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Pulse Code Modulation (PCM)

PULSE CODE MODULATION (PCM)

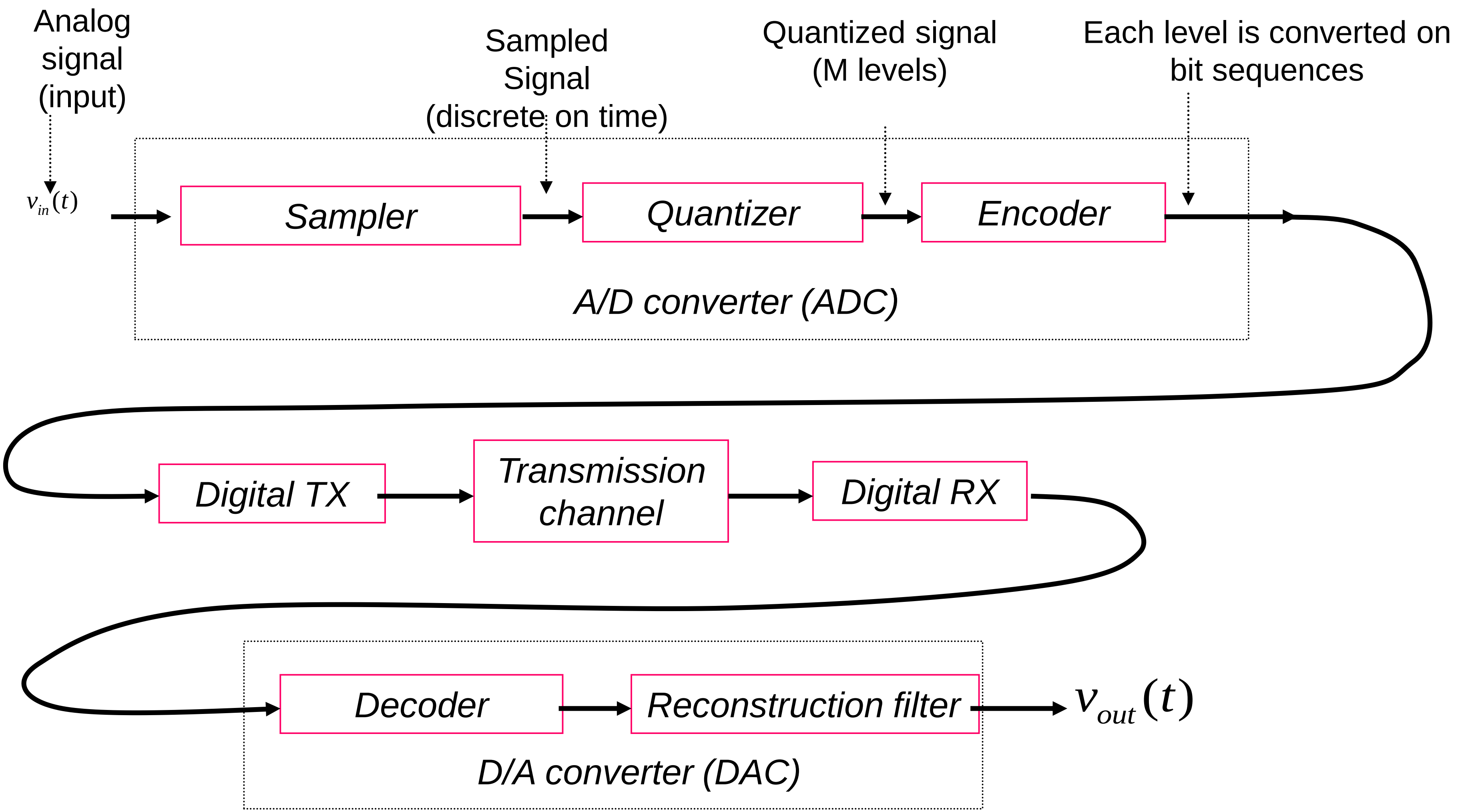
- PCM is a technique used to represent and transmit analog signals with a digital stream
- It is the standard form of digital audio compact discs, digital telephony and other audio applications
- PCM principle: the magnitude of the analog signal is sampled regularly at uniform intervals, and each sample is quantized and encoded in a digital word
- PCM has two basic properties that determine fidelity to the original analog signal: sampling rate and bit n depth (resolution)

PCM

A/D conversion involves the following three operations:

1. **Sampling**
2. **Quantization**
3. **Encoding on bit sequences**

PCM block diagram



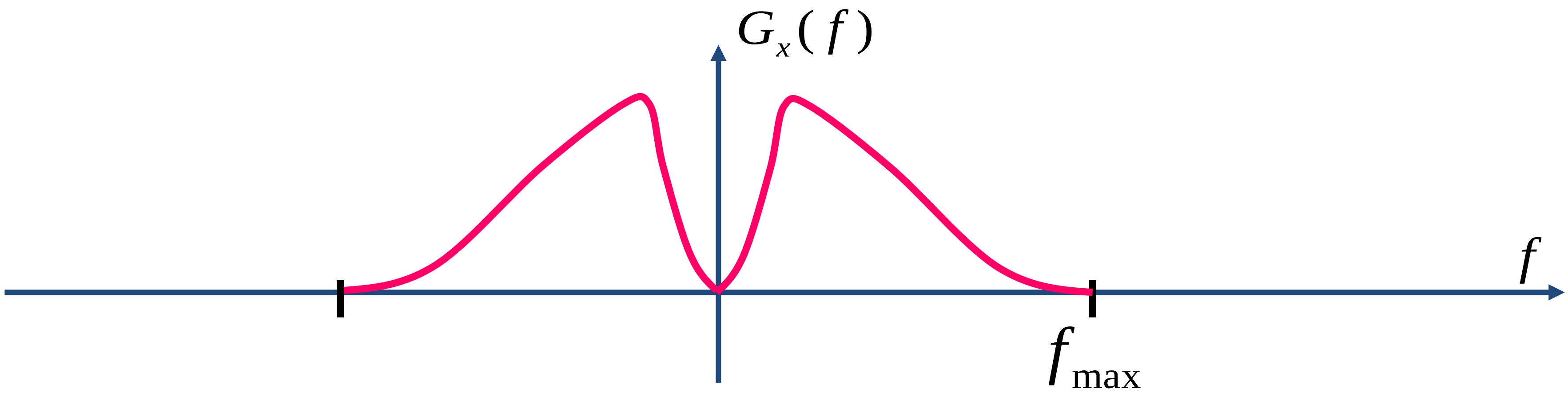
PCM goals

- Transmit in a reliable way an analog signal

- PCM systems are designed to deliver at the output an analog signal $v_{out}(t)$ that is as closer as possible to $v_{in}(t)$

SAMPLING THEOREM

- It is necessary to know the frequency of highest spectral component of the signal



- For a signal with a maximum frequency f_{max}, the sampling frequency must fullfill the following rule:

$$f_c \geq 2 \cdot f_{max}$$

QUANTIZATION

- For each sample, the signal level v_{in} (that can assume any value) must be mapped on finite number of levels

- Sampling is the operation that convert a signal $w(t)$ in a sequence of real numbers $w(i \cdot T_c)$

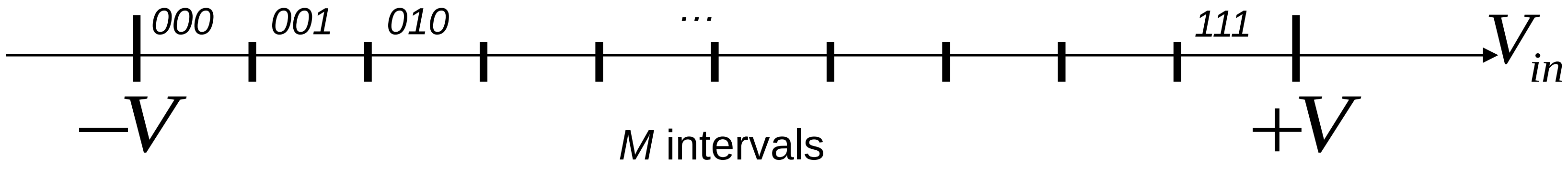
- Quantization is the operation that convert each of the real numbers $w(i \cdot T_c)$ into a discrete and finite number of values

UNIFORM QUANTIZATION

- The simplest (and most common) approach is to use a "uniform quantization"
- Assuming the analog signal to be converted is limited in the interval *[-V,+V]*
- At the ADC, the range *[-V,+V]* is divided in *M* intervals having same amplitude
- Then at each interval is associated a string of bits
  - For practical reason, *M* is always selected as a power of 2

$$M = 2^{n_{bit}}$$

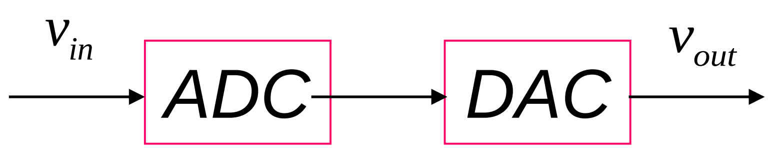
  - n_{bit} is the number of bits needed to "count" in binary the M intervals



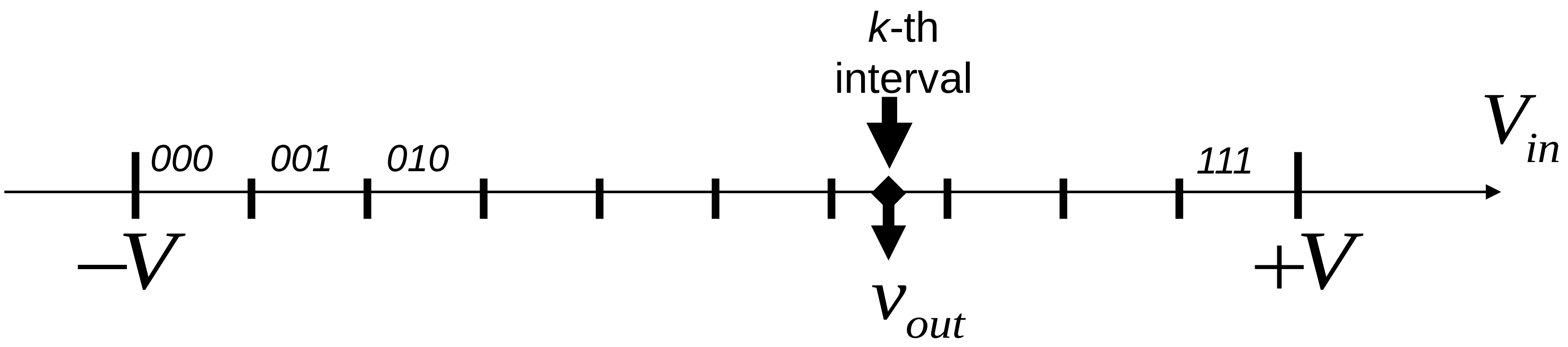
UNIFORM QUANTIZATION

- At the DAC each bit string is used the reconstruct a level (typically a voltage level)
- For reason that will be clear in the following, each bit string is always used to map the level of the center point of the interval in the ADC

