POLITECNICO DI TORINO

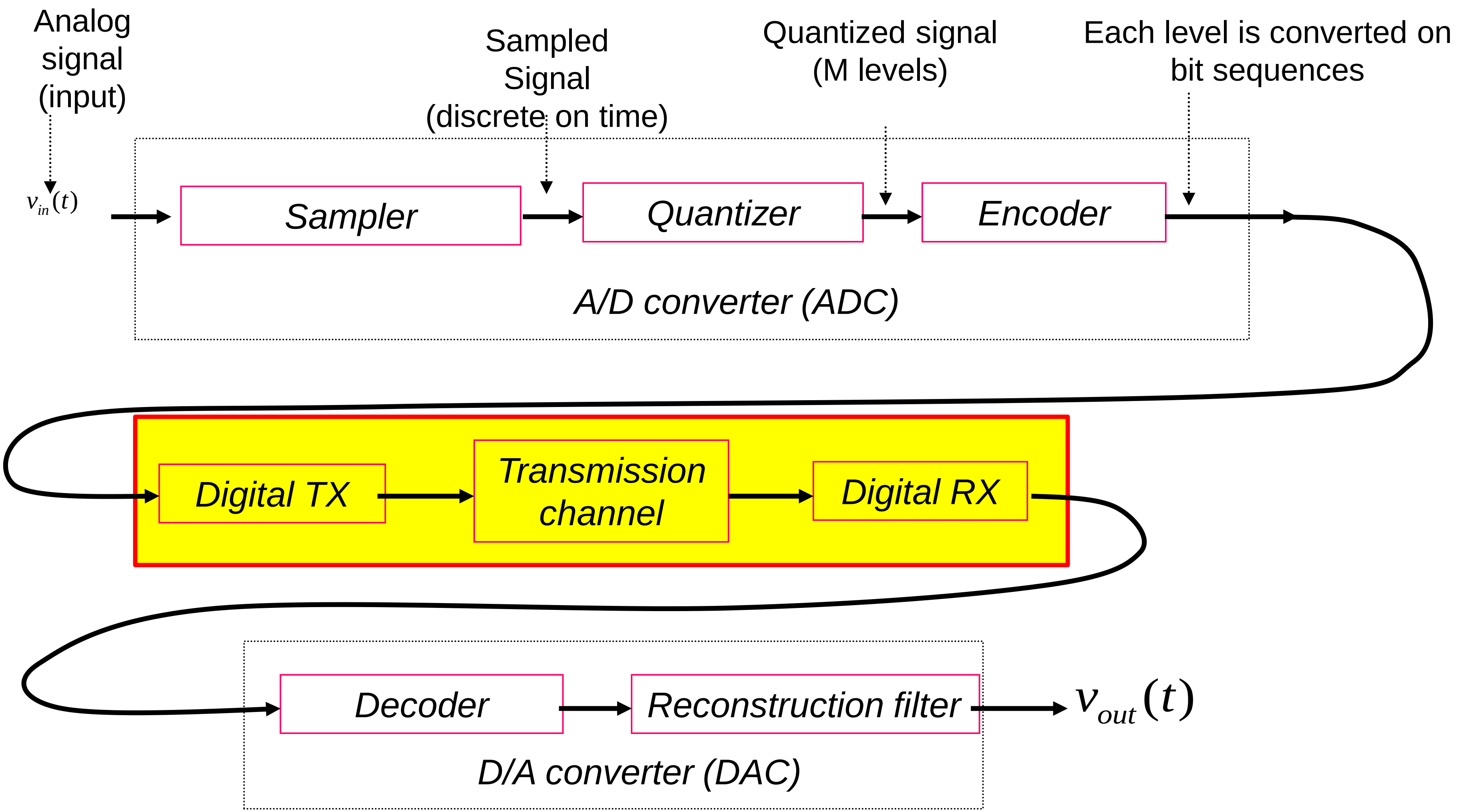
DET

Department of Electronics and Telecommunications

# Digital Transmission

# Context

# Communication system: block diagram

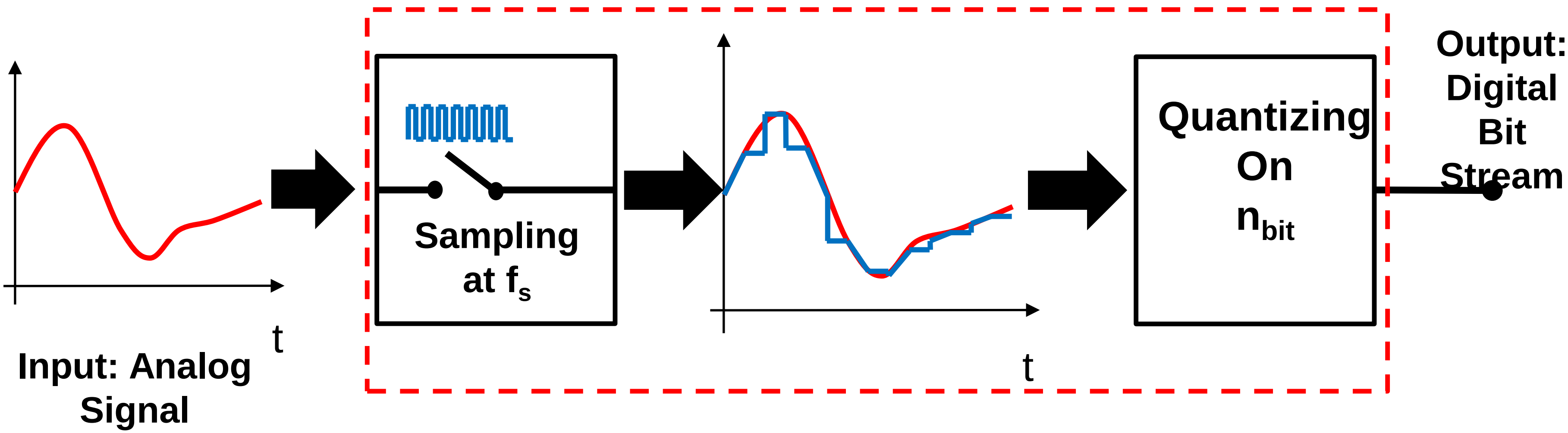


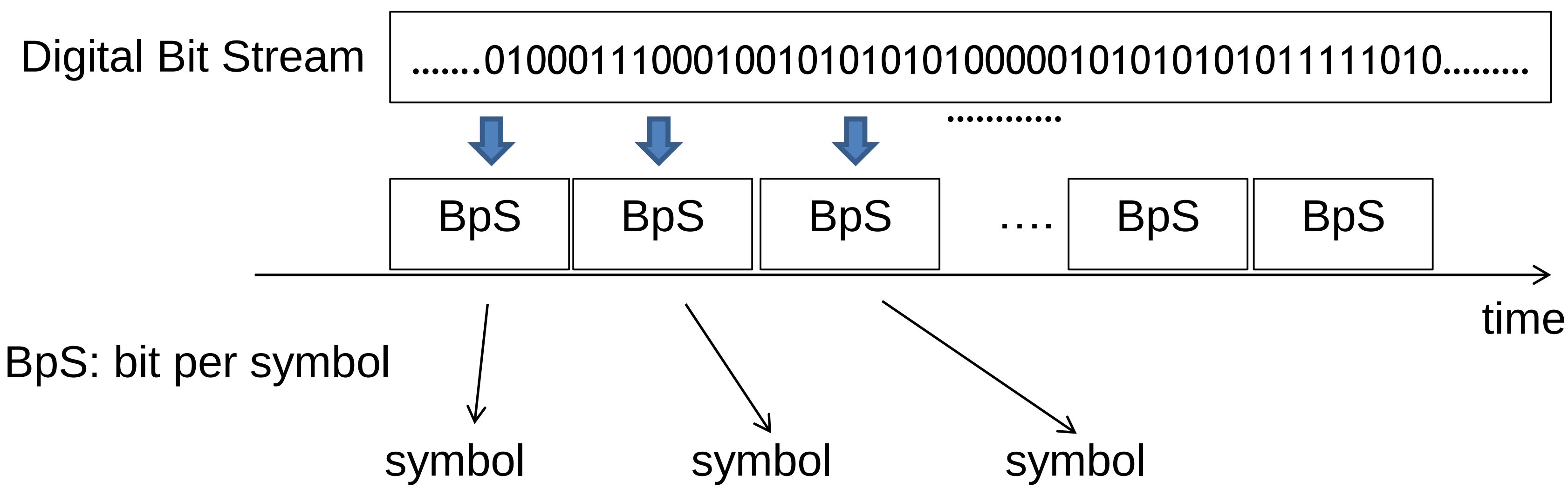
# Digital transmission

# Basic concept

- Group bits in symbols:  the symbol length BpS is called number of bit per symbol
- The number of symbols $K=2^{BpS}$ is called constellation dimension
- Map symbol onto an electromagnetic signal

# Symbol generation



# Symbol rate

- We can think the digital source as a source using an alphabet X with $M=2^{BpS}$ symbols
- Example, if n=3, X={000,001,010,011,100,101,110,111}
- In this case the speed at the TX output is the symbol rate defined as

$$R_s = R_b / \mathrm{BpS} \quad [\text{symbol/s=baud}]$$

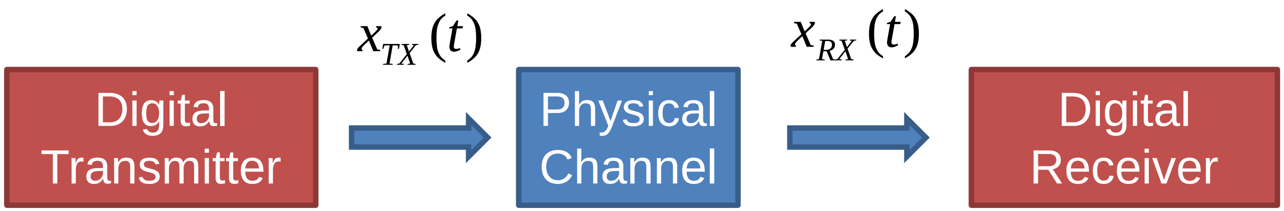
- Note that the unit for the symbol-rate is *baud* corresponding to *symbol/s*

# Symbol time

$$T_b = 1/R_b \qquad T_S = 1/R_S = BpS \cdot T_b$$

- If BpS=1, $T_S = T_b$ ⇨ binary transmission
- BpS>1 ⇨ multilevel transmission

# Digital communication system




- $x_{TX}(t)$ and $x_{RX}(t)$ are analog signals
- The digital transmitter maps each symbol on a proper signal, that must be compatible with characteristic of the physical channel
- Physical channel: ether, wired (copper/fiber)
- The digital receiver should implement the reverse operation compared to the transmitter

# Mapping symbols on signals

- The simplest way to map digital symbols onto an analog signal is to use AMPLITUDE levels
  - EXAMPLE: BINARY MODULATION
    - '1' presence of signal
    - '0' absence of signal

