SFU Macm 316

Final Exam: April 22, 2023

Instructions: 3 Hours. Closed book. Two sided cheat sheet permitted. Answer all 9 questions.

EXPLAIN ALL ANSWERS. Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation. Mark very clearly on the question if you use the back of any page.

1. (4 marks) Given $g(x) = \cos(x)$, use a theorem from the course to show that the the fixed-point sequence $p_n = g(p_{n-1})$ will converge to the unique fixed-point of g for any p_0 in $[0, 1]$.

2. (4 marks) Determine a quadratic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$ and satisfies $S'(0) = 2$.

3. (4 marks) Solve the linear system

$$\begin{cases} 0.211x_1 + 0.811x_2 &= \quad 1.51 \\ 1.71x_1 - 1.06x_2 &= -0.512 \end{cases}$$

using 3 digit rounding arithmetic and Gaussian elimination with scaled partial pivoting.

4. (4 marks) Find the constants a, b and c so that the quadrature formula

$$\int_0^1 f(x)dx = af(1) + bf'(-1) + cf'(1)$$

has the highest possible degree of precision. What is the degree of precision?

5. (5 marks) Let $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$. Given

$$f(x) = -2e^{-x} + \frac{1}{4}x^4 - \frac{1}{120}x^5 + 2x, \quad f'(x) = 2e^{-x} + x^3 - \frac{1}{24}x^4 + 2,$$

$$f''(x) = -2e^{-x} + 3x^2 - \frac{1}{6}x^3, \quad f'''(x) = 2e^{-x} + 6x - \frac{1}{2}x^2,$$

$$f^{(4)}(x) = -2e^{-x} + 6 - x, \quad f^{(5)}(x) = 2e^{-x} - 1.$$

(a) Find the Lagrange Interpolating Polynomial, $P_2(x)$, of degree at most 2 for $f(x)$ using x_0, x_1, x_2. You do not need to simplify your answer.

(b) Use the general error formula for the Lagrange Interpolating Polynomial to find a bound for the absolute error at 0.65. Your answer should be a real number. You may assume that $f'''(x)$ has no relevant critical points.

HINT: The following may be useful: $\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$

6. (3 marks) Show the initial value problem

$$y' = y - t\sin(ty),\ \ 0 \le t \le 2,\ \ y(0) = 2$$

has a unique solution and is also well-posed.

7. (4 marks) The equation $f(x) = 1 - 2x^2\cos(\pi x) = 0$ has at least one solution in the interval [-1,2].

   (a) Verify that the Bisection method can be applied to the function $f(x)$ on [-1,2].

   (b) Using the error formula for the Bisection method, find the number of iterations needed for accuracy 10^{-4}. Do not do the Bisection calculations.

   (c) Compute p_3 for the Bisection method.

8. (4 marks) Suppose

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) Find $||A||_\infty$.

(b) Find $\rho(A)$.

(c) Find an eigenvector of A corresponding to the eigenvalue for which $|\lambda| = \rho(A)$.

9. (4 marks) Suppose a numerical procedure is available to approximate a number M with the approximation $N(h)$, where the parameter of the procedure is $h > 0$. Further, suppose there is an error expansion $M = N(h) + c_1 h + c_3 h^3 + c_5 h^5$. The unknown constants c_1, c_3, and c_5 are independent of h.

   For a given h we have calculated $N(h)$, $N(h/3)$ and $N(h/9)$. Give the formulas to obtain the best possible approximation to M using what has already been calculated along with extrapolation.

MACM 316: Spring 2023 Exam Solns

Note Title 2023-04-08

1 $g(x) = \cos(x)$ is continuous for all x.
$g'(x) = -\sin(x)$ exists for all x.

$|g'(x)| \le |\sin(1)| < 0.85 < 1$ for all $x \in (0,1)$

By the fixed point theorem,
$p_n = g(p_{n-1})$ converges
to the unique fixed point p in $[0,1]$ for any number p_0 in $[0,1]$.

