Candidate No. A28437

№1

a) In equilibrium $p^S(q^*) = p^D(q^*)$

Then $e^{q^*} = 6e^{-q^*} + 1 \quad | \cdot e^{q^*}$

$e^{2q^*} = 6 + e^{q^*}$

$(e^{q^*})^2 - e^{q^*} - 6 = 0$

$D = (-1)^2 - 4 \cdot 1 \cdot (-6) = 25 = 5^2$

$e^{q^*} = \frac{1+5}{2} = 3$ or $e^{q^*} = \frac{1-5}{2} = -2$ – we can not take it, because $e^n > 0$

Then $e^{q^*} = 3$

Then $q^* = \ln 3$ ← equilibrium quantity

Then $p^* = e^{\ln 3} = 3$ ← equilibrium price

b) $0 < T < 6$

$p_T^S(q) = e^q - T \qquad p_T^D(q) = 6e^{-q} + 1$

$e^q - T = 6e^{-q} + 1$

Let $e^q = u$

Then

$u - T = \frac{6}{u} + 1$

$u^2 - uT - 6 - u = 0$

$u^2 - (T+1)u - 6 = 0$

$D = T^2 + 2T + 25$

Then $u = \frac{T+1 \pm \sqrt{T^2+2T+25}}{2}$ → here we take only $u = \frac{T+1+\sqrt{T^2+2T+25}}{2}$, because $T+1 \leq \sqrt{(T+1)^2} = \sqrt{T^2+2T+1} < \sqrt{T^2+2T+25}$, then $e^q = u = \frac{T+1-\sqrt{T^2+2T+25}}{2} < 0$, which can not be true

Then $q = \ln\left(\frac{T+1+\sqrt{T^2+2T+25}}{2}\right)$

equilibrium quantity after tax

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Then we can calculate

$$p_T^* = \cancel{T+2+\dots}\, e^{q^*} - T = \frac{T+1+\sqrt{T^2+2T+25}}{2} - T =$$

$$= \frac{-T+1+\sqrt{T^2+2T+25}}{2}$$

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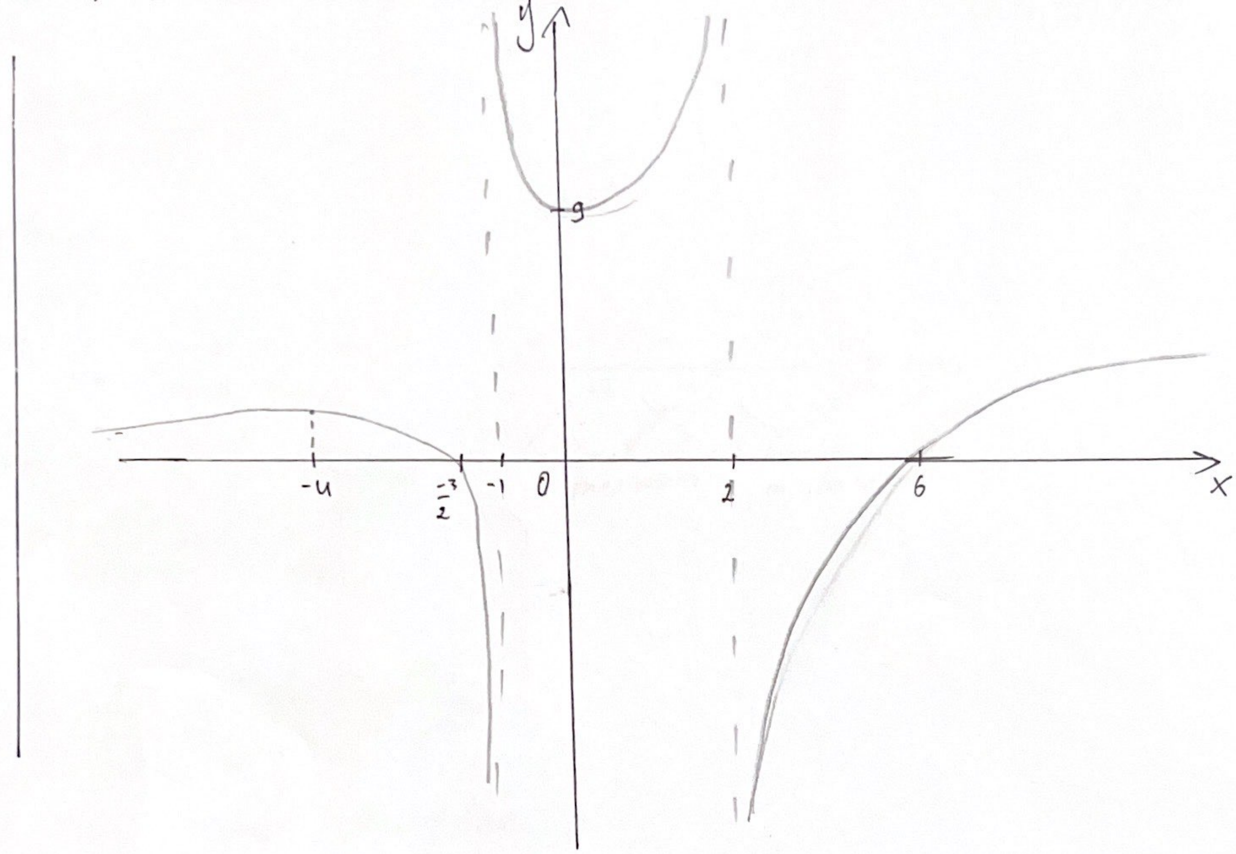
N2

