SFU Macm 316

Midterm Test: March 3, 2023

Instructions: 50 minutes. Answer all 5 questions. Closed book.
One-sided cheat sheet and non-graphing calculator permitted.

Mark very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS.
Do not expect ANY marks for answers that do not provide intermediate work.
Marks may be deducted for poor presentation.

1. (3 marks) Find the rate of convergence of the following sequence as $h \to 0$:

$$\lim_{h \to 0} \left[h\cos(h^2) - \sin(h) \right] = 0$$

Show your steps.

2. (4 marks) We saw in class a theorem which gives conditions for a fixed-point sequence to converge to a unique fixed point.

   Suppose $g(x) = \frac{5}{x^2} + 2$. Use the theorem to show that the fixed-point method $p_n = g(p_{n-1})$ will converge to the unique fixed point of g for any p_0 in $[2.5, 3]$. Make sure to verify all hypotheses of the theorem.

3. (4 marks) Let $PA = LU$ where

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}, \text{ and } P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Solve the linear system $Ax = b$ where $b = [0\ {-2}]^T$ using the given factorization.

(b) What is the determinant of A in terms of $\det(L)$ and $\det(U)$?

4. (4 marks) Show that if A is strictly diagonally dominant, then $||T_j||_\infty < 1$, where T_j is the iteration matrix for Jacobi's method. (This tells us that Jacobi's method converges for any diagonally dominant matrix.)

5. (3 marks) Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the spectral radius of matrix A.

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1 $$h\cos(h^2) - \sin(h)$$

$$= h\left(1 - \frac{(h^2)^2}{2} + O(h^8)\right) - \left(h - h^3/6 + O(h^5)\right)$$

$$= \frac{h^3}{6} + O(h^5)$$

$$= O(h^3)$$

2 g is continuous away from zero, so $g \in C[2.5, 3]$.

g is a decreasing function over the interval. So

$$\max_{x \in [2.5,3]} g(x) = \frac{5}{(5/2)^2} + 2$$

$$= \frac{4}{5} + 2 < 3$$

$$\min_{x \in [2.5,3]} g(x) = \frac{5}{3^2} + 2 > 2.5$$

Thus $g(x) \in [2.5, 3]$.

We compute

$$g'(x) = -\frac{10}{x^3}$$

$$|g'(x)| \le \max_{x \in [2.5,3]} |g'(x)|$$

$$= \frac{10}{(5/2)^3} = \frac{16}{25} < 1$$

Thus the sequence converges to the unique fixed pt.

3a We have $PAx = Pb$

$LUx = Pb$

Let $y = Ux$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$\Longrightarrow \quad y_1 = -2$

$2y_1 + y_2 = 0 \Rightarrow y_2 = 4$

$$\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$\Longrightarrow 2x_2 = 4 \Rightarrow \boxed{x_2 = 2}$

$4x_1 + x_2 = -2 \Rightarrow \boxed{x_1 = -1}$

b $PA = LU$

$A = P^{-1}LU$

$\det A = \det(P^{-1})\det(L)\det(U)$

$= -\det(L)\det(U).$

4

$$A = D - L - U$$

$$T_j = D^{-1}(L+U)$$

$$= \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots \\ & & \ddots & \end{pmatrix}$$

$$\|T_j\|_\infty = \max_j \sum_{\substack{i=1 \\ i \neq j}}^{n} \left| \frac{a_{ji}}{a_{jj}} \right|$$

But $\sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ji}| < |a_{jj}|$ for all j

Thus $\frac{1}{|a_{jj}|} \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ji}| < 1$ for all j

$\Leftrightarrow \|T_j\|_\infty < 1.$

5

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)(2-\lambda) - 2(2-\lambda)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2 - 2)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda)$$

$$= (2-\lambda)(\lambda)(\lambda - 3)$$

Thus the spectral radius is $\max(|2|, |0|, |3|)$

$= 3$