Assignment 1

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1) NAND gate pretty much opposite of AND gate

1i) NOT as NAND & fanout

X	NOT(X)
0	1
1	0

Want this with NAND & fanout

$\Rightarrow$ NOT(X) $\equiv$ NAND(fanout(X))

X	NAND(fanout(X))
0	1
1	0

✓

1ii) WTS: from {NAND, fanout}, we can construct {AND, OR, NOT, fanout}

From 1i) we showed that NOT(X) $\equiv$ NAND(fanout(X)). Now we can use NOT in our implementation to mean NAND(fanout(X)).
So NOT is covered ✓ And fanout is part of original set ✓

Now lets make OR.

X	Y	OR(X,Y)
0	0	0
0	1	1
1	0	1
1	1	1

$\Rightarrow$

X	Y	NAND(NOT(X), NOT(Y))
0	0	0
0	1	1
1	0	1
1	1	1

✓ $\Rightarrow$ OR(X,Y) $\equiv$ NAND(NOT(X), NOT(Y))

Finally AND(X,Y) $\equiv$ NAND(NOT(X), NOT(Y)), ✓ $\Rightarrow$ {NAND, NOT} is universal

2:ii

2) let $|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

2:i) $|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{-i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix}$, $|\phi\rangle = \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

2:ii) $\langle\psi|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$

$\langle\phi|\phi\rangle = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix} = 0+\frac{1}{2}+\frac{1}{2} = 1$

$\langle\psi|\phi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix} = \frac{-i}{\sqrt{6}}+\frac{i}{\sqrt{6}} = 0$

$|\psi\rangle\langle\phi| = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}}\\ 0 & \frac{i}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\\ 0 & -\frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}}\end{pmatrix}$

$|\psi\rangle\otimes|\phi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{i}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix}\otimes\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

2ii)

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ -\frac{i}{\sqrt{6}} \\ 0 \\ \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \end{pmatrix}$$

2iii)

$$|\psi\rangle + |\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} - \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}i+\sqrt{3}}{\sqrt{6}} \\ \frac{-\sqrt{2}-\sqrt{3}i}{\sqrt{6}} \end{pmatrix}$$

$$\| |\psi\rangle + |\phi\rangle \| = \sqrt{\left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{\sqrt{2}i+\sqrt{3}}{\sqrt{6}}\right|^2 + \left|\frac{-\sqrt{2}-\sqrt{3}i}{\sqrt{6}}\right|^2}$$

$$= \sqrt{\frac{1}{3} + \frac{2+3}{6} + \frac{2+3}{6}} = \sqrt{\frac{2}{6} + \frac{5}{6} + \frac{5}{6}} = \sqrt{\frac{12}{6}}$$

$$= \sqrt{2}$$

2iii) $\| |\psi\rangle + |\phi\rangle \| = \sqrt{2} \neq 1$, it's not a unit vector

To make it a unit vector, divide each component with its magnitude

$$\frac{|\psi\rangle + |\phi\rangle}{\| |\psi\rangle + |\phi\rangle \|} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{2} + \frac{1}{\sqrt{6}} \\ -\frac{i}{2} - \frac{1}{\sqrt{6}} \end{pmatrix} \checkmark$$

3) $|\psi_3\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\theta}}{\sqrt{2}} \end{pmatrix}$, $\theta \in \mathbb{R}$

3i) $|0\rangle \to \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$, $|1\rangle \to \left|\frac{e^{i\theta}}{\sqrt{2}}\right|^2 = \frac{|e^{2i\theta}|}{2} = \frac{1}{2}$

3ii) $$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{e^{i\theta}}{2} \\ \frac{1}{2} - \frac{e^{i\theta}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+e^{i\theta}}{2} \\ \frac{1-e^{i\theta}}{2} \end{pmatrix}$$

$$|0\rangle \to \left|\frac{1+e^{i\theta}}{2}\right|^2 = \frac{|1+2e^{i\theta}+e^{2i\theta}|}{4} = \frac{1}{4} + \frac{|2e^{i\theta}+e^{2i\theta}|}{4}$$

$$|1\rangle \to \left|\frac{1-e^{i\theta}}{2}\right|^2 = \left|\frac{1-2e^{i\theta}+e^{2i\theta}}{4}\right| = \frac{1}{4} + \frac{|-2e^{i\theta}+e^{2i\theta}|}{4}$$

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4ii) $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Y|+Y\rangle = |+Y\rangle$. let $|+Y\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \; a = -ib \;\&\; b = ia$$

$\Rightarrow b = i \;\&\; a = 1$ as $a = -ib$, $1 = -i \cdot i$ & $b = ia$, $i = i \cdot 1$, both true

$\rightarrow |+Y\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow$ scale by $\frac{1}{\sqrt{2}}$ to normalize, $|+Y\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ ✓

Now let $|-Y\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}, \; a = ib \;\&\; b = -ia$$

$\Rightarrow b = 1 \;\&\; a = i$ as $a = ib$, $i = i \cdot 1$ & $b = -i \cdot a$, $1 = -i \cdot i$ both true

$\rightarrow |-Y\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix} \Rightarrow$ scale by $\frac{1}{\sqrt{2}}$, $|-Y\rangle = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

4iii) $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $UU^\dagger = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$U^\dagger U = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, $UU^\dagger = U^\dagger U = I$, unitary ✓

iii) $\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$