Department for
Applied Mathematics

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Basics in Applied Mathematics

Sheet 3 – 28.10.2025 (submission until 4.11.2025)

Task 1 (8 points)**.** Let $A, B \in \mathbb{R}^{m\times n}$ and $(v_1, v_2, \dots, v_n) \subset \mathbb{R}^n$ be a basis of $\mathbb{R}^n$. Show that from $Av_i = Bv_i$ for $i = 1, 2, \dots, n$ the equality $A = B$ follows.

Task 2 (8 points)**.** Determine a singular value decomposition of the matrix

$$A = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 & -3 \\ -1 & -3 & 3 & 1 \end{bmatrix}^\mathsf{T}.$$

Calculate A^+ using the singular value decomposition as well as the identity $A^+ = (A^\mathsf{T} A)^{-1} A^\mathsf{T}$. Use A^+ to solve the least squares problem defined by A and $b = [4, 1, 2, 3]^\mathsf{T}$.

Task 3 (8 points)**.** (i) Determine the Gershgorin circles of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

(ii) Let $A \in \mathbb{R}^{n\times n}$ be strictly diagonally dominant, i.e., we have $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for $i = 1, 2, \dots, n$, and symmetric. Provide an explicit upper bound for the condition number $\operatorname{cond}_2(A)$, i.e., an upper bound in terms of entries of A.

Task 4 (8 points)**.** (i) Let $A \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and let $v_1 \in \mathbb{R}^n \setminus \{0\}$ be an eigenvector corresponding to the eigenvalue λ_1. Show that

$$\lambda_2 = \max_{\substack{x\in\mathbb{R}^n\setminus\{0\} \\ x\cdot v_1 = 0}} \frac{x^\mathsf{T} A x}{\|x\|_2^2}.$$

(ii) Show that the vector $x^* \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector of the symmetric matrix $A \in \mathbb{R}^{n\times n}$ if and only if $\nabla r(x^*) = 0$ holds with the function

$$r : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \frac{x^\mathsf{T} A x}{\|x\|_2^2}.$$

Project 1 (4 bonus points)**.** In Matlab the singular value decomposition of a matrix A can be calculated with the command `svd`. For an image defined by the file `img.jpg`, a compression of the grayscale representation can be defined with the lines shown in Figure 1. Choose as an image, for example, the section from Albrecht Dürer's picture *Melancolia I*, which shows the magic square. Explain the individual lines of the program and extend it by a calculation of the approximation error $\|X - X_{comp}\|_\mathcal{F}$. How do you assess the ratio of quality loss to reduction of storage requirements for different values of k? Test the program for another image.

```
RGB = imread('img.jpg');
G = rgb2gray(RGB);
D = double(G);
X = D/max(max(D));
figure(1);
subplot(1,2,1); imshow(X); title('original');
[U,S,V] = svd(X);
for k = 5:5:size(U,1)
    X_comp = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';
    subplot(1,2,2); imshow(X_comp);
    title('compressed'); pause
end
```

Abbildung 1. Image compression using singular value decomposition