac{18/10}{8/10} = \frac{225}{2}$$

if we use the fact, given in the question, that $(1.05)^{12}$ is approximately 1.8. That is, the required maximum value of I is approximately 112.50 (in dollars).

(b) To find the stationary points of the function, we note that:

$$f_x(x, y) = 6x^2 - 6x - 12 \quad \text{and} \quad f_y(x, y) = 2y + 6$$

so that we can solve the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$. The first of these equations gives us:

$$x^2 - x - 2 = 0 \quad \Rightarrow \quad (x - 2)(x + 1) = 0 \quad \Rightarrow \quad x = -1, 2$$

and the second equation gives us $y = -3$. Hence, to satisfy both equations, we must have $(-1, -3)$ and $(2, -3)$ as the stationary points.

To classify these stationary points, we note that:

$$f_{xx}(x, y) = 12x - 6, \quad f_{xy}(x, y) = 0 = f_{yx}(x, y) \quad \text{and} \quad f_{yy}(x, y) = 2$$

so that the Hessian is:

$$H(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = (12x - 6)(2) - 0^2 = 12(2x - 1).$$

Therefore, evaluating this at each of the stationary points, we find that:

- at $(-1, -3)$, we have $H(-1, -3) = 12(-3) = -36 < 0$
- at $(2, -3)$, we have $H(2, -3) = 12(3) = 36 > 0$ and $f_{xx}(2, -3) = 18 > 0$.

Therefore, $(-1, -3)$ is a saddle point and $(2, -3)$ is a local minimum.

Question 4

(a) Evaluate the definite integral:

$$\int_0^{\pi/8} x \sin(2x)\, \mathrm{d}x.$$

(b) Two consumers, Anne and Brian, have \$10 each to spend on cats and dogs. A cat costs \$2 and a dog costs \$1.

i. If x is the number of cats and y is the number of dogs, sketch the budget set for these consumers.

ii. Anne has a utility function given by:

$$u_A(x, y) = x^2 + y^2.$$

Sketch a few of Anne's indifference curves and indicate the bundle (x, y) of cats and dogs which will maximise her utility subject to her budget constraint. What is this bundle?

iii. Brian has a utility function given by:

$$u_B(x, y) = x^2y^3.$$

Sketch a few of Brian's indifference curves and indicate the bundle (x, y) of cats and dogs which will maximise his utility subject to his budget constraint.

Why can we use the method of Lagrange multipliers to find this bundle?

Use the method of Lagrange multipliers to find this bundle.

Reading for this question

For (a), Block 4: Integration by parts. For (b), Block 6: Constrained optimisation.

Approaching the question

(a) We can use integration by parts to see that:

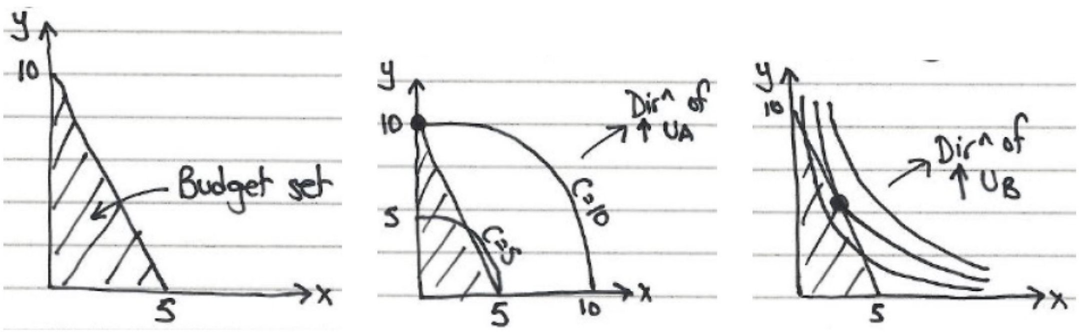
$$\begin{aligned}\int_0^{\pi/8} x\sin(2x)\,\mathrm{d}x &= \left[x\left(-\frac{\cos(2x)}{2}\right)\right]_0^{\pi/8} - \int_0^{\pi/8}\left(-\frac{\cos(2x)}{2}\right)\mathrm{d}x \\ &= -\frac{\pi}{16}\cos\left(\frac{\pi}{4}\right) + \left[\frac{\sin(2x)}{4}\right]_0^{\pi/8} \\ &= -\frac{\pi}{16}\cos\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin\left(\frac{\pi}{4}\right) \\ &= \frac{4-\pi}{16}\end{aligned}$$

is the final answer.

(b) i. Given that each consumer has \$10 to spend on cats and dogs at a cost of \$2 and \$1 respectively, their budget constraint is:

$$2x + y \leq 10$$

where, of course, $x, y \geq 0$ as they are quantities. This means that, sketching the budget set, we get something like the shaded triangle in Figure 1(a).



(a) (b) (c)

Figure 1: The sketches for Question 4(b).

ii. Anne's utility function is $u_A(x,y) = x^2 + y^2$ and so a typical indifference curve, say $u_A(x,y) = c$, will be a circle of radius $\sqrt{c}$ centred on the origin. Sketching two such indifference curves and noting the direction in which her utility is increasing, as in Figure 1(b), it should be clear that the bundle $(0, 10)$ – indicated by a • in the sketch – will maximise Anne's utility subject to her budget constraint.

iii. Brian's utility function is $u_B(x,y) = x^2y^3$ and so a typical indifference curve, say $u_B(x,y) = c$, will look a bit like a rectangular hyperbola. Sketching three such indifference curves and noting the direction in which his utility is increasing, as in Figure 1(c), it should be clear that the bundle indicated by a • in the sketch will maximise his utility subject to his budget constraint.

As this optimal occurs where an indifference curve is both **on** and **tangential** to the budget line $2x + y = 10$, we can use the method of Lagrange multipliers to find it.

To find this bundle using the method of Lagrange multipliers, we construct the Lagrangean:

$$\mathcal{L}(x,y,\lambda) = x^2y^3 - \lambda(2x + y - 10)$$

whose first-order derivatives are given by:

$$\mathcal{L}_x(x,y,\lambda) = 2xy^3 - 2\lambda, \quad \mathcal{L}_y(x,y,\lambda) = 3x^2y^2 - \lambda \quad \text{and} \quad \mathcal{L}_\lambda(x,y,\lambda) = -(2x + y - 10)$$

and solve the equations $\mathcal{L}_x(x, y, \lambda) = 0$, $\mathcal{L}_y(x, y, \lambda) = 0$ and $\mathcal{L}_\lambda(x, y, \lambda) = 0$, i.e. we have:

$$2xy^3 - 2\lambda = 0, \quad 3x^2y^2 - \lambda = 0 \quad \text{and} \quad 2x + y - 10 = 0$$

to find the required point. We start by eliminating λ from the first two equations to see that:

$$\lambda = xy^3 = 3x^2y^2 \quad \Rightarrow \quad y = 3x$$

and, solving this simultaneously with the third equation, we get:

$$2x + y = 10 \quad \Rightarrow \quad 2x + 3x = 10 \quad \Rightarrow \quad x = 2$$

so that using:

$$y = 3x \quad \text{again, we get} \quad y = 3(2) = 6.$$

Therefore, Brian's optimal bundle is $(2, 6)$.