$$f(x) = \frac{2x^2-9x-18}{x^2-x-2}$$

a)



$$\frac{2x^2-9x-18}{x^2-x-2} = 0$$

$2x^2-9x-18 = 0$

$2x^2-6x-3x-18=0$

$2x^2-12x+3x-18=0$

$2x(x-6)+3(x-6)=0$

$(x-6)(2x+3)=0$

Then $x_1 = 6$, $x_2 = -\frac{3}{2}$ -

Then poins $(6;0)$ and $(-\frac{3}{2};0)$ are in the graph of this function

if $x=0$

$\frac{-18}{-2} = 9$

Then point $(0;9)$ is in the graph of this function

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$$\frac{d}{dx}\left(\frac{2x^2-9x-18}{x^2-x-2}\right) = \frac{(x^2-x-2)(-18-9x+2x^2)' - (-18-9x+2x^2)(x^2-x-2)'}{(-2-x+x^2)^2} =$$

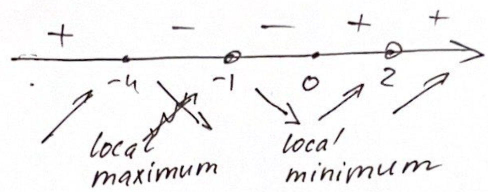
$$= \frac{(x^2-x-2)(4x-9) - (-18-9x+2x^2)(-1+2x)}{(-2-x-x^2)^2} = \frac{(-9+4x)(-2-x+x^2) - (-1+2x)(-18-9x+2x^2)}{(x^2-x-2)^2} =$$

$$= \frac{7x(4+x)}{(-2-x+x^2)^2}$$

$$\frac{7x(4+x)}{(-2-x+x^2)^2} = 0$$

zeroes: $x=0$ and $x=-4$

$x \neq -1$ and $x \neq 2$



let's find second derivative

$$\left(\frac{7x(4+x)}{(-2-x+x^2)^2}\right)' = \frac{(7x(4+x))'(-2-x+x^2)^2 - (7x(4+x))((-2-x+x^2)^2)'}{(-2-x+x^2)^4} =$$

