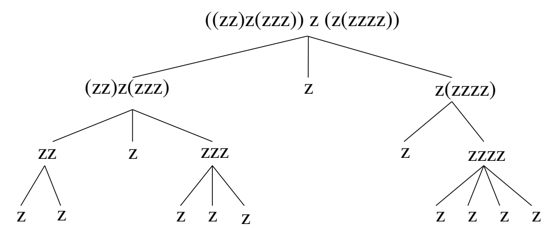
1. [See Lecture 6] Here is a combinatorial class that you have not yet seen. Informally, a *bracketing* is any "legal" way to place parentheses on a non-empty sequence of symbols. For example, there are exactly 11 bracketings of the sequence $zzzz$.

$$zzzz,\ (zz)zz,\ (zzz)z,\ z(zz)z,\ z(zzz),\ zz(zz),\ ((zz)z)z,\ (z(zz))z,\ z((zz)z),\ z(z(zz)),\ (zz)(zz).$$

More formally, the atomic symbol z is itself a bracketing; and any sequence of two or more consecutive bracketings enclosed by a pair of parentheses is a bracketing. To simplify notation, we have written z instead of (z) and we have also removed the outermost parentheses. For example, the bracketing $zz(zz)$ should interpreted to be $((z)(z)((z)(z)))$. The construction tree for the bracketing $((zz)z(zzz))z(z(zzzz))$ is shown on the right.



Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$ be the combinatorial class where $\mathcal{B}_n$ is the set of bracketings of the sequence $\overbrace{zz\ldots z}^{n}$, for $n \geq 1$.

(a) Find a recursive specification for $\mathcal{B}$ in terms of the atomic class $\mathcal{Z} = \{z\}$, the neutral class $\mathcal{E}$, and the operators '+', '$\times$' and 'Seq'.
Solution:

$$\begin{aligned}\mathcal{B} &= \mathcal{Z} + \texttt{Seq}_{\geq 2}(\mathcal{B}) \\ &= \mathcal{Z} + \mathcal{B} \times \mathcal{B} \times \texttt{Seq}(\mathcal{B}).\end{aligned}$$

(b) Use your specification to show the the OGF for $\mathcal{B}$ is $B(z) = \frac{1}{4}\left(1 + z - \sqrt{1 - 6z + z^2}\right)$.
Solution:

$$\begin{aligned}B(z) &= z + \frac{B(z)^2}{1 - B(z)} \\ B(z)(1 - B(z)) &= z(1 - B(z)) + B(z)^2 \\ B(z) - B(z)^2 &= z - zB(z) + B(z)^2 \\ 0 &= 2B(z)^2 - (z+1)B(z) + z \\ B(z) &= \frac{z + 1 \pm \sqrt{(z^2 + 2z + 1) - 8z}}{4} \\ &= \frac{z + 1 - \sqrt{1 - 6z + z^2}}{4}\end{aligned}$$

We discarded the the plus sign, since there are no bracketings of size zero, so the 1's must cancel in the series expansion of the square root.

(c) Use the formula from part (b) and the following table to find B_5, the number of bracketings of $zzzz$. You may leave your answer in a "basic-calculator ready" form, such as $B_5 = \frac{3}{8} \cdot$

$\left(\frac{13}{35} - \frac{13\cdot 7}{8} + \frac{17\cdot 4}{3}\right)$.

k	$\binom{1/2}{k}$	$(z^2-6z)^k$
1	$\frac{1}{2}$	$z^2 - 6\,z$
2	$\frac{-1}{8}$	$z^4 - 12\,z^3 + 36\,z^2$
3	$\frac{1}{16}$	$z^6 - 18\,z^5 + 108\,z^4 - 216\,z^3$
4	$\frac{-5}{128}$	$z^8 - 24\,z^7 + 216\,z^6 - 864\,z^5 + 1296\,z^4$
5	$\frac{7}{256}$	$z^{10} - 30\,z^9 + 360\,z^8 - 2160\,z^7 + 6480\,z^6 - 7776\,z^5$
6	$\frac{-21}{1024}$	$z^{12} - 36\,z^{11} + 540\,z^{10} - 4320\,z^9 + 19440\,z^8 - 46656\,z^7 + 46656\,z^6$