Question 5

(a) A curve is given by the equation:

$$\boldsymbol{y^3 + 3xy + 2x^3 = 9.}$$

Show that the point $(2, -1)$ is on this curve and find the Cartesian equation of the tangent line to this curve at this point.

(b) Suppose consumers anticipate market trends according to the demand function:

$$\boldsymbol{q^D(p(t)) = 8 - 2p(t) - 2p'(t) + p''(t)}$$

and the supply function is $q^S(p) = p - 1$.

Use the condition for equilibrium to find $p(t)$ given that $p(0) = 1$ and $p'(0) = 2$.

Describe the behaviour of $p(t)$ as t increases.

Reading for this question

For (a), Block 2: Tangent lines and linear approximations and Block 5: The chain rule. For (b), Block 8: Market trends.

Approaching the question

(a) Given the point $(2, -1)$, we can see that it does indeed lie on the curve given by the equation:

$$y^3 + 3xy + 2x^3 = 9$$

as it satisfies this equation since:

$$(-1)^3 + 3(2)(-1) + 2(2^3) = -1 - 6 + 16 = 9.$$

To find the gradient of this curve, we can write it as $g(x, y) = c$ with:

$$g(x, y) = y^3 + 3xy + 2x^3 = 9 \quad \text{and} \quad c = 9$$

so that we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial g/\partial x}{\partial g/\partial y} = -\frac{3y + 6x^2}{3y^2 + 3x} = -\frac{y + 2x^2}{y^2 + x}$$

and so, at the point $(2, -1)$, the gradient of the curve is $-7/3$. Therefore, we see that:

$$-\frac{7}{3} = \frac{y - (-1)}{x - 2} \quad \Rightarrow \quad -7(x - 2) = 3(y + 1) \quad \Rightarrow \quad y = -\frac{7}{3}x + \frac{11}{3}$$

is the Cartesian equation of the tangent line to the curve at the given point.

(b) At equilibrium, we have a price, $p(t)$, where the amount supplied is equal to the amount demanded, i.e. we have:

$$8 - 2p(t) - 2p'(t) + p''(t) = p(t) - 1$$

and so, we get the non-homogeneous second-order differential equation:

$$p''(t) - 2p'(t) - 3p(t) = -9.$$

To solve this, we start by solving the auxiliary equation, which is:

$$m^2 - 2m - 3 = 0 \quad \Rightarrow \quad (m-3)(m+1) = 0 \quad \Rightarrow \quad m = 3, -1.$$

Therefore, as we have two distinct solutions to the auxiliary equation, the complementary function is:

$$pf(t) = A\mathrm{e}^{3t} + B\mathrm{e}^{-t}$$

for some arbitrary constants A and B. To find a particular integral, as the right-hand side is a constant, we try something of the form $p(t) = \alpha$ where α is a constant to be determined. So, as $p'(t) = 0$ and $p''(t) = 0$, we can substitute these into the differential equation, to get:

$$0 - 2(0) - 3\alpha = -9 \quad \Rightarrow \quad \alpha = 3.$$

Hence the particular integral is $p(t) = 3$ and, adding this to the complementary function, we get:

$$p(t) = A\mathrm{e}^{3t} + B\mathrm{e}^{-t} + 3$$

as the general solution of our differential equation. Indeed, since $p(0) = 1$, we have:

$$1 = A + B + 3 \quad \Rightarrow \quad A + B = -2$$

and, since $p'(0) = 2$, we find that:

$$p'(t) = 3A\mathrm{e}^{3t} - B\mathrm{e}^{-t} \quad \Rightarrow \quad 2 = 3A - B.$$

Then, solving these two equations simultaneously, we find that $A = 0$ and $B = -2$ so that:

$$p(t) = 3 - 2\mathrm{e}^{-t}.$$

Indeed, we can see that as t increases, this function increases to 3.

(Observe, in particular, that the *behaviour* of $p(t)$ is that it is *increasing* to 3 as t increases.)

Examiners' commentaries 2021

MT1186 Mathematical methods

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2020–21. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the online subject guide. You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

General remarks

Learning outcomes

At the end of this half course and having completed the Essential reading and activities you should have:

- used the concepts, terminology, methods and conventions covered in the course to solve mathematical problems in this subject
- the ability to solve unseen mathematical problems involving the understanding of these concepts and application of these methods
- seen how calculus can be used to solve problems in economics and related subjects
- demonstrated knowledge and understanding of the underlying principles of calculus.

Showing your working

We start by emphasising that you should **always** include your working. This means two things. First, you should not simply write down the answer in the examination script, but you should explain the method by which it is obtained. Second, you should include rough working (even if it is messy!). The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined. We also stress that if you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if you have written down a wrong answer and nothing else, no marks can be awarded. So it is certainly in your interests to include all your workings.

Covering the syllabus and choosing questions

You should ensure that you have covered the syllabus in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: you should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, **any** topic could appear in the examination questions.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the candidate does indeed know the methods, understands them, and is able to use them, and not merely whether they vaguely remember them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different, from previous years' questions. You should **expect** to be surprised by some of the questions. Of course, you will only be examined on material in the syllabus, so all questions can be answered using the material of the course. There will be enough, routine, familiar content in the examination so that a candidate who has achieved competence in the course will pass, but, of course, for a high mark, more is expected: you will have to demonstrate an ability to solve new and unfamiliar problems.

Answer the question

Please do read the questions carefully. You might be asked to use specific methods, even when others could be used. The purpose of the examination is to test that you know certain methods, so the examiners might occasionally ask you to use a specific technique. In such circumstances, only limited partial credit can be given if you do not use the specified technique. It is also worth reading the question carefully so that you do not do more than is required (because it is unlikely that you would get extra marks for doing so). For instance, if a question asked you only to find the critical points of a function, but not their natures, then you should not determine their natures. Be careful to read all questions carefully because, although they may look like previous examination questions on first glance, there can be subtle differences.

Calculators

You are reminded that calculators are **not** permitted in the examination for this course, under any circumstances. The examiners know this, and so they set questions that do not require a calculator. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this course.

Examination revision strategy

Many candidates are disappointed to find that their examination performance is poorer than they expected. This may be due to a number of reasons, but one particular failing is '**question spotting**', that is, confining your examination preparation to a few questions and/or topics which have come up in past papers for the course. This can have serious consequences.

We recognise that candidates might not cover all topics in the syllabus in the same depth, but you need to be aware that examiners are free to set questions on **any aspect** of the syllabus. This means that you need to study enough of the syllabus to enable you to answer the required number of examination questions.

The syllabus can be found in the Course information sheet available on the VLE. You should read the syllabus carefully and ensure that you cover sufficient material in preparation for the examination. Examiners will vary the topics and questions from year to year and may well set questions that have not appeared in past papers. Examination papers may legitimately include questions on any topic in the syllabus. So, although past papers can be helpful during your revision, you cannot assume that topics or specific questions that have come up in past examinations will occur again.

If you rely on a question-spotting strategy, it is likely you will find yourself in difficulties when you sit the examination. We strongly advise you not to adopt this strategy.

Examiners' commentaries 2021

MT1186 Mathematical methods

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2020–21. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

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Comments on specific questions – Zone A

Candidates should answer all **FIVE** questions. All questions carry equal marks.

Question 1

A market has supply and demand functions given by

$$q^S(p) = p - 1 \quad \text{and} \quad q^D(p) = 3 - p.$$

respectively.

(a) Find the equilibrium price and quantity for this market.

