Dyck paths and specifications

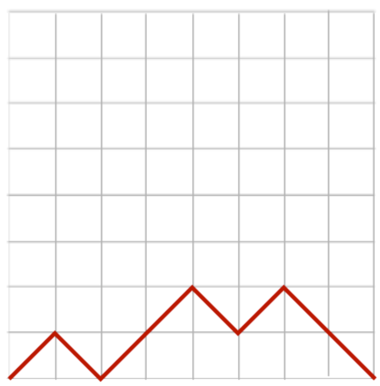
Contents

1 Dyck Paths

1.1 Decomposing Dyck paths

Definition. A **Dyck path** is a path on $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$ that never steps below the line $y=0$ with steps from the set $\{(1,1),(1,-1)\}$. The weight of a Dyck path is the total number of steps.

Here is a Dyck path of length 8:



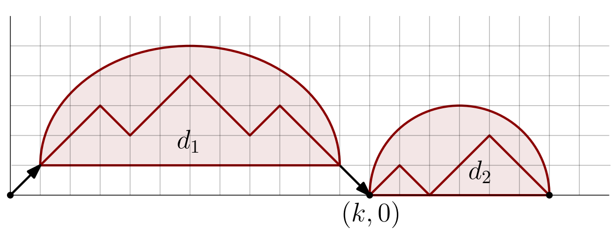
Let $\mathcal{D}$ be the combinatorial class of Dyck paths. Every Dyck path is a sequence comprised of two atoms

$$\mathcal{Z}_{\nearrow} = (1,1) - \text{step} \qquad \mathcal{Z}_{\searrow} = (1,-1) - \text{step}$$

The *trivial* Dyck path is a walk of length 0, which we represent by the symbol ϵ. Every nontrivial Dyck path must begin with a $\mathcal{Z}_{\nearrow}$ and must end with a $\mathcal{Z}_{\searrow}$. There are a few ways to decompose a Dyck path.

Here is the first way to do this. Let d be a Dyck path. If $d \neq \epsilon$, then there is at least place where d touches the x-axis after a $\mathcal{Z}_{\searrow}$-step. We break d at the first point that it returns to the x-axis, say at $(k,0)$. This breaks the Dyck path into two blocks.

The first block must start with $\mathcal{Z}_{\nearrow}$, end with $\mathcal{Z}_{\searrow}$, and never goes below $x=1$ between these steps. Between these two steps is another Dyck path, call it d_1, but it is shifted up and to the right by one unit, starting at $(1,1)$ and ending at $(k-1,1)$. The second block is another Dyck path, call it d_2, that is shifted to the right by k units.

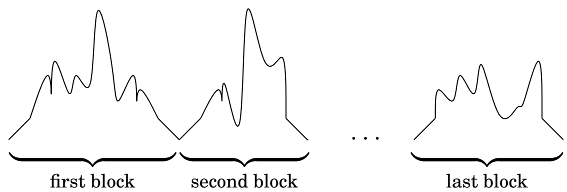


It is evident that if $d \neq \epsilon$ then d is specified uniquely by the 4-tuple $(\mathcal{Z}_{\nearrow}, d_1, \mathcal{Z}_{\searrow}, d_2)$, Moreover any pair (d_1, d_2) of Dyck paths results in a non-trivial Dyck path by this construction.

The above discussion justifies the following combinatorial equivalence.

$$\mathcal{D} \cong \epsilon + (\mathcal{Z}_{\nearrow} \times \mathcal{D} \times \mathcal{Z}_{\searrow} \times \mathcal{D}) \tag{1}$$

A second way to decompose a Dyck path d is to break it whereever d touches the x-axis. This breaks d into a sequence of blocks, where each blocks begins with $\mathcal{Z}_{\nearrow}$, ends with $\mathcal{Z}_{\searrow}$ and never touches the x-axis strictly inside a block.



Every Dyck path is uniquely specified by a sequence of such blocks. This gives a second specification for Dyck paths.

$$\mathcal{D} \cong \textsc{Seq}(\mathcal{Z}_{\nearrow} \times \mathcal{D} \times \mathcal{Z}_{\searrow}) \tag{2}$$

1.2 Connection to trees

Either of the above specifications (1) and (2) leads us to a quadratic equation that must be satisfied by the OGF $D(z)$ for Dyck paths. We find $D(z)$ by solving the equation in the usual manner. For example, (2) gives

$$\begin{aligned} D(z) &= \frac{1}{1 - z^2 D(z)} \\ z^2 D(z)^2 - D(z) + 1 &= 0 \\ D(z) &= \frac{1 - \sqrt{1 - 4z^2}}{2z^2}. \end{aligned}$$

As was the case for Catalan numbers, we chose "$-$" in the quadratic formula to eliminate the term of the form $\frac{A}{z^2}$. You may recognize the similarity between $D(z)$ and the generating function for binary plane trees that we saw in class[1].

