MACM 316 Midterm: 12:30-13:20 Fri Feb 28, 2014

Answer all 6 questions. Closed book. One-sided cheat sheet and calculator permitted. Explain all steps: Do not expect marks to be granted for a correct answer if the intermediate steps are omitted.

1. (2 marks) Suppose that three-digit chopping arithmetic was used to compute the sum $\sum_{i=1}^{10}(1/i^2)$ first by $\frac{1}{1}+\frac{1}{4}+\cdots+\frac{1}{100}$ and then by $\frac{1}{100}+\frac{1}{81}+\cdots+\frac{1}{1}$. Does it matter which series is used? Explain your answer. Indicate which series is more accurate if the results differ.

2. (3 marks) Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h\to 0} \cos(h) + \frac{1}{2}\sin(h^2) = 1$$

   Show your steps.

3. (3 marks) Does the following sequence converge linearly or quadratically to $p = 0$? Use the definition of linear/quadratic convergence to justify your answer.

$$p_n = \frac{1}{n^2}. \quad n \geq 1$$

4. (4 marks) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

   (a) Compute $||A||_\infty$.

   (b) Compute $||A||_2$.

   (c) On $\mathbb{R}^n$ define the norm $||\mathbf{x}||_1 = \sum_{i=1}^n |x_i|$. Define the matrix norm $||\cdot||_1$ by

$$||A||_1 = \max_{||\mathbf{x}||_1=1} ||A\mathbf{x}||_1$$

   Using the definition, compute $||A||_1$.

5. (3 marks) Suppose that

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 8x_3 &= 5 \\ 6x_1 + \alpha x_2 + 10x_3 &= 5 \end{aligned}$$

   with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

   (a) $\alpha = 6$

   (b) $\alpha = 9$

   (c) $\alpha = -3$

6. (2 marks) Find a sharp bound on the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$. Assume that the bisection method is used and that the initial interval is $[1, 4]$.

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1. The second series is more accurate.

With 3 digit chopping small numbers should be added to other small numbers otherwise the sums will not be accurately ~~exactly~~ calculated.

2. $\cos(h) + \frac{1}{2}\sin(h^2) - 1$

$$= 1 - \frac{h^2}{2} + \frac{h^4}{4!} + \frac{1}{2}\left(h^2 - \frac{h^6}{3!}\right) - 1 + O(h^6)$$

$$= \frac{h^4}{4!} + O(h^6)$$

$$= O(h^4)$$

The rate of convergence is

$$O(h^4)$$

3 Consider

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)^2}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

By the definition of linear convergence p_n converges linearly

4a $\|A\|_\infty = \max\{|1|+|2|, |0|+|4|\}$
$= \max\{3, 4\}$
$= 4$

b $A^T A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 2 & 20 \end{bmatrix}$

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 20-\lambda \end{bmatrix}$

$\Rightarrow \lambda^2 - 21\lambda + 16 = 0$

Thus $\lambda = 0.79$ or $\lambda = 20.2082$

Thus the spectral radius is 20.2082

& $\|A\|_2 = \sqrt{20.2082}$
$= 4.4954$

c/ $$\max_{\|(x_1,x_2)^T\|_1=1} \left\| \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_1$$

$$= \max_{|x_1|+|x_2|=1} |x_1+2x_2| + |4x_2|$$

We observe x_1 & x_2 have the same sign to achieve a maximum, assume $x_1, x_2 \geq 0$

$$= \max_{\substack{x_1+x_2=1 \\ x_1, x_2 \geq 0}} x_1 + 6x_2$$

$$= \max_{0 \leq x_2 \leq 1} 1 + 5x_2$$

$$= 6.$$

5/ $2x_1 + x_2 + 3x_3 = 8$

Scales

$s_1 = 3$

$4x_1 + 6x_2 + 8x_3 = 5$ $s_2 = 8$

$6x_1 + \alpha x_2 + 10x_3 = 5$ $s_3 = 10$

$2/s_1 = 2/3, \quad 4/s_2 = 1/2, \quad 6/s_3 = 6/10$

So 2 is the largest scaled pivot. Use it as the pivot.

$$2x_1 + x_2 + 3x_3 = 8$$
$$4x_2 + 2x_3 = -11$$
$$(\alpha - 3)x_2 + x_3 = -19$$

a/ $\alpha = 6$: $4/s_2 = 1/2, \quad 3/s_3 = 3/10$

So 4 is the largest scaled pivot: <u>no pivoting</u>

b/ $\alpha = 9$: $4/s_2 = 1/2, \quad 6/s_3 = 6/10$

So $(\alpha - 3)$ is the largest scaled pivot: pivoting required.

c/ $\alpha = -3$: $4/s_2 = 1/2, \quad -6/s_3 = -6/10$

So $(\alpha - 3)$ is the largest pivot (in abs value) : pivoting required.

6/ From THM 2.1

$$|p_n - p| \le \frac{b-a}{2^n}$$

We want $\frac{b-a}{2^n} < 10^{-3}$

$$2^n > (4-1)10^3$$

$$2^n > 3 \times 10^3$$

$$\ln(2^n) > \ln(3 \times 10^3)$$

$$n > \ln(3 \times 10^3) / \ln(2)$$

$$= 11.55$$

So $n \ge 12$ itns.