    OR
    - '1' high level
    - '0' low level

# Baseband digital signals: PAM

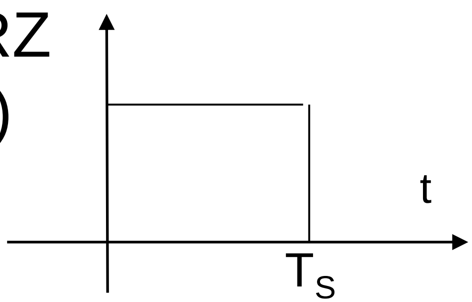
## **PAM: Pulse Amplitude Modulation**

$$x_{TX}(t) = \sum_{k=-\infty}^{+\infty} \alpha_k s(t - kT_S)$$

$\alpha_k$ are the symbols transmitted (real valued)
s(t) is the pulse shape

EXAMPLES

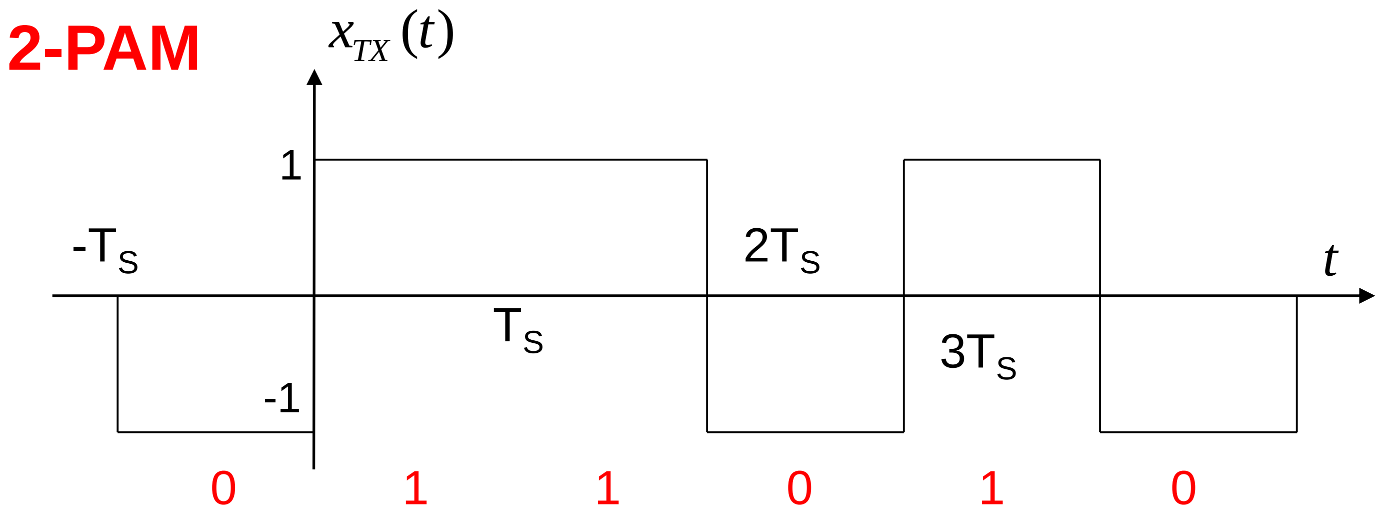
NRZ
s(t)




Binary
$\alpha_k$={0,1}
$\alpha_k$={-1,1}

Unipolar vs. Antipodal

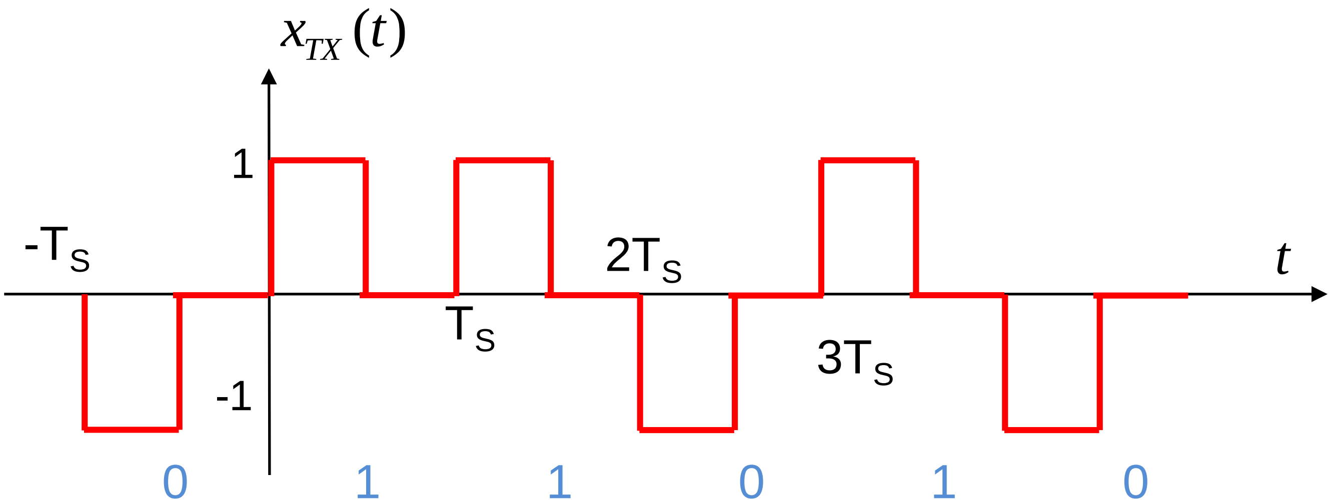
# Examples: 2-PAM + NRZ pulses



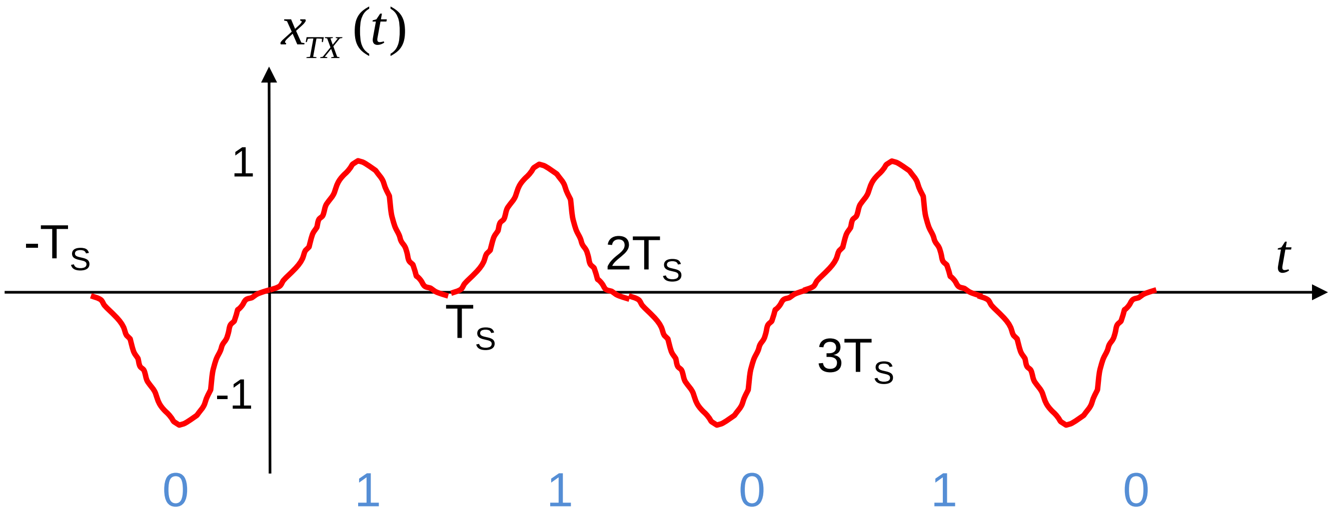

ANTIPODAL: it generates a zero average signal

# Examples: 2-PAM + RZ pulses

**RZ-Rectangular**




**RZ-Gaussian**



# Examples: 4-PAM

**4-PAM**

$$x_{TX}(t) = \sum_{k=-\infty}^{+\infty} \alpha_k s(t - kT_S)$$

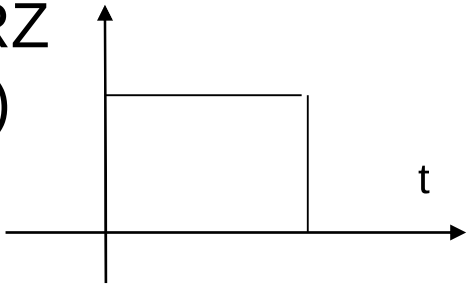
Multilevel: 4 levels

$\alpha_k$={0,1,2,3}

$\alpha_k$={-3,-1,1,+3}

Unipolar vs. Antipodal

NRZ
s(t)

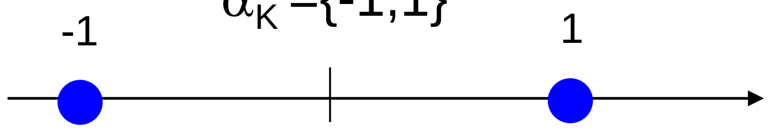


# Baseband digital systems

All baseband signals can carry ONLY monodimensional constellations

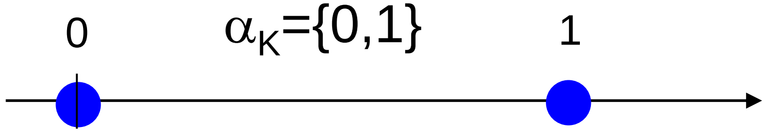
2-PAM ANTIPODAL

$\alpha_K = \{-1, 1\}$



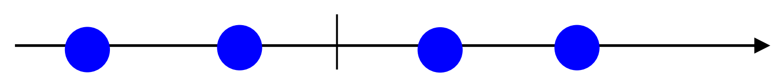

2-PAM UNIPOLAR

$\alpha_K = \{0, 1\}$



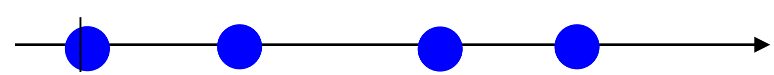

4-PAM ANTIPODAL

$\alpha_K = \{-3, -1, +1, +3\}$



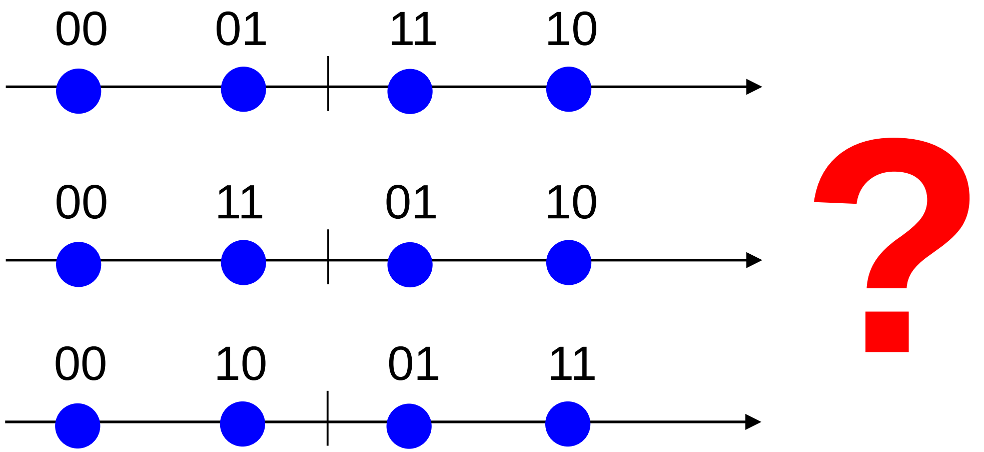
4-PAM UNIPOLAR

$\alpha_K = \{0, +1, +3, +5\}$



# Bit mapping / Labelling

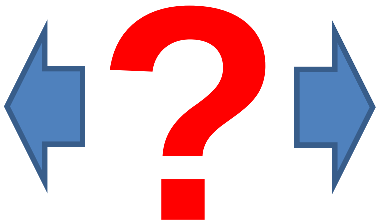
- For multilevel modulation there is one further degree of freedom, beside the constellation choice: bit mapping / labelling



# Examples: 4-PAM and Bit mapping

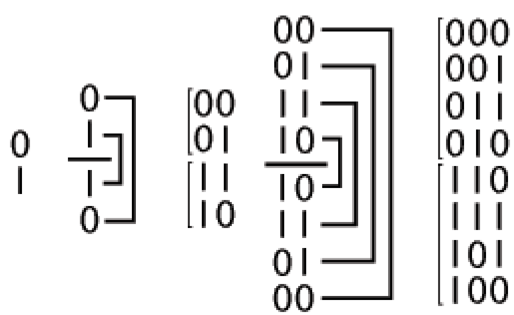
- Bit mapping
  - It's a degree of freedom in the design of systems
  - It has an impact on systems performances
  - How to map symbols to levels?

