Assignment 1

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1/ NAND gate pretty much opposite of AND gate

1.i/ NOT as NAND & fanout

X	NOT(X)
0	1
1	0

Want this with NAND & fanout

$$\Rightarrow NOT(X) \equiv NAND(fanout(X))$$

X	NAND(fanout(X))
0	1
1	0

✓

1.ii/ WTS: From {NAND, fanout}, we can construct {AND, OR, NOT, fanout}

From 1.i/ we showed that $NOT(X) \equiv NAND(fanout(X))$. Now we can use NOT in our implementation to mean $NAND(fanout(X))$. So NOT is covered ✓ And fanout is part of original set ✓

Now lets make OR.

X	Y	OR(X,Y)
0	0	0
0	1	1
1	0	1
1	1	1

$$\Rightarrow$$

X	Y	NAND(NOT(X), NOT(y))
0	0	0
0	1	1
1	0	1
1	1	1

✓

$$\Rightarrow OR(X,y) \equiv NAND(NOT(x), NOT(y))$$

Finally $AND(X,y) \equiv NAND(NOT(x), NOT(y))$ ✓ $\Rightarrow$ {NAND, NOT} is univer...

2) let $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

2 i) $|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$, $|\phi\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

2 iii) $\langle\psi|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

$\langle\phi|\phi\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = 0 + \frac{1}{2} + \frac{1}{2} = 1$

$\langle\psi|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \frac{-i}{\sqrt{6}} - \frac{i}{\sqrt{6}} = \frac{-2i}{\sqrt{6}} = -\sqrt{\frac{2}{3}}\, i$

$|\psi\rangle\langle\phi| = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{i}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} \end{pmatrix}$

$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

2.ii)

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{6}} \\ \dfrac{-i}{\sqrt{6}} \\ 0 \\ \dfrac{i}{\sqrt{6}} \\ \dfrac{1}{\sqrt{6}} \\ \dfrac{-1}{\sqrt{6}} \\ 0 \\ \dfrac{-1}{\sqrt{6}} \\ \dfrac{i}{\sqrt{6}} \end{pmatrix}$$

2.iii)

$$|\psi\rangle + |\phi\rangle = \begin{pmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{i}{\sqrt{3}} \\ \dfrac{-1}{\sqrt{3}} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{2}} \\ \dfrac{-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{i}{\sqrt{3}} + \dfrac{1}{\sqrt{2}} \\ \dfrac{-1}{\sqrt{3}} + \dfrac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{\sqrt{2}\,i + \sqrt{3}}{\sqrt{6}} \\ \dfrac{-\sqrt{2} - \sqrt{3}\,i}{\sqrt{6}} \end{pmatrix}$$

$$\big\| |\psi\rangle + |\phi\rangle \big\| = \sqrt{\left|\dfrac{1}{\sqrt{3}}\right|^2 + \left|\dfrac{\sqrt{2}\,i + \sqrt{3}}{\sqrt{6}}\right|^2 + \left|\dfrac{-\sqrt{2} - \sqrt{3}\,i}{\sqrt{6}}\right|^2}$$

$$= \sqrt{\dfrac{1}{3} + \dfrac{-2 + 2\sqrt{6}\,i + 3}{6} + \dfrac{2 + 2\sqrt{6}\,i - 3}{6}}$$

$$= \sqrt{\dfrac{2 + 1 + 4\sqrt{6}\,i - 1}{6}}$$

$$= \sqrt{\dfrac{1 + 2\sqrt{3}\,i}{\,}}$$

2iii) $\| |\psi\rangle + |\phi\rangle \| = \dfrac{\sqrt{1+2\sqrt{3}i}}{\sqrt{3}} \neq 1$, it's not a unit vector

To make it a unit vector, divide each component with its magnitude

$$\frac{|\psi\rangle + |\phi\rangle}{\| |\psi\rangle + |\phi\rangle \|} = \begin{pmatrix} \dfrac{\sqrt{1+2\sqrt{3}i}}{3} \\[2mm] \dfrac{|\sqrt{2}i + \sqrt{3}| \cdot \sqrt{1+2\sqrt{3}i}}{\sqrt{18}} \\[2mm] \dfrac{|-\sqrt{2} - \sqrt{3}i| \cdot \sqrt{1+2\sqrt{3}i}}{\sqrt{18}} \end{pmatrix} \qquad \checkmark$$

3) $|\psi_3\rangle = \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\[2mm] \dfrac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} , \quad \theta \in \mathbb{R}$

3i) $|0\rangle \rightarrow \left| \dfrac{1}{\sqrt{2}} \right|^2 = \dfrac{1}{2} , \quad |1\rangle \rightarrow \left| \dfrac{e^{i\theta}}{\sqrt{2}} \right|^2 = \dfrac{|e^{2i\theta}|}{2}$

3ii) $\dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\[1mm] \dfrac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\[1mm] \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\[1mm] \dfrac{e^{i\theta}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} + \dfrac{e^{i\theta}}{2} \\[1mm] \dfrac{1}{2} - \dfrac{e^{i\theta}}{2} \end{pmatrix} = \begin{pmatrix} \dfrac{1+e^{i\theta}}{2} \\[1mm] \dfrac{1-e^{i\theta}}{2} \end{pmatrix}$

$|0\rangle \rightarrow \left| \dfrac{1+e^{i\theta}}{2} \right|^2 = \dfrac{|1+2e^{i\theta}+e^{2i\theta}|}{4} = \dfrac{1}{4} + \dfrac{|2e^{i\theta}+e^{2i\theta}|}{4}$

$|1\rangle \rightarrow \left| \dfrac{1-e^{i\theta}}{2} \right|^2 = \left| \dfrac{1-2e^{i\theta}+e^{2i\theta}}{4} \right| = \dfrac{1}{4} + \dfrac{|-2e^{i\theta}+e^{2i\theta}|}{4}$

4i) $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Y|+Y\rangle = |+Y\rangle$. let $|+Y\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, $a = -ib$ & $b = ia$

$\Rightarrow b = i$ & $a = 1$ as $a = -ib$, $1 = -i \cdot i$ & $b = ia$, $i = i \cdot 1$ $\quad$ both true

$\rightarrow |+Y\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$ ✓

Now let $|-Y\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$, $a = ib$ & $b = -ia$

$\Rightarrow b = 1$ & $a = i$ as $a = ib$, $i = i \cdot 1$ & $b = -ia$, $1 = -i \cdot i$ $\quad$ both true

$\rightarrow |-Y\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix}$ ✓

4ii) $U = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $UU^\dagger = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq I$

$UU^\dagger \neq I \Rightarrow$ not unitary

iii) $\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$