Department for
Applied Mathematics

WiSe 2025/2026

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Basics in Applied Mathematics

Sheet 5 – 11.11.2025 (submission until 18.11.2025)

Task 1 (8 points). (i) Let $f \in C^2([a,b])$ with the property $f(a) = f(b)$ and $f'(a) = f'(b) = 0$. Provide an optimal lower bound for the number of roots of f''.
(ii) For nodes $x_0 < x_1 < \cdots < x_n$, let $w(x) = \prod_{j=0}^{n}(x - x_j)$ be the node polynomial and L_i, $i = 0, 1, \ldots, n$, the i-th Lagrange basis polynomial. Show that

$$L_i(x) = \frac{w(x)}{(x - x_i)w'(x_i)}.$$

Task 2 (8 points). (i) Provide a method using as few as possible basic arithmetic operations for evaluating the polynomial $(x+3)^{16}$.
(ii) Compare the effort of the direct evaluation of the polynomial $p(x) = a_0 + a_1x_1 + \cdots + a_nx^n$ with that of using the equivalent representation

$$p(x) = a_0 + x(a_1 + x(a_2 + \ldots x(a_{n-2} + x(a_{n-1} + xa_n))\ldots)).$$

Task 3 (8 points). (i) Let $f(x) = \sin(\pi x)$ for $x \in [0,1]$, $x_0 = 0$ and $x_i = i/n$, $i = 0, 1, \ldots, n$ if $n > 0$. Calculate and sketch the interpolation polynomial of f for $n = 0, 1, \ldots, 4$.
(ii) Which n is sufficient in (i) to obtain a maximal absolute interpolation error of at most 0.01?

Task 4 (8 points). Assume that the quadrature formula $Q : C^0([a,b]) \to \mathbb{R}$ is exact of degree $2q$ and the associated weights $(w_i)_{i=0,\ldots,n}$ and nodes $(x_i)_{i=0,\ldots,n}$ are symmetrically arranged with respect to the interval midpoint $(a+b)/2$. Show that Q is exact of degree $2q+1$.

Project 1 (4 bonus points). Use the composite trapezoidal and Simpson rules, as well as a composite Gaussian 3-point quadrature formula, to approximate the integrals in the interval $[0,1]$ of the functions

$$f(x) = \sin(\pi x)e^x, \quad g(x) = x^{1/3}$$

with step sizes $h = 2^{-\ell}$, $\ell = 1, 2, \ldots, 10$. Calculate the error e_h in each case and determine an experimental convergence rate γ from the approach $e_h \approx c_1 h^\gamma$ and the resulting formula

$$\gamma \approx \frac{\log(e_h/e_H)}{\log(h/H)}$$

for two successive step sizes $h, H > 0$. Compare the experimental convergence rates with the theoretical convergence rates of the methods and comment on your results. Display the pairs (h, e_h) for the different quadrature formulas comparatively as polygonal chains graphically in logarithmic axis scaling using the Matlab command `loglog`.