| Levels | Symbol |
|---|---|
| -3 | 00 |
| -1 | 01 |
| +1 | 10 |
| +3 | 11 |



| Levels | Symbol |
|---|---|
| -3 | 00 |
| -1 | 01 |
| +1 | 11 |
| +3 | 10 |

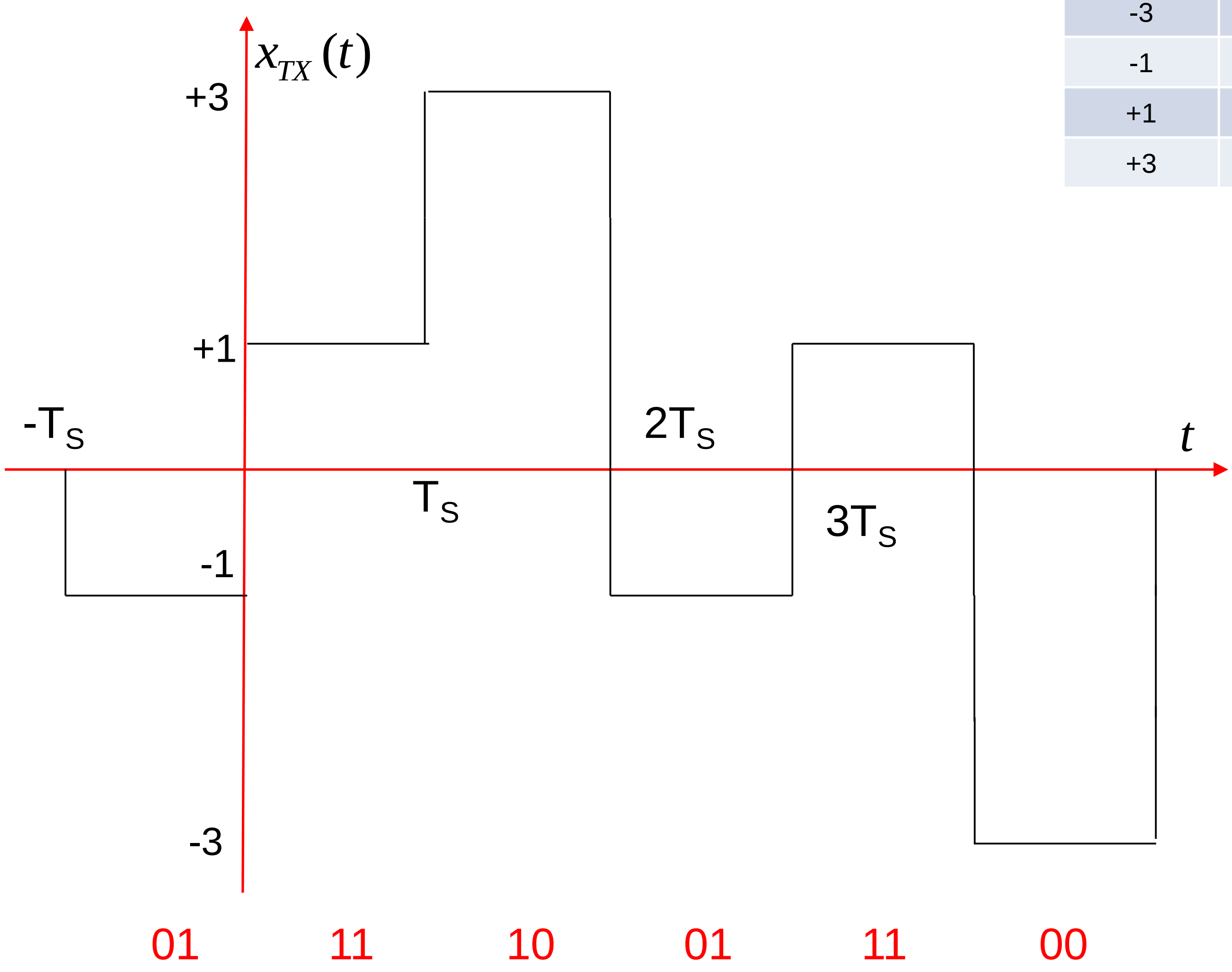
GRAY CODING



# Examples: 4-PAM + Gray Coding + NRZ pulses

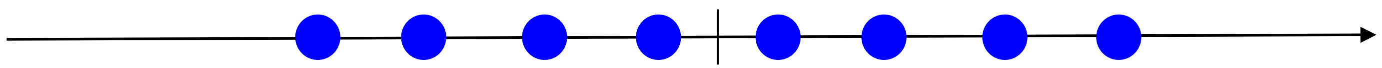
## 4-PAM

| Levels | Symbol |
|---|---|
| -3 | 00 |
| -1 | 01 |
| +1 | 11 |
| +3 | 10 |

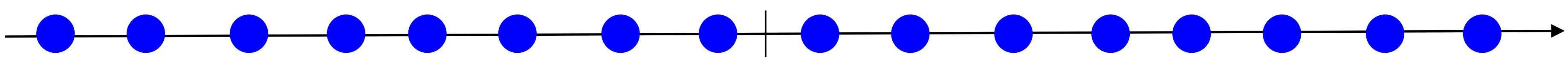


# M-PAM

**8-PAM**



**16-PAM**



Which is the advantage to move toward higher orders?

$$R_s = R_b / BpS \quad \text{[symbol/s=baud]}$$

Symbol rate is reduced, so…

# Spectral evaluation

Considering a signal carrying digital information as $x_{TX}(t) = \sum_{k=-\infty}^{+\infty} \alpha_k s(t - kT_S)$

where s(t) is the pulse employed and $\alpha_k$ are the symbols transmitted

Applying properly Fourier transform to it, we can evaluate the PSD of $x_{TX}(t)$:

$$G_x(f) = D\left|S(f)\right|^2 \sigma_\alpha^2 + (m_\alpha D)^2 \sum_{n=-\infty}^{+\infty} \left|S(nD)\right|^2 \delta\left(f - nD\right)$$

where:

- D is the symbol rate ($R_S$)
- $m_\alpha = E\left[\alpha_n\right]$
- $\sigma_\alpha^2 = E\left[(\alpha_n - m_\alpha)^2\right]$

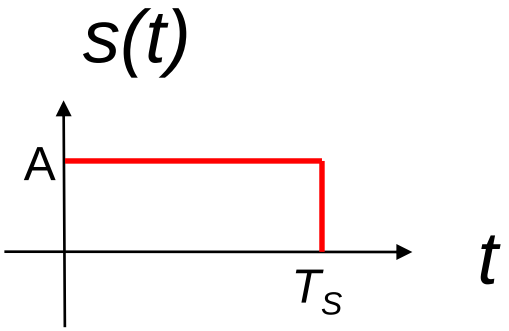
# Spectrum examples: 2-PAM

- NRZ antipodal

$\alpha_k = \{-1, +1\}$ equiprobable

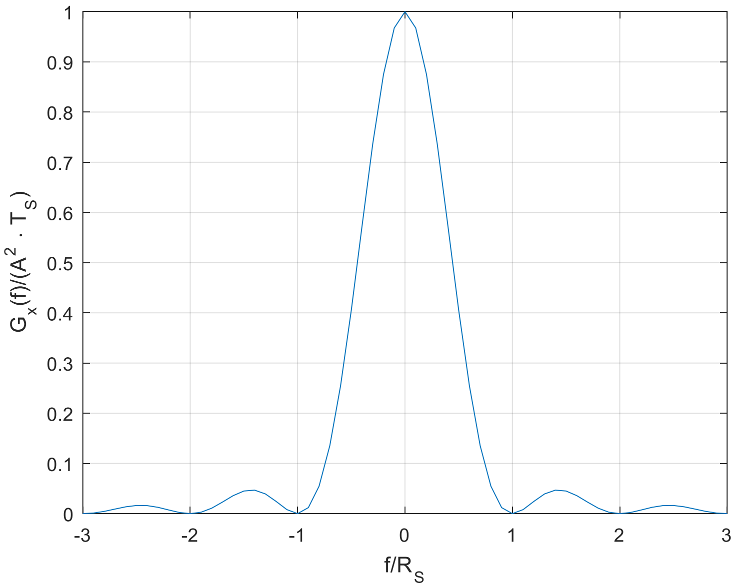
$m_\alpha = 0$

$\sigma_\alpha^2 = 1$




$$|S(f)|^2 = A^2 \left( \frac{\sin(\pi f T_S)}{\pi f} \right)^2 = A^2 T_S^2 Sinc^2(fT_S)$$

$$G_x(f) = D|S(f)|^2 = A^2 T_S Sinc^2(fT_S)$$

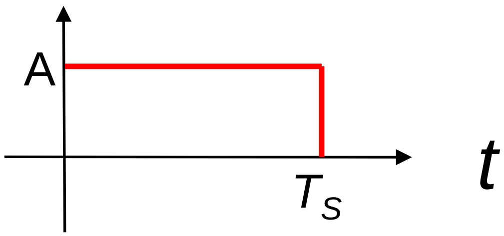


# Spectrum examples – 2-PAM

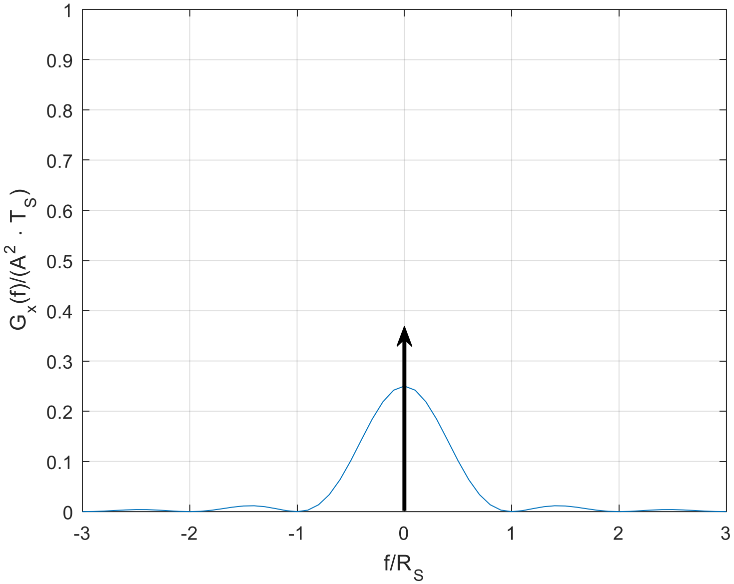
- NRZ unipolar

$\alpha_k = \{0,+1\}$ equiprobable

$m_\alpha = 0.5 \qquad \sigma_\alpha^2 = 0.25$




$$|S(f)|^2 = A^2\left(\frac{\sin(\pi f T_S)}{\pi f}\right)^2 = A^2 T_S^2 Sinc^2(fT_S)$$

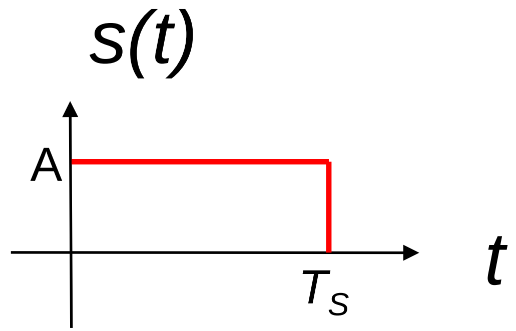



$$G_x(f) = D|S(f)|^2\sigma_\alpha^2 + (m_\alpha D)^2 \sum_{n=-\infty}^{+\infty} |S(nD)|^2 \delta(f-nD) =$$

$$= \frac{A^2 T_S}{4} Sinc^2(fT_S) + \frac{A^2}{4} \sum_{n=-\infty}^{+\infty} Sinc^2(n)\,\delta(f-nD) =$$

$$= \frac{A^2 T_S}{4} Sinc^2(fT_S) + \frac{A^2}{4}\,\delta(f)$$

# Spectrum examples: M-PAM




- NRZ antipodal

$\alpha_k$ ={-3,-1,+1,+3} equiprobable

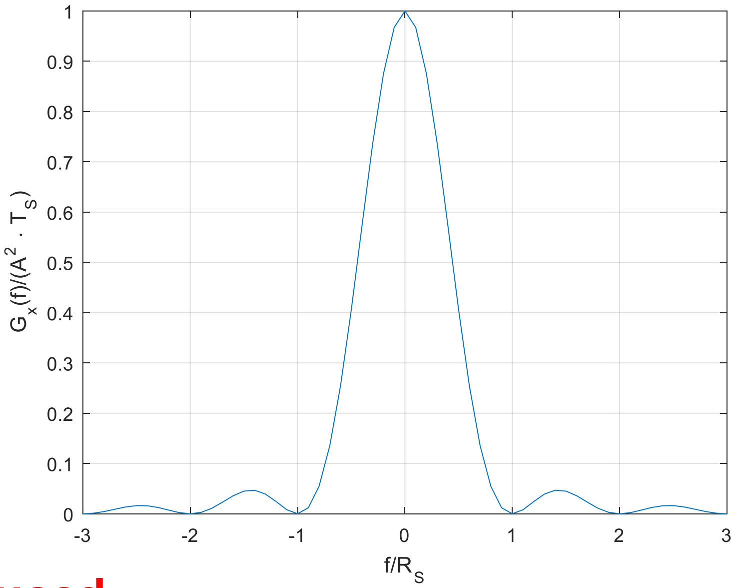
$$m_\alpha = 0$$

$$\sigma_\alpha^2 = 1$$

$$|S(f)|^2 = A^2 \left( \frac{\sin(\pi f T_S)}{\pi f} \right)^2 = A^2 T_S^2 Sinc^2(f T_S)$$