Solution:
(This question is similar to, but harder than #4 above. Probably it is too long for an exam.) We must extract the coefficient of z^5 in $B(z) = \frac{1}{4}\left(1 + z - \sqrt{1-6z+z^2}\right)$. Using the extended binomial theorem, we find

$$\begin{aligned} B_5 &= [z^5]\frac{1}{4}\left(1+z-\sqrt{1-6z+z^2}\right) \\ &= 0 + 0 - \frac{1}{4}[z^5]\sqrt{1-6z+z^2} \\ &= -\frac{1}{4}[z^5]\left(1+(z^2-6z)\right)^{1/2} \\ &= -\frac{1}{4}[z^5]\left(\sum_{k\geq 0}\binom{1/2}{k}(z^2-6z)^k\right). \end{aligned}$$

From the table provided, we see that the term z^5 only appears in the expansion of $(z^2-6z)^k$ when $3 \leq k \leq 5$. Using this table, we find that the number of bracketings of $zzzzz$ is

$$\begin{aligned} B_5 &= -\frac{1}{4}\left[\binom{1/2}{3}[z^5](z^2-6z)^3 + \binom{1/2}{4}[z^5](z^2-6z)^4 + \binom{1/2}{5}[z^5](z^2-6z)^5\right] \\ &= -\frac{1}{4}\left[\frac{1}{16}(-18) + \frac{-5}{128}(-864) + \frac{7}{256}(-7776)\right] \qquad \text{(for full grade)}. \end{aligned}$$

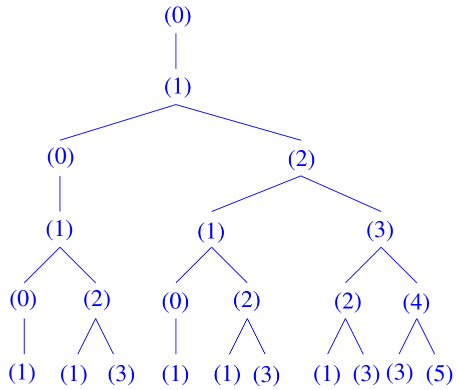
We may optionally simplify this expression by hand.

$$\begin{aligned} B_5 &= \frac{1}{4}\left[\frac{9}{8} - \frac{5\cdot 27}{4} + \frac{7\cdot 243}{8}\right] \\ &= \frac{1}{32}[9 - 270 + 1701] \\ &= \frac{1440}{32} \\ &= 45. \end{aligned}$$

2. [See Lecture 17] Draw the first six levels of the generating tree specified by the rule $[(0); \{(k) \to (k-1)(k+1)\}]$.

Solution:

The rule for the root node is actually $(0) \to (1)$, not $(0) \to (-1)(1)$, since negative labels are not allowed.



3. [Lecture 16] The Johnson-Trotter minimal change order for the permutations of $\{1,2,3\}$ are the successive rows of the matrix shown at right. Write down the **four** permutations of $\{1,2,3,4\}$ that come immediately after the permutation $\pi = [3\ 1\ 4\ 2]$ in the Johnson-Trotter order.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Solution:
The permutation π belongs to either the block $\sigma^{\rightarrow}$ or the block $\sigma^{\leftarrow}$ where $\sigma = [3\ 1\ \ \ 2]$. Since σ is the third row of the matrix shown, π appears in the block $\sigma^{\leftarrow}$, so the "4" is shifting to the left on successive steps until it can not move farther, then we continue by listing the elements of $\sigma'^{\rightarrow}$ where $\sigma' = [3\ 2\ 1]$ is the successor of σ. This gives the next four permutations

$$\begin{bmatrix} 3 & 1 & 4 & 2 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

4. [Lecture 7] Write a recursive specification for $\mathcal{T}$, the set of plane trees with red and blue nodes, where

- every blue node has no blue children and even number of red children, and
- every red node has either no children or it has exactly one blue child and exactly one red child, in either order.

