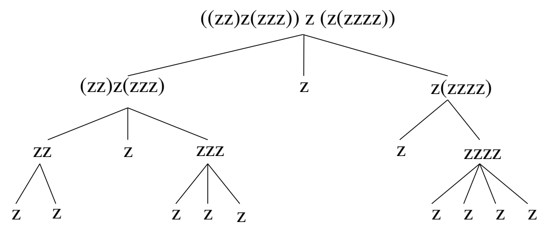
MT 2 Fri: Nov 15.

1. [See Lecture 6] Here is a combinatorial class that you have not yet seen. Informally, a *bracketing* is any "legal" way to place parentheses on a non-empty sequence of symbols. For example, there are exactly 11 bracketings of the sequence $zzzz$.

$$zzzz,\ (zz)zz,\ (zzz)z,\ z(zz)z,\ z(zzz),\ zz(zz),\ ((zz)z)z,\ (z(zz))z,\ z((zz)z),\ z(z(zz)),\ (zz)(zz).$$

More formally, the atomic symbol z is itself a bracketing; and any sequence of two or more consecutive bracketings enclosed by a pair of parentheses is a bracketing. To simplify notation, we have written z instead of (z) and we have also removed the outermost parentheses. For example, the bracketing $zz(zz)$ should interpreted to be $((z)(z)((z)(z)))$. The construction tree for the bracketing $((zz)z(zzz))z(z(zzzz))$ is shown on the right.



Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$ be the combinatorial class where $\mathcal{B}_n$ is the set of bracketings of the sequence $\overbrace{zz\ldots z}^{n}$, for $n \geq 1$.

(a) Find a recursive specification for $\mathcal{B}$ in terms of the atomic class $\mathcal{Z} = \{z\}$, the neutral class $\mathcal{E}$, and the operators '+', '$\times$' and 'Seq'.

(b) Use your specification to show the the OGF for $\mathcal{B}$ is $B(z) = \frac{1}{4}\left(1 + z - \sqrt{1 - 6z + z^2}\right)$.

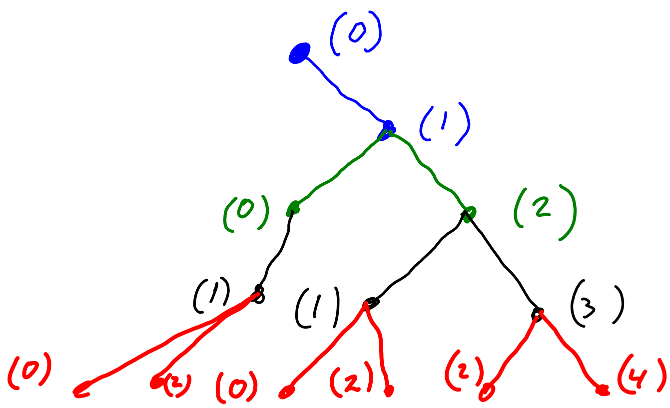
(c) Use the formula from part (b) and the following table to find B_5, the number of bracketings of $zzzzz$. You may leave your answer in a "basic-calculator ready" form, such as $B_5 = \frac{3}{8} \cdot \left(\frac{13}{35} - \frac{13\cdot 7}{8} + \frac{17\cdot 4}{3}\right)$.

k	$\binom{1/2}{k}$	$(z^2 - 6z)^k$
1	$\frac{1}{2}$	$z^2 - 6\,z$
2	$\frac{-1}{8}$	$z^4 - 12\,z^3 + 36\,z^2$
3	$\frac{1}{16}$	$z^6 - 18\,z^5 + 108\,z^4 - 216\,z^3$
4	$\frac{-5}{128}$	$z^8 - 24\,z^7 + 216\,z^6 - 864\,z^5 + 1296\,z^4$
5	$\frac{7}{256}$	$z^{10} - 30\,z^9 + 360\,z^8 - 2160\,z^7 + 6480\,z^6 - 7776\,z^5$
6	$\frac{-21}{1024}$	$z^{12} - 36\,z^{11} + 540\,z^{10} - 4320\,z^9 + 19440\,z^8 - 46656\,z^7 + 46656\,z^6$

2. [See Lecture 17] Draw the first six levels of the generating tree specified by the rule $[(0); \{(k) \to (k-1)(k+1)\}]$.

