SFU Macm 316

Final Exam: April 26, 2022

Instructions: 3 Hours. Closed book. Two sided cheat sheet permitted. Answer all 10 questions.

EXPLAIN ALL ANSWERS. Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation. Mark very clearly on the question if you use the back of any page.

1. (4 marks) Use a theorem from the course to show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

2. (4 marks) A natural cubic spline S is defined by

$$S = \begin{cases} S_0(x) &= 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \leq x < 2, \\ S_1(x) &= 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3 & \text{if } 2 \leq x \leq 3 \end{cases}$$

If S interpolates the data (1, 1), (2, 1), and (3, 0), find B, D, b and d.

3. (4 marks) Are the following true or false? Provide justification.

   (a) Suppose Jacobi's method is used to solve $Ax = b$ and the matrix A is diagonal and invertible. Then Jacobi's method will converge in one iteration from *any* initial guess.

   (b) Choleski's method may be applied to solve $Ax = b$ for any diagonal, invertible matrix A.

   (c) The number 3 approximates π to one significant digits.

   (d) Suppose the data $\{(x_i, f(x_i))\}_i^n$ lie on a straight line. Then we know the free cubic spline for the function f is a straight line.

4. (4 marks) Find the constants x_0, x_1 and c_0 so that the quadrature formula

$$\int_{-1}^{0} f(x)dx = c_0 f(x_0) + \frac{1}{2} f(x_1)$$

has the highest possible degree of precision. What is the degree of precision?

5. (4 marks) Let $P(x)$ be the Lagrange interpolating polynomial for the data (0, 0), (0.5, y), (1, 3) and (2, 2). The coefficient of x^3 in $P(x)$ is 6. Find y.

6. (4 marks) Initial Value Problems.

   (a) Show that the following initial value problem is well-posed

$$y' = e^t \sin(y), \ \ 0 \le t \le 2, \ \ y(0) = -1.$$

   (b) Approximate $y(1)$ using one of the methods that was covered in this course. Use a step size of $h = 1$.

7. (4 marks) Consider the sequence

$$p_n = 3^{-(c^n)}$$

where c is a positive real number. Clearly the sequence converges to zero for all $c > 1$ as n tends to infinity.

(a) Suppose $c = 2$. Does the sequence converge linearly ($\alpha = 1$), quadratically ($\alpha = 2$), cubically ($\alpha = 3$), or with some other order of convergence α? Use the definition of order of convergence to justify your answer.

(b) Does there exist a c for which the sequence converges to zero linearly? Explain.

8. (4 marks) Derive a three-point formula to approximate $f''(x_0)$ that uses $f(x_0), f(x_0+h)$ and $f(x_0+3h)$. For full marks, simplify your answer in terms of h.

   The error takes the form $O(h^p)$. What is p? (for the error, you may just give an answer without derivation).

9. (4 marks) Suppose $N(h)$ is an approximation to M for every $h > 0$ and that

$$M = N(h) + K_2 h^2 + K_4 h^4 + \cdots,$$

for some constants $K_2, K_4 \ldots$. Use the values $N(h)$ and $N(h/3)$ to produce an $O(h^4)$ approximation to M.

10. (4 marks) Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h \to 0} \cos(h^2) = 1.$$

Show your steps.

Note Title 2022-04-13

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Q1 See midterm solns.

Q2

$$S_0(2) = 1 = S_1(2) \Rightarrow 1 + B - D = 1$$

$$S_0'(2) = S_1'(2) \Rightarrow B - 3D = b$$

$$S_0''(2) = S_1''(2) \Rightarrow -6D = -\tfrac{3}{2}$$

$$\Rightarrow D = 1/4 \Rightarrow B = \frac{1}{4} \Rightarrow b = -\frac{1}{2}$$

Natural cubic spline $\Rightarrow -\frac{3}{2} + 6d = 0$

$\Rightarrow d = + 1/4$

Q3 a/ $x^{(n+1)} = D^{-1}(\underbrace{L+U}_{=0})x^{(n)} + D^{-1}b = D^{-1}b$

TRUE

b/ Cholesky: fails if some $a_{ii} < 0$.

FALSE.

c/ $\frac{|3-3.14159\cdots|}{3} < .05$

$\Rightarrow$ 2 significant digits

FALSE.

d/ TRUE. We have

$S''(a) = 0 = S''(b)$

$\leftarrow$ S is C^2.

