Assignment 1

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1) NAND gate pretty much opposite of AND gate

1i) NOT as NAND & fanout

X	NOT(X)
0	1
1	0

Want this with NAND & fanout

$\Rightarrow$ NOT(X) $\equiv$ NAND(fanout(X))

X	NAND(fanout(X))
0	1
1	0

✓

1ii) WTS: From {NAND, fanout}, we can construct {AND, OR, NOT, fanout}

From 1i) we showed that NOT(X) $\equiv$ NAND(fanout(X)). Now we c
use NOT in our implementation to mean NAND(fanout(X)).
So NOT is covered ✓ And fanout is part of original set ✓

Now lets make OR.

X	Y	OR(X,Y)
0	0	0
0	1	1
1	0	1
1	1	1

$\Rightarrow$

X	Y	NAND(NOT(X), NOT(Y))
0	0	0
0	1	1
1	0	1
1	1	1

$\Rightarrow$ OR(X,Y) $\equiv$ NAND(NOT(X), N

Finally AND(X,Y) $\equiv$ NAND(NOT(X), NOT(Y)), ✓ $\Rightarrow$ {NAND, NOT} is univ

2) let $|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

2i) $|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix}$, $|\phi\rangle = \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

2ii) $\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{3}}\ \frac{i}{\sqrt{3}}\ -\frac{1}{\sqrt{3}}\right)\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix} = \frac{1}{3} - \frac{1}{3} + \frac{1}{3} = \frac{1}{3}$

$\langle\phi|\phi\rangle = \left(0\ \frac{1}{\sqrt{2}}\ -\frac{i}{\sqrt{2}}\right)\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix} = 0 + \frac{1}{2} - \frac{1}{2} = 0$

$\langle\psi|\phi\rangle = \left(\frac{1}{\sqrt{3}}\ \frac{i}{\sqrt{3}}\ -\frac{1}{\sqrt{3}}\right)\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix} = \frac{i}{\sqrt{6}} + \frac{i}{\sqrt{6}} = \frac{2i}{\sqrt{6}} = \sqrt{\frac{2}{3}}\,i$

$|\psi\rangle\langle\phi| = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix}\left(0\ \frac{1}{\sqrt{2}}\ -\frac{i}{\sqrt{2}}\right) = \begin{pmatrix}0 & \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}}\\ 0 & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & -\frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}}\end{pmatrix}$

$|\psi\rangle\otimes|\phi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix}\otimes\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

2ii)

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{i}{\sqrt{6}} \\ 0 \\ \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \end{pmatrix}$$

2iii)

$$|\psi\rangle + |\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} - \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}i + \sqrt{3}}{\sqrt{6}} \\ \frac{-\sqrt{2} - \sqrt{3}i}{\sqrt{6}} \end{pmatrix}$$

$$\| |\psi\rangle + |\phi\rangle \| = \sqrt{\left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{\sqrt{2}i + \sqrt{3}}{\sqrt{6}}\right|^2 + \left|\frac{-\sqrt{2} - \sqrt{3}i}{\sqrt{6}}\right|^2}$$

$$= \sqrt{\frac{1}{3} + \frac{-2 + 2\sqrt{6}i + 3}{6} + \frac{2 + 2\sqrt{6}i - 3}{6}}$$

$$= \sqrt{\frac{2 + 1 + 4\sqrt{6}i - 1}{6}}$$

$$= \sqrt{\frac{1 + 2\sqrt{3}i}{3}}$$

2iii) $|| |\psi\rangle + |\phi\rangle || = \frac{\sqrt{1+2\sqrt{3}i}}{\sqrt{3}} \neq 1$, it's not a unit vector

To make it a unit vector, divide each component with its magnitude

$$\frac{|\psi\rangle + |\phi\rangle}{|| |\psi\rangle + |\phi\rangle ||} = \begin{pmatrix} \frac{\sqrt{1+2\sqrt{3}i}}{3} \\ \frac{(\sqrt{2}i + \sqrt{3}) \cdot \sqrt{1+2\sqrt{3}i}}{\sqrt{18}} \\ \frac{(-\sqrt{2} - \sqrt{3}i) \cdot \sqrt{1+2\sqrt{3}i}}{\sqrt{18}} \end{pmatrix} \checkmark$$

3) $|\psi_3\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\theta}}{\sqrt{2}} \end{pmatrix}$, $\theta \in \mathbb{R}$

3i) $|0\rangle \rightarrow \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$, $|1\rangle \rightarrow \left|\frac{e^{i\theta}}{\sqrt{2}}\right|^2 = \frac{|e^{2i\theta}|}{2}$

3ii) $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{e^{i\theta}}{2} \\ \frac{1}{2} - \frac{e^{i\theta}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+e^{i\theta}}{2} \\ \frac{1-e^{i\theta}}{2} \end{pmatrix}$

$|0\rangle \rightarrow \left|\frac{1+e^{i\theta}}{2}\right|^2 = \frac{|1+2e^{i\theta}+e^{2i\theta}|}{4} = \frac{1}{4} + \frac{|2e^{i\theta}+e^{2i\theta}|}{4}$

$|1\rangle \rightarrow \left|\frac{1-e^{i\theta}}{2}\right|^2 = \left|\frac{1-2e^{i\theta}+e^{2i\theta}}{4}\right| = \frac{1}{4} + \frac{|-2e^{i\theta}+e^{2i\theta}|}{4}$

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4i) $Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $Y|+Y\rangle=|+Y\rangle$. let $|+Y\rangle=\begin{pmatrix}a\\b\end{pmatrix}$

$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$, $\begin{pmatrix}-ib\\ia\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$, $a=-ib$ & $b=ia$

$\Rightarrow b=i$ & $a=1$ as $a=-i\cdot b$, $1=-i\cdot i$ & $b=ia$, $i=i\cdot 1$ both true

$\rightarrow |+Y\rangle=\begin{pmatrix}1\\i\end{pmatrix}$ ✓

Now let $|-Y\rangle=\begin{pmatrix}a\\b\end{pmatrix}$

$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=-\begin{pmatrix}a\\b\end{pmatrix}$, $\begin{pmatrix}-ib\\ia\end{pmatrix}=\begin{pmatrix}-a\\-b\end{pmatrix}$, $a=ib$ & $b=-ia$

$\Rightarrow b=1$ & $a=i$ as $a=ib$, $i=i\cdot 1$ & $b=-i\cdot a$, $1=-i\cdot i$ both true

$\rightarrow |-Y\rangle=\begin{pmatrix}i\\1\end{pmatrix}$ ✓

4ii) $U=\begin{pmatrix}1&i\\i&1\end{pmatrix}$, $UU^\dagger=\begin{pmatrix}1&i\\i&1\end{pmatrix}\begin{pmatrix}1&-i\\-i&1\end{pmatrix}=\begin{pmatrix}2&0\\0&2\end{pmatrix}\neq I$

$UU^\dagger\neq I \Rightarrow$ not unitary

iii) $\begin{pmatrix}1&-i\\-i&1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&i\\i&1\end{pmatrix}=\begin{pmatrix}1&-i\\-i&1\end{pmatrix}\begin{pmatrix}1&-i\\i&-1\end{pmatrix}=\begin{pmatrix}2&0\\0&-2\end{pmatrix}$

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