SFU Macm 316

Midterm Test: Feb 24, 2016

Last Name: ____________________________

First Name: ____________________________

Email: ____________________________

ID: ____________________________

Instructions: 50 minutes. Answer all 5 questions. Closed book. One-sided cheat sheet and non-graphing calculator permitted.

Mark very clearly on the question if you use the back of any page.

EXPLAIN ALL ANSWERS.
Do not expect ANY marks for answers that do not provide intermediate work. Marks may be deducted for poor presentation.

1. (3 marks) Find the rate of convergence of the following sequence as $h \to 0$:

$$\lim_{h\to 0} \cos(h) + \frac{1}{2}h^2 = 1$$

Show your steps.

$$\cos(h) + \frac{1}{2}h^2 - 1$$

$$= 1 - \frac{h^2}{2} + \frac{h^4}{4!} + \frac{h^2}{2} - 1 + O(h^6)$$

$$= \frac{h^4}{4!} + O(h^6)$$

$$= O(h^4)$$

2. (3 marks) Suppose A is an $n \times n$ matrix. Use the definition of matrix norm to show that $||\cdot||_*$, defined by

$$||A||_* = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|$$

is a matrix norm.

ii) $||A||_* = 0 \iff A = 0$:

$$\boxed{||A||_* = 0 = \sum\sum |a_{ij}| \Rightarrow a_{ij} = 0 \ \forall i,j \Rightarrow A = 0}$$

$$\boxed{A = 0 \Rightarrow a_{ij} = 0 \ \forall i,j \Rightarrow \sum\sum |a_{ij}| = 0 = ||A||_*}$$

i) $||A||_* \geq 0$: $||A_*|| = \sum\sum \underbrace{|a_{ij}|}_{\geq 0} \geq 0$

iii) $||\alpha A||_* = |\alpha|\,||A||_*$: $||\alpha A||_* = \sum\sum |\alpha a_{ij}| = \sum\sum |\alpha|\,|a_{ij}|$

$$= |\alpha| \sum\sum |a_{ij}|$$
$$= |\alpha|\,||A||_*$$

(iv) $||A+B||_* \leq ||A||_* + ||B_*||$:

$$||A+B||_* = \sum\sum |a_{ij} + b_{ij}| \leq \sum\sum |a_{ij}| + |b_{ij}| = ||A||_* + ||B||_*$$

v) $||AB||_* \leq ||A||_*\,||B||_*$:

$$||AB||_* = \sum\sum (a_{i1}\ a_{i2} \cdots a_{in}) \cdot (b_{1j}\ b_{2j} \cdots b_{nj})$$
$$\leq \sum\sum ||(a_{i1}\ a_{i2} \cdots a_{in})||_2\ ||(b_{1j}\ b_{2j} \cdots b_{nj})||_2$$
$$\leq \sum\sum ||(a_{i1}\ a_{i2} \cdots a_{in})||_1\ ||(b_{1j}\ b_{2j} \cdots b_{nj})||_1$$
$$= \sum_i ||(a_{i1}\ a_{i2} \cdots a_{in})||_1 \sum_j ||(b_{1j}\ b_{2j} \cdots b_{nj})||_1$$
$$= ||A||_*\ ||B||_*$$

by Cauchy Schwarz

3. (3 marks) Use Gaussian Elimination with scaled partial pivoting to solve the following system:

$$\begin{aligned} 4x + 40y &= 60 \\ 2x + y &= 2 \end{aligned}$$

Scale Row 1: 40
Scale Row 2: 2

Pivot choice: $4/40 < 2/2$

So $E_1 \Leftrightarrow E_2$

$2x + y = 2$
$4x + 40y = 60$

$-2E_1 + E_2 \rightarrow E_2$:

$2x + y = 2$
$38y = 56$

$y = \frac{56}{38} = 1.4737$

$x = \frac{2 - \frac{56}{38}}{2} = .2632$

4. (3 marks) Recall that iterative methods for solving the linear system $Ax = b$ take the form $x^{(n)} = Tx^{(n-1)} + c$ for some initial guess $x^{(0)}$.

   (a) What are T and c for Jacobi's method when

$$A = \begin{bmatrix} 5 & -2 & 0 \\ 0 & 6 & -3 \\ -4 & 0 & 8 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

   (b) Suppose you were to solve $Mx = b$ using Jacobi's method for

$$M = \begin{bmatrix} 0 & 6 & -3 \\ -4 & 0 & 8 \\ 5 & -2 & 0 \end{bmatrix}$$

   What would you use for your iteration matrix?

a/ $$T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2/5 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{bmatrix}$$

$$c = \begin{bmatrix} 5 & & \\ & 6 & \\ & & 8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 0 \\ 1/2 \end{bmatrix}$$

b/ Interchange rows

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} M$$

We can use T from part a/.

5. (3 marks) Let A be a given positive constant and $g(x) = 2x - Ax^2$

a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $t = 1/A$.

b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in the interval.

a/ $x = 2x - Ax^2$

$-1 = -Ax$

$x = 1/A.$

b/ Use THM 2.4: g is cts (g is a polynomial)

$g'(x) = 2 - 2Ax$

$|g'| < 1 \Rightarrow -1 < g' < 1$

$2 - 2Ax > -1, \quad 2 - 2Ax < 1$

$x < \frac{3}{2A}, \quad x > \frac{1}{2A}.$

Does $g(x) \in [\frac{1}{2A}, \frac{3}{2A}]$ for all $x \in [\frac{1}{2A}, \frac{3}{2A}]$?

$g(\frac{1}{2A}) = \frac{3}{4A} \in [\frac{1}{2A}, \frac{3}{2A}], \quad g(\frac{3}{2A}) = \frac{3}{A} - \frac{9}{4A} = \frac{3}{4A} \in [\frac{1}{2A}, \frac{3}{2A}]$

extrema at $x = 1/A$. $g(\frac{1}{A}) = \frac{1}{A} \in [\frac{1}{2A}, \frac{3}{2A}]$

Thus $g(x) \in [\frac{1}{2A}, \frac{3}{2A}]$ for all $x \in [\frac{1}{2A}, \frac{3}{2A}]$.

By the THM 2.4, the iteration converges for $p_0 \in [\frac{1}{2A} + \varepsilon, \frac{3}{2A} - \varepsilon]$ for some small $\varepsilon > 0$.