Exercise 6.7. Linear-quadratic regulator[1]

Consider the state space system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

and the cost function

$$J(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt$$

It can be shown that a feedback control law that minimizes the value of the cost is given by $\mathbf{u} = -\mathbf{K}\mathbf{x}$ where

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

where $\mathbf{P}$ is found by solving the continuous time algebraic Riccati equation given by

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - (\mathbf{P}\mathbf{B})\mathbf{R}^{-1}(\mathbf{B}^T\mathbf{P}) + \mathbf{Q} = \mathbf{0}.$$

We assume this result that can be found using optimal control theory. The corresponding controller is called Linear Quadratic Regulator (LQR). The main advantage of the LQR is that it allows us to give weights to the required dynamics. In general, the matrices are chosen as diagonal. Consider the inverted pendulum of Exercise 6.6 with the length of the rod $l = 1$. We assume that the state is measured without any error.

1) Find a value for $\mathbf{K}$ such that the system initialized at $\mathbf{x}(0) = (2, 0, 0, 0)^T$ reaches $\mathbf{0}$ in less than 10 s with an error of less than 0.2 m for the position of the cart x_1 whereas u is always inside the interval $[-1, 1]$. What are the poles of the closed loop system?

2) Compare with the pole-placement method.

3) Find a value for $\mathbf{K}$ such that the system initialized at $\mathbf{x}(0)$ reaches 0 in less than 5 s with an error of less than 0.2 m for x_1.

4) We now have $\mathbf{x}(0) = (50, 0, 0, 0)^T$. How do we have to choose the controller $\mathbf{K}$ to be able to come back to zeroes without getting unstabilized?

Solve with the `lqr` predefined funcion of `Matlab`

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

Solution of Exercise 6.7

In this exercise the linear-quadratic regulator will be introduced. In this regulator the control action is determined in such a way a cost functional is minimized. Since this cost function is quadratic and the regulator is for linear systems, it is called linear-quadratic regulator. This approach is a first example of optimal control theory.

The cost function associates a state trajectory $\mathbf{x}(\cdot)$ and a control function $\mathbf{u}(\cdot)$ a real number

$$J(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt$$

In the cost function both $\mathbf{Q}$ and $\mathbf{R}$ are positive definite matrices.

A $n \times n$ real matrix M is said to be positive-definite if the scalar $x^T Mx$ is strictly positive for every non-zero vector $x \in \mathbb{R}^n$. We can assume that $\mathbf{Q}$ and $\mathbf{R}$ are diagonal matrices with positive coefficients. This cost is always a positive number.

The control objective is to drive the system to the state $\mathbf{x} = \mathbf{0}$ using the control input of the form

$$\mathbf{u} = -\mathbf{K}\mathbf{x}$$

that minimizes the cost function. Solving this problem entails finding the matrix $\mathbf{K}$ which is a quite complex problem from the mathematical point of view. The solution is

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

Since we know the matrices $\mathbf{R}$ and $\mathbf{B}$ we only have to determine the positive definite matrix P. This matrix can be found solving this equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - (\mathbf{P}\mathbf{B})\mathbf{R}^{-1}(\mathbf{B}^T\mathbf{P}) + \mathbf{Q} = \mathbf{0}.$$

called the Riccati equation.

We said before that we can assume that $\mathbf{Q}$ and $\mathbf{R}$ are diagonal matrices with positive coefficients. Each of the coefficients on the diagonal represent a weight of the corresponding element of the state or of the control vector. For example, choosing a large weight for one of the state variables makes this state variable converge to zero faster than others. Choosing larger weights for the control variables than for the state variables, we get a solution with a lower control effort in which the convergence of the state to zero slower. On the contrary, choosing larger weights for the state variables than for the control variables, we get a solution with a higher control effort in which the convergence of the state to zero faster. The possibility to choose the matrices $\mathbf{Q}$ and $\mathbf{R}$ is the main advantage of this technique with respect to the pole placement technique.