Revolving door ordering

Contents

1 Minimal Change Ordering for subsets

1.1 Subsets of an n-set

Fix n, and consider the class of subsets of an n-set. We want to generate all elements of this class with a minimum change. For example, between consecutive elements in a listing, we perhaps there is a difference of a single element.

We have already explored binary strings, and we have also already explored the connection between generating binary strings and generating a subset. A Gray code for binary strings implicitly describes a minimal change exhaustive generation scheme for the set of subsets of an n-set.

Well, that was easy! Now let us make it harder.

1.2 Generating k-subsets

Let us consider a restriction: all possible $\binom{n}{k}$ k-subsets of an n-set. We can formuate this in terms of binary strings: the difference between two binary strings must be at least two, and ideally we would like to describe a scheme in which the difference is exactly 2 bits.

First, we describe a natural minimal change order for k-subsets of an n-set which we denote by $R_k(n)$. The recursive definition of $R_k(n)$ is analogous to the RBC $R(n)$. Let us use some information that we have at our disposal:

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, 1 \leq k \leq n.$$

This identity actually suggests a minimal change order for k-subsets, defined in a recursive way. RBC can be viewed as an interpretation of the identity

$$2^n = 2^{n-1} + 2^{n-1},$$

and we let this guide us. We will represent $R_k(n)$ by a matrix with n columns and $\binom{n}{k}$ rows with entries from $\{0, 1\}$.

Definition. For integers n and k with $0 \leq k \leq n$ we define the matrix $R_k(n)$ recursively as follows

$$R_0(n) = [\underbrace{0\,0\,\ldots 0}_{n}], \qquad R_n(n) = [\underbrace{1\,1\,\ldots 1}_{n}], \qquad R_k(n) = \begin{bmatrix} \mathbf{0}\; R_k(n-1) \\ \mathbf{1}\; R_{k-1}(n-1)^R \end{bmatrix} \text{ (if } 0 < k < n).$$

Here $\mathbf{0}\,M$ (resp. $\mathbf{1}\,M$) is the matrix obtained from M by adding a new column of 0's (resp. 1's) on the left. Also M^R denotes the matrix M with its rows listed in reverse order.

Example. We have $R_0(1) = [\,0\,]$, $R_0(2) = [\,0\,0\,]$, $R_1(1) = [\,1\,]$, and $R_2(2) = [\,1\,1\,]$, so

$$R_1(2) = \begin{bmatrix} \mathbf{0}\; R_1(1) \\ \mathbf{1}\; R_0(1)^R \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R_1(3) = \begin{bmatrix} \mathbf{0}\; R_1(2) \\ \mathbf{1}\; R_0(2)^R \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_2(3) = \begin{bmatrix} \mathbf{0}\; R_2(2) \\ \mathbf{1}\; R_1(2)^R \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and so

$$R_2(4) = \begin{bmatrix} \mathbf{0}\ R_2(3) \\ \mathbf{1}\ R_1(3)^R \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

If we interpret the binary strings occurring in $R_k(n)$ as k-subsets of $[n]$ with the first bit representing the appearance or nonappearance of n, the second bit $n-1$ and so on, then each subset is listed in decreasing order, and we can rephrase this definition as follows

$$R_0(n) = \emptyset, \qquad R_n(n) = \{n, n-1, \dots, 1\}, \qquad R_k(n) = R_k(n-1),\ \{n\} \cup R_{k-1}(n-1)^R \ \text{ (if } 0 < k < n\text{)}.$$

Here $\{n\} \cup R_{k-1}(n-1)^R$ denotes the sequence obtained by adjoining n to every subset in the reversal of the sequence $R_{k-1}(n-1)$.

Example. Rewriting part of the last example in terms of subsets, we have

$$R_1(3) = \{1\}, \{2\}, \{3\}, \quad R_2(3) = \{2,1\}, \{3,2\}, \{3,1\}$$

and so

$$R_2(4) = R_2(3),\ \{4\} \cup R_1(3)^R = \{2,1\}, \{3,2\}, \{3,1\},\ \{4,3\}, \{4,2\}, \{4,1\}.$$

1.3 Proofs

We want to prove that $R_k(n)$ defines a cyclic minimal change ordering of all k-subsets of $[n]$. This order is called the Revolving door order. First we need a proposition to help us understand the beginning and the end of the order

Proposition. *For $k > 0$, the first row of $R_k(n)$ is $[0 \cdots 01 \cdots 1]$ and the last row of $R_k(n)$ is $[10 \cdots 01 \cdots 1]$.*

Proof. By induction on n. The base case is $n = 1$, hence $k = 1$ which was can observe directly is true.

Assume the result holds for $n = m \geq 1$, and for any $0 \leq k \leq m$. Now consider $n = m + 1$.

If $k = n$ then the result can again be observed directly. Assume $1 \leq k \leq n - 1$. Then the first row of $R_k(n)$ is the first row of $R_k(n-1)$ with an extra 0 at the beginning which by induction is as it should be. The last row of $R_k(n)$ is the first row of $R_{k-1}(n-1)$ with an extra 1 at the beginning, which by induction is as it should be. $\square$

Theorem. *The rows of $R_k(n)$ represent a cyclic minimal change ordering of all k-subsets of $[n]$.*

Proof. We prove this by induction on n. Again $n = 1$ gives $k = 1$ and so can be observed directly.

Assume the result holds for $n = m \geq 1$, and for any $0 \leq k \leq m$. Now consider $n = m + 1$.

If $k = n$ or $k = 0$ then the result can again be observed directly. Assume $1 \leq k \leq n - 1$.

Consider the distance between adjacent rows of $R_k(n)$. The adjacent differences among the first $\binom{n-1}{k}$ rows are the same as the adjacent differences between the rows of $R_k(n-1)$ and so by induction are all 2. Likewise the adjacent differences among the last $\binom{n-1}{k-1}$ rows are the same as the adjacent differences between the rows of $R_{k-1}(n-1)$ and so by induction are all 2. It remains to check the difference between the $\binom{n-1}{k}$th row and its successor, and between the first and last rows.

The $\binom{n-1}{k}$th row of $R_k(n)$ is the $\binom{n-1}{k}$th row of $R_k(n-1)$ with an extra 0 at the beginning. The next row of $R_k(n)$ is the $\binom{n-1}{k-1}$th row of $R_{k-1}(n-1)$ with an extra 1 at the beginning. By the proposition these two rows of $R_k(n)$ are the following two rows

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{bmatrix}$$

which are distance 2 apart.

The first row of $R_k(n)$ is the first row of $R_k(n-1)$ with an extra 0 at the beginning. The last row of $R_k(n)$ is the first row of $R_{k-1}(n-1)$ with an extra 1 at the beginning. By the previous proposition these rows are

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{bmatrix}$$

which are distance 2 apart. This completes the proof. $\square$

This generation scheme has a major drawback– To compute $R_k(n)$ you need to compute $R_{k-1}(n-1), R_k(n-1)$ and thus also $R_{k-2}(n-2), R_{k-2}(n-1), R_{k-1}(n-2), R_k(n-1)\ldots$ which will either represent a lot of repeated calculation, or a lot of storage.

Next time we'll look at a successor function for this ordering, so that we never need to generate it by the definition.