1. For $n = 0, 1, 2\ldots$, find the coefficient of z^n in the series expansion of $\dfrac{1+5z-2z^2}{1-2z}$.

2. Suppose that the combinatorial family $\mathcal{F}$ has ordinary generating function $F(x) = \dfrac{1+3z}{1-6z+9z^2}$. Find F_n, the number of objects in $\mathcal{F}$ having size n.

3. Describe the first step in extracting the coefficient of z^n in the series of the following rational function? (Do **not** attempt to find the coefficient, just show and/or describe what must be done.)

$$G(z) = \frac{2z-3}{(1-3z)^3(1+z)(1-4z)}$$

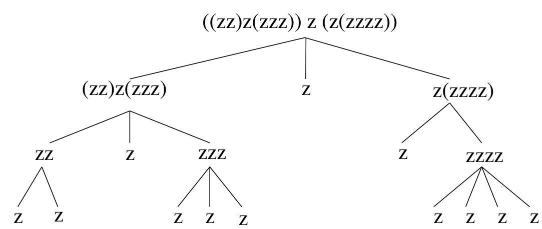
4. How many objects of the combinatorial class $\mathcal{A}$ have size one, if its ordinary generating function is

$$A(z) = \frac{1+5z^2-\sqrt{1-6z^2+z^4}}{4z} \quad ?$$

5. [See Lecture 6] Here is a combinatorial class that you have not yet seen. Informally, a *bracketing* is any "legal" way to place parentheses on a non-empty sequence of symbols. For example, there are exactly 11 bracketings of the sequence $zzzz$.

$$zzzz,\ (zz)zz,\ (zzz)z,\ z(zz)z,\ z(zzz),\ zz(zz),\ ((zz)z)z,\ (z(zz))z,\ z((zz)z),\ z(z(zz)),\ (zz)(zz).$$

More formally, the atomic symbol z is itself a bracketing; and any sequence of two or more consecutive bracketings enclosed by a pair of parentheses is a bracketing. To simplify notation, we have written z instead of (z) and we have also removed the outermost parentheses. For example, the bracketing $zz(zz)$ should interpreted to be $((z)(z)((z)(z)))$. The construction tree for the bracketing $((zz)z(zzz))z(z(zzzz))$ is shown on the right.



Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$ be the combinatorial class where $\mathcal{B}_n$ is the set of bracketings of the sequence $\overbrace{zz\ldots z}^{n}$, for $n \geq 1$.

(a) Find a recursive specification for $\mathcal{B}$ in terms of the atomic class $\mathcal{Z} = \{z\}$, the neutral class $\mathcal{E}$, and the operators '$+$', '$\times$' and 'Seq'.

(b) Use your specification to show the the OGF for $\mathcal{B}$ is $B(z) = \frac{1}{4}\left(1 + z - \sqrt{1-6z+z^2}\right)$.

(c) Use the formula from part (b) and the following table to find B_5, the number of bracketings of $zzzz$. You may leave your answer in a "basic-calculator ready" form, such as $B_5 = \frac{3}{8} \cdot \left(\frac{13}{35} - \frac{13\cdot7}{8} + \frac{17\cdot4}{3}\right)$.

k	$\binom{1/2}{k}$	$(z^2-6z)^k$
1	$\frac{1}{2}$	$z^2 - 6\,z$
2	$\frac{-1}{8}$	$z^4 - 12\,z^3 + 36\,z^2$
3	$\frac{1}{16}$	$z^6 - 18\,z^5 + 108\,z^4 - 216\,z^3$
4	$\frac{-5}{128}$	$z^8 - 24\,z^7 + 216\,z^6 - 864\,z^5 + 1296\,z^4$
5	$\frac{7}{256}$	$z^{10} - 30\,z^9 + 360\,z^8 - 2160\,z^7 + 6480\,z^6 - 7776\,z^5$
6	$\frac{-21}{1024}$	$z^{12} - 36\,z^{11} + 540\,z^{10} - 4320\,z^9 + 19440\,z^8 - 46656\,z^7 + 46656\,z^6$

6. [See Lecture 17] Draw the first six levels of the generating tree specified by the rule $[(0); \{(k) \to (k-1)(k+1)\}]$.