- Let’s now consider the effect of the two cascaded processes ADC+DAC, assuming an ideal transmission system in between



UNIFORM QUANTIZATION

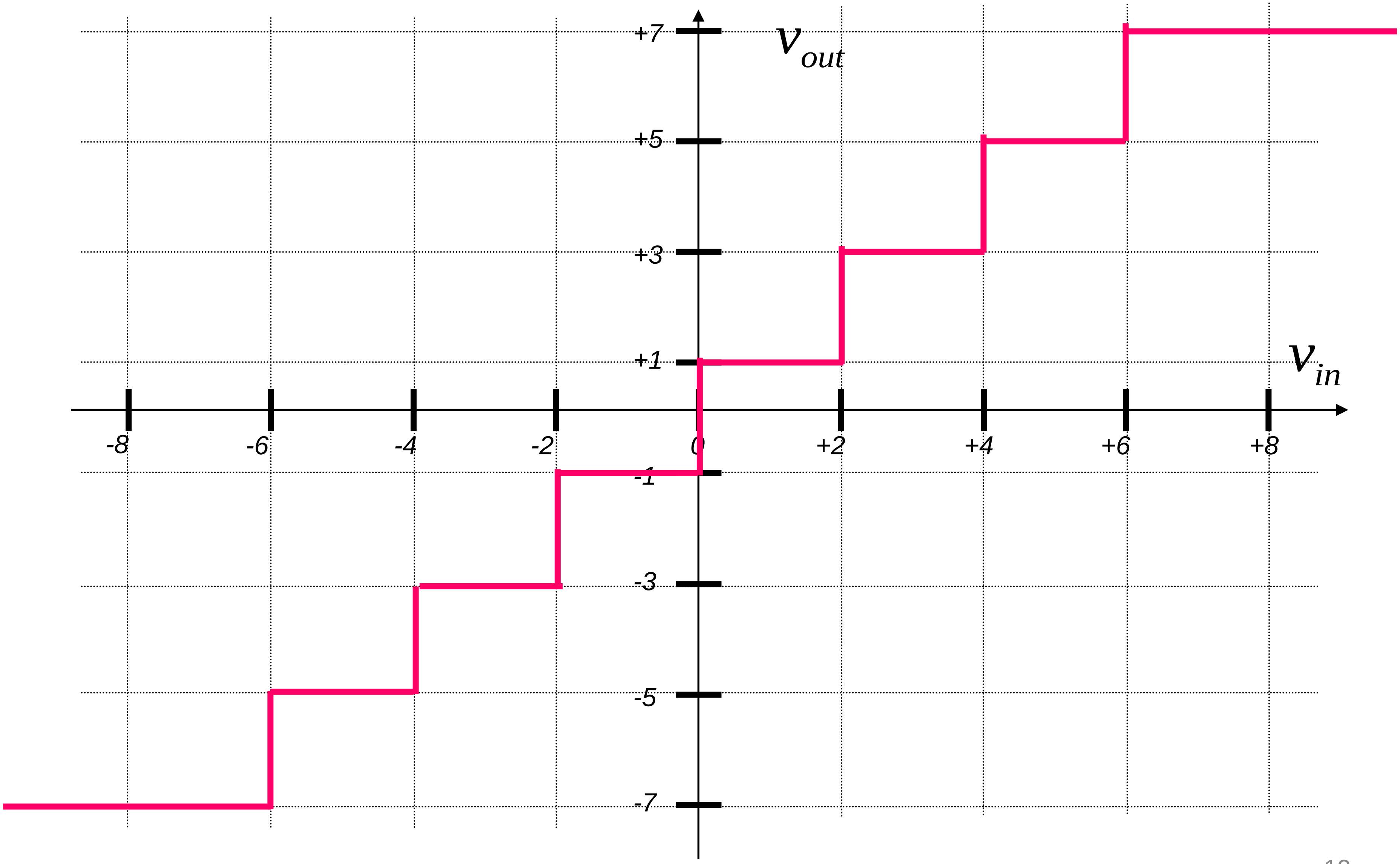


Joint effect of ADC and DAC

- The cascade of and ADC and a DAC is equivalent to a system (non-linear, without memory) operating the input-output function represented in the following slide

- Assuming:
  - V_{in} in the range [-8,+8]
  - Quantized over $M=8$ levels

Joint effect of ADC and DAC



QUANTIZATION

- Example: if V_{in} belong to the interval [0,2] it follows that$V_{out}=1$

- Note that the whole interval is mapped to a single value
  - As a consequence, V_{in} is in general always different with respect to V_{out} (except for some particular cases)

- While sampling theoretically does not introduce any loss of information (signal integrity), quantization always add a certain amount of "quantization error" that can be calculated and kept under control

Comments on quantization

- The division of the ADC process in three separate actions
  - Sampling
  - Quantization
  - Encoding

  is needed only from a theoretical point of view

- In practice, in most applications all three operations are applied at same time and in the same chip

Performance analysis: quantization effect

- Goal of this analysis is to evaluate the impact of quantization process, due to the cascade of the ADC and DAC conversion

- This is important because from results will follow rules and criteria to properly dimension ADC parameters
  - In particular, it will follow the criteria to determine the number of levels M

QUANTIZATION ERROR

Let's define the quantization error as:

$$e_q = V_{in} - V_{out}$$

Being Δ the associated to each level, it follows that:

$$max\ |e_q| = \Delta/2$$

(for the uniform quantizer)

SNR DUE TO QUANTIZATION

- Quantization error can be analyzed as a noise, added to the signal
- Let's evaluate the SNR due to quantization
- SNR is the ration between signal power and the power of the noise due to quantization
- It can be obtained as the ratio between variances:

$$\left(\frac{S}{N}\right)_Q = \frac{\sigma_{V_{in}}^2}{\sigma_{e_q}^2}$$

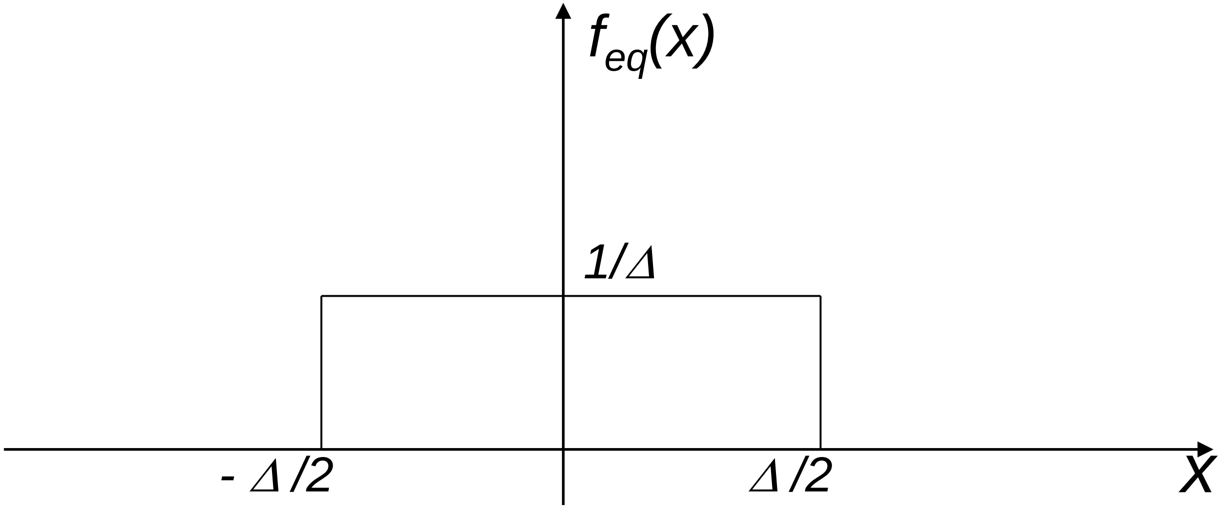
SNR DUE TO QUANTIZATION

- HYPOTESIS:
  - Uniform quantizer with M levels (all steps are equal) : $\Delta=2V/M$
  - Signal with uniform density probability in the same range of the quantizer:

$$[-V, +V].$$

SNR DUE TO QUANTIZATION

As a consequence $e_q = V_{in} - V_{out}$ is also uniformly distributed between $-\Delta/2$ a $+\Delta/2$ (being Δ the interval width)



SNR DUE TO QUANTIZATION

- Note that the quantization error, on each interval, has a zero mean value
- Moreover, the distribution is the same over all intervals
- It is possible to evaluate the quadratic mean, equivalent in this case to the variance

$$E\left[e_q^{\ 2}\right]=\int_{-\infty}^{+\infty} x^2 f_{eq}(x)dx=$$

$$E\left[e_q^{\ 2}\right]=\frac{\Delta^2}{12}=\frac{(2V/M)^2}{12}=\frac{V^2}{3M^2}$$

SNR DUE TO QUANTIZATION

- Signal power is:

$$E\left[V_{in}^2\right]=\int_{-V}^{+V} x^2 \frac{1}{2V}dx=\int_{-V}^{+V} x^2 \frac{1}{2V}\,dx=\frac{1}{2V}[\frac{x^3}{3}]_{-V}^{+V}=\frac{1}{2V}\frac{V^3}{3}2=\frac{V^2}{3}$$

- It can be defined the SNR due to quantization only as:

$$\boxed{\left(\frac{S}{N}\right)_Q=\frac{V^2/3}{V^2/3M^2}=M^2}$$

SNR DUE TO QUANTIZATION

- Assuming an encoding based on binary symbols (bits), it is better to have $M=2^n$.
- So:

$$\left(\frac{S}{N}\right)_Q = 2^{2n}$$

- Converting in dB:

$$\boxed{\left.\left(\frac{S}{N}\right)_Q\right|_{dB} = 10\cdot\log_{10}\left(2^{2n}\right) = 2n\cdot 10\cdot\log_{10}(2) \cong 6n}$$

SNR DUE TO QUANTIZATION

- VERY IMPORTANT RESULTS

$$\left.\left(\frac{S}{N}\right)_Q\right|_{dB} \cong 6n\,[dB]$$

- In general, even in more complex situations, adding 1 bit of resolution, improve of 6dB performances in term of $(S/N)_Q$
- This important results is usually called “6-dB law”

Example: PCM for telephony

Voice signal, telephone quality

- Forcing the spectral occupation between 300 and 3400 Hz.
  - Minimum sampling frequency: f_c=2*3400=6.8 kHz
  - Adding some margin, it has been standardized: f_c= 8 kHz

- Number of bits to have an acceptable performance: n=8 (256 levels)
  - Total resulting bit rate is $B_R=8 \cdot f_c$=64 kbit/s
  - $(S/N)_Q$=48 dB
    - In real application, for telephony PCM is based on a non uniform quantization (see following slides)

Example: PCM for telephony

- Due to its vast popularity, PCM pushed for the “standardization” also in other audio application of this two parameters:
  - Sampling at 8 KHz (periodicity of 125 μs)
  - Quantization of 8 bit

Example: audio signal for music

Standard CD audio: signal up to 20 kHz.

