1. (3 marks) Find the rate of convergence of the following sequence as $n \to \infty$:

$$\lim_{n\to\infty} n^2 \sin^2(1/n) = 1$$

Show your steps.

Let $h = 1/n$.

$$\frac{1}{h^2}\sin^2 h - 1$$

$$= \frac{1}{h^2}\left(h - \frac{h^3}{6}\right)^2 - 1 + O(h^4)$$

$$= \left(1 - \frac{h^2}{6}\right)^2 - 1 + O(h^4)$$

$$= -\frac{h^2}{3} + O(h^4)$$

$$= O(h^2)$$

2. (4 marks) State whether the following are TRUE or FALSE. For any FALSE statement, explain why the statement is FALSE. No explanation is required for TRUE statements.

(a) You can round a real number x to the nearest integer by taking the integer part of $(x + 0.5)$.

(b) The function x^2 has precisely one fixed point over the set of real numbers.

(c) The matrix A given below is symmetric and positive definite

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 6 & 1/2 \\ 3 & 1/2 & -1 \end{bmatrix}$$

(d) If Gaussian elimination with scaled partial pivoting is applied to the following augmented system,

$$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 4 \\ 2 & 0 & 2 & 1 & 8 \\ 7 & 0 & 0 & 14 & 3 \end{array}\right]$$

the first pivot value is 2.

a/ Apply rounding to mantissa. So $fl(-1.5) = -2$ $\boxed{\text{FALSE.}}$

b/ fixed points given by $p = p^2 \Rightarrow p = 0$ or 1 $\boxed{\text{FALSE}}$

c/ All diagonal entries must be positive for a positive def matrix. $\boxed{\text{FALSE}}$

d/ Scale 1: 4, Scale 2: 2, Scale 3: 2
Scale 4: 14

$2/2 > 3/4$, $2/2 > 1/2$, $2/2 > 7/14$ $\boxed{\text{TRUE}}$

3. (6 marks) Some questions on direct methods for solving linear systems follow. Suppose

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 3 & 9 \\ 2 & 3 & 5 \end{bmatrix}$$

(a) Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1's on its diagonal and U is upper triangular. Do not find L or U.

(b) We wish to solve $Ax = b$. Express x as a product of the matrices $L, L^{-1}, U, U^{-1}, P, P^{-1}$ and the vector b. Do not evaluate any of the matrix inverses or matrix products.

(c) Could we solve $Ax = b$ using Choleski factorization? Briefly explain why/why not.

(d) Factor B into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

a/ $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\ldots \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b/ $PAx = Pb \Rightarrow LUx = Pb$

$x = U^{-1} L^{-1} P b$

c/ No. Not positive definite.

d/ $\begin{bmatrix} 1 & -1 & 1 \\ 3 & 3 & 9 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & 6 \\ 0 & 5 & 3 \end{bmatrix}$

$\xrightarrow{R_3 - \frac{5}{6} R_2 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & 6 \\ 0 & 0 & -2 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{6} & 1 \end{bmatrix}$

4. (6 marks) Suppose

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & -7 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 0 \\ -7 \end{bmatrix}.$$

(a) Compute the spectral radius of the matrix A.

(b) Suppose that we wish to solve $Ax = b$. Find the first two iterations of Gauss Seidel's method starting with the zero vector as $x^{(0)}$.

(c) Starting from the zero vector, will Gauss-Seidel converge for this problem? Explain.

(d) Does Gauss-Seidel converge faster, slower or the same as Jacobi's method for this problem? Explain.

a/ e-values are 3, 6, −7. Largest in absolute value is −7. Thus $\rho(A) = 7$.

b/ $T_{GS} = (D-L)^{-1}U = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/7 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & -1/3 \\ 0 & 0 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$

$C = (D-L)^{-1}b = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/7 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$x^{(1)} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $x^{(2)} = T_{GS}x^{(1)} + C = \begin{bmatrix} -1/3 \\ -1/3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 1 \end{bmatrix}$

c/ A is diagonally dominant, so GS converges for any $x^{(0)}$.

d/ Because $L=0$, both methods are the same. They converge at the same rate & give the same answers.

5. (3 marks) Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. Prove that the bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}$$

when $n \geq 1$.

For each $n \geq 1$ we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \& \quad p \in (a_n, b_n).$$

Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \geq 1$, it follows that

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$$

$\blacksquare$