Assignment 5

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1i) Let $|x+S\rangle = \frac{1}{\sqrt{|S|}} \sum_{s \in S} |x+s\rangle$

$$H^{\otimes n}|x+S\rangle = H^{\otimes n} \frac{1}{\sqrt{|S|}} \sum_{s \in S} |x+s\rangle$$

$$\frac{1}{\sqrt{|S|}} \sum_{s \in S} \sum_{z \in \{0,1\}^n} (-1)^{(x+s)\cdot z} |z\rangle$$

$|S|$ - elements in S
$\rightarrow \forall s \in S \Rightarrow$ index will be from 1 to $|S|$

$$* \quad \frac{1}{\sqrt{|S|}} \frac{|S|}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{(x+s_1)\cdot z} |z\rangle + (-1)^{(x+s_2)\cdot z} |z\rangle + \dots + (-1)^{(x+s_{|S|})\cdot z} |z\rangle$$

WTS: hint, exactly half of $s \in S$ is orthogonal to z

$s = a_1\vec{b}_1 + \dots + a_n\vec{b}_n$ for $a_i \in \mathbb{C}$, $\vec{b}_i$ a basis vector of S

$\vec{z}\cdot\vec{s} = \vec{z}\cdot a_1\vec{b}_1 + \dots + \vec{z}\cdot a_n\vec{b}_n$

if $\vec{z}\cdot\vec{s} \neq 0$ then at least one of $\vec{z}\cdot a_i\vec{b}_i = 1$, let $\vec{b}_m$ be one of these vectors, focus on $\vec{b}_m$ for now

Remove $\vec{b}_i$ from basis set $\Rightarrow B = \text{span}\{\vec{b}_1, \dots, \vec{b}_n\}$
$B' = \text{span}\{\vec{b}_1, \dots, \vec{b}_{m-1}, \vec{b}_{m+1}, \dots, \vec{b}_n\}$

Since $\vec{b}_m$ is the only basis vector that $\vec{z}\cdot\vec{b}_m = 1$ & all others have $\vec{z}\cdot\vec{b}_i = 0$, $\vec{b}_m$ is linearly independant from all $\vec{b}_i$

Meaning for set $B' + \vec{b}_m = \text{span}\{\vec{b}_1 + \vec{b}_m, \dots, \vec{b}_{m-1} + \vec{b}_m, \vec{b}_{m+1} + \vec{b}_m, \dots, \vec{b}_n + \vec{b}_m\}$ is disjoint from B

2i) $5^r \equiv 1 \bmod 21$

$\underbrace{5\cdot 5\cdot\ldots\cdot 5}_{r \text{ times}} \equiv 1 \bmod 21$

r times $\Rightarrow r=6$

$5^6 = 15625$, $\frac{15625}{21} = 744\frac{1}{21}$

$\Rightarrow 5^6 \bmod 21 \equiv 1$

ii) $GCD(5^{\frac{6}{2}}+1, 21)$, $GCD(5^{\frac{6}{2}}-1, 21)$

$GCD(126, 21)$, $GCD(124, 21)$

21 $\quad\quad 1$

$21 \mid 5^{\frac{r}{2}}+1$, $21 \nmid 5^{\frac{r}{2}}-1 \Rightarrow 124, 21$ coprime

$126 \bmod 21 \equiv 21 \bmod 21 \equiv 0 \bmod 21$

$124 \bmod 21 \equiv 19 \bmod 21 \not\equiv 0 \bmod 21$

$5^{\frac{r}{2}} \pm 1$ is either a multiple of 21 or not (coprime with 21)

iii) $2^x \bmod 21 \Rightarrow 2^r \equiv 1 \bmod 21 \rightarrow 2^6 = 64 \equiv 1 \bmod 21 \rightarrow r=6$

$GCD(2^{\frac{6}{2}}+1, 21)$, $GCD(2^{\frac{6}{2}}-1, 21)$

3 $\quad\quad 7$

prime factors: $21 = 3\cdot 7$

ii) take $\vec{b}' \in B'$ where $B = \text{span}\{\vec{b}_1, \ldots, \vec{b}_{m-1}, \vec{b}_{m+1}, \ldots \vec{b}_n\}$

$$\vec{z} \cdot \vec{b}' = \vec{z} \cdot a_1 \vec{b}_1 + \ldots + \vec{z} a_{m-1} \vec{b}_{m-1} + \vec{z}\, a_{m+1} \vec{b}_{m+1} + \ldots + \vec{z} \cdot a_n \vec{b}_n$$

