MATH308 D100, Spring 2016

12. First summary

(based on notes from Dr. J. Hales)

Ladislav Stacho

SFU Burnaby

We Should (Could (Might)) Know...

LP Problems

- Intuitive definition of LP problem—too vague; many problems can be formulated as an LP problem (e.g., circular disk problem) but this 'modelling' can be difficult.
- Canonical form of an LP problem—starting point to 'LP World'

$$\begin{aligned} \textbf{maximize } & f(\boldsymbol{x}) = \boldsymbol{c}^\intercal \boldsymbol{x} - d \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \leqslant 0 \\ & \boldsymbol{x} \geqslant \boldsymbol{0} \end{aligned} \qquad \begin{aligned} \textbf{minimize } & g(\boldsymbol{x}) = \boldsymbol{c}^\intercal \boldsymbol{x} - d \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \geqslant 0 \\ & \boldsymbol{x} \geqslant \boldsymbol{0} \end{aligned}$$

Math Behind

- Constraint set of a canonical LP problem is a polyhedral convex set—intersection of finitely many closed half-spaces. Even more—if it is non-empty then there is always an extreme point.
- Constraint set may be empty $\Rightarrow$ infeasible LP problem.
- Classification of LP problems:
  (i) infeasible LP problems
  (ii) unbounded LP problems
  (iii) LP problems having bounded constraint set for which the optimal value of the objective function is attained at an extreme point
  (iv) LP problems having unbounded constraint set for which the optimal value of the objective function is attained at an extreme point
- If there is an optimal solution for an LP problem then it is attained in some extreme point.

Thus the idea of finding an optimal solution is simple: We can focus on extreme points.

Geometric Method

- Having n variables and m main constraints we are dealing with $n + m$ hyperplanes.
- We have to decide whether the LP problem is bounded or not.
- Taking n equations at a time out of all $n + m$ we get $\binom{n+m}{n}$ systems of n linear equations with n unknowns.
- We disregard inconsistent systems (i.e., with no solution) and systems with more than one (i.e., infinitely many) solutions.
- We obtain set of extreme point candidates.
- We eliminate infeasible points and get the set of all extreme points.
- By plugging then into the objective function we find an optimal solution.

There is a better way!

Simplex Algorithm for Maximization LP Problems

- Standard (equational) form, slack variables, canonical slack form, Tucker tableaux.
- Basic (dependent) variables, non-basic (independent) variables.
- Basic solution (extreme point candidate), basic feasible solution (extreme point).

Pivot Transformation on TT

- One variable enters the basis (entering variable), another leaves the basis (leaving variable).
- We require the basis to define a non-singular matrix (in fact an identity matrix) hence the pivot has to be non-zero.

$$
\begin{array}{r}
x_1 \leqslant 3 \\
-x_1 + x_2 \leqslant 2 \\
x_1 + 2x_2 \leqslant 1 \\
\hline
-x_1 + 2x_2 + 3 = f
\end{array}
\longrightarrow
\begin{array}{cc|c|l}
x_1 & x_2 & -1 & \\
\hline
1 & 0 & 3 & = -x_3 \\
-1 & 1 & 2 & = -x_4 \\
1 & 2^* & 1 & = -x_5 \\
\hline
-1 & 2 & -3 & = f
\end{array}
\longrightarrow
\begin{array}{cc|c|l}
x_1 & x_5 & -1 & \\
\hline
1 & 0 & 3 & = -x_3 \\
-3/2 & -1/2 & 3/2 & = -x_4 \\
1/2 & 1/2 & 1/2 & = -x_2 \\
\hline
-2 & -1 & -4 & = f
\end{array}
$$

And now we perform the same operation to LP in standard equational form:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$
$$\left|\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 3 \\ -1 & 1 & 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 3 \end{array}\right|$$

Theorem

Let i, j, B, B' be as stated above. Let $k \in \{1, 2, \dots, m\}$ be such that $a_{ki} = 1$. For B' to define an identity matrix we need to perform following elementary operations on the system $\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b}$

