Department for
Applied Mathematics

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Basics in Applied Mathematics

Sheet 1 – 16.10.2025 (submission until 21.10.2025)

Task 1 (8 points)**.** Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with $a, b, c \in \mathbb{R}$, such that $\det A \neq 0$. Determine $\operatorname{cond}_1(A)$, $\operatorname{cond}_2(A)$ and $\operatorname{cond}_\infty(A)$ and discuss for which ratios of a, b and c the corresponding linear equation systems are ill conditioned.

Task 2 (8 points)**.** Let $A \in \mathbb{R}^{n\times n}$ be invertible and let $\|\cdot\|$ be an induced operator norm on $\mathbb{R}^{n\times n}$. Show that

$$\|A^{-1}\| = \big(\inf_{\|x\|=1} \|Ax\|\big)^{-1}$$

and $\|A^{-1}\| \geq \|A\|^{-1}$ hold.

Task 3 (8 points)**.** Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix, i.e. $x^\mathsf{T} Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.
(i) Show that A is regular.
(ii) Show that for all $1 \leq k \leq n$ the $k \times k$ submatrix $A_k = (a_{ij})_{1\leq i,j\leq k}$ is also positive definite.
(iii) Show that all real eigenvalues of A are positive.

Task 4 (8 bonus points)**.** Let $A \in \mathbb{R}^{n\times n}$, a lower triangular matrix L and an upper triangular matrix U with $A = LU$ be given. Show that, for $k = 1, 2, \ldots, n$ and the left, upper $k \times k$ submatrices A_k, L_k and U_k of A, L and U respectively, the decomposition $A_k = L_k U_k$ also holds.

Project 1 (4 bonus points)**.** Write a program that calculates the operator norm $\|\cdot\|_\infty$ of a matrix $A \in \mathbb{R}^{m\times n}$. Measure manually, for the Hilbert matrix $H \in \mathbb{R}^{n\times n}$ with entries $h_{ij} = 1/(i+j-1)$, $1 \leq i, j \leq n$, the runtimes of the programs for $n = 10^k$, $k = 1, 2, \ldots, 4$.