SFU Macm 316

Final Exam: April 17, 2016

Last Name: ____________________________________

First Name: ____________________________________

Email: ____________________________________

ID: ____________________________________

Instructions: 3 hours. Answer all 12 questions. Closed book.
Two-sided cheat sheet and non-graphing calculator permitted.

Mark very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS.
Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation.

1. (3 marks) Find the rate of convergence of the following sequence as $h \to 0$:

$$\lim_{h\to 0} \exp(h) - \sin(h) = 1$$

Show your steps.

$\exp(h) - \sin(h) - 1$

$= 1 + h + \frac{h^2}{2} + \frac{h^3}{3!} + O(h^4)$

$- \left[h - \frac{h^3}{3!} + \frac{h^5}{5!} + O(h^7)\right] - 1$

$= \frac{h^2}{2} + O(h^3)$

$= O(h^2).$

2. (3 marks) Suppose A is an $n \times n$ matrix. Use the definition of matrix norm to show that $||\cdot||_*$, defined by

$$||A||_* = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

is a matrix norm.

ii/ $||A||_* = 0 \Leftrightarrow A = 0$:

$$\boxed{||A||_* = 0 = \Sigma\Sigma |a_{ij}| \Rightarrow a_{ij} = 0 \ \forall i,j \Rightarrow A = 0}$$

$$\boxed{A = 0 \Rightarrow a_{ij} = 0 \ \forall i,j \Rightarrow \Sigma\Sigma |a_{ij}| = 0 = ||A||_*}$$

i/ $||A||_* \geq 0$: $||A_*|| = \Sigma\Sigma \underbrace{|a_{ij}|}_{\geq 0} \geq 0$

iii/ $||\alpha A||_* = |\alpha| \, ||A||_*$: $||\alpha A||_* = \Sigma\Sigma |\alpha a_{ij}| = \Sigma\Sigma |\alpha| |a_{ij}|$
$= |\alpha| \Sigma\Sigma |a_{ij}|$
$= |\alpha| \, ||A||_*$

(iv) $||A+B||_* \leq ||A||_* + ||B_*||$:
$||A+B||_* = \Sigma\Sigma |a_{ij} + b_{ij}| \leq \Sigma\Sigma |a_{ij}| + |b_{ij}| = ||A||_* + ||B||_*$

v/ $||AB||_* \leq ||A||_* \, ||B||_*$:

$$
\begin{aligned}
||AB||_* &= \Sigma\Sigma (a_{i1} \ a_{i2} \cdots a_{in}) \cdot (b_{1j} \ b_{2j} \cdots b_{nj}) \\
&\leq \Sigma\Sigma ||(a_{i1} \ a_{i2} \cdots a_{in})||_2 \ ||(b_{1j} \ b_{2j} \cdots b_{nj})||_2 \\
&\leq \Sigma\Sigma ||(a_{i1} \ a_{i2} \cdots a_{in})||_1 \ ||(b_{1j} \ b_{2j} \cdots b_{nj})||_1 \\
&= \sum_i ||(a_{i1} \ a_{i2} \cdots a_{in})||_1 \ \sum_j ||(b_{1j} \ b_{2j} \cdots b_{nj})||_1 \\
&= ||A||_* \ ||B||_*
\end{aligned}
$$

by Cauchy Schwarz

3. (3 marks) Suppose $f(t, y) = \cos(yt) + t\sin(y)$.

   (a) Does $f(t, y)$ satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$? Explain.

   (b) Is the initial value problem

$$y' = f(t, y),\ 0 \le t \le 1, y(0) = 1.$$

well-posed? Explain.

a) $\left|\frac{\partial f}{\partial y}\right| = \left| -\sin(yt)\cdot t + t\cos(y) \right|$

$\le |t \sin(yt)| + |t\cos y|$

≤ 2 for $t \in [0, 1]$

ANS: yes.

b) f is CTS

(sum of CTS fns).

& satisfies a Lipschitz condition

$\Rightarrow$ well posed.

4. (3 marks) A clamped cubic spline S for a function f is defined by

$$S(x) = \begin{cases} S_0(x) &= 1 + Bx + 2x^2 - 2x^3 & \text{on } [0,1) \\ S_1(x) &= 1 + b(x-1) + c(x-1)^2 + 7(x-1)^3 & \text{on } [1,2] \end{cases}$$

Find B, b, c, $f'(0)$ and $f'(2)$.

