MACM 316 Midterm: 12:30-13:20 Mon Feb 20, 2012

Answer all 9 questions. Closed book. One-sided cheat sheet and calculator permitted. Unless otherwise specified, explain all steps: Do not expect marks to be granted for a correct answer if the intermediate steps are omitted.

1. (2 marks) Suppose 3 digit rounding arithmetic is used. What is the result of the following pseudocode?

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set SUM = 0;
for k=1...100
     set SUM = SUM + (1/k)^3;
end;
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2. (3 marks) Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h\to 0} e^h - \sin(h) = 1$$

Show your steps.

3. (3 marks) Factor the following matrix into the LU decomposition using the LU factorization Algorithm with $l_{ii} = 1$ for all i:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -4 & 7 \\ -1 & 2 & 0 \end{bmatrix}$$

4. (2 marks) Suppose

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & 3 & 7 \\ -1 & -1 & 1 & 5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1's on its diagonal and U is upper triangular.
Do not find L or U.

5. (2 marks) Express the Jacobi iteration method for a linear system $Ax = b$ in the form $x^{(k)} = Tx^{(k-1)} + c$. Without using any theorems on the convergence of Jacobi's method, show that the spectral radius of the Jacobi iteration matrix T is less than 1 for any strictly diagonally dominant matrix, A.

You may use the following fact: for an $n \times n$ matrix M we have $\rho(M) \leq ||M||$ for any natural norm $||\cdot||$.

Multiple choice. Choose the correct statement, or if there is more than one correct statement choose the best answer. No explanation is required.

6. (1 mark) If Gaussian elimination with scaled partial pivoting is applied to the following augmented system,

$$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 4 \\ 2 & 0 & 2 & 1 & 8 \\ 7 & 0 & 0 & 14 & 3 \end{array}\right]$$

what is the first pivot value?

(a) 3

(b) 1

(c) 2

(d) 7

7. (1 mark) Suppose P is a permutation matrix. Then

(a) $P = P^t$

(b) $P^{-1} = P^t$

(c) P can be factored into LU where L is lower triangular and U is upper triangular

(d) none of the above

8. (1 mark) Choleski's method may be applied to

(a) any symmetric, strictly diagonally dominant matrix

(b) any symmetric, positive definite matrix

(c) any symmetric triangular matrix

(d) none of the above

9. (1 mark) Suppose we wish to solve $Lx = b$ where L is the lower triangular matrix given by

$$L = \begin{bmatrix} 1 & & & \\ 1 & 2 & & \\ & \ddots & \ddots & \\ & & 1 & 99 \end{bmatrix}$$

Suppose Gauss-Seidel was applied to solve the system. Then Gauss-Seidel

(a) converges for some values of b, and diverges for other values.

(b) converges for some initial approximations, $x^{(0)}$, and diverges for other values.

(c) converges for any b and any initial approximation, but the convergence is slow.

(d) converges for any b and any initial approximation, and the convergence is fast.

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1. $k=1$ $\quad SUM = 1$

$k=2$ $\quad SUM = fl(1 + fl(0.5^3))$

$= fl(1 + 0.125)$

$= 1.13$

$k=3$ $\quad SUM = fl(1.13 + fl(fl(\frac{1}{3})^3))$

$= fl(1.13 + fl(.333^3))$

$= fl(1.13 + 0.0369)$

$= 1.17$

$k=4$ $\quad SUM = fl(1.17 + fl(fl(\frac{1}{4})^3))$

$= fl(1.17 + fl(.25^3))$

$= fl(1.17 + 0.0156)$

$= 1.19$

$k=5$ $\quad SUM = fl(1.19 + fl(.2^3))$

$= fl(1.19 + .00800)$

$= 1.20$

$k=6$ and higher have no effect on SUM.

Thus sum $= 1.20$.

2) $e^h - \sin h - 1$

$$= 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + O(h^4)$$

$$- \left(h - \frac{h^3}{3!} + O(h^5)\right)$$

$$- 1$$

$$= h^2/2 + O(h^3)$$

$$= O(h^2).$$

3/

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -4 & 7 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3-(-1)R_1 \to R_3]{R_2-(2)R_1 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 7 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-(-\frac{1}{2})R_2 \to R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 7 \\ 0 & 0 & 7/2 \end{bmatrix} \equiv U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}$$

4/

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & 3 & 7 \\ -1 & -1 & 1 & 5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

$$\xrightarrow[R_4 - 2R_1 \to R_4]{\substack{R_2 - 2R_1 \to R_2 \\ R_3 + R_1 \to R_3}} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 1 & 6 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

5/ $T = D^{-1}(L+U)$

$$= \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\|T\|_\infty = \max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$$

$$= \max_i \left| \frac{1}{a_{ii}} \right| \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

But $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

(strict diag dom)

Thus $\|T\|_\infty < 1$

Using $\rho(T) \leq \|T\|$

we have $\rho(T) < 1$.

6) scaled value for various rows

row 1: 3/4

row 2: 1/2

row 3: 1 $\leftarrow$ ANS = (c)

row 4: 1/2

7) (b)

8) (a) is false. ~~[illegible]~~ consider $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) is true (THM).

(c) is false. consider $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) is false

9) $x^{(k)} = (D-L)^{-1} U x^{(k-1)} + (D-L)^{-1} b.$

with $U = 0 \Rightarrow$ converges in one step.

ANS = (d).