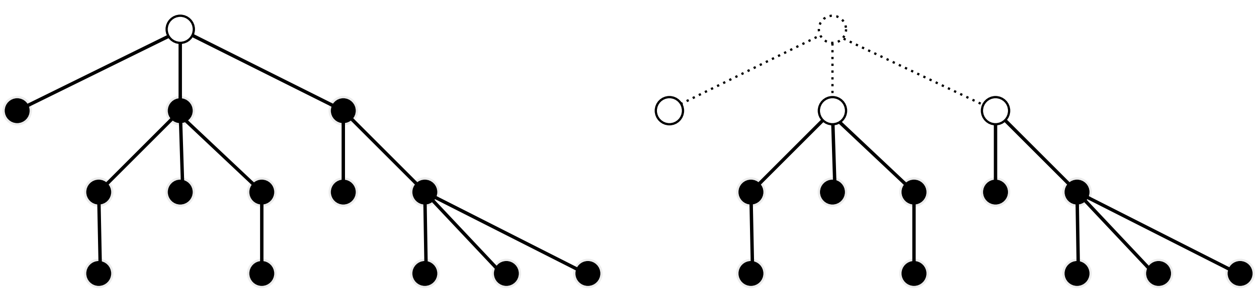
Trees and Lagrange inversion

Contents

1 Recursive specifications

The prototypical *recursive* structure is a tree.

Definition. A plane tree is the embedding of a graph without cycles into the plane. Such a tree is rooted if one of its vertices is specified (the root vertex). Since the tree is embedded in the plane, the children of each node have a unique ordering (say clockwise). If the node has a child, then one of its children is specified to be the left-most child, So its children go from left to right. The size of a rooted plane tree is the number of vertices it contains.



The enumeration of trees is best done recursively (there are other sneakier ways). Take any tree you like — and delete its root. One is left with a "forest" of trees — possibly empty. This forest consists of a (possibly empty) sequence of trees — each rooted at the vertex which was attached to the original root.

1.1 Plane Binary Trees $\mathcal{B}$

A **binary tree** is a rooted tree where every node has either 0 or 2 children. Let $\mathcal{B}$ as the combinatorial class of all **non-empty plane binary trees**, where the size of a tree is its number of nodes.

1.1.1 A recursive specification

We can describe $\mathcal{B}$ with a recursive combinatorial description. A binary tree is either a node, or a node and an ordered pair of subtrees. This gives the following specification.

$$\mathcal{B} \equiv \mathcal{Z} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B} \qquad \text{i.e.} \quad \mathcal{B} \equiv \mathcal{B} \times \left(\mathcal{E} + \mathcal{B}^2\right).$$

By the sum and the product rules, the generating function $B(z)$ for $\mathcal{B}$ satisfies

$$B(z) = Z(z) + Z(z)B(z)B(z) = z + zB(z)^2.$$

We can solve for $B(z)$ with the quadratic formula.

$$zB(z)^2 - B(z) + z = 0$$
$$B(z) = \frac{1 \pm \sqrt{1-4z^2}}{2z}.$$

Maple shows that only one of these two solutions expands to the desired power series $B(z)$.

```
> series((1+sqrt(1-4*z^2))/(2*z),z,15);
```
$$z^{-1} - z - z^3 - 2z^5 - 5z^7 - 14z^9 - 42z^{11} - 132z^{13} + O(z^{15}) \quad (1)$$
```
> series((1-sqrt(1-4*z^2))/(2*z),z,15);
```
$$z + z^3 + 2z^5 + 5z^7 + 14z^9 + 42z^{11} + 132z^{13} + O(z^{15}) \quad (2)$$

In Lecture 2, we applied the extended binomial theorem to find an explicit expression for the coefficients of a similar power series. There we defined the k-th Catalan number to be

$$C_k = [z^k]\frac{1-\sqrt{1-4z}}{2z} = \frac{1}{k+1}\binom{2k}{k}.$$

The power series $B(z)$ is obtained from this by replacing z with z^2 and multiplying by z.

$$C_k = [z^{2k}]\frac{1-\sqrt{1-4z^2}}{2z^2}$$
$$= [z^{2k+1}]\frac{1-\sqrt{1-4z^2}}{2z}$$

Therefore the number of plane binary trees with n nodes is

$$B_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ C_k = \frac{1}{k+1}\binom{2k}{k} & \text{if } n = 2k+1 \end{cases}.$$

Exercise. Compute b_{2m+1} for small m, and verify that it gives the right number of binary trees.

