1. Which of the following is a combinatorial class? Give reasons.

   (a) $\mathcal{S} = \text{SEQ}(\{0,1,2,3,\dots\})$, where $\text{SIZE}((s_1,s_2,\dots,s_k)) = \sum_{i=1}^k s_i$.
   No. The family contains an infinite set $\{0, 00, 000, 0000, 00000, \dots\}$ of words in $\mathcal{S}$ all size 0.

   (b) $\mathcal{T} = \text{SEQ}(\{0,1,2,3\dots\})$, where $\text{SIZE}((s_1,s_2,\dots,s_k)) = k$.
   No. We have $\{(0),(1),(2),(3),\dots,\} \subseteq \mathcal{S}$. An inifinity numbe of words in $\mathcal{S}$ have size 1.

   (c) $\mathcal{U} = \text{SEQ}(\{0,1,2,3,\dots\})$, where $\text{SIZE}((s_1,s_2,\dots,s_k)) = k + \sum_{i=1}^k s_k$.
   Yes. Every word size is a non-negative integer. It remains to show that, for every integer $n \geq 0$, there are a finite number of words in $\mathcal{U}_n$. Let $n \geq 0$, each word $(s_1,s_2,\dots,s_k)$ of size n satisfies $k \leq n$. Also the largest entry in that word satisfies

$$\max\{s_1,s_2,\dots,s_l\} \leq \sum_{i=1}^{k} s_k \leq n.$$

   We have shown that every word of size n is a sequence of length at most n with entries from $\{0,1,2,\dots,n\}$. That is,

$$\mathcal{U}_n \subseteq \text{SEQ}_{\leq n}(\{0,1,2,\dots,n\})$$

   There are $(n+1)^k$ sequences in $\text{SEQ}_k((\{0,1,2,\dots,n\}))$. Therefore

$$U_n \leq 1 + (n+1) + (n+1)^2 + \cdots + (n+1)^n,$$

   which is a finite number.

2. Suppose that the OGF of a combinatorial set $\mathcal{G}$ is

$$G(z) = \frac{1+3z}{1-6z+9z^2}$$

Find G_n, for all $n \geq 0$.
Solution 1: (with partial fractions)
(Note: In an exam, the steps below would probably be separated and tested independently.)
The numerator has lower degree than the denominator, so can already decompose into partial fractions. The denominator of $F(z)$ factors as $(1-3z)^2$ so we seek a partial fraction decomposition of the following form

$$\frac{1+3z}{(1-3z)^2} = \frac{A}{(1-3z)^2} + \frac{B}{1-3z}.$$

Expanding this equation and comparing coefficients gives

$$\begin{aligned} 1+3z = A + B(1-3z) &= (A+B) \ - \ 3B\,z \\ 1 = A+B \qquad &\text{and} \qquad 3 = -3B. \\ A = 2 \qquad &\text{and} \qquad B = -1. \end{aligned}$$

So

$$\begin{aligned} G_n &= [z^n]C(z) \\ &= [z^n]\left(\frac{2}{1-3z} - \frac{1}{1-3z}\right) \\ &= 2[z^n]\frac{1}{(1-3z)^2} - [z^n]\frac{1}{1-3z} \\ &= 2\binom{n+2-1}{2-1}3^n - 3^n \qquad \text{(by the extended binomial theorem)} \\ &= 2(n+1)3^n - 3^n \\ &= (2n+1)3^n. \end{aligned}$$

Solution 2 (more direct, with extended binomial theorem): The denominator of $F(z)$ factors as $(1-3z)^2$. We can just expand the numerator and do every term separately.

$$\begin{aligned}
G_n &= [z^n]G(z) \\
&= \left([z^n]\frac{1}{(1-3z)^2}\right) + \left([z^n]3z\frac{1}{(1-3z)^2}\right) \\
&= \left(\binom{n+2-1}{2-1}3^n\right) + 3\left([z^{n-1}]\frac{1}{(1-3z)^2}\right) \\
&= (n+1)3^n + 3\cdot\binom{(n-1)+2-1}{2-1}3^{n-1} && \text{(by the extended binomial theorem)} \\
&= (n+1)3^n + n3^n \\
&= (2n+1)3^n.
\end{aligned}$$

3. Find the coefficient

$$[z^3](1-8z)^{\frac{1}{4}}$$

The binomial theorem gives

$$\begin{aligned}
[z^3](1-8z)^{\frac{1}{4}} &= \binom{\frac{1}{4}}{3}(-8)^3 \\
&= \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)\cdot\left(\frac{1}{4}-2\right)}{3!}(-8)^3 \\
&= \frac{\frac{1}{4}\cdot\left(-\frac{3}{4}\right)\cdot\left(-\frac{7}{4}\right)}{3!}(-8)^3 \\
&= \frac{21}{4^3\cdot 6}(-8)^3 \\
&= \frac{7}{2}(-2)^3 \\
&= -28.
\end{aligned}$$

4. Suppose that the ordinary generating function of combinatorial class $\mathcal{C}$ is $C(z) = \dfrac{7z+1}{1-z-6z^2}$. Find the number C_n of objects having size n.