(a) Use the notation $\mathcal{T} = \mathcal{T}_r + \mathcal{T}_b$ where $\mathcal{T}_r$ are those trees with a red root. Your specification should be three equations which involve $\mathcal{T}_b$, $\mathcal{T}_r$ and $\mathcal{T}$.
Solution:
Since the trees are plane trees, there are two ways to arrange the red and black children of a red node. We have

$$\begin{aligned} \mathcal{T}_b &= \mathcal{Z} \times \textsc{Seq}(\mathcal{T}_r^2) \\ \mathcal{T}_r &= \mathcal{Z} \times (\mathcal{E} + \mathcal{T}_r \times \mathcal{T}_b + \mathcal{T}_b \times \mathcal{T}_r). \\ \mathcal{T} &= \mathcal{T}_b \cup \mathcal{T}_r \end{aligned}$$

(b) Find a set of equations which, if solved, give the generating function $T(z)$ for $\mathcal{T}$. Do not solve the equations!
Solution:

$$\begin{aligned} T_b(z) &= z\frac{1}{1 - T_r(z)^2} \\ T_r(z) &= z(1 + 2T_r(z)T_b(z)). \\ T(z) &= T_b(z) + T_r(z) \end{aligned}$$

5. [Lecture 6] Let $\mathcal{W}^{(k)}$ be the class of binary strings counted by length, which have no more than k consecutive 0s. Show that the generating function for $\mathcal{W}^{(k)}$ is

$$W^{(k)}(z) = \frac{1 - z^{k+1}}{1 - 2z + z^{k+2}}.$$

Solution:

We start from a specification for this family. If we break a string from $\mathcal{W}^{(k)}$ right after each block of zeros, then all blocks except possibly the first and last belong to the class $\mathcal{B}$ of words consisting of a a positive number of ones followed by between 1 and k zeros.

$$\begin{aligned}\mathcal{B} &= 1(1)^* \left(0 + 00 + 000 + \cdots + \overbrace{00\ldots0}^{k}\right) \\ &= 1(1)^* 0(\epsilon + 0 + 00 + \cdots + \overbrace{00\ldots0}^{k-1}).\end{aligned}$$

The first block is a sequence of zeros whose length belongs to $\{0, 1, \ldots, k\}$. The last block is a block of ones of any length (possibly empty).

$$\mathcal{W}^{(k)} = (\epsilon + 0 + 00 + \cdots + \overbrace{00\ldots0}^{k})\, \mathcal{B}^*\, 1^*.$$

The generating function for $(\epsilon + 0 + 00 + \cdots + \overbrace{00\ldots0}^{k})$ is $1 + z + z^2 + \cdots + z^k = \dfrac{1 - z^{k+1}}{1 - z}$ (this is a partial geometric series). Another way to see this is to notice that

$$(\epsilon + 0 + 00 + \cdots + \overbrace{00\ldots0}^{k}) = 0^* - 0^{k+1}\, 0^*.$$

Therefore the generating function for $\mathcal{B}$ is

$$B(z) = z \cdot \frac{1}{1 - z} \cdot z \cdot \frac{1 - z^k}{1 - z} = \frac{z^2 - z^{k+2}}{(1 - z)^2}.$$

So the generating function for $\mathcal{W}^{(k)}$ is

$$\begin{aligned}W^{(k)}(z) &= \frac{1 - z^{k+1}}{1 - z} \frac{1}{1 - B(z)} \frac{1}{1 - z} \\ &= \frac{1 - z^{k+1}}{(1 - z)^2} \frac{1}{1 - \frac{z^2 - z^{k+2}}{(1-z)^2}} \qquad \text{(There is no need to go beyond here in your solution.)} \\ &= \frac{1 - z^{k+1}}{(1 - z)^2 - (z^2 - z^{k+2})} \\ &= \frac{1 - z^{k+1}}{1 - 2z + z^2 - z^2 + z^{k+2}} \\ &= \frac{1 - z^{k+1}}{1 - 2z + z^{k+2}}.\end{aligned}$$

6. [Lecture 6] Let $\mathcal{T}$ be the set of non-empty rooted planar trees where the number of children that each node has belongs to the set $\Omega = \{0, 2, 4, 6, \dots\}$ (any even number). The size of a tree in $\mathcal{T}$ is the number of nodes in the tree. Use Lagrange inversion to compute T_n, the number of trees in $\mathcal{T}$ with size n.
Solution:
The recursive combinatorial specification $\mathcal{T} = \mathcal{Z} \times \textsc{Seq}(\mathcal{T}^2)$ implies that the OGF satisfies

$$T(z) = \frac{z}{1-(T(z))^2}.$$

We use Lagrange Inversion as follows.