The government wants to impose an excise tax of T per unit on the market. That is, if the price including the tax is p_T, then the equilibrium price is now determined by the equation

$$q^S(p_T - T) = q^D(p_T).$$

(b) Find the equilibrium price and quantity for this market in the presence of this tax.

How much of the tax would be passed onto the consumer?

For what values of T will this equilibrium be economically meaningful?

(c) Write down an expression for the government's tax revenue.

Find the value of T that will maximise this revenue.

A firm supplies an equilibrium quantity of goods to this market at the equilibrium price and has a cost function given by $C(q) = 4q^2$.

(d) **Find the firm's gross (i.e. pre-tax) profit as a function of T.**
Sketch this profit function.
What value of T would the firm like the government to choose?

Reading for this question

Section 1.6 and Chapter 3 of the subject guide.

Approaching the question

(a) The equilibrium price occurs when $q^S(p) = q^D(p)$ and so

$$p - 1 = 3 - p \quad \Longrightarrow \quad p = 2$$

and this gives us an equilibrium quantity of $q = q^D(2) = 3 - 2 = 1$.

(b) In the presence of the excise tax, the equilibrium price occurs when $q^S(p - T) = q^D(p)$ and so

$$p - T - 1 = 3 - p \quad \Longrightarrow \quad p = 2 + \frac{T}{2}$$

and this gives us an equilibrium quantity of $q = q^D(2 + T/2) = 1 - T/2$.

From the equilibrium price, we can see that half the tax has been passed onto the consumer and, from the equilibrium quantity (which must be non-negative) we can see that the economically meaningful values of T are $0 \leq T \leq 2$.

(c) The government's tax revenue is

$$R_T = qT = \left(1 - \frac{T}{2}\right) T = T - \frac{T^2}{2}.$$

As $R'_T = 1 - T = 0$ when $T = 1$ and $R''_T = -1 < 0$ for all T, we can see that the tax revenue is maximised when $T = 1$.

(d) Working at the equilibrium with $C(q) = 4q^2$, the firm's gross profit is given by

$$\pi(T) = \left(2 + \frac{T}{2}\right)\left(1 - \frac{T}{2}\right) - 4\left(1 - \frac{T}{2}\right)^2 = -\frac{5}{4}T^2 + \frac{7}{2}T - 2$$

and this is only economically meaningful if $0 \leq T \leq 2$.

To sketch the profit function, we note that

- the π-intercept is $\pi = -2$
- the T-intercepts occur when

$$5T^2 - 14T + 8 = 0 \quad \Longrightarrow \quad (5T - 4)(T - 2) = 0 \quad \Longrightarrow \quad T = \frac{4}{5},\ 2.$$

- as $\pi'(T) = -5T/2 + 7/2 = 0$ when $T = 7/5$ and $\pi''(T) = -5/2 < 0$ for all T, we can see that the curve has a maximum at $T = 7/5$ and $\pi(7/5) = 9/20$.

With this information, we get the sketch in Figure 1. Based on this, the firm would like the government to choose $T = 7/5$ as that would maximise their profit.

Question 2

A town can accommodate a population of K people and, $N(t)$, the population of the town at time $t \geq 0$ satisfies the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{K - N}{K} N,$$

with an initial population of $K/10$.

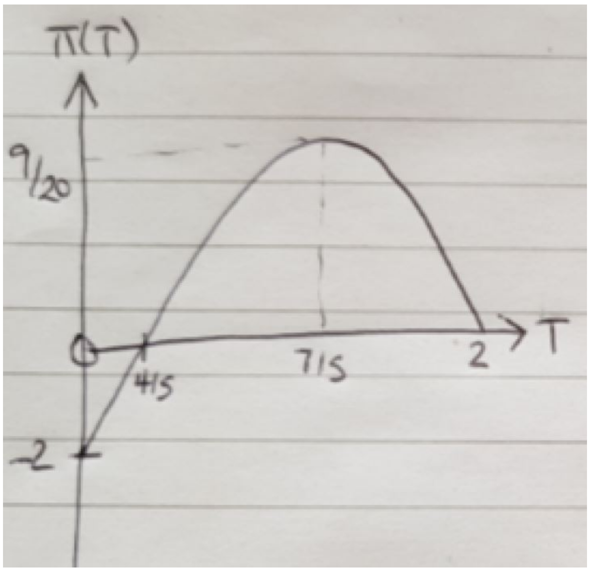

Figure 1: The sketch of $\pi(T)$ against T for Question 1 part (d).

(a) **Find $N(t)$ and describe its behaviour.**

(b) **Show that the population is increasing most rapidly when $N(t) = K/2$. At what time does this occur?**

(c) **Sketch the function $N(t)$ for $t \geq 0$.**

Reading for this question

Section 8.2.1 and Chapter 3 of the subject guide.

Approaching the question

(a) The given differential equation is separable and so, using the standard method, we have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{K-N}{K}N \quad \Longrightarrow \quad \int \frac{K\,\mathrm{d}N}{(K-N)N} = \int \mathrm{d}t.$$

So, using partial fractions on the left-hand side, we have

$$\int \frac{K\,\mathrm{d}N}{(K-N)N} = \int \left(\frac{1}{K-N} + \frac{1}{N}\right) \mathrm{d}N$$

and so we get

$$-\ln|K-N| + \ln|N| = t + c \quad \Longrightarrow \quad \ln\left|\frac{N}{K-N}\right| = t + c \quad \Longrightarrow \quad \frac{N}{K-N} = A\mathrm{e}^t.$$

Now, using the initial condition $N(0) = K/10$, we see that

$$\frac{K/10}{9K/10} = A\mathrm{e}^0 \quad \Longrightarrow \quad A = \frac{1}{9}$$

and so we have

$$\frac{N}{K-N} = \frac{\mathrm{e}^t}{9} \quad \Longrightarrow \quad 9N = K\mathrm{e}^t - N\mathrm{e}^t \quad \Longrightarrow \quad N(t) = \frac{K\mathrm{e}^t}{9+\mathrm{e}^t}$$

as the solution we seek.

To describe how the population of the town is changing with time, we note that $K > 0$ and

$$N(t) = \frac{Ke^t}{9 + e^t} = K - \frac{9K}{9 + e^t}$$

which means that, as t increases, we see that $N(t)$ will be increasing to K.

(b) The population of the town is increasing most rapidly when $N'(t)$ is maximised. So, as

$$N'(t) = \frac{9Ke^t}{(9 + e^t)^2}$$

we can see that this occurs when

$$N''(t) = 9Ke^t\,\frac{9 - e^t}{(9 + e^t)^3} = 0$$

which gives us $t = \ln(9)$. Note that this is indeed a maximum as $N''(t)$ is positive for $0 \leq t < \ln(9)$ and negative for $t > \ln(9)$. Consequently, we can see that when $t = \ln(9)$, the population of the town is increasing most rapidly and at this time

$$N(t) = \frac{Ke^{\ln(9)}}{9 + e^{\ln(9)}} = \frac{9K}{18} = \frac{K}{2}$$

as required. (This can, of course, also be found directly from the differential equation.)

(c) To sketch $N(t)$, we notice from (a) that $N(0) = K/10$, $N(t) > 0$ and $N(t)$ is increasing to K as t increases. Furthermore, from (b), we know that $N(t)$ is convex for $0 \leq t \leq \ln(9)$ and concave for $t \geq \ln(9)$. Using this information, we get the sketch in Figure 2.

