Chapter 6
Robotics and Automatic Geometric Theorem Proving

In this chapter we will consider two applications of concepts and techniques from algebraic geometry in areas of computer science. First, continuing a theme introduced in several examples in Chapter 1, we will develop a systematic approach that uses algebraic varieties to describe the space of possible configurations of mechanical linkages such as robot "arms." We will use this approach to solve the forward and inverse kinematic problems of robotics for certain types of robots.

Second, we will apply the algorithms developed in earlier chapters to the study of automatic geometric theorem proving, an area that has been of interest to researchers in artificial intelligence. When the hypotheses of a geometric theorem can be expressed as polynomial equations relating the Cartesian coordinates of points in the Euclidean plane, the geometrical propositions deducible from the hypotheses will include all the statements that can be expressed as polynomials in the ideal generated by the hypotheses.

§1 Geometric Description of Robots

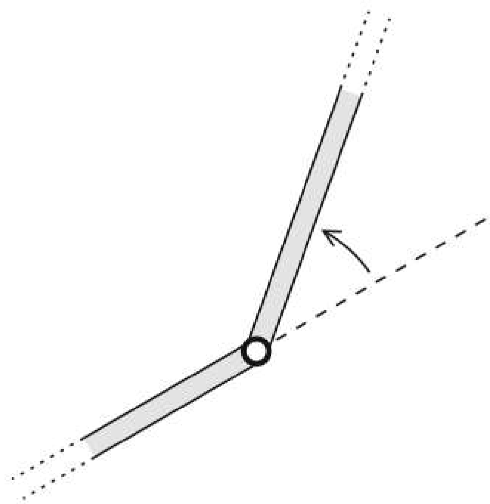
To treat the space of configurations of a robot geometrically, we need to make some simplifying assumptions about the components of our robots and their mechanical properties. We will not try to address many important issues in the engineering of actual robots (such as what types of motors and mechanical linkages would be used to achieve what motions, and how those motions would be controlled). Thus, we will restrict ourselves to highly idealized robots. However, within this framework, we will be able to indicate the types of problems that actually arise in robot motion description and planning.

We will always consider robots constructed from rigid links or segments, connected by joints of various types. For simplicity, we will consider only robots in which the segments are connected *in series*, as in a human limb. One end of our robot "arm" will usually be fixed in position. At the other end will be the "hand" or "effector," which will sometimes be considered as a final segment of the robot. In actual robots, this "hand" might be provided with mechanisms for grasping objects or with tools for performing some task. Thus, one of the major goals is to be able to describe and specify the position and orientation of the "hand."

Since the segments of our robots are rigid, the possible motions of the entire robot assembly are determined by the motions of the joints. Many actual robots are constructed using

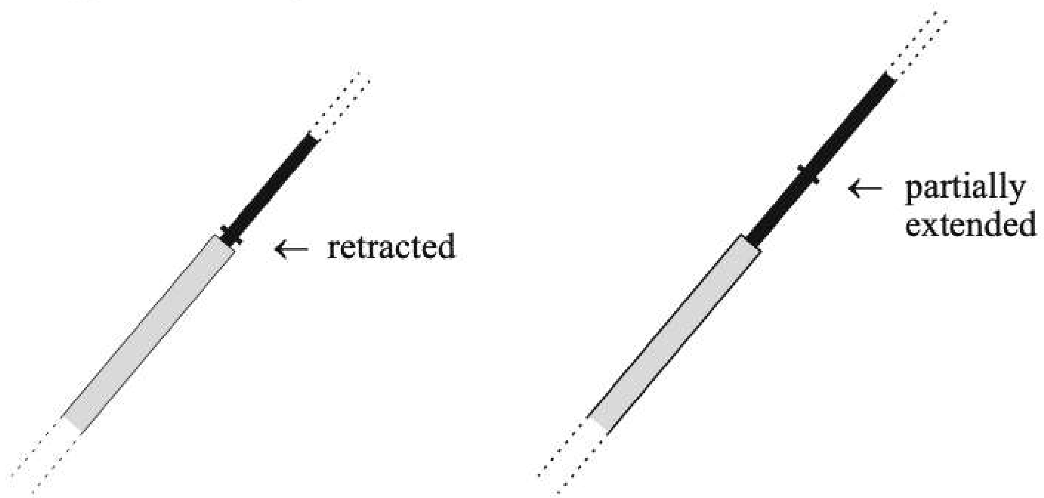
- planar revolute joints, and
- prismatic joints.

A planar revolute joint permits a *rotation* of one segment relative to another. We will assume that both of the segments in question lie in one plane and all motions of the joint will leave the two segments in that plane. (This is the same as saying that the axis of rotation is perpendicular to the plane in question.)



a revolute joint

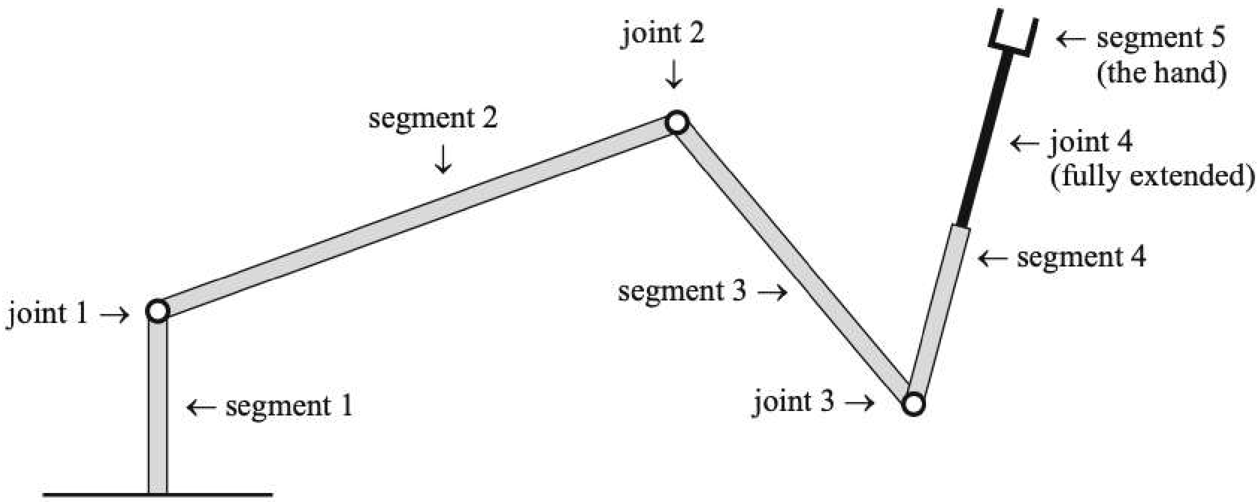
A prismatic joint permits one segment of a robot to move by sliding or *translation* along an axis. The following sketch shows a schematic view of a prismatic joint between two segments of a robot lying in a plane. Such a joint permits translational motion along a line in the plane.



a prismatic joint

If there are several joints in a robot, we will assume for simplicity that the joints all lie in the same plane, that the axes of rotation of all revolute joints are perpendicular to that plane, and, in addition, that the translation axes for the prismatic joints all lie in the plane of the joints. Thus, all motion will take place in one plane. Of course, this leads to a very restricted class of robots. Real robots must usually be capable of 3-dimensional motion. To achieve this, other types and combinations of joints are used. These include "ball" joints allowing rotation about any axis passing through some point in $\mathbb{R}^3$ and helical or "screw" joints combining rotation and translation along the axis of rotation in $\mathbb{R}^3$. It would also be possible to connect several segments of a robot with planar revolute joints, but with *nonparallel* axes of rotation. All of these possible configurations can be treated by methods similar to the ones we will present, but we will not consider them in detail. Our purpose here is to illustrate how affine varieties can be used to describe the geometry of robots, not to present a treatise on practical robotics. The planar robots provide a class of relatively uncomplicated but illustrative examples for us to consider.