- It has been set: f_c = 44.1 kHz
- Quantization *: n* = 16 bit (65536 levels)
- Total bit rate = $B_R = 16 \cdot f_c$ = 705.6 kbit/s (per channel, when using stereo signals is doubled : 1.4112 kbit/s = 1.4 Mbit/s)
- *$(S/N)_Q$= 96 dB*
  - Basic standard for audio PCM assume uniform quantization

Example: audio signal for music

Note that without any specific coding the requested bit rate is very high

Let's calculate the net bit rate:

- Bit rate= 44.1 ksamples/s*32 bit/sample
- Bit rate=1.41 Mbit/s

- All modern application for music signals are based on compression
  - As an example, using the MP3 format very high compression ratios are achievable: typical values are between 4 and 12

Audio CD

- Let's evaluate the amount of data to be written on CD containing 80 minutes of audio:

$$44.1\left[\frac{Ksample}{\sec}\right]\cdot 32\left[\frac{bit}{sample}\right]\cdot 60[\sec]\cdot 80[\min]=$$

$$=6.774\cdot 10^{9}\,[bit]=846.7\cdot 10^{6}\,[byte]\cong 807.5[Mbyte]$$

<u>Curiosity</u>:
A CD-R is specified for 80 minutes of audio and 700 Mbyte of data.
We have calculate that 80 minutes of audio using the CD standard is equal to about 800 Mbyte.
The difference is due to different standards for data and audio.
In particular, the CD-ROM standard for data has a "stronger" protection against errors with respect to CD-AUDIO: it adds bigger redundancy reducing the available space.

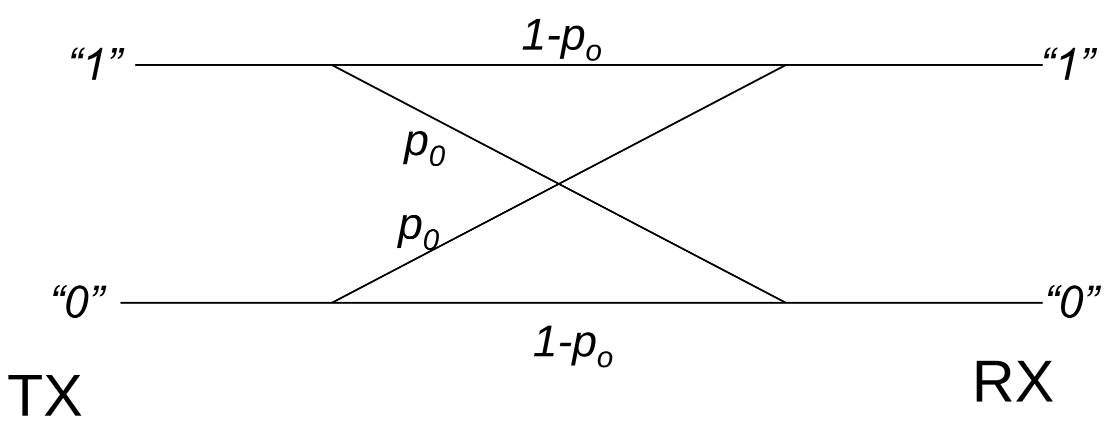
The BSC: PCM and errors

- In previous slides it has been found a formula for SNR due to quantization
- Signal integrity is also degraded by error introduced during transmission over the channel
- Let’s use the simplest model for the transmission channel
  - The symmetric binary channel (BSC)

BSC

- p_0 is the "transition probabilty" of the BS,C defined as:

$p_0 = P(0RX|1TX)$ fro symmetry $p_0 = P(1RX|0TX)$.



PERFORMANCE OF THE BSC

- Error probability is:

$$P_b(e) = P(e|1TX)P(1TX) + P(e|0TX)P(0TX)$$

- If the bit source is emitting symbols with same probability

$$P(1TX) = P(0TX)=1/2$$

- it follows:

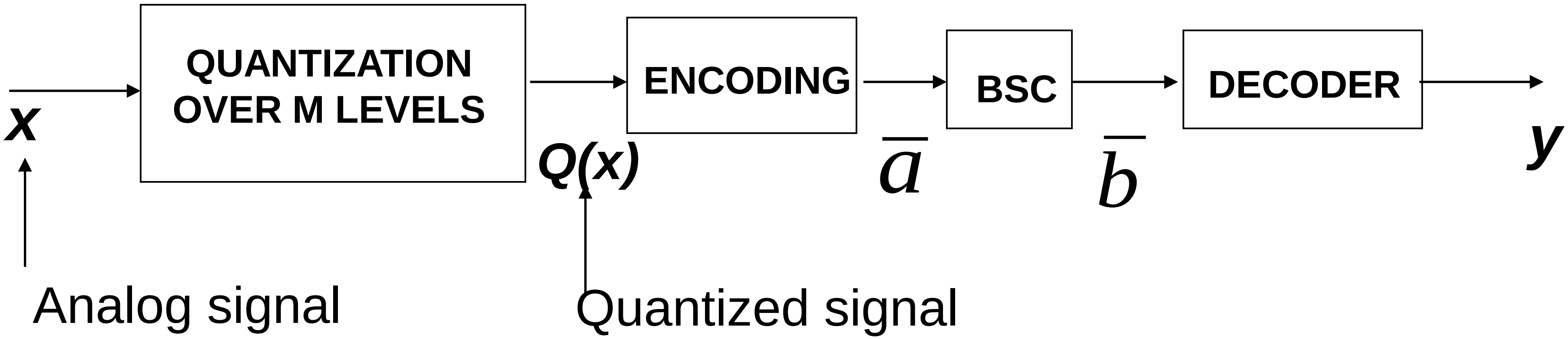
$$P_b(e) =P_e= p_0$$

➢ This is the only parameter used to characterize the transmission channel

SNR DUE TO THE TRANSMISSION CHANNEL

- We want to calculate the SNR resulting after the transmission of a PCM bit stream over a BSC with an error probability $P_b(e)$.

- This is the block diagram of the system:



SNR DUE TO THE TRANSMISSION CHANNEL

- We need an analytical expression to link the quantized signal *Q(x)* (over *M* levels) and the corresponding *n*-uple of bits to be transmitted over the BSC
- Each sample is represented by a “vector” of *n* bits as:

$$\overline{a} = \left[a_1, \ldots, a_N\right]$$

  - For simplicity we assume each elements of the “vector” to be *+1* o *-1* (not just “0” e “1”)

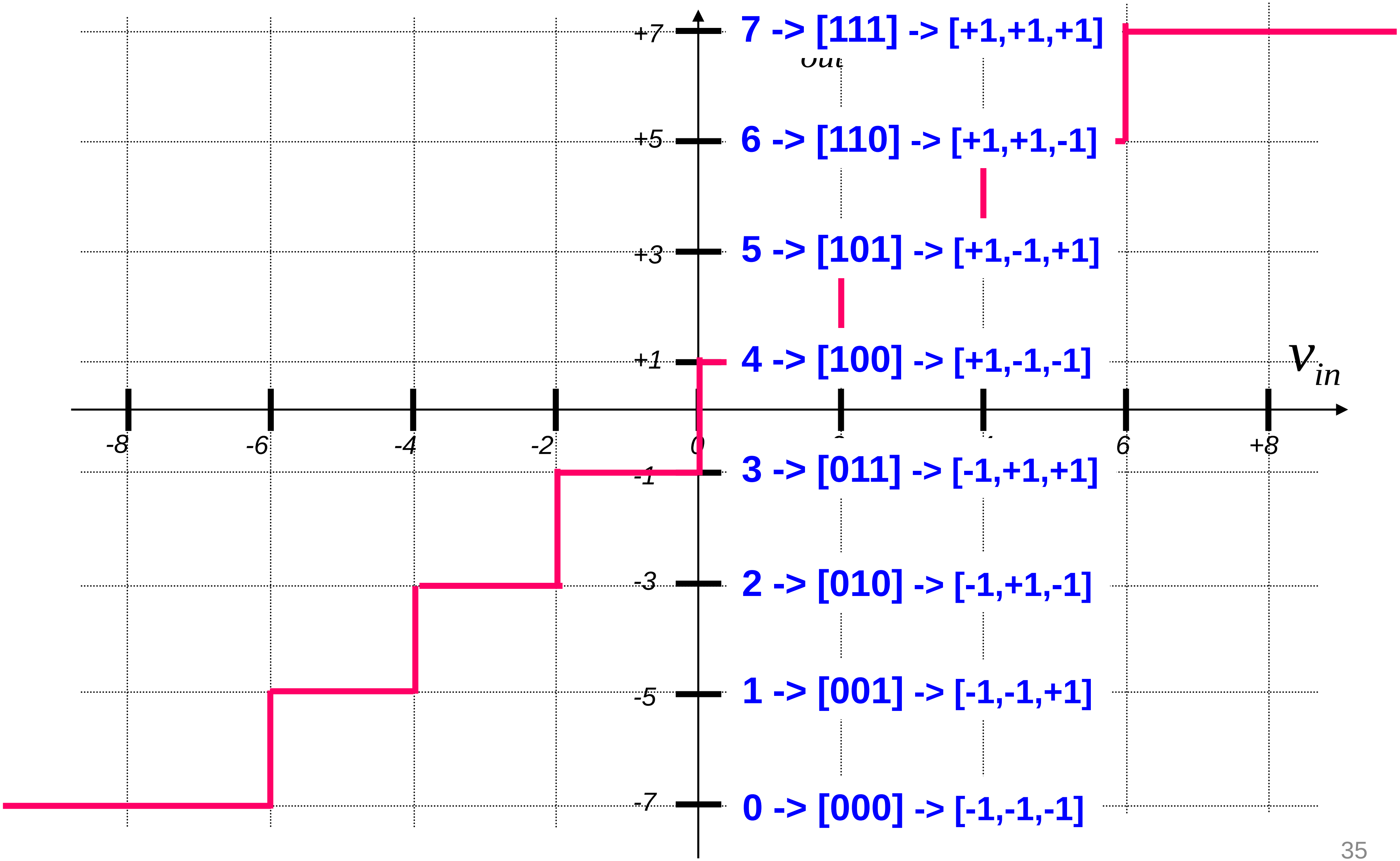
Analysis

- The simplest encoding is the following:

$$\boxed{Q(x) = V \cdot \sum_{j=1}^{n} a_j (1/2)^j}$$

- This formula is important: using the plot in next slide we can better understand it
- Let’s assume: M=8, [-V,+V]=[-8,+8]
  - First we “count” the levels in decimal and then we convert them in binary, finally we further convert bits into -1/+1
    - From 0->[000] to 7->[111]
    - Then convert to -1 e +1

