SFU Macm 316 Final Exam: April 16, 2024

Instructions: 3 Hours. Closed book. Two sided cheat sheet permitted. Answer all 13 questions. State very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS. Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation.

1. (4 marks) A natural cubic spline S for a function f is defined on $[1,3]$ by

$$S = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \leq x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3 & \text{if } 2 \leq x \leq 3 \end{cases}$$

If S interpolates the data (1,1), (2,1), and (3,0), find B, D, b and d.

2. (4 marks) Derive an $O(h^2)$ three-point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, and $f(x_0 + 2h)$.

3. (4 marks) Suppose Newton's method is used to solve the equation

$$0 = \sin(x)(\cos(x) - 1).$$

(a) What is the value of p_1 using an initial guess of $p_0 = 0.1$?

(b) Why do we expect Newton's method to give less than quadratic convergence in this example?

(c) Give a modified algorithm that would give quadratic convergence.

4. (3 marks) Consider the problem of evaluating

$$\frac{\sqrt{x^2+1}-1}{x^2}$$

for small x.

(a) Explain why this expression can lead to significant round-off errors for small values of x.

(b) Propose an alternative form of the given expression to reduce round-off errors when x is small. Justify your alternative form theoretically, showing how it minimizes the potential for error.

5. (3 marks) Let $||A||$ be any natural matrix norm of matrix A. Use the definition of natural matrix norm to show that

$$|\lambda| \leq ||A||$$

for any nonsingular matrix A and any eigenvalue λ of A. You must use the definition of natural matrix norm to receive any marks for this question.

6. (4 marks) Given $g(x) = \frac{2-x^3+2x}{3}$, use the Fixed Point Theorem to show that the fixed-point sequence $p_{n+1} = g(p_n)$ will converge to the unique fixed point of g for any p_0 in $[-1, 1.1]$.

7. (4 marks) Suppose a numerical procedure is available to approximate a number N with the approximation $N(h)$, where the parameter of the procedure is $h > 0$. Further, suppose there is an error expansion $N = N(h) + h + 2h^3$. For a given h we have calculated $N(h)$, $N(h/2)$ and $N(h/4)$. Derive the formulas to obtain the best possible approximation to N using what has already been calculated along with extrapolation.

Multiple choice. Circle the correct statement, or if there is more than one correct statement choose the best answer. No explanation is required.

8. (1 mark) Recall the weighted mean value theorem for integrals

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

for some number c in (a, b). The theorem requires that the Riemann integral of g exists on $[a, b]$. What else is required of the function g?

(a) Nothing else is required.

(b) g must not change sign on $[a, b]$.

(c) g must be continuous.

(d) g must be C^1.

(e) g must satisfy a Lipschitz condition.

9. (1 mark) Suppose that the $n \times n$ matrix A is tridiagonal, and strictly diagonally dominant. Then the following is true:

(a) A^{-1} is tridiagonal.

(b) A^{-1} is sparse.

(c) A^{-1} is strictly diagonally dominant.

(d) A^{-1} consists entirely of nonnegative entries.

(e) none of the above.

10. (1 mark) Which of the following statements best describes the local truncation error of a numerical method for solving initial value problems (IVPs) for ordinary differential equations?

(a) It is the error made in one step of the method, assuming all previous steps were executed without error.

(b) It is the cumulative error accumulated over all steps of the method up to a specific point in time.

(c) It is the difference between the numerical solution and the exact solution of the IVP at the final step only.

(d) It refers to the error introduced due to rounding off numbers during the computations at each step.

(e) It is the error in the initial condition given to the IVP, affecting the accuracy of the numerical method.

Multiple choice. Circle the correct statement, or if there is more than one correct statement choose the best answer. No explanation is required.

11. (1 mark) Suppose P is a permutation matrix. Then

(a) $\det(P) = 1$.

(b) $P = P^T$.

(c) P is the identity matrix with two rows interchanged.

(d) All of the above.

(e) None of the above.

12. (1 mark) The degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$$

is equal to

(a) 1/2

(b) 1

(c) 2

(d) 3

(e) none of the above

13. (1 mark) Aitken's method will accelerate convergence of the following

(a) bisection method.

(b) Newton's method.

(c) secant method.

(d) all of the above.

(e) none of the above.

MACM 316 Final Exam Solns
Spring 2024

1 $S_0(1) = 1$ ✓

$S_0(2) = 1 + B - D = 1 \Rightarrow B = D$ ①

$S_1(2) = 1$ ✓

$S_1(3) = 1 + b - \frac{3}{4} + d = 0$

$\Rightarrow b + d = -\frac{1}{4}$ ②

$S_0''(1) = 0 \Rightarrow -6D(0) = 0$ ✓

$S_1''(3) = 0 \Rightarrow -\frac{3}{2} + 6d = 0$

$d = \frac{1}{4}$

into ②: $b = -\frac{1}{2}$

$S_0'(2) = S_1'(2) \Rightarrow B - 3D = b = -\frac{1}{2}$ ③

apply ①: $-2B = -\frac{1}{2}$

$B = \frac{1}{4}$

into ①: $D = \frac{1}{4}$

Note $S_0''(2) = -6D = -\frac{3}{2}$

$S_1''(2) = -\frac{3}{2}$ ✓

We have $B = D = \frac{1}{4}$

$b = -\frac{1}{2}, \ d = \frac{1}{4}$

2 $$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3) \quad (1)$$

$$f(x_0+2h) = f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + O(h^3) \quad (2)$$