Example 1. Consider the following planar robot "arm" with three revolute joints and one prismatic joint. All motions of the robot take place in the plane of the page.



For easy reference, we number the segments and joints of a robot in increasing order out from the fixed end to the hand. Thus, in the above figure, segment 2 connects joints 1 and 2, and so on. Joint 4 is prismatic, and we will regard segment 4 as having variable length, depending on the setting of the prismatic joint. In this robot, the hand of the robot comprises segment 5.

In general, the position or setting of a revolute joint between segments i and $i+1$ can be described by measuring the angle θ (counterclockwise) from segment i to segment $i+1$. Thus, the totality of settings of such a joint can be parametrized by a *circle* S^1 or by the interval $[0, 2\pi]$ with the endpoints identified. (In some cases, a revolute joint may not be free to rotate through a full circle, and then we would parametrize the possible settings by a subset of S^1.)

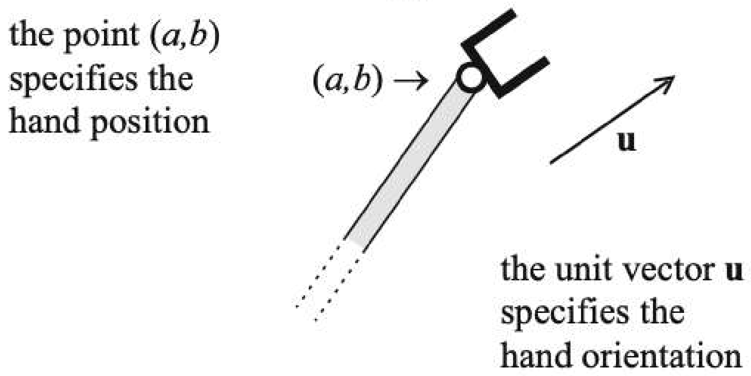
Similarly, the setting of a prismatic joint can be specified by giving the distance the joint is extended or, as in Example 1, by the total length of the segment (i.e., the distance between the end of the joint and the previous joint). Either way, the settings of a prismatic joint can be parametrized by a finite interval of real numbers.

If the joint settings of our robot can be specified independently, then the possible settings of the whole collection of joints in a planar robot with r revolute joints and p prismatic joints can be parametrized by the Cartesian product

$$\mathcal{J} = S^1 \times \cdots \times S^1 \times I_1 \times \cdots \times I_p,$$

where there is one S^1 factor for each revolute joint, and I_j gives the settings of the j-th prismatic joint. We will call $\mathcal{J}$ the *joint space* of the robot.

We can describe the space of possible configurations of the "hand" of a planar robot as follows. Fixing a Cartesian coordinate system in the plane, we can represent the possible positions of the "hand" by the points (a, b) of a region $U \subseteq \mathbb{R}^2$. Similarly, we can represent the orientation of the "hand" by giving a unit vector aligned with some specific feature of the hand. Thus, the possible hand orientations are parametrized by vectors $\mathbf{u}$ in $V = S^1$. For example, if the "hand" is attached to a revolute joint, then we have the following picture of the hand configuration:



We will call $C = U \times V$ the *configuration space* or *operational space* of the robot's hand.

Since the robot's segments are assumed to be rigid, each collection of joint settings will place the "hand" in a uniquely determined location, with a uniquely determined orientation. Thus, we have a function or mapping

$$f : \mathcal{J} \longrightarrow \mathcal{C}$$

which encodes how the different possible joint settings yield different hand configurations.

The two basic problems we will consider can be described succinctly in terms of the mapping $f : \mathcal{J} \to \mathcal{C}$ described above:

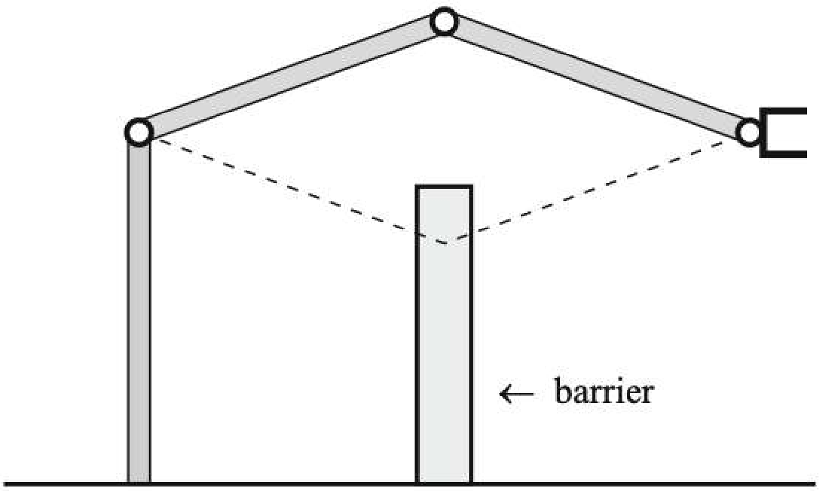
- (Forward Kinematic Problem) Can we give an explicit description or formula for f in terms of the joint settings (our coordinates on $\mathcal{J}$) and the dimensions of the segments of the robot "arm"?
- (Inverse Kinematic Problem) Given $c \in \mathcal{C}$, can we determine one or all the $j \in \mathcal{J}$ such that $f(j) = c$?

In §2, we will see that the forward problem is relatively easily solved. Determining the position and orientation of the "hand" from the "arm" joint settings is mainly a matter of being systematic in describing the relative positions of the segments on either side of a joint. Thus, the forward problem is of interest mainly as a preliminary to the inverse problem. We will show that the mapping $f : \mathcal{J} \to \mathcal{C}$ giving the "hand" configuration as a function of the joint settings may be written as a polynomial mapping as in Chapter 5, §1.