Analysis



+7
+5
+3
+1
-1
-3
-5
-7
-8
-6
-4
-2
0
6
+8
v_{in}
7 -> [111] -> [+1,+1,+1]
6 -> [110] -> [+1,+1,-1]
5 -> [101] -> [+1,-1,+1]
4 -> [100] -> [+1,-1,-1]
3 -> [011] -> [-1,+1,+1]
2 -> [010] -> [-1,+1,-1]
1 -> [001] -> [-1,-1,+1]
0 -> [000] -> [-1,-1,-1]

Analysis

- For example:

  if $a = [1,1,\ldots\ldots,1]$

$$Q(x) = V \cdot \sum_{j=1}^{n} a_j (1/2)^j$$

$$Q(x)=V(1/2+1/4+1/8+\ldots.+1/2n)=V-V/2^n$$

- In general, we have:

$$\boldsymbol{V / 2^n = V / (\text{number of levels})=}$$
$$\boldsymbol{=V / M = \Delta / 2=}$$
$$\boldsymbol{=(\text{quantization step})/ 2}$$

Analysis

In this case:

- Δ=2,
- V=8
- Highest level= 8-2/2 = 7

Similar calculations can be done for all other levels.

Analysis

- The "vector" $\overline{a}$ when transmitted over BSC become $\overline{b}$
- Due to errors added by the transmission channel (BSC), the two "vectors" can be different
  - This is the effect we want to study

- The reconstructed signal will be:

$$y = V \cdot \sum_{j=1}^{n} b_j \left(\frac{1}{2} \right)^j$$

Analysis

- Let’s comparing *Q(x)* and *y* to evaluate the “noise” due to error added by the channel

$$e_b = y - Q(x)$$

- We can evaluate the quadratic mean of the error:

$$E[e_b^2] = E[(y - Q(x))^2]$$

Analysis

$$E[e_b^2] = E[V^2\left(\sum_{j=1}^{n}(b_j - a_j)\left(\frac{1}{2}\right)^j\right)^2] =$$

$$= V^2 E[\sum_{j=1}^{n}(b_j - a_j)2^{-j} \cdot \sum_{k=1}^{n}(b_k - a_k)2^{-k} =$$

$$= V^2 \sum_{j=1}^{n}\sum_{k=1}^{n} 2^{-j-k}\left(E[(b_j - a_j)(b_k - a_k)]\right) =$$

$$= V^2 \sum_{j=1}^{n}\sum_{k=1}^{n} 2^{-j-k}\left(E[b_j b_k] - E[a_j b_k] - E[b_j a_k] + E[a_j a_k]\right)$$

Analysis

- For all terms with *j* different from *k*, thanks to the statistical independence, is:

$$E[b_j b_k]=E[b_j]E[b_k]=0$$
$$E[a_j a_k]=E[a_j]E[a_k]=0$$
$$E[b_j a_k]=E[b_j]E[a_k]=0$$

being all mean equal to zero:

$$E[b_j]=E[b_k]= E[a_j]=E[a_k] = 0$$

Analysis

- When $j=k$, we have:

$$E[b_j b_j]=1$$

- It remains only a term in the sum, when $j=k$ from 1 to n

- Moreover, for $j=k$ it is important to analyze the mixed terms :

$$E[a_j b_j]$$

- Let’s analyze in detail all four combinations

Analysis

a_j	b_j	$P(a_j b_j)$	$a_j b_j$
1	1	$\frac{1}{2}(1-P_e)$	$+1$
1	-1	$\frac{1}{2}P_e$	-1
-1	1	$\frac{1}{2}P_e$	-1
-1	-1	$\frac{1}{2}(1-P_e)$	$+1$

$$E[a_j b_j] = (+1)1/2(1-P_e) + (-1)1/2P_e + (-1)1/2P_e + (+1)1/2(1-P_e) = (1-P_e-P_e) = 1-2P_e$$

Analysis

- Putting together all results:

$$E[e_b^2]=V^2\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}2^{-j-k}\left(E[b_jb_k]-E[a_jb_k]-E[b_ja_k]+E[a_ja_k]\right)$$

$$=V^2\cdot\sum_{j=1}^{n}2^{-2j}\left(E[b_j^2]-2E[a_jb_j]+E[a_j^2]\right)=$$

$$=V^2\cdot\sum_{j=1}^{n}2^{-2j}\left(1-2+4\cdot P_e+1\right)=4\cdot V^2\cdot P_e\cdot\sum_{j=1}^{n}\left(\frac{1}{4}\right)^j$$

Using the geometric series sum we get:

$$E[e_b^2]=\frac{4}{3}\cdot V^2\cdot P_e\,\frac{M^2-1}{M^2}$$

SNR DUE TO THE TRANSMISSION CHANNEL

- Signal power

$$E\left[V_{in}^2\right] = \int_{-V}^{+V} x^2 \frac{1}{2V} dx = \frac{V^2}{3}$$

- Putting together signal and noise we obtain:

$$\left(\frac{S}{N}\right)_e = \frac{V^2/3}{\frac{4}{3} V^2 P_e \frac{M^2-1}{M^2}} = \frac{1}{4P_e \frac{M^2-1}{M^2}}$$

SNR DUE TO THE TRANSMISSION CHANNEL

$$\left(\frac{S}{N}\right)_e = \frac{1}{4P_e \frac{M^2-1}{M^2}}$$

- If M>>1, we can have the following approximation:

$$\left(\frac{S}{N}\right)_e \cong \frac{1}{4P_e}$$

Comment: the SNR due to transmission channel depends only on P_e

Comments

- We have considered two effects impacting PCM performances
  - Quantization: $\boxed{\left(\frac{S}{N}\right)_Q = M^2}$
  - Transmission channel $\boxed{\left(\frac{S}{N}\right)_e \cong \frac{1}{4P_e}}$
- Let's put together the two effects

SNR total

- Assuming the two «noises» as independent and uncorrelated, we can sum up quadratic means

$$E[e_{out}^2] = E[e_b^2] + E[e_q^2] = \frac{4}{3} V^2 P_e \frac{(M^2-1)}{M^2} + \frac{V^2}{3M^2}$$

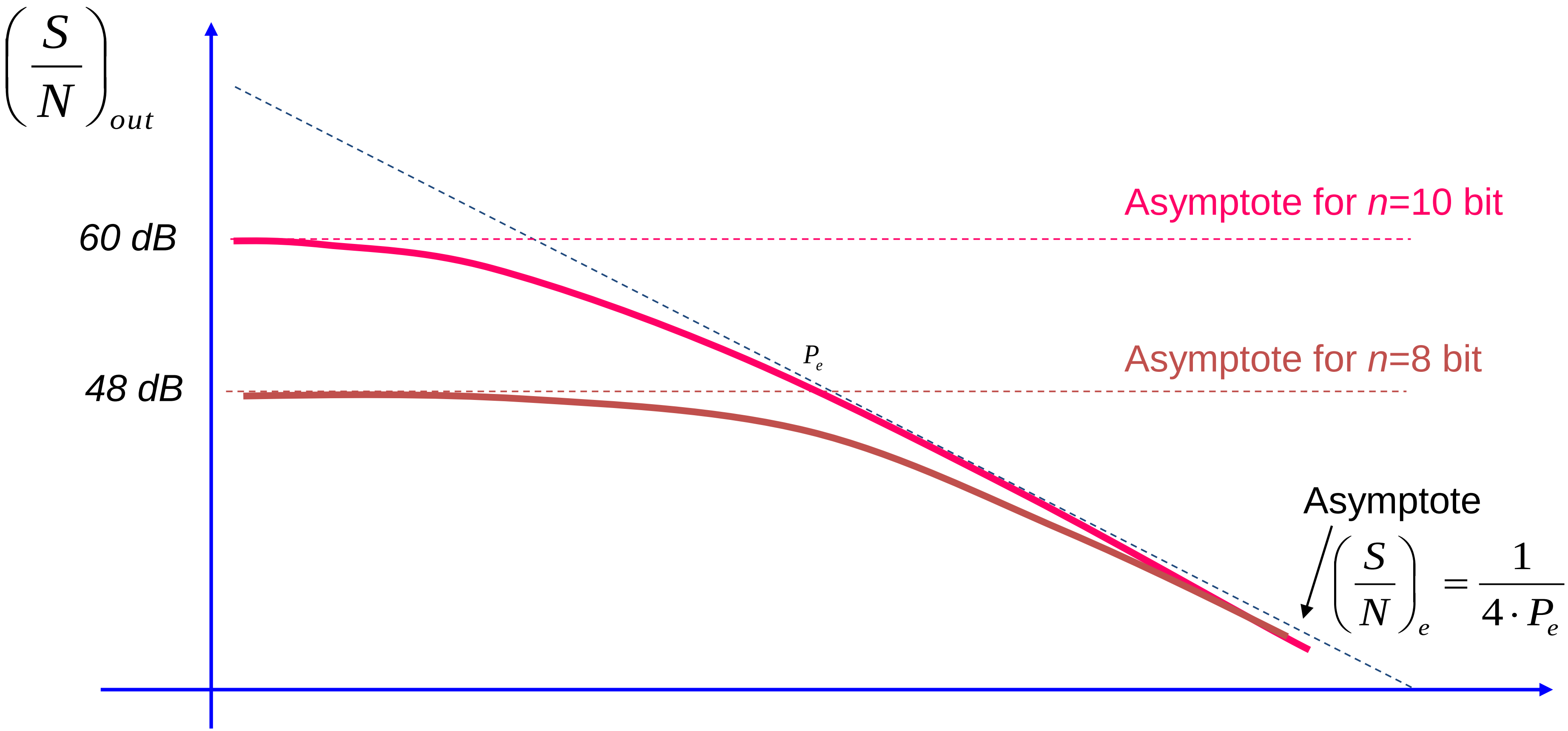
$$E[e_{out}^2] = \frac{V^2}{3M^2}(4P_e(M^2-1)+1)$$