7. [Lectures 18, 20] Suppose we are using a backtrack search with pruning to solve an instance of

$$\texttt{Knapsack}(p_1, p_2, \dots, p_n; w_1, w_2, \dots, w_n; M): \quad \max \sum_{i=1}^{n} p_i x_i \ \text{ subject to } \ \sum_{i=1}^{n} w_i x_i \le M$$
$$\text{and} \quad x_i \in \{0, 1\}, \quad i = 1, 2, \dots n.$$

with $n = 7$ and input data $((p_i), (w_i), M)$ sorted so that $\frac{p_1}{w_1} \ge \frac{p_2}{w_2} \ge \cdots \ge \frac{p_n}{w_n}$, as given below. Suppose the algorithm is at currently executing at depth $m = 3$ of the search tree with the current partial solution $X = [x_1, x_2, x_3] = [1, 0, 1]$. The current best solution found so far, `OptX`, is also given in the table below.

Maximum Weight: $M = 100$

Item	i	1	2	3	4	5	6	7
Profit	p_i	90	65	50	20	10	9	1
Weight	w_i	30	60	50	25	15	15	5
Current Best Solution `OptX`	$\texttt{OptX}_i$	1	1	0	0	0	0	1
Current Partial Solution `X`	x_i	1	0	1				
Current search depth: $m = 3$				$\uparrow$				

(a) Find `OptP`, the profit realized by `OptX`.

(b) Assume that we are pruning infeasible solutions. Find the choice set $C[3] \subseteq \{0, 1\}$, the set of values for x_4 that will keep $[x_1, x_2, x_3, x_4]$ feasible. Give a reason for your answer

(c) Suppose we continueed our search at depth $m = 4$ by selecting $x_4 = 0$, so now $X = [1,\ 0,\ 1,\ 0\ ]$. Use the bounding function we saw in class,

$$\texttt{B} = \sum_{i=1}^{m} x_i p_i + \texttt{RationalKnapsack}(p_{m+1}, p_{m+2}, \dots, p_n;\ w_{m+1}, w_{m+2}, \dots, w_n;\ M - \texttt{CurW})$$

to decide whether or not to prune the current node X from further exploration. For full marks,

i. write explicitly the data that will be input into `RationalKnapsack`,

ii. find an optimal solution to this instance of `RationalKnapsack` and the value of `B`.

iii. show how to decide whether or not the current node $X = [1,\ 0,\ 1,\ 0\ ]$ should be pruned from the search tree.

8. [Lecture 16] The Johnson-Trotter minimal change order for the permutations of $\{1, 2, 3\}$ are the successive rows of the matrix shown at right. Write down the **four** permutations of $\{1, 2, 3, 4\}$ that come immediately after the permutation $\pi = [3\ 1\ 4\ 2]$ in the Johnson-Trotter order.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

9. [Lecture 7] Write a recursive specification for $\mathcal{T}$, the set of plane trees with red and blue nodes, where

   - every blue node has no blue children and even number of red children, and
   - every red node has either no children or it has exactly one blue child and exactly one red child, in either order.

   (a) Use the notation $\mathcal{T} = \mathcal{T}_r + \mathcal{T}_b$ where $\mathcal{T}_r$ are those trees with a red root. Your specification should be three equations which involve $\mathcal{T}_b$, $\mathcal{T}_r$ and $\mathcal{T}$.

   (b) Find a set of equations which, if solved, give the generating function $T(z)$ for $\mathcal{T}$. Do not solve the equations!

10. [Lecture 6] Let $\mathcal{W}^{(k)}$ be the class of binary strings counted by length, which have no more than k consecutive 0s. Show that the generating function for $\mathcal{W}^{(k)}$ is

$$W^{(k)}(z) = \frac{1 - z^{k+1}}{1 - 2z + z^{k+2}}.$$

11. [Lecture 6] Let $\mathcal{T}$ be the set of non-empty rooted planar trees where the number of children that each node has belongs to the set $\Omega = \{0, 2, 4, 6, \dots\}$ (any even number). The size of a tree in $\mathcal{T}$ is the number of nodes in the tree. Use Lagrange inversion to compute T_n, the number of trees in $\mathcal{T}$ with size n.

12. [Lecture 14] Find the rank of the word $s = 101101$ in the reflected binary code $000000, 000001, 000011, \dots$.

