Partial Fractions

Contents

1 Partial fractions

1.1 Relevant facts about polynomials

Partial fractions is useful whenever you want to reduce a rational function (that is one polynomial divided by another) to a sum of minimal pieces. We first need two facts about polynomials.

By convention we will say that the degree of the constant zero polynomial is $-\infty$.

Proposition (Division algorithm for polynomials)**.** *Let f and g be polynomials. Then there exist polynomials q and r with $\deg r < \deg g$ and*

$$f(z) = g(z)q(z) + r(z)$$

The proof is by applying the procedure of *long division of polynomials.*

Proposition (Extended Euclidean algorithm for polynomials)**.** *Let f and g be polynomials with no common factor then there exist polynomials s and t such that*

$$f(z)s(z) + g(z)t(z) = 1$$

The proof of this is too involved to summarize in a sentence, but it works the same way as the extended Euclidean algorithm for integers: For positive integers a, b with no common factors, there exist unique integers x, y such that

$$ax + by = 1, \qquad |x| \le b/2, \qquad |y| \le a/2.$$

As in the integer case, there is a good algorithm for computing $s(z)$ and $t(z)$. In general the algorithm is $O(n^2)$ where n is the maximum degree of f and g, but key parts of it, which are sufficient for many purposes, are faster (see `http://planetmath.org/encyclopedia/HalfGCDAlgorithm.html`).

1.2 The partial fractions theorem

Now we are ready for the partial fraction decomposition, first for two factors and then in general. You can find many presentations of this result online, this presentation is based on `http://marasingha.org/mathspages/partialfrac/html/node2.html`.

Proposition. *Let f_1, f_2, and g be polynomials such that f_1 and f_2 have no common factor and $\deg g < \deg f_1 + \deg f_2$. Then there exist polynomials g_1 and g_2 with*

$$\frac{g(z)}{f_1(z)f_2(z)} = \frac{g_1(z)}{f_1(z)} + \frac{g_2(z)}{f_2(z)}$$

and $\deg g_1 < \deg f_1$, $\deg g_2 < \deg f_2$.

Proof. By the extended Euclidean algorithm we can find polynomials h_1 and h_2 such that

$$1 = f_1(z)h_2(z) + f_2(z)h_1(z)$$

Thus
$$g(z) = f_1(z)(h_2(z)g(z)) + f_2(z)(h_1(z)g(z))$$
By the division algorithm we can also write
$$h_1(z)g(z) = f_1(z)q(z) + g_1(z)$$
where $\deg(g_1) < \deg(f_1)$. Let
$$g_2(z) = h_2(z)g(z) + q(z)f_2(z)$$
then
$$\begin{aligned} g(z) &= f_1(z)g_2(z) - f_1(z)f_2(z)q(z) + f_2(z)f_1(z)g(z) + f_2(z)g_1(z) \\ &= f_1(z)g_2(z) + f_2(z)g_1(z) \end{aligned}$$
so
$$\frac{g_1(z)}{f_1(z)} + \frac{g_2(z)}{f_2(z)} = \frac{f_2(z)g_1(z) + f_1(z)g_2(z)}{f_1(z)f_2(z)} = \frac{g(z)}{f_1(z)f_2(z)}$$
It remains to show that $\deg(g_2) < \deg(f_2)$. Towards a contradiction suppose that $\deg(g_2) \geq \deg(f_2)$. Then
$$\deg(f_1g_2) \geq \deg(f_1f_2)$$
but
$$\deg(f_2g_1) < \deg(f_1f_2)$$
So in $f_1(z)g_2(z) + f_2(z)g_1(z)$ the f_1g_2 term dominates, so
$$\deg(g) = \deg(f_1g_2 + f_2g_1) = \deg(f_1g_2) \geq \deg(f_1f_2) = \deg f_1 + \deg f_2$$
which is a contradiction, completing the proof. □

Theorem (partial fraction decomposition). *Let f and g be polynomials and write $f = f_1^{a_1}f_2^{a_2}\cdots f_r^{a_t}$ where f_i and f_j have no common factors for all $i \neq j$. Suppose $\deg g < \deg f$, then we can write*
$$\frac{g(z)}{f(z)} = \sum_{i=1}^{t}\sum_{j=1}^{a_i} \frac{g_{ij}(z)}{(f_i(z))^j}$$
for some polynomials g_{ij} with $\deg g_{ij} < \deg f_i$.

