MATH 450
Assignment 1

Please submit using your Crowdmark link for this assignment by the due date.

(1) Show that the topologies on $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

(2) Let X be a set.

 (a) Let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Show that $\mathcal{T}_c$ is a topology on X.

 (b) Let $\mathcal{T}_\infty$ be the collection of all subsets U of X such that $X - U$ is infinite or empty or all of X. Is $\mathcal{T}_\infty$ a topology on X?

(3) For $a, b \in \mathbb{Z}$, $b > 0$, let $N_{a,b} = \{a + nb : n \in \mathbb{Z}\}$. Let $\mathcal{T}$ be the collection of sets $O \subseteq \mathbb{Z}$ such that either O is empty or for every $a \in O$ there is some $b > 0$ such that $N_{a,b} \subseteq O$.

 (a) Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

 (b) Show that any non-empty open set is infinite and the set $N_{a,b}$ is closed.

 (c) Any $x \in \mathbb{Z}$, $x \neq 1, -1$ has a prime divisor p, and hence is contained in $N_{0,p}$. Thus, $\mathbb{Z} - \{-1, 1\} = \cup_p N_{0,p}$, where p runs through all primes of $\mathbb{Z}$. Use this and the previous parts to prove there are infinitely many primes.

(4) Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X which contain $\mathcal{A}$.

(5) Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$? Justify your answers.

 (a) $A = \left\{x : \frac{1}{2} < |x| < 1\right\}$

 (b) $B = \left\{x : \frac{1}{2} < |x| \leq 1\right\}$

 (c) $C = \left\{x : \frac{1}{2} \leq |x| < 1\right\}$

 (d) $D = \left\{x : \frac{1}{2} \leq |x| \leq 1\right\}$

 (e) $E = \left\{x : 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{N}\right\}$

(6) Show that the countable collection

$$\{(a, b) \times (c, d) : a < b, c < d, a, b, c, d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$.