Exercise 4.12 Canonical control form[1]

Let us consider the order 3 linear system with a single input and a single output, described by the following differential equation:

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{u} + b_1\dot{u} + b_0 u$$

1) Calculate its transfer function $G(s)$.
2) By noticing that this system can be obtained by placing the two transfer function systems in series:

$$\begin{aligned} G_1(s) &= \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}, \text{ and} \\ G_2(s) &= b_2 s^2 + b_1 s + b_0, \end{aligned}$$

   deduce a wiring system with only three integrators, some adders and amplifiers.
3) Give the state equations associated with this wiring.

Solution

1) Calculate its transfer function $G(s)$.

   The transfer function of our system is:

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

2) Deduce a wiring system with only three integrators, some adders and amplifiers.

   If $\hat{y}(s)$ and $\hat{u}(s)$ represent the Laplace transforms of the signals $y(t)$ and $u(t)$, we have:

$$\hat{y}(s) = (b_2 s^2 + b_1 s + b_0)\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}\ \hat{u}(s)$$

   which can be written in the form:

$$\begin{aligned} \hat{x}_1(s) &= \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}\hat{u}(s) \\ \hat{y}(s) &= (b_2 s^2 + b_1 s + b_0)\hat{x}_1(s). \end{aligned}$$

   which is equivalent to

$$\begin{aligned} \hat{x}_1(s)(s^3 + a_2 s^2 + a_1 s + a_0) &= \hat{u}(s) \\ \hat{y}(s) &= (b_2 s^2 + b_1 s + b_0)\hat{x}_1(s). \end{aligned}$$

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

Thus,

$$
\begin{aligned}
s^3\hat{x}_1(s) &= \hat{u}(s) - a_2 s^2\hat{x}_1(s) - a_1 s\hat{x}_1(s) - a_0\hat{x}_1(s) \\
\hat{y}(s) &= b_2 s^2\hat{x}_1(s) + b_1 s\hat{x}_1(s) + b_0\hat{x}_1(s).
\end{aligned}
$$

Let us now draw the wiring associated with these two equations. The only differential operator we are allowed to use is the integrator, with transfer function $\frac{1}{s}$. First of all, we build a chain of 3 integrators in order to create $s^3\hat{x}_1$, $s^2\hat{x}_1$ and $s\hat{x}_1$ as shown on top of Figure 1.

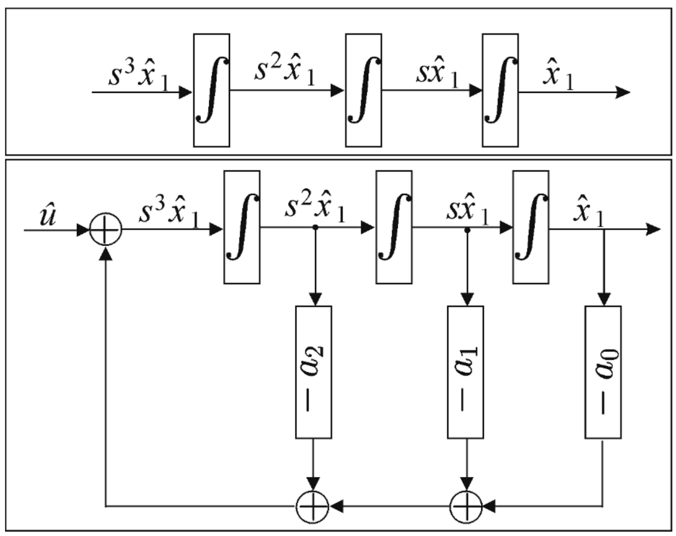

Figure 1: Wiring system.

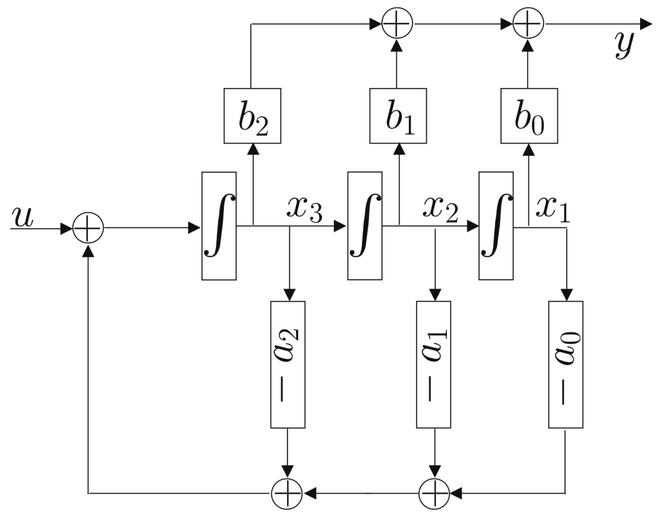

Figure 2: Wiring system.

Then, we wire the equation $s^3\hat{x}_1 = \hat{u} - a_2 s^2 \hat{x}_1 - a_1 s\hat{x}_1 - a_0\hat{x}_1$ (see bottom of Figure 1). We then wire the second equation and obtain the wiring of Figure 2.

3) Give the state equations associated with this wiring.

The state variables of this wiring system are the values x_1, x_2, x_3 memorized by each of the integrators (the adders and amplifiers do not memorize anything). By reading the diagram, we can directly write the state equations of this system:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \\
y &= \begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\end{aligned}$$

This reasoning can be applied in order to find the state representation of any monovariate linear system of any order n. This particular form for the state representation, which involves the coefficients of the transfer function in the matrices is called canonical form of control. Thus, generally, in order to obtain the canonical form of control equivalent to a given monovariate linear system, we have to calculate its transfer function in the form

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Then, knowing a_0, a_1, a_2 from the denominator of $G(s)$, $s^3 + a_2s^2 + a_1s + a_0$, we can immediately write its canonical form of control. In this way we computed a state space representation of a linear system from its transfer function representation.

A comprehensive treatment of the `Matlab` predefined functions to convert a spate space representation of a linear systems into a canonical state-space representation can be found in the `Matlab` documentation[2]

[2]https://es.mathworks.com/help/control/ug/canonical-state-space-realizations.html