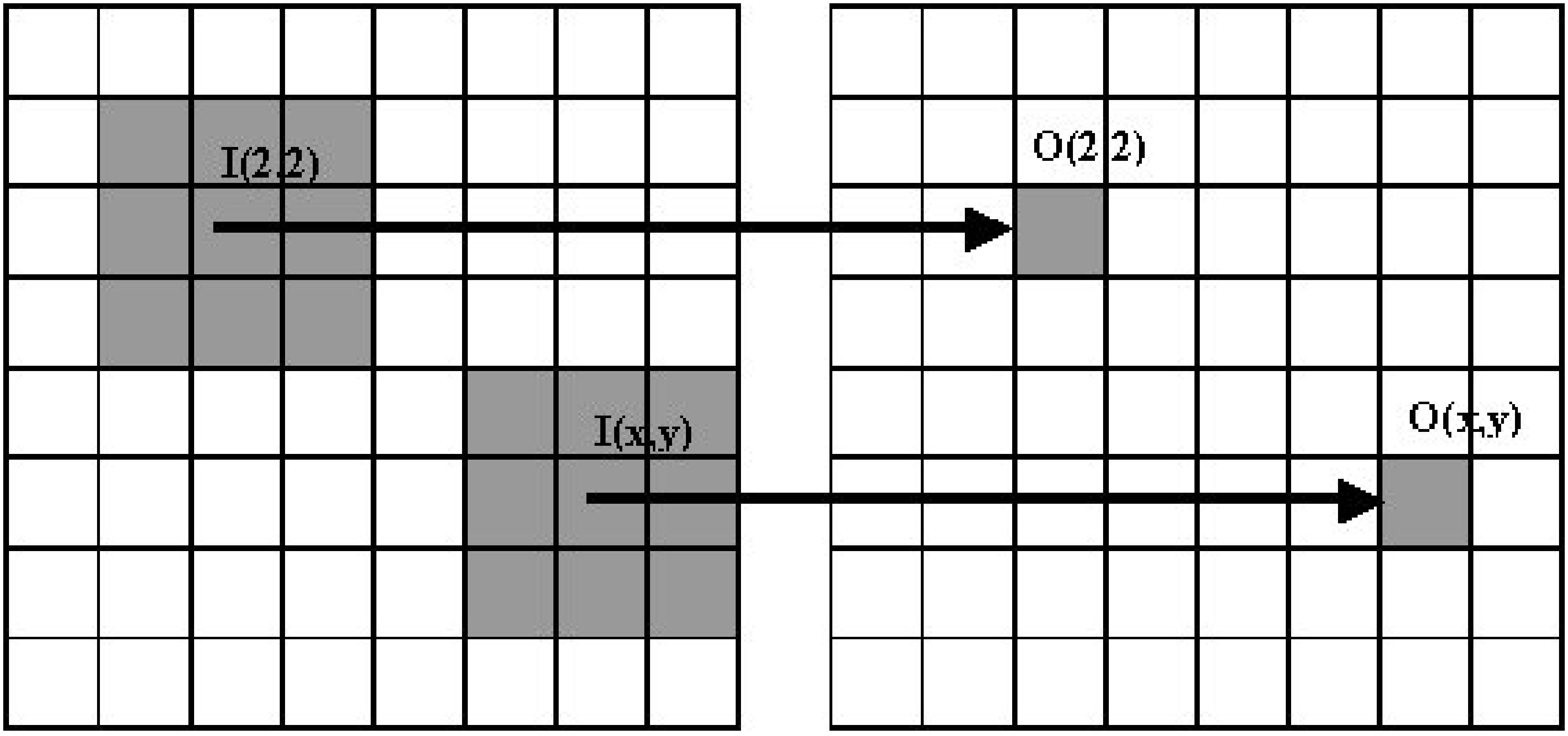
Digital Image Processing

Image Enhancement : Spatial Filtering

Suyash P. Awate

Spatial Filtering

- Key idea
  - Intensity at pixel "p" in output image is a function of the pixel intensities in the neighborhood of "p" in input image