4(1) − (2):

$$4f(x_0-h) - f(x_0+2h)$$
$$= 4f(x_0) - f(x_0) - 4h f'(x_0) - 2hf'(x_0) + O(h^3)$$

$$f'(x_0) = \frac{4f(x_0-h) - 3f(x_0) - f(x_0+2h)}{-6h} + O(h^2)$$

$$= \frac{-\frac{2}{3}f(x_0-h) + \frac{1}{2}f(x_0) + \frac{1}{6}f(x_0+2h)}{h} + O(h^2)$$

3a $f(x) = \sin(x)(\cos x - 1)$

$f'(x) = \cos(x)(\cos x - 1)$

$- \sin^2(x)$

$p_1 = p_0 - f(p_0)/f'(p_1)$

≈ 0.06661

b There is a root of multiplicity 2 at $x=0$

c Define $\mu(x) = f(x)/f'(x)$

and apply Newton's method to $\mu(x)$.

4a For small x,

$$\sqrt{x^2+1} \approx 1$$

and we have cancellation error in the numerator.

b Rationalize the numerator:

$$\frac{\sqrt{x^2+1}-1}{x^2} \times \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$$

$$= \frac{x^2+1-1}{x^2\left(\sqrt{x^2+1}+1\right)} = \frac{1}{\sqrt{x^2+1}+1}$$

There is no subtraction of similar numbers.

No cancellation error.

5 $\|A\| = \max_{\|x\|=1} \|Ax\|$

$\geq \max_{\|x\|=1} \|Ax\|$

x is an eigenvector with eigenvalue λ

$= \max_{\|x\|=1} \|\lambda x\|$

$= \max_{\|x\|=1} |\lambda| \, \|x\|$

$= |\lambda|$

6 g is CTS (a polynomial)

$\underset{[-1,1.1]}{\min/\max}\, g = ?$

$g' = -x^2 + 2/3 = 0 \Rightarrow x = \pm\sqrt{2/3}$

$g\left(\sqrt{2/3}\right) = \dfrac{2 + \frac{2\sqrt{2}}{3\sqrt{3}} - 2\sqrt{2}}{3} \approx 1.03$

$g\left(-\sqrt{2/3}\right) \; \dfrac{2 - \frac{2\sqrt{2}}{3\sqrt{3}} + 2\sqrt{2}}{3} \approx 0.3$

$g(-1) = \dfrac{2+1-2}{3} = 1/3$

$g(1.1) \approx .956$

We see $g \in [-1, 1.1]$

$|g'(x)| = |-x^2 + 2/3| < 2/3$

$\Rightarrow$ $p_{n+1} = g(p_n)$ converges to the unique fixed pt.

7

$$N = N(h) + h + 2h^3$$

$$N = N\left(\tfrac{h}{2}\right) + \tfrac{h}{2} + \tfrac{1}{4}h^3$$

$$N = N\left(\tfrac{h}{4}\right) + \tfrac{h}{4} + \tfrac{1}{32}h^3$$

$$N = 2N\left(\tfrac{h}{2}\right) - N(h) + \tfrac{1}{2}h^3 - 2h^3$$

$$= 2N\left(\tfrac{h}{2}\right) - N(h) - \tfrac{3}{2}h^3 \qquad (*)$$

$$N = 2N\left(\tfrac{h}{4}\right) - N\left(\tfrac{h}{2}\right) + \tfrac{1}{16}h^3 - \tfrac{1}{4}h^3$$

$$= 2N\left(\tfrac{h}{4}\right) - N\left(\tfrac{h}{2}\right) - \tfrac{3}{16}h^3 \qquad (**)$$

$8(**) - (*)$:

$$7N = 16N\left(\tfrac{h}{4}\right) - 8N\left(\tfrac{h}{2}\right) - 2N\left(\tfrac{h}{2}\right) + N(h)$$

$$N = \frac{16N\left(\tfrac{h}{4}\right) - 10N\left(\tfrac{h}{2}\right) + N(h)}{7}$$

8 (b)

9 (e)

10 (a)

11 (e)

12 Try $f(x) = 1$: $LHS = \int_{-1}^{1} f(x)dx = 2$

$RHS = \frac{1}{3} + \frac{4}{3} + \frac{1}{3} = 2$ ✓

$f(x) = x$: $LHS = \int_{-1}^{1} x\,dx = 0$

$RHS = -\frac{1}{3} + \frac{1}{3} = 0$ ✓

$f(x) = x^2$: $LHS = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$RHS = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ ✓

$f(x) = x^3$: $LHS = \int_{-1}^{1} x^3 dx = 0$

$RHS = -\frac{1}{3} + \frac{1}{3} = 0$ ✓

$f(x) = x^4$: $LHS = \int_{-1}^{1} x^4 dx = \frac{2}{5}$

$RHS = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ ✗

Degree of precision 3 $\Rightarrow$ (d)

13 (e)