Last one :)

Alen Assignment 6 301476201

1) $\hat{H} = \theta_1 (Z \otimes X) + \theta_2 (X \otimes Z)$

i) Show that $(Z \otimes X)(X \otimes Z) = (X \otimes Z)(Z \otimes X)$

$$
\begin{aligned}
(Z \otimes X)(X \otimes Z) &= ZX \otimes XZ \\
&= -XZ \otimes -ZX \\
&= (-1)(-1) XZ \otimes ZX \\
&= XZ \otimes ZX \\
&= (X \otimes Z)(Z \otimes X) \checkmark
\end{aligned}
$$

ii) $e^{A+B} = e^A e^B$ when A & B commute

$$
\begin{aligned}
U(t) = e^{-i\hat{H}t} &= e^{-i(\theta_1 (Z\otimes X) + \theta_2 (X \otimes Z))t} \\
&= e^{-i\theta_1 t (Z \otimes X) + (-i\theta_2 t (X \otimes Z))} \\
&= e^{-i\theta_1 t (Z \otimes X)} e^{-i\theta_2 t (X \otimes Z)}
\end{aligned}
$$

$i\theta t$ are just constants so $Z \otimes X$ & $X \otimes Z$ commute

Are $(Z \otimes X)$ & $(X \otimes Z)$ normal? $(AA^\dagger = A^\dagger A)$

$$
\begin{array}{ll}
(Z \otimes X)(Z \otimes X)^\dagger & (Z \otimes X)^\dagger (Z \otimes X) \\
(Z \otimes X)(Z^\dagger \otimes X^\dagger) & (Z^\dagger \otimes X^\dagger)(Z \otimes X) \\
ZZ^\dagger \otimes XX^\dagger & Z^\dagger Z \otimes X^\dagger X \\
I \otimes I = I_4 & I \otimes I = I_4 \checkmark
\end{array}
$$

$(Z \otimes X)$ is normal

$$
\begin{array}{ll}
(X \otimes Z)(X \otimes Z)^\dagger & (X \otimes Z)^\dagger (X \otimes Z) \\
XX^\dagger \otimes ZZ^\dagger & X^\dagger X \otimes Z^\dagger Z \\
I \otimes I & I \otimes I \checkmark
\end{array}
$$

$(X \otimes Z)$ is normal

Meaning we can use matrix exponentials on $e^{-i\theta_1 t (Z \otimes X)} e^{-i\theta_2 t (X \otimes Z)}$

$$
\Rightarrow P_{ZX} \begin{pmatrix} e^{-i\theta_1 t \lambda_{1ZX}} & & \\ & \ddots & \\ & & e^{-i\theta_1 \lambda_{nZX}} \end{pmatrix} P_{ZX}^\dagger \, P_{XZ} \begin{pmatrix} e^{-i\theta_2 t \lambda_{1XZ}} & & \\ & \ddots & \\ & & e^{-i\theta_2 \lambda_{nXZ}} \end{pmatrix} P_{XZ}^\dagger \quad \text{where}
$$

$P_\#$ has columns that are eigenvectors of #

λ are eigenvalues of their individual matrices (ZX or XZ)

iii) Putting all this together, finding P_{XZ} & P_{ZX}

$$X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Using symbolab for my own sanity

$\lambda_1 = 1, \lambda_2 = -1$, eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

has $\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_1 = 1 \quad \lambda_2 = -1$

$$Z \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$\lambda_1 = 1, \lambda_2 = -1$, eigenvectors

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

has $\lambda_1 = 1, \lambda_2 = -1, \lambda_1 = 1, \lambda_2 = -1$

Putting this all together, $P_{ZX} \begin{pmatrix} e^{-i\omega_1 t \lambda_{1ZX}} & & \\ & \ddots & \\ & & e^{-i\omega_1 t \lambda_{nZX}} \end{pmatrix} P_{ZX}^{+} P_{XZ} \begin{pmatrix} e^{-i\omega_2 t \lambda_{1XZ}} & & \\ & \ddots & \\ & & e^{-i\omega_2 t \lambda_{nXZ}} \end{pmatrix} P_{XZ}^{+}$