$$G_x(f) = D|S(f)|^2 = A^2 T_S Sinc^2(f T_S)$$

$T_S = log_2(M) \cdot T_b$ $\qquad$ $R_S = Rb/log2(M)$




**Comparing with 2-PAM at same $R_b$, $R_S$ is reduced**

# Spectral efficiency

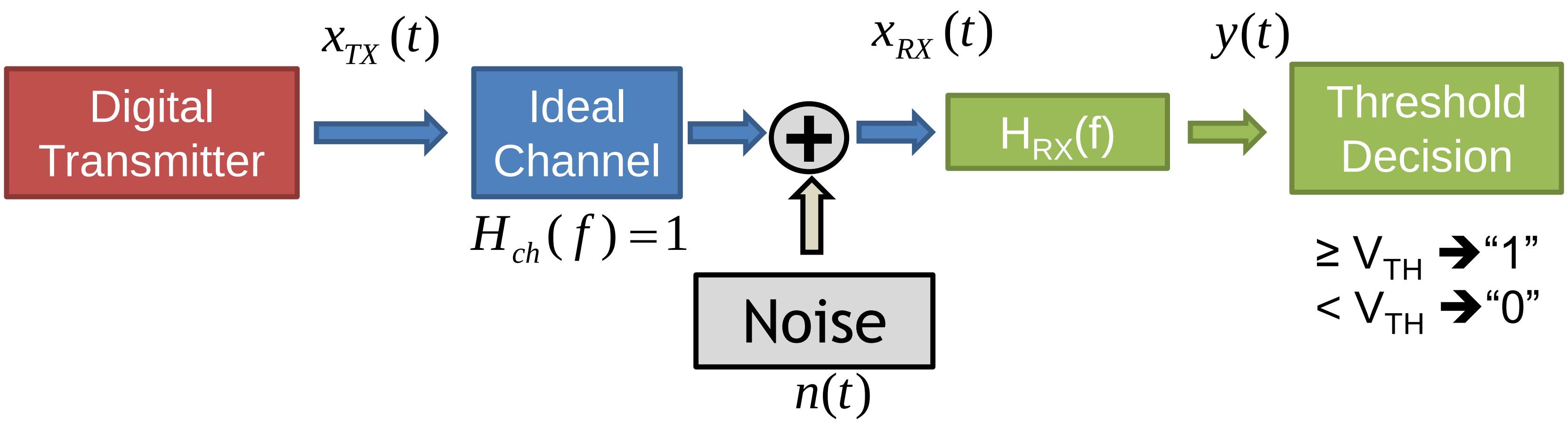
DEFINITION $$\eta = \frac{R_b}{W} \quad \left[\frac{bit}{s \cdot Hz}\right]$$

- W is the bandwidth occupied
  - Let's define the bandwidth to the first notch

- 2-PAM: $W=R_S=R_b$ $\quad \eta=1$ [bit/s/Hz]
- 4-PAM: $W=R_S=R_b/2$ $\quad \eta=2$ [bit/s/Hz]
- 8-PAM: $W=R_S=R_b/3$ $\quad \eta=3$ [bit/s/Hz]
- 16-PAM: $W=R_S=R_b/4$ $\quad \eta=4$ [bit/s/Hz]

# BER evaluation for 2-PAM

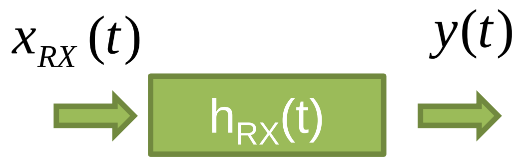
- Hypothesis:
  - Binary isolated bits
  - Additive White Gaussian Noise
  - Ideal channel
  - Receiver based on a filter (to limit noise) and a threshold decision




Isolated bits

$$x_{RX}(t) \begin{cases} \alpha_1 s(t) + n(t) \\ \alpha_0 s(t) + n(t) \end{cases}$$

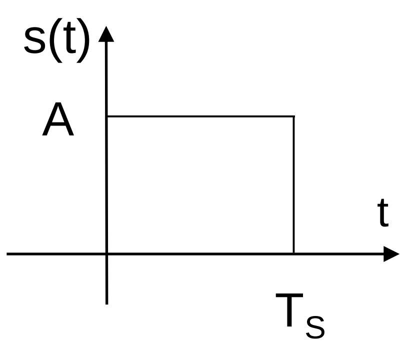
# Receiver filtering

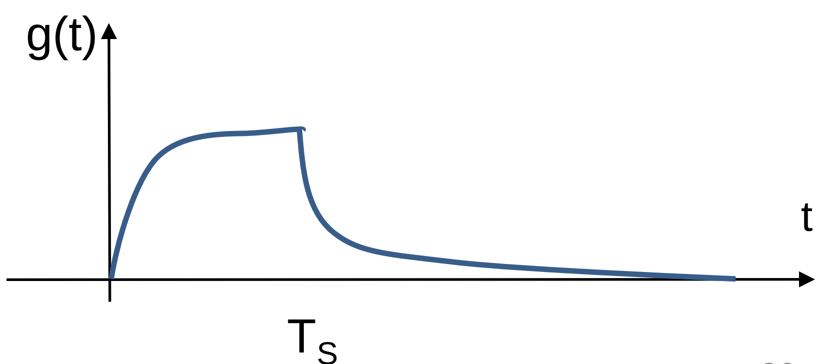



$$y(t) = x_{RX}(t) * h_{RX}(t) = \underbrace{\alpha_i (s(t) * h_{RX}(t))}_{g(t)} + \underbrace{(n(t) * h_{RX}(t))}_{n_F(t)}$$

$$y(t) = g(t) + n_F(t)$$

EXAMPLE: single pole filter (RC filter)



# Optimum sampling instant

$$y(t_0) = g(t_0) + n_F(t_0)$$

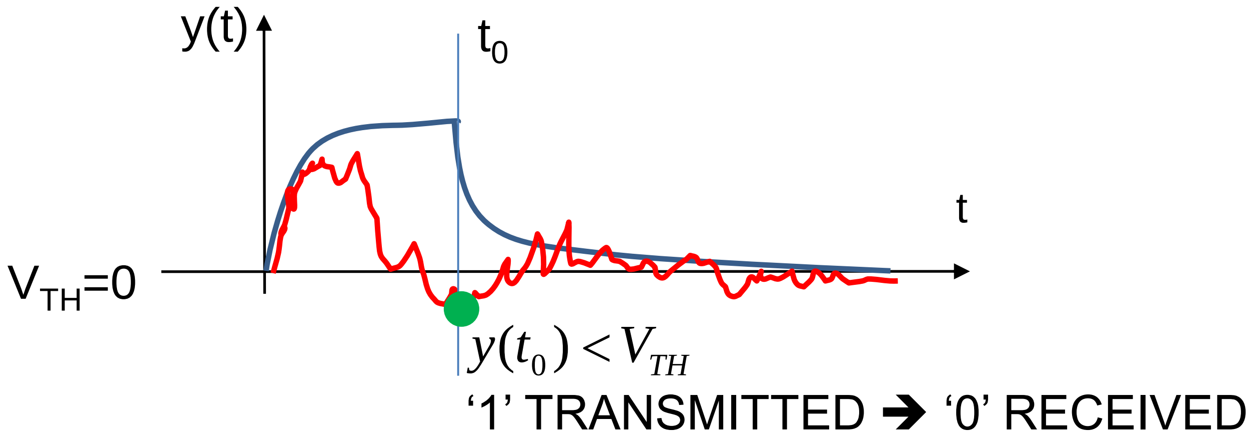
- We want to maximize $g(t_0)$
- Selecting t, we obtain a random variable

$$y_0 = \alpha_i g_0 + n_{F0}$$

- Where:
  - $\alpha_i$ is a binary random variable
  - $g_0$ is the signal, a deterministic value
  - $n_{F0}$ is a random variable with zero mean ($E[n_{F0}]=0$ ) and variance

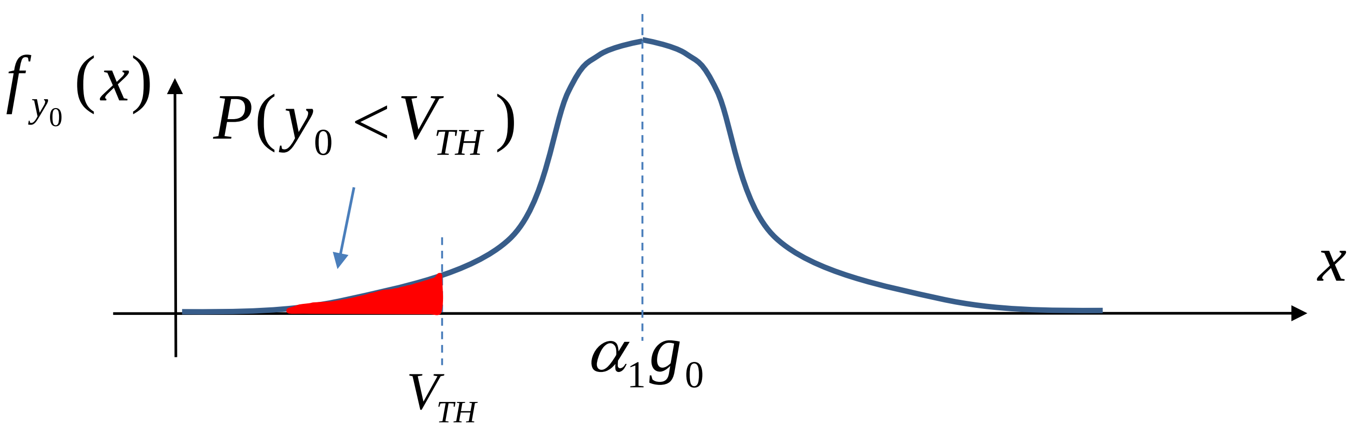
$$\sigma^2_{n_{F0}} = \frac{N_0}{2} \int_{-\infty}^{+\infty} \left| H_{RX}(f) \right|^2 df$$

# Bit errors




$$P(e|1TX) = P(y_0 < V_{TH}|1TX) = P(\alpha_1 g_0 + n_{F0} < V_{TH})$$

- Being $\alpha_1 g_0 + n_{F0}$ a Gaussian random variable with mean $\alpha_1 g_0$



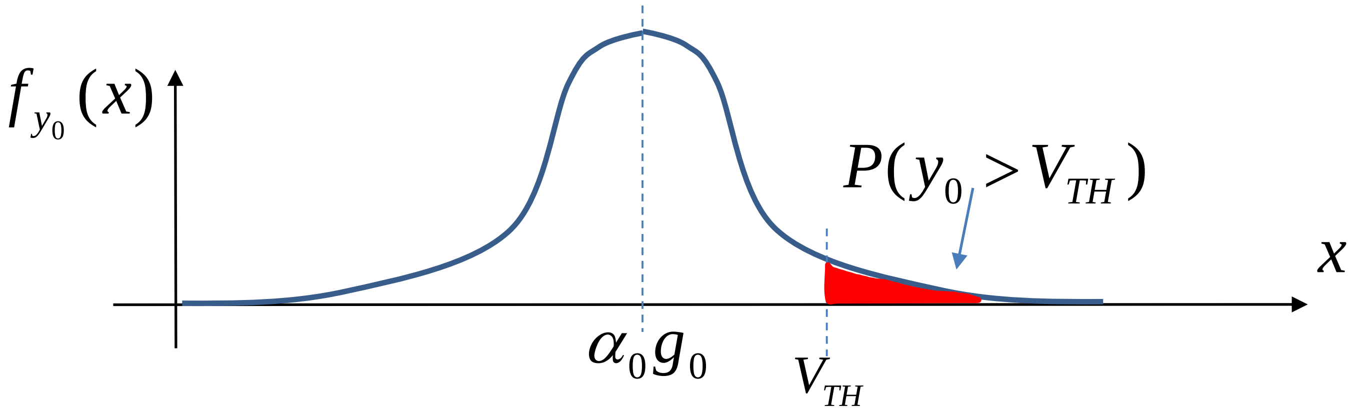
# BER

- We obtain

$$P(e|1TX) = \frac{1}{2} erfc\left( \frac{\alpha_1 g_0 - V_{TH}}{\sqrt{2}\sigma_{n_{F0}}} \right)$$