Proof. We apply the above proposition $t-1$ times, first with $f_1^{a_1}$ and $f_2^{a_2}\cdots f_r^{a_t}$ in place of f_1 and f_2, respectively, then with $f_2^{a_2}$ and $f_3^{a_3}\cdots f_r^{a_t}$ in place of f_1 and f_2, and so on. This results in polynomials $h_1, h_2, \ldots, h_t$, with $\deg(h_i) < \deg(f_i^{a_i}) = a_i \deg(f_i)$, for $i = 1, 2, \ldots, t$, and satisfying
$$\frac{g(z)}{f(z)} = \sum_{i=1}^{t} \frac{h_i(z)}{(f_i(z))^{a_i}}.$$

To continue the decomposition, we consider any one of the terms, say
$$\frac{h_i(z)}{f_i(z)^{a_i}}.$$
If $a_i = 1$, then we define $g_{i1} = h_i$ and we are done with this value of i. Assume $a_i > 1$. By the division algorithm, there exists polynomials q_1 and r_1 with $\deg r_1 < \deg f_i$ such that
$$h_i(z) = q_1(z)f_i(z) + r_1(z). \tag{1}$$
Therefore
$$\frac{h_i(z)}{f_i(z)^{a_i}} = \frac{q_1(z)}{f_i(z)^{a_i-1}} + \frac{r_1(z)}{f_i(z)^{a_i}}. \tag{2}$$

Furthermore either $q_1 = 0$ or $q_1(z)f_i(z)$ is the dominant term in (1), in which case $\deg h_i = \deg(q_0 f_i) = \deg q_1 + \deg f_i$. But $\deg h_i < a_i \deg f_i$ so $\deg q_1 < (a_i - 1)\deg f_i$. We repeat (1) and (3) with $q_1(z)$ in place of $h_i(z)$ to obtain $q_2(z)$ and $r_2(z)$ with $\deg q_2 < (a_i - 2)\deg f_i$ such that

$$\frac{h_i(z)}{f_i(z)^{a_i}} = \left(\frac{q_2(z)}{f_i(z)^{a_i-2}} + \frac{r_2(z)}{f_i(z)^{a_i-1}}\right) + \frac{r_1(z)}{f_i(z)^{a_i}}. \tag{3}$$

After continuing in this way a_i times, we have determined polynomials $r_1(z)$, $r_2(z)$, ..., $r_{a_1}(z)$, each one having degree less than $\deg f_i$, for which

$$\frac{h_i(z)}{f_i(z)^{a_i}} = \frac{r_1(z)}{f_i(z)^{a_i}} + \frac{r_2(z)}{f_i(z)^{a_i-1}} + \cdots + \frac{r_{a_1}(z)}{f_i(z)^1}. \tag{4}$$

For this value of i we define $g_{ij} = r_j$ for $j = 1, 2, \ldots, a_i$, and the expansion in equation (4) becomes

$$\sum_{j=1}^{a_i} \frac{g_{ij}(z)}{(f_i(z))^j}.$$

The above argument holds for every value of $i \in \{1, 2, \ldots, t\}$, so the result follows. $\square$

Example. Rewrite $\frac{z}{(1+z)(1-z)^2}$ by partial fractions, then extract the coefficient $[z_n]\frac{z}{(1+z)(1-z)^2}$.

