Exercise 6.3. Control of a first-order non-linear system[1]

Let us consider the non-linear system given by the state equations:

$$\begin{aligned} \dot{x} &= 2x^2 + u, \\ y &= 3x, \end{aligned}$$

which we wish to stabilize around the state $\bar{x} = 2$. At equilibrium, we would like y to be equal to its setpoint w. Moreover, we would like all the poles of the looped system to be equal to -1:

1) Give the state equations of the controller using pole placement which satisisfies these constraints.

2) What are the state equations of the looped system?

Solution of Exercise 6.3

At equilibrium, we have $\mathbf{f}(\bar{x}, \bar{u}) = 0$. Thus the equilibrium states are described by

$$\begin{aligned} 0 &= 2\bar{x}^2 + \bar{u}, \\ \bar{y} &= 3\bar{x}, \end{aligned}$$

For $\bar{x} = 2$, we need to have $\bar{u} = -8$. The corresponding output is $\bar{y} = 6$.

Let us linearize the system around this equilibrium point. Since

$$\begin{aligned} \mathbf{A} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\bar{\mathbf{x}}, \bar{\mathbf{u}}), \qquad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\bar{\mathbf{x}}, \bar{\mathbf{u}}), \\ \mathbf{C} &= \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\bar{\mathbf{x}}, \bar{\mathbf{u}}), \qquad \mathbf{D} = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}(\bar{\mathbf{x}}, \bar{\mathbf{u}}), \end{aligned}$$

we get $\mathbf{A} = 8, \mathbf{B} = 1, \mathbf{C} = 3, \mathbf{D} = 0$. Thus, the linearized system around $\bar{x} = 2$ is

$$\begin{aligned} \dot{\widetilde{x}} &= 8\widetilde{x} + \widetilde{u}, \\ \widetilde{y} &= 3\widetilde{x}, \end{aligned}$$

where $\widetilde{x} = x - \bar{x}$ and $\widetilde{u} = u - \bar{u}$. If we want that at equilibrium $\bar{y} = \bar{w} = 6$, we have to take $E = 3$.

For the control of the linearized system we use the output feedback scheme with an observer of the state. To have all the poles of the looped system at -1, to find matrices K and L, we need to solve the two polynomial equations

$$\begin{aligned} \det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) &= s + 1, \\ \det(s\mathbf{I} - \mathbf{A}^T + \mathbf{C}^T\mathbf{L}^T) &= s + 1, \end{aligned}$$

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

which in this case become

$$
\begin{aligned}
s - 8 + K &= s + 1, \\
s - 8 + 3L &= s + 1,
\end{aligned}
$$

Therefore, $K = 9$ and $L = 3$. The precompensator H is

$$\mathbf{H} = -\left(\mathbf{E}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}\right)^{-1} = -\left(3\,(8-9)^{-1}\,1\right)^{-1} = \frac{1}{3}.$$

For $\overline{x} = 2$, we have $\overline{u} = -8$ and $\overline{y} = 6$. Therefore $\overline{w} = 6 = 3\overline{x}$.

The state equation of the observer is

$$\frac{d}{dt}\hat{\mathbf{x}} \quad = \quad (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{BH}(\mathbf{w} - \overline{\mathbf{w}}) + \mathbf{L}(\mathbf{y} - \overline{\mathbf{y}}),$$

which in this case becomes

$$
\begin{aligned}
\frac{d}{dt}\hat{x} \quad &= \quad (8 - 1 \cdot 9 - 3 \cdot 3)\,\hat{x} + 1 \cdot \frac{1}{3}(w - 6) + 3(y - 6) \\
&= \quad -10\,\hat{x} + \frac{1}{3}(w - 6) + 3(y - 6).
\end{aligned}
$$

The input u of the controlled system is

$$\mathbf{u} \quad = \quad \overline{\mathbf{u}} - \mathbf{K}\hat{\mathbf{x}} + \mathbf{H}(\mathbf{w} - \overline{\mathbf{w}}),$$

which in this case becomes

$$u \quad = \quad -8 - 9\hat{x} + \frac{1}{3}(w - 6).$$

Simulate the behaviour of the system with Euler and Runge-Kutta methods