- And similarly

$$P(e|0TX) = P(y_0 > V_{TH}|0TX) = P(\alpha_0 g_0 + n_{F0} > V_{TH})$$




$$P(e|0TX) = \frac{1}{2} erfc\left( \frac{V_{TH} - \alpha_0 g_0}{\sqrt{2}\sigma_{n_{F0}}} \right)$$

# BER

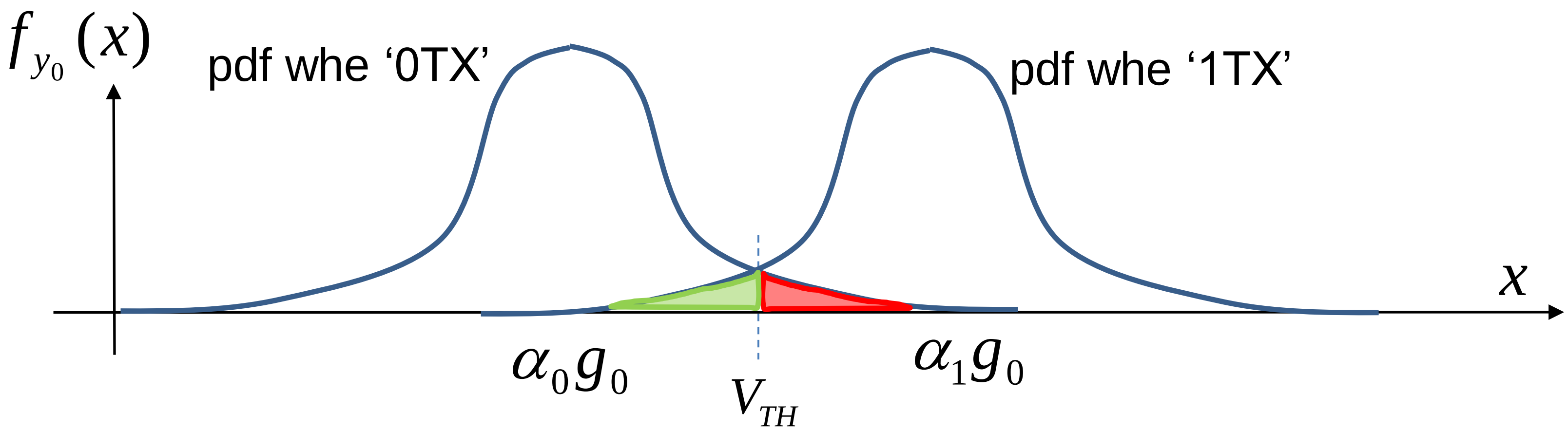
- Assuming equiprobable symbols $P(1TX) = P(0TX) = \frac{1}{2}$

- We get

$$BER = \frac{1}{4} erfc\left( \frac{\alpha_1 g_0 - V_{TH}}{\sqrt{2}\sigma_{n_{F0}}} \right) + \frac{1}{4} erfc\left( \frac{V_{TH} - \alpha_0 g_0}{\sqrt{2}\sigma_{n_{F0}}} \right)$$

- This formula is valid for: binary systems, mono-dimensional s(t), isolated bits, AWGN, generic $H_{RX}(f)$

# Threshold optimization




- Threshold must be between $\alpha_0 g_0$ and $\alpha_1 g_0$

- Optimization gives the obvious solution: it's exactly in the middle

$$V_{TH} = \frac{\alpha_0 + \alpha_1}{2} g_0$$

- It minimizes BER and at same time gives $P(e|1TX) = P(e|0TX)$

**It is a BINARY SYMMETRIC CHANNEL!**

# BER formula

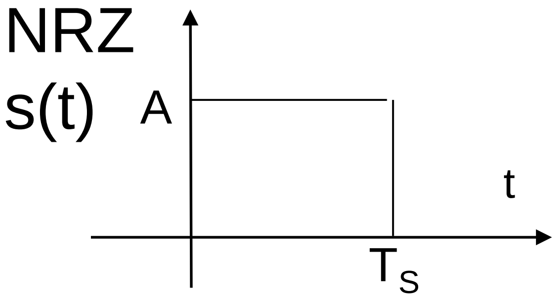
$$BER = \frac{1}{2} erfc\left( \frac{(\alpha_1 - \alpha_0) g_0}{2\sqrt{2}\sigma_{n_{F0}}} \right)$$

- Conditions: generic filter, generic s(t), generic alfa's

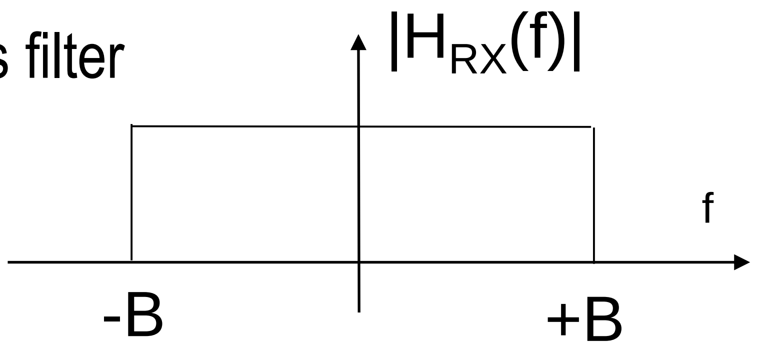
- COMMENTS
- Being erfc a monotonic decreasing function, we want to maximize its argument
- The formula depends on:
  - The transmitted elementary signal s(t), hidden in $g_0$
  - The filter $H_{RX}(f)$ which determines $g_0$ and $\sigma_{nF}$
  - The two levels $\alpha_0$ and $\alpha_1$

# FIRST CASE - I

- 2-PAM antipodal symbols: $\alpha_k=\{-1,1\}$
- Rectangular s(t), NRZ



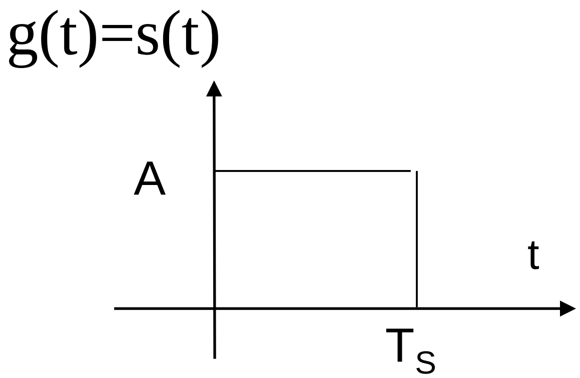

- Ideal low pass filter




$B=R_B=1/T_B$

# FIRST CASE - II

**unfiltered**




$$g(t) = s(t) * h_{RX}(t) \cong s(t)$$

$$g_0 = g(t_0) = A$$

$$\alpha_1 - \alpha_0 = 2$$

$$n(t) \Rightarrow \boxed{h_{RX}(t)} \Rightarrow n_F(t)$$

$$\sigma_{n_F}^2 = N_0 R_B$$

$$BER = \frac{1}{2} erfc\left( \frac{(\alpha_1 - \alpha_0) g_0}{2\sqrt{2}\sigma_{n_{F0}}} \right) = \frac{1}{2} erfc\left( \frac{2A}{2\sqrt{2 N_0 R_B}} \right) = \frac{1}{2} erfc\left( \sqrt{\frac{A^2 T_B}{2 N_0}} \right)$$

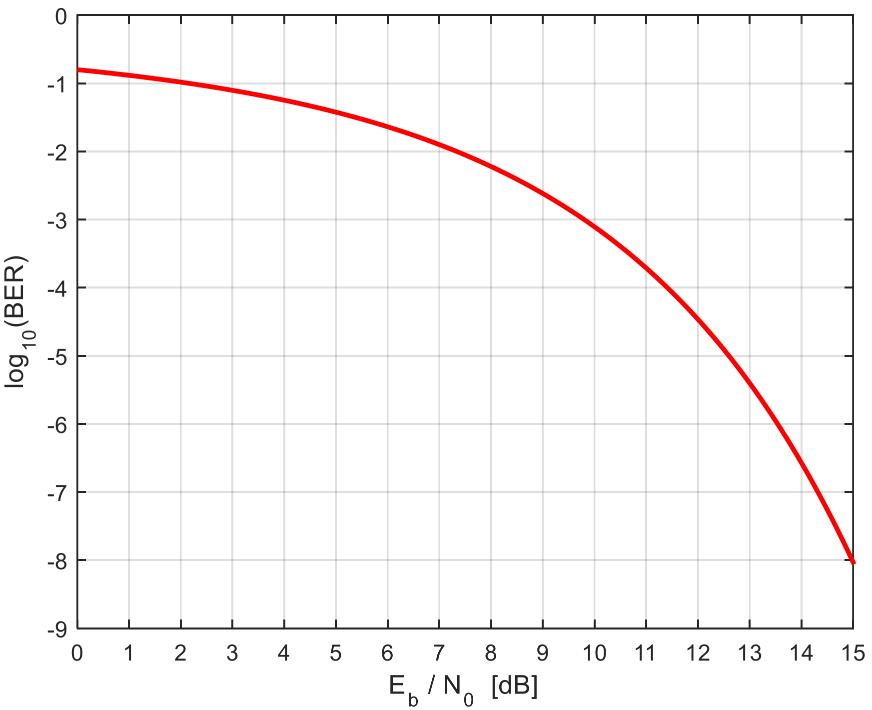
# FIRST CASE - III

$$BER = \frac{1}{2} erfc\left(\sqrt{\frac{A^2 T_B}{2N_0}}\right)$$

$$E_S = \int_{-\infty}^{+\infty} s^2(t)dt = \int_0^t s^2(t)dt = A^2 T_B$$

$$E_b = \frac{E_{b1} + E_{b0}}{2} = \frac{\alpha_1^2 E_S + \alpha_0^2 E_S}{2} = E_S = A^2 T_B$$

$$BER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$



# $E_b/N_0$

- What's $E_b/N_0$?

$$\frac{E_b}{N_0} = \frac{P_{RX} \cdot T_B}{N_0} = \frac{P_{RX}}{N_0 R_B} = \frac{P_{RX}}{P_N} = SNR$$

- SNR over a band equal to $R_B$

- COMMENTS

  For a given BER that require a given $E_b/N_0$, increasing $R_B$ requires higher $P_{RX}$ for the same $N_0$

# OPTIMUM FILTER: THE MATCHED FILTER

$$BER = \frac{1}{2} erfc\left( \frac{(\alpha_1 - \alpha_0)}{2\sqrt{2}} \sqrt{\frac{g_0^2}{\sigma_{n_{F0}}^2}} \right)$$

- Goal: maximize $\frac{g_0^2}{\sigma_{n_{F0}}^2}$ properly selecting $H_{RX}(f)$ given $s(t)$

MATCHED FILTER CONDITIONS

$$H_{RX}(f) = k \cdot S^*(f) e^{-j2\pi f t_0}$$

$$h_{RX}(t) = k \cdot s^*(t_0 - t)$$

# THE MATCHED FILTER

$$\frac{g_0^2}{\sigma_{n_{F0}}^2} = \frac{E_S}{N_0}$$

$$BER = \frac{1}{2} erfc\left( \frac{(\alpha_1 - \alpha_0)}{2} \sqrt{\frac{E_S}{N_0}} \right)$$