$[(0); \{(k) \rightsquigarrow (k-1)(k+1)\}]$

T



1
1
2
3
6 ...
10

$T(z) = 1 + z + z z^2 + 3z^3 + 6z^4 + \ldots$

is algebraic

ie $\exists$ polynomial $p(x)$, $p(T(z)) = 0$

3. [Lecture 16] The Johnson-Trotter minimal change order for the permutations of $\{1, 2, 3\}$ are the successive rows of the matrix shown at right. Write down the **four** permutations of $\{1, 2, 3, 4\}$ that come immediately after the permutation $\pi = [3\ 1\ 4\ 2]$ in the Johnson-Trotter order.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$\sigma(123) = 1$

$4 \sigma(132) = -1$

4 4 4 4 $\sigma(\pi) = -1$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Sign: $\sigma(\pi) = 1$ even: $123 = (1)(2)(3)$, $312 = (132)$, $231 = (123)$

odd: $132 = (1)(23)$, $321 = (2)(13)$, $213 = (3)(12)$

# of inversions in π

$\sigma(\pi) = (-1)$

312

132 or 321

$= (-1)^{\text{\# of even cycles in } \pi}$

$= (-1)^{\text{\# of transpositions needed to get to } 12 \ldots n}$

$3\ 1\ \cancel{4}\ 2$ or $3\ 1\ \cancel{4}\ 2$

$3\ 1\ 2\ 4$

$3\ \cancel{4}\ 1\ 2$

$4\ \boxed{3\ 1\ 2}$

$4\ \boxed{3\ 2\ 1}$

$3\ 4\ 2\ 1$

$\det(a_{ij}) = \sum_{\pi \in S_n} \sigma(\pi) \prod_{1 \le i \le n} a_{i\pi(i)}$

$\sigma\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 4 & 2 \end{pmatrix} = 1$

(136254)

$\sigma((12 \ldots n)) = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases}$

$\sigma\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$

$(136)(254)$

odd odd

0 0 $\to$ 0

$1\ 2\ 3\ 4$ → 4 $1\ 2\ 3$ since $\sigma(123) = 1$

$4\ 1\ 3\ 2$ → $1\ 3\ 2\ 4$ $1\ 3\ 2$ since $\sigma(132) = -1$

$\pi =$ (odd) (odd) (odd) (even) (even)

1 1 $= 0$

$a_1 a_2 \cdots a_{n-1} 4$

$4\ a_1 \ldots a_{n-1}$

if $\sigma(a_1 \ldots a_n) = 1$

replace $a_1 a_2 \ldots a_{n-1}$ with $(a_1 a_2 \ldots, a_{n-1})$

$(a_1 \ldots a_{n-1})$

2 3 4 1

2 3 1 4

if

$\sigma(2\ 3\ 1) = -1$

4

2 3 1 4

$\boxed{2\ 3\ 4\ 1}$

if

$\sigma(2\ 3\ 1) = 1$

$\sigma((1\ 2\ 3)) = 1$

2 3 1

1 3 2

1 2 3

4. [Lecture 7] Write a recursive specification for $\mathcal{T}$, the set of plane trees with red and blue nodes, where

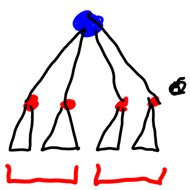
   - every blue node has no blue children and even number of red children, and
   - every red node has either no children or it has exactly one blue child and exactly one red child, in either order.

   (a) Use the notation $\mathcal{T} = \mathcal{T}_r + \mathcal{T}_b$ where $\mathcal{T}_r$ are those trees with a red root. Your specification should be three equations which involve $\mathcal{T}_b$, $\mathcal{T}_r$ and $\mathcal{T}$. all

$\mathcal{E}$ = null class $\to 1$

- $\mathcal{T} = \mathcal{T}_r + \mathcal{T}_b$

Atoms $\begin{cases} \bullet = \mathcal{Z}_b & \to z \\ \bullet = \mathcal{Z}_r & \to z \end{cases}$



- $\mathcal{T}_b = \mathcal{Z}_b \times Seq\left(\mathcal{T}_r^{\,2}\right)$
- $\mathcal{T}_r = \mathcal{Z}_r \times \left(\mathcal{E} + \mathcal{T}_b \times \mathcal{T}_r + \mathcal{T}_r \times \mathcal{T}_b\right)$

   (b) Find a set of equations which, if solved, give the generating function $T(z)$ for $\mathcal{T}$. Do not solve the equations!

Let $T_b(z)$, $T_r(z)$ be OGFs for $\mathcal{T}_r$, $\mathcal{T}_r$ (resp.)