$$\begin{aligned}
T(z) &= z\,\phi(T(z)), \qquad \text{where} \quad \phi(u) = \frac{1}{1-u^2} \\
\phi(u)^n &= \frac{1}{(1-u^2)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} u^{2k} \qquad\qquad \text{(extended binomial theorem)} \\
[u^t]\phi(u)^n &= \begin{cases} \binom{n+k-1}{n-1} = \binom{n+t/2-1}{n-1} & \text{if } t = 2k\text{, for some integer } k \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

By Lagrange inversion we have

$$T_n = \frac{1}{n}[u^{n-1}]\phi(u)^n = \begin{cases} \frac{1}{n}\binom{n+(n-1)/2-1}{n-1} & \text{if } n-1 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

7. [Lecture 14] Find the rank of the word $s = 101101$ in the reflected binary code $000000, 000001, 000011, \dots$.

Solution:
The rank, when written in base 2 is the binary sequence b whose ith element is the mod 2 sum of the first i digits of the sequence s.

$$\begin{aligned} s &= 101101 \\ b &= 110110 \end{aligned}$$

We convert this to decimal, to find that the rank of s is

$$32 + 16 + 4 + 2 = 54.$$

8. [Lecture 9] Let $\mathcal{P}_{2n}(x, y)$ be the set of lattice paths that start at (x, y) and end at $(2n, 0)$ using *up-steps* $\nearrow = (1, 1)$ and *down-steps* $\searrow = (1, -1)$. Let $\mathcal{B}_{2n}(x, y)$ be the set of paths in $\mathcal{P}_{2n}(x, y)$ which at some point steps down to the line $y = -1$.

   (a) Show that $|\mathcal{B}_{2n}(0, 0)| = |\mathcal{P}_{2n}(0, -2)|$.
   Solution:
   We define a function $\phi : \mathcal{B}_{2n}(0, 0) \to \mathcal{P}_{2n}(0, -2)$ and show that ϕ is a bijection. For any path P in $\mathcal{B}_{2n}(0, 0)$ we reflect about the line $y = -1$ the part of P that precedes the first time that P touches the line $y = -1$. The resulting path belongs to $\mathcal{P}_{2n}(0, -2)$. Applying the same construction to any path $P' \in \mathcal{P}_{2n}(0, -2)$ results in a path in $P \in \mathcal{B}_{2n}(0, 0)$ for which $\phi(P) = P'$ Therefore ϕ is reversible and onto, so ϕ is a bijection and $|\mathcal{B}_{2n}(0, 0)| = |\mathcal{P}_{2n}(0, -2)|$.

(b) Recall that a *Dyck path* of length $2n$ is any lattice path in $\mathcal{P}_{2n}(0,0)$ that does not touch the line $y=-1$. Use part (a) and the formula $|\mathcal{P}_{2n}(x,y)| = \binom{2n}{n-\frac{x+y}{2}}$ and perhaps a bit of algebra to show that the number of Dyck paths of length $2n$ equals the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Solution:
The Dyck paths are precisely the paths in $\mathcal{P}_{2n}(0,0)$ which are not in $\mathcal{B}_{2n}(0,0)$. The number of such Dyck paths is

$$\begin{aligned}
|\mathcal{P}_{2n}(0,0)| - |\mathcal{B}_{2n}(0,0)| &= |\mathcal{P}_{2n}(0,0)| - |\mathcal{P}_{2n}(0,-2)| \\
&= \binom{2n}{n} - \binom{2n}{n-\frac{0-2}{2}} \\
&= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!} \\
&= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{n!\,n!}\frac{n}{n+1} \\
&= \left(1-\frac{n}{n+1}\right)\frac{(2n)!}{n!\,n!} \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

(c) Recall that a *totally balanced word* of length $2n$ is any binary sequence $W=(w_1,w_2,\dots,w_{2n})$ which can be obtained from a Dyck path of length $2n$ by writing "0" for each up-step and "1" for each down-step. In other words a binary word W is totally balanced if

