SFU MACM 316

Final Exam: Aug 13, 2018

Last Name: ______________________________

First Name: ______________________________

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Instructions: 3 hours. Answer all 10 questions. Closed book.
Two-sided cheat sheet and non-graphing calculator permitted.

Mark very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS.
Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation.

1. (4 marks) Find the rate of convergence of the following sequence as $h \to 0$:

$$\lim_{h\to 0} \left(\exp(-h^2/2) - \cos(h)\right) = 0$$

Show your steps.

$$1 - \frac{h^2}{2} + \frac{1}{2}\left(\frac{h^2}{2}\right)^2 + O(h^6)$$

$$-\left(1 - \frac{h^2}{2} + \frac{h^4}{4!} + O(h^6)\right)$$

$$= \frac{1}{12} h^4 + O(h^6)$$

$$= O(h^4)$$

2. (4 marks) Suppose A is an $n \times n$ matrix. Use the definition of matrix norm to show that $||\cdot||_*$, defined by

$$||A||_* = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$$

is a matrix norm.

i/ $\|A\|_* \geq 0$: $\|A_*\| = \sum\sum \underbrace{|a_{ij}|}_{\geq 0} \geq 0$

ii/ $\|A\|_* = 0 \Leftrightarrow A = 0$

$\|A\|_* = 0 = \sum\sum |a_{ij}| \Rightarrow a_{ij} = 0 \Rightarrow A = 0$

$A = 0 \Rightarrow a_{ij} = 0 \ \forall i,j. \Rightarrow \sum\sum |a_{ij}| = 0 = \|A\|_*$

iii/ $\|\alpha A\|_* = |\alpha| \|A\|_*$: $\|\alpha A\|_* = \sum\sum|\alpha a_{ij}| = \sum\sum|\alpha||a_{ij}|$

$= |\alpha| \sum\sum |a_{ij}|$

$= |\alpha| \, \|A\|_*$ ■

iv/ $\|A+B\|_* \leq \|A\|_* + \|B\|_*$:

$\|A+B\|_* = \sum\sum|a_{ij}+b_{ij}| \leq \sum\sum |a_{ij}|+|b_{ij}| = \|A\|_* + \|B\|_*$

v/ $\|AB\|_* = \sum\sum (a_{i1}\ a_{i2} \cdots a_{in}) \cdot (b_{1j}\ b_{2j} \cdots b_{nj})$

$\leq \sum\sum \|(a_{i1}\ a_{i2} \cdots a_{in})\|_2 \|(b_{1j}\ b_{2j} \cdots b_{nj})\|_2$ by Cauchy Schwartz

$\leq \sum\sum \|(a_{i1}\ a_{i2} \cdots a_{in})\|_1 \|(b_{1j}\ b_{2j} \cdots b_{nj})\|_1$

$= \sum_i \|(a_{i1}\ a_{i2} \cdots a_{in})\|_1 \sum_j \|(b_{1j}\ b_{2j} \cdots b_{nj})\|_1$

$= \|A\|_* \, \|B\|_*$

3. (4 marks) A clamped cubic spline S for a function f is defined on $[1, 3]$ by

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{on } [1, 2) \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{on } [2, 3] \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c, and d.

$S_0(2) = S_1(2)$

$3 + 2 - 1 = 4 = a$

$S_0'(2) = S_1'(2)$

$3 + 4 - 3 = 4 = b$

$S_0''(2) = S_1''(2)$

$4 - 6 = 2c$

$c = -1$

$S_0'(1) = S_1'(3)$

$3 = b + 2c + 3d$

$d = \dfrac{3 - 4 + 2}{3} = 1/3$

4. (4 marks) Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = a \cdot f(-1) + b \cdot f(0) + c \cdot f(1) + d \cdot f'(-1)$$

that has the highest possible degree of precision. What is the degree of precision?