- $T(z) = T_r(z) + T_b(z)$

- $T_b(z) = z \dfrac{1}{1 - \left(T_r(z)\right)^2}$

- $T_r(z) = z\left(1 + 2T_r(z)\,T_b(z)\right)$

5. [Lecture 6] Let $\mathcal{W}^{(k)}$ be the class of binary strings counted by length, which have no more than k consecutive 0s. Show that the generating function for $\mathcal{W}^{(k)}$ is

$$W^{(k)}(z) = \frac{1-z^{k+1}}{1-2z+z^{k+2}}.$$

$\mathrm{Seq}(\mathcal{A}) = (\mathcal{E} + \mathcal{A} + \mathcal{A}^2 + \cdots)$

$\varepsilon \quad \wedge\, \underbrace{00\cdots0}_{\le k}\, 11\cdots1 \;\wedge\, \underbrace{0\cdots0}_{\le k}\, 11\cdots1 \;\wedge\; \varepsilon,\ 0,\ 00,\ \underbrace{0\cdots0}_{k}$

Decomposition rule.

cut after every run of 1's and before any run of 0's

$11 \in 10^0 1$

$\mathcal{F}$ = class of first parts $= 1^* = (\varepsilon + 1 + 11 + \cdots)$

$\mathcal{M}$ = class of middle parts $= \mathrm{Seq}_{1 \le\ \le k}(0) \times \mathrm{Seq}_{\ge 1}(1)$

$0 + 00 + \cdots + \underbrace{00\cdots0}_{k}$

$z + z^2 + z^3 + \cdots + z^k$

$= 0(\varepsilon + 0 + 00 + \cdots + 0^{k-1})11^*$

$= 0(0^* \setminus 0^k 0^*)11^*$

$\mathcal{L}$ = class of last parts

$= 0^* \setminus (0^{k+1} 0^*)$

$$W(z) = \frac{\frac{1}{1-z} \cdot \frac{1-z^{k+1}}{1-z}}{1 - \left(z\,\frac{1-z^k}{1-z}\,\frac{z}{1-z}\right)} = \frac{1-z^k}{(1-z)^2 - z^2 - z^{k+2}}$$

$$= \frac{1-z^k}{1-2z-z^{k+2}}$$

$$\mathrm{Seq}_{\le k}(W) \longrightarrow \frac{1-W^{k+1}}{1-W}$$

$$z + z^2 + z^3 \ldots + z^k$$

$$= z\left(1 + z^2 + \ldots + z^{k-1}\right)$$

$$= z\left(1 + z^2 + \ldots \ldots - \boxed{z^k + z^{k+1} + \ldots}\right)$$

$$= z\left(\frac{1}{1-z} - z^k \frac{1}{1-z}\right)$$

$$= O\left(O^* - O^k O^*\right)$$

$$= OO^* - O^{k+1} O^*$$

$$= \frac{z}{1-z} - \frac{z^{k+1}}{1-z} = \frac{z - z^{k+1}}{1-z}$$

$$1 + 3 + 9 + \cdots + 3^{10}$$

$$= \frac{3^{11} - 1}{3 - 1}$$

$$1 + 10 + 100 + \cdot\ 1000000$$

$$= 1111111$$

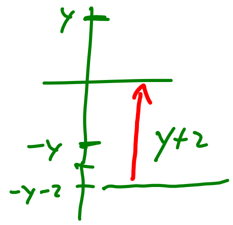
$$= \frac{10^{7} - 1}{10 - 1} = \frac{9999999}{9}$$

$$= 1{,}111{,}111$$

6. [Lecture 6] Let $\mathcal{T}$ be the set of non-empty rooted planar trees where the number of children that each node has belongs to the set $\Omega = \{0, 2, 4, 6, \dots\}$ (any even number). The size of a tree in $\mathcal{T}$ is the number of nodes in the tree. Use Lagrange inversion to compute T_n, the number of trees in $\mathcal{T}$ with size n.

$$\binom{2n-x}{n+1-\frac{x-y}{2}}$$

$2n-x$ steps



$$= \binom{2n-x}{n-1-\frac{x+y}{2}}$$

$$u+d = 2n-x$$

$$u-d = y+2$$

$$2u = 2n+y-x+2$$

$$u = n+1-\frac{x-y}{2}$$

$$2d = 2n-x-y-2 \qquad d = n-1-\frac{x+y}{2}$$

$$\binom{u+d}{d} = \binom{u+d}{u}$$

7. [Lecture 14] Find the rank of the word $s = 101101$ in the reflected binary code $000000, 000001, 000011, \ldots$.