1.2 All Plane Rooted Trees

We now consider the class $\mathcal{T}$ of all non-empty rooted plane trees.

1.2.1 Counting

Every plane rooted tree is described as a root vertex with a (possibly empty) finite sequence of rooted plane subtrees trees. We can count these trees similarly to the plane binary trees.

$$\mathcal{T} = \mathcal{Z} \times \textsc{Seq}(\mathcal{T})$$
$$T(z) = \frac{z}{1-T(z)}$$
$$T(z)^2 - T(z) + z = 0$$
$$T(z) = \frac{1 \pm \sqrt{1-4z}}{2}$$

Again the "+" results in negative coefficients. Again get the Catalan numbers (shifted by 1).

$$T(z) = z\,\frac{1-\sqrt{1-4z}}{2z} = z\sum_{n\geq 0} C_n z^n$$
$$T_n = [z^n]T(z) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

That is, the number of plane trees with n nodes equals the $(n-1)$th Catalan number C_{n-1}.

1.2.2 Generation

We can step through this recursion slowly to generate the elements of $\mathcal{T}$. We rewrite the recursion as

$$\mathcal{T}^{[m+1]} = \mathcal{Z} \times \textsc{Seq}(\mathcal{T}^{[m]}).$$

We start by taking $\mathcal{T}^{[0]} = \{\circ\}$, then

$$\mathcal{T}^{[1]} = \{\circ, \circ(\circ), \circ(\circ,\circ), \circ(\circ,\circ,\circ), \dots\}$$
$$\mathcal{T}^{[2]} = \{\circ, \circ(\circ), \circ(\circ,\circ), \circ(\circ(\circ)), \circ(\circ,\circ(\circ)), \circ(\circ(\circ),\circ)\dots\}$$

Note that $\mathcal{T}^{[m]}$ encodes the set of plane trees trees of depth at most m, and

$$\mathcal{T} = \cup_{m\geq 0}\mathcal{T}^{[m]}.$$

Exercise. We have seen that $[z^{2n-1}]B(z) = [z^n]T(z)$ (the$(n-1)$th Catalan number). That is, the number of plane binary trees with $2n-1$ nodes equals the number of plane rooted trees with n nodes. Can you find a natural combinatorial bijection between B_{2n-1} and T_n?

2 Simple trees and Lagrange Inversion

2.1 Restricting the out-degree

Above, we concidered the plane binary trees and the plane rooted trees. In each case a tree is specified by a root vertex whose deletion leaves an ordered sequence of (smaller) rooted trees. This allowed us find their generating functions and extract the coefficients.

$$\mathcal{B} = \mathcal{Z} \times \left(\mathcal{E} + \mathcal{B}^2\right) \qquad\qquad \mathcal{T} = \mathcal{Z} \times \textsc{Seq}(\mathcal{T})$$
$$B(z) = z\left(1 + B(z)^2\right) \qquad\qquad T(z) = z\,\frac{1}{1-T(z)}$$
$$B(z) = \frac{1-\sqrt{1-4z^2}}{2z} \qquad\qquad T(z) = \frac{1-\sqrt{1-4z}}{2}$$
$$B_{2k} = 0 \;\;\&\;\; B_{2k+1} = C_k \qquad\qquad T_n = C_{n-1}$$

Both of these derivations take the following form.

$$\mathcal{A} = \mathcal{Z} \times \psi(\mathcal{A}) \tag{1}$$
$$A(z) = z\,\phi(A(z)) \tag{2}$$
$$A(z) = f(z) \tag{3}$$
$$A_n = [z^n]f(z). \tag{4}$$