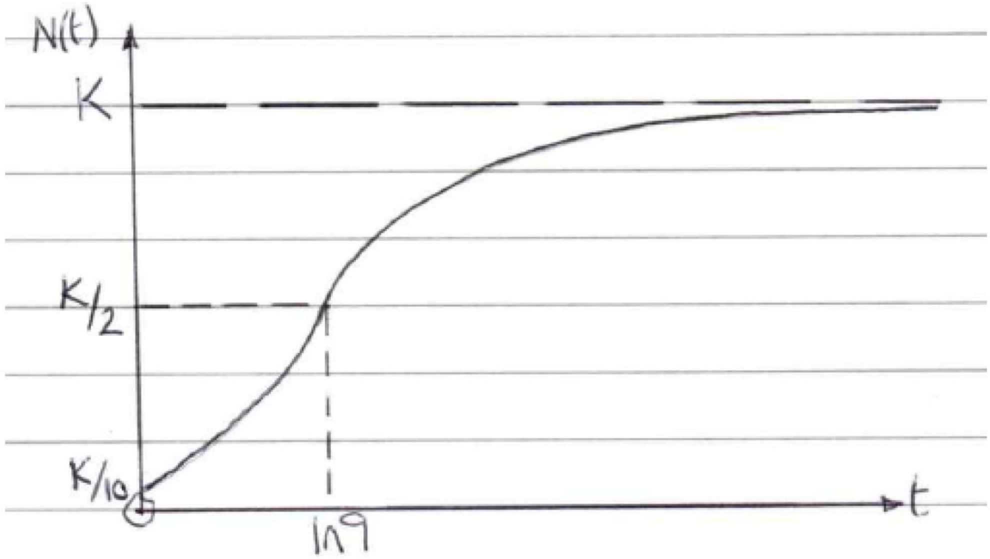

Figure 2: The sketch of $N(t)$ against t for Question 2 part (c).

Question 3

(a) Suppose that, for $n \in \mathbb{N}$, I_n is given by

$$\boldsymbol{I_n = \int_0^{\pi/2} \cos^n(\theta)\, d\theta.}$$

For $n \geq 2$, show that

$$\boldsymbol{I_n = \frac{n-1}{n}\, I_{n-2}}$$

and hence find I_2 and I_3.

Deduce the value of I_n for $n \geq 2$.

(b) **Find and classify the stationary points of the function**

$$\boldsymbol{f(x,y) = 3x^2 - 2x^3y + y^2.}$$

Reading for this question

Chapter 4 of the subject guide for part (a); Section 6.2 of the subject guide for part (b).

Approaching the question

(a) For $n \geq 2$, we can use integration by parts to see that

$$\begin{aligned}
I_n &= \int_0^{\pi/2} \cos^{n-1}(\theta)\cos(\theta)\,\mathrm{d}\theta \\
&= \Big[\cos^{n-1}(\theta)\sin(\theta)\Big]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2}(\theta)\sin^2(\theta)\,\mathrm{d}\theta \\
&= (n-1)\int_0^{\pi/2} \cos^{n-2}(\theta)[1-\cos^2(\theta)]\,\mathrm{d}\theta \\
&= (n-1)(I_{n-2} - I_n).
\end{aligned}$$

Rearranging this then gives us

$$I_n = \frac{n-1}{n}\, I_{n-2}$$

as required. Using this, we then see that

$$I_2 = \frac{1}{2}\, I_0 = \frac{\pi}{4} \quad \text{as} \quad I_0 = \int_0^{\pi/2} 1\,\mathrm{d}\theta = \Big[\theta\Big]_0^{\pi/2} = \frac{\pi}{2}$$

and also that

$$I_3 = \frac{2}{3}\, I_1 = \frac{2}{3} \quad \text{as} \quad I_1 = \int_0^{\pi/2} \cos(\theta)\,\mathrm{d}\theta = \Big[\sin(\theta)\Big]_0^{\pi/2} = 1.$$

Indeed, we can see that for even n, we have

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}$$

and for odd n we have

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{2}{3} \times 1$$

if we apply our formula multiple times.

(b) To find the stationary points of the function, we note that

$$f_x(x,y) = 6x - 6x^2y \quad \text{and} \quad f_y(x,y) = -2x^3 + 2y$$

so that we can solve the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ simultaneously. The second equation gives us $y = x^3$ so that the first equation becomes

$$x - x^5 = 0 \quad \Longrightarrow \quad x(1-x^4) = 0.$$

Thus, $x = 0, \pm 1$ and, using $y = x^3$ again, we get the stationary points $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

To classify these stationary points, we note that

$$f_{xx}(x, y) = 6 - 12xy, \quad f_{xy}(x, y) = -6x^2 = f_{yx}(x, y) \quad \text{and} \quad f_{yy}(x, y) = 2$$

so that the Hessian is

$$H(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6 - 12xy)(2) - (-6x^2)^2 = 12(1 - 2xy - 3x^4)$$

which means that

- at $(0, 0)$, $H(0, 0) = 12 > 0$ and $f_{xx}(0, 0) = 6 > 0$ so this is a local minimum
- at $(1, 1)$, $H(1, 1) = 12(-4) < 0$ and so this is a saddle point
- at $(-1, -1)$, $H(-1, -1) = 12(-4) < 0$ and so this is a saddle point too.

Question 4

Consider the function

$$\boldsymbol{g(x, y) = \frac{y}{x},}$$

for $x > 0$ and the circle with equation $(x - 4)^2 + y^2 = 4$.

(a) On the same axes, sketch the circle and some contours $g(x, y) = c$ for positive and negative values of c.

(b) By referring to your sketch in (a), explain why the method of Lagrange multipliers can be used to find *both* the maximum *and* the minimum values of $g(x, y)$ subject to the constraint that the point (x, y) lies on the circle.

(c) Hence use the method of Lagrange multipliers to find the maximum and minimum values of $g(x, y)$ subject to the constraint that the point (x, y) lies on the circle.

(d) Find the value of the Lagrange multiplier at the points found in (c).

Hence describe how the maximum and minimum values found in (c) change if the radius of the circle increases by 1/2.

Reading for this question

Section 6.3 of the subject guide.

Approaching the question

(a) The circle is centred on the point $(4, 0)$ with radius 2 and the contours $g(x, y) = c$ are straight lines through the origin given by $y = cx$. This gives us (in black) the required sketch in Figure 3.

(b) Looking at our sketch in Figure 3, we see (in green) the direction of increasing c. From this, we see that the maximum and minimum values of $g(x, y)$ subject to the constraint occur at the points A and B, respectively. As these are points where contours $g(x, y) = c$ are **tangent** to the circle, they can be found using the method of Lagrange multipliers.