if $\vec{b}_m$ was the only vector that $\vec{z} \cdot a_m \vec{b}_m = 1$, then $\vec{z} \cdot \vec{b}' = 0$
if $\vec{b}_m$ wasn't the only vector that $\vec{z} \cdot a_m \vec{b}_m = 1$, then $\vec{z} \cdot \vec{b}' = 1$

take $(\vec{b}' + \vec{b}_m) \cdot \vec{z} \Rightarrow \vec{z} \cdot \vec{b}' + \vec{z} \cdot \vec{b}_m = 1$ or 2, depending on which if statement above you take

taking any $\vec{b}' \in S$, we can map $\vec{b}' \to \vec{b}' + \vec{b}_m$ as they have equal amounts of vectors $(|B'|)$

$\Rightarrow \exists$ a 1:1 mapping & onto mapping $\to$ bijective map

$\Rightarrow \exists$ equal amounts of vectors s.t. $\vec{z} \cdot \vec{b}' = 0$ & $\vec{z} \cdot \vec{b}' = 1$

$\Rightarrow$ for half of the vectors in S, $s \cdot \vec{z} = 0$. WTS ✓

Now back to $*$ $\frac{\sqrt{|S|}}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{(x+s_1) \cdot z} |z\rangle + (-1)^{(x \cdot s_2) \cdot z} |z\rangle + \ldots + (-1)^{(x+s_{|S|}) z} |z\rangle$

$$\Rightarrow \sqrt{\frac{|S|}{2^n}} \sum_{z \in \mathbb{Z}_2^n} (-1)^{x \cdot z + s \cdot z} |z\rangle \quad \text{using WTS}$$

$$\Rightarrow \sqrt{\frac{|S|}{2^n}} \left| \sum_{z | z \cdot s \neq 0} (-1)^{x \cdot z + s \cdot z} |z\rangle + \sum_{z | z \cdot s = 0} (-1)^{x \cdot z + s \cdot z} |z\rangle \right|$$

Only z's that don't destructively interfere are z's orthogonal to s

$$\Rightarrow \sqrt{\frac{|S|}{2^n}} \sum_{z \in S^\perp} (-1)^{x \cdot z} |z\rangle$$

ii) Given a linear subspace S of $\mathbb{Z}_2^n$,
$f(x) = f(y)$ iff $x = y \oplus s$ for some $s \in S$

- We have our state prepared in $\sqrt{\frac{|S|}{2^n}} \sum_{z \in S^\perp} (-1)^{x \cdot z} |z\rangle$ with $H^{\otimes n}$

- After U_f implementation we have $|f(x)\rangle$ tensored as last qubit

- Measure each ancilla to get $f(x) = c_1, \ldots, f(x) = c_n$

- Apply hadamard to our top states again (not ancillas)

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3i) $QFT_{2^n}|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} \omega_N^{x\cdot y}|y\rangle$

$$QFT_{2^n}(QFT_{2^n}|x\rangle) = QFT_{2^n}\left(\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} \omega_N^{x\cdot y}|y\rangle\right)$$

$$= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} \omega_N^{x\cdot y}\left(\frac{1}{\sqrt{N}}\sum_{z=0}^{N-1} \omega_N^{y\cdot z}|z\rangle\right)$$

$$= \frac{1}{N}\sum_{y=0}^{N-1}\sum_{z=0}^{N-1} \omega_N^{x\cdot y+y\cdot z}|z\rangle$$

ii) We have $\frac{1}{N}\sum_{y=0}^{N-1}\sum_{z=0}^{N-1} \omega_N^{x\cdot y+y\cdot z}|z\rangle$

to get the correct bit string in a classical sense we can't use destructive interference or measurement

Applying QFT^{-1} will cancel out QFT, so we know our correct result is $QFT_{2^n}^{-1}(QFT_{2^n}|x\rangle)$

4i) $X^3 = X\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$

$= \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I_3$

$Z^3 = Z\begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^2\end{pmatrix}\begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^2\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^2\end{pmatrix}\begin{pmatrix}1&0&0\\0&\omega_3^2&0\\0&0&\omega_3^4\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&\omega_3^3&0\\0&0&\omega_3^6\end{pmatrix}$, $\omega_3^3 = \left(e^{i\frac{2\pi}{3}}\right)^3 = e^{i2\pi} = 1$

$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\cdot1\end{pmatrix} = I_3$ ✓

ii) $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^2\end{pmatrix}$ $\omega_3^2\begin{pmatrix}1&0&0\\0&\omega_3&0\\0&0&\omega_3^2\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$