The inverse problem is somewhat more subtle since our explicit formulas will not be linear if revolute joints are present. Thus, we will need to use the general results on systems of polynomial equations to solve the equation

$$f(j) = c. \tag{1}$$

One feature of nonlinear systems of equations is that there can be several different solutions, even when the entire set of solutions is finite. We will see in §3 that this is true for a planar robot arm with three (or more) revolute joints. As a practical matter, the potential nonuniqueness of the solutions of the systems (1) is sometimes very desirable. For instance, if our real world robot is to work in a space containing physical obstacles or barriers to movement in certain directions, it may be the case that some of the solutions of (1) for a given $c \in \mathcal{C}$ correspond to positions that are not physically reachable:



To determine whether it is possible to reach a given position, we might need to determine *all* solutions of (1), then see which one(s) are feasible given the constraints of the environment in which our robot is to work.

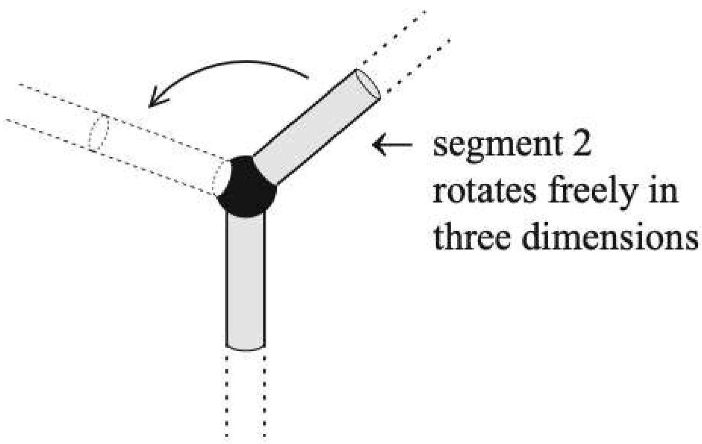
EXERCISES FOR §1

1. Give descriptions of the joint space $\mathcal{J}$ and the configuration space $\mathcal{C}$ for the planar robot picture in Example 1 in the text. For your description of $\mathcal{C}$, determine a bounded subset of $U \subseteq \mathbb{R}^2$ containing all possible hand positions. Hint: The description of U will depend on the lengths of the segments.
2. Consider the mapping $f : \mathcal{J} \to \mathcal{C}$ for the robot pictured in Example 1 in the text. On geometric grounds, do you expect f to be a *one-to-one* mapping? Can you find two different ways to put the hand in some particular position with a given orientation? Are there more than two such positions?

The text discussed the joint space $\mathcal{J}$ and the configuration space $\mathcal{C}$ for planar robots. In the following problems, we consider what $\mathcal{J}$ and $\mathcal{C}$ look like for robots capable of motion in three dimensions.

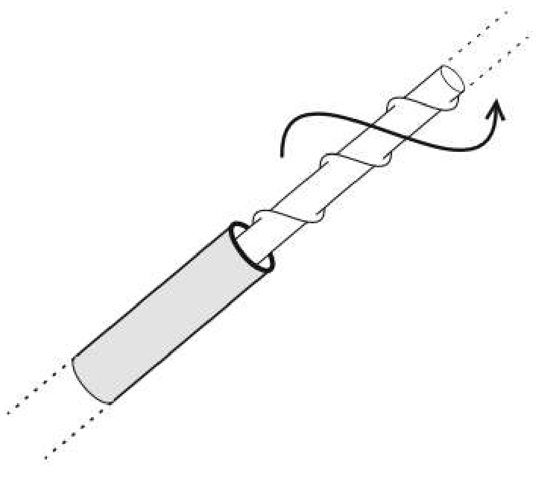
3. What would the configuration space $\mathcal{C}$ look like for a 3-dimensional robot? In particular, how can we describe the possible hand orientations?

4. A "ball" joint at point B allows segment 2 in the robot pictured below to rotate by any angle about any axis in $\mathbb{R}^3$ passing through B. (Note: The motions of this joint are similar to those of the "joystick" found in some computer games.)



a ball joint

   a. Describe the set of possible joint settings for this joint mathematically. Hint: The distinct joint settings correspond to the possible direction vectors of segment 2.
   b. Construct a one-to-one correspondence between your set of joint settings in part (a) and the unit sphere $S^2 \subseteq \mathbb{R}^3$. Hint: One simple way to do this is to use the spherical angular coordinates ϕ, θ on S^2.
5. A helical or "screw" joint at point H allows segment 2 of the robot pictured below to extend out from H along the line L in the direction of segment 1, while rotating about the axis L.



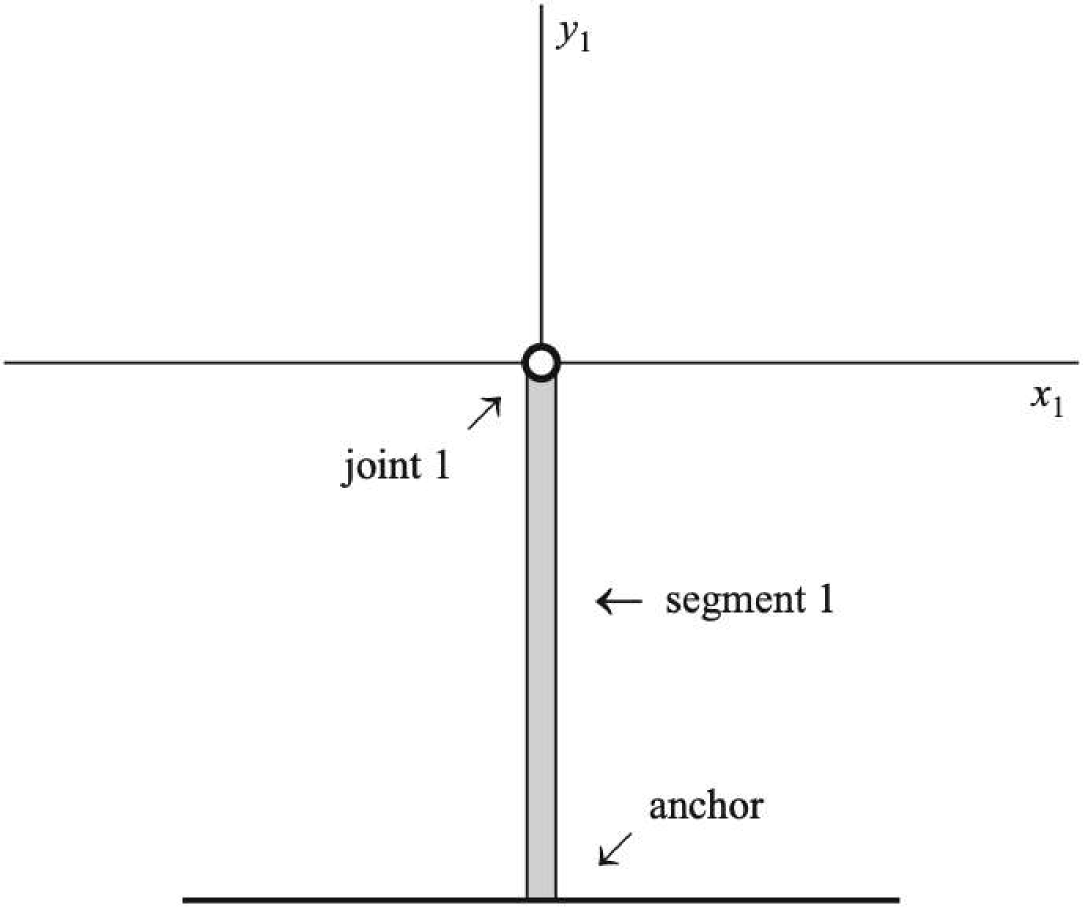
a helical or "screw" joint

   The rotation angle θ (measured from the original, unextended position of segment 2) is given by $\theta = l \cdot \alpha$, where $l \in [0, m]$ gives the distance from H to the other end of segment 2 and α is a constant angle. Give a mathematical description of the space of joint settings for this joint.
6. Give a mathematical description of the joint space $\mathcal{J}$ for a 3-dimensional robot with two "ball" joints and one helical joint.

