Department for
Applied Mathematics

WiSe 2025/2026

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Basics in Applied Mathematics

Sheet 4 – 4.11.2025 (submission until 11.11.2025)

Task 1 (8 points)**.** Show that the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$ of the $n \times n$ matrix

$$A = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$$

is given by $p(\lambda) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)$.

Task 2 (8 points)**.** Determine the k-th iterate of the power method for the matrix

$$A = \begin{bmatrix} 0 & 2 & & \\ & \ddots & \ddots & \\ & & \ddots & 2 \\ 2 & & & 0 \end{bmatrix}$$

with the starting vectors $x_0 = [1, 0, \dots, 0]^\mathsf{T}$ and $x_0 = [1, 1, \dots, 1]^\mathsf{T}$ and discuss the validity of the assumptions of the convergence result.

Task 3 (8 points)**.** Construct a matrix $M \in \mathbb{R}^{2\times 2}$, which is a contraction with respect to an operator norm and not with respect to another.

Task 4 (8 points)**.** Perform 5 steps of the Richardson, Jacobi and Gauss-Seidel methods for

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

with $\omega = 1$ and $\omega = 1/10$ and $x^0 = [1, 1, 1]^\mathsf{T}$ respectively. Compare the iterates with the exact solution of the system of equations.

Project 1 (4 bonus points)**.** To illustrate Google's PageRank algorithm, a model internet with N pages is considered. Let n_i be the number of links leading from the i-th page to other pages. The variable $x_i \geq 0$ is supposed to indicate the relevance of the i-th page and for each link leading from the j-th to the i-th page, it increases by the value x_j/n_j. In the sketch shown in Figure 1, for example, we have

$$x_1 = \frac{0}{2}x_2 + \frac{1}{3}x_3 + \frac{2}{4}x_4.$$

Overall, a system of linear equations for determining the vector $x = [x_1, x_2, \dots, x_N]^\mathsf{T}$, which describes a balance state of the proportional page accesses of a group of users, is

established when they repeatedly switch between pages at random.
(i) Show that the determination of a solution of the system of equations can be formulated as an eigenvalue problem $\lambda x = Ax$ with $\lambda = 1$.
(ii) Determine the Gerschgorin circles for A^{T}, to show that $|\lambda| \leq 1$ for all eigenvalues of A, and prove that $\lambda = 1$ is an eigenvalue of A^{T} or A.
(iii) Determine with the help of Matlab an eigenvector x of the matrix A for the eigenvalue 1 with $x_i \geq 0$, $i = 1, 2, \ldots, N$, and $\|x\|_1 = 1$ for the model internet shown in Figure 1.
(iv) Perform 5 steps of the power method with the starting vector $x_0 = [1, 1, 1, 1]^{\mathsf{T}}/4$ and normalise with respect to the norm $\|\cdot\|_1$.
(v) Discuss whether the matrix A can be assumed to be sparse in reality and whether the effort can be reduced by using suitable storage formats and algorithms for matrix-vector multiplication.

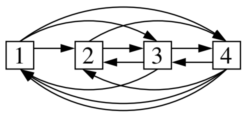

ABBILDUNG 1. Links in a model internet