By the above theorems we know this can be written

$$\frac{z}{(1+z)(1-z)^2} = \frac{A}{1+z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

How should we find A, B, and C? We could follow the proof and use the Euclidean algorithm, but for small hand examples it is usually easier to multiply out

$$\begin{aligned} z &= A(1-z)^2 + B(1+z)(1-z) + C(1+z) \qquad (5)\\ 0 + z + 0z^2 &= A(1 - 2z + z^2) + B(1 - z^2) + C(1+z) \end{aligned}$$

Now this must be true for all z, so it must be true coefficient by coefficient. That is

$$\begin{aligned} 0 &= A + B + C && \text{(coefficient of } z^0)\\ 1 &= -2A + C && \text{(coefficient of } z^1)\\ 0 &= A - B. && \text{(coefficient of } z^2) \end{aligned}$$

This is a system of linear equations which you can solve by your favorite method, say Gaussian elimination

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0\\ -2 & 0 & 1 & 1\\ 1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0\\ 0 & 2 & 3 & 1\\ 0 & -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0\\ 0 & 2 & 3 & 1\\ 0 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & -\frac{1}{4}\\ 0 & 1 & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

Thus

$$\frac{z}{(1+z)(1-z)^2} = \frac{-1/4}{1+z} + \frac{-1/4}{1-z} + \frac{1/2}{(1-z)^2}. \tag{6}$$

Note: There is a faster way to determine some of the constants A, B and C. The trick is to substitute values of z that make most of the terms in equation (5) disappear. For example, setting $z = -1$ and then $z = 1$ gives the two equations

$$-1 = (-2)^2 A + 0 + 0, \qquad 1 = 0 + 0 + 2C.$$

So $A = -1/4$, $C = 1/2$, and value of B is easily determined from comparing, say, the coefficients of z^2.

Now we are ready to extract the coefficient of z^n. To do this, we use the handy formula we learned in section 3 of lecture 2.

$$[z^n]\frac{c}{(1-mz)^r} = c\binom{n+r-1}{r-1}m^n.$$

We only need to convert each denominator in equation (6) into the form $(1-az)^r$ (notice a change of sign).

$$\begin{aligned}
[z^n]\frac{z}{(1+z)(1-z)^2} &= [z^n]\frac{-1/4}{1-(-1)z} + [z^n]\frac{-1/4}{1-z} + [z^n]\frac{1/2}{(1-z)^2} \\
&= -\frac{1}{4}\binom{n}{0}(-1)^n - \frac{1}{4}\binom{n}{0}1^n + \frac{1}{2}\binom{n+1}{1}1^n \\
&= -\frac{(-1)^n}{4} - \frac{1}{4} + \frac{n+1}{2} \\
&= \frac{2n+1-(-1)^n}{4}.
\end{aligned}$$

$$\frac{z}{(1+z)(1-z)^2} = z + z^2 + 2z^3 + 2^4 + 3z^5 + 3z^6 + 4z^7 + \cdots.$$

1.3 Algorithmic concerns

What is the runtime of partial fractions? If we do it by solving the system of equations then the system of equations is $n \times n$ where $n = \deg(f)$. By Gaussian elimination this is $O(n^3)$.

Strassen's algorithm lets one multiply two $n \times n$ matrices in $O(n^{2.81})$; it also gives a matrix inverse in the same time, and hence a system solve in the same time. There have been improvements along similar lines and the current best matrix multiplication is $O(n^{2.3727})$ (according to Wikipedia). However Strassen's algorithm is only faster than a well optimized naive approach for $n > 1000$ and the newer ones are impractical for any matrix you could actually store in a current computer. Certainly no such approach could be better than $O(n^2)$ since one needs to use all n^2 coefficients of the matrix.

However partial fractions has more structure. The proof suggests a different algorithm, one using the extended Euclidean algorithm. In fact partial fractions doesn't even need the full power of the extended Euclidean algorithm, and so with this basic approach and some additional cleverness one can obtain $O(\log n M(n))$ where $M(n)$ is the runtime needed to multiply two (slightly special) polynomials of degree n. Naively $M(n)$ would be n^2, which already gets us better that the system solving approach, but using fast fourier transforms one can obtain $M(n) = O(n \log n)$ and hence partial fractions can be done in $O(n(\log n)^2)$. The details of all of this are beyond the scope of this course. Reference for the algorithm: Kung, H. T. and Tong, D. M., "Fast algorithms for partial fraction decomposition" (1976). Computer Science Department. Paper 1675. `http://repository.cmu.edu/compsci/1675`.