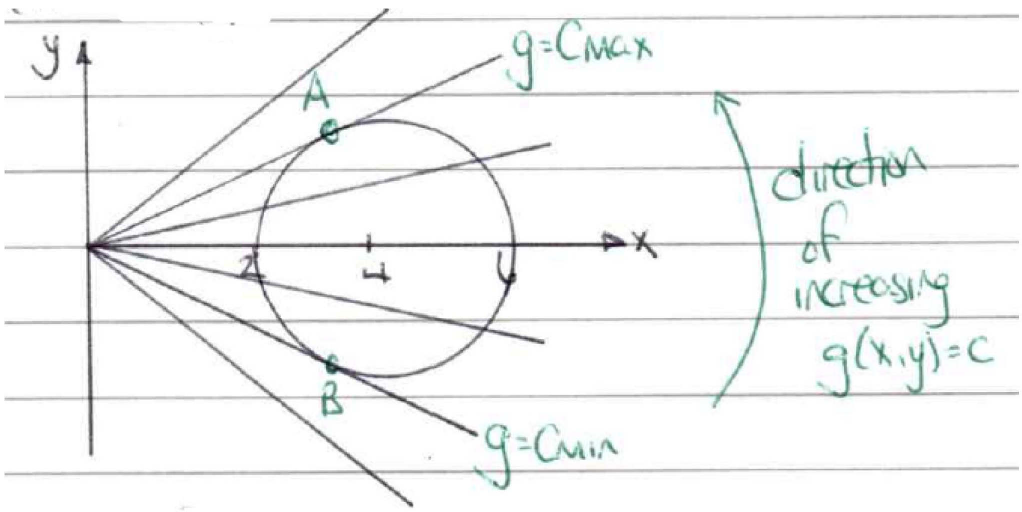

Figure 3: The sketch of some contours and the circle for Question 4 part (a).

(c) The Lagrangean is

$$\mathcal{L}(x,y,\lambda) = \frac{y}{x} - \lambda\left[(x-4)^2 + y^2 - 4\right]$$

and we seek the points where $\mathcal{L}_x(x,y,\lambda) = 0$, $\mathcal{L}_y(x,y,\lambda) = 0$ and $\mathcal{L}_\lambda(x,y,\lambda) = 0$ simultaneously. That is, we need to solve the equations

$$-\frac{y}{x^2} - 2\lambda(x-4) = 0, \quad \frac{1}{x} - 2\lambda y = 0 \quad \text{and} \quad (x-4)^2 + y^2 - 4 = 0$$

simultaneously. The first two equations give us

$$2\lambda = \frac{-y}{x^2(x-4)} = \frac{1}{xy} \quad \Longrightarrow \quad y^2 = -x(x-4)$$

and, substituting this into the third equation, we get

$$(x-4)^2 - x(x-4) - 4 = 0 \quad \Longrightarrow \quad -4x + 12 = 0 \quad \Longrightarrow \quad x = 3.$$

Thus, using the third equation again, we get $y^2 = 3$ so that our points are $(3, \sqrt{3})$ and $(3, -\sqrt{3})$. Consequently, we see that

- point A is $(3, \sqrt{3})$ and the constrained maximum value of $g(x,y)$ is $g(3,\sqrt{3}) = 1/\sqrt{3}$
- point B is $(3, -\sqrt{3})$ and the constrained minimum value of $g(x,y)$ is $g(3,-\sqrt{3}) = -1/\sqrt{3}$.

(d) Using $\lambda = 1/(2xy)$ from part (c), we see that at

- point A $(3, \sqrt{3})$ we have $\lambda = 1/6\sqrt{3}$
- point B $(3, -\sqrt{3})$ we have $\lambda = -1/6\sqrt{3}$.

If the radius of the circle increased by 1/2, our constraint goes from

$$(x-4)^2 + y^2 = 2^2 \quad \text{to} \quad (x-4)^2 + y^2 = \left(2 + \frac{1}{2}\right)^2$$

and so, for the **approximation**

$$\lambda = \frac{\mathrm{d}g}{\mathrm{d}M} \simeq \frac{\Delta g}{\Delta M} \quad \Longrightarrow \quad \Delta g \simeq \lambda \Delta M$$

we have

$$\Delta M = \left(2 + \frac{1}{2}\right)^2 - 2^2 = \left(2 + \frac{1}{2} + 2\right)\left(2 + \frac{1}{2} - 2\right) = \frac{9}{4}$$

and so

- the constrained maximum value will **increase** by approximately $3/8\sqrt{3}$
- the constrained minimum value will **decrease** by approximately $3/8\sqrt{3}$.

Question 5

(a) **For all values of the constants a and b, <u>use a matrix method</u> to determine the number of solutions to the system of equations**

$$\boldsymbol{x + y + 2z = a,}$$

$$\boldsymbol{2x + 2y + 4z = b,}$$

$$\boldsymbol{3x + 3y + 6z = 3.}$$

For any values of a and b that give at least one solution, find the solution(s) to this system of equations and express your answer in vector form.

(b) **For $t \geq 0$, the sequence x_t satisfies the difference equation**

$$\boldsymbol{x_{t+2} - 6x_{t+1} + 8x_t = 24(8^t)}$$

with $x_0 = 3$ and $x_1 = 14$. Find x_t.

Reading for this question

Section 7.2 of the subject guide for part (a); Section 9.3 of the subject guide for part (b).

Approaching the question

(a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 2 & 2 & 4 & b \\ 3 & 3 & 6 & 3 \end{array}\right)$$

where a and b are constants and, in particular, it is easy to reduce this to its row-echelon form by performing the following row operations.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 2 & 2 & 4 & b \\ 3 & 3 & 6 & 3 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 0 & 0 & b - 2a \\ 0 & 0 & 0 & 3 - 3a \end{array}\right).$$

From this, we can see that when

- $a \neq 1$ or $b \neq 2a$, there are no solutions
- $a = 1$ and $b = 2$, there are an infinite number of solutions.

Indeed, when $a = 1$ and $b = 2$, we can set $y = s \in \mathbb{R}$ and $z = t \in \mathbb{R}$ so that

$$x + y + 2z = 1 \quad \text{gives us} \quad x = 1 - s - 2t$$

which gives us

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - s - 2t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

in vector form.

(b) To solve the second-order difference equation

$$x_{t+2} - 6x_{t+1} + 8x_t = 24(8^t)$$

we start with the auxiliary equation which is

$$m^2 - 6m + 8 = 0 \quad \Longrightarrow \quad (m-2)(m-4) = 0 \quad m = 2, 4$$

and so, as we get two distinct real roots, the complementary sequence is

$$x_t = A(2^t) + B(4^t)$$

for some arbitrary constants A and B. To find a particular sequence, as the right-hand side is $24(8^t)$, we try something of the form $x_t = \alpha(8^t)$ where α is a constant to be determined. So, as $x_{t+1} = 8\alpha(8^t)$ and $x_{t+2} = 64\alpha(8^t)$, we can substitute these in to the given difference equation to get

$$64\alpha(8^t) - 48\alpha(8^t) + 8\alpha(8^t) = 24(8^t) \quad \Longrightarrow \quad 24\alpha(8^t) = 24(8^t) \quad \Longrightarrow \quad \alpha = 1$$

giving us the particular sequence $x_t = 8^t$ and, adding this to the complementary sequence, we get

$$x_t = A(2^t) + B(4^t) + 8^t$$

as the general solution of the given difference equation. Indeed, since $x_0 = 3$, we have

$$3 = A + B + 1 \quad \Longrightarrow \quad A + B = 2$$

and, since $x_1 = 14$, we also have

$$14 = 2A + 4B + 8 \quad \Longrightarrow \quad A + 2B = 3$$

which, solving simultaneously, gives us $A = 1$ and $B = 1$. Thus

$$x_t = 2^t + 4^t + 8^t$$

is the sought-after particular solution.