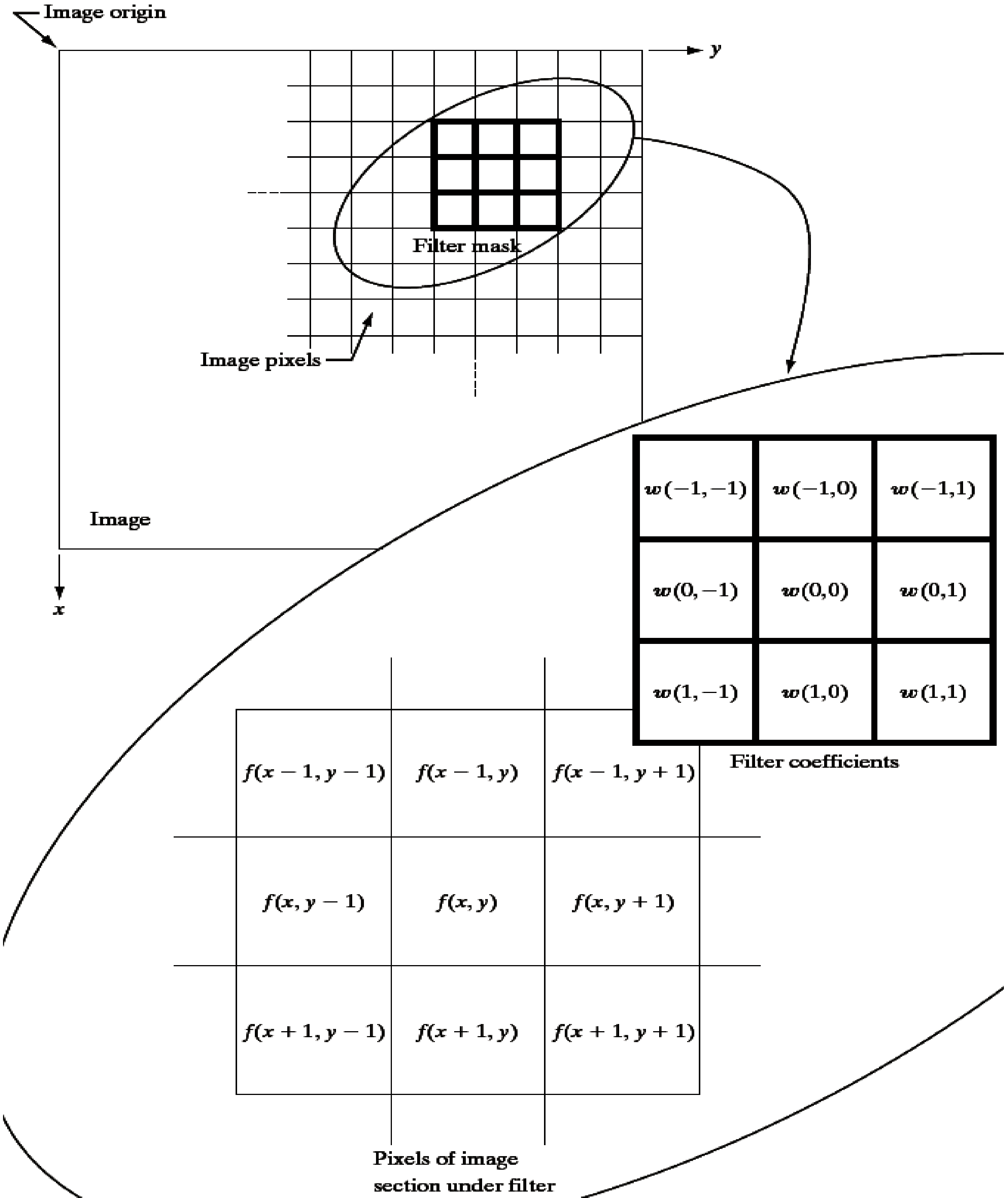


Spatial Filtering

- **Linear** spatial filtering
  - Spatial filtering where the **output** pixel intensity = **linear function** of the **input** pixel intensities

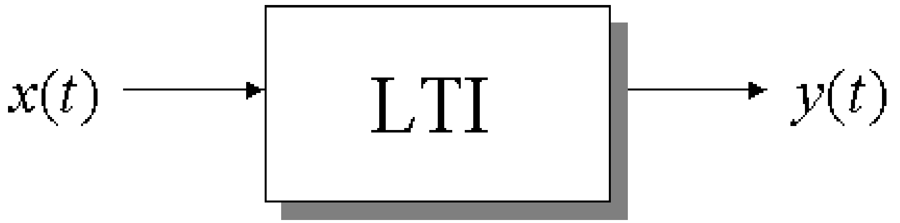
Spatial Filter

- Linear spatial filter
  - g (x,y) typically designed as = $\sum_{i=-1,0,1} \sum_{j=-1,0,1}$ w (i,j) * f (x+i,y+j)
- w = **filter mask**
- w(i,j) = filter **coefficients**
- Linear spatial filter performs **“convolution”**