$\begin{pmatrix}0&0&\omega_3^2\\1&0&0\\0&\omega_3&0\end{pmatrix}$ $\omega_3^2\begin{pmatrix}0&0&1\\\omega_3&0&0\\0&\omega_3^2&0\end{pmatrix}$

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ii) $\omega_3^3 = 1 \Rightarrow \begin{pmatrix} 0 & 0 & \omega_3^2 \\ 1 & 0 & 0 \\ 0 & \omega_3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \omega_3^2 \\ 1 & 0 & 0 \\ 0 & \omega_3 & 0 \end{pmatrix}$

L.S. = R.S. ✓

$XZ = \omega_3^2 ZX$

Now find K s.t. $X^i Z^j = \omega_3^K Z^j X^i$ whenever $i, j \in \{0, 1, 2\}$

for $i = j = 1 \Rightarrow XZ = \omega_3^K ZX \Rightarrow K = 2$

for $i = 0, j = 1 \Rightarrow Z = \omega^K Z \Rightarrow K = 0$

for $i = 1, j = 0 \Rightarrow X = \omega_3^K X \Rightarrow K = 0$

for $i = j = 0 \Rightarrow I = \omega_3^K I \Rightarrow K = 0$

for $i = 2, j = 1 \Rightarrow XXZ = \omega_3^K ZXX$

$X^3 Z = X \omega_3^K ZXX \quad |X^3 = Z^3 = I|$

$Z = X\omega_3^K ZXX$

$ZX = X\omega_3^K Z \Rightarrow ZX = \omega_3^K XZ \Rightarrow ZX = \omega_3^{K+2} ZX$

$\Rightarrow K = 1$

for $i = 1, j = 2 \Rightarrow XZZ = \omega_3^K ZZX$

$X = \omega_3^K ZZXZ$

$ZX = \omega_3^K XZ \Rightarrow ZX = \omega_3^{K+2} ZX \Rightarrow K = 1$

for $i = 2, j = 2 \Rightarrow XXZZ = \omega_3^K ZZXX$

$ZZ = \omega_3^K XZZXX$

$ZZX = \omega_3^K XZZ$

$X = \omega_3^K ZXZZ$

$XZ = \omega_3^K ZX \Rightarrow K = 2$

iii) $H^{\dagger} Z H = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega_3^* & \omega_3^{*2}\\ 1 & \omega_3^{*2} & \omega_3^{*}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \omega_3 & 0\\ 0 & 0 & \omega_3^2\end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega_3 & \omega_3^2\\ 1 & \omega_3^2 & \omega_3\end{pmatrix}$

$$= \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega_3^* & \omega_3^{*2}\\ 1 & \omega_3^{*2} & \omega_3^{*}\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ \omega_3 & \omega_3^2 & \omega_3^3\\ \omega_3^2 & \omega_3^4 & \omega_3^3\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega_3^* & \omega_3^{*2}\\ 1 & \omega_3^{*2} & \omega_3^{*}\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ \omega_3 & \omega_3^2 & 1\\ \omega_3^2 & \omega_3 & 1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1+\omega_3+\omega_3^2 & 1+\omega_3^2+\omega_3 & 3\\ 1+\omega_3^*\omega_3+\omega_3^{*2}\omega_3^2 & 1+\omega_3^*\omega_3^2+\omega_3^{*2}\omega_3 & 1+\omega_3^*+\omega_3^{*2}\\ 1+\omega_3^{*2}\omega_3+\omega_3^*\omega_3^2 & 1+\omega_3^{*2}\omega_3^2+\omega_3^*\omega_3 & 1+\omega_3^{*2}+\omega_3^*\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1+e^{i\frac{2\pi}{3}}+e^{i\frac{4\pi}{3}} & 1+e^{i\frac{4\pi}{3}}+e^{i\frac{2\pi}{3}} & 3\\ 1+e^{-i\frac{2\pi}{3}}e^{i\frac{2\pi}{3}}+e^{-i\frac{4\pi}{3}}e^{i\frac{4\pi}{3}} & 1+e^{-i\frac{2\pi}{3}}e^{i\frac{4\pi}{3}}+e^{-i\frac{4\pi}{3}}e^{i\frac{2\pi}{3}} & 1+e^{-i\frac{2\pi}{3}}+e^{-i\frac{4\pi}{3}}\\ 1+e^{-i\frac{4\pi}{3}}e^{i\frac{2\pi}{3}}+e^{i\frac{2\pi}{3}}e^{i\frac{4\pi}{3}} & 1+e^{-i\frac{4\pi}{3}}e^{i\frac{4\pi}{3}}+e^{i\frac{2\pi}{3}}e^{i\frac{2\pi}{3}} & 1+e^{-i\frac{4\pi}{3}}+e^{-i\frac{2\pi}{3}}\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}} & 1+e^{-i\frac{2\pi}{3}}+e^{i\frac{2\pi}{3}} & 3\\ 1+1+1 & 1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}} & 1+e^{-i\frac{2\pi}{3}}+e^{i\frac{2\pi}{3}}\\ 1+e^{-i\frac{2\pi}{3}}+e^{i\frac{2\pi}{3}} & 1+1+1 & 1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\end{pmatrix}$$