$$= \frac{(7(4+x)+7x)(-2-x+x^2)^2 - (7x(4+x)\cdot 2(2x-1)(-2-x+x^2)}{(-2-x+x^2)^4}$$

$$= \frac{(28+7x+7x)(-2-x+x^2)^2 - (28x+7x^2)\cdot(4x-2)(-2-x+x^2)}{-(2-x+x^2)^4} =$$

$$= \frac{(28+14x)(-2-x+x^2) - (28x+7x^2)(4x-2)}{-(2-x+x^2)^3} = \frac{-14x^3-84x^2-56}{(x^2-x-2)^3}$$

$$\lim_{(x\to -1-)} \frac{2x^2-9x-18}{x^2-x-2} = -\infty \qquad \lim_{(x\to 2+)} \frac{2x^2-9x-18}{x^2-x-2} = -\infty$$

$$\lim_{(x\to -1+)} \frac{2x^2-9x-18}{x^2-x-2} = +\infty \qquad \lim_{(x\to 2-)} \frac{2x^2-9x-18}{x^2-x-2} = +\infty$$

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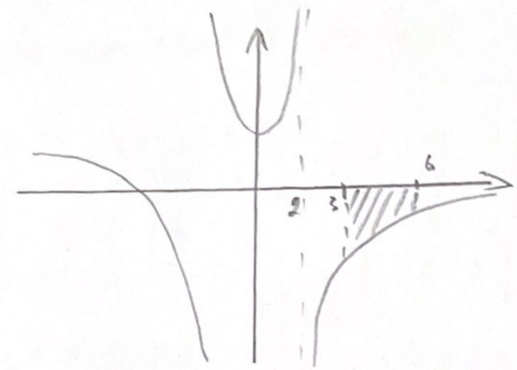
That is, if x tends to -1 from the left, y tends to $-\infty$

if x tends to -1 from the right, y tends to $+\infty$

if x tends to 2 from the left, y tends to $+\infty$

if x tends to 2 from the right, y tends to $-\infty$

b)



$$\int_3^6 \frac{2x^2-9x-18}{x^2-x-2}\,dx = \int_3^6 \frac{2x^2-2x-4-7x-14}{x^2-x-2}\,dx =$$

$$= \int_3^6 \left(2 + \frac{-7x-14}{x^2-x-2}\right)dx = \int_3^6 \left(2 + \frac{-7x-14}{(x+1)(x-2)}\right)dx =$$

$$= \int_3^6 \left(2 + \frac{7}{3(x+1)} - \frac{28}{3(x-2)}\right)dx = \int_3^6 2\,dx + \int_3^6 \frac{7}{3(x+1)}\,dx - \int_3^6 \frac{28}{3(x-2)}\,dx =$$

$$= 2\int_3^6 1\,dx + \frac{7}{3}\int_3^6 \frac{1}{x+1}\,dx - \frac{28}{3}\int_3^6 \frac{1}{x-2}\,dx = \left|\begin{matrix} u = x+1 \\ v = x-2 \end{matrix}\right| =$$

Let $\frac{-7x-14}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} = \frac{(B+A)x+B-2A}{(x-2)(x-1)}$

Then $\left.\begin{matrix} B+A=-7 \\ B-2A=-14 \end{matrix}\right\}$ then $A = \frac{7}{3}$ and $B = -\frac{28}{3}$

$$= 2\int_3^6 1\,dx + \frac{7}{3}\log(u)\Big|_{3+1}^{6+1} + \frac{28}{3}\log(v)\Big|_{3-2}^{6-2} = 2\cdot x\Big|_3^6 + \frac{7}{3}\log u\Big|_4^7 - \frac{28}{3}\log v\Big|_1^4 =$$

$$= 12-6+\frac{7}{3}\log 7 - \frac{7}{3}\log 4 - \frac{28}{3}\log 5 - \frac{28}{3}\log 1 = 6 + \frac{7}{3}\ln\frac{7}{4} - \frac{28}{3}\ln 4 =$$