$S_0(1) = 1 + B = 1 = S_1(1)$

$\Rightarrow B = 0$

$S_0'(1) = 4(1) - 6(1)$

$= -2 = b = S_1'(1)$

$S_0''(1) = 4 - 12(1) = -8$

$= 2c$

$\Rightarrow c = -4$

$f'(0) = S_0'(0) = B = 0$

$f'(2) = S_1'(2) = b + 2c + 3 \times 7$

$= -2 - 8 + 21 = 11$

5. (3 marks) Suppose a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$.

HINT: Recall the error formula for the Lagrange polynomial of degree n is

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

want $\left| \frac{f^{(2)}(\xi)}{2!} (x - x_i)(x - x_{i+1}) \right| < 10^{-6}$

$f^{(2)}(x) = e^x$, $|f^{(2)}(x)| \leq e$ over $[0,1]$

Thus, we want

$\frac{e}{2} \left| (x - x_i)(x - x_{i+1}) \right| < 10^{-6}$

max occurs when $x = \frac{x_i + x_{i+1}}{2}$

Thus $\frac{e}{2}\left(\frac{h}{2}\right)^2 < 10^{-6}$

$h < \sqrt{\frac{8 \times 10^{-6}}{e}} \approx .0017$

6. (3 marks) Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_{-1}^{0} f(x)dx = c_0 f(-1) + c_1 f(x_1)$$

has the highest possible degree of precision.

$$\int_{-1}^{0} 1 = 1 = c_0 + c_1$$

$$\int_{-1}^{0} x = \frac{1}{2}x^2 \Big|_{-1}^{0} = -\frac{1}{2} = -c_0 + c_1 x_1$$

$$\int_{-1}^{0} x^2 = \frac{1}{3}x^3 \Big|_{-1}^{0} = \frac{1}{3} = c_0 + c_1 x_1^2$$

$$\frac{1}{2} = c_1(1+x_1), \quad -\frac{2}{3} = -c_1(1-x_1)(1+x_1)$$

$$\frac{4}{3} = (1-x_1) \Rightarrow x_1 = -\frac{1}{3}$$

$$c_1 = \frac{1/2}{2/3} = 3/4$$

$$c_0 = 1/4.$$

7. (3 marks) Consider the sequence

$$p_n = \exp(-3^n).$$

Clearly the sequence converges to zero as n tends to infinity. Does the sequence converge linearly ($\alpha = 1$), quadratically ($\alpha = 2$), or cubically ($\alpha = 3$)? Use the definition of order of convergence to justify your answer.

$$\frac{|p_{n+1} - p|}{|p_n - p|^3} = \frac{e^{-3^{n+1}}}{\left(e^{-3^n}\right)^3} = 1$$

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} = 1$$

$$\alpha = 3.$$

8. (3 marks) Derive the Trapezoid Rule with its error term.

$$\int_{x_0}^{x_1}\left[P_{01}(x) + \frac{f^{(2)}(\xi)}{2}(x-x_0)(x-x_1)\right]$$

$$= \int_{x_0}^{x_1}\left[\frac{(x-x_1)}{(x_0-x_1)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}f(x_1) + \frac{f^{(2)}(\xi)}{2}(x-x_0)(x-x_1)\right]$$

$$= \frac{1}{2}\frac{(x-x_1)^2}{(x_0-x_1)}f(x_0)\Big|_{x_0}^{x_1} + \frac{1}{2}\frac{(x-x_0)^2}{(x_1-x_0)}f(x_1)\Big|_{x_0}^{x_1}$$

weighted MVT

$$+ \frac{f^{(2)}(\tilde{\xi})}{2}\int_{x_0}^{x_1}(x^2 - xx_0 - xx_1 + x_0x_1)$$

$$= \frac{h}{2}\left[f(x_0)+f(x_1)\right] \leftarrow \text{trapezoid Rule}$$

$$+ \frac{f^{(2)}(\tilde{\xi})}{2}\left(\frac{x^3}{3} - \frac{1}{2}x^2(x_0+x_1) + x\,x_0x_1\right)\Big|_{x_0}^{x_1} \leftarrow \text{Error.}$$

$$\text{Error} = -\frac{f^{(2)}(\tilde{\xi})}{2}\left(\frac{1}{6}\right)(x_1-x_0)^3 = -\frac{1}{12}f^{(2)}(\tilde{\xi})\,h^3$$