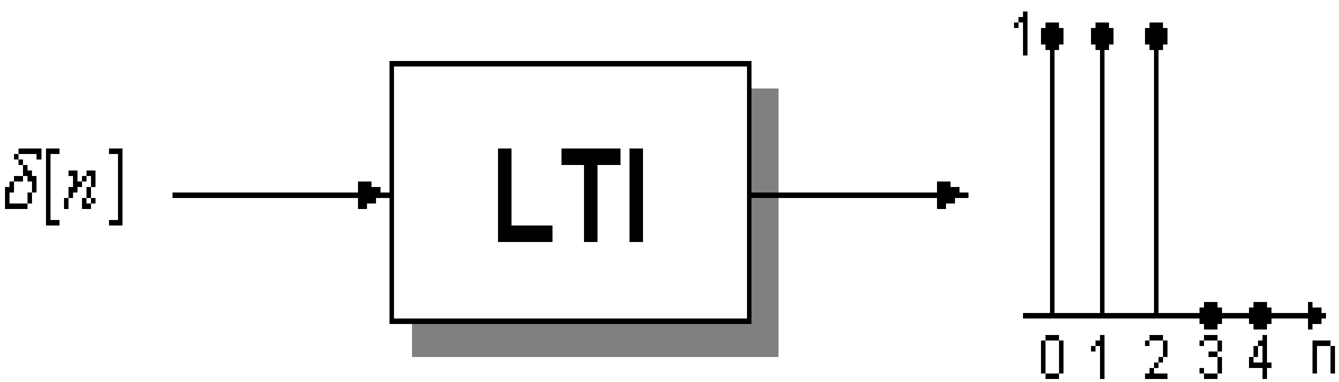


Spatial Filtering

- Consider a black-box system
  - Takes input x(t) → Produces output y(t)

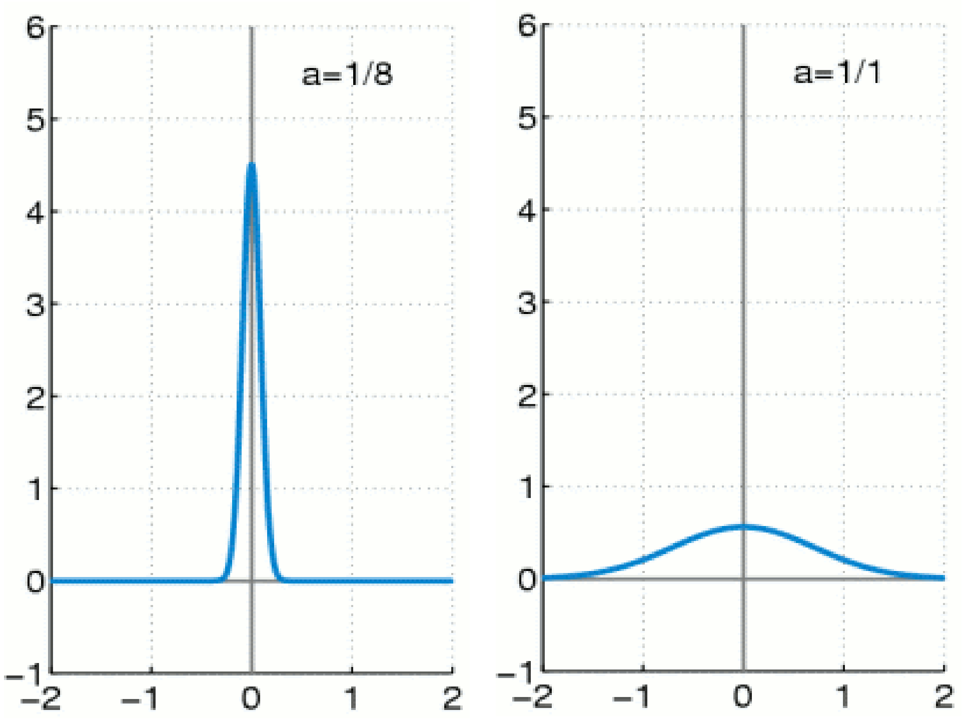


  - Let input = **(discrete) unit-impulse function** at time t=0
    - **Kronecker delta**: δ(t) = 1 if t =0; Otherwise δ(t) = 0
    - **Continuous** impulse function = **Dirac delta**
      - Derivative of the unit-step function
      - Rectangular pulse with width → 0, height → ∞, area = 1
      - Limit of a Gaussian PDF as variance → 0
  - For input = δ(t), output = h(t) = **impulse response**



Spatial Filtering

- Continuous impulse function (Dirac delta)
  - Definition : $\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$ as $a \to 0$.

