Assignment 1

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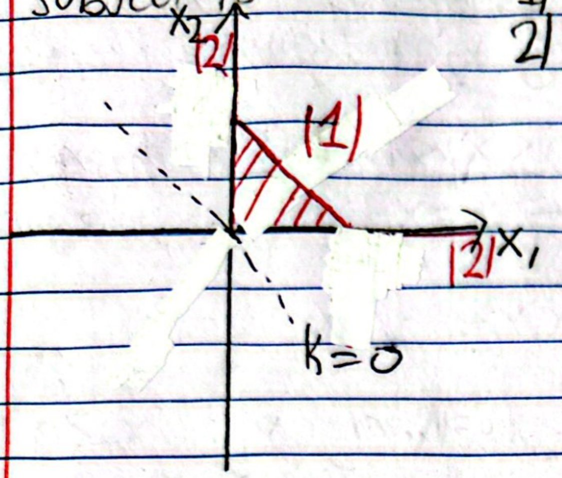
1) Maximize $f(x_1, x_2) = x_1 + x_2$

subject to

1) $sx_1 + tx_2 \leq 1$ (1) makes a line bound

2) $x_1, x_2 \geq 0$ (2) locks in 1^{st} quadrant



Set $K = x_1 + x_2$

a) We have optimal solution when K crosses line set by (1) only once

$\Rightarrow$ K will overlap line set by (1) when $s = t = 1 \rightarrow K = 1$ has infinitely many sol's

So make s or $t \neq 1$ to ensure only one intersection, and we hav optimal solution

Specific example: $s = 2, t = 1$, $2x_1 + x_2 \leq 1$

$\Rightarrow x_1 + x_1 + x_2 \leq 1$

$x_1 + K \leq 1$, x_1 can be 0

$K \leq 1$

$K = 1$ is max for $s = 2, t = 1$

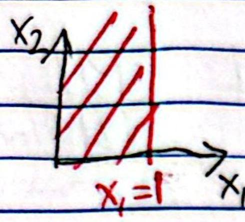
opitmal solution $(x_1, x_2) = (0, 1)$

b) We will have an infeasible solution when K can't cross line set by (1 or enter bounding triangle of (1) & (2)

This ain't possible

c) When we have $t=0$, $s=1$ we have $x_1 \leq 1$

$\Rightarrow$



As we go up x_2, it's unbounded. We can keep increasing x_2 (and K) while keeping $x_1 \leq 1$

So $0 \leq x_1 \leq 1$, $0 \leq x_2$ has no upper bound on $x_2 \Rightarrow K = x_1 + x_2$ is also unbounded

2) Maximize obj. $f(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$
subject to $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ $(i = 1..m)$, $x_1, \ldots, x_n \geq 0$ is unbounded

Meaning $\nexists$ a max M s.t. $f(x_1, \ldots, x_n) \leq M$ according to constriants

$\Rightarrow f(x_1, \ldots, x_n)$ will keep growing & isn't bounded

As $f(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$, $\exists$ K s.t. $1 \leq K \leq n$ x_K isn't upper bounded.
(If all x_i for $1 \leq i \leq n$ were bounded, f would be bounded so taking contrapositive of statement shows existance of K)

Since x_K isn't bounded according to constriants, $f(x_1, \ldots, x_n) = x_K$ also is unbounded according to constriants ✓

3) Minimize $g(x_1, x_2) = 2x_1 - 5x_2$

subject to: 1) $x_1 + x_2 \geq 2$

2) $x_1 - 2x_2 \leq 0$

3) $-2x_1 + x_2 \leq 1$

4) $x_1, x_2 \geq 0$

Consider $\vec{v} = (x_1, x_2)$, where $x_1 = x_2$ & $x_1, x_2 \geq 2$

This line satisfies (1): Minimum of $x_1 + x_2 = 4$, $4 \geq 2$ ✓

(2): $x_1 - 2x_2 \leq 0$
$-x_2 \leq 0$, $-2 \leq 0$ ✓

(3): $-2x_1 + x_2 \leq 1$
$-x_1 \leq 1$, $-2 \leq 1$ ✓

(4): $2, 2 \geq 0$ ✓

satisfies constriants ✓

This $\vec{v}$ is a feasible solution, but it's unbounded
$\Rightarrow$ Constriant set is unbounded