Indeed we shall see that Dyck paths are quite closely related to the class $\mathcal{B}$ of binary plane trees: If we substitute $t = z^2$ in the formula for $D(z)$, then it becomes identical to the OGF for binary plane trees.

$$\frac{1 - \sqrt{1 - 4t}}{2t} = B(t).$$

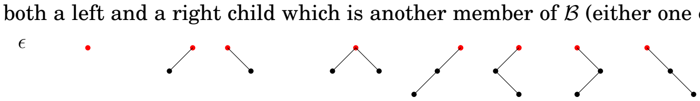
This tells us a few things.

First since $D(z)$ depends only on z^2, all powers of z appearing in the generating function must be even. So there can only be Dyck paths of even length. Looking back at the definition of Dyck paths can you see why this must be true?

Second since $D(z) = B(z^2)$, we have $[z^{2n}]D(z) = [z^n]B(z^2)$, so there are the same number of Dyck paths of length $2n$ as there are binary plane trees with n vertices. We might ask whether there is a

[1]Here it is again: Let $\mathcal{B}$ be the class of plane rooted binary trees, including the empty tree ϵ, where size is the number of nodes. Specifically, each node is an atom $\mathcal{Z}$ which has both a left and a right child which is another member of $\mathcal{B}$ (either one or both of these subtrees can be the empty tree ϵ).

Here are all the binary trees of size at most 3:



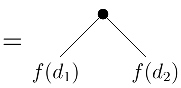
Every tree in $\mathcal{B}$ except for ϵ uniquely decomposes like this, so $\mathcal{B} \cong \mathcal{E} + \mathcal{B} \times \mathcal{Z} \times \mathcal{B}$. The OGF thus satisfies $B(z) = 1 + zB(z)^2$, which solves to $B(z) = \frac{1-\sqrt{1-4z}}{2z}$. This is the OGF for Catalan numbers that we saw in Lecture 2, so the number of such trees with n nodes is $B_n = [z^n]B(z) = \frac{1}{n+1}\binom{2n}{n}$.

nice way to see this directly from the combinatorial objects. Is there a natural bijection between Dyck paths of length $2n$ and binary plane trees with n nodes? Here is one such bijection which relates to the combinatorial decomposition (1) above.

Let d be a Dyck path. Define the binary plane tree $f(d)$ inductively as follows. If $d = \epsilon$, then $f(d)$ is the empty tree. Otherwise, decompose

$$d = (\mathcal{Z}_{\nearrow}, d_1, \mathcal{Z}_{\searrow}, d_2)$$

as per (1). Here d_1 and d_2 have strictly smaller length than d. We define

$f(d) =$ 

Now we just need to check that f is a bijection. We could do this in the usual way: check it is one-to-one and onto, but here is another approach which is often useful for these combinatorial bijections: describe what should be the inverse of f. Namely given a binary plane tree t, we define a Dyck path $g(t)$ by

$$g(t) = \begin{cases} \epsilon & \text{if } t = \epsilon \\ \mathcal{Z}_{\nearrow}\ \ g(t_1)\ \ \mathcal{Z}_{\searrow}\ \ g(t_2) & \text{if } t \text{ consists of a root with left child } t_1 \text{ and right child } t_2. \end{cases}$$

Then it suffices to check that $g(f(d)) = d$ for all Dyck paths d, and that $f(g(t)) = t$ for all binary plane trees t. We also verify that if d has $2n$ steps, then $f(d)$ has n nodes. We are relying here on the following set-theoretic proposition using (applied to $\mathcal{D}_{2n}$ and $\mathcal{T}_n$).

Proposition. *Let A and B be sets. Let $f : A \to B$ and $g : B \to A$. If $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$, then f is a bijection (and g is its inverse bijection).*

Proof. By symmetry it suffices to prove that f is a bijection.
Suppose $f(a_1) = f(a_2)$ with $a_1, a_2 \in A$. Then $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$ so f is one-to-one.
Take $b \in B$. Then $f(g(b)) = b$ and $g(b) \in A$, so f is onto. Thus f is a bijection. □

Exercise. For $f : \mathcal{D} \to \mathcal{T}$ and $g : \mathcal{T} \to \mathcal{D}$ as defined before the proposition, prove that $f(\mathcal{D}_{2n}) \subseteq \mathcal{T}_n$ and that $g(f(d)) = d$ for all Dyck paths $w \in \mathcal{D}$ and $f(g(t)) = t$ for all binary plane trees $t \in \mathcal{B}$.

2 Combinatorial specifications

2.1 Recall

Elemental classes: Let $\mathcal{E}$ denote the **neutral class** that consists of a single object ϵ of size zero called the **neutral element**, so $\mathcal{E} = \{\epsilon\}$ and $E(z) = 1$. Let $\mathcal{Z}_0, \mathcal{Z}_1, \dots$ denote the **atomic classes**, each of which that consists of a single object z_i of size 1 called an **atom**, so $\mathcal{Z}_i = \{z_i\}$ and $Z_i(z) = z$, for $i = 0, 1, 2, \dots$.

