Generating functions

Contents

1 Enumeration

Perhaps the most fundamental question about a combinatorial class, is "how many?". The enumeration problem can be vital for solving other problems, and our approach will use generating functions since they are so extremely powerful. We will only just scratch the surface of their potential.

Our initial approach will build up a small toolbox of combinatorial constructions

- union
- cartesian product
- sequence
- set and multiset
- cycle
- pointing and substitution

which translate to simple (and not so simple) operations on the generating functions.

1.1 Generating functions

We now jump head first into generating functions. Wilf's book *generatingfunctionology* is a great first text if you want more details and examples.

Definition 1.1. The **ordinary generating function** (**OGF**) of a sequence (A_n) is the formal power series

$$A(z) = \sum_{n=0}^{\infty} A_n z^n.$$

We extend this to say that the OGF of a class $\mathcal{A}$ is the generating function of its counting sequence $A_n = \text{cardinality}(\mathcal{A}_n)$. Equivalently, we can write the OGF as

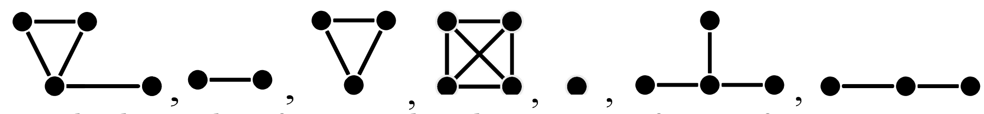
$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}.$$

This is because each object $\alpha \in \mathcal{A}$ *contrinutes* 1 to the coefficient of $z^{|\alpha|}$ in $A(z)$. In this context we say that the variable z **marks** the size of the underlying objects.

Exercise 1.2. Prove that these two forms for $A(z)$ are equivalent.

Hint. How does each object $\alpha \in \mathcal{A}$ *contribute* to each of the sums $\sum_{n=0}^{\infty} A_n z^n$ and $\sum_{\alpha \in \mathcal{A}} z^{|\alpha|}$?

Example 1.3. Consider the class consisting of the seven (unlabeled) connected graphs $\mathcal{H}$ given here

$\mathcal{H} = \{$  $\}$

If we define size to be the number of vertices, then the generating function for $\mathcal{H}$ is

$$H(z) = z^4 + z^2 + z^3 + z^4 + z^1 + z^4 + z^3 = z + z^2 + 2z^3 + 3z^4.$$

Alternatively, if we define size to be the number of edges, then the OGF is

$$H(z) = z^4 + z^1 + z^3 + z^6 + z^0 + z^3 + z^2 = 1 + z + z^2 + 2z^3 + z^4 + z^6.$$

Two examples from Lecture 1 have the following OGFs:

Binary words (size is word length): $\quad W(z) = 1 + 2z + 4z^2 + 8z^3 + \cdots = \sum_{n=0}^{\infty} 2^n z^n$

Permutations (size is number of symbols): $\quad P(z) = 1 + z + 2z^2 + 6z^3 + \cdots = \sum_{n=0}^{\infty} n!\, z^n$

2 Manipulating formal power series

2.1 Definitions

The formal power series turns out to be a very convenient way to store a sequence. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. We call the (possibly infinite) sum $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \ldots$ a **formal power series**. Here z is a variable, but we do not care if the sum makes sense for any value of z aside from 0. Despite this we often treat $A(z)$ as if it were a function of z, and apply operations and notations that we have seen in calculus. For example, we recognize the above OGF for binary words $W(z)$ to be a geometric series, which can be written as

$$W(z) = \sum_{n=0}^{\infty} (2z)^n = \frac{1}{1-2z}.$$

The second power series $P(z)$ does not have any simple expression in terms of standard functions. Indeed $P(z)$ does not converge (as an analytic object) for any complex number z other than 0. However, we can still manipulate it as a *purely formal* algebraic object.

The number a_k is called the **coefficient** of z^k in $A(z)$. We refer this coefficient with the following notation.

$$[z^k]A(z) = a_k.$$

Using the above two examples, we have that $[z^3]W(z) = 8$, and $[z^4]P(z) = 24$.

Two power series $\sum_{n \geq 0} a^n z^n$ and $\sum_{n \geq 0} b_n z^n$ are equal if and only if $a_n = b_n$ for all $n \geq 0$. Two special power series are $0 = 0 + 0z + 0z^2 + \ldots$ and $1 = 1 + 0z + 0z^2 + \ldots$.