In (1), the expression $\psi(\mathcal{A})$ involves $\mathcal{A}$ and admimissible operators like $+$, $\times$ and $\textsc{Seq}()$. In (2), $\phi(u)$ is an algebraic expression corresponding to ψ and evaluated at $u = A(z)$. In both cases, we were able to find the function f in (3) with the quadratic formula, but in general $f(z)$ is not easy to find. In the next section, we present a method called Lagrange Inversion for deriving (4) directly from (2) that avoids (3) altogether.

Each of the constructions above consist of a root node and a sequence of subtrees, where each node has its out-degree restricted to a specific subset

$$\Omega \subseteq \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}.$$

We can define the class of Ω-restricted trees $\mathcal{T}^{\Omega}$ to be the set of finite non-empty rooted plane trees whose out-degrees all belong to Ω. If we take $\Omega = \mathbb{Z}^+$, then we get all plane rooted trees. If we take $\Omega = \{0, 2\}$ we ge the plane binary trees.

The function ϕ appearing in (2) depends only on Ω. In particular,

$$\phi(u) = \phi_\Omega(u) = \sum_{\omega\in\Omega} u^\omega.$$

Here are expressions for Ω, ψ and ϕ for some classes $\mathcal{T} = \mathcal{T}^\Omega$ of Ω-restricted plane trees:

Type of plane tree	Ω	$\psi(\mathcal{T})$	$\phi(u)$
full binary	$\{0,2\}$	$\mathcal{E}+\mathcal{T}^2$	$1+u^2$
unary-binary	$\{0,1,2\}$	$\mathcal{E}+\mathcal{T}+\mathcal{T}^2$	$1+u+u^2$
all plane trees	$\mathbb{Z}_{\geq 0}$	$\textsc{Seq}(T)$	$1/(1-u)$
even outdegrees	$\{0,2,4,\dots\}$	$\textsc{Seq}(T^2)$	$1/(1-u^2)$

For Ω-restrcted trees, equation (2) specializes as follows.

Lemma. *The OGF, $T^\Omega(z)$, for the class of Ω-restricted plane trees satisfies the following equation*

$$T^\Omega(z) = z\,\phi(T^\Omega(z))$$

where $\phi(u) = \sum_{\omega\in\Omega} u^\omega$.

A class of trees that satisfies such an equation is called a simple variety of trees.

2.2 Lagrange Inversion

The functional relation $T(z) = z\,\phi(T(z))$ can be written

$$z = \frac{T(z)}{\phi(T(z))}.$$

That is, the function T takes some number z and turns it into $T(z)$. If $T(z) = z\,\phi(T(z))$, then you can recover z from $T(z)$ (ie. functional inverse of ϕ) by computing $T(z)/\phi(T(z))$. In this case, the functional form can be exploited in order to get an exact expression for the coefficients T_n of $T(z)$ — this uses something called the Lagrange inversion formula.

Theorem (Lagrange inversion)**.** *The coefficients of an inverse function and all of its powers are determined by the coefficients of powers of the forward function. In particular, if $z = T(z)/\phi(T(z))$, then*

$$[z^n]\,T(z) = \frac{1}{n}[u^{n-1}]\phi(u)^n$$

More generally, for any positive integer k,

$$[z^n]\,T(z)^k = \frac{k}{n}[u^{n-k}]\phi(u)^n.$$

Applying this to a simple variety $\mathcal{T}^\Omega$ of trees, the number of trees with size n is

$$T_n^\Omega = \frac{1}{n}[u^{n-1}]\phi(u)^n, \quad \text{where } \phi(u) = \sum_{\omega\in\Omega} u^\omega.$$

Considerably more general forms of this theorem exist, but this suffices for our purposes.