Since the numerator of $G(z)$ already has smaller degree than the denominator, we do not have to do long division. We factor $1-z-6z^2 = (1-3z)(1+2z)$. So the partial fraction expansion takes the form

$$G(z) = \frac{7z+1}{(1-3z)(1+2z)} = \frac{A}{1-3z} + \frac{B}{1+2z}.$$

Expanding this equation and comparing coefficients gives

$$\begin{aligned}
7z+1 &= A(1+2z) + B(1-3z) = (A+B) \;+\; (2A-3B)\,z \\
1 &= A+B \qquad \text{and} \qquad 7 = 2A-3B. \\
A &= 2 \qquad \text{and} \qquad B = -1.
\end{aligned}$$

So

$$\begin{aligned}G_n &= [z^n]G(z)\\ &= [z^n]\left(\frac{2}{1-3z} - \frac{1}{1+2z}\right)\\ &= 2[z^n]\frac{1}{1-3z} - [z^n]\frac{1}{1-(-2)z}\\ &= 2\cdot 3^n - (-2)^n.\end{aligned}$$

5. Let $\mathcal{W}$ be the set of binary words where every nonempty block of 1s has length 2, 3, or 5.
   (a) Describe the class $\mathcal{W}$ with a regular expression, using the symbols $0, 1, +, (\cdot)^*, \epsilon$.
   (b) Find the ordinary generating function $W(z)$.
   (c) Using a computer, we find the factorization

$$1 - z - z^3 - z^4 - z^6 = (1-az)(1-bz)(1-cz)(1-dz)(1-ez)$$

We present one of several correct solutions for this question.

(a) If we cut each word just before each block of 1s (plus one more cut right after the word, if it ends with a 0), then the first block is any sequence of zeros (this includes the empty string). Each middle block is a word from $(11+111+11111)$ followed by at least one zero, and the last block is either empty or a word from $(11+111+11111)$. Therefore

$$\mathcal{W} = 0^* \left((11+111+11111)00^*\right)^* (\epsilon + 11 + 111 + 11111).$$

(b) Each of "0" and "1" is an atom, so from part (a) we have,

$$\begin{aligned}W(z) &= \frac{1}{1-z}\,\frac{1}{1-\left((z^2+z^3+z^5)z\frac{1}{1-z}\right)}(1+z^2+z^3+z^5)\\ &= \frac{1+z^2+z^3+z^5}{(1-z)-(z^3+z^4+z^6)}\\ &= \frac{1+z^2+z^3+z^5}{1-z-z^3-z^4-z^6}.\end{aligned}$$

The last two equations are optional, as I did not ask you to simplify the solution.

6. Let $\mathcal{H} = \textsc{Seq}(\{0,1,2\})$ the class of ternary sequences with $\textsc{Size}((h_1, h_2, \ldots, h_k)) = k$.
   (a) Find the OGF $H(z)$.
   The OGF for $\{0,1,2\}$ is $3z$, so the OGF for $\mathcal{H}$ is $H(z) = \dfrac{1}{1-3z}$.
   (b) Use $H(z)$ to find a formula for H_n.
   We have $H_n = [z^n]H(z) = 3^n$.

7. For $n = 0, 1, 2\ldots$, find the coefficient of z^n in the series expansion of $\dfrac{1+5z-2z^2}{1-2z}$.
   Using long division (at right) we find

$$\frac{1+5z-2z^2}{1-2z} = -2 + z + \frac{3}{1-2z}.$$

$$\begin{array}{r|l} & z - 2\\ \hline 2z-1 & -2z^2+5z+1\\ & -2z^2 + z\\ \hline & 4z+1\\ & 4z-2\\ \hline & 3\end{array}$$

Since $[z^n]\dfrac{3}{1-2z} = 3 \cdot 2^n$, we have

$$[z^n]\frac{-2+z-2z^2}{1-2z} = \begin{cases} 3 \cdot 2^0 - 2 = 1 & \text{if } n = 0 \\ 3 \cdot 2^1 + 1 = 7 & \text{if } n = 1 \\ 3 \cdot 2^n & \text{if } n \geq 2. \end{cases}$$

8. Describe the first step in extracting the coefficient of z^n in the series of the following rational function? (Do **not** attempt to find the coefficient, just show and/or describe what must be done.)

