Lecture 5

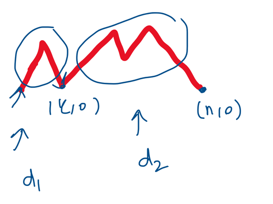
Friday, October 11, 2024 9:42 AM

Dyck Paths

(= ↗ (1,1)) = ↘ (1,−1)

- last one should be $y=0$

combinatorial → - size non-negative integer
- finite



- every time it catches 0, there's a block.

$\mathcal{P}$

null class

dyck path length zero

Atoms $Z_{\nearrow}$, $Z_{\searrow}$, ε

$$\mathcal{P} \cong seq(Z_{\nearrow} \times \mathcal{P} \times Z)$$

not same as 1's since it's not smaller parts can't decompose

$$D(Z) = \frac{1}{1 - (Z_{up}\, P(z)\, Z_{down})}$$

$$D(z) - z^2 D^2_{(z)} = 1$$

$$z^2 D^2(z) - D(z) + 1 = 0$$

$$D(z) = \frac{1 \pm \sqrt{1-4z^2}}{2z^2} = B(t) \quad \text{let } t = z^2$$

generating function

(−) bc to have positive answer

if $z = t$

$$\frac{1 - \sqrt{1-4t^2}}{2t^2}$$

same

$$\mathcal{P} \cong \varepsilon + Z_{\nearrow} \times \mathcal{P} \times Z_{\searrow} \times \mathcal{P}$$

$$D(z) = 1 + z^2 D(z)$$

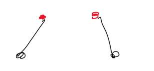
- Dyck Paths have even length

Plane binary trees

size = # of nodes

empty

ε



Combinatorial class B

ε + plane b.t

empty

$$B \cong \varepsilon + B \times Z \times B$$

left branch, single node, right branch

- Same as dyck paths

specification

$A = A^{(1)}, \ldots A^{(r)}$ multiple classes not only one (Dyck path or binary trees)

Dyck Path → recursive one formulas $A = \phi(\varepsilon, Z, A)$ A included

→ iterative $A = \phi(\varepsilon, Z_0, Z_1)$ A not included

∴ constructable when we can have $A = \phi(\varepsilon, Z_0, Z_1)$ we can find equations

$$z B(z)^2 - B(z) + z = 0$$

$$z^2 D(z)^2 - D(z) + 1 = 0$$

$$D(z) = \frac{1 - \sqrt{1-2z^2}}{2z^2}$$

$$B = \frac{1 - \sqrt{1-4z^2}}{2z} = D(z) \cdot z$$

shifted to right

$$B(z) = z + z^3 + 2z^5 + 5z^7 + \ldots$$

$$D(z) = \frac{1 - \sqrt{1-4z^2}}{2z^2}$$

how to extract coefficient?

let $t = z^2$

$$D(\sqrt{t}) = \frac{1 - \sqrt{1-4t}}{2t}$$

$$\frac{1}{2} t^{-1} - \frac{1}{2} t^{-1} (1-4t)^{\frac{1}{2}}$$

if $n=0$

$$[t^n] D(\sqrt{t}) = -\frac{1}{2} [t^{n+1}] (1-4t)^{\frac{1}{2}}$$

by binomial thm

$$= -\frac{1}{2} \binom{\frac{1}{2}}{n+1} (-4)^{n+1}$$

$$= -\frac{1}{2} \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{2n-1}{2})}{n+1!} (-4)^n$$

⋮

$$= \frac{1}{n+1} \binom{2n}{n}$$

$$b_n = b_{2m+1} = \frac{1}{m+1} \binom{2m}{m}$$

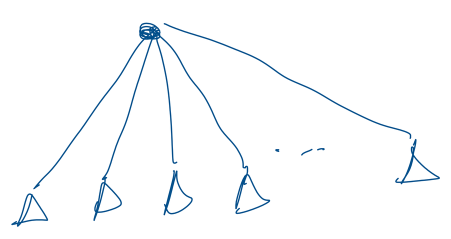
# of dyck paths of length $2m$ is $\frac{1}{m+1}\binom{2m}{m}$

Catalan sequence

$= 1, 1, 2, 5, 14, \ldots$

Catalan number

all rooted plane trees



none - empty

not ε bc infinite # of empties don't work

$$T = Z \times SEQ(T)$$

specification

$$T(z) = \frac{z}{1 - T(z)}$$

generating function