Note Title 2023-04-14

2 $S(x) = \begin{cases} S_0, & 0 \le x < 1 \\ S_1, & 1 < x \le 2 \end{cases}$

$S_0(x) = a_0 + b_0 x + c_0 x^2$

$S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2$

$f(0) = 0 \Rightarrow S_0(0) = 0 \Rightarrow a_0 = 0$

$S'(0) = 2 \Rightarrow S_0'(0) = 2 \Rightarrow b_0 = 2$

$f(1) = 1 \Rightarrow S_0(1) = 1 \Rightarrow 2 + c_0 = 1$

$\Rightarrow c_0 = -1$

$S_0'(1) = b_0 + 2c_0(1) = 0$

$S_1'(0) = S_0'(1) = 0$

$\Rightarrow b_1 = 0.$

$f(1) = 1 \Rightarrow S_1(1) = 1 \Rightarrow a_1 = 1.$

$f(2) = 2 \Rightarrow S_1(2) = 2$

$\Rightarrow 1 + c_1(1)^2 = 2$

$\Rightarrow c_1 = 1.$

Thus,

$a_0 = 0$	$a_1 = 1$
$b_0 = 2$	$b_1 = 0$
$c_0 = -1$	$c_1 = 1$

Note Title 2023-04-14

3 Scale 1 = 0.811
Scale 2 = 1.71

$0.211/\text{Scale 1} < 1.71/\text{Scale 2}$

$\Longrightarrow$ exchange rows.

$\boxed{m_{21} = 0.123}$

$$1.71x_1 - 1.06x_2 = -0.512$$
$$0.211x_1 + 0.811x_2 = 1.51$$

$$E_2 - fl\left(\frac{0.211}{1.71}\right)E_1 \rightarrow E_2:$$

$$fl(fl(m_{21}\cdot(-1.06)) + 0.811) = 0.681$$
$$fl(fl(m_{21}\cdot(-.512)) + 1.51) = 1.57$$

We have
$$1.71x_1 - 1.06x_2 = -0.512$$
$$0.941x_2 = 1.57.$$

Back substitution: $x_2 = fl\left(\frac{1.57}{0.941}\right) = 1.67$

$$x_1 = fl\left(\frac{fl(-0.512 + fl(+1.06\cdot x_2))}{1.71}\right)$$
$$= fl\left(\frac{fl(-0.512 + 1.77)}{1.71}\right)$$
$$= fl\left(1.26/1.71\right)$$
$$= 0.737$$

Note Title 2023-04-14

4 $f = 1: \int_0^1 1 = 1 = a$

$f = x: \int_0^1 x = \frac{1}{2} = a + b + c$

$f = x^2: \int_0^1 x^2 = \frac{1}{3} = a - 2b + 2c$

Solve : $b + c = -1/2$

$-2b + 2c = -2/3$

$\Rightarrow -b + c = -1/3$

$2c = -5/6$

$c = -5/12$

$b = -1/12$

Check $f = x^3$:

$LHS = \int_0^1 x^3 = \frac{1}{4}$. $RHS = a + 3b + 3c$

$LHS \neq RHS$.

Degree of precision is 2.

Note Title 2023-04-14

5(a)
$$P_2(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2$$
$$= \frac{(x-\frac{1}{2})(x-1)}{(1/2)}(-2)$$
$$+ \frac{(x-0)(x-1)}{(1/2)(-1/2)}\left(-2e^{-1/2} + \frac{1}{4}\left(\frac{1}{2}\right)^4 - \frac{1}{120}\left(\frac{1}{2}\right)^5 + 1\right)$$
$$+ \frac{(x-0)(x-1/2)}{(1/2)}\left(-2e^{-1} + \frac{1}{4} - \frac{1}{120} + 2\right)$$