2.2 Sum, Product and Power

A formal power series is a mathematical object which behaves essentially like an infinite polynomial. We define addition and multiplication of formal power series in a familiar manner. Let $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$. Then

$$A(z) + B(z) := \sum_{n\geq 0} (a_n + b_n)\, z^n$$

$$A(z) \cdot B(z) := \sum_{n\geq 0} \left(\sum_{0\leq k\leq n} a_k b_{n-k} \right) z^n.$$

Remarks: In both cases, we have completely specified the coefficient of z^n to be a finite number for each n, and so the sum and product are well defined. Each coefficient can be computed in finite time given $A(z)$ and $B(z)$. We sometimes write $A(z)B(z)$ instead of $A(z) \cdot B(z)$. We also interpret exponentiation in the usual way

$$A(z)^k := \underbrace{A(z) \cdot A(z) \cdot \ \ldots \ \cdot A(z)}_{k \text{ terms}}.$$

2.2.1 Interpretation of addition and multiplication for OGFs

Suppose $\mathcal{A}$ and $\mathcal{B}$ are disjoint combinatorial sets with OGFs $A(x)$ and $B(x)$, respectively. Then $[z^n](A(z)+B(z))$ is the number of objects of size n in the **union**, $\mathcal{A} \cup \mathcal{B}$. That is, $A(z) + B(z)$ is the OGF of the combinatorial set $\mathcal{A} \cup \mathcal{B}$. **Caution!** If there is an object in both $\mathcal{A}$ and $\mathcal{B}$, then it will be *counted twice* by $A(z) + B(z)$. For this reason, we usually prefer that $\mathcal{A}$ and $\mathcal{B}$ are **disjoint** sets (that is, $\mathcal{A} \cap \mathcal{B} = \emptyset$), when adding their OGFs.

Each coefficient of the product $A(z)B(z)$ is often called a **convolution** of the two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$. For example $[z^4](A(z)B(z)) = a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0$ is a convolution. The **cross product** of two combinatorial sets, written $\mathcal{A} \times \mathcal{B}$, is the set of ordered pairs (a, b) where $a \in \mathcal{A}$ and $b \in \mathcal{B}$, where the **size** of a pair (a, b) is defined to be the size of a (in $\mathcal{A}$) plus the size of b (in $\mathcal{B}$). Thus $A(z)B(z)$ is the OGF of the combinatorial set $\mathcal{A} \times \mathcal{B}$.

For example, where $\mathcal{W}$ is the combinatorial set of binary words, we have

$$\begin{aligned}\mathcal{W} \times \mathcal{W} = \{&(\epsilon,\epsilon),\\ &(\epsilon,0),(\epsilon,1),(0,\epsilon),(1,\epsilon),\\ &(\epsilon,00),(\epsilon,01),(\epsilon,10),(\epsilon,11),(0,0),(0,1),(1,0),(1,1),(00,\epsilon),(01,\epsilon),(10,\epsilon),(11,\epsilon),\\ &\ldots\}\end{aligned}$$

so the OGF for $\mathcal{W} \times \mathcal{W}$ is $1 + 4z + 12z^2 + \ldots$. Since $W(z) = 1 + 2z + 4z^2 + 8z^3 + \cdots$, we can compute the number of pairs in $\mathcal{W} \times \mathcal{W}$ having size 3 without listing them out by computing the convolution

$$[z^3](W(z)^2) = 1 \cdot 8 + 2 \cdot 4 + 4 \cdot 2 + 8 \cdot 1 = 32.$$

As we had written the OGF for $\mathcal{W}$ in the compact form $W(z) = \frac{1}{1-2z}$, the OGF for $\mathcal{W} \times \mathcal{W}$ may be written compactly as $\frac{1}{(1-2z)^2}$.

Exercise 2.1. Can you find a formula for $[z^n](W(z)^2)$? This equals the number of ordered pairs of binary strings, where the total length of the two strings equals n. For more of a challenge, find a formula for $[z^n](W(z)^3)$. This is the number of ordered triples of binary strings, with total length n.

2.3 Multiplicative inverse $A(z)^{-1}$

The multiplicative inverse of a formal power series $A(z)$ is the formal power series $C(z) = \sum_{n\geq 0} c_n z^n$ that satisfies

$$A(z) \cdot C(z) = 1.$$

Thus,

$$\sum_{n\geq 0}\sum_{k\geq n} a_k c_{n-k} z^n = 1 + 0z + 0z^2 + \dots.$$

Recall that two formal power series are equal if and only all of their coefficients are the same. This leads to the system of equations:

$$\begin{aligned} a_0c_0 &= 1 && (1)\\ a_1c_0 + a_0c_1 &= 0 && (2)\\ a_2c_0 + a_1c_1 + a_0c_2 &= 0 && (3)\\ &\vdots && (4) \end{aligned}$$

This is a triangular system, and hence we can compute c_k once we know of $c_0, \dots, c_{k-1}$. We see from (1) that $c_0 = 1/a_0$, and provided that $a_0 \neq 0$, this is well-defined. The multiplicative inverse of $A(z)$ is written as $A(z)^{-1}$ or $\frac{1}{A(z)}$.