$$= \begin{array}{c} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ v_1 v_2 v_3 v_4 \end{array} \begin{pmatrix} e^{-i\omega_1 t} & 0 & 0 & 0 \\ 0 & e^{i\omega_1 t} & 0 & 0 \\ 0 & 0 & e^{-i\omega_1 t} & 0 \\ 0 & 0 & 0 & e^{i\omega_1 t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{array}{c} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ v_1 v_2 v_3 v_4 \end{array} \begin{pmatrix} e^{-i\omega_2 t} & 0 & 0 & 0 \\ 0 & e^{i\omega_2 t} & 0 & 0 \\ 0 & 0 & e^{-i\omega_2 t} & 0 \\ 0 & 0 & 0 & e^{i\omega_2 t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

is our two qubit circuit for $U(t)$

Scanned with CamScanner

iii) After approximating to Clifford+T gate set $c \approx 1.5$ with error $\epsilon = 10^{-17}$

$$\Rightarrow \left(\log_2\left(\frac{1}{10^{-17}}\right)\right)^{1.5} \hat{\approx} 1 \text{ (very close to 1)}$$

iv) $(Z \otimes X)|V_i\rangle = \lambda_{a,i}|V_i\rangle$

$(X \otimes Z)|V_i\rangle = \lambda_{b,i}|V_i\rangle$, $\langle V_i|V_j\rangle = 0 \;\forall i \neq j$

$(Z \otimes X)|V_i\rangle = \lambda_{a,i}|V_i\rangle$

$(X \otimes Z)(Z \otimes X)|V_i\rangle = (X \otimes Z)\lambda_{a,i}|V_i\rangle$

$(X \otimes Z)(Z \otimes X)|V_i\rangle = \lambda_{a,i}(X \otimes Z)|V_i\rangle$

$(X \otimes Z)(Z \otimes X)|V_i\rangle = \lambda_{a,i}\lambda_{b,i}|V_i\rangle$

$$(X \otimes Z)(Z \otimes X) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

eigenvectors of this matrix: $V_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $V_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $V_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_1 = 1$ $\quad \lambda_2 = -1$ $\quad \lambda_1 = 1$ $\quad \lambda_2 = -1$

Make them unit: multiply by $\frac{1}{\sqrt{2}}$. $V_i \cdot V_j = 0 \;\forall i \neq j$ ✓

Therefore, orthonormal basis for eigenspace:

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

v) Λ_X is matrix of eigenvectors of X

$Z \otimes X = U \Lambda_{ZX} U^{\dagger}$, $X \otimes Z = U \Lambda_{XZ} U^{\dagger}$

"as $Z \otimes I$ & $I \otimes Z$"

$$Z \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$I \otimes Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$Z \otimes X =$

Scanned with CamScanner

2) Given 32-bit hash H of a message M with two-One protocol

$H = h(M)$ where $h: \{0,1\}^{32} \to \{0,1\}^{32}$ & h has the property that for every $H = h(M)$ $\exists$ exactly one collision $M' \neq M$ s.t. $h(M') = H$

i) input: $h: \{0,1\}^{32} \to \{0,1\}^{32}$ s.t. $\forall H = h(M)$ $\exists!$ $M' \neq M$ s.t. $h(M') = H$

goal: find $M' \in \{0,1\}^{32}$ s.t. $h(M') = M$, $M' \neq M$

here, our search function $f = h$

Implementation: Prepare $\frac{1}{\sqrt{2^{32}}} \sum_{x \in \{0,1\}^{32}} |x\rangle$. All states have amplitude $\frac{1}{2^{16}}$

Apply phase oracle $U_f^{\pm}: |x\rangle \to (-1)^{f(x)} |x\rangle$ and Apply U_{diff} for amplitude manipulation

Measure when solution state has high prob

ii) Each M has a pair that has same image under h

Meaning for any M $\exists$ 2 hashs, one of which is M, the other is M'

Our search function def^n excludes $M' \neq M$ meaning we have one sol

$|s\rangle$ will have angle θ with $|\psi_{bad}\rangle$, $\sin\theta = \frac{1}{2^{16}}$ (very small meaning $\sin\theta \approx \theta$)