P1 W has exactly n zeros and n ones,

P2 no prefix $(w_1,w_2,\dots,w_i)$ of W has more ones than zeros.

i. [1 point] What is the last totally balanced word of length $2n$ in the lexicographic order?
Solution:
We want to move each "1" as far to to the left as possible in the last word W. We must not violate condition **P1**, so W must begin $(0,1,\dots)$. For the same reason, W must continue as $(0,1,0,1,\dots)$, and then as $(0,1,0,1,0,1,\dots)$, and so on. Therefore, the last totally balanced word is $0101\dots10$

ii. [2 points] Find the totally balanced word that is the lexicographic successor of $(0,1,0,0,0,1,1,1,0,1)$.

Solution:
We seek the rightmost "0" that can be changed to a "1" without violating condition **P2**. The fifth "0" fails, since changing it results in a word with the prefix $(0,1,0,0,0,1,1,1,1)$, with four 0s and five 1s. However the fourth "0" can be changed to "1", without violating **P2**, giving a word of the form $(0,1,0,0,1,\cdot,\cdot,\cdot,\cdot,\cdot)$. After this change, by **P1**, the last five digits must have two 0s and three 1s, where as many 0s as possible appear before the first "1". That is, the successor ends with $(\dots,0,0,1,1,1)$. Therefore the successor of $(0,1,0,0,0,1,1,1,0,1)$ is $(0,1,0,0,1,0,0,1,1,1)$.

9. [See Lecture 8 notes] Let $\mathcal{L}_{n,k}$ be the listing of the k-element subsets of $[n]$ in *reverse lexicographic order*. That is, we represent each k subset by a decreasing list $M = (m_1, m_2, \ldots, m_k)$, $m_1 > m_2 > \cdots > m_k$, and these decreasing lists are sorted lexicographically. The corank of M is the number of subsets that precede M in the reverse lexicographic order. For example

$$\mathcal{L}_{5,3} = (3,2,1),\ \ (4,2,1),(4,3,1),(4,3,2),\ \ (5,2,1),(5,3,1),(5,3,2),(5,4,1),(5,4,2),(5,4,3)$$

so corank$((5,3,2) = 6$.

Let $M = (m_1, m_2, \ldots, m_k)$ be a list in $\mathcal{L}_{n,k}$. For $i = 1, 2, \ldots, k$, let $c_i(M)$ be the number of lists in $\mathcal{L}_{n,k}$ which begin with $m_1, m_2, \ldots, m_{i-1}$ and have all of its remaining elements less than m_i. For example $c_1((5,3,2)) = 4$, and $c_2((5,3,2)) = 1$, and $c_3((5,3,2)) = 1$.

(a) Find a formula for $c_i(M)$ that depends only on the numbers k, i and m_i.
Solution:
We want to count the lists $(m_1, m_2, \ldots, m_{i-1}, \ell_i, \ell_{i+1}, \ldots \ell_k)$ in $\mathcal{L}_{n,k}$which satisfy

$$m_i > \ell_i > \ell_{i+1} > \cdots > \ell_k \geq 1.$$

Here $\{\ell_i, \ell_{i+1}, \cdots, \ell_k\}$ can be any $(k-i+1)$-subset of $\{1, 2, \ldots, m_i - 1\}$. So $c_i = \binom{m_i-1}{k-i+1}$.

(b) Use your solution to part (a) to find a formula for corank(M), for any $M = (m_1, m_2, \ldots, m_k)$ in $\mathcal{L}_{n,k}$.
Solution:
Each predecessor of M has a smallest index i, with $1 \leq i \leq k$, where its ith entry is less than m_i. This predecessor is counted by $c_i(M)$, so the number of predecessors of M equals

$$\text{corank}(M) = \sum_{i=1}^{k} c_i = \sum_{i=1}^{k} \binom{m_i - 1}{k - i + 1}.$$