(b)
$$\text{error} = \frac{f^{(3)}(\xi)}{3!}(x-0)(x-\tfrac{1}{2})(x-1)$$
$$|f^{(3)}(\xi)| = |\underbrace{2e^{-\xi} + 6\xi - \tfrac{1}{2}\xi^2}_{\text{an increasing fn.}}|$$
$$|f^{(3)}(\xi)| \le |2e^{-1} + 6(1) - \tfrac{1}{2}(1)^2|$$
$$= 6.2358$$
$$\Rightarrow |\text{error}| \le \left|\frac{6.2358}{6}(.65-0)(.65-\tfrac{1}{2}) \cdot (.65-1)\right|$$
$$= 0.0355.$$

Note Title 2023-04-14

6 $\left|\frac{\partial\, y - t\sin(ty)}{\partial y}\right| = \left|1 - t^2 \cos ty\right|$

$\leq |1 + t^2|$

≤ 5

$\Rightarrow$ f satisfies a Lipschitz condition on D with Lipschitz constant 5.

Since f is continuous on D, the problem is well-posed.

Note Title 2023-04-14

7 (a) $f(x) = 1 - 2x^2\cos(\pi x)$ is continuous.

$f(-1) = 1 - 2(-1)^2\cos(-\pi)$
$= 1 + 2 > 0$

$f(2) = 1 - 2(2)^2\cos(2\pi) = -7 < 0$

$\therefore$ Bisection may be applied.

(b) $|p_n - p| = \frac{b-a}{2^n} < 10^{-4}$

$3/2^n < 10^{-4}$

$2^n > 3\times10^4$

$n\log 2 > \log(3\times10^4)$

$n > 14.87$

But n is an integer

$\Rightarrow n \geq 15.$

(c) $p_1 = \frac{b+a}{2} = 0.5$

$f(0.5) = 1 > 0 \Rightarrow$ Look for zero over $[0.5, 2]$

$p_2 = 1.25$

$f(1.25) = 3.21\ldots > 0$
$\Rightarrow$ Look for zero over $[1.25, 2]$

$\boxed{p_3 = 1.625}$

Note Title 2023-04-14

8 (a) $\|A\|_\infty = \max\{|2|+|1|, |1|+|2|+|2|\} = 3$

(b) $0 = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 - (2-\lambda)$

$= (2-\lambda)(4-4\lambda+\lambda^2-1)$

$= (2-\lambda)(\lambda-3)(\lambda-1)$

Eigen values are 1, 2, 3

Spectral radius is 3.

(c) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.

Note Title 2023-04-14

9 ① $M = N(h) + c_1 h + c_3 h^3 + c_5 h^5$

② $M = N(h/3) + c_1(h/3) + c_3(h/3)^3 + c_5(h/3)^5$

③ $M = N\left(\frac{h}{9}\right) + c_1\left(\frac{h}{9}\right) + c_3\left(\frac{h}{9}\right)^3 + c_5\left(\frac{h}{9}\right)^5$

$\frac{3 \cdot ② - ①}{2}$: $M = \frac{3N\left(\frac{h}{3}\right) - N(h)}{2} + \tilde{c}_3 h^3 + \tilde{c}_5 h^5$

Let $N_1(h) = \frac{3N\left(\frac{h}{3}\right) - N(h)}{2}$

④ $M = N_1(h) + \tilde{c}_3 h^3 + \tilde{c}_5 h^5$

⑤ $M = N_1\left(\frac{h}{3}\right) + \tilde{c}_3\left(\frac{h}{3}\right)^3 + \tilde{c}_5\left(\frac{h}{3}\right)^5$

where $N_1\left(\frac{h}{3}\right) = \frac{3N\left(\frac{h}{9}\right) - N(h/3)}{2}$

$\frac{27 \cdot ⑤ - ④}{26} = M = \frac{27N_1\left(\frac{h}{3}\right) - N_1(h)}{26} + O(h^5).$

Best approximation is $\frac{27N_1\left(\frac{h}{3}\right) - N_1(h)}{26}$.