§2 The Forward Kinematic Problem

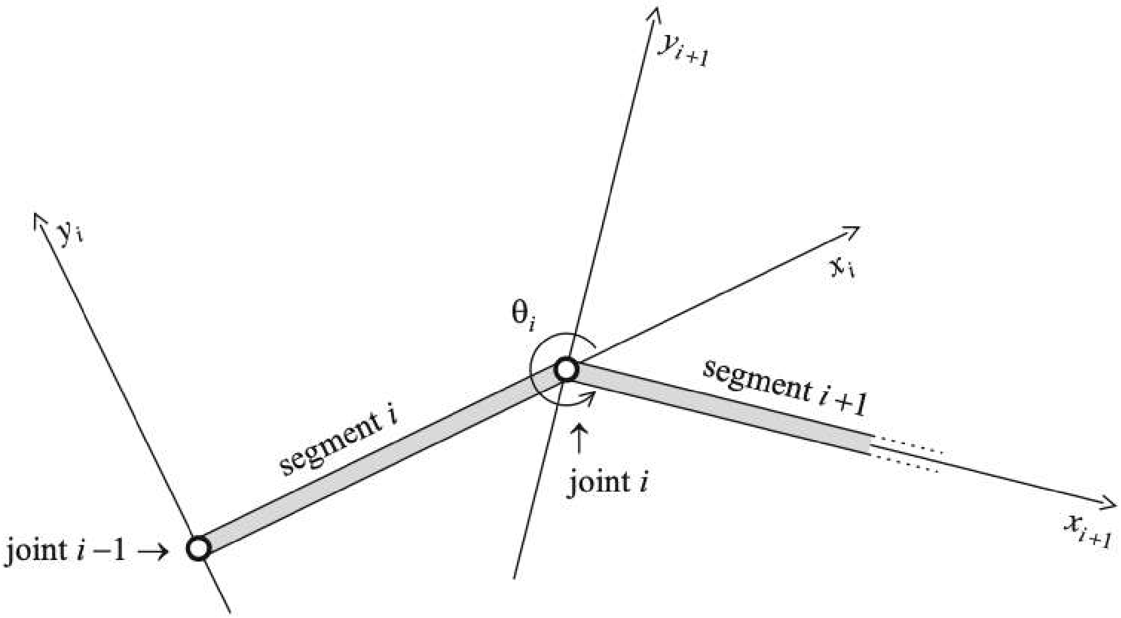
In this section, we will present a standard method for solving the forward kinematic problem for a given robot "arm." As in §1, we will only consider robots in $\mathbb{R}^2$, which means that the "hand" will be constrained to lie in the plane. Other cases will be studied in the exercises.

All of our robots will have a first segment that is anchored, or fixed in position. In other words, there is no movable joint at the initial endpoint of segment 1. With this convention, we will use a standard rectangular coordinate system in the plane to describe the position and orientation of the "hand." The origin of this coordinate system is placed at joint 1 of the robot arm, which is also fixed in position since all of segment 1 is. For example:



In addition to the global (x_1, y_1) coordinate system, we introduce a local rectangular coordinate system at each of the *revolute joints* to describe the relative positions of the segments meeting at that joint. Naturally, these coordinate systems will *change* as the position of the "arm" varies.

At a revolute joint i, we introduce an (x_{i+1}, y_{i+1}) coordinate system in the following way. The origin is placed at joint i. We take the positive x_{i+1}-axis to lie along the direction of segment $i + 1$ (in the robot's current position). Then the positive y_{i+1}-axis is determined to form a normal right-handed rectangular coordinate system. Note that for each $i \geq 2$, the (x_i, y_i) coordinates of joint i are $(l_i, 0)$, where l_i is the length of segment i.



Our first goal is to relate the (x_{i+1}, y_{i+1}) coordinates of a point with the (x_i, y_i) coordinates of that point. Let θ_i be the counterclockwise angle from the x_i-axis to the x_{i+1}-axis. This is the same as the joint setting angle θ_i described in §1. From the diagram above, we see that if a point q has (x_{i+1}, y_{i+1}) coordinates

$$q = (a_{i+1}, b_{i+1}),$$

then to obtain the (x_i, y_i) coordinates of q, say

$$q = (a_i, b_i),$$

we rotate by the angle θ_i (to align the x_i- and x_{i+1}-axes), and then translate by the vector $(l_i, 0)$ (to make the origins of the coordinate systems coincide). In the exercises, you will show that rotation by θ_i is accomplished by multiplying by the rotation matrix

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}.$$

It is also easy to check that translation is accomplished by adding the vector $(l_i, 0)$. Thus, we get the following relation between the (x_i, y_i) and (x_{i+1}, y_{i+1}) coordinates of q:

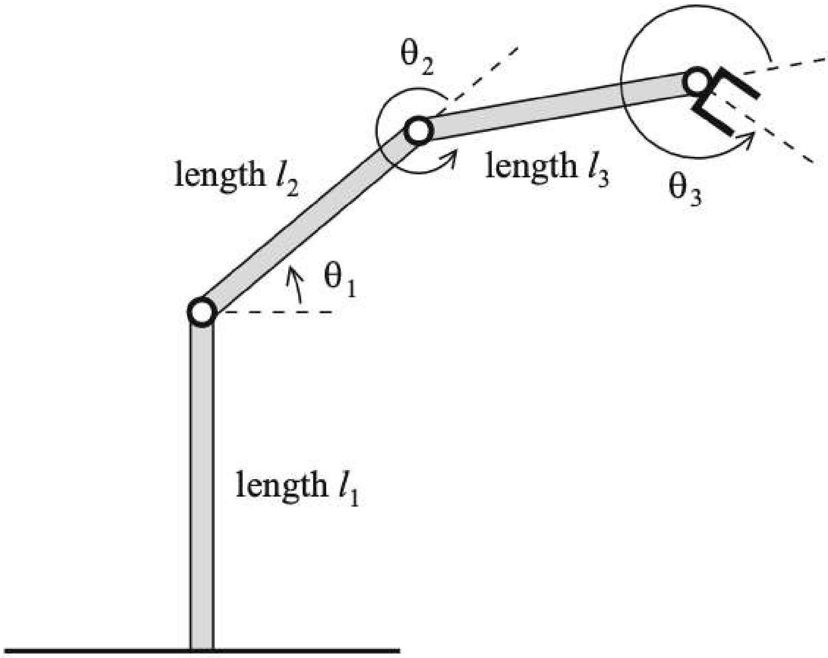
$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix}.$$

This coordinate transformation is also commonly written in a shorthand form using a 3×3 matrix and 3-component vectors:

$$(1) \qquad \begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & l_i \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} = A_i \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}.$$

This allows us to combine the rotation by θ_i with the translation along segment i into a single 3×3 matrix A_i.