4 $\int_{-1}^{0} 1\,dx = 1 = c_0 + \frac{1}{2} \Rightarrow c_0 = \frac{1}{2}$

$\int_{-1}^{0} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{0} = -\frac{1}{2} = c_0 x_0 + \frac{1}{2} x_1$

$\Rightarrow -1 = x_0 + x_1$

$\int_{-1}^{0} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{0} = \frac{1}{3} = c_0 x_0^2 + \frac{1}{2} x_1^2$

$\frac{1}{3} = \frac{1}{2} x_0^2 + \frac{1}{2} x_1^2$

$x_1 = -x_0 - 1$

$\frac{1}{3} = \frac{1}{2}(x_0)^2 + \frac{1}{2}(x_0+1)^2$

$\frac{1}{3} = x_0^2 + x_0 + 1/2$

$= x_0^2 + x_0 + 1/6$

$x_0 = \dfrac{-1 \pm \sqrt{1 - 2/3}}{2}$

$= \dfrac{-1 \pm \sqrt{1/3}}{2}$

$x_1 = \dfrac{-1 \mp \sqrt{1/3}}{2}$

Degree of precision is 3. To verify we can check formula is exact for x^3 but not so for x^4.

$(x_0, x_1) = \left(\dfrac{-1+\sqrt{1/3}}{2}, \dfrac{-1-\sqrt{1/3}}{2}\right)$

or $(x_0, x_1) = \left(\dfrac{-1-\sqrt{1/3}}{2}, \dfrac{-1+\sqrt{1/3}}{2}\right)$

same result.

5 We have

$f[x_0, x_1, x_2, x_3] = 6$

x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	
0	0			
$1/2$	y	$2y$		
1	3	$2(3-y)$	$-4y+6$	
2	2	-1	$\frac{-7+2y}{3/2}$	$\frac{8}{3}y - \frac{16}{3}$

$\frac{y-0}{1/2}$

$\frac{3-y}{1-1/2}$ $\quad \frac{2-3}{2-1}$

$\frac{2(3-y)-2y}{1-0}$

$\frac{-1-6+2y}{2-1/2}$

$$\frac{\frac{-7+2y}{3/2} - \frac{-4y+6}{1}}{2}$$

$= -\frac{7}{3} + \frac{2y}{3} - (-2y+3)$

$= \frac{8}{3}y - \frac{16}{3}$

We have

$\frac{8}{3}y - \frac{16}{3} = 6$

$4y - 8 = 9$

$y = \frac{17}{4}$

6(a) $f(t,y) = e^t \sin y$ is CTS (product of CTS functions)

$$\left|\frac{\partial f}{\partial y}\right| = \left|e^t \cos y\right| < e^2$$

$\Rightarrow$ satisfies a Lipschitz condition.

$\Rightarrow$ Well-posed

(b) Use Euler's method:

$$y_1 = y_0 + h f(t_0, y_0)$$
$$= -1 + 1 e^0 \sin(-1)$$
$$= -1 - \sin 1.$$

7 $\lim_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|^{\alpha}}$

$= \lim_{n\to\infty} \frac{3^{-2^{n+1}}}{(3^{-2^n})^{\alpha}}$

$= \lim_{n\to\infty} \frac{3^{-2^{n+1}}}{3^{-2^{n+1}}} = 1$ with $\alpha = 2$

Quadratic.

(b) No. Convergence to zero iff $c > 1$.

$\alpha = c$ for all $c > 1$.

8 Find quadratic approximating f:

$$P(x) = L_0 f_0 + L_1 f_1 + L_2 f_2$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$P''(x_0) = \frac{2}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{2}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{2}{(x_2-x_0)(x_2-x_1)} f_2$$

$$= \frac{2}{(-h)(-3h)} f_0 + \frac{2}{h(-2h)} f_1 + \frac{2}{3h(2h)} f_2$$

Error is $O(h)$.

9

$$M = N(h) + K_2 h^2 + K_4 h^4 + \cdots$$

$$M = N\left(\frac{h}{3}\right) + K_2 \frac{h^2}{9} + K_4 \frac{h^4}{81} + \cdots$$

$$9M - M = 9N\left(\frac{h}{3}\right) - N(h) + O(h^4).$$

$$M = \frac{9N\left(\frac{h}{3}\right) - N(h)}{8} + O(h^4)$$

10 $\cos h^2 - 1$

$$= 1 - \frac{(h^2)^2}{2} + \frac{(h^2)^4}{4!} + O(h^{12}) - 1$$

$$= O(h^4).$$