MATH 308 - Assignment 3

1. Consider the following maximization LP problem in an equational form:
Maximize $f(x_1, x_2, x_3, x_4) = x_1 - x_2$, subject to

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 18 \\ 4x_1 - x_2 + x_4 &= 18 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- What are the matrix $\boldsymbol{A}^s$ and vectors $\boldsymbol{c}^s$ and $\boldsymbol{b}$ of this formulation?
- Verify that matrix $\boldsymbol{A}^s$ has rank 2 (system of linear equations is consistent).
- Write an initial Tucker tableau for this problem.
  Hint: Do not introduce any additional variables, you just need four variables already presented.
- Using pivot transformations find all (both feasible and infeasible) basic solutions of this problem.
  Hint: $\binom{4}{2} = 6$
- Report all feasible basic solutions. Is there any optimal solution?

Hint. Look at Lecture 6, slide 5.

2. We have described a way how to perform a pivot transformation on a Tucker tableaux using the Pivot Transformation Algorithm. Show that these rules are equivalent to the transformation of the linear program using simple algebra. This is a generalization of a similar question from the previous assignment.

How to show this: Assume that we start with the Tucker tableaux

$$\begin{array}{cccc|c|l} x_1 & x_2 & \dots & x_n & -1 & \\ \hline a_{11} & a_{12} & \dots & a_{1n} & b_1 & = -t_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 & = -t_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m & = -t_m \\ \hline c_1 & c_2 & \dots & c_n & d & = f \end{array} \tag{1}$$

and we want to perform a pivot step with x_i as the entering variable and t_j as the leaving variable using the Pivot Transformation Algorithm.

Express the Tucker tableau as a linear program with independent (non-basic) variables $x_1, \ldots, x_n$ and dependent (basic) variables $t_1, \ldots, t_m$. Perform algebraic modifications on the corresponding system of linear equations which will add x_i to the dependent variables and t_j to the independent variables. Show that the new LP formulation corresponds to the Tucker tableau we would obtain by applying the Pivot Transformation Algorithm.

3. Given a system of linear inequalities $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ $(i = 1, 2, \ldots, m)$. Formulate an LP problem which has an optimal solution if and only if the syslem of linear inequalities has a solution. Justify your answer.(*Hint.* This is easier than you think...)

4. Solve the following LP problem using the simplex algorithm for maximum tableaus:

$$\begin{array}{ll} \text{Maximize:} & f(x_1, x_2, x_3) = x_1 + 2x_2 + 2x_3 \\ \text{subject to} & x_1 + 2x_2 + 2x_3 \leq -4 \\ & 2x_1 + 3x_3 \leq 5 \\ & 2x_1 - x_2 + x_3 \leq -4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

5. Solve the following LP problem using the simplex algorithm for maximum tableaus:

$$\begin{array}{ll} \text{Maximize:} & f(x_1, x_2) = 3x_1 + x_2 \\ \text{subject to} & x_1 - x_2 \leq -1 \\ & -x_1 - x_2 \leq -3 \\ & 2x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

6. Each day, workers at a local medical clinic work two 4-hour shifts chosen from 12am–4am, 4am-8am, 8am–12pm, 12pm–4pm, 4pm-8pm, 8pm–12am (every worker works the same type shifts every day). The following number of workers are needed during each shift: 12am–4am 5 workers, 4am-8am 4 workers, 8am–12pm 15 workers, 12pm–4pm 10 workers, 4pm-8pm 20 workers and 8pm–12am 8 workers. Workers whose two shifts are consecutive are paid \$25 per hour (Shifts 8pm–12am and 12am-4am are considered consecutive); workers whose shifts are not consecutive are paid \$32 per hour.

Formulate an LP problem in canonical minimization form that can be used to minimize the cost of the medical clinic while meeting its daily work-force demands. You do not need to solve the problem.

Strayer, Chapter 2, Exercise 5c, 5d (pp.65), Chapter 3, Exercise 1a, 1b, 1g, 1h (pp. 83, 84), Chapter 3, Exercise 2a, 2b (p. 84)