Example 1. With this notation in hand, let us next consider a general plane robot "arm" with three revolute joints:



We will think of the hand as segment 4, which is attached via the revolute joint 3 to segment 3. As before, l_i will denote the length of segment i. We have

$$A_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

since the origin of the (x_2, y_2) coordinate system is also placed at joint 1. We also have matrices A_2 and A_3 as in formula (1). The key observation is that the global coordinates of any point can be obtained by starting in the (x_4, y_4) coordinate system and working our way back to the global (x_1, y_1) system one joint at a time. In other words, we multiply the (x_4, y_4) coordinate vector of the point by A_3, A_2, A_1 in turn:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1A_2A_3 \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}.$$

Using the trigonometric addition formulas, this equation can be written as

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2)+l_2\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & l_3\sin(\theta_1+\theta_2)+l_2\sin\theta_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}.$$

Since the (x_4, y_4) coordinates of the hand are $(0,0)$ (because the hand is attached directly to joint 3), we obtain the (x_1, y_1) coordinates of the hand by setting $x_4 = y_4 = 0$ and computing the matrix product above. The result is

$$(2) \qquad \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} l_3\cos(\theta_1+\theta_2)+l_2\cos\theta_1 \\ l_3\sin(\theta_1+\theta_2)+l_2\sin\theta_1 \\ 1 \end{pmatrix}.$$

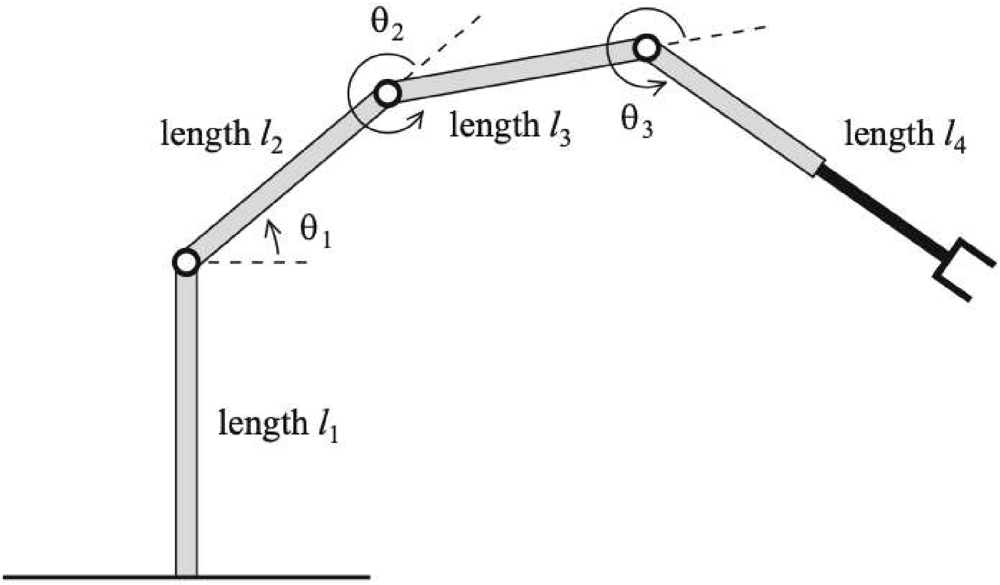
The hand orientation is determined if we know the angle between the x_4-axis and the direction of any particular feature of interest to us on the hand. For instance, we might simply want to use the direction of the x_4-axis to specify this orientation. From our computations, we know that the angle between the x_1-axis and the x_4-axis is simply $\theta_1 + \theta_2 + \theta_3$. Knowing the θ_i allows us to also compute this angle.

If we combine this fact about the hand orientation with the formula (2) for the hand position, we get an explicit description of the mapping $f : \mathcal{J} \to C$ introduced in §1. As a function of the joint angles θ_i, the configuration of the hand is given by

$$(3) \qquad f(\theta_1 + \theta_2 + \theta_3) = \begin{pmatrix} l_3 \cos(\theta_1 + \theta_2) + l_2 \cos\theta_1 \\ l_3 \sin(\theta_1 + \theta_2) + l_2 \sin\theta_1 \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}.$$

The same ideas will apply when any number of planar revolute joints are present. You will study the explicit form of the function f in these cases in Exercise 7.

Example 2. Prismatic joints can also be handled within this framework. For instance, let us consider a planar robot whose first three segments and joints are the same as those of the robot in Example 1, but which has an additional prismatic joint between segment 4 and the hand. Thus, segment 4 will have variable length and segment 5 will be the hand.



The translation axis of the prismatic joint lies along the direction of segment 4. We can describe such a robot as follows. The three revolute joints allow us exactly the same freedom in placing joint 3 as in the robot studied in Example 1. However, the prismatic joint allows us to change the length of segment 4 to any value between $l_4 = m_1$ (when retracted) and $l_4 = m_2$ (when fully extended). By the reasoning given in Example 1, if the setting l_4 of the prismatic joint is known, then the position of the hand will be given by multiplying the product matrix $A_1 A_2 A_3$ times the (x_4, y_4) coordinate vector of the hand, namely $(l_4, 0)$. It follows that the configuration of the hand is given by

$$(4) \quad g(\theta_1,\theta_2,\theta_3,l_4) = \begin{pmatrix} l_4\cos(\theta_1+\theta_2+\theta_3)+l_3\cos(\theta_1+\theta_2)+l_2\cos\theta_1 \\ l_4\sin(\theta_1+\theta_2+\theta_3)+l_3\sin(\theta_1+\theta_2)+l_2\sin\theta_1 \\ \theta_1+\theta_2+\theta_3 \end{pmatrix}.$$

As before, l_2 and l_3 are constant, but $l_4 \in [m_1, m_2]$ is now another variable. The hand orientation will be given by $\theta_1+\theta_2+\theta_3$ as before since the setting of the prismatic joint will not affect the direction of the hand.

