MATH 469/769 ASSIGNMENT #2

Assignment Due: Monday February 17, 2025, 5:30pm

The total number of marks possible marks for the assignment is 70. All students should attempt all questions. Make sure you show all your work, and make sure that your work is your own. Your reasoning and work is more important than your answer.

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1. The "friendship paradox" says that on average, your friends have more friends than you do, because a friend is reached by following an edge, and more edges are connected to nodes of higher degree. This question is about the *generalized friendship paradox*.

   Suppose we have a quantity x (e.g. wealth, virus count ..) on each node in a configuration model network. Denote the value of x on node i by x_i .

   (a) Choose a *random edge* on the network, and consider the node at the end of this edge. Show that the average value of x at that node (let's call this $\bar{x}_{\text{edge}}$) is $\frac{1}{2m}\sum_{i=1}^{n} k_i x_i$.

   (b) Show that $\bar{x}_{\text{edge}} - \bar{x} = \frac{\text{cov}(x,k)}{\bar{k}}$, where cov$(x, k)$ is the *covariance* between x and the degree k, and $\bar{k}$ is the average degree.[1]

   (c) Interpret your answer: do your friends tend to have higher values of x than you? When? When don't they?

   [10]

2. In class we played a game on a lattice, in which each player aimed to get their initial in the most boxes, and boxes were filled in a process like bond percolation.

   Here, let's look at the lattice as a very large graph (ignoring the game, initialling and box-filling), with degree distribution

$$p_k = \begin{cases} 1 \text{ if } k = 4, \\ 0 \text{ otherwise} \end{cases} .$$

   (a) If the lattice were a configuration model, find the critical occupation probability, ϕ_c. You can ignore the nodes at the top, sides about bottom of the lattice (where the degree is not 4).

   (b) If $\phi = 0.5$, find the size of the giant percolation cluster, as a fraction of the total lattice size. (You may need to use a numerical method. Any approximate solution is fine, including using a series of educated guesses, just explain what you did).

   (c) Is the lattice a configuration model? Why or why not? How do you expect this to matter in the percolation process?

   [10]

[1] cov$(X, Y) = E(X - E[X])(Y - E[Y])$ where E is the average.

3. Giant components (GC) in configuration models.

   (a) Consider a random node i in a configuration model network. Choose one of its edges, and follow that edge to node j. Let u be the probability that j is not in the GC. Use conditioning to get a consistency equation for u in terms of a probability generating function.

   (b) What is the trivial (boring) solution to your equation? When is there a non-trivial solution, and hence, a GC?

   (c) If i has k neighbours, what's the probability that *all* of them are *not connected* to the GC?

   (d) Condition on the degree again to get the probability of not being in the GC in terms of a probability generating function.

   (e) What's the expected size of the GC, as a fraction of the size of the network?

   (f) Compare this to what we obtained for the giant percolation cluster. What's the main difference? Argue that any network with a giant percolation cluster also has a giant component.

[15]

4. What about the small components? If a node is not in the giant component, we will say that it's in a "small component" (which could be of size 1). Assume a configuration model network with degree distribution p_k.

   (a) Use Bayes' theorem to relate (1) the probability that a node has degree k given that it is in a small component and (2) the probability that a node is in a small component given that is has degree k.

   (b) Find an expression, in terms of a probability generating function, for the average degree of nodes in small components.

   (c) Compare the average degree in small components to the average degree in the whole network, for a network with Poisson degree distribution with a mean of 3. (You will need to use a numerical method to find u. Any approximate solution is fine, including using a series of educated guesses, just explain your logic. You will also find it useful to know that $g_1(u) = g_0'(u)/E(k)$).

[15]

5. Degrees in the giant component. Consider a configuration model with degree distribution p_k.

   (a) Write the expression for the probability that a node of degree k IS in the giant component, in terms of u (the probability that it is not connected to the GC via a specific one of its edges).

   (b) Hence derive an expression for the probability that a node in the GC has degree k.

   (c) Show that the average degree of a node in the GC is $E(k)(1-u^2)/S$ where S is the expected size of the GC (as a fraction of the number of nodes).

[20]