1. (3 marks) Find the rate of convergence of the following sequence as $h \to 0$:

$$\lim_{h\to 0} \left[h\sin(h) - \sin^2(h)\right] = 0$$

Show your steps.

2. (3 marks) Let $||\mathbf{x}||$ be any vector norm. Show that $|\ ||\mathbf{x}|| - ||\mathbf{y}||\ | \leq ||\mathbf{x} - \mathbf{y}||$ for any vectors $\mathbf{x}$ and $\mathbf{y}$. (Hint: $\mathbf{x} = \mathbf{x} - \mathbf{y} + \mathbf{y}$. Use norm properties.)

3. (4 marks) Suppose

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

(a) Factor A into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

(b) Using the factorization obtained in (a) compute the determinant of A.

4. (3 marks) Given the linear system $Ax = b$ where

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

(a) Will the Jacobi and/or the Gauss-Seidel methods converge for this linear system? Why or why not?

(b) Find the first iteration using the Jacobi method with $x^{(0)} = (1, 1, 1)^t$.

5. (4 marks) Use a theorem from the course to show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

MT 2022 Solns

Note Title 2022-03-07

1 $$h\sin(h) - \sin^2(h)$$
$$= h\left(h - h^3/3!\right) - \left(h - h^3/3!\right)^2 + O(h^6)$$
$$= h^2 - \frac{h^4}{3!} - \left(h^2 - 2\frac{h^4}{3!}\right) + O(h^6)$$
$$= h^4/3! + O(h^6)$$
$$= O(h^4).$$

2 $\|x\| = \|x-y+y\|$

$\|x\| \leq \|x-y\| + \|y\|$

$\|x\| - \|y\| \leq \|x-y\|$ ①

Exchange x & y in this expression

$\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$ ②

① & ② together give

$\big| \|x\| - \|y\| \big| \leq \|x-y\|$

3a $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

$$\xrightarrow[R_3-(-1)R_1 \to R_3]{R_2-(2)R_1 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3-(\frac{1}{2})R_2 \to R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1/2 & 1 \end{bmatrix}$$

(b) $\det(A) = \det(L)\det(U)$

$= 1 \cdot (1 \cdot 4 \cdot 1/2)$

$= 2.$

4a Yes. A is strictly diagonally dominant.

b $x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b$

$$D^{-1}(L+U)$$

$$= \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2/3 & 0 \\ 1/2 & 0 & 1/4 \\ 0 & -1/2 & 0 \end{pmatrix}$$

$$c = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ 1/4 \\ -1 \end{pmatrix}$$

$$x^{(1)} = \begin{pmatrix} 2/3 \\ 3/4 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1/4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1 \\ -3/2 \end{pmatrix}$$

5 $a = \sqrt{2}$, $b = \infty$

$g(x) = \frac{1}{2}x + \frac{1}{x}$

is CTS for all $x > 0$.

$g'(x) = \frac{1}{2} - \frac{1}{x^2}$

$|g'(x)|$ is strictly increasing for $x > \sqrt{2}$.

$|g'(x)| < \frac{1}{2}$ for $x > \sqrt{2}$.

Further, $\max_{x \in [\sqrt{2}, \infty)} g(x)$

$\geq \min_{x \in [\sqrt{2}, \infty)} g(x)$

extrema when $\frac{1}{2} - \frac{1}{x^2} = 0$

$x = \sqrt{2}$

$\Longrightarrow$ min $g(x)$ at $x = \sqrt{2}$ for $x \in [\sqrt{2}, \infty)$

Thus $g(x) \in [\sqrt{2}, \infty)$ for all $x \in [\sqrt{2}, \infty)$.

Since $x_0 > \sqrt{2}$ we obtain convergence to the unique fixed pt.

The fixed pt p is given by

$$p = \frac{1}{2}p + \frac{1}{p}$$

$$\Rightarrow \frac{1}{2}p = \frac{1}{p}$$

$$p = \sqrt{2}.$$