Exercise. Control of a planar RR robot manipulator in the operational space (with feedback linearization method).[1]

Case 1: without gravity

Consider the planar horizontal *RR* robot manipulator represented in Figure 1

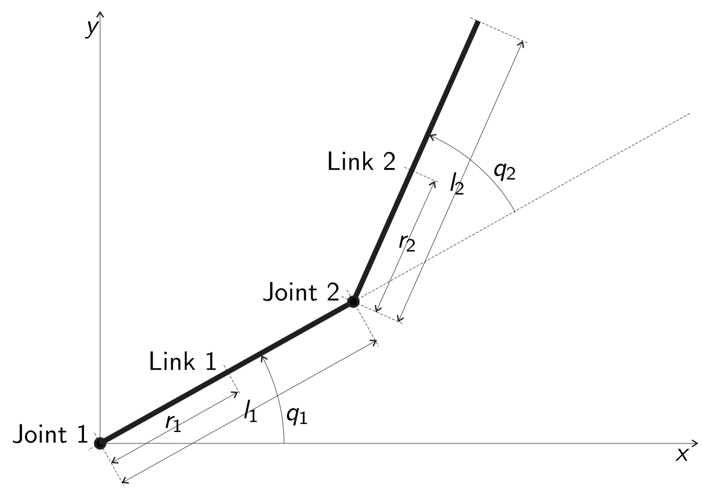

Figure 1: RR robot manipulator that moves in a plane.

The planar *RR* manipulator is composed by two homogeneous links and two actuated joints moving in a horizontal plane $\{x, y\}$, as shown in Figure 1, where l_i is the length of link i, r_i is the distance between joint i and the mass center of link i, m_i is the mass of link i, and I_{z_i} is the barycentric inertia with respect to a vertical axis z of link i, for $i = 1, 2$. Since the robot manipulator moves in a horizontal plane, the dynamic model of this robotic system is represented by the second order differential equation

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{u},$$

where the two matrices $\mathbf{B}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ have the following expressions

$$\begin{aligned}
\mathbf{B}(\mathbf{q}) &= \begin{bmatrix} \alpha + 2\beta\cos(q_2) & \delta + \beta\cos(q_2) \\ \delta + \beta\cos(q_2) & \delta \end{bmatrix}, \\
\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) &= \begin{bmatrix} -\beta\sin(q_2)\dot{q}_2 & -\beta\sin(q_2)(\dot{q}_1 + \dot{q}_2) \\ \beta\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}.
\end{aligned}$$

[1] From B. Siciliano, L. Sciavicco, L. Villani, G. Oriolo, Robotics: Modelling, Planning and Control, Springer, 2009.

$\mathbf{q} = (q_1, q_2)^T$ is the vector of configuration variables, where q_1 is the angular position of link 1 with respect to the x axis of the reference frame $\{x, y\}$ and q_2 is the angular position of link 2 with respect to link 1 as illustrated in Figure 1. The vector $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2)^T$ is the vector of angular velocities, where $\dot{q}_1$ and $\dot{q}_2$ are the angular velocities at joint 1 and joint 2, respectively. The vector $\ddot{\mathbf{q}} = (\ddot{q}_1, \ddot{q}_2)^T$ is the vector of accelerations, where $\ddot{q}_1$ and $\ddot{q}_2$ are the accelerations at joint 1 and joint 2, respectively. The control inputs of the system are $\mathbf{u} = (u_1, u_2)$, where u_1 is the torque applied by the actuator at joint 1, and u_2 is the torque applied by the actuator at joint 2. The parameters α, β and δ have the following expressions

$$
\begin{aligned}
&\alpha = I_{z_1} + I_{z_2} + m_1 r_1^2 + m_2(l_1^2 + r_2^2), \\
&\beta = m_2 l_1 r_2, \\
&\delta = I_{z_2} + m_2 r_2^2.
\end{aligned}
$$

The parameters of the dyanamic model of the robot are $I_{z_1} = 1$ kg m^2, $I_{z_2} = 1$ kg m^2, $m_1 = 1$ kg, $m_2 = 1$ kg, $l_1 = 1$ m, $l_2 = 1$ m, $r_1 = \frac{l_1}{2}$ m, $r_2 = \frac{l_2}{2}$ m.

The robotic problem we study is a tracking problem: follow a setpoint $\mathbf{w}$ describing a target circle centred at $(1, 1)$ m with radius 0.5 m (on the right of Figure 1) which moves counterclockwise with angular velocity 1 rad/s. Suppose that the initial configuration of the robot manipulator is $\mathbf{q}_I = (q_{1_I}, q_{2_I})^T = (0, 0)$.