Examiners' commentaries 2021

MT1186 Mathematical methods

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2020–21. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the online subject guide. You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions – Zone B

Candidates should answer all **FIVE** questions. All questions carry equal marks.

Question 1

The supply and demand sets for a market are

$$S = \{(q,p) \mid p - 3q = 6\} \quad \text{and} \quad D = \{(q,p) \mid p + 2q = 16\}.$$

(a) **Find the equilibrium price and quantity for this market.**

(b) **The cobweb model dictates that, for $t \geq 1$, a market will evolve according to the formulae**

$$q_t = q^S(p_{t-1}) \quad \text{and} \quad p_t = p^D(q_t).$$

Find a recurrence equation which describes how the price changes with time if this market evolves in accordance with the cobweb model.

If the initial price is ten, solve this equation and describe how p_t behaves as t increases.

Also find q_t as an explicit function of t.

(c) **If the government imposes a percentage [of the price] tax of $100r\%$ on the sale of goods in this market, the price including the tax is p_r and the equilibrium price is now determined by the equation**

$$q^S(p_r - rp_r) = q^D(p_r).$$

Find the equilibrium price and quantity for this market in the presence of this tax.

For what values of r will this equilibrium be economically meaningful?

Reading for this question

Sections 1.6 and 9.5.2 of the subject guide.

Approaching the question

(a) Given the supply and demand sets

$$S = \{(q, p) \mid p - 3q = 6\} \quad \text{and} \quad D = \{(q, p) \mid p + 2q = 16\}$$

we can see that the equilibrium price and quantity occurs when we solve the equations $p - 3q = 6$ and $p + 2q = 16$ simultaneously. Doing this, by (say) eliminating p, we find that

$$6 + 3q = 16 - 2q \quad \Longrightarrow \quad 5q = 10 \quad \Longrightarrow \quad q = 2$$

is the equilibrium quantity and, as such, $p = 6 + 3(2) = 12$ is the equilibrium price.

(b) From the given supply and demand sets, we find that

$$q^S(p) = \frac{p - 6}{3} \quad \text{and} \quad p^D(q) = 16 - 2q$$

and so, using the cobweb model, we have

$$q_t = \frac{p_{t-1} - 6}{3} \quad \text{and} \quad p_t = 16 - 2q_t.$$

Thus, eliminating q_t from these two expressions, the recurrence equation we seek is

$$p_t = 16 - 2\left(\frac{p_{t-1} - 6}{3}\right) = 20 - \frac{2}{3}\,p_{t-1}.$$

As a, the coefficient of p_{t-1}, is not equal to one, this has a time-independent solution, p^*, given by

$$p^* = 20 - \frac{2}{3}\,p^* \quad \Longrightarrow \quad \frac{5}{3}\,p^* = 20 \quad \Longrightarrow \quad p^* = 12$$

so that, using the formula $p_t = p^* + (p_0 - p^*)a^t$, we have the solution

$$p_t = 12 + (10 - 12)\left(-\frac{2}{3}\right)^t = 12 - 2\left(-\frac{2}{3}\right)^t$$

as $p_0 = 10$. Indeed, we can see that p_t oscillates decreasingly towards 12 as t increases. As $p_t = 16 - 2q_t$, we also have

$$q_t = 8 - \frac{1}{2}\left[12 - 2\left(-\frac{2}{3}\right)^t\right] = 2 + \left(-\frac{2}{3}\right)^t$$

which tells us q_t as an explicit function of $t \geq 1$.

(c) From the given supply and demand sets, we find that

$$q^S(p) = \frac{p - 6}{3} \quad \text{and} \quad q^D(q) = \frac{16 - p}{2}$$

and so, in the presence of the tax, the equilibrium price is now determined by the equation

$$\frac{p - rp - 6}{3} = \frac{16 - p}{2} \quad \Longrightarrow \quad 2p - 2rp - 12 = 48 - 3p \quad \Longrightarrow \quad (5 - 2r)p = 60 \quad \Longrightarrow \quad p = \frac{60}{5 - 2r}$$

and the equilibrium quantity in the presence of the tax is then given by

$$q = q^D\left(\frac{60}{5 - 2r}\right) = 8 - \frac{30}{5 - 2r} = \frac{10 - 16r}{5 - 2r}.$$

For this equilibrium to be economically meaningful, we must have $5 - 2r > 0$ (or $r < 5/2$) so that the equilibrium price is non-negative and $10 - 16r \geq 0$ (or $r \leq 5/8$) so that the equilibrium quantity is non-negative as well. That is, as $r \geq 0$, we must have $0 \leq r \leq 5/8$.

Question 2

A town can accommodate a population of K people and, $N(t)$, the population of the town at time $t \geq 0$ satisfies the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N-K}{K}N,$$

with an initial population of $9K/10$.

(a) Find $N(t)$ and describe its behaviour.

(b) Show that the population is decreasing most rapidly when $N(t) = K/2$. At what time does this occur?

(c) Sketch the function $N(t)$ for $t \geq 0$.

Reading for this question

Section 8.2.1 and Chapter 3 of the subject guide.

Approaching the question

(a) The given differential equation is separable and so, using the standard method, we have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N-K}{K}N \quad \Longrightarrow \quad \int \frac{K\,\mathrm{d}N}{(N-K)N} = \int \mathrm{d}t.$$

So, using partial fractions on the left-hand side, we have

$$\int \frac{K\,\mathrm{d}N}{(N-K)N} = \int \left(\frac{1}{N-K} - \frac{1}{N}\right)\mathrm{d}N$$

and so we get

$$\ln|N-K| - \ln|N| = t + c \quad \Longrightarrow \quad \ln\left|\frac{N-K}{N}\right| = t + c \quad \Longrightarrow \quad \frac{N-K}{N} = A\mathrm{e}^t.$$

Now, using the initial condition $N(0) = 9K/10$, we see that

$$\frac{-K/10}{9K/10} = A\mathrm{e}^0 \quad \Longrightarrow \quad A = -\frac{1}{9}$$

and so we have

$$\frac{N-K}{N} = -\frac{\mathrm{e}^t}{9} \quad \Longrightarrow \quad 9N - 9K = -N\mathrm{e}^t \quad \Longrightarrow \quad N(t) = \frac{9K}{9+\mathrm{e}^t}$$

as the solution we seek.

To describe how the population of the town is changing with time, we note that $K > 0$ and so, as t increases, we see that $N(t)$ will be decreasing to 0.

(b) The population of the town is decreasing most rapidly when $N'(t)$ is minimised. So, as

$$N'(t) = -\frac{9K\mathrm{e}^t}{(9+\mathrm{e}^t)^2}$$

we can see that this occurs when

$$N''(t) = -9K\mathrm{e}^t\,\frac{9-\mathrm{e}^t}{(9+\mathrm{e}^t)^3} = 0$$

which gives us $t = \ln(9)$. Note that this is indeed a minimum as $N''(t)$ is negative for $0 \leq t < \ln(9)$ and positive for $t > \ln(9)$. Consequently, we can see that when $t = \ln(9)$, the population of the town is decreasing most rapidly and at this time

$$N(t) = \frac{9K}{9 + \mathrm{e}^{\ln(9)}} = \frac{9K}{18} = \frac{K}{2}$$

as required. (This can, of course, also be found directly from the differential equation.)