$$e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}} = \left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)+\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= \left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)+\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$$

$= 2\cdot\cos\frac{2\pi}{3} = -1$ meaning $e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}} = -1$

$\Rightarrow 1+e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}} = 0$

ii) for $i=0, j=2 \Rightarrow ZZ = \omega_3^k ZZ \Rightarrow k=0$

for $i=2, j=0 \Rightarrow XX = \omega_3^k XX \Rightarrow k=0$

$i \backslash j$	0	1	2
0	$k=0$	$k=0$	$k=0$
1	$k=0$	$k=2$	$k=1$
2	$k=0$	$k=1$	$k=2$

function mapping

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iii) $\Rightarrow \frac{1}{3}\begin{vmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = X$ ✓

iv) $\det|X - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$

$$= 0\begin{vmatrix} 0 & 1 \\ -\lambda & 0 \end{vmatrix} + 1\begin{vmatrix} -\lambda & 1 \\ 1 & 0 \end{vmatrix} - \lambda\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$-1 - \lambda(\lambda^2) = 0$$

$$-\lambda^3 = 1$$

$$\lambda^3 = -1$$

$\lambda = 1, \omega_3, \omega_3^2$ cube roots of unity

$|X - \lambda I|V = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$\lambda = 1 \Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} -x_1 + x_3 = 0 \\ x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \end{matrix} \rightarrow \begin{matrix} x_1 = x_3 \\ x_1 = x_2 \\ x_2 = x_3 \end{matrix}$

iv) for $\lambda=1$, all enteries are equal, so for it to be unit, each entry is $\frac{1}{\sqrt{3}}$ as $\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2=1$

$$\Rightarrow \quad \vec{V}=\begin{pmatrix}\sqrt{3}^{-1}\\ \sqrt{3}^{-1}\\ \sqrt{3}^{-1}\end{pmatrix} \text{ eigenvector for } \lambda=1$$

$$\lambda=\omega_3 \Rightarrow \begin{pmatrix}-\omega_3 & 0 & 1\\ 1 & -\omega_3 & 0\\ 0 & 1 & -\omega_3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{aligned}-e^{i\frac{2\pi}{3}}x_1+x_3&=0\\ x_1-e^{i\frac{2\pi}{3}}x_2&=0\\ x_2-e^{i\frac{2\pi}{3}}x_3&=0\end{aligned}$$

Solution: $x_1=1,\ x_2=e^{i\frac{4\pi}{3}},\ x_3=e^{i\frac{2\pi}{3}}$

Non-normalized: $\vec{V}=\begin{pmatrix}1\\ \omega_3\\ \omega_3^2\end{pmatrix} \Rightarrow \|\vec{V}\|=\sqrt{1^2+|\omega_3|^2+|\omega_3^2|^2}$

$$=\sqrt{1+\omega_3\omega_3^*+\omega_3^2\omega_3^{*2}}$$

$$=\sqrt{1+\left(\cos\tfrac{2\pi}{3}+i\sin\tfrac{2\pi}{3}\right)\left(\cos\tfrac{2\pi}{3}-i\sin\tfrac{2\pi}{3}\right)+\left(\cos\tfrac{4\pi}{3}+i\sin\tfrac{4\pi}{3}\right)\left(\cos\tfrac{4\pi}{3}-i\sin\tfrac{4\pi}{3}\right)}$$

$$=\sqrt{1+\left(\cos^2\tfrac{2\pi}{3}+\sin^2\tfrac{2\pi}{3}\right)+\left(\cos^2\tfrac{4\pi}{3}+\sin^2\tfrac{4\pi}{3}\right)}$$

$$=\sqrt{1+1+1}=\sqrt{3}$$

Normalized: $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \omega_3\\ \omega_3^2\end{pmatrix}$ for $\lambda=\omega_3$

iv) $\lambda = \omega_3^2 \Rightarrow \begin{pmatrix} -\omega_3^2 & 0 & 1 \\ 1 & -\omega_3^2 & 0 \\ 0 & 1 & -\omega_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -e^{i\frac{4\pi}{3}}x_1 + x_3 = 0 \\ x_1 - e^{i\frac{4\pi}{3}}x_2 = 0 \\ x_2 - e^{i\frac{4\pi}{3}}x_3 = 0 \end{matrix}$