Example. All Plane Trees: Here $\mathcal{T} = \mathcal{T}^\Omega$ where $\Omega = \{0,1,2,\dots\}$. The recursive combinatorial specification $\mathcal{T} = z \times \textsc{Seq}(\mathcal{T})$ implies that the OGF satisfies

$$T(z) = \frac{z}{1-T(z)} = z\,\phi(T(z)), \qquad \text{where} \quad \phi(u) = \frac{1}{1-u}.$$

That is, We proceed as follows

$$\begin{aligned}
\phi(u)^n &= \frac{1}{(1-u)^n} = \sum_{t=0}^{\infty}\binom{n+t-1}{t}u^t && \text{(extended binomial theorem)}\\
[u^t]\phi(u)^n &= \binom{n+t-1}{t}\\
[u^{n-1}]\phi(u)^n &= \binom{2n-2}{n-1}\\
T_n &= \frac{1}{n}[u^{n-1}]\phi(u)^n = \frac{1}{n}\binom{2n-2}{n-1}. && \text{(the } (n-1)\text{th Catalan number)}
\end{aligned}$$

Example. Binary Plane Trees: Here $\mathcal{B} = T^{\Omega}$ where $\Omega = \{0, 2\}$.

$$\begin{aligned}
\mathcal{B} &= \mathcal{Z}\times\left\{\mathcal{E}+\mathcal{B}^2\right\}\\
B(z) &= z\,(1+B(z)^2)\\
B(z) &= z\,\phi(B(z)), \qquad \text{where} \quad \phi(u) = 1+u^2\\
\phi(u)^n &= (1+u^2)^n = \sum_{k=0}^{n}\binom{n}{k}u^{2k}\\
[u^t]\phi(u)^n &= \begin{cases} 0 & \text{if } t \text{ is odd}\\ \binom{n}{k} & \text{if } t = 2k\end{cases}\\
B_n &= \frac{1}{n}[u^{n-1}]\phi(u)^n = \begin{cases} 0 & \text{if } n-1 \text{ is odd (i.e. } n \text{ is even)}\\ \frac{1}{2k+1}\binom{2k+1}{k} & \text{if } n-1 = 2k \ \ \text{(i.e. } n = 2k+1).\end{cases} && \text{(use } t = n-1)
\end{aligned}$$

This last expression is the kth Catalan number in disguise.

$$B_{2k+1} = \frac{1}{2k+1}\frac{(2k+1)!}{k!(k+1)!} = \frac{1}{k+1}\frac{(2k)!}{k!k!} = \frac{1}{k+1}\binom{2k}{k} = C_k.$$

Example. Unary-Binary Plane Trees: $\mathcal{U} = T^{\Omega}$ where $\Omega = \{0, 1, 2\}$. This one is more challenging. The trinomial theorem is needed here.

$$(a+b+c)^n = \sum_{\substack{0\le i,j,k\le n\\ i+j+k=n}}\frac{n!}{i!\,j!\,k!}a^ib^jc^k = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!\,j!\,(n-i-j)!}a^ib^jc^{n-i-j}$$

When $n = i+j+k$ we may write this using the coefficient extraction operator.

$$[a^ib^jc^k](a+b+c)^n = \frac{n!}{i!\,j!\,k!}.$$

The last expression above is often written as $\binom{i+j+k}{i,\,j,\,k}$ and is called a *trinomial coefficient*. It counts the ways of putting n labeled balls into three labeled boxes containing i, j and k balls, repectively.

We proceed as before.

$$\begin{aligned}
\mathcal{U} &= \mathcal{Z}\times\left\{\mathcal{E}+\mathcal{U}+\mathcal{U}^2\right\}\\
U(z) &= z\,(1+U(z)+U(z)^2)\\
U(z) &= z\,\phi(U(z)) \qquad \text{where} \quad \phi(u) = 1+u+u^2\\
\phi(u)^n &= (u^2+u+1)^n = \sum_{\substack{0\le i,j,k\le n\\ i+j+k=n}}\binom{n}{i,\,j,\,k}\left(u^2\right)^i u^j 1^j && \text{(using } a = u^2, b = u, c = 1)\\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!\,j!\,(n-i-j)!}u^{2i+j}.
\end{aligned}$$