Example: The inverse of $A(z) = 1 - z$. The above system simplifies to:

$$\begin{aligned} c_0 &= 1\\ -c_0 + c_1 &= 0\\ -c_1 + c_2 &= 0\\ &\vdots\\ -c_k + c_{k+1} &= 0\\ &\vdots \end{aligned}$$

This has the unique solution: $c_n = 1$ for all n. We should recognize this as the *geometric series*:

$$(1-z)^{-1} = \sum_{n\geq 0} z^n = 1 + z + z^2 + z^3 + \dots.$$

2.4 Quotient

The quotient $C(z) = \dfrac{A(z)}{B(z)}$ of formal power series $A(z)$ and $B(z)$ is the power series $C(z) = A(z)\cdot B(z)^{-1}$. The terms of $C(z) = \sum_{n\geq 0} c_n z_n$ can be obtained by solving the triangular system

$$\begin{aligned} b_0c_0 &= a_0 && (5)\\ b_1c_0 + b_0c_1 &= a_1 && (6)\\ b_2c_0 + b_1c_1 + b_0c_2 &= a_2 && (7)\\ &\vdots && (8) \end{aligned}$$

The first few terms of $C(z)$ can be efficiently found by hand using “long division” starting with the constant term. For example, we compute the first few terms of $\dfrac{1+2z^2+4z^4+6z^6+\dots}{1-3z+z^2}$ as follows.

$$
\begin{array}{r l}
 & 1+3z+10z^2+27z^3+\dots \\ \cline{2-2}
1-3z+z^2\,\big) & 1+0z+2z^2+0z^3+4z^4+0z^5+6z^6+\dots \\
 & 1-3z+z^2 \\ \cline{2-2}
 & \quad 3z+z^2+0z^3 \\
 & \quad 3z-9z^2+3z^3 \\ \cline{2-2}
 & \qquad 10z^2-3z^3+4z^4 \\
 & \qquad 10z^2-30z^3+10z^4 \\ \cline{2-2}
 & \qquad\qquad 27z^3-6z^4+0z^5 \\
 & \qquad\qquad 27z^3-81z^4+27z^5 \\ \cline{2-2}
 & \qquad\qquad\qquad 75z^4-27z^5+6z^6 \\
 & \qquad\qquad\qquad \dots
\end{array}
$$

2.5 Composition

We define composition of formal power series as follows:

$$A(B(z)) := \sum_{n\geq 0} a_n \cdot B(z)^n.$$

It is a bit tedious to expand out the first three terms of this series.

$$
\begin{aligned}
A(B(z)) &= a_0 + a_1(b_0+b_1z+b_2z^2+\dots) + a_2(b_0+b_1z+b_2z^2+\dots)^2+\dots \\
&= a_0 + a_1(b_0+b_1z+b_2z^2+\dots) + a_2(b_0^2+(b_0b_1+b_1b_0)z+(b_0b_2+b_1^2+b_2b_1)z^2+\dots)+\dots \\
&= (a_0+a_1b_0+a_2b_0^2+a_3b_0^3+\dots)+(a_1b_1+2a_2b_0b_1+3a_3b_0^2b_1)z+(a_1b_2+a_2b_1^2+2a_2b_0b_2+3a_3b_0b_1^2+3a_3b_0^2b_2)z^2+\dots
\end{aligned}
$$

Is the composition well defined? Because of the infinite sum in the constant term, the composition results in a formal power series if and only if $a_0+a_1b_0+a_2b_0^2+a_3b_0^3+\dots$ is a finite number! Thus the composition $A(B(z))$ is well defined **if and only if** either $b_0 = 0$ or a finite number the coefficients of $A(z)$ are not zero.

2.6 Differentiation

We can develop a complete theory of derivatives and integrals of formal power series using the power rule $\frac{d}{dz}z^k = kz^{k-1}$, and essentially generalizing the derivative of a polynomial. Remark, this does not involve a limit and so the usual properties (sum rule, product rule, chain rule) should be proved from the definition of formal power series.

- Definition: $\frac{d}{dz}A(z) = \sum_{n\geq 0}(n+1)a_{n+1}z^n$;
- Product rule: $\frac{d}{dz}\left(A(z)B(z)\right) = \left(\frac{d}{dz}A(z)\right)B(z) + A(z)\left(\frac{d}{dz}B(z)\right)$;
- Chain rule for powers: $\frac{d}{dz}(A(z))^n = n(A(z))^{n-1}\frac{d}{dz}A(z)$.