- Signal power is: $E\left[V_{in}^2\right] = \int_{-V}^{+V} x^2 \frac{1}{2V} dx = \frac{V^2}{3}$
- We get:

$$\boxed{\left(\frac{S}{N}\right)_{out} = \frac{M^2}{1+4(M^2-1)P_e}}$$

Fundamental results for PCM

Overall performance for PCM



Overall performance for PCM

- Given *n* bit of quantization, it is defined the quantity:

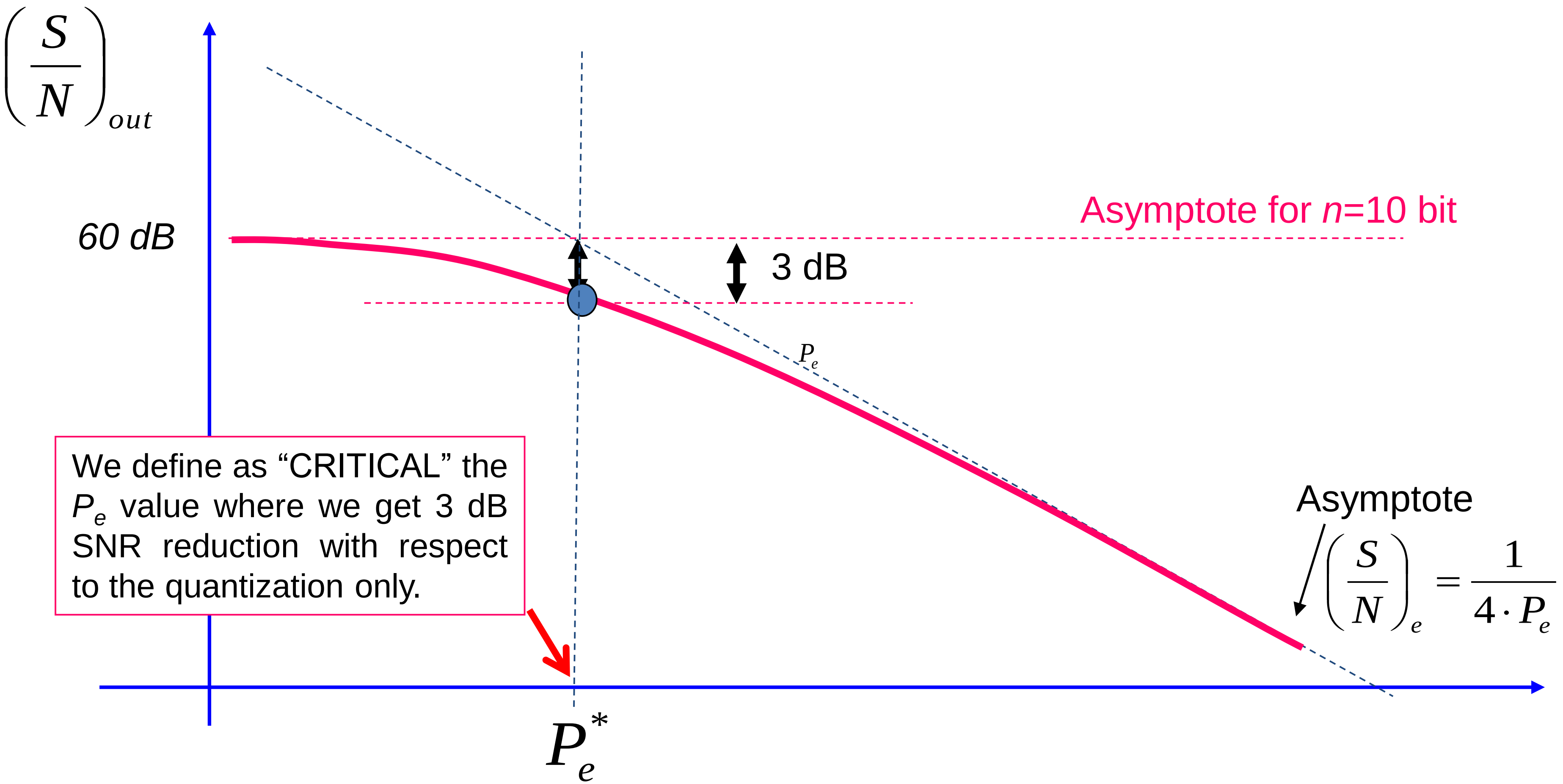
$$\left(\frac{S}{N}\right)_Q = 6 \cdot n$$

- Looking the plot, we observe that:
  - below a given P_e, performances are independent of P_e.
  - above a given threshold of P_e, performance $(S/N)_{out}$ start to worsen drastically

Overall performance for PCM

- PCM systems have a threshold behavior with respect to P_e
  - For all P_e lower than a given “critical” threshold, performances are independent on P_e and coincide with those given by quantization
  - For all P_e larger than a given “critical” threshold, performances depends only on P_e
    - Moreover, they drastically decrease when increasing P_e: PCM in these conditions does not work

Threshold definition



We define as "CRITICAL" the P_e value where we get 3 dB SNR reduction with respect to the quantization only.

Threshold definition

- We define the threshold as the P_e that give a 3 dB worsening, i.e. a factor ½ in linear scale

$$\left.\left(\frac{S}{N}\right)_{out}\right|_{P_e^*} = \frac{1}{2}\left(\frac{S}{N}\right)_Q$$

$$\Rightarrow \frac{M^2}{1+4(M^2-1)P_e^*} = \frac{1}{2}M^2$$

Threshold definition

- We get:

$$\boxed{P_e^* = \frac{1}{4(M^2 - 1)}}$$

- Concluding, a PCM system is
  - "**above threshold**" if $P_e < P_e^*$
    - PCM works properly
  - "**below threshold**" if $P_e > P_e^*$
    - PCM does not work properly

Threshold effect

- As most digital systems, PCM systems have a threshold behavior
- They work perfectly up to a given P_e threshold, then performance fall down drastically

- Values for critical error probability, P_e^* :
  - 8 bit -> $P_e^* = 3.8 \ 10^{-6}$
  - 16 bit -> $P_e^* = 5.8 \ 10^{-11}$
  - Note that increasing the number of quantization bits, the required P_e^* decrease

NON-UNIFORM QUANTIZATION

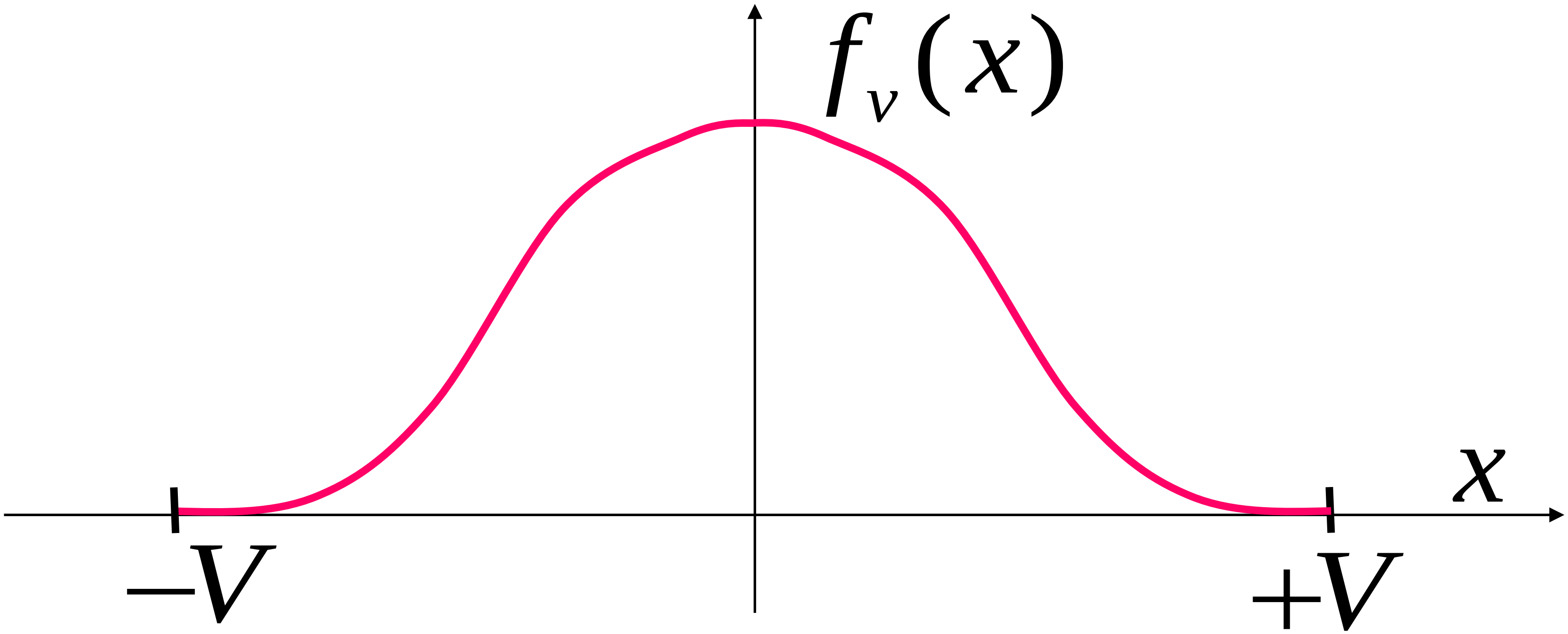
- Assuming uniform quantization and a signal with a uniform pdf, we found:

$$\left(\frac{S}{N}\right)_Q = M^2$$

- Uniform quantization is the simplest, but it is not always the best solution
- In particular, non-uniform quantization is important for signal with non-uniform pdf

PDF of signals

- Some signals, like voice, have a non-uniform pdf with higher probabilities around the average value



Comment

- In the following we need to clearly separate two concepts:
  - pdf of the input signal
  - quantization technique applied

UNIFORM QUANTIZATION

- It can be shown that when signal pdf is not uniform, SNR is lower than the case of uniform pdf
- Analytically, in case of uniform quantization we have:
  - The quantization error variance is independent of the input signal
  - At constant range, however, the power of the signal is lower