To extract a particular coefficient, say $[u^t]\phi(u)^n$, we need to select from the double-sum those index pairs (i, j) , $i, j \geq 0$ for which $2i + j = t$. Here i can be any number in $\{0, 1, \ldots, \lfloor t/2 \rfloor\}$, which gives $j = t - 2i$ and $n - i - j = n - i - (t - 2i) = n - t + i$.

$$[u^t]\phi(u)^n = \sum_{i=0}^{\lfloor t/2 \rfloor} \frac{n!}{i!\,(t-2i)!\,(n-t+i)!}$$

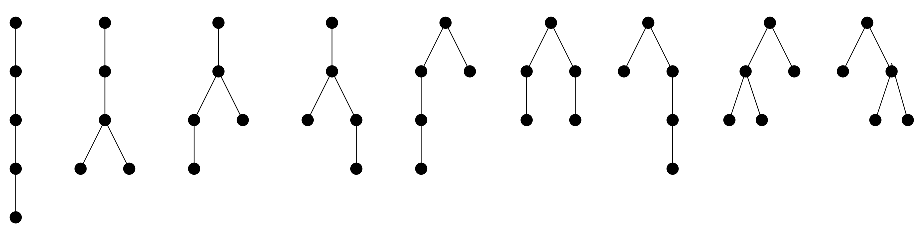
Now we can apply the Lagrange inversion formula.

$$\begin{aligned} U_n &= \frac{1}{n}[u^{n-1}]\phi(u)^n \\ &= \frac{1}{n} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{n!}{i!\,(n-1-2i)!\,(n-(n-1)1+i)!} \\ &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{(n-1)!}{i!\,(n-1-2i)!\,(i+1)!}. \end{aligned}$$

This formula can not really be further simplified. It is remarkable that Maple can express this last sum in terms of “hypergeometric functions”.

We can check number of unary-binary plane trees with 5 nodes.

$$\begin{aligned} U_5 &= \sum_{i=0}^{2} \frac{4!}{i!\,(4-2i)!\,(i+1)!} \\ &= \frac{4!}{0!\,4!\,1!} + \frac{4!}{1!\,2!\,2!} + \frac{4!}{2!\,0!\,3!} \\ &= 1 + 6 + 2 \\ &= 9. \end{aligned}$$



```
> #######################
  # Unary-Binary Trees #
  #######################
> # The generating function U(z)

  solve(U=z*(1+U+U^2), U);
  series([%][2],z,13);
```

$$\frac{1}{2}\,\frac{-z+1+\sqrt{-3z^2-2z+1}}{z},\ -\frac{1}{2}\,\frac{z-1+\sqrt{-3z^2-2z+1}}{z}$$

$$z+z^2+2z^3+4z^4+9z^5+21z^6+51z^7+127z^8+323z^9+835z^{10}+2188z^{11}+O(z^{12})$$

```
> # Check whether summation formula that we found by hand
  # using Lagrange Inversion appears to be correct:

  Usum:= n -> add((n-1)!/i!/(i+1)!/(n-1-2*i)!,i=0..(n-1)/2);
  seq( Usum(n), n=0..12 );
```

$$Usum := n \rightarrow add\left(\frac{(n-1)!}{i!\,(i+1)!\,(n-1-2i)!}, i=0..\frac{1}{2}n-\frac{1}{2}\right)$$

$$0, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798$$

```
> # Ask Maple to evaluate this sum in closed form
  # and check whether it appears to evaluate correctly, to 4 decimal places

  Sum((n-1)!/i!/(i+1)!/(n-1-2*i)!,i=0..(n-1)/2) = sum((n-1)!/i!/(i+1)!/(n-1-2*i)!,i=0..((n-1)/2));
  Uhyper:=unapply(rhs(%),n);
  seq( evalf(g(n),4), n=0..12);
```

