Exercise 5.11. PID control[1]

We consider a second-order system of the form described by the following differential equation:

$$\ddot{y} + a_1\dot{y} + a_0 y = u$$

1) Give the state equation of the system in matrix form. We will take as state vector $\mathbf{x} = \begin{pmatrix} y & \dot{y} \end{pmatrix}^T$.

2) Let w be a setpoint that we will assume constant. We would like $y(t)$ to converge toward w. We define the error by: $e(t) = w - y(t)$. We suggest controlling our system by the following proportional-derivative-integral (PID) controller:

$$u(t) = \alpha_{-1}\int_0^t e(\tau)d\tau + \alpha_0 e(t) + \alpha_1 \dot{e}(t),$$

where the α_i are the coefficients of the controller. This is a state feedback controller, where we assume that $\mathbf{x}(t)$ is measured. Give the state equations of this PID controller with the inputs $\mathbf{x}$, w and output u. A state variable

$$z(t) = \int_0^t e(\tau)d\tau$$

will have to be created in order to take into account the integrator of the control.

3) Draw the wiring diagram of the looped system. This diagram will only be composed of integrators, adders and amplifiers. Encircle the controller on the one hand, and the system to be controlled on the other hand.

4) Give the state equations of the looped system in matrix form.

5) How do we choose the coefficients α_i of the control (as a function of the a_i) so as to have a stable looped system in which all the poles are equal to -1?

6) We slightly change the value of the parameters a_0 and a_1 while keeping the same controller. We assume that this modification does not destabilize our system. The new values for a_0, a_1 are denoted by a_0', a_1'. For a value $\overline{w}$ for given w, what value $\overline{y}$ does converge y to? What do you conclude from this?

Solution of Exercise 5.11

1) Give the state equation of the system in matrix form. We will take as state vector $\mathbf{x} = \begin{pmatrix} y & \dot{y} \end{pmatrix}^T$.

Since $\mathbf{x} = \begin{pmatrix} y & \dot{y} \end{pmatrix}^T$, the state equations are written as:

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a_1 x_2 - a_0 x_1 + u, \\
y &= x_1.
\end{aligned}
$$

2) Let w be a setpoint that we will assume constant. We would like $y(t)$ to converge toward w. We define the error by: $e(t) = w - y(t)$. We suggest controlling our system by the following proportional-derivative-integral (PID) controller:

$$u(t) = \alpha_{-1} \int_0^t e(\tau) d\tau + \alpha_0 e(t) + \alpha_1 \dot{e}(t),$$

where the α_i are the coefficients of the controller. This is a state feedback controller, where we assume that $\mathbf{x}(t)$ is measured. Give the state equations of this PID controller with the inputs $\mathbf{x}$, w and output u. A state variable

$$z(t) = \int_0^t e(\tau) d\tau$$

will have to be created in order to take into account the integrator of the control.

Since w is constant, $\dot{w} = 0$, and

$$
\begin{aligned}
u &= \alpha_{-1} z + \alpha_0 (w - x_1) + \alpha_1 (\dot{w} - \dot{x}_1) \\
&= \alpha_{-1} z + \alpha_0 w - \alpha_0 x_1 - \alpha_1 x_2.
\end{aligned}
$$

Since $z(t) = \int_0^t e(\tau) d\tau$, we have $\dot{z} = e(t) = w - x_1$. The state equations of the controller are therefore:

$$
\begin{aligned}
\dot{z} &= -x_1 + w, \\
u &= \alpha_{-1} z + \alpha_0 w - \alpha_0 x_1 - \alpha_1 x_2.
\end{aligned}
$$

3) Draw the wiring diagram of the looped system. This diagram will only be composed of integrators, adders and amplifiers. Encircle the controller on the one hand, and the system to be controlled on the other hand.

The wiring diagram of the looped system is given in Figure 1.

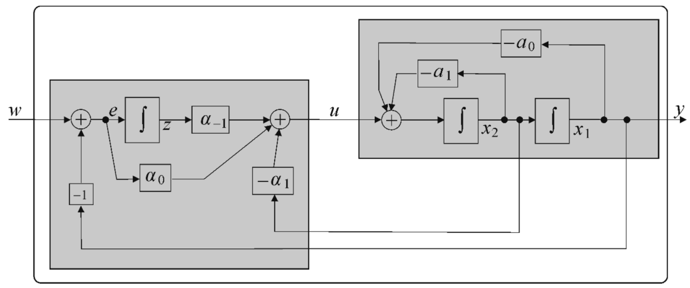

Figure 1: PID controller.

4) Give the state equations of the looped system in matrix form.

$$
\begin{aligned}
\dot{x}_2 &= -a_1 x_2 - a_0 x_1 + \alpha_{-1} z + \alpha_0 w - \alpha_0 x_1 - \alpha_1 x_2 \\
&= -(\alpha_0 + a_0) x_1 - (\alpha_1 + a_1) x_2 + \alpha_{-1} z + \alpha_0 w.
\end{aligned}
$$

The state equations of the closed-loop system are therefore:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -(\alpha_0 + a_0) x_1 - (\alpha_1 + a_1) x_2 + \alpha_{-1} z + \alpha_0 w, \\
\dot{z} &= -x_1 + w, \\
y &= x_1,
\end{aligned}
$$

which in matrix form become

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -(\alpha_0 + a_0) & -(\alpha_1 + a_1) & \alpha_{-1} \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_0 \\ 1 \end{pmatrix} w, \\
y &= x_1.
\end{aligned}
$$

5) How do we choose the coefficients α_i of the control (as a function of the a_i) so as to have a stable looped system in which all the poles are equal to -1?

```
syms s a0 a1 alpham1 alpha0 alpha1

A = [0 1 0;
    -(alpha0+a0) -(alpha1 +a1) alpham1;
    -1 0 0];

simplify(det(s*eye(3)-A))
```

```
ans =
alpham1 + a0*s + alpha0*s + a1*s^2 + alpha1*s^2 + s^3
```

The characteristic polynomial is $P(s) = \det(s\mathbf{I} - \mathbf{A}) = s^3 + (a_1 + \alpha_1)s^2 + (a_0 + \alpha_0)s + \alpha_{-1}$.

In order to have poles at -1, we need to have $P(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$. Therefore:

$$\begin{aligned} \alpha_{-1} &= 1, \\ \alpha_0 &= 3 - a_0, \\ \alpha_1 &= 3 - a_1. \end{aligned}$$

Taking $a_0 = a_1 = 2$, we get $\alpha_0 = \alpha_1 = 1$.

Simulate the step response of the system using Euler and Runge-Kutta methods

6) We slightly change the value of the parameters a_0 and a_1 while keeping the same controller. We assume that this modification does not destabilize our system. The new values for a_0, a_1 are denoted by a_0', a_1'. For a value $\overline{w}$ for given w, what value $\overline{y}$ does converge y to? What do you conclude from this?

At equilibrium, we have:

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -(\alpha_0 + a_0') & -(\alpha_1 + a_1') & \alpha_{-1} \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{z} \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_0 \\ 1 \end{pmatrix} \overline{w}, \\ \overline{y} &= \overline{x}_1. \end{aligned}$$

Therefore $\overline{y} = \overline{x} = \overline{w}$. Regardless of the value of the parameters, if the system is stable, given the integral term, we will always have $\overline{y} = \overline{w}$. Adding an integral term allows for a robust control relative to any kind of constant disturbance.

Simulate again the step response of the system using Euler and Runge-Kutta methods taking $a_0 = a_1 = 2.5$