The state space representation of the dynamics of the manipulator in which $\mathbf{x} = (x_1, x_2, x_3, x_4)^T = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$ can be calculated as follows. From

$$
\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{u},
$$

since the matrix $\mathbf{B}(\mathbf{q})$ is always invertible, we have

$$
\ddot{\mathbf{q}} = -\mathbf{B}^{-1}(\mathbf{q})\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{B}^{-1}(\mathbf{q})\mathbf{u}.
$$

From the definition of the state space vector, we have $\dot{x}_1 = x_3$, $\dot{x}_2 = x_4$. Since $\ddot{q}_1 = \dot{x}_3$ and $\ddot{q}_2 = \dot{x}_4$ we can write

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \\
\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} &= -\mathbf{B}^{-1}(x_1, x_2)\mathbf{C}(\mathbf{x}) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mathbf{B}^{-1}(x_1, x_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\end{aligned}
$$

The relation between the configuration variables $\mathbf{q} = (q_1, q_2)^T$ and the position of the tip is $\mathbf{p} = (p_1, p_2)^T$ with components

$$
\begin{aligned}
p_1 &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2), \\
p_2 &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2),
\end{aligned}
$$

The Jacobian matrix of this transformation is

```
syms l1 l2 q1 q2

p1 = (l1*cos(q1) +l2*cos(q1 + q2));
p2 = (l1*sin(q1) +l2*sin(q1 + q2));

J = jacobian([p1, p2], [q1, q2])
```

```
J =

[- l2*sin(q1 + q2) - l1*sin(q1), -l2*sin(q1 + q2)]
[  l2*cos(q1 + q2) + l1*cos(q1),  l2*cos(q1 + q2)]
```

$$\mathbf{J}(\mathbf{q}) = \begin{pmatrix} -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) & l_2 \cos(q_1 + q_2) \end{pmatrix}.$$

It is clear that

$$\dot{\mathbf{p}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}.$$

Using the state variables, the Jacobian matrix can be rewritten as

$$\mathbf{J}(x_1, x_2) = \begin{pmatrix} -l_2 \sin(x_1 + x_2) - l_1 \sin(x_1) & -l_2 \sin(x_1 + x_2) \\ l_2 \cos(x_1 + x_2) + l_1 \cos(x_1) & l_2 \cos(x_1 + x_2) \end{pmatrix}.$$

The time derivative of the Jacobian matrix is

```
syms l1 l2 x1(t) x2(t)

J = [-l1 * sin(x1) - l2 * sin(x1 + x2), -l2 * sin(x1 + x2);
  l1 * cos(x1) + l2 * cos(x1 + x2),  l2 * cos(x1 + x2)];

Jdot = diff(J,t)
```

```
Jdot(t) =

[- l2*cos(x1(t) + x2(t))*(diff(x1(t), t) + diff(x2(t), t)) - l1*cos(x1(t))*diff(x1(t), t), ...
       -l2*cos(x1(t) + x2(t))*(diff(x1(t), t) + diff(x2(t), t))]
[- l2*sin(x1(t) + x2(t))*(diff(x1(t), t) + diff(x2(t), t)) - l1*sin(x1(t))*diff(x1(t), t), ...
       -l2*sin(x1(t) + x2(t))*(diff(x1(t), t) + diff(x2(t), t))]
```

$$\dot{\mathbf{J}}(\mathbf{x}) = \begin{pmatrix} -l_1 \cos x_1 x_3 - l_2 \cos(x_1 + x_2)(x_3 + x_4) & -l_2 \cos(x_1 + x_2)(x_3 + x_4) \\ -l_1 \sin x_1 x_3 - l_2 \sin(x_1 + x_2)(x_3 + x_4) & -l_2 \sin(x_1 + x_2)(x_3 + x_4) \end{pmatrix}.$$

Using the configuration variables, the time derivative of the Jacobian matrix can be rewritten as

$$\dot{\mathbf{J}}(\mathbf{q}) = \begin{pmatrix} -l_1 \cos q_1 \dot{q}_1 - l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) & -l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \\ -l_1 \sin q_1 \dot{q}_1 - l_2 \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) & -l_2 \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \end{pmatrix}.$$

Suppose that the output variables are

$$\begin{array}{rcl} y_1 & = & q_1 \\ y_2 & = & q_2 \end{array}$$

we have

$$\ddot{\mathbf{y}} = -\mathbf{B}^{-1}(\mathbf{q})\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{B}^{-1}(\mathbf{q})\mathbf{u}.$$

