Homework 6: Optimization problems

Hand in: 25.11.2025 (Tuesday)

Please follow the submission instructions from the webpage of the course.

Correction: tutorial session on 27.11.2025 (Thursday)

Exercise 1: 1D Optimization problems (8 points)

In this exercise, we study the solutions of some optimization problems in the following form:

$$\underset{x\in\mathbb{R}}{\text{minimize}} \quad f(x) \tag{1}$$

For each of the following functions $f(x)$, find *all* local minima, and specify which ones are global minima (when they exist) for the problem (1).

1. $f(x) := x^2 + 4x + 10$
2. $f(x) = x^3 - 3x^2 + 1$
3. $f(x) = x^2 + \frac{1}{x^2+1}$
4. $f(x) = |x| + 2\,|x-1|$
5. $f(x) = e^{-x^2}$

Hint: For smooth functions, use first- and second-order optimality conditions to find and classify stationary points. More precisely, when f is smooth, look at the points where $f'(x) = 0$ holds, and check if $f''(x) > 0$ also holds for these points.

Exercise 2: A function approximation problem (8 points)

In this exercise, we will study a function approximation problem. More precisely, a function $g : \mathbb{R} \to \mathbb{R}$ is unknown, but we have access to some of its values: $y_j = g(t_j)$ for some points $t_1 < t_2 < \cdots < t_m$.

The goal is to approximate $g(\cdot)$ by a function $\Phi(\cdot; x)$ of the following form:

$$\Phi(t;x) = \sum_{i=1}^{n} \varphi_i(t) x_i \tag{2}$$

where $\varphi_1(\cdot), \ldots, \varphi_n(\cdot)$ are known functions and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are the parameters to be determined.

We formulate the approximation problem as a least-square problem:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{j=1}^{m} \big(y_j - \Phi(t_j; x)\big)^2 \tag{3}$$

1. Show that if the function $g(\cdot)$ is already in the form: $g(t) = \Phi(t; x^\star)$ for some $x^\star \in \mathbb{R}^n$, then the point $x^\star$ is a solution of the problem (3).
2. Show that the problem (3) can be written in the following general form:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \|y - Ax\|^2 \tag{4}$$

where $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ is a matrix that you need to specify.

3. Show that the problem (4) can be reformulated in the general QP form:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}x^\top Q x - c^\top x + r \tag{5}$$

where $Q \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$ and $r \in \mathbb{R}$ are matrices/vectors/values that you need to specify.

4. Assume that the matrix $A^\top A$ is invertible. Then show that the problem (4) has a unique solution, and specify it as a function of A and y.

5. Discuss a necessary condition on the number of samples m, and the number of parameters n for $A^\top A$ to be invertible. Also provide an intuitive explanation of this condition.

Exercise 3: The Fourier transform (8 points)

This exercise is also a follow-up from the last lecture of the first part of this course on Numerics, where we introduced the Fourier transform.

As a special case of the previous exercise, we study function approximation problem where we approximate a function by a sum of sinus functions. More precisely, consider a function $f : [0,1] \mapsto \mathbb{R}$, which we approximate you, and the following approximation problem, similar to (3)

$$\underset{x\in\mathbb{R}^2}{\text{minimize}} \quad \sum_{j=1}^{m} \big(f(t_j) - \Phi(t_j;x)\big)^2, \tag{6}$$

with $t_j = \frac{j}{m}$, and where the function $\Phi(t_j;x)$ is defined as follows:

$$\Phi(t;x) = x_1 \sin(2\pi t) + x_2 \sin(4\pi t). \tag{7}$$

We make the assumption that $m \geq 5$ Here, the goal compute the solution $x^\star$ of (6), but more explicitly than before.

