Q5. $Ce = \lambda e$

$$f^T f = 1, \quad e^T f = 0 \text{ or } f^T e = 0$$

$$J(f) = f^T C f$$

maximise $J(f)$ under above constraint using Lagrange multiplier

$$J(f) = f^T C f - \lambda_1 (f^T f - 1) - \lambda_2 e^T f$$

Taking derivative of $J(f)$ w.r.t. f and setting it zero.

$$2Cf - 2\lambda_1 f - \lambda_2 e = 0 \quad \rightarrow (1)$$

premultiply (1) with e^T

$\Rightarrow 2 e^T C f - \lambda_2 = 0$

$\Rightarrow e^T C f = \frac{\lambda_2}{2}$

Taking transpose

$\Rightarrow f^T C^T e = \frac{\lambda_2}{2}$

Since $C^T = C \Rightarrow f^T C e = \frac{\lambda_2}{2}$

$\Rightarrow f^T (\lambda e) = \frac{\lambda_2}{2} \quad (Ce = \lambda e)$

$\Rightarrow \frac{\lambda_2}{2} = 0 \Rightarrow \lambda_2 = 0 \quad \rightarrow (2)$

from (1) & (2)

$Cf = \lambda_1 f \Rightarrow f$ is eigenvector of C with eigen value λ_1.

hence $f^T C f = \lambda_1$, λ_1 is second highest eigen value, because highest eigenvalue corresponds to 'e'.

Q6(a)

$$y^T P y = y^T A^T A y = (Ay)^T Ay \rightarrow ①$$

Ay is $m\times 1$ vector

① is dot product of vector with itself
hence ① ≥ 0

Similarly

$$z^T Q z = z^T A A^T z = (A^T z)^T A^T z \rightarrow ②$$

$A^T z$ is $n\times 1$ vector
② is dot product of vector with itself
hence ② ≥ 0

Since $Z^T Q Z$ and $Y^T P Y$ are ≥ 0
and $Z_1^T Q Z_1 = \lambda_Q$, $Y_1^T P Y_1 = \lambda_P$
Z_1 and Y_1 are eigenvectors of Q and P
respectively Hence $\lambda_Q \geq 0$ $\lambda_P \geq 0$.

Q6(b) Given: $\cancel{u} P u = \lambda u$

$\Rightarrow \cancel{u^T A^T A u \Rightarrow} A^T A u = \lambda u$

$\cancel{\Rightarrow u u^T A^T A u = A u} \Rightarrow A A^T A u = \lambda A u$

$\cancel{\Rightarrow \text{Since } u^T u = u u^T =}$

$\Rightarrow \cancel{A}\, Q (A u) = \lambda (A u)$

hence Au is eigenvector of Q

no. of elements in $u = n$

Given: $Q v = \mu v$

$\Rightarrow A A^T v = \mu v$

$\Rightarrow A^T A A^T v = \mu A^T v$

$\Rightarrow P (A^T v) = \mu (A^T v)$

hence $A^T v$ is eigenvector of P

no. of elements in $v = m$

Q6 (C) $Q v_i = \lambda_i v_i$ where λ_i is eigenvalue corresponding to eigenvector v_i.

$\Rightarrow A A^T v_i = \lambda_i v_i$

$\Rightarrow A \dfrac{(A^T v_i)}{\|A^T v_i\|_2} = \dfrac{\lambda_i}{\|A^T v_i\|_2} v_i$

$\Rightarrow A U_i = \gamma_i v_i$ where $\gamma_i = \dfrac{\lambda_i}{\|A^T v_i\|_2}$

Q6 (d) We know from (b) ~~A^T~~

that $A^T v_i$ is eigenvector of P

$\Rightarrow A^T v_i = \lambda_i u_i$

for all $i \in 1$ to m

this can be written as ~~$A^T[v_1|v_2|--|v_m] = [\lambda_1 u_1| - \lambda_m$~~

$$A^T[v_1|v_2|--|v_m] = [u_1|u_2|--|u_m]\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & 0 & - & - & \lambda_m \end{bmatrix}$$

$\Rightarrow A^T U = V\Sigma$

Premultiply with U^T

$A^T U U^T = V \Sigma U^T$

Since $U^T U = U U^T = I$

$\Rightarrow$ ~~A^T~~ $A^T = V \Sigma U^T$

Transpose both side

$A = U \Sigma^T V^T$

Since Σ^T is diagonal matrix

$A = U \Sigma V$

Hence proved.