One final algorithmic concern. What if the factorization of f is not given? This is a whole different story; fast polynomial factorization, say over the rationals, is done by looking modulo primes and then putting it back together. The good news is that it is polynomial time. In the case we're interested in we want to factor down to linear factors so we actually need to factor over the complex numbers, but not all polynomials can have their roots expressed in terms of square roots, cube roots etc, which leads in to the very interesting world of Galois theory.

Factoring Quadratics It is very likely that you know how to factor we can factor a monic quadratic polynomial (*monic* means that the highest degree coefficient equals 1)

$$z^2 + bz + c = (z-\alpha)(z-\beta) \quad \text{where } \alpha, \beta = \frac{-b \pm \sqrt{b^2-4c}}{2}.$$

For the purposes of coefficient extraction using partical fractions, we are working with polynomials where the *constant term* equals 1. Here we use a variant of the quadratic formula.

$$1 + bz + az^2 = (1-\gamma z)(1-\delta z) \quad \text{where } \gamma, \delta = \frac{-b \pm \sqrt{b^2-4a}}{2}. \tag{7}$$

To see this, we observe that γ, δ should satisfy

$$-\gamma - \delta = b \quad \text{and} \quad (-\gamma)(-\delta) = a.$$

These two equations are easily solved to give (7)

1.4 Algorithm for coefficient extraction in rational functions

Given a rational function of the form $R(z) = \frac{p(z)}{q(z)}$ we can extract the coefficients as follows.

1. Long division If $\deg p \geq \deg q$ we can use long division to write $R(z)$ in the form $t(z) + \frac{p'(z)}{q(z)}$, where $t(z)$ and $p'(z)$ are polynomials with $\deg p' < \deg q$.

2. Factor the denominator We want to factor denominator completely into the form

$$q(z) = (1 - m_1 z)^{a_1}(1 - m_2 z)^{a_2} \dots (1 - m_t z)^{a_t}$$

where all of the numbers $m_1, m_2, \dots, m_t$ are distinct complex numbers.

3. Partial fraction Decomposition Use partial fractions to decompose the rational function. This results in an expression of the following form, where each $\alpha_{i,j}$ is a polynomial of degree less than 1 (i.e. each $\alpha_{i,j}$ a constant).

$$\frac{p'(z)}{q(z)} = \sum_{i=1}^{t} \sum_{j=1}^{a_i} \frac{\alpha_{i,j}}{(1 - m_i z)^j}$$

4. Binomial Theorem Extract coefficients using the sum rule and applying the binomial theorem to each term.

$$[z^n]\frac{p'(z)}{q(z)} = \sum_{i=1}^{t} \sum_{j=1}^{a_i} [z^n] \frac{\alpha_{i,j}}{(1 - m_i z)^j} = \sum_{i=1}^{t} \sum_{j=1}^{a_i} \alpha_{ij} \binom{n+j-1}{j-1} m_i^n$$

5. Adjust initial terms The polynomial $t(z)$ from step **1.** will affect the only the initial terms.

$$[z^n]R(z) = \begin{cases} [z^n]t(z) + \sum_{i=1}^{t} \sum_{j=1}^{a_i} \alpha_{ij} \binom{n+j-1}{j-1} m_i^n & \text{if } n \leq \deg(t) \\ \sum_{i=1}^{t} \sum_{j=1}^{a_i} \alpha_{ij} \binom{n+j-1}{j-1} m_i^n & \text{if } n > \deg(t) \end{cases}$$