- 2-PAM antipodal

$$\alpha_1 - \alpha_0 = 2$$

$$E_S = E_b$$

$$BER = \frac{1}{2} erfc\left( \sqrt{\frac{E_b}{N_0}} \right)$$

**INDEPENDENT OF THE SHAPE s(t)**

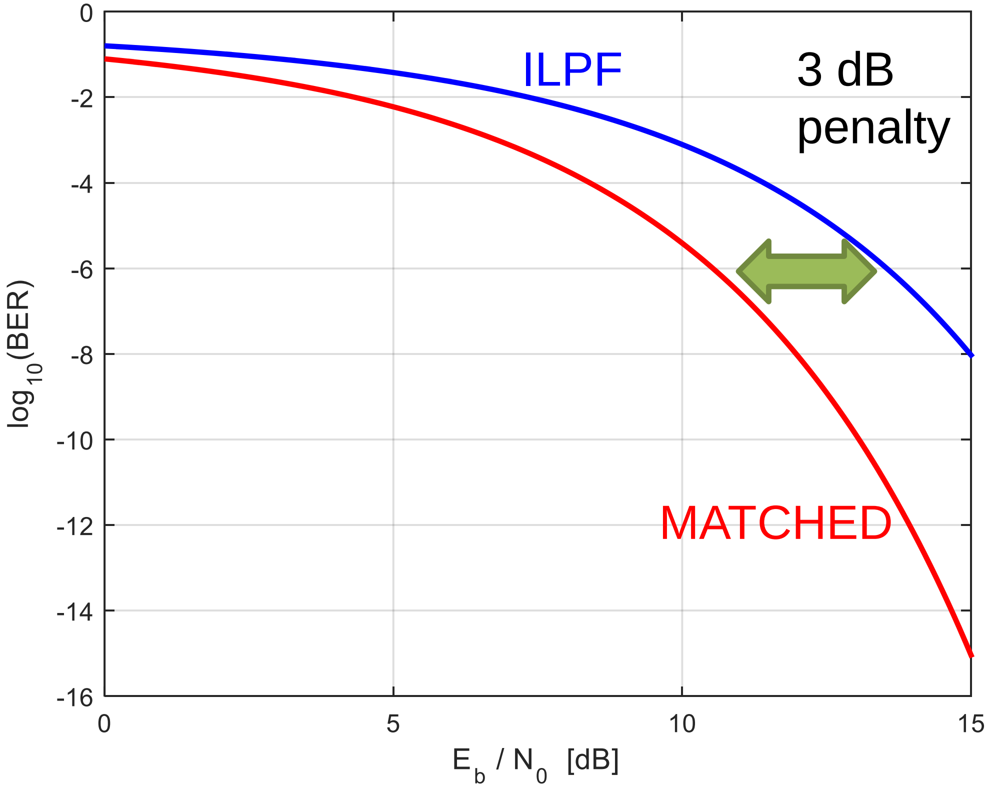
# COMPARING FILTERS

- Matched filter

$$BER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- Ideal low pass filter (ILPF)

$$BER = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$



# M-PAM: BER formulas

- General and EXACT formula for M-PAM

$$BER = \frac{M-1}{M} erfc\left(\sqrt{\frac{3\log_2(M)}{M^2-1}\frac{E_b}{N_0}}\right)$$

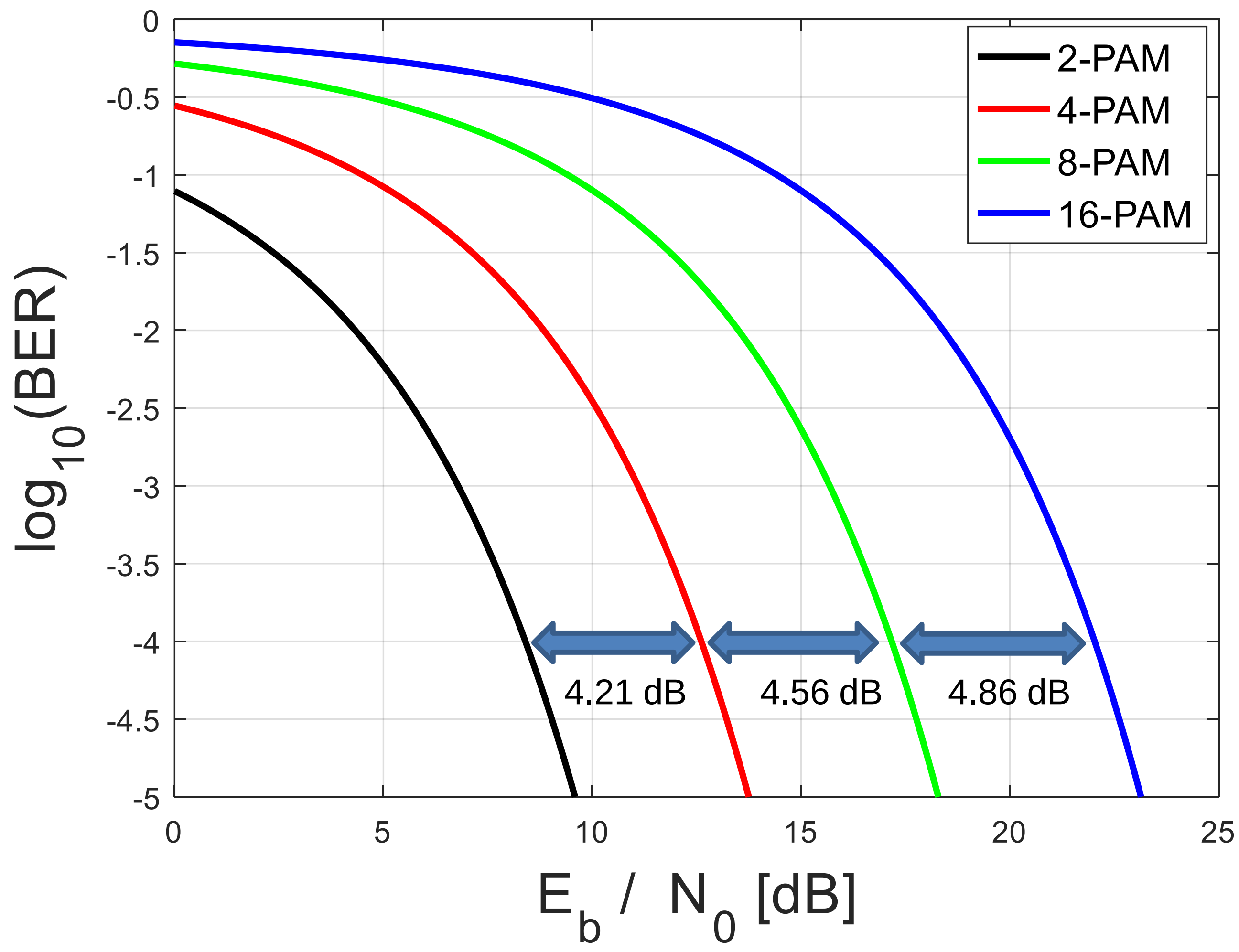
- First factor, for M>4, become independent on M, it tends to 1
- The factor in front of $E_b/N_0$, strongly depends on M

$$\gamma_{M-PAM} = \frac{3\log_2(M)}{M^2-1}$$

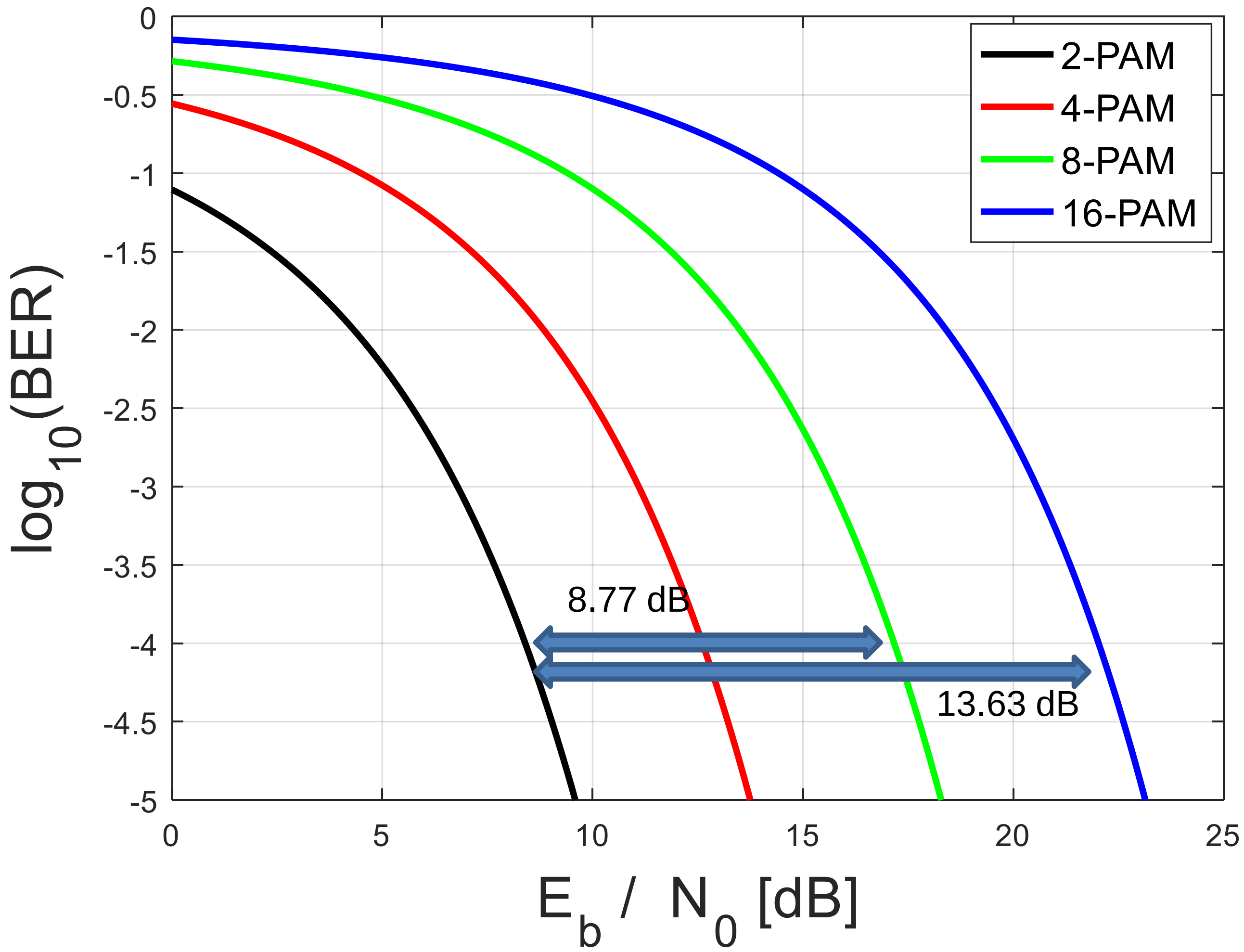
- Considering 2-PAM as a reference, we have:
  - M=4 ➔ $\gamma_{4\text{-PAM}}$=0.400 $\rightarrow$ -3.97 dB
  - M=8 ➔ $\gamma_{8\text{-PAM}}$=0.142 $\rightarrow$ -8.45 dB
  - M=16 ➔ $\gamma_{16\text{-PAM}}$=0.005 $\rightarrow$ -13.27 dB