Exercise 2.2. Prove the above product and chain rules.

2.7 Two special series: $\exp$, $\log$

The formal power series $\sum_{n\geq 0} z^n/n!$ is reminiscent of the Taylor series for the exponential, and hence we define

$$\exp(z) := \sum_{n\geq 0}\frac{z^n}{n!}.$$

In fact, it satisfies all of the usual properties of $\exp$, but again, since it is *defined* as the sum, these properties should be proved from the basic formal power series properties:

- $\frac{d}{dz}\exp(z) = \exp(z)$;
- $\exp(F(z) + G(z)) = \exp(F(z)) \cdot \exp(G(z))$;
- $\exp(F(z))^{-1} = \exp(-F(z))$.

Similarly, we can define a series

$$\log\left((1-z)^{-1}\right) := \sum_{n\geq 1} \frac{z^n}{n}.$$

Exercise 2.3. Prove the following properties which we can derive from formal power series definitions:

1. $\frac{d}{dz}\log\left((1-z)^{-1}\right) = (1-z)^{-1}$;
2. $\log(\exp(A(z))) = A(z)$;
3. $\exp(\log(A(z))) = A(z)$, if a_0=1;
4. $\log(A(z) \cdot B(z)) = \log(A(z)) + \log(B(z))$, if $a_0 = b_0 = 1$.

Solution to 1.

$$\frac{d}{dx}\sum_{n\geq 1}\frac{z^n}{n} = \sum_{n\geq 1}\frac{d}{dx}\frac{z^n}{n} = \sum_{n\geq 1}\frac{nz^{n-1}}{n} = \sum_{n\geq 0} z^n.$$

3 The basic toolbox for coefficient extraction

There are several key theorems to aid with coefficient extraction. For any constant p and any nonnegative integers m and n we have the following. If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, then $[z^n]A(z) := a_n$.

Linearity rule

$$[z^n]\, pA(z) = p[z^n]A(z)$$

Power rule

$$[z^n]A(pz) = p^n[z^n]A(z)$$

Reduction rule

$$[z^n]z^m A(z) = [z^{n-m}]A(z)$$

Sum rule

$$[z^n]\,(A(z) + B(z)) = [z^n]A(z) + [z^n]B(z)$$

Product rule

$$[z^n](A(z) \cdot B(z)) = \sum_{k=0}^{n}\left([z^k]A(z)\right) \cdot \left([z^{n-k}]B(z)\right)$$

Binomial theorem Let n and r be positive integers.

$$[z^n](1+mz)^r = \binom{r}{n}a^n = \frac{r!}{(r-n)!n!}m^n.$$

Extended binomial theorem Let k be a positive integer, and r be a real number. Then define

$$\binom{r}{k} := \frac{r(r-1)(r-2)\dots(r-k+1)}{k!}.$$

Then

$$[z^k](1+mz)^r = \binom{r}{k} m^n.$$

A Useful Binomial Formula

$$\begin{aligned}\binom{-r}{k} &= \frac{(-r)(-r-1)\dots(-r-k+1)}{k!} \\ &= (-1)^k\binom{r+k-1}{r-1} = (-1)^k\binom{r+k-1}{k}.\end{aligned}$$

This formula is usually applied via the following handy formula that you should memorize.

$$[z^n]\frac{1}{(1-mz)^r} = \binom{n+r-1}{r-1} m^n.$$

Example 3.1.

$$\frac{1}{(1-3z)^2} = \binom{0+2-1}{2-1}3^0+\binom{1+2-1}{2-1}3^1z+\binom{2+2-1}{2-1}3^2z^2+\binom{3+2-1}{2-1}3^3z^3+\dots = 1+2\cdot 3z+3\cdot 3^2z^2+4\cdot 3^3z^3+\dots$$

$$[z^n]\frac{1}{(1-3z)^2} = \binom{n+2-1}{2-1}3^n = (n+1)3^n$$

$$\frac{1}{(1-5z)^3} = \binom{0+3-1}{2-1}5^0+\binom{1+3-1}{3-1}5^1z+\binom{2+3-1}{2-1}5^2z^2+\dots = 1+\binom{3}{2}\cdot 5z+\binom{4}{2}\cdot 5^2z^2+\binom{5}{2}\cdot 5^3z^3+\dots$$

$$[z^n]\frac{1}{(1-5z)^3} = \binom{n+3-1}{3-1}5^n = \binom{n+2}{2}5^n = \frac{(n+2)(n+1)}{2}5^n$$

Example 3.2 (Catalan numbers)**.**
The generating function

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}$$

is a "famous" generating function. The coefficients are called Catalan numbers. We will revisit them shortly as a fundamental counting sequence. They count, for example, rooted plane trees with n edges, full binary trees with $n+1$ leaves, and the ways to completely parenthesize a product of $n+1$ factors.