6. Asymptotics As $n \to \infty$, the growth of the coefficient $[z^n]R(z)$ is dominated by the root m_i with the highest complex norm. For this i and for any fixed j with $1 \leq j \leq a_i$, $\binom{n+j-1}{j-1}$ is a polynomial of degree $j-1$, so the largest value of j, for which $\alpha_{ij} \neq 0$ will dominate the growth of the coefficient. Provided that $(1 - m_i z)$ is not a common factor of the denominator $q(z)$ and the numerator $p(z)$ of $R(z)$, we are assured that $\alpha_{ia_i} \neq 0$. Summarizing this, if $|m_1| > |m_2| > \cdots > |m_t|$, then as $n \to \infty$ we have

$$[z^n]R(z) \sim \alpha_{1a_1} \binom{n + a_1 - 1}{a_1 - 1} m_1^n \sim \frac{\alpha_{1a_1}}{(a_1 - 1)!} n^{a_1 - 1} m_1^n$$

Example (Extracting coefficient of a rational function)**.** Let $R(z) = \frac{1152\, z^6 - 2208\, z^5 + 1736\, z^4 - 718\, z^3 + 167\, z^2 - 24\, z + 2}{1 - 18\, z + 129\, z^2 - 460\, z^3 + 816\, z^4 - 576\, z^5}$. Compute $[z^n]R(z)$.

1. Since the degree of the numerator is not less than the degree of the denominator, we first use long division to obtain

$$R(z) = 1 - 2z + \frac{2\, z^2 - 4\, z + 1}{1 - 18\, z + 129\, z^2 - 460\, z^3 + 816\, z^4 - 576\, z^5}.$$

2. We factor the denominator. This is hard to do in general. We can do this with the Maple command `factor`. We find that

$$1-18\,z+129\,z^2-460\,z^3+816\,z^4-576\,z^5=(1-3z)^2(1-4z)^3.$$

3. We solve for constants $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,1}$, $a_{2,2}$, in

$$\frac{0\,z^4+0\,z^3+2\,z^2-4\,z+1}{(1-3z)^2(1-4z)^3}=\frac{a_{1,1}}{1-3\,z}+\frac{a_{1,2}}{(1-3\,z)^2}+\frac{a_{2,1}}{1-4\,z}+\frac{a_{2,2}}{(1-4\,z)^2}+\frac{a_{2,3}}{(1-4\,z)^3}.$$

After expanding and comparing coefficients, we get 5 equations in 5 unknowns.

$$\begin{aligned}
0&=-208\,a_{1,1}-64\,a_{1,2}-168\,a_{2,1}-36\,a_{2,2}\\
0&=192\,a_{1,1}+144\,a_{2,1}\\
2&=84\,a_{1,1}+48\,a_{1,2}+73\,a_{2,1}+33\,a_{2,2}+9\,a_{2,3}\\
-4&=-15\,a_{1,1}-12\,a_{1,2}-14\,a_{2,1}-10\,a_{2,2}-6\,a_{2,3}\\
1&=a_{1,1}+a_{1,2}+a_{2,1}+a_{2,2}+a_{2,3}
\end{aligned}$$

This has the solution $[a_{1,1},a_{1,2},a_{2,1},a_{2,2},a_{2,3}]=[12,3,-16,0,2]$. The Maple command `solve` can do this. We have found

$$R(z)=1-2z+\frac{12}{1-3\,z}+\frac{3}{(1-3\,z)^2}+\frac{-16}{1-4\,z}+\frac{2}{(1-4\,z)^3}. \tag{8}$$

The Maple command `convert(R, parfrac)` can do all three above steps at once.

4. We apply the extended binomial theorem term by term to extract the coefficients. The first two terms of (8) only contribute to $[z^n]R(z)$ for $n=0$ and $n=1$. For $n\geq 2$ we have

$$\begin{aligned}
[z^n]R(z)&=[z^n]\frac{12}{1-3\,z}+[z^n]\frac{3}{(1-3\,z)^2}+[z^n]\frac{-16}{1-4\,z}+[z^n]\frac{2}{(1-4\,z)^3}\\
&=12\cdot 3^n+3\cdot\binom{n+1}{1}3^n-16\cdot 4^n+2\cdot\binom{n+2}{2}4^n\\
&=(12+3(n+1))\cdot 3^n+(-16+(n+1)(n+2))\cdot 4^n.
\end{aligned}$$