$= 6 + \frac{7}{3}\ln\left(\frac{7}{4}\right) - \frac{28}{3}\ln 4 \leftarrow$ area of the given region

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N3

a) $x+2y+z=a$

$x-y+z=2a$

$x+3y+2z=b$

$3x-y+z=c$

Let's use matrix method

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & -1 & 1 & -2a \\ 1 & 3 & 2 & b \\ 3 & -1 & 1 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & -1 & 1 & c \\ 1 & -1 & 1 & -2a \\ 1 & 3 & 2 & b \\ 1 & 2 & 1 & a \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2a \\ 0 & -3 & 0 & 3a \\ 0 & 1 & 1 & b-a \\ 0 & -7 & -2 & c-3a \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & b \\ 0 & 0 & -2 & c-10a \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & c-10a+2b \end{array}\right)$$

In order for this system to have a solution

$c-10a+2b=0$

Then $c = 10a-2b$

Then $\left.\begin{array}{l} x = a-b \\ y = -a \\ z = b \end{array}\right\}$ only one solution

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N3

б) $f(x,y) = x^3 - x^2 - x - y^2 + 2y$

$f_x(x,y) = 3x^2 - 2x - 1$ $\quad f_{xy}(x,y) = 0$

$f_{xx}(x,y) = 6x - 2$

$f_y(x,y) = -2y + 2$

$f_{yy}(x,y) = -2$

Then $H = \begin{vmatrix} 6x-2 & 0 \\ 0 & -2 \end{vmatrix} = -2\cdot(6x-2) = -12x + 4$ ← Hessian

Let's solve $f_x(x,y) = 0$ and $f_y(x,y) = 0$

$3x^2 - 2x - 1 = 0$ $\qquad -2y + 2 = 0$

$D = 4 + 12 = 16 = 4^2$ $\qquad -2y = -2$

$x_1 = \frac{2+4}{6} = 1$ or $x_2 = \frac{2-4}{6} = -\frac{1}{3}$ $\qquad y = 1$

Then saddle points are $(1,1)$; $(-\frac{1}{3}; 1)$

Then, evaluating ~~the~~ Hessian at each of the stationary points we find that

1) at $(1,1)$ we have $H(1,1) = -12\cdot 1 + 4 = -8 < 0$,
Then $(1,1)$ is a saddle point ~~saddle~~

2) at $(-\frac{1}{3}, 1)$ we Have $H(-\frac{1}{3}; 1) = 4 + 4 = 8$
$f_{xx}(-\frac{1}{3}, 1) = -2 - 2 = -4$
Then $(-\frac{1}{3}, 1)$ is local maximum

Stationary points:
$(1,1)$ - saddle point
$(-\frac{1}{3}, 1)$ - local maximum

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N4

$q(x,y) = ((x+1)^3 + y^3)^{\frac{1}{3}}$ - a firm can produce quantity in kg

According to Lagrange multiplier method

$L(x,y,\lambda) = rx + sy - \lambda(((x+1)^3+y^3)^{\frac{1}{3}} - Q)$

$$\begin{cases} \frac{\partial L}{\partial x} = r - \lambda(x+1)^2((x+1)^3+y^3)^{-\frac{2}{3}} = 0 \\ \frac{\partial L}{\partial y} = s - \lambda(y^2)((x+1)^3+y^3)^{-\frac{2}{3}} = 0 \\ \frac{\partial L}{\partial \lambda} = -((x+1)^3+y^3)^{1/3} + Q = 0 \end{cases} \Rightarrow \text{then } \frac{r}{s} = \frac{(x+1)^2}{y^2}, \text{ then}$$

$y = x+1\sqrt{\frac{s}{r}}$, let's substitute it into one of the equations

$((x+1)^3 \cdot \left(\frac{s}{r}\right)^{\frac{3}{2}} + (x+1)^3)^{\frac{1}{3}} = Q$, then $(x+1)^3((\frac{s}{r})^{\frac{3}{2}}+1) = Q^3$