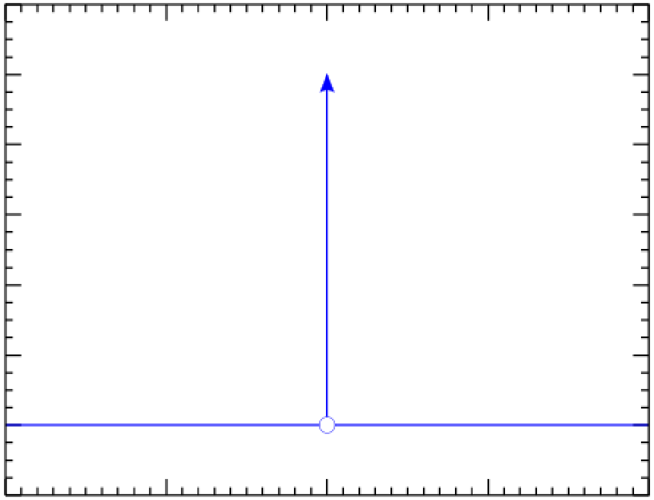


    - δ(x) = 0 for all |x| > 0
    - δ(x) integrates to 1

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

  - Sifting / sampling property

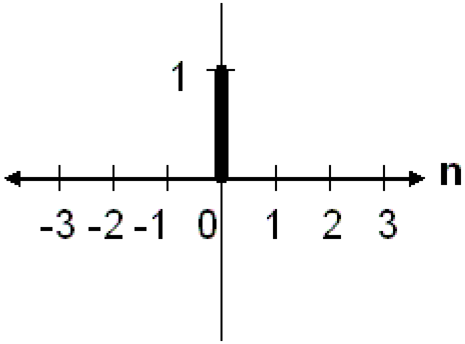
$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\, dt = f(T)$$



Spatial Filtering

- Discrete impulse function (Kronecker delta)
  - Definition :

