Exercise 2.5 Dynamic modeling of an inverted rod pendulum[1]

Let us consider the so-called inverted rod pendulum system, composed of a pendulum of length l placed in an unstable equilibrium on a carriage, as represented in Figure 1. The value u is the force exerted on the carriage of mass m_c, s indicates the position of the carriage, θ is the angle between the pendulum and the vertical axis and $\mathbf{r}$ is the force exerted by the carriage on the pendulum. At the extremity $\mathbf{b}$ of the rod a point mass m_r is fixated. We may ignore the mass of the rod. Finally, $\mathbf{a}$ is the point of articulation between the rod and the carriage and $\omega = \dot{\theta}\mathbf{k}$ is the rotation vector associated with the rod.

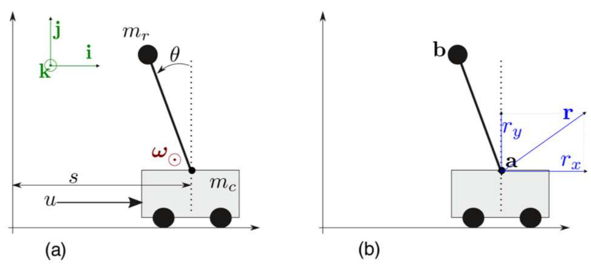

Figure 1: Inverted rod pendulum.

1) Write the fundamental principle of dynamics as applied on the carriage and the pendulum.

2) Show that the speed vector at point $\mathbf{b}$ is expressed by the relation $\dot{\mathbf{b}} = (\dot{s} - l\dot{\theta}\cos\theta)\,\mathbf{i} - l\dot{\theta}\sin\theta\,\mathbf{j}$. Calculate the acceleration $\ddot{\mathbf{b}}$ of $\mathbf{b}$.

3) In order to model the inverted pendulum, we will take the state vector $\mathbf{x} = (s, \theta, \dot{s}, \dot{\theta})$. Justify this choice.

Solution of Exercise 2.5

1) Write the fundamental principle of dynamics as applied on the carriage and the pendulum.

   We assume that the rod is massless. The system is an articulated system composed by a carriage and a rod which rotates about an axis that passes through the point $\mathbf{a}$. We have to model the translation of the carriage, the translation of the rod, and the rotation of the rod.

   Since the system is an articulated system, we have an internal force $\mathbf{r}$ whose components are r_x and r_y.

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

- The translation of the carriage occurs in $\mathbf{i}$ direction. Thus, considering r_x a reaction force we take it with negative sign, and

$$(u - r_x)\mathbf{i} = m_c \ddot{s}\ \mathbf{i}.$$

- The translation of the rod occurs in both $\mathbf{i}$ and $\mathbf{j}$ directions.

$$r_x \mathbf{i} + r_y \mathbf{j} - m_r g \mathbf{j} = m_r \ddot{\mathbf{b}}.$$

- To describe the rotation of the rod, we can observe that, since the mass of the rod is concentrated in the point $\mathbf{b}$, the moment of inertia with respect to an axis that passes through this point is zero. Thus, making reference to a rotation axis that passes through point $\mathbf{b}$, we can write

$$r_x l \cos\theta + r_y l \sin\theta = 0\ \ddot{\theta}.$$

2) Show that the speed vector at point $\mathbf{b}$ is expressed by the relation $\dot{\mathbf{b}} = (\dot{s} - l\dot{\theta}\cos\theta)\ \mathbf{i} - l\dot{\theta}\sin\theta\ \mathbf{j}$. Calculate the acceleration $\ddot{\mathbf{b}}$ of $\mathbf{b}$.

   For the sake of simplicity the height of the carriage is assumed to be zero. Thus the position of point $\mathbf{b}$ is

$$\mathbf{b} = (s - l\sin\theta)\ \mathbf{i} + l\cos\theta\ \mathbf{j}.$$

   Differentiating with respect to time we obtain

$$\dot{\mathbf{b}} = (\dot{s} - l\cos\theta\dot{\theta})\ \mathbf{i} - l\sin\theta\dot{\theta}\ \mathbf{j}.$$

   Differentiating with respect to time once more we obtain

$$\ddot{\mathbf{b}} = (\ddot{s} - l\cos\theta\ddot{\theta} + l\sin\theta\dot{\theta}^2)\ \mathbf{i} - (l\cos\theta\dot{\theta}^2 + l\sin\theta\ddot{\theta})\ \mathbf{j}.$$

3) In order to model the inverted pendulum, we will take the state vector $\mathbf{x} = (s, \theta, \dot{s}, \dot{\theta})$. Justify this choice.

