Exercise 3.2. Simple pendulum[1]

Let us consider the simple pendulum of Figure 1 described by the following state equations:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\sin x_1.\end{aligned}$$

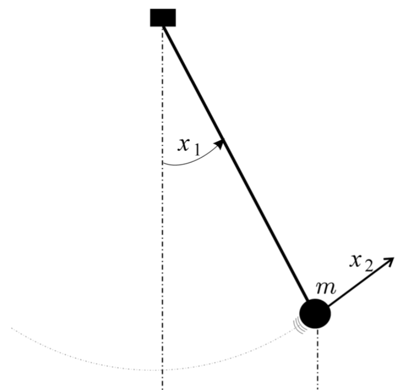

Figure 1: Simple pendulum with state vector $\mathbf{x} = (x_1, x_2) = (\theta, \dot{\theta})$.

The vector field associated with the evolution function $\mathbf{f}(x)$ is drawn in Figure 2.

[1]Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

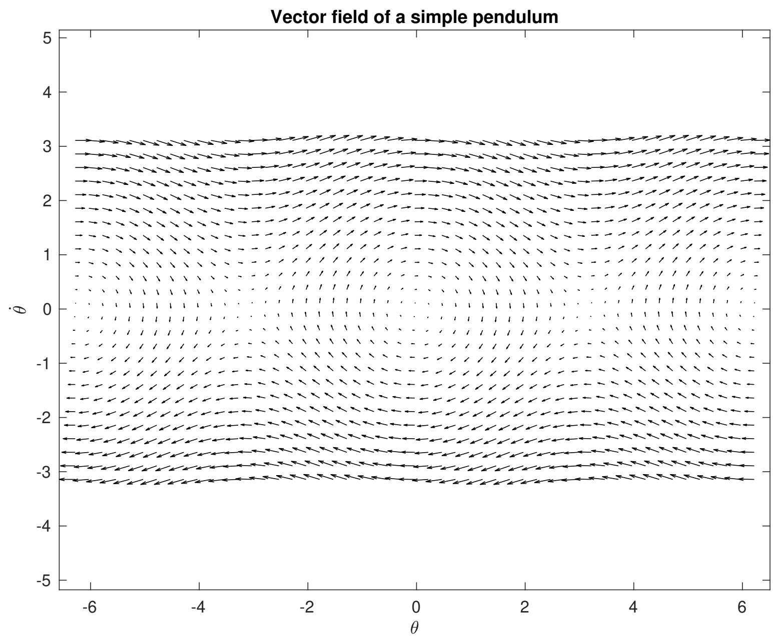

Figure 2: Vector field associated with the simple pendulum in the phase plane $(x_1, x_2) = (\theta, \dot{\theta})$.

1) From the vector field, give the stable and unstable points of equilibrium.
2) Program an Euler integration and draw the trajectory on the state space.
3) Compare the Euler integration with a Runge-Kutta method.

Solution of Exercise 3.2.

1) From the vector field, give the stable and unstable points of equilibrium.

   The state equations of the system are

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -g \sin x_1, \end{aligned}$$

   which is in the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. For the sake of simplicity we will assume that the gravity acceleration is $g = 1$. The equilibrium states are those states for which $\dot{\mathbf{x}} = 0$. So, they can be found solving the equation

$$\mathbf{f}(\mathbf{x}) = 0,$$

that is, the set of equations

$$\begin{aligned} x_2 &= 0, \\ -\sin x_1 &= 0, \end{aligned}$$

From the second equation we have the solution

$$x_1 = k\pi, k \in \mathbb{Z}$$

with $\mathbb{Z}$ the set of integer numbers, that is $k = 0, \pm 1, \pm 2, \ldots$. This means that the equilibrium states are those in which the angular velocity $x_2 = 0$, and the angular position of the pendulum is $x_1 = k\pi, k = 0, \pm 1, \pm 2, \ldots$. We have to distinguish two types of equilibrium states: stable (not asintotically, because there is not friction in the model) or unstable.

- For $x_1 = 0 + 2k\pi$, the equilibrium state is stable (down position of the peundulum)
- For $x_1 = \pi + 2k\pi$, the equilibrium state is unstable (top position of the pendulum)

Let's draw the vector field of this system using `Matlab`.

We get Figure 2 from which we will try to guess the trajectories.

Go to https://en.wikipedia.org/wiki/Pendulum.