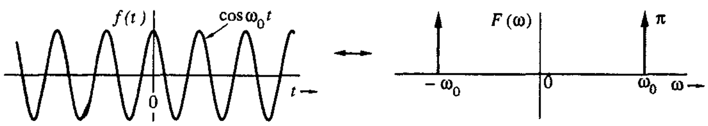

Fig. 4.13 A cosine signal and its Fourier spectrum.

Adding Eqs. (4.26a) and (4.26b), and using the above formula, we obtain

$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \tag{4.27}$$

The spectrum of cos $\omega_0 t$ consists of two impulses at ω_0 and $-\omega_0$, as shown in Fig. 4.13. The result also follows from qualitative reasoning. An everlasting sinusoid cos $\omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore the Fourier spectrum consists of only two components of frequencies ω_0 and $-\omega_0$. ■

■ Example 4.7

Find the Fourier transform of the unit step function $u(t)$.

Trying to find the Fourier transform of $u(t)$ by direct integration leads to an indeterminate result, because

$$U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-j\omega t}\,dt = \frac{-1}{j\omega}e^{-j\omega t}\bigg|_0^{\infty}$$

Observe that the upper limit of $e^{-j\omega t}$ as $t \to \infty$ yields an indeterminate answer. So we approach this problem by considering $u(t)$ as a decaying exponential $e^{-at}u(t)$ in the limit as $a \to 0$ (Fig. 4.14a). Thus

$$u(t) = \lim_{a \to 0} e^{-at}u(t)$$

and

$$U(\omega) = \lim_{a \to 0} \mathcal{F}\{e^{-at}u(t)\} = \lim_{a \to 0} \frac{1}{a + j\omega} \tag{4.28a}$$

Expressing the right-hand side in terms of its real and imaginary parts yields

$$\begin{aligned} U(\omega) &= \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2} - j\frac{\omega}{a^2 + \omega^2}\right] \\ &= \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] + \frac{1}{j\omega} \end{aligned} \tag{4.28b}$$

The function $a/(a^2 + \omega^2)$ has interesting properties. First, the area under this function (Fig. 4.14b) is π regardless of the value of a

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + \omega^2}\,d\omega = \tan^{-1}\frac{\omega}{a}\bigg|_{-\infty}^{\infty} = \pi$$

Second, when $a \to 0$, this function approaches zero for all $\omega \neq 0$, and all its area (π) is concentrated at a single point $\omega = 0$. Clearly, as $a \to 0$, this function approaches an impulse of strength π.† Thus

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \tag{4.29}$$

†The second term on the right-hand side of Eq. (4.28b), being an odd function of ω, has zero area regardless of the value of a. As $a \to 0$, the second term approaches $1/j\omega$.

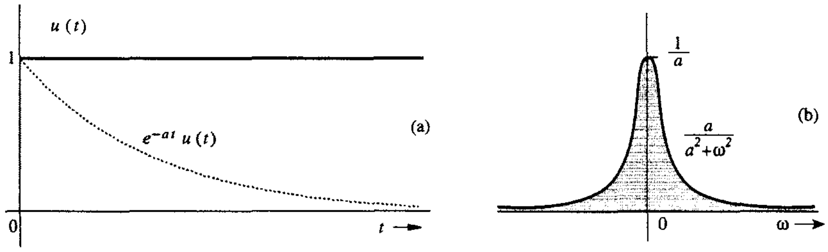

Fig. 4.14 Derivation of the Fourier transform of the step function.

Note that $u(t)$ is not a "true" dc signal because it is not constant over the interval $-\infty$ to ∞. To synthesize a "true" dc, we require only one everlasting exponential with $\omega = 0$ (impulse at $\omega = 0$). The signal $u(t)$ has a jump discontinuity at $t = 0$. It is impossible to synthesize such a signal with a single everlasting exponential $e^{j\omega t}$. To synthesize this signal from everlasting exponentials, we need, in addition to an impulse at $\omega = 0$, all frequency components, as indicated by the term $1/j\omega$ in Eq. (4.29). ■

△ **Exercise E4.2**
Show that the Fourier transform of the sign function sgn(t) depicted in Fig. 4.15a is $2/j\omega$. Hint: Note that sgn(t) shifted vertically by 1 yields $2u(t)$ ▽.

△ **Exercise E4.3**
Show that the inverse Fourier transform of $F(\omega)$ illustrated in Fig. 4.15b is $f(t) = \frac{\omega_0}{\pi}\,\text{sinc}\,(\omega_0 t)$. Sketch $f(t)$. ▽.

△ **Exercise E4.4**
Show that $\cos(\omega_0 t + \theta) \Longleftrightarrow \pi[\delta(\omega + \omega_0)e^{-j\theta} + \delta(\omega - \omega_0)e^{j\theta}]$.

Hint: $\cos(\omega_0 t + \theta) = \frac{1}{2}[e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}]$ ▽

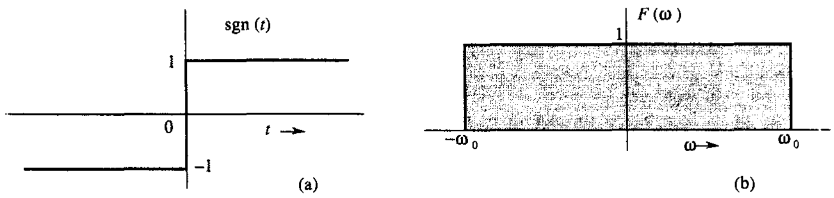

Fig. 4.15

4.3 Some properties of the Fourier Transform

We now study some of the important properties of the Fourier transform and their implications as well as applications. Before embarking on this study, we shall explain an important and pervasive aspect of the Fourier transform: the time-frequency duality.