8. [Lecture 9] Let $\mathcal{P}_{2n}(x, y)$ be the set of lattice paths that start at (x, y) and end at $(2n, 0)$ using *up-steps* $\nearrow = (1, 1)$ and *down-steps* $\searrow = (1, -1)$. Let $\mathcal{B}_{2n}(x, y)$ be the set of paths in $\mathcal{P}_{2n}(x, y)$ which at some point steps down to the line $y = -1$.

   (a) Show that $|\mathcal{B}_{2n}(0, 0)| = |\mathcal{P}_{2n}(0, -2)|$.

(b) Recall that a *Dyck path* of length $2n$ is any lattice path in $\mathcal{P}_{2n}(0,0)$ that does not touch the line $y = -1$. Use part (a) and the formula $|\mathcal{P}_{2n}(x,y)| = \binom{2n}{n - \frac{x+y}{2}}$ and perhaps a bit of algebra to show that the number of Dyck paths of length $2n$ equals the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

(c) Recall that a *totally balanced word* of length $2n$ is any binary sequence $W = (w_1, w_2, \ldots, w_{2n})$ which can be obtained from a Dyck path of length $2n$ by writing "0" for each up-step and "1" for each down-step. In other words a binary word W is totally balanced if

P1 W has exactly n zeros and n ones,

P2 no prefix $(w_1, w_2, \ldots, w_i)$ of W has more ones than zeros.

i. [1 point] What is the last totally balanced word of length $2n$ in the lexicographic order?

ii. [2 points] Find the totally balanced word that is the lexicographic successor of $(0,1,0,0,0,1,1,1,0,1)$.

9. [See Lecture 8 notes] Let $\mathcal{L}_{n,k}$ be the listing of the k-element subsets of $[n]$ in *reverse lexicographic order*. That is, we represent each k subset by a decreasing list $M = (m_1, m_2, \ldots, m_k)$, $m_1 > m_2 > \cdots > m_k$, and these decreasing lists are sorted lexicographically. The corank of M is the number of subsets that precede M in the reverse lexicographic order. For example

$$\mathcal{L}_{5,3} = (3,2,1),\ \ (4,2,1),(4,3,1),(4,3,2),\ \ (5,2,1),(5,3,1),(5,3,2),(5,4,1),(5,4,2),(5,4,3)$$

so corank$((5,3,2) = 6$.

Let $M = (m_1, m_2, \ldots, m_k)$ be a list in $\mathcal{L}_{n,k}$. For $i = 1, 2, \ldots, k$, let $c_i(M)$ be the number of lists in $\mathcal{L}_{n,k}$ which begin with $m_1, m_2, \ldots, m_{i-1}$ and have all of its remaining elements less than m_i. For example $c_1((5,3,2)) = 4$, and $c_2((5,3,2)) = 1$, and $c_3((5,3,2)) = 1$.

(a) Find a formula for $c_i(M)$ that depends only on the numbers k, i and m_i.

(b) Use your solution to part (a) to find a formula for corank(M), for any $M = (m_1, m_2, \ldots, m_k)$ in $\mathcal{L}_{n,k}$.

(c) Let $(\ell_1, \ldots, \ell_k) \subseteq [n]$ with $\ell_1 < \ell_2 < \cdots < \ell_k$. Let $L = (\ell_1, \ell_2, \ldots, \ell_k)$ and let the *reflection* of L be the list

$$\tilde{L} = (n+1-\ell_1, n+1-\ell_2, \ldots, n+1-\ell_k).$$

Then the elements of $\tilde{L}$ are listed in decreasing order. Write a formula (that we learned in class) that relates rank(L) (in the lexicographic order) and corank$(\tilde{L})$. You do not have to prove the formula.

(d) State an advantage that the formula from part (c) has, compared to the naïve ranking formula

$$\text{rank}(\ell_1, \ldots, \ell_k) = \sum_{i=1}^{k} \sum_{a=\ell_{i-1}+1}^{\ell_i - 1} \binom{n-a}{k-i}, \qquad \text{where } \ell_1 < \ell_2 < \cdots < \ell_k.$$

10. [See Lecture 10] Find and draw the spanning tree with vertex set $\{1, 2, 3, \ldots, n\}$, for some n, whose Prüfer sequence is $L = (4, 6, 1, 4)$.