Solution: $x_1 = e^{i\frac{2\pi}{3}}, x_2 = e^{i\frac{4\pi}{3}}, x_3 = 1$

Non-normalized: $\vec{v} = \begin{pmatrix} \omega_3 \\ \omega_3^2 \\ 1 \end{pmatrix}$, Same enteries as for $\vec{v}$ when $\lambda = \omega_3$

$\Rightarrow \|\vec{v}\| = \sqrt{1^2 + |\omega_3|^2 + |\omega_3^2|^2} = \sqrt{3}$

Normalized: $\frac{1}{\sqrt{3}} \begin{pmatrix} \omega_3 \\ \omega_3^2 \\ 1 \end{pmatrix}$ for $\lambda = \omega_3^2$

v) Given $f: \mathbb{Z}_3 \to \mathbb{Z}_3$ s.t. f is either constant or balanced

balanced: $\forall y \in \mathbb{Z}_3 \ \exists!\, x \in \mathbb{Z}_3$ s.t. $f(x) = y$

Goal: Determine whether f is constant or balanced

1) Set up n-qutrit string: $H^{\otimes n}|x_1 x_2 \ldots x_n\rangle = \frac{1}{\sqrt{3}} \sum_{z_1 \in \mathbb{Z}_3} \omega_3^{x_1 y_1} |z_1\rangle \otimes \ldots \otimes \frac{1}{\sqrt{3}} \sum_{z_n \in \mathbb{Z}_3} \omega_3^{x_n y_n} |z_n\rangle$

$$= \frac{1}{\sqrt{3}^n} \sum_{z_1 \ldots z_n} \omega_3^{x_1 y_1 + \ldots x_n y_n} |z_1 \ldots z_n\rangle$$

$$= \frac{1}{\sqrt{3}^n} \sum_{z \in \{0,1,2\}^n} \omega_3^{x \cdot y} |z\rangle$$

$x \cdot y$ is dot product over $\{0,1,2\}^n = \mathbb{Z}_3^n$

2) final state: $H^{\otimes n}\left(\frac{1}{\sqrt{3}^n}\sum_{x\in\{0,1,2\}^n} \omega_3^{f(x)}|x\rangle\right)$

$$=\frac{1}{\sqrt{3}^n}\sum_{x\in\{0,1,2\}^n}\omega_3^{f(x)}\left(\frac{1}{\sqrt{3}^n}\sum_{z\in\{0,1,2\}^n}\omega_3^{x\cdot y}|z\rangle\right)$$

$$=\frac{1}{3^n}\sum_{x,z\in\{0,1,2\}^n}\omega_3^{f(x)+x\cdot y}|z\rangle$$

Which paths interfere for this amplitude?

f constant: $\frac{1}{3^n}\sum_{x\in\{0,1,2\}}\omega_3^{f(x)}|00..0\rangle=\frac{1}{3^n}\sum_x \omega_3^c|00..0\rangle$

$$=\omega_3^n|00...0\rangle \text{ for } n\in\{0,1,2\}$$

f balanced: $\frac{1}{3^n}\sum_x \omega_3^{f(x)}|00...0\rangle$

$$=\frac{1}{3^n}\left(\sum_{x|f(x)=0}\omega_3^0|00..0\rangle+\sum_{x|f(x)=1}\omega_3^1|00..0\rangle+\sum_{x|f(x)=2}\omega_3^2|00..0\rangle\right)$$

$$=\frac{3^{n-1}}{3^n}|00..0\rangle+\frac{\omega_3 3^{n-1}}{3^n}|00..0\rangle+\frac{\omega_3^2 3^{n-1}}{3^n}|00..0\rangle$$

$$=\frac{3^{n-1}}{3^n}(1+\omega_3+\omega_3^2)|00...0\rangle, \quad 1+\omega_3+\omega_3^2=0 \left(\begin{matrix}\text{as shown in}\\ 4.3\end{matrix}\right)$$

$$=0$$

If f constant, $|00...0\rangle$ with 100% prob
If f balanced, $|00...0\rangle$ with 0% prob