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

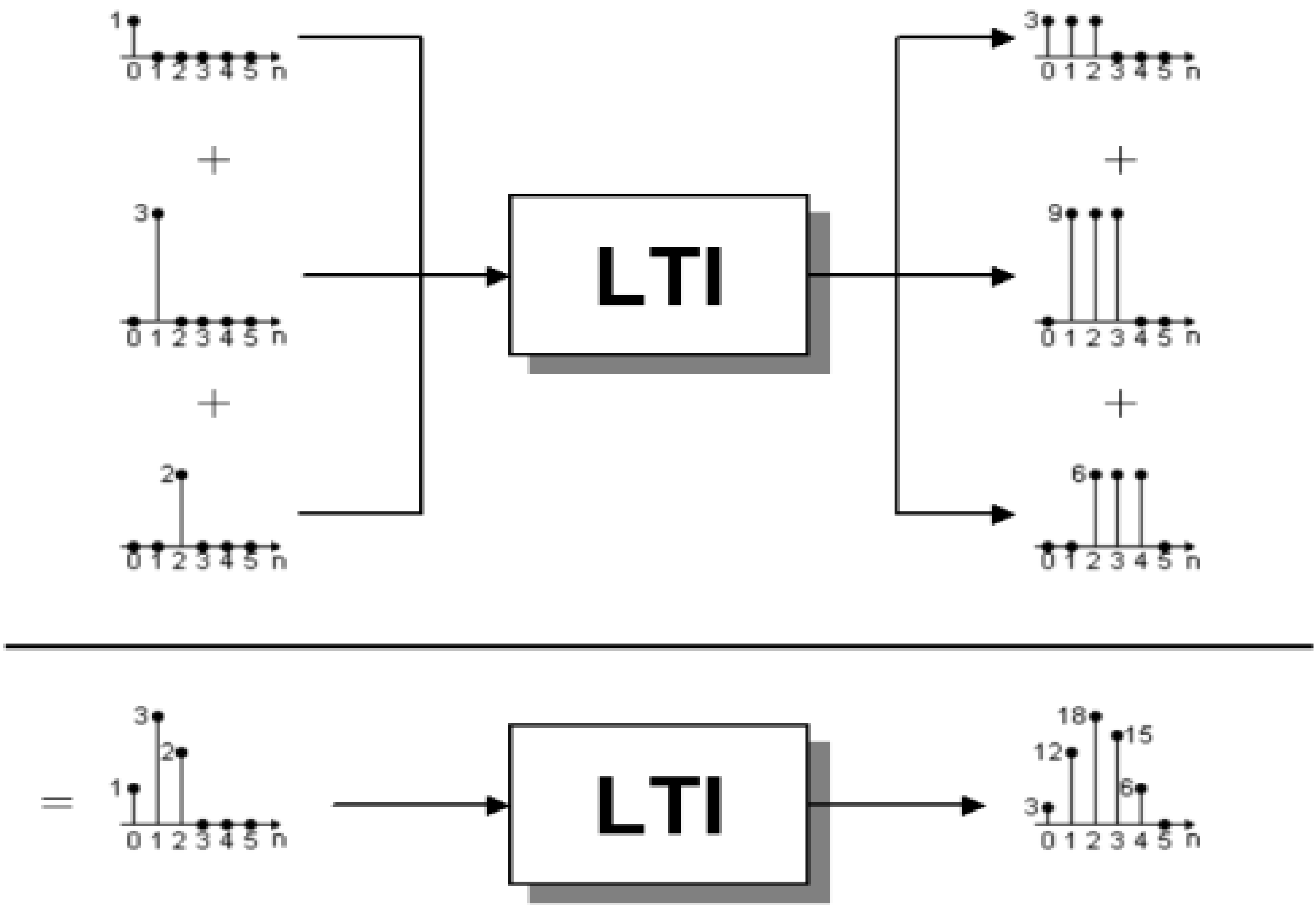


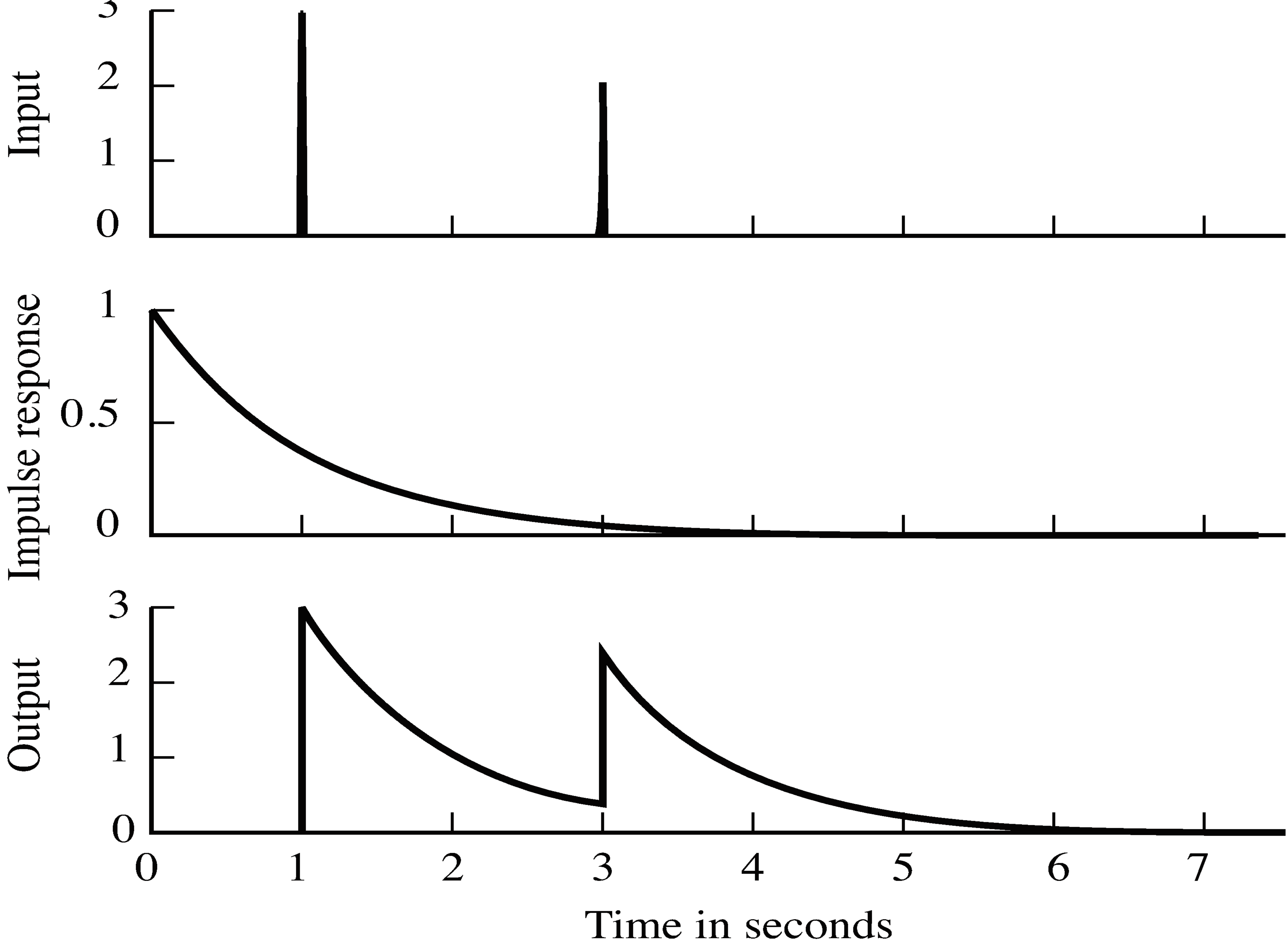
  - Sifting property

$$\sum_{i=-\infty}^{\infty} a_i \delta_{ij} = a_j$$

Spatial Filtering

- Input-output system
  - For a time-invariant system:
    - When input gets **delayed** / **shifted** by some time interval, output gets shifted by the same time interval
      - $\delta(t-t') \rightarrow h(t-t')$
  - For a linear system:
    - When input gets **scaled** by a factor, output gets scaled by same factor
      - $a\,\delta(t) \rightarrow a\,h(t)$
    - When the input is a **sum** of 2 (shifted) impulse functions, the output is the sum of 2 (shifted) impulse responses
      - $\delta(t) + \delta(t-t') \rightarrow h(t) + h(t-t')$
    - See picture on next slide ...


1
0 1 2 3 4 5 n
3
0 1 2 3 4 5 n
2
0 1 2 3 4 5 n
LTI
3
0 1 2 3 4 5 n
9
0 1 2 3 4 5 n
6
0 1 2 3 4 5 n
=
3
2
1
0 1 2 3 4 5 n
LTI
18
15
12
6
3
0 1 2 3 4 5 n



Input
3
2
1
0
Impulse response
1
0.5
0
Output
3
2
1
0
0
1
2
3
4
5
6
7
Time in seconds

Spatial Filtering

- Linear time-invariant system
  - f (shifted candle)
    = shifted (f (candle))
  - f (candle1 + candle2)