9. (3 marks) Suppose that

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 8x_3 &= 5 \\ 6x_1 + \alpha x_2 + 10x_3 &= 5 \end{aligned}$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

(a) $\alpha = 6$

(b) $\alpha = 9$

(c) $\alpha = -3$

$s_1 = 3,\ s_2 = 8,\ s_3 = 10.$

$2/s_1 > 4/s_2,\ 2/s_1 > 6/s_3$. Use 2 as pivot.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 8 \\ 4x_2 + 2x_3 &= -11 \\ (\alpha-3)x_2 + x_3 &= -19 \end{aligned}$$

a) $\alpha = 6$.

$\frac{4}{s_2} > \frac{3}{s_3}$

No pivoting

b) $\alpha = 9$

$\frac{4}{s_2} < \frac{6}{s_3}$

~~pivot~~

exchange R_2 & R_3

c) $\alpha = -3$

$\frac{2}{s_2} < \left|\frac{-6}{s_3}\right|$

exchange R_2 & R_3

10. (3 marks) Apply a theorem from class to prove that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

$g(x) = \frac{1}{2}x + \frac{1}{x}$

$g'(x) = \frac{1}{2} - \frac{1}{x^2} > 0$ for $x > \sqrt{2}$

Thus $\min_{x \geq \sqrt{2}} g(x) = g(\sqrt{2}) = \sqrt{2}$

& $g(x) \in [\sqrt{2}, \infty)$ for $x \in [\sqrt{2}, \infty)$.

$|g'(x)| \leq \frac{1}{2} < 1$ for all $x \in [\sqrt{2}, \infty)$

& g is CTS for all $x \geq \sqrt{2}$. (sum of CTS fn).

Thus the iteration converges to the unique fixed pt.

Noting $x = \frac{1}{2}x + \frac{1}{x}$

$\frac{1}{2}x = \frac{1}{x}$

$x^2 = 2 \Rightarrow$ fixed pt is $\sqrt{2}$.

11. (3 marks) Use Newton's method to solve

$$\sin(p/2) = 1$$

to two decimal places using an initial guess of $p_0 = 3$. Do you observe linear or quadratic convergence? Explain why this order of convergence is observed.

Newton:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$= p_n - \frac{\sin(p_n/2) - 1}{\frac{1}{2}\cos(p_n/2)}$$

$p_0 = 3$ error $\approx .14$

$p_1 = 3.0708$ error $\approx .07$

$p_2 = 3.1062$ error $\approx .035$

$p_3 = 3.1239$

$p_4 = 3.1327$ $\boxed{\text{we see linear convergence}}$

$p_5 = 3.1372$

$\downarrow$

$p_\infty = \pi$

$f(p) = \sin(\frac{p}{2}) - 1$

$f(\pi) = 0$

$f'(\pi) = \cos(\pi/2) = 0$.

$f''(\pi)$

linear because we have a root of multiplicity 2.

12. (3 marks) Using

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{1}{2}f''(x_0)h^2+\frac{1}{6}f'''(x_0)h^3+\frac{1}{24}f^{(4)}(\xi_1)h^4 \qquad 1$$

and

$$f(x_0-h) = f(x_0)-f'(x_0)h+\frac{1}{2}f''(x_0)h^2-\frac{1}{6}f'''(x_0)h^3+\frac{1}{24}f^{(4)}(\xi_{-1})h^4 \qquad 2$$

where $x_0 - h < \xi_{-1} < x_0 < \xi_1 < x_0 + h$, derive an approximation for $f''(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$ and $f(x_0 + h)$.

As part of your derivation, find an expression for the error of your approximation.

Add 1 and 2

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + f''(x_0)h^2 + \frac{1}{24}f^{(4)}(\xi_1)h^4 + \frac{1}{24}f^{(4)}(\xi_{-1})h^4$$

re-arrange

formula

$$f''(x_0) = \boxed{\frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}}$$

$$\underbrace{- \frac{1}{24}f^{(4)}(\xi_1)h^2 - \frac{1}{24}f^{(4)}(\xi_{-1})h^2}$$

error $\boxed{-\frac{1}{12}f^{(4)}(\xi)h^2}$ assuming $f^{(4)}$ is cts!