We can use this in $g(x_1, x_2) = 2x_1 - 5x_2$
$= 2x_2 - 5x_2$
$= -3x_2$

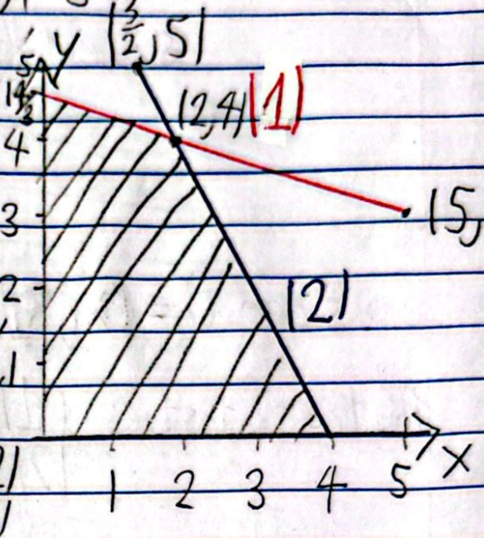
$g(x_1, x_2)$ will go to neg. inf. with $x_1, x_2 \geq 2$ & $x_1 = x_2$
$\Rightarrow$ Obj func unbounded on constriant set
$\Rightarrow$ LP unbounded

Scanned with CamScanner

Thm^{23}: If constriant set of canonical LP problem is unbounded,
Max: $\exists M \in \mathbb{R}$ s.t. $f \leq M$ (bound above), max(f) is at extreme point
Min: $\exists M \in \mathbb{R}$ s.t. $f \geq M$ (bound below), min(f) is at extreme point

4) Thm^{22}: If constriant set of canonical LP problem is bounded, then max/min is at extreme point of S

a) Maximize: $f(x,y) = 5x+2y$
subject to: 1) $x+3y \leq 14$
2) $2x+y \leq 8$
3) $x \geq 0$, 4) $y \geq 0$



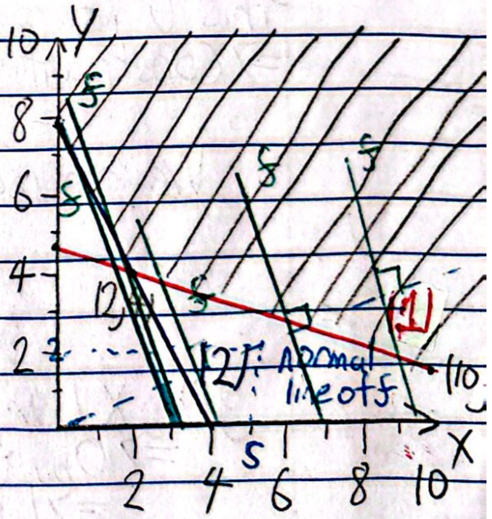
${}_4C_2 = 6$

Thm^{22}
S is bounded

(1)(2): $x+3y=14$, $2x+y=8$. (2,4) ✓
(1)(3): $x+3y=14$, $x=0$. $(0, \frac{14}{3})$ ✓
(1)(4): $x+3y=14$, $y=0$. (14,0) X (2)
(2)(3): $2x+y=8$, $x=0$. (0,8) X (1)
(2)(4): $2x+y=8$, $y=0$. (4,0) ✓
(3)(4): $x=0$, $y=0$. (0,0) ✓

Out of (1)(2), (1)(3), (2)(4), (3)(4), (2)(4) is optimal solution
$\Rightarrow f(4,0) = 20$ is max of LP

b) Minimize: $g(x,y) = 5x+2y$
subject to: 1) $x+3y \geq 14$
2) $2x+y \geq 8$
3) $x \geq 0$
4) $y \geq 0$



Thm^{23}
S is unbounded
f is bounded below by 16 f(0,8)

(1)(2): $x+3y=14$, $2x+y=8$. (2,4) ✓
(1)(3): $x+3y=14$, $x=0$. $(0, \frac{14}{3})$ X (2)
(1)(4): $x+3y=14$, $y=0$. (14,0) ✓
(2)(3): $2x+y=8$, $x=0$. (0,8) ✓
(2)(4): $2x+y=8$, $y=0$. (4,0) X (1)
(3)(4): $x=0$, $y=0$. (0,0) X (1),(2)

Out of (1)(2), (1)(4), (2)(3), (2)(3) is optimal solution
$\Rightarrow f(0,8) = 16$ is min of LP

5) Let f_1 be first fourmulation, f_2 second, f_3 third.
Let f be reverue of company
Let V_c be vitamin content & m_c be mineral content

$$f(f_1, f_2, f_3) = 1\cdot f_1 + 2\cdot f_2 + 3\cdot f_3$$
$$= f_1 + 2f_2 + 3f_3$$

Manufacturing: $V_c \leq 100 \Rightarrow f_1 + \frac{3}{4}f_2 + \frac{1}{2}f_3 \leq 100$
$m_c \leq 75 \Rightarrow \frac{1}{4}f_2 + \frac{1}{2}f_3 \leq 75$

By thm 22, this is bounded. With natural constriants of $f_1, f_2, f_3 \geq 0$
We have 3 variables, 5 constriants. Let's index them.