   To describe the state of this system, which has two degrees of freedom, we need two variables s, θ to describe its configuration. Besides these configuration variables, we need their derivatives $\dot{s}, \dot{\theta}$ which represent the linear velocity of the carriage and the angular velocity of the row. The configuration variables and their derivatives completely describe the state of the system which is holonomic. Indeed, given the control input u and an initial state, for instance, $(s, \theta, \dot{s}, \dot{\theta}) = (2\ [m], 1\ [rad], -10\ [m/s], 20\ [rad/s])$ we are able to compute the evolution of the system.

4) Find the state equations of the inverted rod pendulum.

   We have the vector equation we obtained in 1)

$$(u - r_x)\mathbf{i} = m_c \ddot{s}\ \mathbf{i},$$

$$r_x\mathbf{i} + r_y\mathbf{j} - m_r g\mathbf{j} = m_r\ddot{\mathbf{b}},$$

$$r_x l\cos\theta + r_y l\sin\theta = 0\ \ddot{\theta},$$

and the equation we obtained in 2)

$$\ddot{\mathbf{b}} = (\ddot{s} - l\cos\theta\ddot{\theta} + l\sin\theta\dot{\theta}^2)\ \mathbf{i} - (l\cos\theta\dot{\theta}^2 + l\sin\theta\ddot{\theta})\ \mathbf{j}.$$

Unit vector $\mathbf{i}$ can be removed from both members of quation $(u - r_x)\mathbf{i} = m_c\ddot{s}\ \mathbf{i}$. We obtain

$$m_c\ddot{s} = (u - r_x).$$

From the vector equation

$$r_x\mathbf{i} + r_y\mathbf{j} - m_r g\mathbf{j} = m_r\ddot{\mathbf{b}},$$

we consider first the $\mathbf{i}$ component and then the $\mathbf{j}$ component. It is a scalar decomposition.

Considering the $\mathbf{i}$ component we obtain

$$r_x = m_r(\ddot{s} - l\cos\theta\ddot{\theta} + l\sin\theta\dot{\theta}^2).$$

Considering the $\mathbf{j}$ component we obtain

$$r_y - m_r g = -m_r(l\cos\theta\dot{\theta}^2 + l\sin\theta\ddot{\theta}).$$

The last equation is already scalar

$$r_x l\cos\theta + r_y l\sin\theta = 0.$$

To find the state equation of the system we should obtain an expression of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}),$$

with $\mathbf{x} = (s, \theta, \dot{s}, \dot{\theta})$.

Rewriting the previous equations in matrix form we obtain

$$\begin{pmatrix} m_c & 0 & 1 & 0 \\ -m_r & m_r l\cos\theta & 1 & 0 \\ 0 & m_r l\sin\theta & 0 & 1 \\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \ddot{s} \\ \ddot{\theta} \\ r_x \\ r_y \end{pmatrix} = \begin{pmatrix} u \\ m_r l\dot{\theta}^2\sin\theta \\ m_r g - m_r l\dot{\theta}^2\cos\theta \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} \ddot{s} \\ \ddot{\theta} \\ r_x \\ r_y \end{pmatrix} = \begin{pmatrix} m_c & 0 & 1 & 0 \\ -m_r & m_r l\cos\theta & 1 & 0 \\ 0 & m_r l\sin\theta & 0 & 1 \\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix}^{-1} \begin{pmatrix} u \\ m_r l\dot{\theta}^2\sin\theta \\ m_r g - m_r l\dot{\theta}^2\cos\theta \\ 0 \end{pmatrix}.$$

To select the $\ddot{s}$ and the $\ddot{\theta}$ components of the vector

$$\begin{pmatrix} \ddot{s} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_c & 0 & 1 & 0 \\ -m_r & m_r l \cos\theta & 1 & 0 \\ 0 & m_r l \sin\theta & 0 & 1 \\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix}^{-1} \begin{pmatrix} u \\ m_r l \dot{\theta}^2 \sin\theta \\ m_r g - m_r l \dot{\theta}^2 \cos\theta \\ 0 \end{pmatrix}.$$

Compute the right hand side of this expression with the `Matlab` Symbolic Toolbox and upload the `Matlab` code to Aula Virtual