    = f (candle1) + f (candle2)

Spatial Filtering

- **Linear time-invariant system**
  - In equations:
    - Impulse response = H(t)
    - Input = F(t)
    - Output = G(t)
  - We can **rewrite any input F(t)** as

    $$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$

    - F(t) = F(0) δ(t) + F(1) δ(t-1) + F(2) δ(t-2) + …
  - By **definition of impulse response**,
    if input = δ(t),
    then output = H(t)
  - By **definition of LTI system**,
    if input is F(t) = sum of scaled shifted δ(t) functions,
    then output G(t) = sum of scaled shifted H(t) responses

Spatial Filtering

- **Linear time-invariant system**
  - By the definition of the LTI system,
    output G(t) = sum of scaled shifted impulse responses
    - $G(t) = F(0)\, H(t) + F(1)\, H(t-1) + F(2)\, H(t-2) + \ldots$
    - $G(t) = \sum_{k=0,1,2} F(k)\, H(t-k)$
  - Output G(t)
    = **convolution** of input F(t) with impulse response H(t)
  - In general
    - Input function can be arbitrarily long
    - Impulse response function can be arbitrarily long
    - $\mathbf{G(t) = \sum_{k=-\infty,\ldots,\infty} F(k)\, H(t-k)}$
  - $\mathbf{G = F * H}$
  - $\mathbf{G(t) = (F * H)(t)}$

