Department for
Applied Mathematics

WiSe 2025/2026
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Basics in Applied Mathematics

Sheet 0 – 14.10.2025 (no submission)

Homepage: `https://uni-freiburg.de/stochastik/schmidt/basics-am/`

Task 1 (0 points)**.** State four different characterizations for the unique solvability of the linear system $Ax = b$ for every right-hand side $b \in \mathbb{R}^n$ and discuss their equivalences.

Task 2 (0 points)**.** For $p > 0$, $\beta > 1$ and $j = 1, 2, 3, 4$ let the sequences $(a_n^{(j)})_{n\in\mathbb{N}}$ be defined by

$$a_n^{(1)} = n^p, \quad a_n^{(2)} = \beta^n, \quad a_n^{(3)} = n!, \quad a_n^{(4)} = \log_2 n.$$

For which pairs $1 \le i, j \le 4$ does $a_n^{(i)} = \mathcal{O}(a_n^{(j)})$ hold?

Task 3 (0 points)**.** For $1 \le p < \infty$, a norm is defined on $\mathbb{R}^\ell$ by $\|x\|_p = \left(\sum_{j=1}^{\ell} |x_j|^p\right)^{1/p}$. The induced operator norm is also denoted by $\|\cdot\|_p$.
(i) Show that $\|A\|_1 = \max_{k=1,\dots,n} \sum_{j=1}^{m} |a_{jk}|$ holds for all $A \in \mathbb{R}^{m\times n}$.
(ii) For the symmetric matrix $B \in \mathbb{R}^{n\times n}$, let

$$\rho(B) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } B\}.$$

Show, that $\|A\|_2 = \sqrt{\rho(A^\mathsf{T} A)}$ holds for all $A \in \mathbb{R}^{m\times n}$.

Task 4 (0 points)**.** (i) Let $A \in \mathbb{R}^{n\times n}$. Show that

$$\|A\|_2^2 \le \|A\|_1 \|A\|_\infty$$

holds and verify the statement explicitly for $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
(ii) Show that for every matrix $A \in \mathbb{R}^{n\times n}$ the estimates

$$\begin{aligned} n^{-1/2}\|A\|_2 \le &\|A\|_1 \le n^{1/2}\|A\|_2, \\ n^{-1}\|A\|_\infty \le &\|A\|_1 \le n\|A\|_\infty \end{aligned}$$

hold and provide matrices $A \in \mathbb{R}^{n\times n}$ that show that the estimates cannot be improved.

Project 1 (0 points)**.** The functions $f, g : \mathbb{R}_{>0} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x} - \frac{1}{x+1}, \quad g(x) = \frac{1}{x(x+1)}$$

agree, but motivate two different methods for numerical computation. Determine for $x_k = 10^k$, $k = 1, 2, \dots, 15$, the expression

$$\delta_k = \frac{|f(x_k) - g(x_k)|}{|g(x_k)|}$$

in a programming language and arrange the results in a table. What do you observe and how do you explain the observations?