$\to$ opt. interations are $\frac{\pi}{4}\sqrt{\frac{2^{32}}{1}} - \frac{1}{2} \approx 51471$

iii) After M reps, angle with $|\psi_{bad}\rangle$ is $(2M+1)\theta$, $\theta \approx \frac{1}{2^{16}}$ assuming small

$\Rightarrow (51471 \cdot 2+1)\left(\frac{1}{2^{16}}\right) = 1.570785522\ldots$ is angle with $|\psi_{bad}\rangle$. We want $\frac{\pi}{2}$ angle

$\frac{\pi}{2} \approx 1.570796327$

Very high similarity with $\frac{\pi}{2}$, approximate similarity: $\frac{1.570785522}{1.570796327}$

$= 0.9999931213\ldots$

Round down to about 99% success

3) $X \otimes X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

eigenvectors of this matrix $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ by inspection

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0 \rightarrow$ orthogonal

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |00\rangle + |11\rangle = \underbrace{|0\rangle_L + |1\rangle_L}_{=|\psi\rangle \text{ with } \alpha=\beta=1}$ $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = |01\rangle + |10\rangle = |0\rangle_L |1\rangle_L + |1\rangle_L |0\rangle_L$

If bit flips are independant we could end up with a non-eigenvector like $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
meaning $(X \otimes X)|\psi\rangle \neq |\psi\rangle \rightarrow$ doesn't work

Scanned with CamScanner

4) Quantum channel: $\rho \to \rho'$

Kraus operators: $\{K_i\}$ s.t. $\sum_i K_i^\dagger K_i = I$

The channel then maps $\rho \to \sum_i K_i \rho K_i^\dagger$. $\{\frac{1}{2}I, \frac{1}{2}X, \frac{1}{2}Y, \frac{1}{2}Z\}$

Assume ρ is 2x2

Depolarizing channel sends $\rho \to \sum_i K_i \rho K_i^\dagger$

$$\sum_i K_i \rho K_i^\dagger = \frac{1}{2}I \rho \frac{1}{2}I^\dagger + \frac{1}{2}X \rho \frac{1}{2}X^\dagger + \frac{1}{2}Y \rho \frac{1}{2}Y^\dagger + \frac{1}{2}Z \rho \frac{1}{2}Z^\dagger$$
$$= \frac{1}{4}I\rho I^\dagger + \frac{1}{4}X\rho X^\dagger + \frac{1}{4}Y\rho Y^\dagger + \frac{1}{4}Z\rho Z^\dagger$$

Let's do this manually. let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $Tr(\rho) = 1 \to a+d=1$ density matrix

I is identity so expression simplifies to

$$\frac{1}{4}\rho + \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\frac{1}{4}\rho + \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & a \\ d & c \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} ib & -ia \\ id & -ic \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

$$\frac{1}{4}\rho + \frac{1}{4}\begin{pmatrix} d & c \\ b & a \end{pmatrix} + \frac{1}{4}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} + \frac{1}{4}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} d & c \\ b & a \end{pmatrix} + \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} + \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} a+d+d+a & b+c-c-b \\ c+b-b-c & d+a+a+d \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 1+1 & 0 \\ 0 & 1+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = 2I$$

5) Steane code encodes one logical qubit through 7 physical qubits

5i) for $X\otimes\ldots\otimes X$, the matrix will be an off diagonal of 1's, 0 elsewhere
meaning it will flip qubits no matter it's position

Comparing $|0\rangle_L$ & $|1\rangle_L$, their qubits are just flipped
meaning $X_L|0\rangle_L = |1\rangle_L \Rightarrow X_L$ acts like $X\otimes X\otimes X\otimes X\otimes X\otimes X\otimes X$ as it
flips each 7 qubit

for $Z\otimes\ldots\otimes Z$, the matrix will be a diagonal of $1, -1, 1, -1, \ldots$ 0 elsewhere
meaning it will flip every $|1\rangle$ qubit, leave $|0\rangle$ qubit alone

