Exercise 6.4. State feedback[1]

Let us consider the system represented by Figure 1.

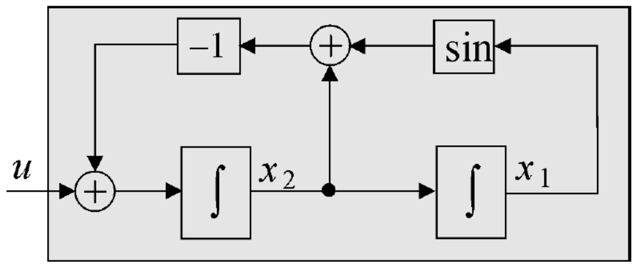

Figure 1: Block diagram of a nonlinear system.

1) Give the state equations of the system.
2) Calculate its equilibrium points.
3) Linearize this system around an equilibrium point x corresponding to $x_1 = \pi$. Is this a stable equilibrium point?
4) Propose a state feedback controller of the form $u = -\mathbf{K}(\mathbf{x} - \bar{\mathbf{x}})$ which stabilizes the system around $\bar{\mathbf{x}}$. Place all poles at -1.

Solution of Exercise 6.4

1) Give the state equations of the system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - \sin x_1 - x_2 \end{aligned}$$

These equations are those of a simplified damped pendulum (see Exercise 3.2 of [2]) with angle $\theta = x_1$ and angular velocity $\dot{\theta} = x_2$.

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf
[2] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

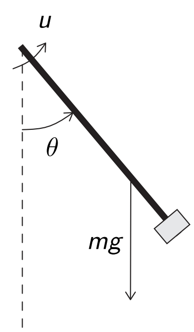

Figure 2: Damped pendulum.

2) Calculate its equilibrium points.

   We solve the $\mathbf{f}(\overline{\mathbf{x}}, \overline{u}) = 0$ with $\overline{u} = 0$. We find:

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} k\pi \\ 0 \end{pmatrix}$$

   with $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$, the set of integer numbers.

3) Linearize this system around an equilibrium point x corresponding to $x_1 = \pi$. Is this a stable equilibrium point?

   The equilibrium point is

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}.$$

   We have

$$\dot{\mathbf{x}} = \mathbf{f}(\overline{\mathbf{x}}, \overline{u}) + \mathbf{A}(\mathbf{x} - \overline{\mathbf{x}}) + \mathbf{B}(\mathbf{x} - \overline{\mathbf{x}}).$$

   Since $\sin \pi = 0$

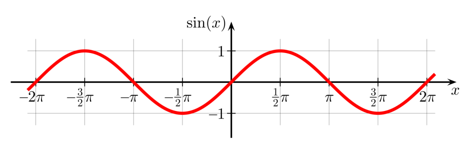

Figure 3: Sine function.

   we obtain

$$\mathbf{f}(\overline{\mathbf{x}}, \overline{\mathbf{u}}) \quad = \quad \begin{pmatrix} 0 \\ 0 - \sin \pi - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Since $\overline{x}_1 = \pi$, $\overline{x}_2 = 0$, and $\cos \pi = -1$

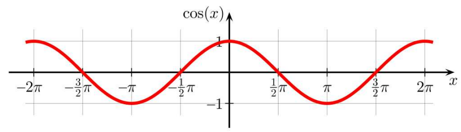

Figure 4: Cosine function.

we obtain

$$\begin{aligned}
\mathbf{A} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\overline{\mathbf{x}}, \overline{\mathbf{u}}) = \begin{pmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{pmatrix}\Bigg|_{\mathbf{x}=\overline{\mathbf{x}}, \mathbf{u}=\overline{\mathbf{u}}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \\
\mathbf{B} &= \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\overline{\mathbf{x}}, \overline{\mathbf{u}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
(\mathbf{x} - \overline{\mathbf{x}}) &= \begin{pmatrix} x_1 - \pi \\ x_2 - 0 \end{pmatrix}, \\
(u - \overline{u}) &= u - 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\overline{\mathbf{x}}, \overline{u}) + \mathbf{A}(\mathbf{x} - \overline{\mathbf{x}}) + \mathbf{B}(\mathbf{u} - \overline{u}) \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - \pi \\ x_2 - 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.
\end{aligned}$$

The characteristic polynomial is $P(s) = s^2 + s - 1$. Its roots are $s = -1.6180$ and $s = 0.61803$. Both are real. One of them is positive The system is therefore unstable.

4) Propose a state feedback controller of the form $u = -\mathbf{K}(\mathbf{x} - \overline{\mathbf{x}})$ which stabilizes the system around $\overline{\mathbf{x}}$. Place all poles at -1.

To calculate K, we solve:

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = (s+1)^2$$

that is

$$\det\left(s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \end{pmatrix}\right) = s^2 + 2s + 1,$$

$$\det\begin{pmatrix} s & -1 \\ k_1 - 1 & s + k_2 - 1 \end{pmatrix} = s^2 + 2s + 1,$$

$$s^2 + s(k_2 + 1) + k_1 - 1 = s^2 + 2s + 1.$$

Thus, $k_1 = 2$, $k_2 = 1$. The controller is therefore:

$$u = (-2, -1)(\mathbf{x} - \overline{\mathbf{x}}) = (-2, -1)\begin{pmatrix} x_1 - \pi \\ x_2 - 0 \end{pmatrix} = -2x_1 + 2\pi - x_2.$$

Simulate the evolution of the system using Euler and Runge-Kutta methods.