Due October 30, 9:00pm. [Total points: 6+11+7+8 = 26].

1. Let $\mathcal{T}$ be the class of binary rooted trees, **not** including the empty tree, where *every node in the tree has either 0 or 2 children.* The *size* of a tree is its number of nodes.

   (a) [1 point] Every tree $\tau \in \mathcal{T}$ is either the (atomic) tree t_1 which consists of one single node, or τ consists of its root (t_1) together with a left and a right subtree, each selected from $\mathcal{T}$. Find a (recursive) specification for $\mathcal{T}$ which involves the combinatorial classes $\mathcal{T}$ and $\mathcal{Z} = \{t_1\}$, and operations such as *cross product* "$\times$" and *set addition* "$+$", $\textsc{Seq}(\cdot)$, etc.

   (b) [1 point] Use part (a) to find an equation which the generating function $T(z)$ must satisfy.

   (c) [1 point] Solve the equation from part (b) for the generating function $T(z)$. You will have to reject one of two possible solutions for $T(z)$ by using a fact such as $[z^{-1}]T(z) = 0$ or $[z^1]T(z) \geq 0$.

   (d) [2 points] Extract the nth coefficient $T_n = [z^n]T(z)$ from the formula for $T(z)$ that you found in part (c). You computation should be similar to finding the formula for the nth Catalan number. You might find a shortcut by reviewing the derivation of $(W_{odd})_n$ at the end of Lecture 4.

   (e) [1 point] How does your formula for T_n compare to that of the the class of all binary rooted trees $\mathcal{B}$ that you saw in Lecture 5?

2. (a) [2 points] Find a specification and the generating function for the class $\mathcal{W}_{odd+0}$ of binary words in which every *nonempty* block of 0s has odd length. This is similar to the class $\mathcal{W}_{odd}$ that we did in class, but a little bit trickier, since words in $\mathcal{W}_{odd+0}$ can have consecutive 1's

$$\mathcal{W}_{odd+0} = \{\epsilon, 0, 1, 11, 01, 10, 111, 110, 101, 011, 000, \dots\}$$

   (b) [2 points] Show that every word in $\mathcal{W}_{odd+0}$ is uniquely generated by your specification.

   (c) [1 point] Write out the generating function $W_{odd+0}(z)$ for $\mathcal{W}_{odd+0}$.

   (d) [2 points] Use Maple's `simplify()` command to simplify your generating function, then use `series()` to find the first 10 elements of the counting sequence.

   (e) [1 point] Notice that in Maple the command `convert(W, parfrac)` fails to make a partial fraction decomposition of your generating function $W = W_{odd+0}(z)$. Why do you think this happens? Also the command `convert(convert(W, parfrac), radical)` gives something ugly. Try use instead `convert(W, parfrac, real)`, similarly to the Maple worksheet `4-343MapleWordsWhose0BlocksHaveEvenLength.mw` on the Lecture Notes Canvas page.

   (f) [2 points] A fraction of the form $\dfrac{a}{bz+c}$ can be rewritten into the form $\dfrac{A}{1+Bz}$ by multiplying the numerator and denominator by $1/c$. This allows us to find a formula for the nth coefficient of its power series:

$$[z^n]\frac{a}{bz+c} = [z^n]\frac{A}{1-(-B)z} = A \cdot (-B)^n$$

   Use this fact, and the output you obtained using `convert(W, parfrac, real)` in part (e), to write a formula for $(W_{odd+0})_n$, the number of words in $\mathcal{W}_{odd+0}$ having length n. It is best if you do this in Maple, because it is tedious and error prone to do this without a calculator. **Do not use the procedure `coeffFromParfrac` provided in the sample worksheet `4-343MapleWordsWhose0BlocksHaveEvenLength.mw` to answer this part.**

   (g) [1 point] Because of round-off error, the formula for $(W_{odd+0})_n$ that you found in part (f) might not evaluate to an integer, even when n is an integer. However, it will be very close to an integer. Enter your formula for $(W_{odd+0})_n$ as an expression in Maple using the variable `n`, and evaluate it for `n = 0,1,2,3,...,9` to check that it agrees with part (d) of this question.

3. This question is about strings using the letters $0, 1, 2$ and counted by length. Let $\mathcal{C}$ be the combinatorial class of such strings with no consecutive 2s.

(a) [3 points] Find $C(z)$.

(b) [4 points] Find an exact formula for the number c_n of words of size n in $\mathcal{C}$. You will have to use the quadratic formula to break $C(z)$ into partial fractions.

4. [4 points] Let $\mathcal{T}$ be the class of plane trees where each node has a number of children which is divisible by three. Use Lagrange Inversion to find a formula for the number T_n of such trees with n nodes.

5. [6 points] Let $\mathcal{S}$ be the class of plane *binary-ternary* rooted trees, where each vertex either zero, two or three children. Similarly to the last example in Lecture 7, use Lagrange inversion to find a formula for the number S_n of such trees of having n nodes.

6. [3 points] Show that the number, $C_n^{(r,k)}$, of compositions of the integer n with exactly k summands, each of which is at most r is given by

$$C_n^{(r,k)} = [z^n]\left(z\frac{1-z^r}{1-z}\right)^k$$

For example when $r=4$, $k=3$ and $n=9$, we have that $C_9^{(4,3)} = 10$ since there are 10 compositions of 9 in the following set.

$$\{1+4+4, 2+3+4, 2+4+3, 3+2+4, 3+3+3, 3+4+2, 4+1+4, 4+2+3, 4+3+2, 4+4+1\}$$