First note that the "1" in the numerator only affects the coefficient of z^{-1}. So for $n \geq 0$ we may ignore the "1" and compute as follows.

$$
\begin{aligned}
[z^n]\frac{1-\sqrt{1-4z}}{2z} &= -\frac{1}{2}[z^n]z^{-1}\sqrt{1-4z} && \text{(provided } n\geq 2 \text{ and by linearity rule)}\\
&= -\frac{1}{2}[z^{n+1}](1-4z)^{1/2} && \text{(reduction rule)}\\
&= -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} && \text{(extended binomial theorem with } m=-4 \text{ and } r=\tfrac{1}{2})\\
&= \frac{-(-4)^{n+1}}{2}\,\frac{\overbrace{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-(n+1)+1\right)}^{n+1\text{ terms}}}{(n+1)!}\\
&= \frac{-(-\cancel{4}^{\,2})^{n+1}}{2}\cancel{\left(\frac{1}{2}\right)^{n+1}}\,\frac{1\cdot\overbrace{(-1)\cdot(-3)\cdot(-5)\dots(-(2n-1))}^{n}}{(n+1)!}\\
&= \frac{-(-2)^{n+1}}{2}(-1)^n\,\frac{1\cdot 1\cdot 3\cdot 5\dots(2n-1)}{(n+1)!}\\
&= 2^n\,\frac{1\cdot 1\cdot 3\cdot 5\dots(2n-1)}{(n+1)!}\cdot\frac{n!}{n!} && \text{(a little trick!)}\\
&= \frac{1\cdot 1\cdot 3\cdot 5\cdots(2n-1)\;\cdot\;\overbrace{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot n)}^{2^n n!}}{(n+1)!\;n!}\\
&= \frac{(2n)!}{(n+1)\,n!\;n!}\\
&= \frac{1}{n+1}\binom{2n}{n}. \quad = n^{th} \text{ catalan number}
\end{aligned}
$$

Fixes the z^0 term

Note that when $n=-1$ we have that

$$-\frac{1}{2}[z^{n+1}](1-4z)^{1/2} = -\frac{1}{2}(-4z)^0 = -\frac{1}{2},$$

so the "1" that we ignored serves to cancel out the annoying term $-\frac{1}{2}z^{-1}$ that appears in the expansion of $\frac{-\sqrt{1-4z}}{2z}$. The following Maple code computes the first eight terms of this power series directly, and also from the above formula.

```
> series( (1-sqrt(1-4*z))/(2*z), z, 8 );
        1 + z + 2 z^2 + 5 z^3 + 14 z^4 + 42 z^5 + 132 z^6 + 429 z^7 + O(z^8)     (1)
> seq( C[n] = binomial(2*n,n)/(n+1), n=0..7 );
        C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, C_7 = 429     (2)
```

Exercise 3.3. Go to the web page: http://oeis.org/ This is the Online Encyclopedia of Integer sequences. The first few terms of this sequence are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862. Enter them into the search engine to find some other combinatorial interpretations of these numbers.

4 A sampling of counting sequences and generating functions

$\mathcal{A}$	Size	A_n	$A(z)$
Permutations	Number of objects	$n!$	$\sum n!\ z^n$
Simple vertex-labelled graphs	Number of vertices	$2^{\binom{n}{2}}$	$\sum 2^{\binom{n}{2}} z^n$
Binary words	Length of word	2^n	$\dfrac{1}{1-2z}$
Rooted plane trees	Number of edges	$\dfrac{1}{n+1}\dbinom{2n}{n}$	$\frac{1-\sqrt{1-4z}}{2z}$

4.1 Advanced Ideas

- **Asymptotic Analysis.** The values for which $A(z)$ is not defined can give us much information about the *asymptotic growth* of the coefficient. Consider $A(z) = \frac{1}{1-2z} = \sum 2^n z^n$. The coefficient a_n grows like 2^n. Remark that $A(z)$ is not defined at $z = 1/2$. In fact, very often we can formalize a connection between the smallest *singularity* of the formal power series ρ (roughly, the smallest value for which $A(z)$ is not defined) and the growth of a_n: $a_n \approx 1/\rho^n$. Again, complex analysis is fundamental in this remarkable theory.