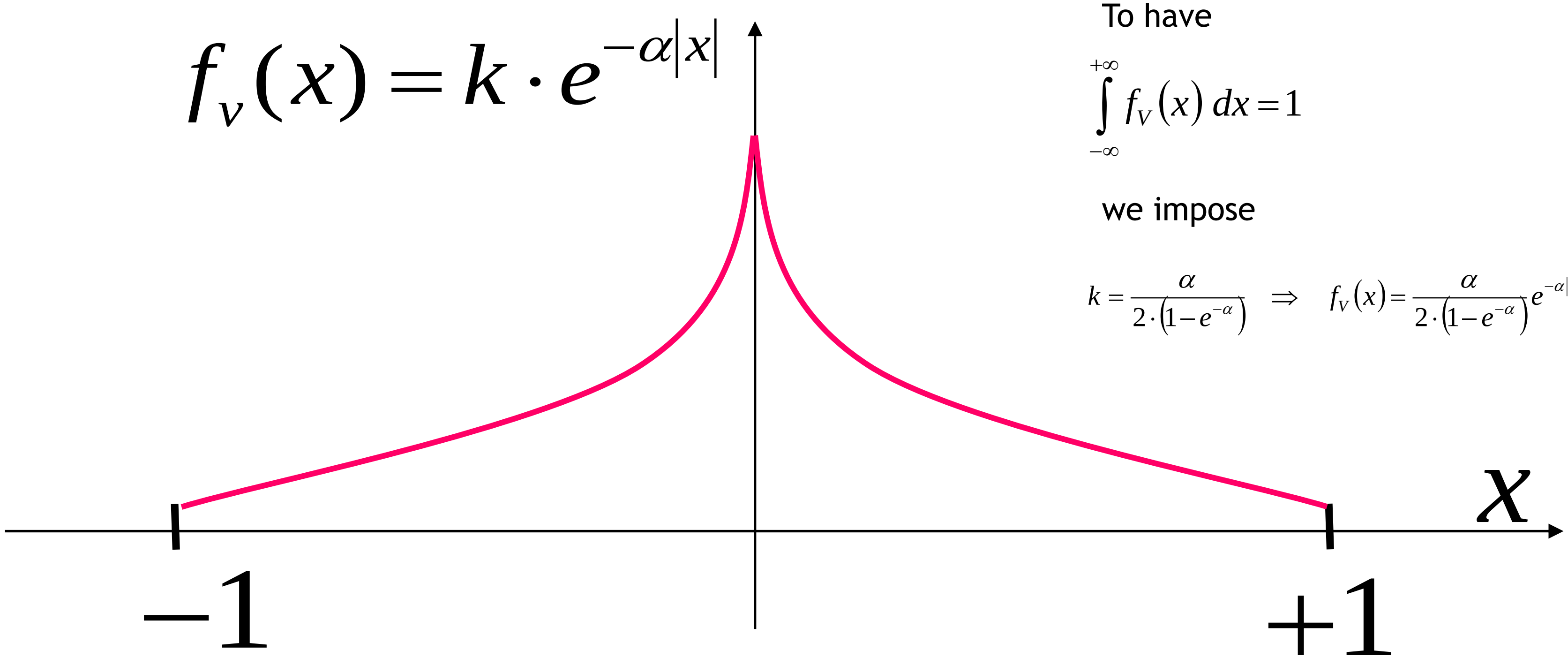
UNIFORM QUANTIZATION

- Let's compare the effect of quantization on two different signals
  – Signal 1, uniform pdf between *[-V,+V]*
  – Signal 2, non-uniform pdf between *[-V,+V]*
- SNR definition: $\left(\dfrac{S}{N}\right)_Q = \dfrac{\sigma^2_{V_{in}}}{\sigma^2_{e_q}}$
  – Same error variance: $\left.\sigma^2_{e_q}\right|_1 = \left.\sigma^2_{e_q}\right|_2$
  – But: $\left.\sigma^2_{V_{in}}\right|_1 > \left.\sigma^2_{V_{in}}\right|_2$
- It follows: $\left.\left(\dfrac{S}{N}\right)_Q\right|_1 > \left.\left(\dfrac{S}{N}\right)_Q\right|_2$

EXAMPLE #1: bilateral truncated exponential

- Let's consider the following pdf: bilateral truncated exponential in [-1,+1]

$$f_v(x) = k \cdot e^{-\alpha|x|}$$



To have

$$\int_{-\infty}^{+\infty} f_V(x)\, dx = 1$$

we impose

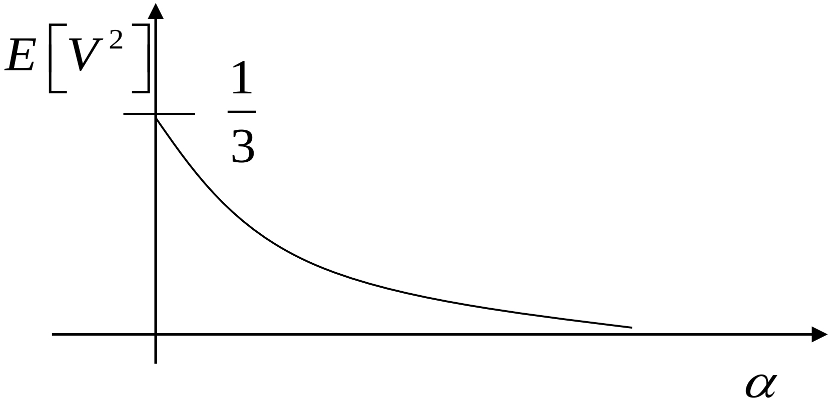
$$k = \frac{\alpha}{2 \cdot \left(1 - e^{-\alpha}\right)} \quad \Rightarrow \quad f_V(x) = \frac{\alpha}{2 \cdot \left(1 - e^{-\alpha}\right)} e^{-\alpha|x|}$$

EXAMPLE #1: bilateral truncated exponential

- After some math, we get the signal power:

$$E\left[V^2\right] = \frac{1}{\alpha^2(1-e^{-\alpha})} \cdot \left(2 - e^{-\alpha}\left(\alpha^2 + 2\alpha + 2\right)\right)$$

  - As a function of α, we have this plot:



EXAMPLE #1: bilateral truncated exponential

- For any value of α, we have:

$$E\left[V^2\right] \leq \frac{1}{3}$$

- On the same range, [-1,+1] a signal with uniform pdf has a power equal to 1/3
- We can see that, comparing signal with same range of values, the uniform pdf signal always has the higher power
  - For any α, all bilateral truncated exponential have lower power

EXAMPLE #2: uniform pdf over narrow interval

- Uniform quantizer over *[-V,+V]*
- Generic signal V_{in} in the range *[-W,+W]*, where *W<V*

- We get:

$$\left(\frac{S}{N}\right)_Q = \frac{3M^2}{V^2} E\left[V_{in}^2\right]$$

EXAMPLE #2: uniform pdf over narrow interval

- Let's assume now that V_{in} has a uniform pdf, but on the range *[-W,+W]* where $W<V$
- Uniform pdf signal, uniform quantizer, but with different ranges
- Being for the signal $E[V_{in}^2]=W^2/3$, we get:

$$\boxed{\left(\frac{S}{N}\right)_Q = M^2\left(\frac{W}{V}\right)^2}$$

EXAMPLE #2: uniform pdf over narrow interval

- Converting in dB, we get:

$$\left(\frac{S}{N}\right)_Q \cong 6n + 10\log_{10}\left(\frac{W}{V}\right)^2 = 6n + 20\log_{10}\left(\frac{W}{V}\right)$$

Being *W*<*V*, this quantity in dB is negative.

It is a “loss” due to the smaller dynamic *[-W,+W]* of the signal compared to the quantizer.

EXAMPLE #2: uniform pdf over narrow interval

- For example, when *W=V/2,*

$$20\cdot\log_{10}(1/2)=-6\ dB$$

we get

$$\left(\frac{S}{N}\right)_{Q|dB}=6\cdot(n-1)$$

It is a loss equivalent to 1 bit of quantization.

Comment

This result can be easily understood: we are not using one of the bit in the ADC.

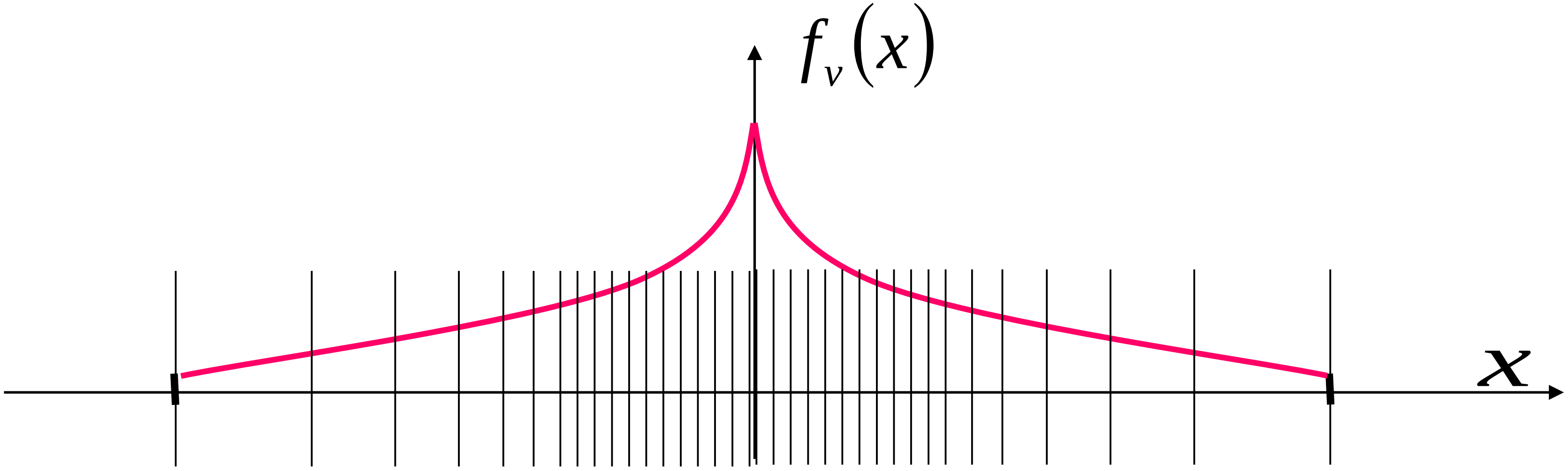
NON-UNIFORM QUANTIZATION

- Let’s now consider a more general case
- We want evaluate the variance of the quantization error in the general case, where:
  - Non-uniform quantizer
  - Generic signal