We will next discuss how formulas such as (3) and (4) may be converted into representations of f and g as polynomial or rational mappings in suitable variables. The joint variables for revolute and for prismatic joints are handled differently. For the revolute joints, the most direct way of converting to a polynomial set of equations is to use an idea we have seen several times before, for example, in Exercise 8 of Chapter 2, §8. Even though $\cos\theta$ and $\sin\theta$ are transcendental functions, they give a parametrization

$$x = \cos\theta,$$
$$y = \sin\theta$$

of the algebraic variety $\mathbf{V}(x^2+y^2-1)$ in the plane. Thus, we can write the components of the right-hand side of (3) or, equivalently, the entries of the matrix $A_1A_2A_3$ in (2) as functions of

$$c_i = \cos\theta_i,$$
$$s_i = \sin\theta_i,$$

subject to the constraints

$$(5) \qquad c_i^2 + s_i^2 - 1 = 0$$

for $i = 1, 2, 3$. Note that the variety defined by these three equations in $\mathbb{R}^6$ is a concrete realization of the joint space $\mathcal{J}$ for this type of robot. Geometrically, this variety is just a Cartesian product of three copies of the circle.

Explicitly, we obtain from (3) an expression for the hand position as a function of the variables $c_1, s_1, c_2, s_2, c_3, s_3$. Using the trigonometric addition formulas, we can write

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = c_1c_2 - s_1s_2.$$

Similarly,

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 = s_1c_2 + s_2c_1.$$

Thus, the (x_1, y_1) coordinates of the hand position are:

$$(6) \qquad \begin{pmatrix} l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{pmatrix}.$$

In the language of Chapter 5, we have defined a polynomial mapping from the variety $\mathcal{J} = \mathbf{V}(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1)$ to $\mathbb{R}^2$. Note that the hand *position* does not depend on θ_3. That angle enters only in determining the hand orientation.

Since the hand orientation depends directly on the angles θ_i themselves, it is *not* possible to express the orientation itself as a polynomial in $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. However, we can handle the orientation in a similar way. See Exercise 3.

Similarly, from the mapping g in Example 2, we obtain the polynomial form

$$(7) \qquad \begin{pmatrix} l_4(c_1(c_2c_3 - s_2s_3) - s_1(c_2s_3 + c_3s_2)) + l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_4(s_1(c_2c_3 - s_2s_3) + c_1(c_2s_3 + c_3s_2)) + l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{pmatrix}$$

for the (x_1, y_1) coordinates of the hand position. Here $\mathcal{J}$ is the subset $V \times [m_1, m_2]$ of the variety $V \times \mathbb{R}$, where $V = \mathbf{V}(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1)$. The length l_4 is treated as another ordinary variable in (7), so our component functions are polynomials in l_4, and the c_i and s_i.

A second way to write formulas (3) and (4) is based on the *rational* parametrization

$$(8) \qquad \begin{aligned} x &= \frac{1 - t^2}{1 + t^2}, \\ y &= \frac{2t}{1 + t^2} \end{aligned}$$

of the circle introduced in §3 of Chapter 1. [In terms of the trigonometric parametrization, $t = \tan(\theta/2)$.] This allows us to express the mapping (3) in terms of three variables $t_i = \tan(\theta_i/2)$. We will leave it as an exercise for the reader to work out this alternate explicit form of the mapping $f : \mathcal{J} \to \mathcal{C}$ in Example 1. In the language of Chapter 5, the variety $\mathcal{J}$ for the robot in Example 1 is birationally equivalent to $\mathbb{R}^3$. We can construct a rational parametrization $\rho : \mathbb{R}^3 \to \mathcal{J}$ using three copies of the parametrization (8). Hence, we obtain a rational mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$, expressing the hand coordinates of the robot arm as functions of t_1, t_2, t_3 by taking the composition of ρ with the hand coordinate mapping in the form (6).

Both of these forms have certain advantages and disadvantages for practical use. For the robot of Example 1, one immediately visible advantage of the rational mapping obtained from (8) is that it involves only three variables rather than the six variables s_i, c_i, $i = 1, 2, 3$, needed to describe the full mapping f as in Exercise 3. In addition, we do not need the three extra constraint equations (5). However, the t_i values corresponding to joint positions with θ_i close to π are awkwardly large, and there is no t_i value corresponding to $\theta_i = \pi$. We do not obtain every theoretically possible hand position in the image of the mapping f when it is expressed in this form. Of course, this might not actually be a problem if our robot is constructed so that segment $i+1$ is not free to fold back onto segment i (i.e., the joint setting $\theta_i = \pi$ is not possible). The polynomial form (6) is more unwieldy, but since it comes from the trigonometric (unit-speed) parametrization of the circle, it does not suffer from the potential shortcomings of the rational form. It would be somewhat better suited for revolute joints that can freely rotate through a full circle.

EXERCISES FOR §2

1. Consider the plane $\mathbb{R}^2$ with an orthogonal right-handed coordinate system (x_1, y_1). Now introduce a second coordinate system (x_2, y_2) by rotating the first counterclockwise by an angle θ. Suppose that a point q has (x_1, y_1) coordinates (a_1, b_1) and (x_2, y_2) coordinates (a_2, b_2). We claim that

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

To prove this, first express the (x_2, y_2) coordinates of q in polar form as

$$q = (a_2, b_2) = (r\cos\alpha,\ r\sin\alpha).$$

   a. Show that the (x_1, y_1) coordinates of q are given by

$$q = (a_1, b_1) = (r\cos(\alpha+\theta),\ r\sin(\alpha+\theta)).$$

   b. Now use trigonometric identities to prove the desired formula.

2. In Examples 1 and 2, we used a 3×3 matrix A to represent each of the changes of coordinates from one local system to another. Those changes of coordinates were rotations, followed by translations. These are special types of *affine transformations*.

   a. Show that any affine transformation in the plane

$$\begin{aligned} x' &= ax + by + e, \\ y' &= cx + dy + f \end{aligned}$$

   can be represented in a similar way:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

   b. Give a similar representation for affine transformations of $\mathbb{R}^3$ using 4×4 matrices.

3. In this exercise, we will reconsider the hand orientation for the robots in Examples 1 and 2. Namely, let $\alpha = \theta_1 + \theta_2 + \theta_3$ be the angle giving the hand orientation in the (x_1, y_1) coordinate system.

   a. Using the trigonometric addition formulas, show that

$$c = \cos\alpha,\ s = \sin\alpha$$

   can be expressed as polynomials in $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. Thus, the whole mapping f can be expressed in polynomial form, at the cost of introducing an extra coordinate function for $\mathcal{C}$.

   b. Express c and s using the rational parametrization (8) of the circle.

4. Consider a planar robot with a revolute joint 1, segment 2 of length l_2, a prismatic joint 2 with settings $l_3 \in [0, m_3]$, and a revolute joint 3, with segment 4 being the hand.

   a. What are the joint and configuration spaces $\mathcal{J}$ and $\mathcal{C}$ for this robot?
   b. Using the method of Examples 1 and 2, construct an explicit formula for the mapping $f : \mathcal{J} \to \mathcal{C}$ in terms of the trigonometric functions of the joint angles.
   c. Convert the function f into a polynomial mapping by introducing suitable new coordinates.