- *multiply row k by $\frac{1}{a_{kj}}$*
- *for each row $\ell \neq k$, add $-a_{\ell j}$ multiple of (new) row k to row ℓ*

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$
$$\left|\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 0 & 3 \\ -1 & 1 & 0 & 1 & 0 & 2 \\ 1/2 & 1 & 0 & 0 & 1/2 & 1/2 \\ 1 & -2 & 0 & 0 & 0 & 3 \end{array}\right|$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$
$$\left|\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 0 & 3 \\ -3/2 & 0 & 0 & 1 & -1/2 & 3/2 \\ 1/2 & 1 & 0 & 0 & 1/2 & 1/2 \\ 1 & -2 & 0 & 0 & 0 & 3 \end{array}\right|$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$
$$\left|\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 0 & 3 \\ -3/2 & 0 & 0 & 1 & -1/2 & 3/2 \\ 1/2 & 1 & 0 & 0 & 1/2 & 1/2 \\ 2 & 0 & 0 & 0 & 1 & 4 \end{array}\right|$$

SA for Maximum Tableaux

1. We have maximum Tucker tableau.
2. If $b_1, b_2, \ldots, b_m \geqslant 0$, go to **Step 6.**
3. Choose $b_i < 0$ such that i is maximal.
4. If $a_{i1}, a_{i2}, \ldots, a_{in} \geqslant 0 \Longrightarrow$ **STOP**; the problem is infeasible.
5. If $i = m$, choose $a_{mj} < 0$, pivot on a_{mj}, and go to **Step 1.**
   If $i < m$, choose $a_{ij} < 0$, compute

   $$\alpha = \min(\{b_i/a_{ij}\} \cup \{b_k/a_{kj} : k > i, a_{kj} > 0\}),$$

   and choose any p with $b_p/a_{pj} = \alpha$. Pivot on a_{pj} and go to **Step 1.**
6. We have MBFT ($b_1, b_2, \ldots, b_m \geqslant 0$)
7. If $c_1, c_2, \ldots, c_n \leqslant 0 \Longrightarrow$ **STOP**; the current basic feasible solution is optimal.
8. Choose any j with $c_j > 0$
9. If $a_{1j}, a_{2j}, \ldots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; the problem is unbounded.
10. Compute

    $$\alpha = \min_{1 \leqslant i \leqslant m} \{b_i/a_{ij} : a_{ij} > 0\}$$

    and choose any p with $b_p/a_{pj} = \alpha$. Pivot on a_{pj} and go to the **Step 6.**

To prevent cycling we can employ anticycling rules; Usually it is not necessary.

Maximize $2x_1 - x_2 + 8x_3$, subject to

$$\begin{aligned} 2x_3 &\leqslant 1 \\ 2x_1 - 4x_2 + 6x_3 &\leqslant 3 \\ -x_1 + 3x_2 + 4x_3 &\leqslant 2 \\ x_1, x_2, x_3 &\geqslant 0 \end{aligned}$$

$$\begin{array}{ccc|cl} x_1 & x_2 & x_3 & -1 & \\ \hline 0 & 0 & 2 & 1 & = -t_1 \\ 2 & -4 & 6 & 3 & = -t_2 \\ -1 & 3 & 4^* & 2 & = -t_3 \\ \hline 2 & -1 & 8 & 0 & = f \end{array} \qquad \begin{array}{ccc|cl} x_1 & x_2 & t_3 & -1 & \\ \hline 1/2 & -3/2 & -1/2 & 0 & = -t_1 \\ 7/2^* & -17/2 & -3/2 & 0 & = -t_2 \\ -1/4 & 3/4 & 1/4 & 1/2 & = -x_3 \\ \hline 4 & -7 & -2 & -4 & = f \end{array}$$