It is easy to see that the differential delay matrix $\mathbf{R}$ is in this case

$$\mathbf{R} = \begin{pmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} \end{pmatrix}$$

and is balanced. Comparing the relation

$$\ddot{\mathbf{y}} = -\mathbf{B}^{-1}(\mathbf{q})\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{B}^{-1}(\mathbf{q})\mathbf{u}$$

with Equation (1.8) of the Lectures Notes[2] we can see that

$$\begin{array}{rcl} \mathbf{A}(\mathbf{q}) & = & \mathbf{B}^{-1}(\mathbf{q}) \\ \mathbf{b}(\mathbf{q}) & = & -\mathbf{B}^{-1}(\mathbf{q})\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} \end{array}$$

Since $\mathbf{A}^{-1}(\mathbf{q}) = \mathbf{B}(\mathbf{q})$, the linearizing and decoupling transformation (1.9) of the Lectures Notes

$$\mathbf{u} = \mathbf{A}^{-1}(\mathbf{q})(\mathbf{v} - \mathbf{b}(\mathbf{q}))$$

is in this case

$$\begin{array}{rcl} \mathbf{u} & = & \mathbf{B}(\mathbf{q})\mathbf{v} - \mathbf{B}(\mathbf{q})\left(-\mathbf{B}^{-1}(\mathbf{q})\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}\right) \\ & = & \mathbf{B}(\mathbf{q})\mathbf{v} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}. \end{array}$$

where $\mathbf{v} = (v_1, v_2)$ is the new control input. Recall that the relation between $\mathbf{v}$ and $\mathbf{y}$ is linear. This simplifies the design of the regulation controller which can be a modified PD controller.

Assuming that $\mathbf{p}_d = (p_{1_d}, p_{2_d})^T$, a regulation law in the operational space is

$$\mathbf{v} = \mathbf{J}^{-1}(\mathbf{q})\left(\ddot{\mathbf{p}}_d + \mathbf{K}_D(\dot{\mathbf{p}}_d - \dot{\mathbf{p}}) + \mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \dot{\mathbf{J}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}\right)$$

which, being $\dot{\mathbf{p}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$, can be rewritten as

$$\mathbf{v} = \mathbf{J}^{-1}(\mathbf{q})\left(\ddot{\mathbf{p}}_d + \mathbf{K}_D(\dot{\mathbf{p}}_d - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}) + \mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \dot{\mathbf{J}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}\right)$$

In terms of the state variables, this regulation law can be expressed as

$$\mathbf{v} = \mathbf{J}^{-1}(x_1,x_2)\left(\ddot{\mathbf{p}}_d + \mathbf{K}_D\left(\dot{\mathbf{p}}_d - \mathbf{J}(x_1,x_2)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}\right) + \mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \dot{\mathbf{J}}(\mathbf{x})\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}\right)$$

[2] https://www.ensta-bretagne.fr/jaulin/robmooc.pdf

As usual, we solve the tracking task by generating a moving desired position of the tip which in this case is

$$\mathbf{p}_d \quad = \quad \mathbf{c} + r \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Thus,

$$\begin{aligned} \dot{\mathbf{p}}_d &= r \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \\ \ddot{\mathbf{p}}_d &= r \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}. \end{aligned}$$

As said before, the position of the tip is $\mathbf{p} = (p_1, p_2)^T$ with components

$$\begin{aligned} p_1 &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2), \\ p_2 &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2), \end{aligned}$$

which in terms of the state variables become

$$\begin{aligned} p_1 &= l_1 \cos x_1 + l_2 \cos(x_1 + x_2), \\ p_2 &= l_1 \sin x_1 + l_2 \sin(x_1 + x_2). \end{aligned}$$

Case 2: with gravity

In this case the RR planar robot manipulator of Figure 1 moves in a vertical plane and the dynamic model becomes

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{N}(\mathbf{q}) = \mathbf{u},$$

in which matrix $\mathbf{N}(\mathbf{q})$, which represents the gravity term, has the following expression

$$\mathbf{N}(\mathbf{q}) = \begin{pmatrix} (m_1 r_1 + m_2 l_1) g \cos q_1 + m_2 r_2 g \cos(q_1 + q_2) \\ m_2 r_2 g \cos(q_1 + q_2) \end{pmatrix}.$$

The other matrices and parameters do not change. In this case

$$\mathbf{u} \quad = \quad \mathbf{B}(\mathbf{q})\mathbf{v} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{N}(\mathbf{q}).$$

and the regulation law in the operational space does not change.

Problems

1) Write a `Matlab` code that implements the controller to execute the task in a horizontal plane. Show, plotting the relevant variables and an animation, that the controllers satisfies the specifications.

2) Write a `Matlab` code that implements the controller to execute the task in a vertical plane. Show, plotting the relevant variables and an animation, that the controllers satisfies the specifications.

Solve