~~Just~~ ~~like in~~ qubits!
It generalizes ✓

5) Hermitian: $H = H^{\dagger}$

$\Rightarrow H|V\rangle = \lambda|V\rangle$ & $H^{\dagger}\langle V| = \lambda\langle V|$ for $|V\rangle$ eigenvector

Take $\lambda\langle V|V\rangle$

$\langle V|\lambda V\rangle$

$\langle V|H|V\rangle$

$(\langle V|H^{\dagger}|V\rangle)^{\dagger}$

$(\lambda\langle V|V\rangle)^{\dagger}$

$\lambda^{*}\langle V|V\rangle = \lambda\langle V|V\rangle$

$\Rightarrow \lambda^{*} = \lambda \Rightarrow \lambda$ real

6i) $\psi(x)$ can be 0 or 1

Given: $\psi(x) = 0 \Rightarrow \sum_{x\in\{0,1\}^n} \sum_{y\in\{0,1\}} (-1)^{y(1\oplus 0)} |x\rangle$

$\sum_{x\in\{0,1\}^n} \sum_{y\in\{0,1\}} (-1)^{y} |x\rangle$

$\sum_{x\in\{0,1\}^n} (-1)^{0}|x\rangle + (-1)^{1}|x\rangle$

$\sum_{x\in\{0,1\}^n} |x\rangle - |x\rangle = 0 \rightarrow$ zero amplitude

i) Given $\varphi(x)=1 \Rightarrow \sum_{x\in\{0,1\}^n}\sum_{y\in\{0,1\}}(-1)^{y(1\oplus 1)}|x\rangle$

$$\sum_{x\in\{0,1\}^n}\sum_{y\in\{0,1\}}(-1)^{y\cdot 0}|x\rangle$$

$$\sum_{x\in\{0,1\}^n}(-1)^{0\cdot 0}|x\rangle+(-1)^{1\cdot 0}|x\rangle$$

$$\sum_{x\in\{0,1\}^n}|x\rangle+|x\rangle=\sum_{x\in\{0,1\}^n}2|x\rangle \Rightarrow \text{non-zero amplitude}$$

i.e. non-zero amplitude iff $\varphi(x)=1$

ii) $|00...0\rangle \mapsto \frac{1}{2\sqrt{2^n}}\sum_{x\in\{0,1\}^n}\sum_{y\in\{0,1\}}(-1)^{y(1\oplus\varphi(x))}|x\rangle$

If $\varphi(x)=0$ then $|00...0\rangle \mapsto \frac{1}{2\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle-|x\rangle=0$, not a unit state

$\Rightarrow$ not a unit state for a specific $\varphi \Rightarrow$ it cannot be implemented using Unitary transformations

iii) This new transform has an extra state $|y\rangle$ at the end

$$\Rightarrow |00...0\rangle \mapsto \frac{1}{2\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{0(1\oplus\varphi(x))}|x\rangle|0\rangle+(-1)^{1(1\oplus\varphi(x))}|x\rangle|1\rangle$$

$$\rightarrow \frac{1}{2\sqrt{2^n}}\sum |x\rangle|0\rangle+(-1)^{1\oplus\varphi(x)}|x\rangle|1\rangle$$

iii) These two new states of $|x\rangle|0\rangle$ & $|x\rangle|1\rangle$ can't be cancelled Out efficienty (interfere while being constant or balanced)

Constant: $|x\rangle|0\rangle \pm |x\rangle|1\rangle \rightarrow |x\rangle \otimes (\pm\sqrt{2}\,|+\rangle)$

Amplitude is always non-zero

Balanced: $\frac{1}{2\sqrt{2^n}}\left(2^{n-1}(|x\rangle|0\rangle + |x\rangle|1\rangle) + 2^{n-1}(|x\rangle|0\rangle - |x\rangle|1\rangle)\right)$

$\frac{1}{2\sqrt{2^n}} 2^n |x\rangle|0\rangle$

$2^{\frac{n}{2}-1}|x\rangle|0\rangle$

Amplitude is always non-zero

In both cases, $\nexists$ a zero amplitude final state (no destructiv interference) so we do not have useful Interference

iv) No

Knowing a classically efficient algorithm doesn't necessarily translate to a corresponding efficient QA.

A lot of QA's offer no significant speedup or a downgrade compared to it's classical counterpart