5. Rewrite the mappings f and g in Examples 1 and 2, respectively, using the rational parametrization (8) of the circle for each revolute joint. Show that in each case the hand position

and orientation are given by rational mappings on $\mathbb{R}^n$. (The value of n will be different in the two examples.)

6. Rewrite the mapping f for the robot from Exercise 4, using the rational parametrization (8) of the circle for each revolute joint.
7. Consider a planar robot with a fixed segment 1 as in our examples in this section and with n revolute joints linking segments of length $l_2, \ldots, l_n$. The hand is segment $n+1$, attached to segment n by joint n.
   a. What are the joint and configuration spaces for this robot?
   b. Show that the mapping $f : \mathcal{J} \to \mathcal{C}$ for this robot has the form

$$f(\theta_1, \ldots, \theta_n) = \begin{pmatrix} \sum_{i=1}^{n-1} l_{i+1} \cos\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{n-1} l_{i+1} \sin\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{n} \theta_i \end{pmatrix}.$$

   Hint: Argue by induction on n.
8. Another type of 3-dimensional joint is a "spin" or nonplanar revolute joint that allows one segment to rotate or spin in the plane perpendicular to the other segment. In this exercise, we will study the forward kinematic problem for a 3-dimensional robot containing two "spin" joints. As usual, segment 1 of the robot will be fixed, and we will pick a global coordinate system (x_1, y_1, z_1) with the origin at joint 1 and segment 1 on the z_1-axis. Joint 1 is a "spin" joint with rotation axis along the z_1-axis, so that segment 2 rotates in the (x_1, y_1)-plane. Then segment 2 has length l_2 and joint 2 is a second "spin" joint connecting segment 2 to segment 3. The axis for joint 2 lies along segment 2, so that segment 3 always rotates in the plane perpendicular to segment 2.
   a. Construct a local right-handed orthogonal coordinate system (x_2, y_2, z_2) with origin at joint 1, with the x_2-axis in the direction of segment 2 and the y_2-axis in the (x_1, y_1)-plane. Give an explicit formula for the (x_1, y_1, z_1) coordinates of a general point, in terms of its (x_2, y_2, z_2) coordinates and of the joint angle θ_1.
   b. Express your formula from part (a) in matrix form, using the 4×4 matrix representation for affine space transformations given in part (b) of Exercise 2.
   c. Now, construct a local orthogonal coordinate system (x_3, y_3, z_3) with origin at joint 2, the x_3-axis in the direction of segment 3, and the z_3-axis in the direction of segment 2. Give an explicit formula for the (x_2, y_2, z_2) coordinates of a point in terms of its (x_3, y_3, z_3) coordinates and the joint angle θ_2.
   d. Express your formula from part (c) in matrix form.
   e. Give the transformation relating the (x_3, y_3, z_3) coordinates of a general point to its (x_1, y_1, z_1) coordinates in matrix form. Hint: This will involve suitably multiplying the matrices found in parts (b) and (d).
9. Consider the robot from Exercise 8.
   a. Using the result of part (c) of Exercise 8, give an explicit formula for the mapping $f : \mathcal{J} \to \mathcal{C}$ for this robot.
   b. Express the hand position for this robot as a polynomial function of the variables $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$.
   c. The orientation of the hand (the end of segment 3) of this robot can be expressed by giving a unit vector in the direction of segment 3, expressed in the global coordinate system. Find an expression for the hand orientation.

§3 The Inverse Kinematic Problem and Motion Planning

In this section, we will continue the discussion of the robot kinematic problems introduced in §1. To begin, we will consider the inverse kinematic problem for the planar robot arm with three revolute joints studied in Example 1 of §2. Given a point $(x_1, y_1) = (a, b) \in \mathbb{R}^2$ and an orientation, we wish to determine whether it is possible to place the hand of the robot at that point with that orientation. If it is possible, we wish to find all combinations of joint settings that will accomplish this. In other words, we want to determine the *image* of the mapping $f : \mathcal{J} \to \mathcal{C}$ for this robot; for each c in the image of f, we want to determine the *inverse image* $f^{-1}(c)$.

It is quite easy to see geometrically that if $l_3 = l_2 = l$, the hand of our robot can be placed at any point of the closed disk of radius $2l$ centered at joint 1—the origin of the (x_1, y_1) coordinate system. On the other hand, if $l_3 \neq l_2$, then the hand positions fill out a closed annulus centered at joint 1. (See, for example, the ideas used in Exercise 14 of Chapter 1, §2.) We will also be able to see this using the solution of the forward problem derived in equation (7) of §2. In addition, our solution will give *explicit formulas* for the joint settings necessary to produce a given hand position. Such formulas could be built into a control program for a robot of this kind.

For this robot, it is also easy to control the hand orientation. Since the setting of joint 3 is independent of the settings of joints 1 and 2, we see that, given any θ_1 and θ_2, it is possible to attain any desired orientation $\alpha = \theta_1 + \theta_2 + \theta_3$ by setting $\theta_3 = \alpha - (\theta_1 + \theta_2)$ accordingly.

To simplify our solution of the inverse kinematic problem, we will use the above observation to ignore the hand orientation. Thus, we will concentrate on the position of the hand, which is a function of θ_1 and θ_2 alone. From equation (6) of §2, we see that the possible ways to place the hand at a given point $(x_1, y_1) = (a, b)$ are described by the following system of polynomial equations:

$$
\begin{aligned}
a &= l_3(c_1c_2 - s_1s_2) + l_2c_1,\\
b &= l_3(c_1s_2 + c_2s_1) + l_2s_1,\\
0 &= c_1^2 + s_1^2 - 1,\\
0 &= c_2^2 + s_2^2 - 1
\end{aligned}
\tag{1}
$$

for c_1, s_1, c_2, s_2. To solve these equations, we first compute a grevlex Gröbner basis with

$$c_1 > s_1 > c_2 > s_2.$$

Our solutions will depend on the values of a, b, l_2, l_3, which appear as symbolic parameters in the coefficients of the Gröbner basis:

$$
\begin{aligned}
&c_1 - \frac{2bl_2l_3}{2l_2(a^2+b^2)}s_2 - \frac{a(a^2+b^2+l_2^2-l_3^2)}{2l_2(a^2+b^2)},\\
&s_1 + \frac{2al_2l_3}{2l_2(a^2+b^2)}s_2 + \frac{b(a^2+b^2+l_2^2-l_3^2)}{2l_2(a^2+b^2)},\\
&c_2 - \frac{a^2+b^2-l_2^2-l_3^2}{2l_2l_3},\\
&s_2^2 + \frac{(a^2+b^2)^2 - 2(a^2+b^2)(l_2^2+l_3^2) + (l_2^2-l_3^2)^2}{4l_2^2l_3^2}.
\end{aligned}
\tag{2}
$$

In algebraic terms, this is the reduced Gröbner basis for the ideal I generated by the polynomials in (1) in the ring $\mathbb{R}(a, b, l_2, l_3)[c_1, s_1, c_2, s_2]$. Note that we allow denominators that depend only on the parameters a, b, l_2, l_3.

