Exercise 5.13. State feedback with integral effect, monovariate case[1]

We consider the system described by the state equations:

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y & = & x_1, \end{array}$$

where u is the input, y the output and $\mathbf{x}$ the state vector.

1) Give the characteristic polynomial of the system. Is the system stable?

2) We loop the system by the following state feedback control

$$u = \alpha \int_0^t (w(\tau) - y(\tau)) d\tau - \mathbf{K}\mathbf{x},$$

with $\mathbf{K} = (k_1, k_2)$, where w is the setpoint. Give the state equations of the controller (we will denote by z the state variable of the controller). What are the poles of the controller?

3) Give the state equations of the looped system.

4) Calculate $\mathbf{K}$ and α in order for all the poles to be equal to -1.

5) We choose a setpoint $w = \overline{w}$ constant in time. What values $\overline{\mathbf{x}}$ and $\overline{z}$, does the state of the system $\mathbf{x}$ and state of the controller z tend to? What value $\overline{y}$ does the output y tend to?

6) We now replace the evolution matrix A by another matrix $\overline{A}$ close to A, while keeping the same controller. What value y will converge to?

Solution of Exercise 5.13

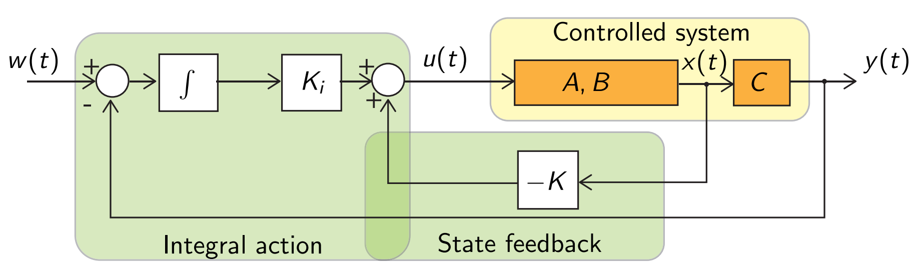

Figure 1: Scheme of the controller with state feedback and integral effect

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

1) Give the characteristic polynomial of the system. Is the system stable?

$$\begin{aligned} P(s) &= \det(s\mathbf{I} - \mathbf{A}) \\ &= \det\left(s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\right) \\ &= \det\begin{pmatrix}s-1 & -1\\ 0 & s-2\end{pmatrix} \\ &= (s-1)(s-2) \\ &= s^2 - 3s + 2. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are $1, 2$. They are real positive numbers and therefore, the system is unstable.

2) We loop the system by the following state feedback control

$$u = \alpha \int_0^t (w(\tau) - y(\tau))d\tau - \mathbf{K}\mathbf{x},$$

with $\mathbf{K} = (k_1, k_2)$, where w is the setpoint. Give the state equations of the controller (we will denote by z the state variable of the controller). What are the poles of the controller?

We introduce a new variable

$$z(t) = \int_0^t (w(\tau) - y(\tau))d\tau.$$

We have $\dot{z}(t) = w(t) - y(t) = -x_1(t) + w(t)$. The state equations of the controller are

$$\begin{aligned} \dot{z} &= -x_1 + w, \\ u &= -k_1x_1 - k_2x_2 + \alpha z \end{aligned}$$

In this case $\mathbf{A} = 0$. This means that $\det(s\mathbf{I} - \mathbf{A}) = s$, and the only pole of the controller is 0.

3) Give the state equations of the looped system.

The looped system has as state equations

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= 2x_2 + u = 2x_2 - k_1x_1 - k_2x_2 + \alpha z \\ \dot{z} &= -x_1 + w, \\ y &= x_1 \end{aligned}$$

or, in matrix form:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 0 \\ -k_1 & 2-k_2 & \alpha \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w \\
y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix}
\end{aligned}$$

4) Calculate $\mathbf{K}$ and α in order for all the poles to be equal to -1.

```
syms s k1 k2 alpha

A = [1 1 0;
    -k1 (2-k2) alpha;
    -1 0 0];

P = simplify(det(s*eye(3)-A))


P =
alpha + 2*s + k1*s - k2*s + k2*s^2 - 3*s^2 + s^3
```

The characteristic polynomial is.

$$P(s) = s^3 + (k_2 - 3)s^2 + (k_1 - k_2 + 2)s + \alpha$$

We would like it to be equal to $s^3 + 3s^2 + 3s + 1$. Thus, it must be

$$\begin{aligned}
\alpha &= 1, \\
k_1 &= 7, \\
k_2 &= 6.
\end{aligned}$$

5) We choose a setpoint $w = \overline{w}$ constant in time. What values $\overline{\mathbf{x}}$ and $\overline{z}$, does the state of the system $\mathbf{x}$ and state of the controller z tend to? What value $\overline{y}$ does the output y tend to?

At equilibrium $\dot{x}_1 = \dot{x}_2 = \dot{z} = 0$. Therefore,

$$\begin{aligned}
\mathbf{0} &= \begin{pmatrix} 1 & 1 & 0 \\ -7 & -4 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overline{w} \\
\overline{y} &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{z} \end{pmatrix}
\end{aligned}$$

Since

```
A=[1 1 0;
   -7 -4 1;
   -1 0 0];

invA = inv(A)
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```
invA =
         0         0   -1.0000
    1.0000    0.0000    1.0000
    4.0000    1.0000   -3.0000
```

we have

$$\begin{aligned}
\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{z} \end{pmatrix} &= -\begin{pmatrix} 1 & 1 & 0 \\ -7 & -4 & 1 \\ -1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overline{w} = -\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overline{w} = \begin{pmatrix} \overline{w} \\ -\overline{w} \\ 3\overline{w} \end{pmatrix} \\
\overline{y} &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{z} \end{pmatrix}
\end{aligned}$$

Thus, $\overline{y} = \overline{x}_1 = \overline{w}$. The role of the integrator is precisely to ensure zero static error and this, even when there are constant disturbances.

Simulate the step response of the system using Euler and Runge-Kutta methods.