Exercise 5.1. Non-observable states, non controllable states[1]

Let us consider the system:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \mathbf{x}(t) + u(t)
\end{aligned}$$

1) Calculate its transfer function.
2) Is this system stable?
3) Draw the associated block diagram and deduce from this the non-observable and non-controllable states.
4) The poles of the system (the eigenvalues of the evolution matrix) are composed of transmission poles (poles of the transfer function) and hidden poles. Give the hidden poles.

Solution of Exercise 5.1

1) Calculate its transfer function.

   The transfer function of the system is given by:

$$\begin{aligned}
G(s) &= \mathbf{C}(s\mathbf{I} - A)^{-1}\mathbf{B} + \mathbf{D} \\
&= \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \left(s \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1.
\end{aligned}$$

   The `Matlab` code to compute it is

```
syms s

I=eye(4);

A = [-1 1 0 0; 0 1 0 0 ; 1 1 1 1; 0 1 0 1];
B=[1; 0; 1; 0];
C=[1 1 0 0];
D=1;

G=C*inv(s*I-A)*B+D
```

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

The output is

```
G =
1/(s + 1) + 1
```

Thus

$$G(s) = \frac{s+2}{s+1}.$$

2) Is this system stable?

We have to find the eigenvalues of matrix A. This can be easily done by the following `Matlab` code

```
A = [-1 1 0 0; 0 1 0 0 ; 1 1 1 1; 0 1 0 1];
eig(A)
```

We obtain

```
ans =

   1
  -1
   1
   1
```

Another possibility find the root of the characteristic polynomial $\det(s\mathbf{I} - \mathbf{A})$. The characteristic polynomial is

```
syms s

A = [-1 1 0 0; 0 1 0 0 ; 1 1 1 1; 0 1 0 1];

Characteristic_polynomial = det(s*I-A)
expand(Characteristic_polynomial)
```

We get

```
ans =

s^4 - 2*s^3 + 2*s - 1
```

To find its roots

```
syms s

A = [-1 1 0 0; 0 1 0 0 ; 1 1 1 1; 0 1 0 1];

Characteristic_polynomial = det(s*I-A)

solve(Characteristic_polynomial)
```

We obtain, as expected:

```
ans =

     1
    -1
     1
     1
```

The characteristic polynomial of the evolution matrix A is $P(s) = s^4 - 2s^3 + 2s - 1$.

Its roots are $-1, 1, 1, 1$. The system is therefore unstable, even though its transfer function $G(s) = \frac{s+2}{s+1}$ having the pole in -1 is stable.

3) Draw the associated block diagram and deduce from this the non-observable and non-controllable states.

The wiring system in Figure 1

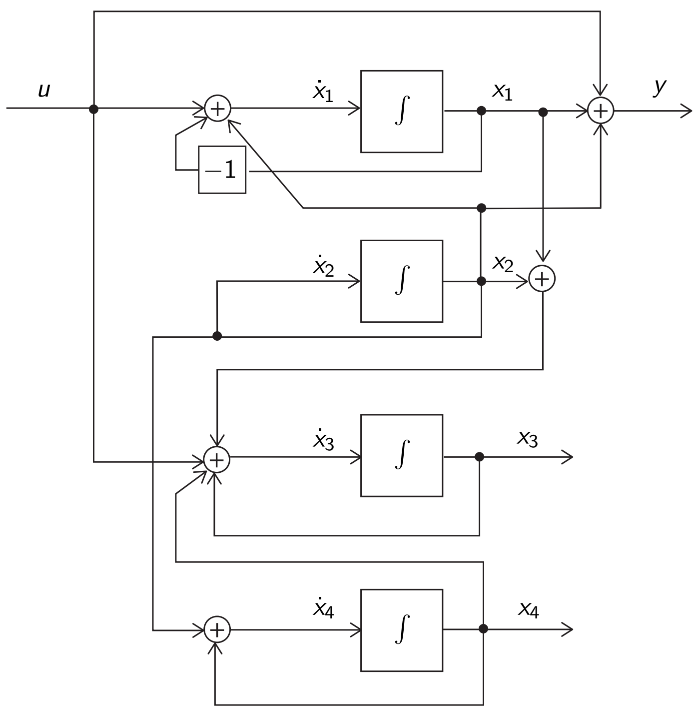

Figure 1: Wiring scheme of the system.

shows that x_1 is observable and controllable, x_2 is observable and non-controllable, x_3 is non-observable and controllable, x_4 is non-observable and non-controllable.

The controllable and observable portion of the system is represented in Figure 2

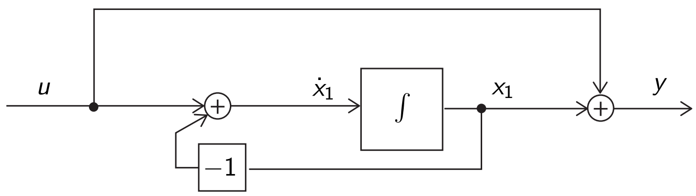

Figure 2: Wiring scheme of the controllable and observable subsystem.

whose transfer function is $G(s) = \frac{s+2}{s+1}$.

4) The poles of the system (the eigenvalues of the evolution matrix) are composed of transmission poles (poles of the transfer function) and hidden poles. Give the hidden poles.

The eigenvalues of the evolution matrix of the system are $1, 1, 1, -1$. The transmission pole is $s = -1$. The hidden modes are described by the eigenvalues $1, 1, 1$.