$$\sum_{i=0}^{\frac{1}{2}n-\frac{1}{2}} \frac{(n-1)!}{i!\,(i+1)!\,(n-1-2i)!} = \mathrm{hypergeom}\left(\left[-\frac{1}{2}n+1, -\frac{1}{2}n+\frac{1}{2}\right], [2], 4\right)$$

$$Uhyper := n \rightarrow \mathrm{hypergeom}\left(\left[-\frac{1}{2}n+1, \frac{1}{2}-\frac{1}{2}n\right], [2], 4\right)$$

$$0.5000 - 0.8660\,I, 1., 1., 2.000, 4.000, 9.000, 21.00, 51.00, 127.0, 323.0, 835.0, 2188., 5798.$$

```
> #This appears to be correct, although the first term is an anomoly.
```

Exercise. Show that when $\Omega = \{0, 1, k\}$ (Unary - k-ary Trees) we get the formula

$$T_n^{\Omega} = \sum_{i=0}^{\lfloor (n-1)/k \rfloor} \frac{(n-1)!}{i!\,(ki-i+1)!\,(n-1-ki)!}.$$

2.3 Trees with coloured nodes

Sometimes we need to consider trees which have nodes of different colours. For example, in computer science we often see a data structure called a *red-black tree*. Here is an example which we use throughout this subsection.

Example. Let $\mathcal{T}$ be the combinatorial class of plane trees where each node is either red or blue, where

- every blue node has at most one blue child and at most one red child,
- every red node has no red children and an even number of blue children.

The size of a tree in $\mathcal{T}$ is its number of nodes, regardless of colour. We wish to find a combinatorial specification for $\mathcal{T}$, so that we can find its OGF and count them.

Evidently we are treating those trees in $\mathcal{T}$ with a red root differently from those trees with a blue root. Therefore we need to define two more combinatorial classes. Let $\mathcal{T}_b$ be those trees in $\mathcal{T}$ having a blue root node (call them *blue trees*), and let $\mathcal{T}_r = \mathcal{T} - \mathcal{T}_b$ be the *red trees*.

Comparing to Section 2.2 of Lecture 5, our task is to find a specification for the triplet of classes

$$\mathcal{A} = (\mathcal{T}, \mathcal{T}_b, \mathcal{T}_r)$$

The specification consists of a set of three combinatorial constructions, one for each of $\mathcal{T}$, $\mathcal{T}_b$ and $\mathcal{T}_r$. Blue nodes and red nodes each contribute 1 toward the size of a tree, but they may need to be treated differently, so the specification will use two atomic classes, say $\mathcal{Z}_b$, $\mathcal{Z}_r$, and the neutral class $\mathcal{E}$.

The first equation comes from observing that every tree in $\mathcal{T}$ is either a blue tree or a red tree, but not both.

$$\mathcal{T} = \mathcal{T}_r + \mathcal{T}_b \tag{5}$$

Each blue tree is uniquely specified to consist of a blue root together with either nothing, or a blue subtree, or a red subtree, or a blue subtree followed by a red subtree, or a red subtree followed by a blue subtree. (The last two cases are different since we are talking about plane trees.)

$$\mathcal{T}_b = \mathcal{Z}_b \times (\mathcal{E} \;+\; \mathcal{T}_b \;+\; \mathcal{T}_r \;+\; \mathcal{T}_b \times T_r \;+\; \mathcal{T}_r \times T_b) \tag{6}$$

Each red tree is uniquely specified to consist of a red root together with a sequence of pairs or blue subtrees.

$$\mathcal{T}_r = \mathcal{Z}_r \times \textsc{Seq}(\mathcal{T}_b \times \mathcal{T}_b). \tag{7}$$

We have completed our specification for $(\mathcal{T}, \mathcal{T}_b, \mathcal{T}_r)$.