This is the first time we have computed a Gröbner basis over a field of rational functions, and one has to be a bit careful about how to interpret (2). Working over $\mathbb{R}(a, b, l_2, l_3)$ means that a, b, l_2, l_3 are abstract variables over $\mathbb{R}$, and, in particular, they are algebraically independent [i.e., if p is a polynomial with real coefficients such that $p(a, b, l_2, l_3) = 0$, then p must be the zero polynomial]. Yet, in practice, we want a, b, l_2, l_3 to be certain specific real numbers. When we make such a substitution, the polynomials (1) generate an ideal $\bar{I} \subseteq \mathbb{R}[c_1, s_1, c_2, s_2]$ corresponding to a specific hand position of a robot with specific segment lengths. The key question is whether (2) remains a Gröbner basis for $\bar{I}$ under this substitution. In general, the replacement of variables by specific values in a field is called *specialization*, and the question is how a Gröbner basis behaves under specialization.

A first observation is that we expect problems when a specialization causes any of the denominators in (2) to vanish. This is typical of how specialization works: things usually behave nicely for most (but not all) values of the variables. In the exercises, you will prove that there is a proper subvariety $W \subseteq \mathbb{R}^4$ such that (2) specializes to a Gröbner basis of $\bar{I}$ whenever a, b, l_2, l_3 take values in $\mathbb{R}^4 \setminus W$. We also will see that there is an algorithm for finding W. The subtle point is that, in general, the vanishing of denominators is not the only thing that can go wrong (you will work out some examples in the exercises). Fortunately, in the example we are considering, it can be shown that W is, in fact, defined by the vanishing of the denominators. This means that if we choose values $l_2 \neq 0, l_3 \neq 0$, and $a^2 + b^2 \neq 0$, then (2) still gives a Gröbner basis of (1). The details of the argument will be given in Exercise 9.

Given such a specialization, two observations follow immediately from the leading terms of the Gröbner basis (2). First, any zero s_2 of the last polynomial can be extended uniquely to a full solution of the system. Second, the set of solutions of (1) is a *finite* set for this choice of a, b, l_2, l_3. Indeed, since the last polynomial in (2) is quadratic in s_2, there can be at most *two* distinct solutions. It remains to see which a, b yield *real* values for s_2 (the relevant solutions for the geometry of our robot).

To simplify the formulas somewhat, we will specialize to the case $l_2 = l_3 = 1$. In Exercise 1, you will show that by either substituting $l_2 = l_3 = 1$ directly into (2) or setting $l_2 = l_3 = 1$ in (1) and recomputing a Gröbner basis in $\mathbb{R}(a, b)[c_1, s_1, c_2, s_2]$, we obtain the same result:

$$
\begin{aligned}
&c_1 - \frac{2b}{2(a^2+b^2)} s_2 - \frac{a}{2},\\
&s_1 + \frac{2a}{2(a^2+b^2)} s_2 + \frac{b}{2},\\
&c_2 - \frac{a^2+b^2-2}{2},\\
&s_2^2 + \frac{(a^2+b^2)(a^2+b^2-4)}{4}.
\end{aligned}
\tag{3}
$$

Other choices for l_2 and l_3 will be studied in Exercise 4. [Although (2) remains a Gröbner basis for any nonzero values of l_2 and l_3, the geometry of the situation changes rather dramatically if $l_2 \neq l_3$.]

It follows from our earlier remarks that (3) is a Gröbner basis for (1) for all specializations of a and b where $a^2 + b^2 \neq 0$, which over $\mathbb{R}$ happens whenever the hand is not at the origin. Solving the last equation in (3), we find that

$$s_2 = \pm\frac{1}{2}\sqrt{(a^2+b^2)(4-(a^2+b^2))}.$$

Note that the solution(s) of this equation are real if and only if $a^2 + b^2 \leq 4$, and when $a^2 + b^2 = 4$, we have a double root. From the geometry of the system, that is exactly what we expect. The distance from joint 1 to joint 3 is at most $l_2 + l_3 = 2$, and positions with $l_2 + l_3 = 2$ can be reached in only one way—by setting $\theta_2 = 0$ so that segment 3 and segment 2 point in the same direction.

Given s_2, the other elements of the Gröbner basis (3) give exactly one value for each of c_1, s_1, c_2. Further, since $c_1^2 + s_1^2 - 1$ and $c_2^2 + s_2^2 - 1$ are in the ideal generated by (3), the values we get for c_1, s_1, c_2, s_2 uniquely determine the joint angles θ_1 and θ_2. Thus, the case where $a^2 + b^2 \neq 0$ is fully understood.

It remains to study s_1, c_1, s_2, c_2 when $a = b = 0$. Geometrically, this means that joint 3 is placed at the origin of the (x_1, y_1) system—at the same point as joint 1. The first two polynomials in our basis (3) are undefined when we substitute $a = b = 0$ in their coefficients. This is a case where specialization fails. In fact, setting $l_2 = l_3 = 1$ and $a = b = 0$ into the original system (1) yields the grevlex Gröbner basis

$$\begin{aligned} &c_1^2 + s_1^2 - 1,\\ &c_2 + 1,\\ &s_2. \end{aligned} \tag{4}$$

With a little thought, the geometric reason for this is visible. There are actually *infinitely many* different possible configurations that will place joint 3 at the origin since segments 2 and 3 have equal lengths. The angle θ_1 can be specified arbitrarily, and then setting $\theta_2 = \pi$ will fold segment 3 back along segment 2, placing joint 3 at $(0, 0)$. These are the only joint settings placing the hand at $(a, b) = (0, 0)$. In Exercise 3 you will verify that this analysis is fully consistent with (4).

Note that the *form* of the specialized Gröbner basis (4) is different from the general form (2). The equation for s_2 now has degree 1, and the equation for c_1 (rather than the equation for s_2) has degree 2. Below we will say more about how Gröbner bases can change under specialization.

This completes the analysis of our robot arm. To summarize, given any (a, b) in (x_1, y_1) coordinates, to place joint 3 at (a, b) when $l_2 = l_3 = 1$, there are

- infinitely many distinct settings of joint 1 when $a^2 + b^2 = 0$,
- two distinct settings of joint 1 when $a^2 + b^2 < 4$,
- one setting of joint 1 when $a^2 + b^2 = 4$,
- no possible settings of joint 1 when $a^2 + b^2 > 4$.

The cases $a^2 + b^2 = 0, 4$ are examples of what are known as *kinematic singularities* for this robot. We will give a precise definition of this concept and discuss some of its meaning below.

In the exercises, you will consider the robot arm with three revolute joints and one prismatic joint introduced in Example 2 of §2. There are more restrictions here for the hand orientation. For example, if l_4 lies in the interval $[0, 1]$, then the hand can be placed in any position in the closed disk of radius 3 centered at $(x_1, y_1) = (0, 0)$. However, an interesting difference is that points on the boundary circle can only be reached with one hand orientation.