$f=1$: $2 = a+b+c$

$f=x$: $\frac{1}{2}x^2\Big|_{-1}^{1} = 0 = -d+c+d$

$f=x^2$: $\frac{1}{3}x^3\Big|_{-1}^{1} = 2/3 = a+c-2d$

$f=x^3$: $\frac{1}{4}x^4\Big|_{-1}^{1} = 0 = -a+c+3d$

$$\boxed{\begin{array}{l} f=x^4 \\ LHS = \int_{-1}^{1} x^4 = 2/5 \\ RHS = a+c = 2/3 \\ LHS \neq RHS \\ \Rightarrow \text{degree of precision } 3 \end{array}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ a \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 2/3 \\ 0 & 0 & 2 & 1 & 2/3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 2/3 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$\Rightarrow d=0, \; c=1/3, \; a=1/3, \; b=4/3$

5. (4 marks) Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that

$$M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

for some constants $K_1, K_2, K_3, \ldots$. Use the values $N(h), N(h/3)$, and $N(h/9)$ to produce an $O(h^6)$ approximation to M.

$$M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

$$M = N(h/3) + K_1 h^2/9 + K_2 h^4/81 + K_3 h^6/3^6 + \cdots$$

$$M = N(h/9) + K_1 h^2/81 + K_2 h^4/9^4 + K_3 h^6/9^6 + \cdots$$

$$M = \frac{9N(h/3) - N(h)}{8} - \frac{K_2}{9} h^4 + O(h^6)$$

$$M = \frac{9N(h/9) - N(h/3)}{8} - \frac{K_2}{9}\frac{h^4}{3^4} + O(h^6)$$

Define $N_2(h) = \dfrac{9N(h/3) - N(h)}{8}$

Now $M = N_2(h) - 1/9\, K_2 h^4 + O(h^6)$

$3^4 M = 3^4 N_2(h/3) - \frac{1}{9} K_2 h^4 + O(h^6)$

$$\Rightarrow M = \underbrace{\frac{3^4 N_2(h/3) - N_2(h)}{3^4 - 1}}_{\text{Our } O(h^4) \text{ approximation}} + O(h^6)$$

6. (4 marks) Derive Euler's method. Compute the local truncation error of Euler's method.

RECALL:
The difference method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, 2, \ldots, N-1 \end{aligned}$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N-1$, where y_i and y_{i+1} denote the solution of the differential equation at t_i and t_{i+1}, respectively.

Let $t_i = ih$, $w_i \approx y(t_i)$.

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i)$$

$\xi_i \in (t_i, t_{i+1})$

Drop error term & replace $y(t_i)$ by w_i, etc:

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$\begin{aligned} LTE &= \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) \\ &= \frac{y_i + y'(t_i)h + \frac{1}{2}y''(\xi_i)h^2 - y_i}{h} - f(t_i, y_i) \\ &= \frac{1}{2}y''(\xi_i)h = O(h) \end{aligned}$$

7. (4 marks) Suppose

$$A = \begin{bmatrix} 7 & -2 & 1 \\ 14 & -7 & -3 \\ -7 & 11 & 18 \end{bmatrix}$$

Factor A into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

$$A \xrightarrow[R_3-(-1R_1)\to R_3]{R_2-(2R_1)\to R_2} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 9 & 19 \end{bmatrix}$$

$$\xrightarrow{R_3-(-3R_2)\to R_3} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

8. (4 marks) Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

$$D^2 = (1-x)^2 + (0-y)^2 = (1-x)^2 + x^4$$

$$\frac{d}{dx} D^2 = \underbrace{-2(1-x) + 4x^3}_{f} = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{-2(1-x_n)+4x_n^3}{2+12x_n^2}$$

Try $x_0 = 1.$

$x_1 = 0.71429$

$x_2 = 0.60517$

$x_3 = 0.59002$

$x_4 = 0.58975$

9. (4 marks) Show that the polynomial interpolating the following data has degree three

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$		
1					
4	3				
11	7	2			
16	5	-1	-1		
13	-3	-4	-1	0	
-4	-17	-7	-1	0	0

$$P_3(x) = 1 + 3(x+2) + 2(x+2)(x+1)$$
$$+ (-1)(x+2)(x+1)(x-0) + 0 + 0 + 0 + \cdots$$

a Cubic.

10. (4 marks) Recall the fixed-point theorem:

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then, for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$.

Replace the assumption "a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a,b)$" with "g satisfies a Lipschitz condition on the interval $[a,b]$ with Lipschitz constant $L < 1$." Show that the conclusions of the theorem are still valid.

$$\begin{aligned} |p_n - p| &= |g(p_{n-1}) - g(p)| \\ &\leq L|p_{n-1} - p| \\ &\leq L^2|p_{n-2} - p| \\ &\vdots \\ &\leq L^n|p_0 - p| \end{aligned}$$

As $n \to \infty$, $L^n \to 0$. Thus

$\lim_{n\to\infty} |p_n - p| = 0$. Converges to unique fixed pt.