SFU Macm 316

Midterm Test: March 4, 2024

Instructions: 50 minutes. Answer all 7 questions. Closed book.
One-sided cheat sheet and non-graphing calculator permitted.

Mark very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS.
Do not expect ANY marks for answers that do not provide intermediate work.
Marks may be deducted for poor presentation.

1. (3 marks) Find the rate of convergence of the following sequence as $h \to 0$:

$$\lim_{h \to 0} \left[\frac{1}{1-h} - 1 - \sin(h) \right] = 0$$

Show your steps.

2. (4 marks) The equation $f(x) = 1 - x^2 \sin x = 0$ has a solution in the interval $[-1, 2]$.
(a) Verify that the Bisection method can be applied to the function $f(x)$ on $[-1, 2]$.
(b) Using the error formula for the Bisection method find the number of iterations needed for accuracy 0.000001. Do not do the Bisection calculations.
(c) Compute the third approximation, p_3, for the Bisection method.

3. (3 marks) Let $||A||$ be any natural matrix norm of matrix A. Use the definition of natural matrix norm to show that

$$|\lambda| \leq ||A||$$

for any nonsingular matrix A and any eigenvalue λ of A.

4. (4 marks) Factor the following matrix into the LU decomposition using the LU factorization Algorithm with $l_{ii} = 1$ for all i.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 3 & 9 \\ -2 & 3 & 4 \end{bmatrix}$$

Multiple choice. Circle the correct statement, or if there is more than one correct statement choose the best answer. No explanation is required.

5. (1 mark) An $n \times n$ strictly diagonally dominant matrix A is tridiagonal. Taking advantage of the zero structure, solving $Ax = b$ requires how many multiplications?

   (a) $O(n)$

   (b) $O(n \log n)$

   (c) $O(n^2)$

   (d) $O(n^3)$

   (e) none of the above

6. (1 mark) Suppose we can factor the matrix A into $A = LL^T$ *via Choleski's method*. What do we know about the matrix A?

   (a) The matrix A is strictly diagonally dominant.

   (b) The matrix A is symmetric.

   (c) The matrix A is tridiagonal.

   (d) All of the above.

   (e) None of the above.

7. (1 mark) Suppose P is a permutation matrix. Then

   (a) $\det(P) = 1$.

   (b) $P = P^T$.

   (c) P is the identity matrix with two rows interchanged.

   (d) All of the above.

   (e) None of the above.

Midterm Solns : MACM 316

Note Title 2024-02-16

Spring 2024

Q1

$$\frac{1}{1-h} - 1 - \sin h$$

$$= 1 + (-1)\frac{1}{(1-0)^2}(-1)h + (-2)\frac{1}{(1-0)^3}(-1)h^2$$

$$- 1 - h + h^3/6 + O(h^3)$$

$$= 1 + h + h^2 - 1 - h + O(h^3)$$

$$= O(h^2)$$

Q2(a) $f(-1) = 1-(-1)^2 \sin(-1) > 0$

$f(2) = 1-2^2 \sin(+2) \approx -2.6 < 0$

(b) $\frac{(b-a)}{2^n} \leq 10^{-6}$

$2^n \geq 10^6 \cdot 3$

$n \geq \log_2 \left(10^6 \cdot 3\right) = 21.5$

Choose $n \geq 22$

(c) $p_1 = \frac{-1+2}{2} = 1/2$

$f(1/2) = .8801 > 0$

$a_2 = 1/2$, $b_2 = 2$

$p_2 = 5/4$

$f(5/4) = -.48 < 0$

$a_3 = 1/2$, $b_3 = 5/4$

$p_3 = 7/8$

⋮

3 $$\|A\| = \max_{\|x\|=1} \|Ax\|$$

$$\geq \|Ax\|$$

where x is an eigenvector s.t. $\|x\| = 1$.

$$= \|\lambda x\| = |\lambda| \|x\|$$

where x is an eigenvalue s.t. $\|x\|=1$

$$= |\lambda|$$

$\blacksquare$

4

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & 3 & 9 \\ -2 & 3 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & 6 \\ 0 & 1 & 6 \end{bmatrix} \qquad \begin{matrix} R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & 6 \\ 0 & 0 & 5 \end{bmatrix} \qquad R_3 - \tfrac{1}{6}R_2 \rightarrow R_3$$

$$= U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1/6 & 1 \end{bmatrix}$$

5 (a) Use Crout's method

6 (b) The matrix is symmetric.

7 (e) We may form counterexamples.

(a) : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) & (c) : $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix}$