Exercise 4.13 Canonical observation form[1]

Let us consider the order 3 linear system with a single input and a single output, described by the following differential equation:

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{u} + b_1\dot{u} + b_0 u$$

1) Show that this differential equation can be written in integral form as:

$$y = \int \left(\int \left(\int (b_2\ddot{u} - a_2\ddot{y} + b_1\dot{u} - a_1\dot{y} + b_0 u - a_0 y)\, dt \right) dt \right) dt$$

2) Deduce from this a wiring system with only three integrators, some adders and amplifiers.

3) Give the state equations associated with this wiring.

4) Compare this with the results obtained in Exercise 4.12.

Solution

1) Show that this differential equation can be written in integral form as:

$$y = \int \left(\int \left(\int (b_2\ddot{u} - a_2\ddot{y} + b_1\dot{u} - a_1\dot{y} + b_0 u - a_0 y)\, dt \right) dt \right) dt.$$

From

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{u} + b_1\dot{u} + b_0 u$$

we can write

$$\dddot{y} = b_2\ddot{u} - a_2\ddot{y} + b_1\dot{u} - a_1\dot{y} + b_0 u - a_0 y$$

By integrating, we can remove the time derivative of y obtaining

$$y = \int \left(\int \left(\int (b_2\ddot{u} - a_2\ddot{y} + b_1\dot{u} - a_1\dot{y} + b_0 u - a_0 y)\, dt \right) dt \right) dt.$$

2) Deduce from this a wiring system with only three integrators, some adders and amplifiers.

From

$$y = \int \left(\int \left(\int (b_2\ddot{u} - a_2\ddot{y} + b_1\dot{u} - a_1\dot{y} + b_0 u - a_0 y)\, dt \right) dt \right) dt.$$

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

since the integral operator is distributive with respect to the sum, we can write

$$y = \int \left\{ b_2 u - a_2 y + \int \left[b_1 u - a_1 y + \int (b_0 u - a_0 y)\, dt \right] dt \right\} dt.$$

By defining

$$\begin{aligned} x_1 &= \int (b_0 u - a_0 y) dt, \\ x_2 &= \int (b_1 u - a_1 y + x_1) dt, \\ x_3 &= \int (b_2 u - a_2 y + x_2) dt, \end{aligned}$$

from this we deduce the wiring of Figure 1

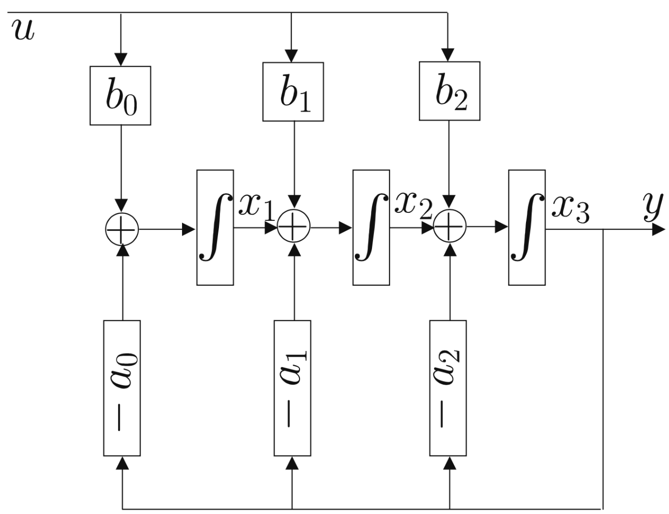

Figure 1: Canonical observation form for a system of order 3.

3) Give the state equations associated with this wiring.

From this diagram, we can directly deduce the state equations of the system

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} u, \\ y &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \end{aligned}$$

The transfer function of our system obtained in Exercise 4.12 is:

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

This particular form for the state representation, which involves the coefficients of the transfer function in the matrices is called canonical observation form. Thus, generally, in order to obtain the canonical form of control equivalent to a given monovariate linear system, we have to calculate its transfer function. Then, knowing a_0, a_1, a_2, we can immediately write its canonical observation form.

4) Compare this with the results obtained in Exercise 4.12.

Let us note that the transformation $A \to A^T$, $B \to C^T$, $C \to B^T$, gives us the canonical form of control of Exercise 4.12.

A comprehensive treatment of the `Matlab` predefined functions to convert a spate space representation of a linear systems into a canonical state-space representation can be found in the `Matlab` documentation[2]

[2] https://es.mathworks.com/help/control/ug/canonical-state-space-realizations.html