Computing the first two terms separately, $[z^0]R(z)=1+(-16+2)\cdot 1+(12+3)\cdot 1=2$ and $[z^1]R(z)=(-2)+(-16+6)\cdot 4+(12+6)\cdot 3=12$. Summarizing we have found that

$$[z^n]R(z)=\begin{cases}2 & \text{if } n=0\\ 12 & \text{if } n=1\\ \left(n^2+3n-14\right)\cdot 4^n+(3n+15)\cdot 3^n & \text{if } n\geq 2.\end{cases}$$

The first few terms of $R(z)$ are as follows.

$$R(z)=2+12\,z+125\,z^2+904\,z^3+5771\,z^4+33914\,z^5+\ldots$$

5. Since $|4|>|3|$ we have that, as $n\to\infty$,

$$[z^n]R(z)\sim 2\binom{n+2}{2}4^n\sim n^24^n$$

Here is an example of how Maple can be employed with the above example.

```
> # Example from Lecture 3
  R := (1152*z^6-2208*z^5+1736*z^4-718*z^3+167*z^2-24*z+2)/(1-18*z+129*z^2-460*z^3+816*z^4-576*z^5);
  Rnumer := numer(R);
  Rdenom := denom(R);
```

$$R := \frac{1152\,z^6 - 2208\,z^5 + 1736\,z^4 - 718\,z^3 + 167\,z^2 - 24\,z + 2}{-576\,z^5 + 816\,z^4 - 460\,z^3 + 129\,z^2 - 18\,z + 1}$$

$$Rnumer := -1152\,z^6 + 2208\,z^5 - 1736\,z^4 + 718\,z^3 - 167\,z^2 + 24\,z - 2$$

$$Rdenom := 576\,z^5 - 816\,z^4 + 460\,z^3 - 129\,z^2 + 18\,z - 1$$

```
> # Long division of polynomials
  Rquo := quo( Rnumer, Rdenom, z);
  Rrem := rem( Rnumer, Rdenom, z);
  R = Rquo + Rrem/Rdenom;
```

$$Rquo := -2\,z + 1$$

$$Rrem := -2\,z^2 + 4\,z - 1$$

$$\frac{1152\,z^6 - 2208\,z^5 + 1736\,z^4 - 718\,z^3 + 167\,z^2 - 24\,z + 2}{-576\,z^5 + 816\,z^4 - 460\,z^3 + 129\,z^2 - 18\,z + 1} = -2\,z + 1 + \frac{-2\,z^2 + 4\,z - 1}{576\,z^5 - 816\,z^4 + 460\,z^3 - 129\,z^2 + 18\,z - 1}$$

```
> #Factoring
  RdenomFactored := factor( Rdenom );
```

$$RdenomFactored := (3\,z - 1)^2\,(4\,z - 1)^3$$

```
> # Set up for partial fractions
  # The symbol % indicates the result of the previous calculation
  parfracForm := a11/(1-3*z) + a12/(1-3*z)^2 + a21/(1-4*z) + a22/(1-4*z)^2 + a23/(1-4*z)^3;
  Rrem/RdenomFactored = parfracForm;
```

$$parfracForm := \frac{a11}{1 - 3\,z} + \frac{a12}{(1 - 3\,z)^2} + \frac{a21}{1 - 4\,z} + \frac{a22}{(1 - 4\,z)^2} + \frac{a23}{(1 - 4\,z)^3}$$

$$\frac{-2\,z^2 + 4\,z - 1}{(3\,z - 1)^2\,(4\,z - 1)^3} = \frac{a11}{1 - 3\,z} + \frac{a12}{(1 - 3\,z)^2} + \frac{a21}{1 - 4\,z} + \frac{a22}{(1 - 4\,z)^2} + \frac{a23}{(1 - 4\,z)^3}$$

```
> # Expand the RHS and compare coefficients to get set of 5 equations|
  simplify(Rdenom*parfracForm);
  collect(%,z);
  equations := { seq( coeff( Rrem,z,k ) = coeff( %,z,k ), k=0..4) };
```