(c) To sketch $N(t)$, we notice from (a) that $N(0) = 9K/10$, $N(t) > 0$ and $N(t)$ is decreasing to 0 as t increases. Furthermore, from (b), we know that $N(t)$ is concave for $0 \leq t \leq \ln(9)$ and convex for $t \geq \ln(9)$. Using this information, we get the sketch in Figure 1.

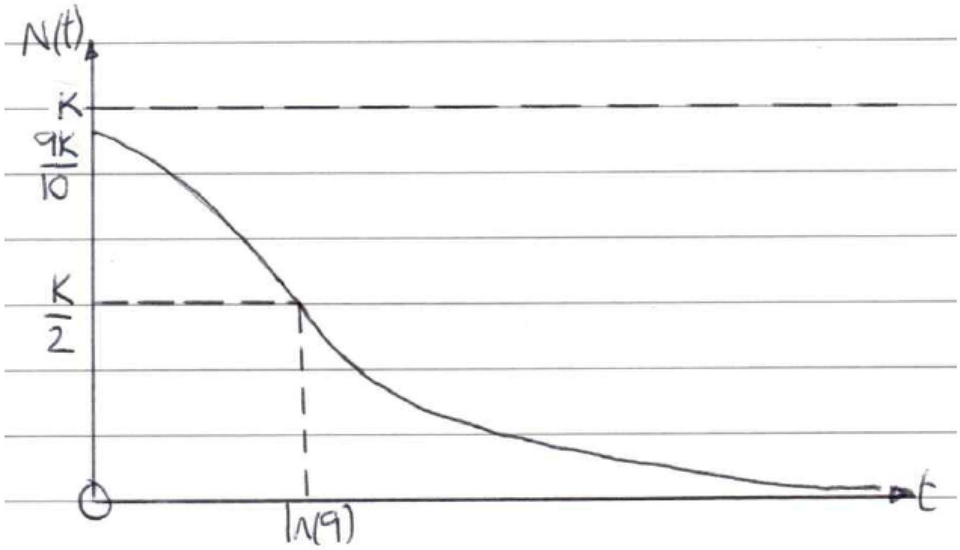

Figure 1: The sketch of $N(t)$ against t for Question 2 part (c).

Question 3

(a) Suppose that, for $n \in \mathbb{N}$, I_n is given by

$$I_n = \int_0^{\pi/2} \sin^n(\theta)\,\mathrm{d}\theta.$$

For $n \geq 2$, show that

$$I_n = \frac{n-1}{n} I_{n-2}$$

and hence find I_2 and I_3.

Deduce the value of I_n for $n \geq 2$.

(b) Find and classify the stationary points of the function

$$f(x, y) = x^2 - 2xy^3 + 3y^2.$$

Reading for this question

Chapter 4 of the subject guide for part (a); Section 6.2 of the subject guide for part (b).

Approaching the question

(a) For $n \geq 2$, we can use integration by parts to see that

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^{n-1}(\theta)\sin(\theta)\,\mathrm{d}\theta \\ &= \Big[-\sin^{n-1}(\theta)\cos(\theta)\Big]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}(\theta)\cos^2(\theta)\,\mathrm{d}\theta \\ &= (n-1)\int_0^{\pi/2} \sin^{n-2}(\theta)[1-\sin^2(\theta)]\,\mathrm{d}\theta \\ &= (n-1)(I_{n-2} - I_n). \end{aligned}$$

Rearranging this then gives us

$$I_n = \frac{n-1}{n}\, I_{n-2}$$

as required. Using this, we then see that

$$I_2 = \frac{1}{2}\, I_0 = \frac{\pi}{4} \quad \text{as} \quad I_0 = \int_0^{\pi/2} 1\,\mathrm{d}\theta = \Big[\theta\Big]_0^{\pi/2} = \frac{\pi}{2}$$

and also that

$$I_3 = \frac{2}{3}\, I_1 = \frac{2}{3} \quad \text{as} \quad I_1 = \int_0^{\pi/2} \sin(\theta)\,\mathrm{d}\theta = \Big[-\cos(\theta)\Big]_0^{\pi/2} = 1.$$

Indeed, we can see that for even n, we have

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}$$

and for odd n we have

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{2}{3} \times 1$$

if we apply our formula multiple times.

(b) To find the stationary points of the function, we note that

$$f_x(x, y) = 2x - 2y^3 \quad \text{and} \quad f_y(x, y) = -6xy^2 + 6y$$

so that we can solve the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously. The first equation gives us $x = y^3$ so that the second equation becomes

$$-y^5 + y = 0 \quad \Longrightarrow \quad y(1 - y^4) = 0.$$

Thus, $y = 0, \pm 1$ and, using $x = y^3$ again, we get the stationary points $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

To classify these stationary points, we note that

$$f_{xx}(x, y) = 2, \quad f_{xy}(x, y) = -6y^2 = f_{yx}(x, y) \quad \text{and} \quad f_{yy}(x, y) = -12xy + 6$$

so that the Hessian is

$$H(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (2)(6 - 12xy) - (-6y^2)^2 = 12(1 - 2xy - 3y^4)$$

which means that

- at $(0, 0)$, $H(0, 0) = 12 > 0$ and $f_{xx}(0, 0) = 2 > 0$ so this is a local minimum
- at $(1, 1)$, $H(1, 1) = 12(-4) < 0$ and so this is a saddle point
- at $(-1, -1)$, $H(-1, -1) = 12(-4) < 0$ and so this is a saddle point too.

Question 4

Consider the function

$$g(x,y) = \frac{y}{x},$$

for $x < 0$ and the circle with equation $(x+4)^2 + y^2 = 4$.

(a) On the same axes, sketch the circle and some contours $g(x,y) = c$ for positive and negative values of c.

(b) By referring to your sketch in (a), explain why the method of Lagrange multipliers can be used to find *both* the maximum *and* the minimum values of $g(x,y)$ subject to the constraint that the point (x,y) lies on the circle.

(c) Hence use the method of Lagrange multipliers to find the maximum and minimum values of $g(x,y)$ subject to the constraint that the point (x,y) lies on the circle.

(d) Find the value of the Lagrange multiplier at the points found in (c).

Hence describe how the maximum and minimum values found in (c) change if the radius of the circle increases by 1/2.

Reading for this question

Section 6.3 of the subject guide.

Approaching the question

(a) The circle is centred on the point $(-4,0)$ with radius 2 and the contours $g(x,y) = c$ are straight lines through the origin given by $y = cx$. This gives us (in black) the required sketch in Figure 2.

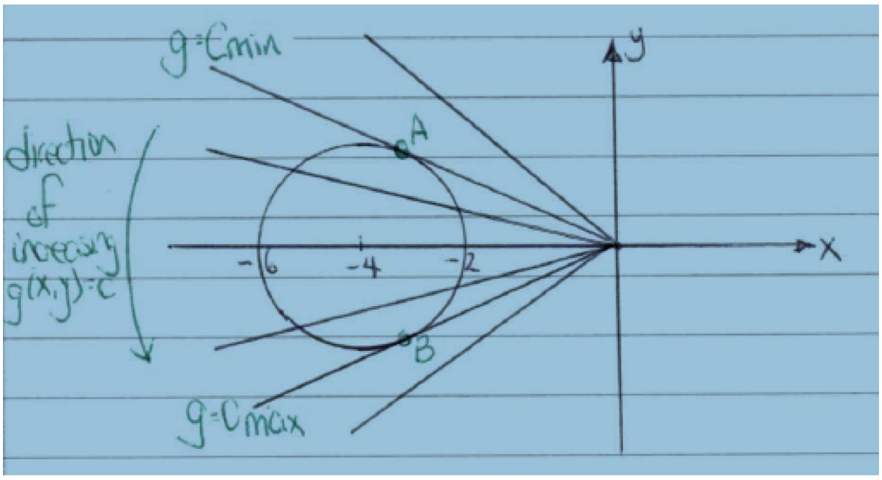

Figure 2: The sketch of some contours and the circle for Question 4 part (a).