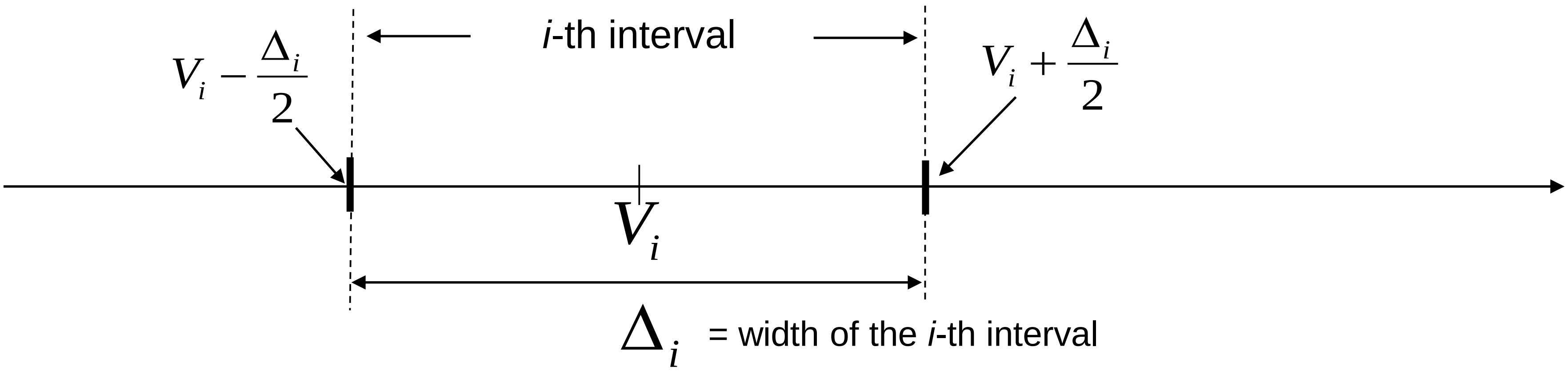
- Assume $M=2^n$ intervals on the input range $[-V,+V]$
- Now each quantization interval has an arbitrary width

NON-UNIFORM QUANTIZATION

- Based on these findings, it is important to study non-uniform quantization
- Basic concept: it is better to have a denser quantization where the signal has higher probability



NON-UNIFORM QUANTIZATION

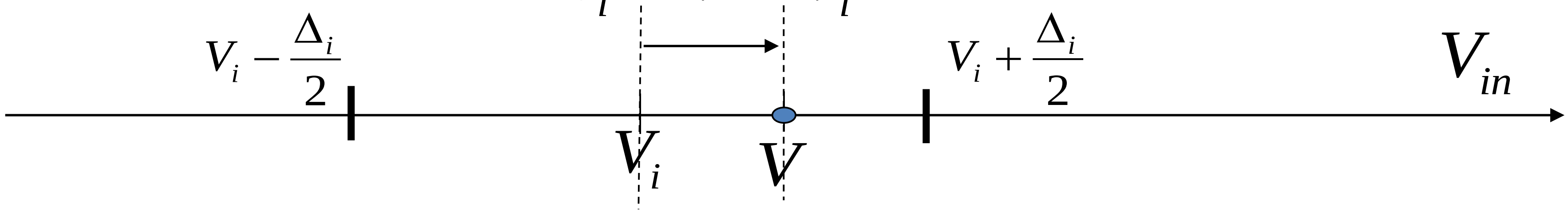


- Parameters of the i-th quantization interval
  - Δ_i = width of the i-th interval
  - V_i = center point of the i-th interval

NON-UNIFORM QUANTIZATION

- Being *V* a generic input signal, if a sample fall in the *i*-th interval

$$e_i = V - V_i$$



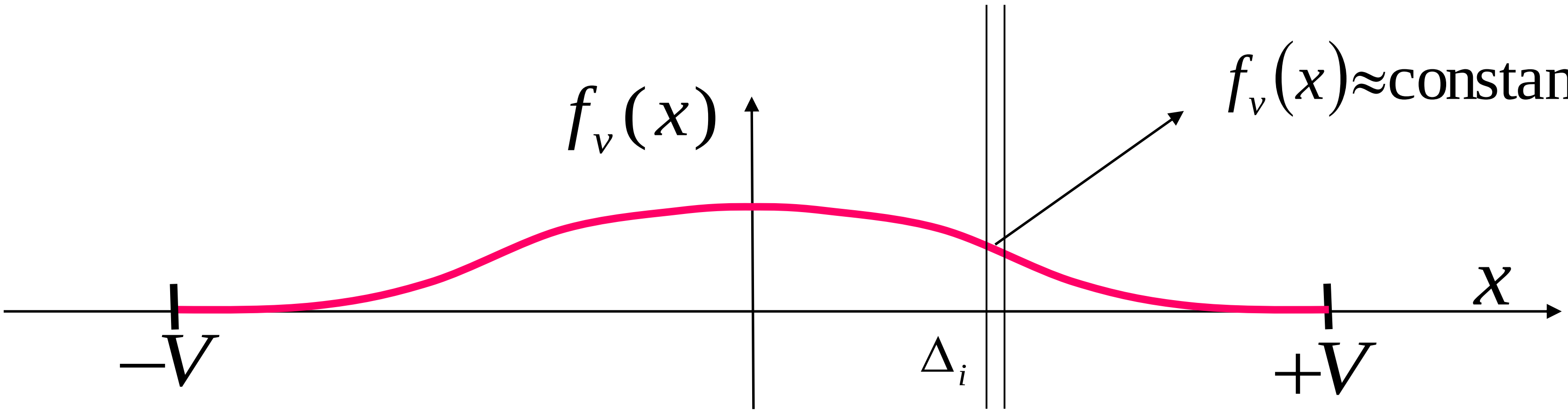
- The quantization error is:

$$e_i = V - V_i \Rightarrow$$

$$E\left[e_i^2\right] = \int_{V_i - \Delta i/2}^{V_i + \Delta i/2} (V - V_i)^2 \, f_v(v \mid v \in i-th)\,dv$$

NON-UNIFORM QUANTIZATION

- If quantization interval are small, we can assume the pdf of the signal constant over each interval



- In particular, the conditional pdf is:

$$f_v(x \mid x \in i-th) = \frac{1}{\Delta_i}$$

NON-UNIFORM QUANTIZATION

$$E[e_i^2] = \int \left(V - V_i\right)^2 \cdot f_v\left(x \mid V \in i-th\right) dV$$

$$= \int_{V_i - \Delta i/2}^{V_i + \Delta i/2} (V - V_i)^2 \frac{1}{\Delta i} dV =$$

$$= \int_{-\Delta i/2}^{+\Delta i/2} x^2 \frac{1}{\Delta i} dx = \frac{1}{\Delta i} \left[\frac{x^3}{3} \right]_{-\Delta i/2}^{+\Delta i/2} = \frac{\Delta_i^2}{12}$$

NON-UNIFORM QUANTIZATION

- Quantization error depends on all $E[e_i^2]$ weighted by the probability of V falling in the i-th interval:

$$E[e_Q^2] = \sum_{i=1}^{M} P(v \in i-th) \cdot \frac{\Delta_i^2}{12}$$

General formula for the quantization error

$$E[e_Q^2] \approx \sum_{i=1}^{M} f_v(v_i)\Delta_i \frac{\Delta_i^2}{12} = \sum_{i=1}^{M} f_v(v_i) \frac{\Delta_i^3}{12}$$

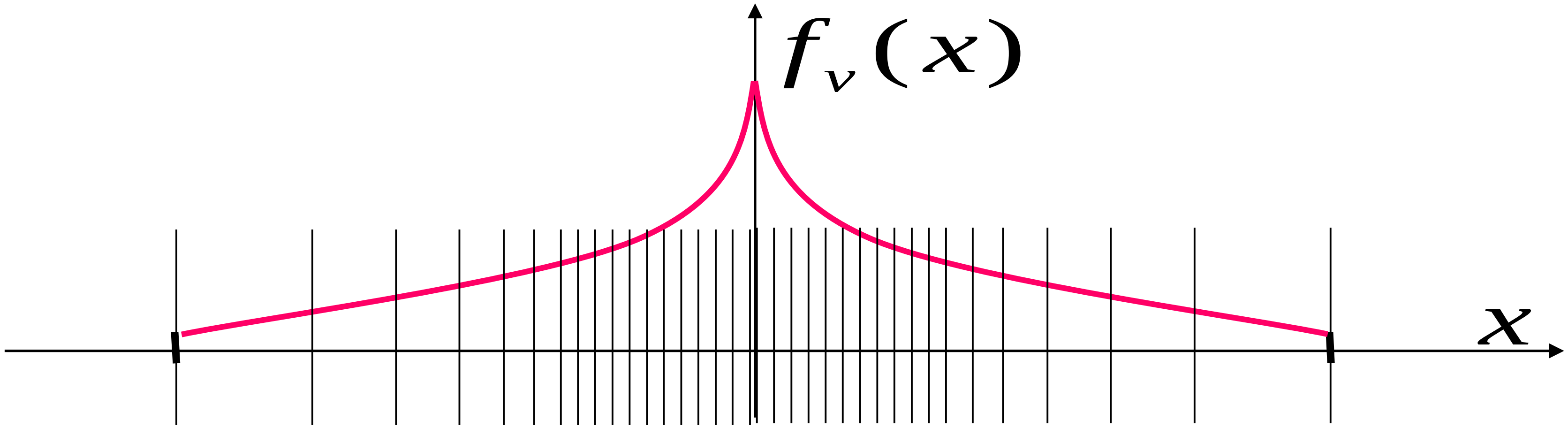
NON-UNIFORM QUANTIZATION

- It can be demonstrated that, independently of M, $(S/N)_Q$ is minimized when all terms in the sum are equal
  - Proof of this results is quite complex, we will not go through it
- In the end, we must have:

$$\boxed{f_v(V_i)\frac{\Delta_i^3}{12} = \text{costant}}$$

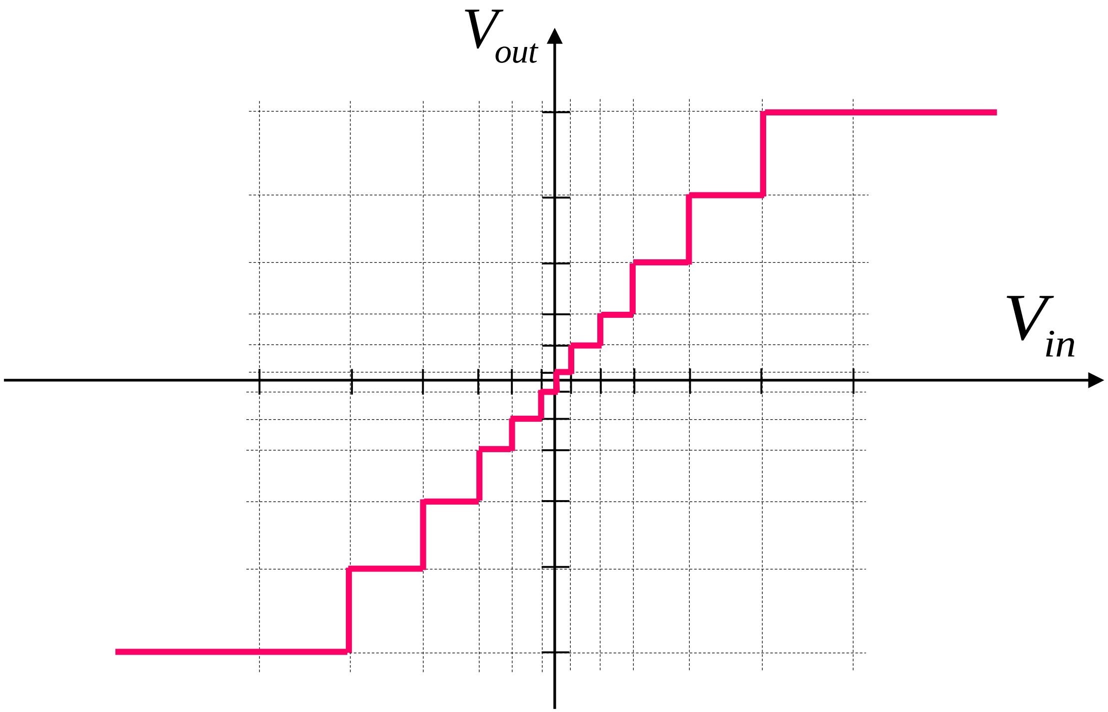
NON-UNIFORM QUANTIZATION

- Very important result: performance is optimized if the quantization is denser where $f_v(v_i)$ is higher



NON-UNIFORM QUANTIZATION

- In practice, quantization curve for the example shown in previous slide is qualitatively the following:



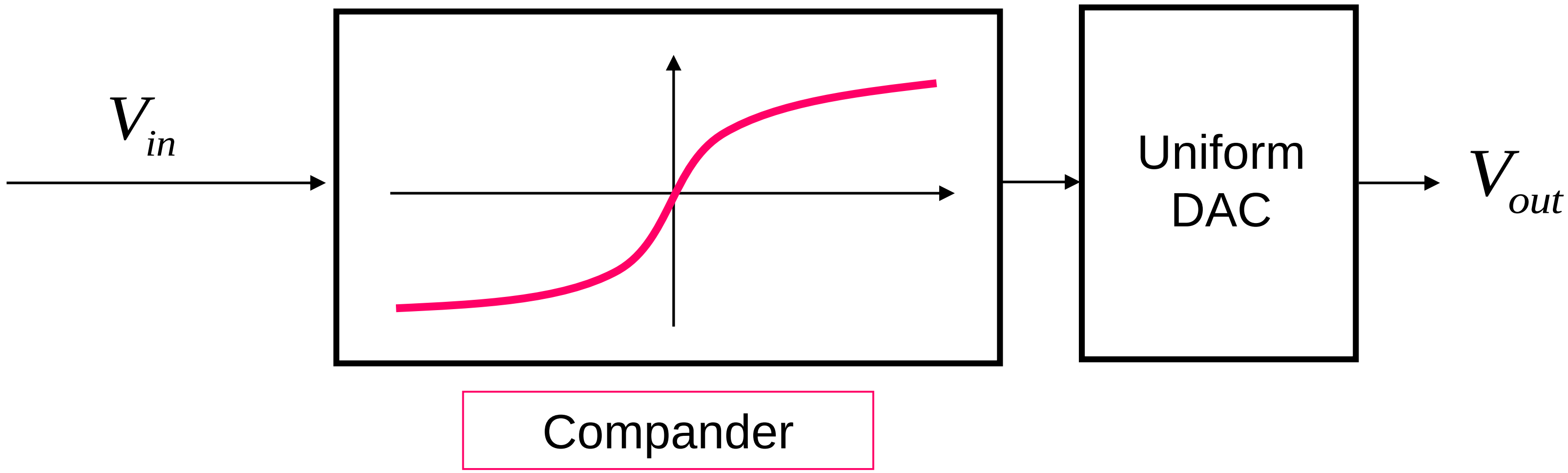
COMPANDING

- Most of commercial ADCs and DACs (in chip) are based on uniform quantization
- To build a non-uniform quantizer using a uniform quantizer, we can use the “companding” technique

- The uniform quantizer in the ADC is preceded by a non-linear component with an input-output relationship carefully selected

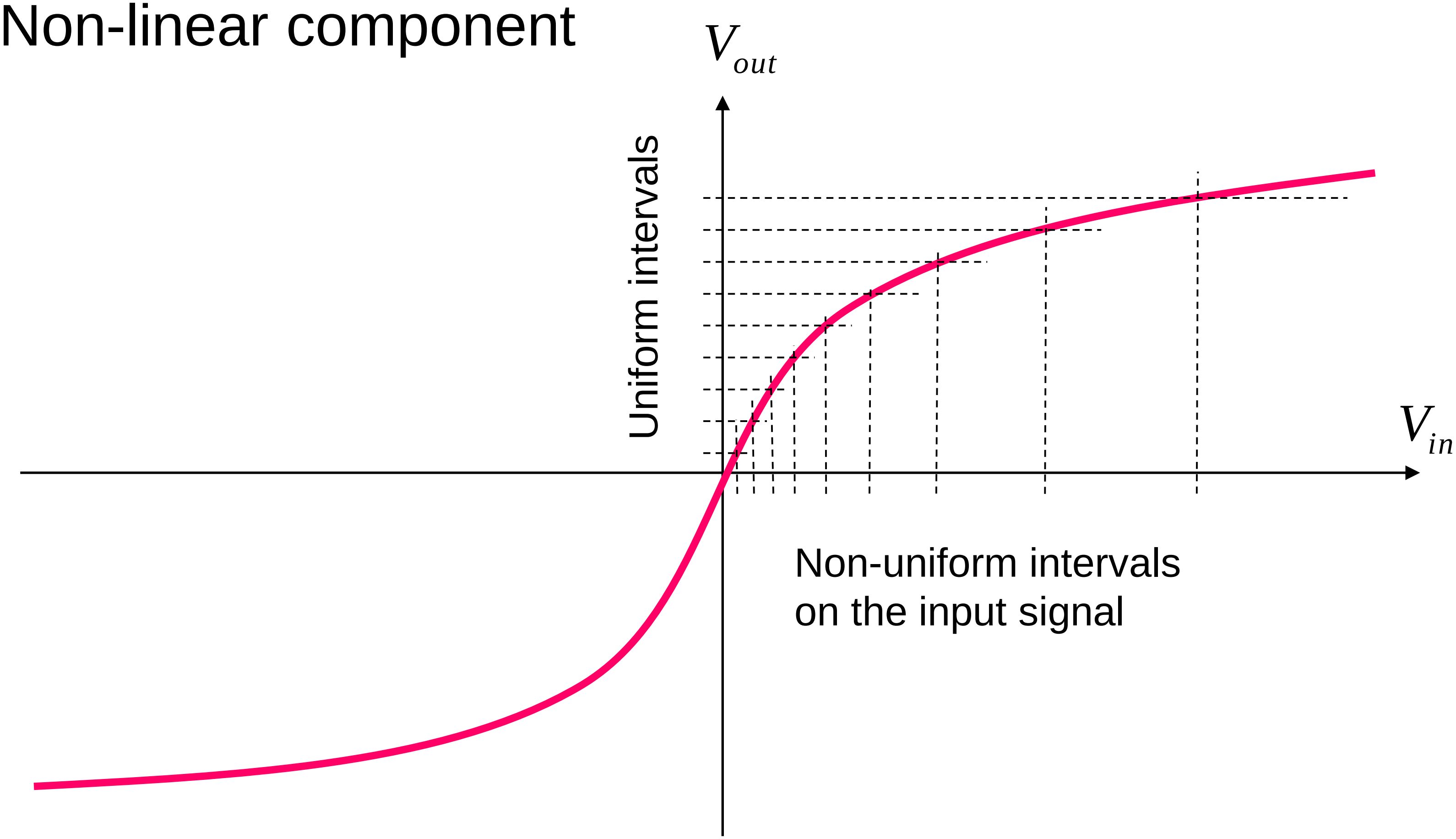
COMPANDING

- At the receiver side, it must be used a non-linear component with a complementary input-output relationship in order to obtain the original signal



COMPANDING

- Non-linear component



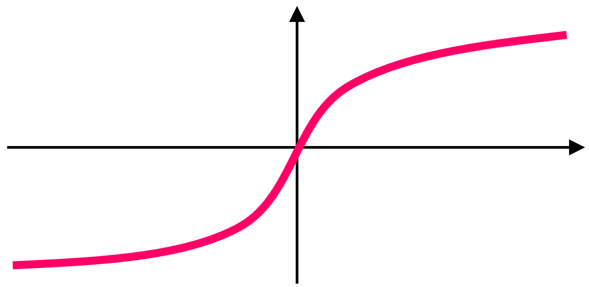
COMPANDING

- Companding is useful also to solve another problem we have yet not discussed that is present in practical application
- Usually, the range of the input signal is not known
  - An example is the signal generated by a microphone: typically it has a bilateral truncated exponential probability density, but the range depends on the volume and on the properties of the microphone

- In this case is better to follow another approach: companding laws where $(S/N)_Q$ is practically independent on signal range

PCM for telephony

- In telephony, PCM always use COMPANDING
- Different companding laws have been standardized in Europe, USA and Japan: all are based on the same principle
- Qualitatively, the law has the following behavior:



- A curve with saturation for high values

PCM for telephony

- “A-law” (Europa)

$$|V_{out}| = \begin{cases} \dfrac{A \cdot |V_{in}|}{1+\log(A)} & \text{per } |V_{in}| \in \left[0, \dfrac{1}{A}\right] \\ \dfrac{1+\log(A \cdot |V_{in}|)}{1+\log(A)} & \text{altrove} \end{cases} \quad \text{dove:} \begin{cases} V_{in} \in [-1,+1] \\ A = 87.56 \end{cases}$$

- “μ-law” (USA)

$$|V_{out}| = \frac{\log(1+\mu \cdot |V_{in}|)}{\log(1+\mu)} \quad \text{dove:} \begin{cases} V_{in} \in [-1,+1] \\ \mu = 255 \end{cases}$$

PCM for telephony

- In general, it can be shown that for any "reasonable" companding technique we get:

$$\boxed{\left.\left(\frac{S}{N}\right)_Q\right|_{dB} = 6n - a}$$

- Where *a* is a parameter that depends on input signal and companding law
- It is still valid the "6 dB law": each bit added to the quantization allow to gain 6 dB in performance