Spatial Filtering

- Convolution
  - What if impulse response $H(t) = \delta(t)$?
    - If, impulse response = impulse function, then $F * H = ?$

Spatial Filtering

- Convolution
  - Convolution is a **symmetric / commutative** operation
  - $F * H = H * F$
  - Observe

$$G(t) = (F * H)(t) = \sum_{k=-\infty,\dots,\infty} F(k)\, H(t-k)$$

$$= \sum_{m=-\infty,\dots,\infty} F(t-m)\, H(m) = (H * F)(t)$$

    - Just a change of variables:

Define $m = t - k$

Then,

$k = t - m$

$k = \infty \quad \rightarrow \quad m = (-\infty)$

$k = (-\infty) \quad \rightarrow \quad m = \infty$

Spatial Filtering

- Convolution
  - Convolution is a **linear** operation on its arguments
  - What if H(t) = A(t) + B(t) ?
    - F * H = F * (A + B) = ?
  - What if H(t) = c A(t) ? where c = constant
    - F * H = F * (c A) = ?

Spatial Filtering

- Convolution
  - Convolution is **associative** (order doesn't matter)
    - (x * h1) * h2 = x * (h1 * h2)
  - Proof: Left hand side (t) = $\int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau * h_2(t)$
  - (expanding) = $\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h_1(\gamma-\tau)d\tau\right]h_2(t-\gamma)d\gamma$
  - (reverse order) = $\int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h_1(\gamma-\tau)h_2(t-\gamma)d\gamma\right]d\tau$

    $\varphi = \gamma - \tau$
  - (substitution) = $\int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h_1(\varphi)h_2(t-\tau-\varphi)d\gamma\right]d\tau$
  - Inner integral = h1 * h2 evaluated at (t − τ)
  - Outer integral = x * (h1 * h2) evaluated at t

Spatial Filtering

- Convolution
  - How to **compute** convolution for discrete images ?
    - Consider time t = 0
      - We can redefine any arbitrary time-point as "0" by translating the coordinate frame for input F (and output G)
      - The origin of the time coordinate frame is NOT important in LTI systems
    - $G(t=0) = \sum_{k=-\infty,\ldots,\infty} F(k)\, H(0-k)$
    - What does this mean algorithmically ?
      - To compute G(0):

        (1) **Flip** the impulse response about the origin $\rightarrow$ H(-k)

        (2) Perform **pointwise multiplication** of F and flipped-H

        (3) **Sum** up
    - How to compute G(t) at some arbitrary timepoint "t" ?