Comparing $|0\rangle_L$ & $|1\rangle_L$, each state in $|0\rangle_L$ has even # of 1's
each state in $|1\rangle_L$ has odd # of 1's

meaning $Z\otimes\ldots\otimes Z$ will make $|0\rangle_L \to |0\rangle_L$ (even amount of -1's multiplied
$|1\rangle_L \to -|1\rangle_L$ (odd amount of -1's multiplied

$\Rightarrow Z_L|0\rangle_L \to |0\rangle_L$, $Z_L|1\rangle \to -|1\rangle_L$

$\Rightarrow Z_L$ acts like $Z\otimes Z\otimes Z\otimes Z\otimes Z\otimes Z\otimes Z$ as it negates $|1\rangle$
and leaves $|0\rangle$

for $S\otimes\ldots\otimes S$, the matrix will leave $|0\rangle$ along $|1\rangle \to i|1\rangle$

Comparing $|0\rangle_L$, $|1\rangle_L$, each state in $|0\rangle_L$ has 4 1's (expect $|0000000\rangle$)
each state in $|1\rangle_L$ has 3 1's (expect $|1111111\rangle$)

meaning $S\otimes\ldots\otimes S$ will make $|0\rangle_L \to |0\rangle_L$ as $i^4 = 1$
$|1\rangle_L \to -i|1\rangle_L$ as $i^3 = -i$

meaning for $S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$, $S^\dagger|0\rangle = |0\rangle$, $S^\dagger|1\rangle = -i|1\rangle$

$\Rightarrow S_L^\dagger$ acts like $S\otimes S\otimes S\otimes S\otimes S\otimes S\otimes S$ as it leaves $|0\rangle$ & adds
factor of ~~-~~i to $|1\rangle$

ii) for $T\otimes\ldots\otimes T$, the matrix will leave $|0\rangle$ alone, $|1\rangle \rightarrow e^{i\frac{\pi}{4}}|1\rangle$

$T\otimes\ldots\otimes T$ will make $|0\rangle_L \rightarrow -|0\rangle_L$ as $(e^{i\frac{\pi}{4}})^4 = -1$

$|1\rangle_L \rightarrow e^{i\frac{3\pi}{4}}|1\rangle_L$ as $(e^{i\frac{\pi}{4}})^3 = e^{i\frac{3\pi}{4}}$

This is not equal to T_L as $T\otimes\ldots\otimes T|0\rangle_L \rightarrow -|0\rangle_L$
$$\neq$$
$$T_L|0\rangle_L \rightarrow |0\rangle_L$$

Different mapping means it is not traversal in Steane code as $T_L \neq T\otimes\ldots\otimes T$

iii) let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Circuit 1

$$T|\psi\rangle = \alpha|0\rangle + \beta e^{i\frac{\pi}{4}}|1\rangle$$

Circuit 2

$$|\psi\rangle|A\rangle = (\alpha|0\rangle + \beta|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|10\rangle + \alpha e^{i\frac{\pi}{4}}|01\rangle + \beta e^{i\frac{\pi}{4}}|11\rangle$$

$$CNOT(|\psi\rangle|A\rangle) = \frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|11\rangle + \alpha e^{i\frac{\pi}{4}}|01\rangle + \beta e^{i\frac{\pi}{4}}|10\rangle$$

measure second qubit $a=0 \rightarrow \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta e^{i\frac{\pi}{4}}|1\rangle)$

apply $S^0 = I \rightarrow \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta e^{i\frac{\pi}{4}}|1\rangle) = |\psi\rangle$

$a=1 \rightarrow \frac{1}{\sqrt{2}}(\beta|1\rangle + \alpha e^{i\frac{\pi}{4}}|0\rangle)$

apply $S^1 \rightarrow \frac{1}{\sqrt{2}}(\beta i|1\rangle + \alpha e^{i\frac{\pi}{4}}|0\rangle)$

$$\frac{1}{\sqrt{2}}(e^{i\frac{\pi}{4}}(\beta e^{i\frac{\pi}{4}}|1\rangle + \alpha|0\rangle)) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}|\psi\rangle$$