13. [Lecture 9] Let $\mathcal{P}_{2n}(x, y)$ be the set of lattice paths that start at (x, y) and end at $(2n, 0)$ using *up-steps* $\nearrow = (1, 1)$ and *down-steps* $\searrow = (1, -1)$. Let $\mathcal{B}_{2n}(x, y)$ be the set of paths in $\mathcal{P}_{2n}(x, y)$ which at some point steps down to the line $y = -1$.

    (a) Show that $|\mathcal{B}_{2n}(0, 0)| = |\mathcal{P}_{2n}(0, -2)|$.

    (b) Recall that a ***Dyck path*** of length $2n$ is any lattice path in $\mathcal{P}_{2n}(0, 0)$ that does not touch the line $y = -1$. Use part (a) and the formula $|\mathcal{P}_{2n}(x, y)| = \binom{2n}{n - \frac{x+y}{2}}$ and perhaps a bit of algebra to show that the number of Dyck paths of length $2n$ equals the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

    (c) Recall that a *totally balanced word* of length $2n$ is any binary sequence $W = (w_1, w_2, \dots, w_{2n})$ which can be obtained from a Dyck path of length $2n$ by writing "0" for each up-step and "1" for each down-step. In other words a binary word W is totally balanced if

    P1 W has exactly n zeros and n ones,

    P2 no prefix $(w_1, w_2, \dots, w_i)$ of W has more ones than zeros.

    i. [1 point] What is the last totally balanced word of length $2n$ in the lexicographic order?

    ii. [2 points] Find the totally balanced word that is the lexicographic successor of $(0, 1, 0, 0, 0, 1, 1, 1, 0, 1)$.

14. [See Lecture 8 notes] Let $\mathcal{L}_{n,k}$ be the listing of the k-element subsets of $[n]$ in *reverse lexicographic order*. That is, we represent each k subset by a decreasing list $M = (m_1, m_2, \ldots, m_k)$, $m_1 > m_2 > \cdots > m_k$, and these decreasing lists are sorted lexicographically. The corank of M is the number of subsets that precede M in the reverse lexicographic order. For example

$$\mathcal{L}_{5,3} = (3,2,1),\quad (4,2,1),(4,3,1),(4,3,2),\quad (5,2,1),(5,3,1),(5,3,2),(5,4,1),(5,4,2),(5,4,3)$$

so corank$((5,3,2) = 6$.

Let $M = (m_1, m_2, \ldots, m_k)$ be a list in $\mathcal{L}_{n,k}$. For $i = 1, 2, \ldots, k$, let $c_i(M)$ be the number of lists in $\mathcal{L}_{n,k}$ which begin with $m_1, m_2, \ldots, m_{i-1}$ and have all of its remaining elements less than m_i. For example $c_1((5,3,2)) = 4$, and $c_2((5,3,2)) = 1$, and $c_3((5,3,2)) = 1$.

(a) Find a formula for $c_i(M)$ that depends only on the numbers k, i and m_i.

(b) Use your solution to part (a) to find a formula for corank(M), for any $M = (m_1, m_2, \ldots, m_k)$ in $\mathcal{L}_{n,k}$.

(c) Let $(\ell_1, \ldots, \ell_k) \subseteq [n]$ with $\ell_1 < \ell_2 < \cdots < \ell_k$. Let $L = (\ell_1, \ell_2, \ldots, \ell_k)$ and let the *reflection* of L be the list

$$\tilde{L} = (n+1-\ell_1, n+1-\ell_2, \ldots, n+1-\ell_k).$$

Then the elements of $\tilde{L}$ are listed in decreasing order. Write a formula (that we learned in class) that relates rank(L) (in the lexicographic order) and corank$(\tilde{L})$. You do not have to prove the formula.

(d) State an advantage that the formula from part (c) has, compared to the naïve ranking formula

$$\text{rank}(\ell_1, \ldots, \ell_k) = \sum_{i=1}^{k} \sum_{a=\ell_{i-1}+1}^{\ell_i - 1} \binom{n-a}{k-i}, \qquad \text{where } \ell_1 < \ell_2 < \cdots < \ell_k.$$

15. [See Lecture 10] Find and draw the spanning tree with vertex set $\{1, 2, 3, \ldots, n\}$, for some n, whose Prüfer sequence is $L = (4, 6, 1, 4)$.