$$-192\,a11\,z^4 - 144\,a21\,z^4 + 208\,a11\,z^3 + 64\,a12\,z^3 + 168\,a21\,z^3 + 36\,a22\,z^3 - 84\,a11\,z^2 - 48\,a12\,z^2 - 73\,a21\,z^2 - 33\,a22\,z^2 - 9\,a23\,z^2 + 15\,a11\,z + 12\,a12\,z + 14\,a21\,z + 10\,a22\,z + 6\,a23\,z - a11 - a12 - a21 - a22 - a23$$

$$(-192\,a11 - 144\,a21)\,z^4 + (208\,a11 + 64\,a12 + 168\,a21 + 36\,a22)\,z^3 + (-84\,a11 - 48\,a12 - 73\,a21 - 33\,a22 - 9\,a23)\,z^2 + (15\,a11 + 12\,a12 + 14\,a21 + 10\,a22 + 6\,a23)\,z - a11 - a12 - a21 - a22 - a23$$

$$equations := \{-2 = -84\,a11 - 48\,a12 - 73\,a21 - 33\,a22 - 9\,a23,\ -1 = -a11 - a12 - a21 - a22 - a23,\ 0 = -192\,a11 - 144\,a21,\ 0 = 208\,a11 + 64\,a12 + 168\,a21 + 36\,a22,\ 4 = 15\,a11 + 12\,a12 + 14\,a21 + 10\,a22 + 6\,a23\}$$

```
> # Solve the equations, assign the values to the partial fraction decomposition of R
  solve(equations);
  assign(%);
  Rparfrac := Rquo + parfracForm;
```

$$\{a11 = 12,\ a12 = 3,\ a21 = -16,\ a22 = 0,\ a23 = 2\}$$

$$Rparfrac := -2\,z + 1 + \frac{12}{1 - 3\,z} + \frac{3}{(1 - 3\,z)^2} - \frac{16}{1 - 4\,z} + \frac{2}{(1 - 4\,z)^3}$$

```
> unassign('a11','a12','a21','a22','a23');
> #Maple provides a couple of ways of doing all of the above in one command.  Notice the format of the denominator is slightly different.
  convert( R, parfrac );
  genfunc:-rgf_pfrac( R,z );
```

$$-2\,z + 1 - \frac{12}{3\,z - 1} + \frac{3}{(3\,z - 1)^2} + \frac{16}{4\,z - 1} - \frac{2}{(4\,z - 1)^3}$$

$$-2\,z + 1 - \frac{12}{3\,z - 1} + \frac{3}{(3\,z - 1)^2} + \frac{16}{4\,z - 1} - \frac{2}{(4\,z - 1)^3}$$

```
> # Directly compute the first 10 terms of the series
  series( R, z, 10);
```

$$2 + 12\,z + 125\,z^2 + 904\,z^3 + 5771\,z^4 + 33914\,z^5 + 187897\,z^6 + 996236\,z^7 + 5105543\,z^8 + 25468222\,z^9 + O(z^{10})$$

```
> # The genfunc package can find a closed formula for the coefficients of Rrem
  # This formula gives the coefficients of R only for n at least 2, because of the terms in Rquo = 1-2z
  coeffFormula :=  genfunc:-rgf_expand( R,z,n);
  add( eval(coeffFormula, n=k)*z^k, k=0..9 );
  genfunc:-rgf_sequence('firstcf',R,z);          # This gives the smallest value of n for which the above formula is valid for R
  genfunc:-rgf_sequence('delta',R,z);            # This gives values that must be added to the above formula (due to Rquo) when n<2
```

$$coeffFormula := 12\,3^n + (3\,n + 3)\,3^n - 16\,4^n + 2\,(n + 1)\left(\frac{1}{2}\,n + 1\right)4^n$$

$$1 + 14\,z + 125\,z^2 + 904\,z^3 + 5771\,z^4 + 33914\,z^5 + 187897\,z^6 + 996236\,z^7 + 5105543\,z^8 + 25468222\,z^9$$

$$2$$

$$0 = 1,\ 1 = -2$$