SFU Macm 316

Final Exam: April 21, 2010

Name: ______________________ ID: ______________________

Instructions: Answer all questions. Closed book. Two sided cheat sheet permitted. Mark very clearly on the question if you use the back of any page. EXPLAIN ALL ANSWERS. Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation.

Time: 3 Hours.

1	/3
2	/3
3	/5
4	/5
5	/4
6	/4
7	/4
8	/5
9	/3
10	/3
11	/4
12	/4
Total	/47

1. (3 marks) Find the rate of convergence of the following function as $h \to 0$:

$$\lim_{h\to 0} \frac{1 - e^h}{h} = -1$$

Show your steps.

2. (3 marks) Recall that iterative methods for solving the linear system

$$Ax = b$$

take the form

$$x^{(n)} = Tx^{(n-1)} + c$$

for some initial guess $x^{(0)}$.

What are T and c for the Jacobi method when

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -4 & 8 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

3. (5 marks) Are the following true or false? GIVE YOUR REASONING.

   (a) When applied to a system with N variables, Gaussian elimination with scaled partial pivoting requires $O(N^2)$ additional floating point operations over Gaussian elimination with partial pivoting.

   (b) Choleski's method can solve $Ax = b$ for any diagonal, invertible matrix A.

   (c) Rounding the number -0.145 to two decimal places gives -0.14.

   (d) Suppose the data $\{(x_i, f(x_i))\}_i^n$ lie on a straight line. Then we know the Lagrange interpolating polynomial for the function f is a straight line.

   (e) The l_∞ norm and the l_2 norm of a diagonal matrix are equal.

4. (5 marks) Are the following true or false? GIVE YOUR REASONING.

   (a) The bisection method can be accelerated using Aitken's method.

   (b) The function e^{-x} has a unique fixed point on $[1/3, 1]$.

   (c) As h tends to zero, composite trapezoid rule becomes inaccurate due to cancellation error.

   (d) As h tends to zero, Romberg integration becomes inaccurate due to cancellation error.

   (e) The following sequence converges linearly to $p = 0$: $p_n = \frac{1}{n^3}, n \geq 1$.

5. (4 marks) In this question, select one of the following as your answer: $O(1)$, $O(\log(n))$, $O(\sqrt{n})$, $O(n)$, $O(n\log(n))$, $O(n^2)$, $O(n^3)$, $O(n^4)$, $O(2^n)$ or $O(n!)$ operations. Explain your reasoning.

Assume we choose the optimal algorithm.

(a) Suppose we solve $A_1x = b$ for x. The matrix A_1 is an $n \times n$ invertible matrix, and none of the entries of A_1 are zero. How many multiplications and divisions are required?

(b) Suppose we compute $\det(A_2)$. The matrix A_2 is an $n \times n$ invertible matrix, and none of the entries of A_2 are zero. How many multiplications and divisions are required?

(c) Suppose we solve $A_3x = b$ for x. The matrix A_3 is a tridiagonal $n \times n$ matrix. It is also strictly diagonally dominant. How many additions and subtractions are required?

(d) Suppose we solve $A_4x = b$ for x. The matrix A_4 is an $n \times n$ invertible matrix. It is also lower triangular, but none of the entries below the main diagonal of A_4 are zero. How many additions and subtractions are required?

6. (4 marks) Suppose we seek a solution to

$$x^2 - 2xe^{-x} + e^{-2x} = 0.$$

(a) Approximate the root by carrying out 2 iterations of Newton's method with an initial guess of 0.5.

(b) In part (a), does Newton's method give linear or quadratic convergence? Justify your answer.

(c) Give the iteration formula for an iterative method that will give improved convergence over Newton's method.

7. (4 marks) Use the Hermite polynomial that agrees with the data listed below to find an approximation to $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959

8. (5 marks) Initial Value Problems.

   (a) Show that the following initial value problem has a unique solution

   $$y' = \sin(y+t), 0 \leq t \leq 2, y(0) = -1$$

   (b) Approximate $y(2)$ using Euler's method with a step size of $h = 1$.

   (c) Euler's method with a step size of 0.1 gives -2.3799 as an approximation of $y(2)$. Euler's method with a step size of 0.05 gives -2.3589 as an approximation of $y(2)$. Use Richardson extrapolation to obtain a more accurate approximation of $y(2)$.

9. (3 marks) Let $f(x) = \cos(\pi x)$. Give a numerical differentiation formula based on the values of $f(x)$ at $x = 0.25, 0.5$, and 0.75 which approximates $f''(0.5)$. Evaluate the formula using your calculator.

Compare the result to the exact value. What error dominates your solution, roundoff error or truncation error? Explain.

10. (3 marks) A clamped cubic spline S for a function f is defined by

$$S = \left\{ \begin{array}{lcll} S_0(x) & = & 1 + Bx + 2x^2 - 2x^3, & \text{if } 0 \leq x < 1, \\ S_1(x) & = & 2 + b(x-1) - 4(x-1)^2 + 7(x-1)^3 & \text{if } 1 \leq x \leq 2 \end{array} \right.$$

Find $f'(0)$ and $f'(2)$.

11. (4 marks) Find the constants x_0, x_1 and c_1 so that the quadrature formula

$$\int_0^1 f(x)dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

12. (4 marks) Prove the following, without using the derivative of g in the proof.

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition, that g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$. Then for any number p_0 in $[a, b]$ the sequence defined by

$$p_n = g(p_{n-1}), n \geq 1$$

converges to the unique fixed point p in $[a, b]$.

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