Department for
Applied Mathematics

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Basics in Applied Mathematics

Sheet 2 – 21.10.2025 (submission until 28.10.2025)

Task 1 (8 points)**.** Let $A \in \mathbb{R}^{m\times n}$.
(i) Show that $(\operatorname{Im} A^\mathsf{T})^\perp = \ker A$ with

$$V^\perp = \{v \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } w \in V\}$$

for $V \subset \mathbb{R}^n$ and conclude $\mathbb{R}^n = \operatorname{Im} A^\mathsf{T} + \ker A$.
(ii) Prove the dimension formula $n = \dim(\operatorname{Im} A) + \dim(\ker A)$ and conclude that $\operatorname{rank} A = \operatorname{rank} A^\mathsf{T}$, where for a matrix M the column rank of M is defined by $\operatorname{rank} M = \dim \operatorname{Im} M$.
(iii) Show that

$$\ker A^\mathsf{T} A = \ker A.$$

Task 2 (8 points)**.** Let $A \in \mathbb{R}^{2\times 1}$ and $b \in \mathbb{R}^2$ be defined by

$$A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Determine graphically the solution of the least squares problem by orthogonally decomposing b into vectors v, w with $v \in \operatorname{Im} A$ and $w \in \ker A^\mathsf{T}$.

Task 3 (8 points)**.** Calculate using the Householder method a QR decomposition for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2}/2 \\ 0 & \sqrt{2} & 5/\sqrt{2} \end{bmatrix}$$

and solve the equation $Ax = b$ for $b = [3\sqrt{2}, -1, 7]^\mathsf{T}$.

Task 4 (8 points)**.** (i) Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$ be an eigenvalue of A. Prove the following statements:
(a) The number $\overline{\lambda}$ is an eigenvalue of A.
(b) If A is symmetric, then the eigenvalues of A are real.
(c) If A is regular, then λ^{-1} is an eigenvalue of A^{-1}.
(d) The matrix A^T has the eigenvalue λ.
(ii) Let $A, B \in \mathbb{R}^{n\times n}$ be matrices with eigenvalues λ and μ. Under what conditions is $\lambda\mu$ an eigenvalue of AB?

Project 1 (4 bonus points)**.** From physics, it is known that bodies exposed only to gravity fly in parabolas. A body has the initial velocity $v = (v_x, v_y)$ and is at point 0 at time $t = 0$. At time t it is then at the location $x = v_x t$, $y = v_y t - \frac{1}{2} g t^2$, where g is the acceleration due to gravity. In a series of experiments, the values given in Table 1 were measured. Formulate a suitable least squares problem and solve it in Matlab or Python using the provided QR routine, to determine the velocity v_y and the acceleration due to gravity g as accurately as possible. Create a graph using a plot command, in which the

i	1	2	3	4	5	6	7
$t_i[s]$	0.1	0.2	0.6	0.9	1.1	1.2	2.0
$x_i[m]$	0.73	1.28	4.24	6.11	7.69	8.21	13.83
$y_i[m]$	0.96	1.81	4.23	5.05	5.15	4.81	0.55

Tabelle 1. Measurement values of an experimental series

measured values and the calculated parabola are listed. To what accuracy is it meaningful to specify the results? What model errors, data errors and measurement errors occur in this experiment?