Basic Admissible Constructions

Cartesian Product: The cartesian product construction applied to two classes $\mathcal{B}, \mathcal{C}$ give the set of ordered pairs, $\mathcal{A} = \mathcal{B} \times \mathcal{C} = \{\alpha = (\beta, \gamma) | \beta \in \mathcal{B}, \gamma \in \mathcal{C}\}$ where the size of a pair is additive, namely $|\alpha|_{\mathcal{A}} = |\beta|_{\mathcal{B}} + |\gamma|_{\mathcal{C}}$. Hence the cartesian product operation is admissible and produces a new combinatorial class $\mathcal{A}$. By considering all possible pairs of total size n, we see that the counting sequences satisfy $A_n = \sum_{k=0}^{n} B_k C_{n-k}$. Thus their generating functions satisfy $A(z) = B(z) \cdot C(z)$.

Union: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be combinatorial classes such that $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ and $\mathcal{B} \cap \mathcal{C} = \varnothing$ with size defined in a compatible way; for $\omega \in A$ we have

$$|\omega|_{\mathcal{A}} = \begin{cases} |\omega|_{\mathcal{B}} & \omega \in \mathcal{B} \\ |\omega|_{\mathcal{C}} & \omega \in \mathcal{C}. \end{cases}$$

Then $\mathcal{A}$ is a combinatorial class and we may write $\mathcal{A} = \mathcal{B} + \mathcal{C}$. The union construction is admissible and we have $A_n = B_n + C_n$ and $A(z) = B(z) + C(z)$.

Sequence: If $\mathcal{B}$ is a class then the sequence class $\textsc{Seq}(\mathcal{B})$ is defined to be the infinite sum

$$\mathcal{A} = \textsc{Seq}(\mathcal{B}) = \mathcal{E} + \mathcal{B} + (\mathcal{B} \times \mathcal{B}) + (\mathcal{B} \times \mathcal{B} \times \mathcal{B}) + \dots$$

Equivalently, $\mathcal{A} = \{(\beta_1, \beta_2, \dots, \beta_\ell) \mid \beta_j \in \mathcal{B}, \ell \geq 0\}$. The size functions satisfy

$$|(\beta_1, \beta_2, \dots, \beta_\ell)|_{\mathcal{A}} = \sum_{i=1}^{\ell} |\beta_i|_{\mathcal{B}}$$

Notice $\textsc{Seq}(\mathcal{B})$ is a combinatorial class **only if** $\mathcal{B}$ **does not contain a neutral element** (Why?).

If we want all sequences that contains exactly k-objects or at least k objects then we might write $\textsc{Seq}_k(\mathcal{B}) = \mathcal{B}^k$ and $\textsc{Seq}_{\geq k}(\mathcal{B}) = \mathcal{B}^k \times \textsc{Seq}(\mathcal{B})$

2.2 Defining a class: A specification

Definition. A **specification** for an r-tuple of classes $\mathcal{A} = (\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(r)})$ is a set of r equations

$$\begin{aligned}\mathcal{A}^{(1)} &= \Phi_1(\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(r)})\\ &\vdots\\ \mathcal{A}^{(r)} &= \Phi_r(\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(r)})\end{aligned}$$

where each Φ_i is an expression obtained by successively applying some of the above admissible constructions to some of the following combinatorial classes:

1. atomic classes $\mathcal{Z}_0, \mathcal{Z}_1, \dots$.
2. the neutral class $\mathcal{E}$, and
3. the combinatorial classes $\mathcal{A}^{(i)}$, $i = 1, \dots, r$

Until now we have only encountered specifications with $r = 1$ (a single equation).

Definition. If every formula Φ_i can be expressed in terms of elemental classes $\mathcal{E}$, $\mathcal{Z}_0, \mathcal{Z}_1, \dots$ then the specification is said to be **iterative**. If at least one expression Φ_i must ultimately refer the class $\mathcal{A}_i$, then the specification is said to be **recursive**.

More precisely, we consider the directed graph D of dependencies among these classes. Here D has vertices $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}, \dots, \mathcal{A}^{(r)}$. For every class $\mathcal{A}^{(i)}$ that appears in a formula Φ_j, we draw a directed edge from $\mathcal{A}^{(i)}$ to $\mathcal{A}^{(j)}$. If D has no directed cycles, then the construction is **iterative**. If D has a directed cycle, then it is **recursive**.

So far we have seen iterative specifications for classes of binary words having the form

$$\mathcal{A} = \Phi(\mathcal{E}, \mathcal{Z}_0, \mathcal{Z}_1).$$

We have seen recursive constructions for Dyck paths and some binary trees which took the form

$$\mathcal{A} = \Phi(\mathcal{E}, \mathcal{Z}, \mathcal{A}).$$

For iterative constructions, the generating functions for $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}, \dots, \mathcal{A}^{(r)}$ can be deduced in a relatively straight-forward manner.

For recursive constructions the generating functions can only sometimes be derived, as was the case with Dyck paths and binary trees.

Definition. A class is said to be **constructible** (or specifiable) iff it admits a specification in terms of admissible operators.