(c) Let $(\ell_1, \ldots, \ell_k) \subseteq [n]$ with $\ell_1 < \ell_2 < \cdots < \ell_k$. Let $L = (\ell_1, \ell_2, \ldots, \ell_k)$ and let the *reflection* of L be the list

$$\tilde{L} = (n+1-\ell_1, n+1-\ell_2, \ldots, n+1-\ell_k).$$

Then the elements of $\tilde{L}$ are listed in decreasing order. Write a formula (that we learned in class) that relates rank(L) (in the lexicographic order) and corank$(\tilde{L})$. You do not have to prove the formula.
Solution:

$$\text{rank}(L) + \text{corank}(\tilde{L}) = \binom{n}{k} - 1.$$

(d) State an advantage that the formula from part (c) has, compared to the naïve ranking formula

$$\text{rank}(\ell_1, \ldots, \ell_k) = \sum_{i=1}^{k} \sum_{a=\ell_{i-1}+1}^{\ell_i - 1} \binom{n-a}{k-i}, \quad \text{where } \ell_1 < \ell_2 < \cdots < \ell_k.$$

Solution:
Using (b) and (c) together to compute rank(L) = corank$(\tilde{L}) = \binom{n}{k} - 1 -$ corank$(\tilde{L})$ results in a ranking algorithm which is $O(k)$-time, whereas the naïve summation is $O(n)$-time (the intervals $[\ell_{i-1} + 1.\ell_i - 1]$ are disjoint, so the inner sum is executed at most $n - k$ times). This is a big improvement if k is substanctally smaller than k

10. [See Lecture 10] Find and draw the spanning tree with vertex set $\{1, 2, 3, \ldots, n\}$, for some n, whose Prüfer sequence is $L = (4, 6, 1, 4)$.
Solution:
The sequence has length $n - 2 = 4$, so the vertex set is $\{1, 2, 3, 4, 5, 6\}$. We write a table listing the degrees of each vertex i (which equals one plus the number of times i appears in L), finding the highest degree leaf x, and adding an edge x, y where y is the first entry of L. We decrement the two degrees, delete the first entry from L and repeat until L is empty. Then add a final edge joining the last two leaves We can put the data in a table initialized with L appearing under column y.

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
							4
							6
							1
							4

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2		4
							6
							1
							4

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2	5	4
							6
							1
							4

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2	5	4
2	1	1	2	0	2	3	6
							1
							4

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2	5	4
2	1	1	2	0	2	3	6
2	1	0	2	0	1	6	1
							4

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2	5	4
2	1	1	2	0	2	3	6
2	1	0	2	0	1	6	1
1	1	0	2	0	0	2	4

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2	5	4
2	1	1	2	0	2	3	6
2	1	0	2	0	1	6	1
1	1	0	2	0	0	2	4
1	0	0	1	0	0		

$\rightarrow$

$d =$						E	
d_1	d_2	d_3	d_4	d_5	d_6	x	y
2	1	1	3	1	2	5	4
2	1	1	2	0	2	3	6
2	1	0	2	0	1	6	1
1	1	0	2	0	0	2	4
1	0	0	1	0	0	1	4

Output:
$E = \{\{5, 4\}, \{3, 6\}, \{6, 1\}, \{2, 4\}, \{1, 4\}\}$.
On the exam, you only have to write out the final table.

11. Find the rank of the permutation $[\,p_1p_2p_3p_4p_5\,] = [\,4\ 2\ 5\ 1\ 3\,] \in S_5$ when S_5 is listed in lexicographic order. You can leave your answer unsimplified if you wish.

    Solution:
    There are $3 \cdot 4!$ permutations which begin $[a \qquad\qquad]$ with $a < p_1 = 4$ (here $a \in \{1, 2, 3\}$).
    There are $1 \cdot 3!$ permutations which begin $[4\ a \qquad\quad]$ with $a < p_2 = 2$ (here $a \in \{1\}$).
    There are $2 \cdot 2!$ permutations which begin $[4\ 2\ a \qquad]$ with $a < p_3 = 5$ (here $a \in \{1, 3\}$).
    There are $0 \cdot 1!$ permutations which begin $[4\ 2\ 5\ a \quad]$ with $a < p_4 = 1$ (here $a \in \emptyset$).
    There are $0 \cdot 0!$ permutations which begin $[4\ 2\ 5\ 1\ a \ ]$ with $a < p_5 = 3$ (again $a \in \emptyset$). This counts every permutation which is listed before $[\,4\ 2\ 5\ 1\ 3\,]$ exactly once, so the rank is

$$3 \cdot 4! + 3! + 2 \cdot 2! = 110.$$