Then $x+1 = \frac{Q}{\sqrt[3]{(\frac{s}{r})^{\frac{3}{2}}+1}}$, then $x = \frac{Q}{\sqrt[3]{(\frac{s}{r})^{3/2}+1}} - 1$

Then we can say that

$$y = \frac{Q}{\sqrt[3]{(\frac{s}{r})^{\frac{3}{2}}+1}} \sqrt{\frac{s}{r}}$$

Then bundle

$$\left(\frac{Q}{\sqrt[3]{(\frac{s}{r})^{\frac{3}{2}}+1}} - 1 > \frac{Q}{\sqrt[3]{(\frac{s}{r})^{\frac{3}{2}}+1}} \sqrt{\frac{s}{r}}\right)$$

is the bundle that minimizes firm's costs if it produces Q kg

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N5

a) $A=\begin{pmatrix} 11 & -4 \\ 24 & -9 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 11-\lambda & -4 \\ 24 & -9-\lambda \end{vmatrix} = 0 = (11-\lambda)(-9-\lambda)-(-4)\cdot 24 = -99-11\lambda+9\lambda+\lambda^2+96=$$
$$=\lambda^2-2\lambda-3=(\lambda+1)(\lambda-3)$$

Then $\lambda_1=3$, $\lambda_2=-1$

$\cancel{\lambda_1=-1: \begin{pmatrix} 12 & -4 \\ 24 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0, \text{ then } 8y-4x=0,\ 2y=x \quad v_1=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$\cancel{\lambda_2=3: \begin{pmatrix} 8 & -4 \\ 24 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ then } 24y-12y=0 \quad v_2=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$

$\lambda_1=-1: \begin{pmatrix} 12 & -4 \\ 24 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$, then $12x-4y=0$, $12x=4y$, $3x=y$ then $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda_2=3: \begin{pmatrix} 8 & -4 \\ 24 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, then $24x-12y=0$, $24x=12y$, $2x=y$, then $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Consequently, if we take $P=\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ and $D\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$

we have an invertible matrix P and diag. matrix D

such that $P^{-1}AP=D$

$$P^{-1}=\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}^{-1}=\frac{1}{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}}\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}=-1\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}=\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$$

Then let's calculate $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 11 & -4 \\ 24 & -9 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}=$

$$=\begin{pmatrix} 2 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

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№5

b) $x_t = Ax_{i-1} + q$

A - diagonalisable

q - constant vector

To show that we can define a vector u, such that

$u_t = Du_{t-1} + P^{-1}q$

Let $B = (PDP^{-1})$

$X_t = \cancel{B^t X_0 + q = (PDP^{-1}) X_0 + q = (}$

$= A^t X_0 + q = \begin{pmatrix} 11 & -4 \\ 24 & -9 \end{pmatrix}^T X_0 + q = \begin{pmatrix} 11 & 24 \\ -4 & -9 \end{pmatrix} X_0 + q =$

$P^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

$= PDP^{-1} X_0 + q =$

$= \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1t & 0 \\ 0 & 3t \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + q =$

$= \begin{pmatrix} -t & 3t \\ -3t & 6t \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + q = \begin{pmatrix} 11t & -4t \\ 24t & -9t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + q =$

$\cancel{= \begin{pmatrix} 11tx - 4ty \\ 24tx - 9ty \end{pmatrix} + q}$

~~So if $D = \begin{pmatrix} 11 \\ 24 \end{pmatrix}$~~

So if

$D = \begin{bmatrix} 11t & -4t \\ 24t & -9t \end{bmatrix}$

$\cancel{D = [\;]}$

$P = [1]$

And $u = \begin{bmatrix} x \\ y \end{bmatrix}$

We can rewrite this system