(b) Looking at our sketch in Figure 2, we see (in green) the direction of increasing c. From this, we see that the minimum and maximum values of $g(x,y)$ subject to the constraint occur at the points A and B, respectively. As these are points where contours $g(x,y) = c$ are **tangent** to the circle, they can be found using the method of Lagrange multipliers.

(c) The Lagrangean is

$$\mathcal{L}(x,y,\lambda) = \frac{y}{x} - \lambda\left[(x+4)^2 + y^2 - 4\right]$$

and we seek the points where $\mathcal{L}_x(x,y,\lambda) = 0$, $\mathcal{L}_y(x,y,\lambda) = 0$ and $\mathcal{L}_\lambda(x,y,\lambda) = 0$ simultaneously. That is, we need to solve the equations

$$-\frac{y}{x^2} - 2\lambda(x+4) = 0, \quad \frac{1}{x} - 2\lambda y = 0 \quad \text{and} \quad (x+4)^2 + y^2 - 4 = 0$$

simultaneously. The first two equations give us

$$2\lambda = \frac{-y}{x^2(x+4)} = \frac{1}{xy} \quad \Longrightarrow \quad y^2 = -x(x+4)$$

and, substituting this into the third equation, we get

$$(x+4)^2 - x(x+4) - 4 = 0 \quad \Longrightarrow \quad 4x + 12 = 0 \quad \Longrightarrow \quad x = -3.$$

Thus, using the third equation again, we get $y^2 = 3$ so that our points are $(-3, \sqrt{3})$ and $(-3, -\sqrt{3})$. Consequently, we see that

- point A is $(-3, \sqrt{3})$ and the constrained minimum value of $g(x,y)$ is $g(-3, \sqrt{3}) = -1/\sqrt{3}$
- point B is $(-3, -\sqrt{3})$ and the constrained maximum value of $g(x,y)$ is $g(-3, -\sqrt{3}) = 1/\sqrt{3}$.

(d) Using $\lambda = 1/(2xy)$ from part (c), we see that at

- point A $(-3, \sqrt{3})$ we have $\lambda = -1/6\sqrt{3}$
- point B $(-3, -\sqrt{3})$ we have $\lambda = 1/6\sqrt{3}$.

If the radius of the circle increased by 1/2, our constraint goes from

$$(x+4)^2 + y^2 = 2^2 \quad \text{to} \quad (x+4)^2 + y^2 = \left(2 + \frac{1}{2}\right)^2$$

and so, for the **approximation**

$$\lambda = \frac{\mathrm{d}g}{\mathrm{d}M} \simeq \frac{\Delta g}{\Delta M} \quad \Longrightarrow \quad \Delta g \simeq \lambda \Delta M$$

we have

$$\Delta M = \left(2 + \frac{1}{2}\right)^2 - 2^2 = \left(2 + \frac{1}{2} + 2\right)\left(2 + \frac{1}{2} - 2\right) = \frac{9}{4}$$

and so

- the constrained minimum value will **decrease** by approximately $3/8\sqrt{3}$
- the constrained maximum value will **increase** by approximately $3/8\sqrt{3}$.

Question 5

(a) **For all values of the constants a and b, <u>use a matrix method</u> to determine the number of solutions to the system of equations**

$$\boldsymbol{x + 2y + z = 1,}$$

$$\boldsymbol{2x + 4y + 2z = a,}$$

$$\boldsymbol{3x + 6y + 3z = b.}$$

For any values of a and b that give at least one solution, find the solution(s) to this system of equations and express your answer in vector form.

(b) Find the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \boldsymbol{a^2} \\ \mathbf{4} & \mathbf{1} \end{pmatrix}$$

where $\boldsymbol{a}$ is a constant.

Hence, for each value of $\boldsymbol{a}$, find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P^{-1}AP = D}$ or explain why this is not possible.

Reading for this question

Section 7.2 of the subject guide for part (a); Section 9.3 of the subject guide for part (b).

Approaching the question

(a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & a \\ 3 & 6 & 3 & b \end{array}\right)$$

where a and b are constants and, in particular, it is easy to reduce this to its row-echelon form by performing the following row operations.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & a \\ 3 & 6 & 3 & b \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & a-2 \\ 0 & 0 & 0 & b-3 \end{array}\right).$$

From this, we can see that when

- $a \neq 2$ or $b \neq 3$, there are no solutions
- $a = 2$ and $b = 3$, there are an infinite number of solutions.

Indeed, when $a = 2$ and $b = 3$, we can set $y = s \in \mathbb{R}$ and $z = t \in \mathbb{R}$ so that

$$x + 2y + z = 1 \quad \text{gives us} \quad x = 1 - 2s - t$$

which gives us

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 2s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

in vector form.

(b) To find the eigenvalues of this matrix, we solve the equation

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 1-\lambda & a^2 \\ 4 & 1-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (1-\lambda)^2 - 4a^2 = 0$$

which, in turn, gives us

$$(1-\lambda)^2 = 4a^2 \quad \Longrightarrow \quad 1 - \lambda = \pm 2a$$

and so the eigenvalues are $1 + 2a$ and $1 - 2a$. To find the corresponding eigenvectors we seek a non-zero vector, $\mathbf{x}$, which is a solution to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$, i.e. we have

- for $\lambda = 1 + 2a$, we solve

$$\begin{pmatrix} -2a & a^2 \\ 4 & -2a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad 4x - 2ay = 0 \quad \Longrightarrow \quad x = \frac{ay}{2} \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ 2 \end{pmatrix}$$

  (or any non-zero multiple of this) is an eigenvector.

- for $\lambda = 1 - 2a$, we solve

$$\begin{pmatrix} 2a & a^2 \\ 4 & 2a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad 4x + 2ay = 0 \quad \Longrightarrow \quad x = -\frac{ay}{2} \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a \\ 2 \end{pmatrix}$$

  (or any non-zero multiple of this) is an eigenvector.

Consequently, if $a \neq 0$, we see that

$$P = \begin{pmatrix} a & -a \\ 2 & 2 \end{pmatrix} \quad \text{is an invertible matrix as} \quad |P| = \begin{vmatrix} a & -a \\ 2 & 2 \end{vmatrix} = 4a \neq 0$$

and so, together with the diagonal matrix

$$D = \begin{pmatrix} 1 + 2a & 0 \\ 0 & 1 - 2a \end{pmatrix}$$

we have $P^{-1}AP = D$.

However, if $a = 0$, we see that $\lambda = 1$ is the only eigenvalue and all of the eigenvectors are non-zero multiples of the vector $(0, 1)^T$. As such, we find that any matrix P we construct will have the form

$$P = \begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix} \quad \text{and this is not an invertible matrix as} \quad |P| = \begin{vmatrix} 0 & 0 \\ r & s \end{vmatrix} = 0.$$

That is, if $a = 0$, there is no invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$.