**Same bit rate**

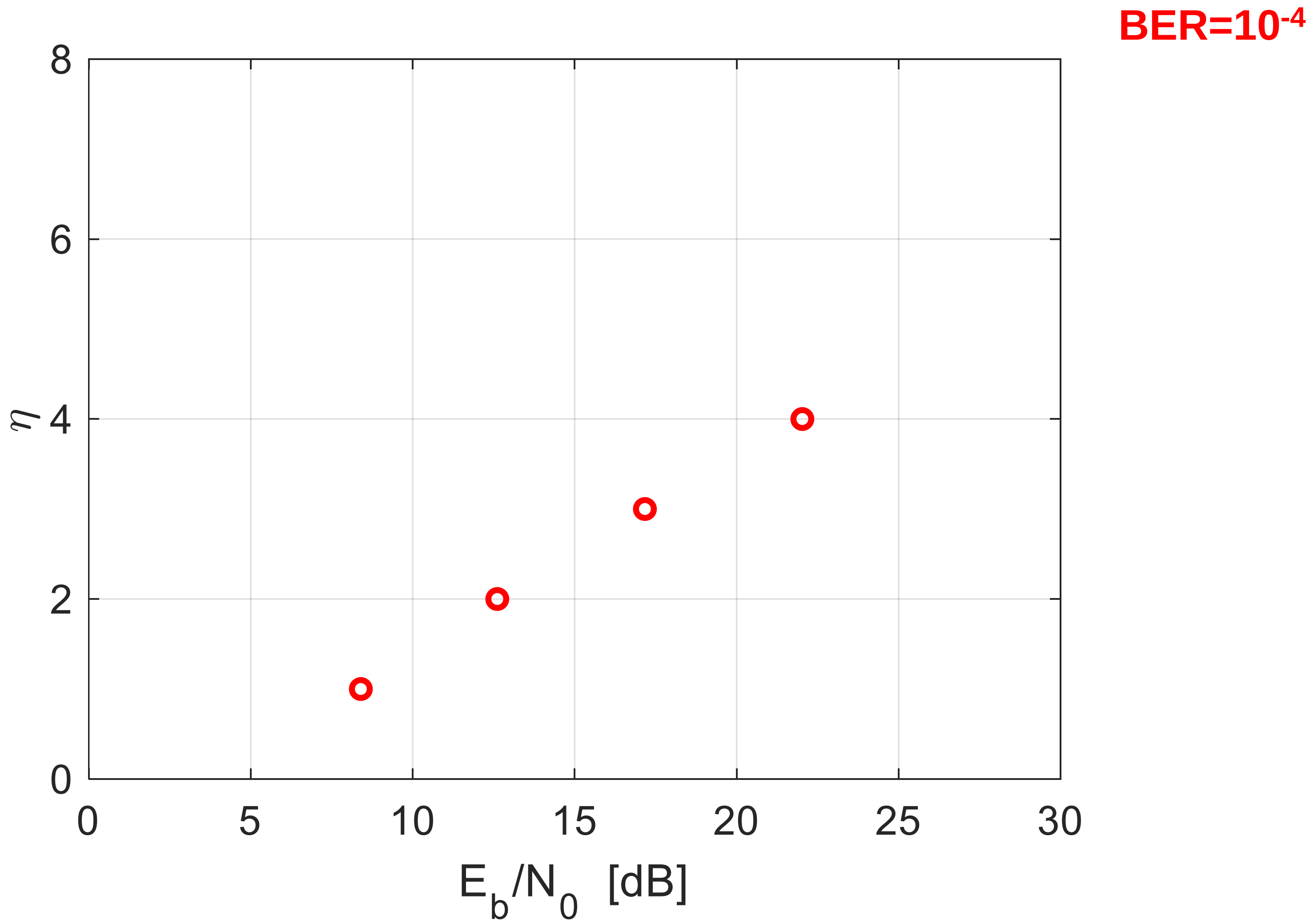
# M-PAM: BER plots



# M-PAM: BER plots



# Spectral efficiency vs. EbN0



# Capacity of a bandlimited channel

- It's the famous Shannon theorem
- It defines the maximum bit-rate that can be sustained over an AWGN channel with arbitrarily low bit-error rate

$$R_b \le W \log_2\left(1+\frac{S}{N}\right)$$

Bound on spectral efficiency

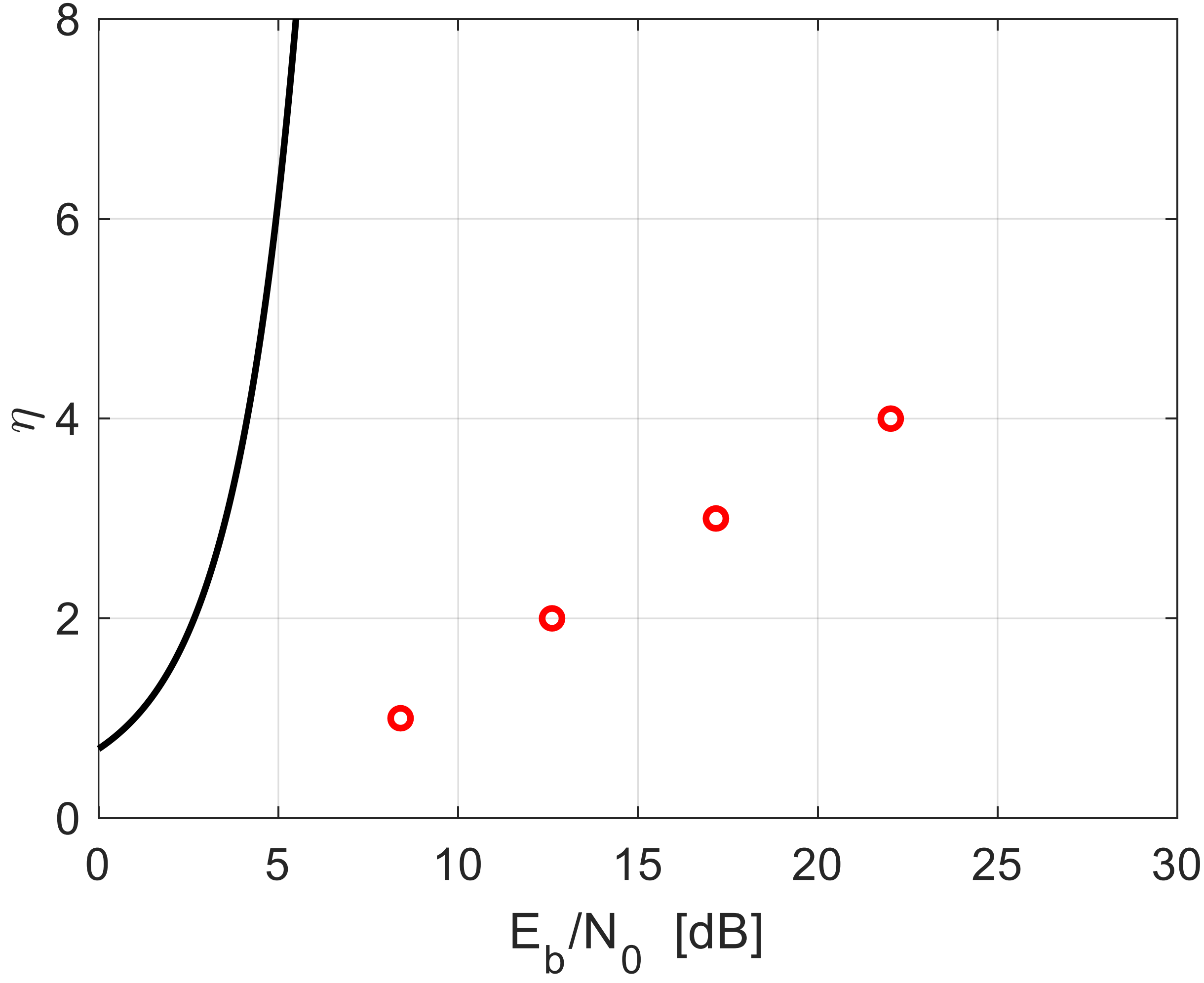
$$\frac{R_b}{W} \le \log_2\left(1+\frac{S}{N}\right)$$