5: 1) $f_1 + \frac{3}{4}f_2 + \frac{1}{2}f_3 \leq 100$
2) $\frac{1}{4}f_2 + \frac{1}{2}f_3 \leq 75$ Z) $f_1, f_2, f_3 \in \mathbb{Z}$
3) $f_1 \geq 0$, 4) $f_2 \geq 0$, 5) $f_3 \geq 0$

${}_5C_3 = 10$.
(1)(2)(3): (0, 50, 125)
(1)(2)(4): (25, 0, 150)
(1)(2)(5): (-125, 300, 0) X (3)
Work done on other paper. (Next page submitted)
(1)(3)(4): (0, 0, 200) X (2)
(1)(3)(5): (0, $\frac{400}{3}$, 0) X (Z)
(1)(4)(5): (100, 0, 0)
(2)(3)(4): (0, 0, 150)
(2)(3)(5): (0, 300, 0) X (1)
(2)(4)(5): essentially just $(f_1, 0, 0)$, which is just (1)(4)(5).
(3)(4)(5): (0, 0, 0), obviously not solution

Work shown on scrap paper

Out of (1)(2)(3), (1)(2)(4), (1)(4)(5), (2)(3)(4)
(1)(2)(3) & (1)(2)(4) are the optimal solutions, with
$f(0, 50, 125) = f(25, 0, 150) = 475$

Optimal solutions are $0\ f_1$'s, $50\ f_2$'s, $125\ f_3$'s & $25\ f_1$'s, $0\ f_2$'s, 150

|1|2|3|: $\frac{3}{4}f_2+\frac{1}{2}f_3=100$, $\frac{1}{4}f_2+\frac{1}{2}f_3=75$

$\frac{3}{4}f_2+75-\frac{1}{4}f_2=100$

$\frac{1}{2}f_2=25$, $f_2=50$

$\frac{50}{4}+\frac{1}{2}f_3=75$, $f_3=\frac{250}{2}$, $|0, 50, \frac{250}{2}|$

|1|2|4|: $f_1+\frac{3}{4}f_2+\frac{1}{2}f_3=100$, $\frac{1}{4}f_2+\frac{1}{2}f_3=75$, $f_2=0$

$f_1+\frac{1}{2}f_3=100$, $\frac{1}{2}f_3=75$. $f_3=150$

$f_1+75=100$, $f_1=25$. $|25, 0, 150|$

|1|2|5|: $f_1+\frac{3}{4}f_2+\frac{1}{2}f_3=100$, $\frac{1}{4}f_2+\frac{1}{2}f_3=75$, $f_3=0$

$\frac{1}{4}f_2=75$, $f_2=300$

$f_1+225=100$, $f_1=-125$. $|-125, 300, 0|$

|1|3|4|: $f_1+\frac{3}{4}f_2+\frac{1}{2}f_3=100$, $f_1=0$, $f_2=0$. $f_3=200$

|1|3|5|: $f_1+\frac{3}{4}f_2+\frac{1}{2}f_3=100$, $f_1=0$, $f_3=0$. $f_2=\frac{400}{3}$

|1|4|5|: $f_1+\frac{3}{4}f_2+\frac{1}{2}f_3=100$, $f_2=f_3=0$. $f_1=100$

|2|3|4|: $\frac{1}{4}f_2+\frac{1}{2}f_3=75$, $f_1=f_2=0$. $f_3=150$

|2|3|5|: $\frac{1}{4}f_2+\frac{1}{2}f_3=75$, $f_1=f_3=0$. $f_2=300$

|2|4|5|:

(1)(2)(3): $f(0, 50, 125) = 475$

(1)(2)(4): $f(25, 0, 150) = 475$

(1)(4)(5): $f(100, 0, 0) = 100$

(2)(3)(4): $f(0, 0, 150) = 450$

b) Extreme points evaluated in (5) revealed that (0, 50, 125) & (25, 0, 150) are the two optimal solutions for the LP.

Call (0, 50, 125) O_1, (25, 0, 150) O_2. $\overrightarrow{O_1O_2} = \begin{pmatrix} 25 \\ -50 \\ 25 \end{pmatrix}$

Parametrize with $0 \le t \le 1$ to be able to go from O_1 to O_2, O_2 to O_1. Start at O_1.

$$\begin{pmatrix} 25 \\ -50 \\ 25 \end{pmatrix} t + \begin{pmatrix} 0 \\ 50 \\ 125 \end{pmatrix} \Rightarrow \begin{matrix} x = 25t \\ y = -50t + 50 \\ z = 25t + 125 \end{matrix}, \quad 0 \le t \le 1$$

Plugging in f, $f(25t, -50t+50, 25t+125) = 25t - 100t + 100 + 75t + 375$
$= 475$ verified