Exercise 2.3 Model a pendulum[1]

Let us consider the pendulum in Figure 1. It is composed by a rigid massless rod, which rotates around a hinge, and a body, which has a mass m attached at its extremity. The input of this system is the torque u exerted on the pendulum around its axis. The output is $y(t)$, the algebraic distance between the mass m and the vertical axis:

1) Determine the state equations of this system using the Newton law.
2) Express the mechanical energy E_m in function of the state of the system. Show that the latter remains constant when the torque u is nil.

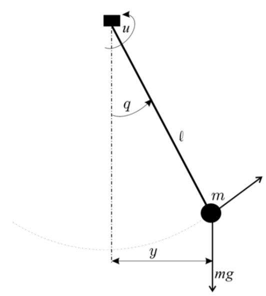

Figure 1: Simple pendulum with state vector $\mathbf{x} = (q, \dot{q})$.

Solution of Exercise 2.3

1) Determine the state equations of this system using the Newton law.

   This is a system with one degree of freedom q. The state of the system can be described by this variable and its time derivative $\dot{q}$. Thus

$$\mathbf{x} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}.$$

   On this system act

   – the control torque u, and
   – the torque generate by the gravity force mg on the mass attached at the end of the rod, which produces a moment $-mgl\sin(q)$.

[1]Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

The resulting torque produces an angular acceleration $\ddot{q}$ of the rod which depends on the moment of inertia J of the system with respect to the hinge.

Thus, the dynamic equation of the system is

$$u(t) - mgl \sin q(t) = J\ddot{q}(t).$$

If we assume that the rod is massless, the moment of inertia of the system with respect to the hinge is $J = ml^2$. Thus,

$$\ddot{q}(t) = \frac{u(t) - mgl \sin q(t)}{ml^2}.$$

Since

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix},$$

we can write

$$\frac{d}{dt}\begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} = \begin{pmatrix} \dot{q}(t) \\ \frac{u(t) - mgl \sin q(t)}{ml^2} \end{pmatrix}.$$

The state space representation of the system is

$$\begin{aligned} \dot{x}_1 &= x_2(t), \\ \dot{x}_2 &= \frac{u(t) - mgl \sin x_1(t)}{ml^2}, \\ y &= l \sin x_1. \end{aligned}$$

The system is not linear. It can be expressed as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}). \end{aligned}$$

2) Express the mechanical energy E_m in function of the state of the system. Show that it remains constant when the momentum u is nil.

The mechanical energy of this system is the sum of the kinetic energy and the potential energy of the system.

- The kinetic energy has the following expression

$$\frac{1}{2} ml^2 \dot{q}^2.$$

- Assuming that the potential energy is zero when $q = 0$, it has the following expression

$$mgl(1 - \cos q).$$

Thus, the mechanical energy of the system is

$$E_m = \frac{1}{2} m l^2 \dot{q}^2 + mgl(1 - \cos q).$$

To show that the mechanical energy E_m is constant when $u(t) = 0$, we can compute its time derivative and show that is it zero. The time derivative of E_m is

$$\dot{E}_m = \frac{1}{2} m l^2 2\dot{q}\ddot{q} + mgl \sin q \; \dot{q}.$$

Since

$$\ddot{q} = -\frac{mgl \sin q}{m l^2},$$

we obtain

$$\dot{E}_m = -\frac{1}{2} m l^2 2\dot{q} \frac{mgl \sin q}{m l^2} + mgl \sin q \; \dot{q} = 0,$$

which is consistent with the fact that the pendulum without friction is a conservative system.