Our next task is to translate these specifications to three equations relating the three OGFs $T(z)$, $T_b(z)$, $T_r(z)$ in the usual way.

$$T(z) = T_b(z) + T_r(z) \tag{8}$$
$$T_b(z) = z(1 + T_b(z) + T_r(z) + 2T_b(z)\,T_r(z)) \tag{9}$$
$$T_r(z) = \frac{z}{1 - T_b(z)^2}. \tag{10}$$

The third task is to solve these equations for $T(z)$, $T_b(z)$ and $T_r(z)$. Substituting (6) into (5) and simplifying gives

$$\begin{aligned}
T_b(z) &= z + zT_b(z) + \frac{z^2}{1 - T_b(z)^2} + \frac{2z^2T_b(z)}{1 - T_b(z)^2} \\
T_b(z) - T_b(z)^3 &= (z - zT_b(z)^2) + (zT_b(z) - zT_b(z)^3) + z^2 + 2z^2T_b(z) \\
0 &= (1 - z)T_b(z)^3 - zT_b(z)^2 + (2z^2 + z - 1)T_b(z) + z^2 + z
\end{aligned}$$

With huge effort this equation can be solved by hand to get

$$\begin{aligned}
T_b(z) = {} & \frac{-2\,z + \sqrt[3]{36\,z^4 - 80\,z^3 - 36\,z^2 + 72\,z + 12\,(z-1)\,\sqrt{(z+1)\,(-96\,z^6 + 9\,z^5 + 129\,z^4 - 72\,z^3 - 36\,z^2 + 60\,z - 12)}}}{6\,z - 6} \\
& + \frac{12\,z^3 - 4\,z^2 - 12\,z + 6}{(3\,z - 3)\sqrt[3]{36\,z^4 - 80\,z^3 - 36\,z^2 + 72\,z + 12\,(z-1)\,\sqrt{(z+1)\,(-96\,z^6 + 9\,z^5 + 129\,z^4 - 72\,z^3 - 36\,z^2 + 60\,z - 12)}}}
\end{aligned}$$

However we are much better off to let Maple handle this part. We find, for example, that exactly 343 trees in $\mathcal{T}$ have 7 nodes.

```
> ##############################################################
  # These plane trees have blue nodes and red nodes.
  # Each blue node has a tmost one black child and at most one red child
  # Each red node has no red children, and an even number of blue children
  unassign('Tb','Tr','T'):

  equations := { T  = Tb + Tr,
                 Tb = z*(1 + Tb + Tr + 2*Tb*Tr),
                 Tr = z/(1-Tb*Tb)
                 };
  solve( equations, {T, Tb, Tr} ):
  assign(op(%));
```

$$equations := \left\{T = Tb + Tr, Tb = z\,(2\,Tb\,Tr + Tb + Tr + 1), Tr = \frac{z}{-Tb^2+1}\right\}$$

```
> # We can print the OGFs for Tb.
  # The first expression is in terms of the roots of a cubic polynomial
  # The second is in radical form, but very complicated
  # The OGF for Tr (not shown here) is even more complicated

  'Tb' = Tb;
  'Tb' = convert(Tb, radical) ;
```

$$Tb = RootOf\left((z-1)\,\_Z^3 + z\,\_Z^2 + \left(-2\,z^2 - z + 1\right)\_Z - z^2 - z\right)$$

$$Tb = \frac{1}{6}\,\frac{1}{z-1}\Big(36\,z^4 - 80\,z^3 + 12\sqrt{-96\,z^7 - 87\,z^6 + 138\,z^5 + 57\,z^4 - 108\,z^3 + 24\,z^2 + 48\,z - 12}\;z - 36\,z^2 - 12\sqrt{-96\,z^7 - 87\,z^6 + 138\,z^5 + 57\,z^4 - 108\,z^3 + 24\,z^2 + 48\,z - 12} + 72\,z\Big)^{1/3} + \frac{2}{3}\left(6\,z^3 - 2\,z^2 - 6\,z + 3\right)\Big/\Big((z-1)\big(36\,z^4 - 80\,z^3 + 12\sqrt{-96\,z^7 - 87\,z^6 + 138\,z^5 + 57\,z^4 - 108\,z^3 + 24\,z^2 + 48\,z - 12}\;z - 36\,z^2 - 12\sqrt{-96\,z^7 - 87\,z^6 + 138\,z^5 + 57\,z^4 - 108\,z^3 + 24\,z^2 + 48\,z - 12} + 72\,z\big)^{1/3}\Big) - \frac{1}{3}\,\frac{z}{z-1}$$

```
> Tb_ser := series( Tb, z, 11);
  Tr_ser := series( Tr, z, 12);
  T_ser  := series( T, z, 12);
```