$$\frac{S}{N} = \frac{P}{N_0 W} = \frac{E_b}{N_0}\frac{R_b}{W}$$

$$\frac{R_b}{W} \le \log_2\left(1+\frac{E_b}{N_0}\frac{R_b}{W}\right)$$

# Spectral efficiency vs. EbN0: The Shannon Limit

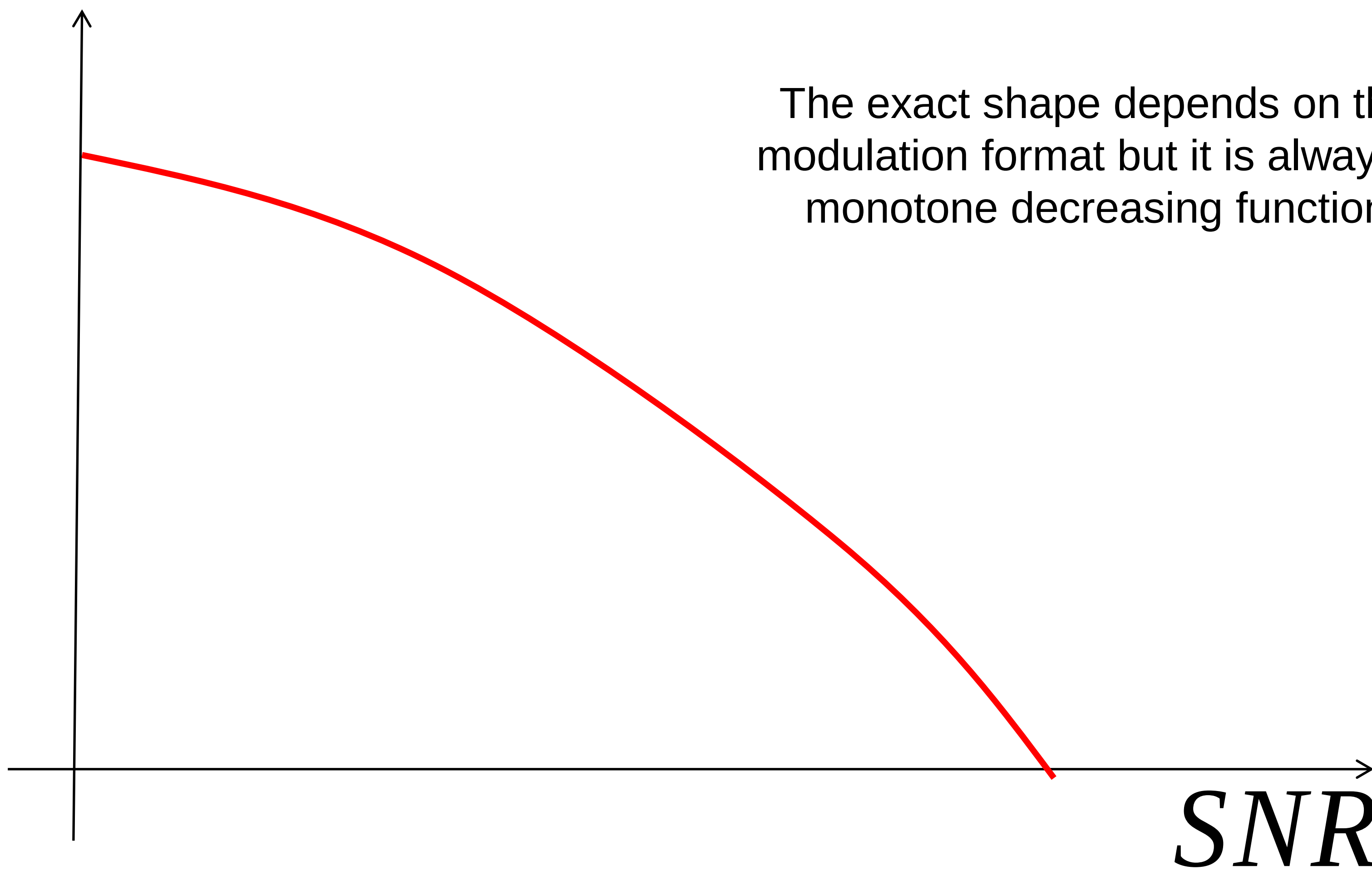
BER=$10^{-4}$



# The sensitivity function

$BER$

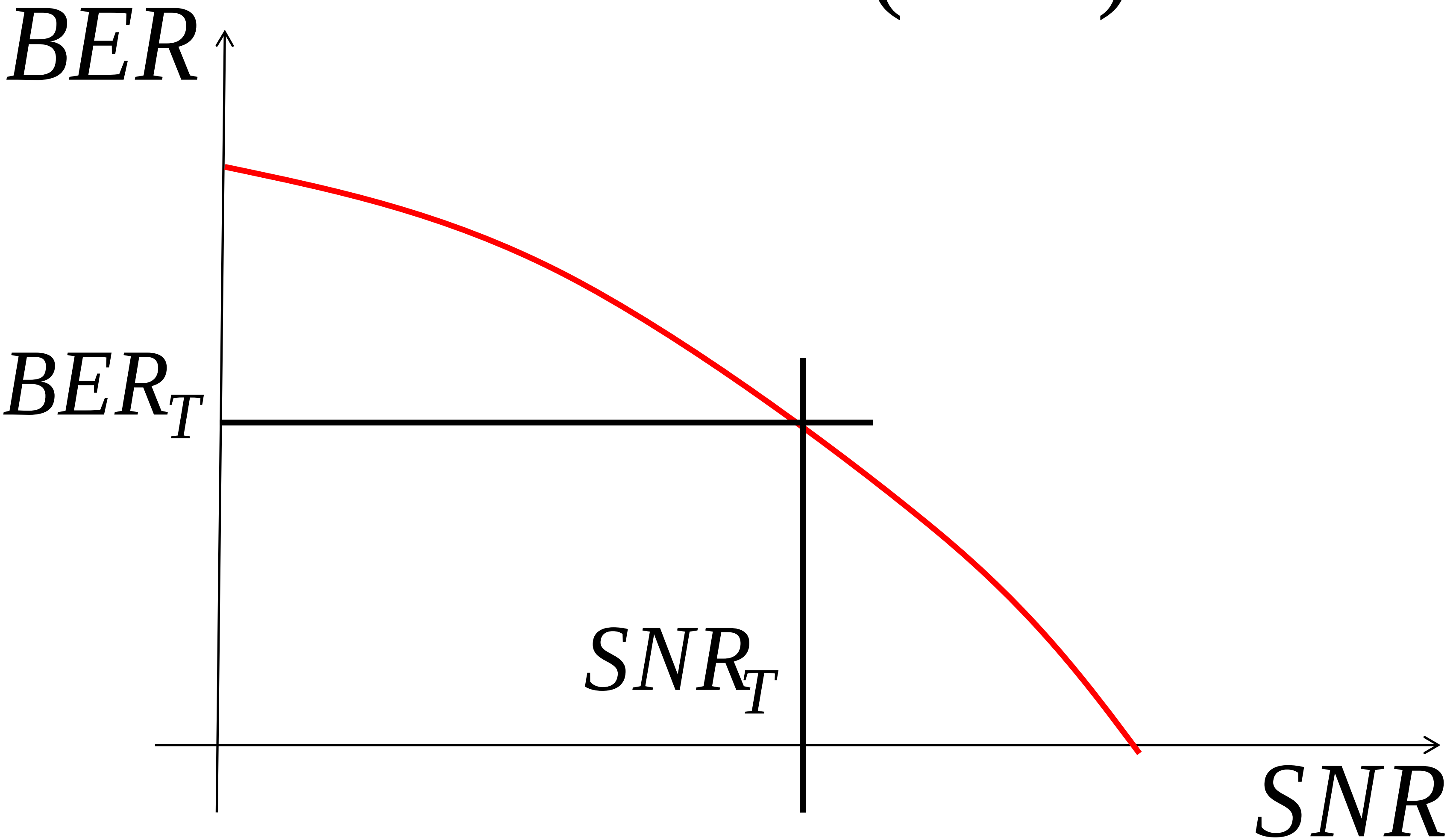
$$BER = \Phi(SNR)$$




The exact shape depends on the modulation format but it is always a monotone decreasing function

# Target SNR

$$BER = \Phi(SNR)$$



# $SNR_T$ and maximum reach

$$SNR_{Rx} = \frac{P_{Rx}}{N_0 B}$$

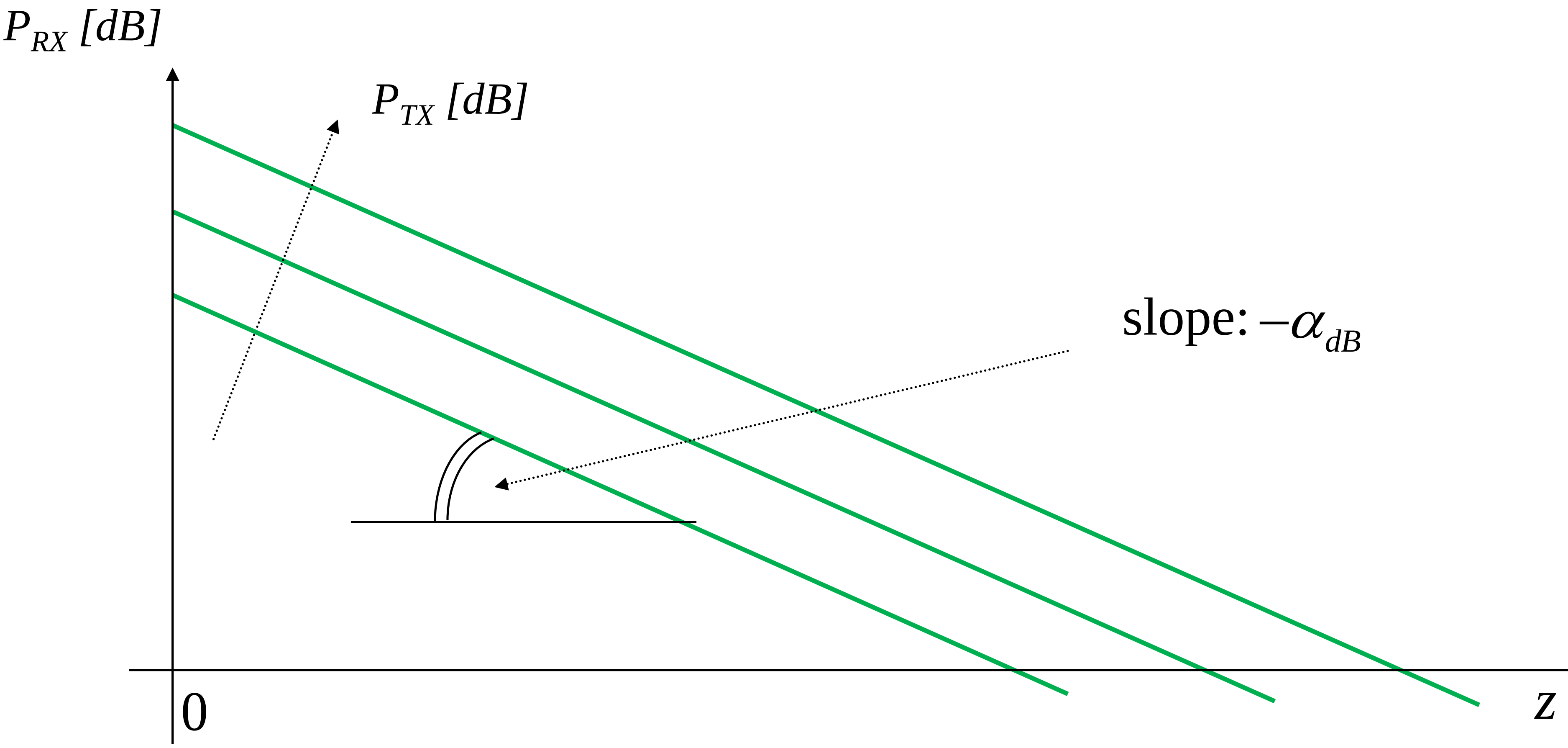
$$SNR_{Rx,dB} = P_{Rx,dB} - 10log_{10}(N_0 B)$$

$$P_{Rx,dB} = P_{Tx,dB} - Loss_{dB}$$

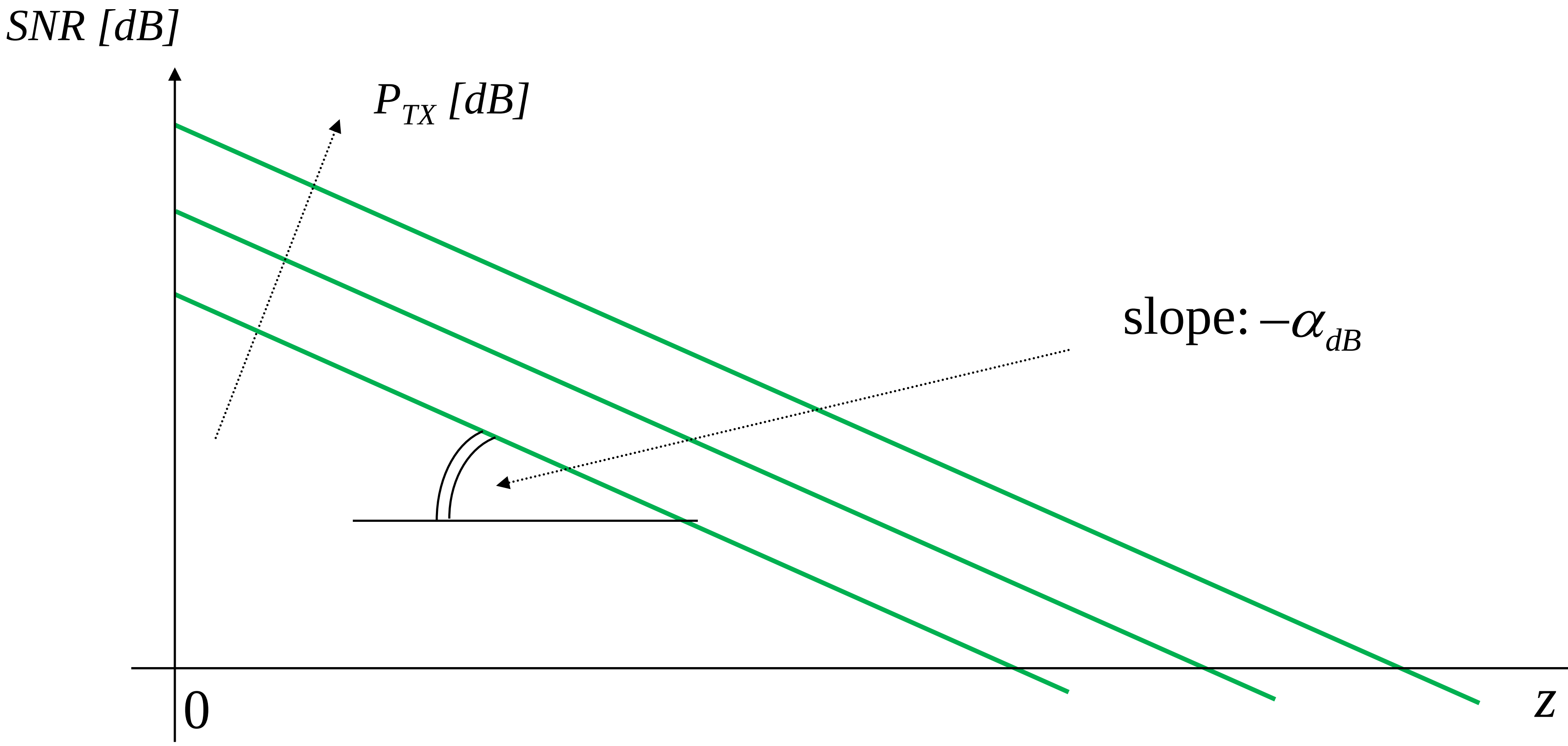
$$Loss_{dB} = \alpha \cdot L \qquad\qquad \alpha \quad \left[\frac{\mathrm{dB}}{\mathrm{km}}\right]$$

$$SNR_{Rx,dB} = P_{Tx,dB} - \alpha L - 10log_{10}(N_0 B)$$

# P vs. L



# SNR vs. L



# mW and dBm $P_{dBm} = 10\log_{10}\left(P_{mW}\right) \quad \left[\mathrm{dBm}\right]$

In order to use losses and gains expressed in dB, typically, power levels are expressed in dBm or dBW

$$P_{dBm} = 10log_{10}(P_{mW}) \;\; [\mathrm{dBm}]$$

$$P_{dBW} = 10log_{10}(P_{W}) \;\; [\mathrm{dBW}]$$

where P is the power expressed in mW / W respectively

$$P_{Rx,mW} = P_{Tx,mW} Loss \quad \left[\mathrm{mW}\right]$$

$$\Downarrow$$

$$P_{Rx,dBm} = P_{Tx,dBm} - Loss_{dB} \quad \left[\mathrm{dBm}\right]$$

# $SNR_T$ and maximum reach

$$SNR_{Rx,dB} \geq SNR_{T,dB}$$

$$P_{Tx,dBm} - \alpha \cdot L - 10\log_{10}(N_0 B) \geq SNR_{T,dB}$$

Given the transmitted power and the target SNR, the maximum reachable distance is

$$L \leq L_{\max} = \frac{P_{Tx,dBm} - SNR_{T,dB} - 10\log_{10}(N_0 B)}{\alpha}$$

# SNR vs. L