$$G(z) = \frac{2z-3}{(1-3z)^3(1+z)(1-4z)}$$

The numerator has lower degree than the denominator, so we are ready to decompose into partial fractions of the form

$$\frac{2z-3}{(1-3z)^3(1+z)(1-4z)} = \frac{A}{(1-3z)^3} + \frac{B}{(1-3z)^2} + \frac{C}{1-3z} + \frac{D}{1+z} + \frac{E}{1-4z}$$

9. How many objects of the combinatorial class $\mathcal{A}$ have size one, if its ordinary generating function is

$$A(z) = \frac{1+5z^2-\sqrt{1-6z^2+z^4}}{4z} \ ?$$

This question is likely just a little too hard for an exam.

$$\begin{aligned}
[z^1]A(z) = [z^1]\frac{1+5z^2-\sqrt{1-6z^2+z^4}}{4z} &= [z^2]\frac{1+5z^2-\sqrt{1-6z^2+z^4}}{4} \\
&= [u^1]\frac{1+5u-\sqrt{1-6u+u^2}}{4} \qquad \text{(where } u = z^2 \text{)} \\
&= \frac{1}{4}\left(5 - [u^1]\left(1+(u^2-6u)\right)^{1/2}\right) \\
&= \frac{1}{4}\left(5 - [u^1]\sum_{k\geq 0}\binom{1/2}{k}(u^2-6u)^k\right) \qquad \text{(extended binomial theorem)}
\end{aligned}$$

The term u^1 only appears in the expansion of $(u^2-6u)^k$ when $k=1$, so

$$\begin{aligned}
[z^1]A(z) &= \frac{1}{4}\left(5 - [u^1]\binom{1/2}{1}(u^2-6u)^1\right) \\
&= \frac{1}{4}\left(5 - \binom{1/2}{1}[u^1](u^2-6u)\right) \\
&= \frac{1}{4}\left(5 - \frac{1}{2}(-6)\right) \\
&= \frac{1}{4}(5+3) \\
&= 2.
\end{aligned}$$

10. Let $\mathcal{W}^{(k)}$ be the class of binary strings counted by length, which have no more than k consecutive 0s. Show that the generating function for $\mathcal{W}^{(k)}$ is

$$W^{(k)}(z) = \frac{1-z^{k+1}}{1-2z+z^{k+2}}.$$

This question might be a bit long for a midterm exam. We find a specification for this family. We can break a $w \in \mathcal{W}^{(k)}$ into blocks by cutting at every location which is either just after a run of zeros or just before a run of ones. The resulting sequence of blocks uniquely specifies w, because the blocks arize from a well-defined decomposition procedure.

The first block $\mathcal{F}$ is a (possibly empty) sequence of zeros having size at most k.

$$\mathcal{F} = \epsilon + 0 + 00 + 0^3 + \cdots + 0^k = 0^* - 0^{k+1}.$$

It generating function is a partial geometric sum.

$$F(z) = 1 + z + z^2 + \cdots + z^k = \frac{1 - z^{k+1}}{1 - z}.$$

The last block $\mathcal{L}$ is a (possibly empty) sequence of ones.

$$\mathcal{L} = 1^* \qquad \text{and} \qquad L(z) = \frac{1}{1-z}$$

All other blocks belong to the class $\mathcal{M}$ of words consisting of a a positive number of ones followed by between 1 and k zeros.

$$\begin{aligned}\mathcal{M} &= 1(1)^* \, (0 + 00 + 000 + \cdots + \overbrace{00\ldots 0}^{k}) \\ &= 1(1)^* 0(\epsilon + 0 + 00 + \cdots + \overbrace{00\ldots 0}^{k-1}).\end{aligned}$$

Its OGF is

$$M(z) = z \cdot \frac{1}{1-z} \cdot z \cdot \frac{1 - z^k}{1-z} = \frac{z^2 - z^{k+2}}{(1-z)^2}.$$

Since $\mathcal{W}^{(k)} = \mathcal{F}\mathcal{M}^*\mathcal{L}$, its generating function is

$$\begin{aligned}W^{(k)}(z) &= F(z) \frac{1}{1 - M(z)} L(z) \\ &= \frac{1 - z^{k+1}}{1 - z} \frac{1}{1 - \frac{z^2 - z^{k+2}}{(1-z)^2}} \frac{1}{1-z} \\ &= \frac{1 - z^{k+1}}{(1-z)^2 - (z^2 - z^{k+2})} \\ &= \frac{1 - z^{k+1}}{1 - 2z + z^2 - z^2 + z^{k+2}} \\ &= \frac{1 - z^{k+1}}{1 - 2z + z^{k+2}}.\end{aligned}$$