1. Show the following complex numbers identity for any $k \in \mathbb{Z}$:

$$\sum_{j=1}^{m} e^{i2\pi k t_j} = \begin{cases} m & \text{if } m \text{ is a divisor of } k, \\ 0 & \text{else.} \end{cases} \tag{8}$$

Remark: *Here, i denotes the imaginary number, i.e. $i^2 = -1$.*

Hint: *Recall the formula for a geometric sum:*

$$\forall z \in \mathbb{C} \text{ such that } z \neq 1, \quad \sum_{j=1}^{m} z^j = z\frac{1-z^m}{1-z}.$$

2. We define the Fourier basis $(\omega_1, \dots, \omega_m) \subset \mathbb{C}^n$ defined by $\omega_k = [e^{i2\pi k t_1}, \dots, e^{i2\pi k t_m}]^\top$, for $k = 1, \dots, m$. Conclude that this basis is orthogonal, i.e. $\omega_{k_1} \cdot \omega_{k_2} = 0$ for $k_1 \neq k_2$.

Remark: *Since we are dealing with complex numbers, the dot product is defined as $z_1 \cdot z_2 := \bar{z_1}^\top z_2$, where $\bar{z_1}$ is the complex conjugate of z_1.*

3. Use the result from the first question to show that:

$$\sum_{j=1}^{m} (\sin(2\pi t_j))^2 = \frac{m}{2}, \tag{9a}$$

$$\sum_{j=1}^{m} (\sin(4\pi t_j))^2 = \frac{m}{2}, \tag{9b}$$

$$\sum_{j=1}^{m} \sin(2\pi t_j)\sin(4\pi t_j) = 0. \tag{9c}$$

Hint: *Recall that for all $\theta \in \mathbb{R}$: $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.*

4. Use the previous result to show that the optimization problem (6) is equivalent to (in the sense that they have the same solution):

$$\underset{x\in\mathbb{R}^2}{\text{minimize}} \quad \frac{x_1^2+x_2^2}{2} - \frac{2x_1}{m}\sum_{j=1}^{m} f(t_j)\sin(2\pi t_j) - \frac{2x_2}{m}\sum_{j=1}^{m} f(t_j)\sin(4\pi t_j). \tag{10}$$

5. Give a closed-form expression of the solution $x^\star = (x_1^\star, x_2^\star)$ to the optimization problem (10).

Exercise 4: A variance estimation problem (8 points)

In this exercise, we will study a famous estimation method called the *Maximum Likelihood Estimation* problem, in the particular case of a variance estimation problem.
Here, the measurements $y_1, \dots, y_m$ are measurements, and are generated through the stochastic model $y_j \sim \mathcal{N}(0, \sigma^2)$, i.e. the probability density function of y_j is given by:

$$p(y_j) = f(y_j; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y_j^2}{2\sigma^2}\right)$$

In order to to estimate the parameter σ from the measurements $y_1, \dots, y_m$, one classical approcah is to solve the following problem:

$$\underset{\sigma\in\mathbb{R}_{>0}}{\text{maximize}} \quad \prod_{j=1}^{m} f(y_j; \sigma^2) \tag{11}$$

In this exercise, we will assume that at least one measurement is such that $y_j \neq 0$.

1. Show that the problem (11) can be reformulated as the following *minimization* problem:

$$\underset{\sigma\in\mathbb{R}_{>0}}{\text{minimize}} \quad \sum_{j=1}^{m} \frac{y_j^2}{\sigma^2} + \log(\sigma^2) \tag{12}$$

   Hint: *Maximizing a function $f(x)$ is equivalent to minimizing the funcion $-2\log(f(x)) + \text{cst}$ for some constant* cst.

2. Let $\bar{v} > 0$ be a positive scalar value. Find the solution(s) of the following optimization problem:

$$\underset{v\in\mathbb{R}_{>0}}{\text{minimize}} \quad \frac{\bar{v}}{v} + \log(v)$$

   Hint: *Sketch a graph of the function $v \to \frac{\bar{v}}{v} + \log(v)$ by looking at its derivative.*

3. Use the two previous questions to find the solution $\hat{\sigma}$ of the problem (11).

Programming tasks (4 bonus points)

Open the jupyter notebook `programming_exercise1.ipynb`, and fill in the missing parts of the code (related to Exercise 2 and Exercise 4).