$$Tb\_ser := z + 2\,z^2 + 4\,z^3 + 9\,z^4 + 23\,z^5 + 66\,z^6 + 204\,z^7 + 661\,z^8 + 2209\,z^9 + 7551\,z^{10} + O(z^{11})$$

$$Tr\_ser := z + z^3 + 4\,z^4 + 13\,z^5 + 42\,z^6 + 139\,z^7 + 472\,z^8 + 1634\,z^9 + 5742\,z^{10} + O(z^{11})$$

$$T\_ser := 2\,z + 2\,z^2 + 5\,z^3 + 13\,z^4 + 36\,z^5 + 108\,z^6 + 343\,z^7 + 1133\,z^8 + 3843\,z^9 + 13293\,z^{10} + O(z^{11})$$

```
> with (combstruct):
  unassign('Tb','Tr','T'):
  RBTrees1 := {T  = Union(Tb,Tr),
               Tb = Prod( Zb, Union( E, Tb, Tr, Prod(Tb,Tr), Prod(Tr,Tb) ) ),
               Tr = Prod( Zr, Sequence( Prod(Tb,Tb) )),
               Zb = Atom, Zr=Atom, E=Epsilon
              };

  OGFs := table(gfsolve( RBTrees1, unlabeled, z)):
  TbGF := OGFs[Tb(z)];
  convert( TbGF, radical);
```

$RBTrees1 := \{E = \mathrm{E}, T = Union(Tb, Tr), Tb = Prod(Zb, Union(E, Tb, Tr, Prod(Tb, Tr), Prod(Tr, Tb))), Tr = Prod(Zr, Sequence(Prod(Tb, Tb))), Zb = Atom, Zr = Atom\}$

$$TbGF := RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)$$

$$\frac{1}{6(z-1)}\left(36z^4 - 80z^3 + 12\sqrt{-96z^7 - 87z^6 + 138z^5 + 57z^4 - 108z^3 + 24z^2 + 48z - 12}\, z - 36z^2 - 12\sqrt{-96z^7 - 87z^6 + 138z^5 + 57z^4 - 108z^3 + 24z^2 + 48z - 12} + 72z\right)^{1/3} + \left(2(6z^3 - 2z^2 - 6z + 3)\right)\Big/\left(3(z-1)\left(36z^4 - 80z^3 + 12\sqrt{-96z^7 - 87z^6 + 138z^5 + 57z^4 - 108z^3 + 24z^2 + 48z - 12}\, z - 36z^2 - 12\sqrt{-96z^7 - 87z^6 + 138z^5 + 57z^4 - 108z^3 + 24z^2 + 48z - 12} + 72z\right)^{1/3}\right) - \frac{z}{3(z-1)}$$

```
> TrGF := OGFs[Tr(z)];
  #convert( TrGF, radical);

  TGF := OGFs[T(z)];
  #convert( TGF, radical);
```

$$TrGF := -\frac{1}{3z}\left(2z\,RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)^2 - 2\,RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)^2 + RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)z - 4z^2 + RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right) - z\right)$$

$$TGF := -\frac{1}{3z}\left(2z\,RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)^2 - 2\,RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)^2 - 2\,RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right)z - 4z^2 + RootOf\left((z-1)\_Z^3 + z\_Z^2 + (-2z^2 - z + 1)\_Z - z^2 - z\right) - z\right)$$

```
> gfseries( RBTrees1, unlabeled, z)[T(z)];
```

$$2z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + O(z^6)$$