Spatial Filtering

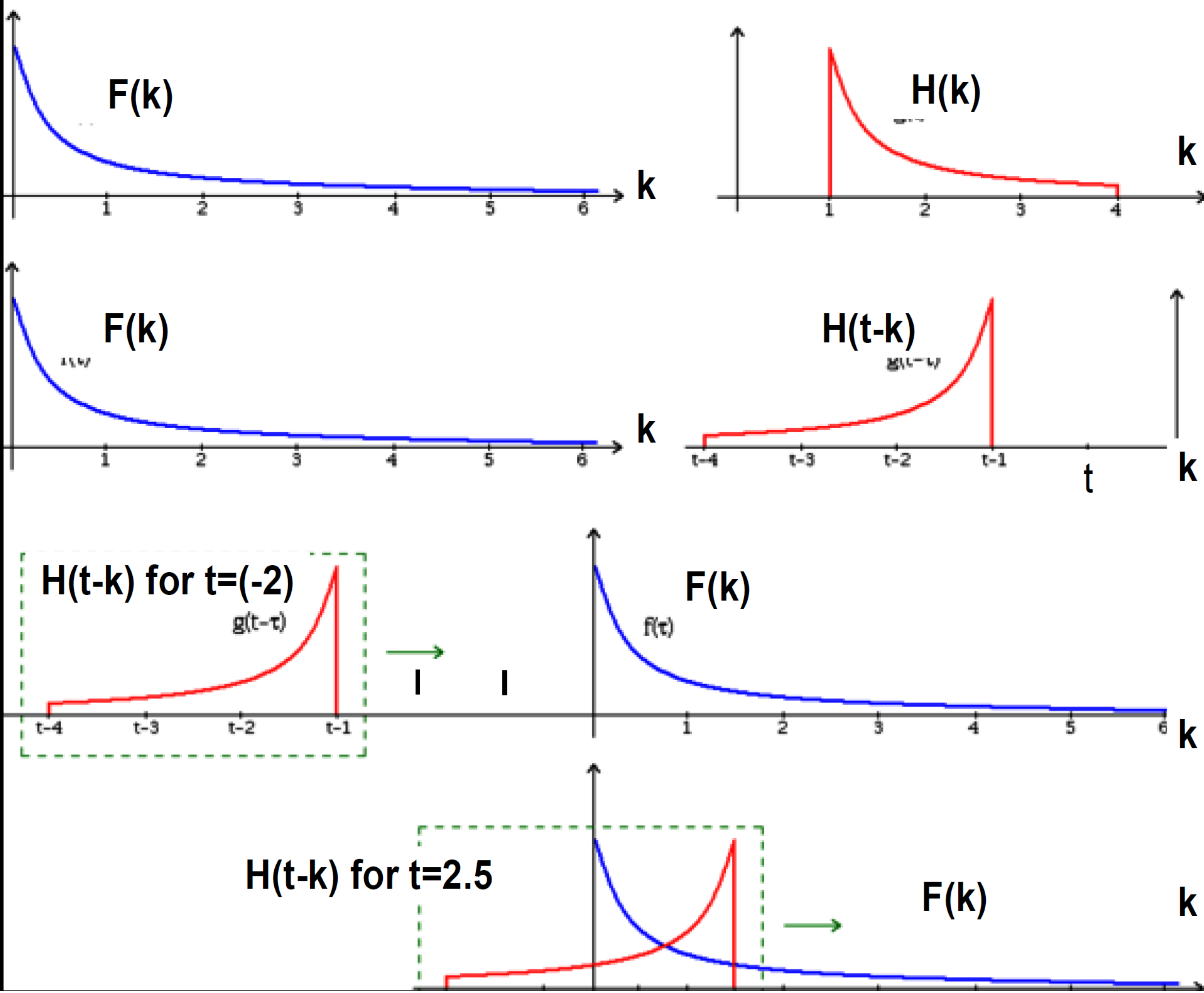
- Convolution
  - What does this mean algorithmically ?
    - We want to compute $G(t) = \sum_{k=-\infty,\ldots,\infty} F(k)\, H(t-k)$ for an arbitrary timepoint t

      (1) **Flip** the impulse response about the origin $\rightarrow$ H(-k)

      (2) **Shift** the flipped impulse response by t $\rightarrow$ H(t-k)

      (3) Perform **pointwise multiplication** of F and shifted-flipped-H

      (4) **Sum** up
  - See next slide for pictures



F(k)
k
1
2
3
4
5
6
H(k)
k
1
2
3
4
F(k)
k
1
2
3
4
5
6
H(t-k)
t-4
t-3
t-2
t-1
t
k
H(t-k) for t=(-2)
g(t-τ)
t-4
t-3
t-2
t-1
F(k)
f(τ)
1
2
3
4
5
6
k
H(t-k) for t=2.5
F(k)
k

Spatial Filteri

convolution

/ˌkɒnvəˈluːʃ(ə)n/

noun

noun: **convolution**; plural noun: **convolutions**; noun: **convolution integral**; plural noun: **convolution integrals**; plural noun: **convolutions integral**

1. a coil or twist.
   "crosses adorned with elaborate convolutions"
   synonyms: twist, turn, coil, spiral, twirl, curl, helix, whorl, loop, curlicue, kink, sinuosity; More
   - the state of being or process of becoming coiled or twisted.
     "the flexibility of the polymer chain allows extensive convolution"
2. a thing that is complex and difficult to follow.
   "the convolutions of farm policy"
   synonyms: complexity, intricacy, complication, twist, turn, entanglement, contortion; More
3. a sinuous fold in the surface of the brain.
4. MATHEMATICS
   a function derived from two given functions by integration which expresses how the shape of one is modified by the other.
   - a method of determination of the sum of two random variables by integration or summation.

Origin

MEDIEVAL LATIN convolvere (roll together) → MEDIEVAL LATIN convolutio

MEDIEVAL LATIN convolutio + ENGLISH convolve → convolution (*mid 16th century*)

mid 16th century: from medieval Latin *convolutio(n-*), from *convolvere* ‘roll together’ (see convolve).

The sign was just as convoluted as the road.