MATH 308 - Assignment 5

1. Consider the dual canonical tableau.

$$
\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\hline
y_1 & a & -a & b & = -t_1 \\
y_2 & -a & a & b & = -t_2 \\
\hline
-1 & c & c & 0 & = f \\
\hline
 & = s_1 & = s_2 & = g &
\end{array}
$$

Prove that

a) If $b > 0$ and $c > 0$, then the primal problem (the maximization problem) is unbounded and the dual problem is infeasible.

b) If $b > 0$ and $c < 0$, then both primal and dual problem have an optimal solution.

c) If $b < 0$ and $c > 0$, then both primal and dual problems are infeasible.

d) If $b < 0$ and $c < 0$, then the primal is infeasible and the dual is unbounded.

2. The table below records data of the daily production and the daily production costs of widgets. The company believes that the daily production and the daily production costs are related as follows

$$\text{production cost} = a \cdot \text{production} + b$$

for some constants a and b. The company wants to find a and b which minimize the **maximum** error (in the absolute value) incurred in estimating daily production costs. For example, if $a = 2$ and $b = -1000$, then the error would be \$2000 for day 1, \$2000 for day 2, \$1000 for day 3, \$0 for day 4 and \$1000 for day 5, hence the maximum error would be \$2000. Formulate an LP that can be used to find the optimal a and b. **You do not need to solve the problem!**

Hint. Your LP should have just 3 variables: a, b and the maximum error E.

Day	Production	Production cost
1	4000	\$5000
2	5000	\$7000
3	7000	\$14000
4	1000	\$1000
5	3000	\$4000

3. Let $A \in \mathbb{R}^{2\times 2}$ be a game matrix. Prove that if A has a saddle point then it can be reduced to a 1×1 matrix by dominance.

4. Consider the following matrix game

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} .$$

a) Verify that A can not be reduced by using domination rules.

b) Find the von Neumann value and the optimal strategies for both players by using the general method you have learned in the class:

   – Write the initial game tableau (dual non-canonical).

   – Find a MBFT by using just two pivot transformations—use the algorithm to find appropriate pivots.

   – Apply SA to find the optimal solutions.

   – Report the optimal strategies and the von Neumann value of the game.

c) Matrix A has a saddle point—find it. Can you directly report the von Neumann value and the optimal strategies for the game? Compare with (b).

5. Player 1 writes an integer between 1 and 15 (including 1 and 15) on a slip of paper. Without showing this slip of paper to Player 2, Player 1 tells Player 2 what they have written. Player 1 may lie or tell the truth. Player 2 must then guess whether or not Player 1 has told the truth. If caught in a lie, Player 1 must pay \$10 to Player 2; if falsely accused of lying, Player 1 collects \$5 from Player 2. If Player 1 tells the truth and Player 2 guesses that Player 1 has told the truth, then Player 1 must pay \$1 to Player 2. If Player 1 lies and Player 2 does not guess that Player 1 has lied, then Player 1 wins \$5 from Player 2. Determine the von Neumann value of the game and optimal strategies for both players.

6. Let $x, y, z \in \mathbb{R}$. Find the von Neumann value and the optimal strategy for each player in the matrix game: $\begin{bmatrix} x & x \\ y & z \end{bmatrix}$.

[Hint. Consider cases how x, y, z are related and do not forget to use domination.]

7. Prove that in every matrix game one of the optimal mixed strategies $\boldsymbol{q} = (q_1, \ldots, q_n)$ of a column player against any fixed row player's mixed strategy $\boldsymbol{p} = (p_1, \ldots, p_m)$ is a column player's pure strategy, i.e. prove that for fixed $p_1, p_2, \ldots, p_m$ so that $\forall i$, $p_i \geq 0$, and $\sum p_i = 1$, we have

$$\min_{\boldsymbol{q}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \min_{1 \leq j \leq n} \sum_{i=1}^{m} a_{ij} p_i \, .$$

In other words, if the row player fixes their mixed strategy $\boldsymbol{p}$, then a best strategy of a column player against this strategy is a pure strategy.

Textbook, Chapter 5, Exercise 1b (p. 135), Textbook, Chapter 5, Exercise 2b (p. 136), Textbook, Chapter 5, Exercise 7a (p. 137)