Model of a pendulum with the Lagrange approach

Calculate the Lagrangian of the system. Deduce the state equations from this.

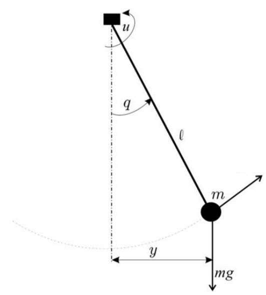

Figure 1: Simple pendulum as a conservative system

Solution

Calculate the Lagrangian of the system. Deduce the state equations from this.

Besides the Hamiltonian method, there is another method available to obtain the dynamic model of a dynamical system: the Lagrangian approach. Whereas the Hamiltonian method is based on the sum of the kinetic and potential energy, the Lagrangian method is based on the difference L between the kinetic energy T and the total potential energy V of the system, which is called Lagrangian, that is,

$$L = T - V.$$

The Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = f_i, \qquad i = 1, 2, \dots, n,$$

where x_i represents the i-th generalised coordinate and f_i is the i-th generalised force applied to the object.

In the case of a simple pendulum system, there is only one x variable, namely the angle between the pendulum rod and the vertical axis and $f = u$. Thus,

$$T = \frac{1}{2} ml^2 \dot{q}^2$$

and

$$V = mgl(1 - cos\ q).$$

After replacing q by x, we get

$$L = T - V = \frac{1}{2} ml^2 \dot{x}^2 - mgl(1 - cos\ x).$$

Now we determine the Lagrange equation for this mechanical system. To compute $\frac{\partial L}{\partial \dot{x}}$ we must consider $\dot{x}$ as a symbol and derive L with respect to it. We get

$$\frac{\partial L}{\partial \dot{x}} = ml^2\dot{x}.$$

To compute $\frac{\partial L}{\partial x}$ we can proceed as usual. We get

$$\frac{\partial L}{\partial x} = -mgl\sin x.$$

Thus, substituting the previous expressions into the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = u,$$

we obtain

$$\frac{d}{dt}\left(ml^2\dot{x}\right) + mglsin\ x = u.$$

Now we have to compute the time derivative of the term $ml^2\dot{x}$. We get

$$\frac{d}{dt}\left(ml^2\dot{x}\right) = ml^2\ddot{x}$$

Substituting this time derivative into the Lagrange equation we get

$$ml^2\ddot{x} + mglsin\ x = u.$$

Finally,

$$\ddot{x} = \frac{u - mglsin\ x}{ml^2}.$$

This differential equation coincides with the model of the pendulum determined using the Netwon approach.

To compute the state-space representation of this system, we can proceed as in Exercise 2.3. We need to introduce two state variables $x_1(t)$ and $x_2(t)$. We define $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, that is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix},$$

and we can write

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \frac{u(t) - mgl\sin x(t)}{ml^2} \end{pmatrix}.$$

The state space representation of the system is thus

$$\begin{aligned} \dot{x}_1 &= x_2(t), \\ \dot{x}_2 &= \frac{u(t) - mgl\sin x_1(t)}{ml^2}, \\ y &= l\sin x_1. \end{aligned}$$

The system is not linear. It can be represented by equations of the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}). \end{aligned}$$