$$\begin{array}{ccc|cl} t_2 & x_2 & t_3 & -1 & \\ \hline -1/2 & -2/7 & -2/7 & 0 & = -t_1 \\ 2/7 & -17/7 & -3/7 & 0 & = -x_1 \\ 1/14 & 1/7^* & 1/7 & 1/2 & = -x_3 \\ \hline -8/7 & 19/7 & -2/7 & -4 & = f \end{array} \qquad \begin{array}{ccc|cl} t_2 & x_2 & t_3 & -1 & \\ \hline 0 & 2 & 0 & 1 & = -t_1 \\ 3/2 & 17 & 2 & 17/2 & = -x_1 \\ 1/2 & 7 & 1 & 7/2 & = -x_3 \\ \hline -5/2 & -19 & -3 & -27/2 & = f \end{array}$$

SA for Minimum Tableaux

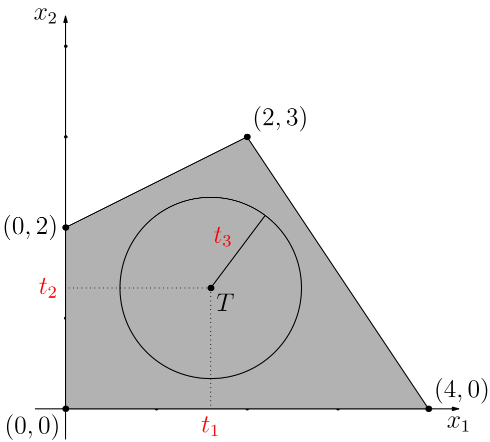
1. We have minimum Tucker tableau.
2. Take the negative transposition of the tableau to obtain a maximum tableau.
3. Apply SA for maximum tableaux.
4. $\min g = -\max(-g)$.

SA vs Geometric Method

- SA detects both infeasibility and unboundedness.
- SA is much more effective:
  For 15 main constraints, 10 variables—$\binom{25}{10} > 3\,200\,000$.
  SA would only require 13 to 50 pivot transformations.
- SA has many refinements, optimized for particular problems.
- SA is easily implemented on computers.

Nice example revisited—Exact vs. approximate solution

Consider we are given 4 points in the plane—$A = (0,0), B = (4,0), C = (2,3), D = (0,2)$. Find the largest circular disk that fits in the quadrangle $ABCD$.



maximize $f(t_1, t_2, t_3) = t_3$, subject to

$$
\begin{aligned}
3t_1 + 2t_2 &\leqslant 12 \\
-t_1 + 2t_2 &\leqslant 4 \\
-t_1 + t_3 &\leqslant 0 \\
-t_2 + t_3 &\leqslant 0 \\
3t_1 + 2t_2 + \sqrt{13}t_3 &\leqslant 12 \\
-t_1 + 2t_2 + \sqrt{5}t_3 &\leqslant 4 \\
t_1, t_2, t_3 &\geqslant 0
\end{aligned}
$$

only approximation:

$$\sqrt{13} \approx 3.605551275$$

$$\sqrt{5} \approx 2.236067977$$

Pivot rule—Largest increase of the objective function

x_1	x_2	x_3	-1	
1	2	1	4	$= -x_4$
2	1	5	5	$= -x_5$
3	2	0	6	$= -x_6$
1	2	3	0	$= f$

x_6	x_2	x_3	-1	
-1/3	4/3	1	2	$= -x_4$
-2/3	-1/3	5	1	$= -x_5$
1/3	2/3	0	2	$= -x_1$
-1/3	4/3	3	-2	$= f$

x_1	x_4	x_3	-1	
1/2	1/2	1/2	2	$= -x_2$
3/2	-1/2	9/2	3	$= -x_5$
2	-1	-1	2	$= -x_6$
0	-1	2	-4	$= f$

x_1	x_2	x_5	-1	
3/5	9/5	-1/5	3	$= -x_4$
2/5	1/5	1/5	1	$= -x_3$
3	2	0	6	$= -x_6$
-1/5	7/5	-3/5	-3	$= f$