Exercise Model of the inverted pendulum on a cart using the Lagrangean approach.

Calculate the Lagrangean of the system. Deduce the state equations from this.

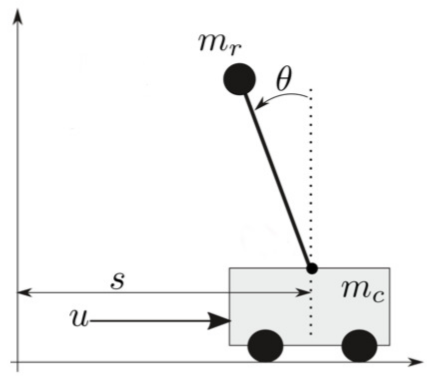

Figure 1: Inverted rod pendulum.

Solution of the Exercise

The dynamical model of the inverted pendulum on a cart are often obtained by applying force analysis using free body diagrams and Newton's second law. However, there are other methods available for achieving a system's dynamical model; for example, the Lagrangean approach which is based on the difference between total kinetic energy T and the total potential energy V of the system, that is, $L = T - V$ and the Hamiltonian approach which is based on the sum of T and L, that is, $H = T + L$.

The Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = f_i, \qquad i = 1, 2, \dots, n,$$

where x_i represents the i-th generalised coordinate and f_i is the i-th generalised force applied to the object.

In the case of the inverted pendulum on a cart, there are two x variables, namely

- the horizontal distance s travelled by cart from the reference and
- the angle θ between the pendulum rod and the vertical axis.

Additionally, there are to $\dot{x}$ variables, namely

- the velocity $\dot{s}$ of the cart along the horizontal axis and

- the angular velocity $\dot{\theta}$ of the rod around the rod-cart hinge.

The total kinetic energy of the pendulum-cart system can be written as:

$$\begin{aligned} T &= \frac{1}{2}m_c\dot{s}^2 + \frac{1}{2}m_r\left(\frac{d}{dt}(s - l\sin\theta)\right)^2 + \frac{1}{2}m_r\left(\frac{d}{dt}(l\cos\theta)\right)^2 \\ &= \frac{1}{2}m_c\dot{s}^2 + \frac{1}{2}m_r\left(\dot{s} - l\cos\theta\,\dot{\theta}\right)^2 + \frac{1}{2}m_r\left(-l\sin\theta\,\dot{\theta}\right)^2 \\ &= \frac{1}{2}(m_c + m_r)\dot{s}^2 - m_r l\dot{\theta}\dot{s}\cos\theta + \frac{1}{2}m_r l^2\dot{\theta}^2, \end{aligned}$$

where m_c and m_r are the masses of the cart and pendulum respectively, l denotes the length of the pendulum and g is the gravity acceleration. In the above expression of the kinetic energy of the system, the term

$$\frac{1}{2}m_c\dot{s}^2$$

represents the kinetic energy of the cart due to the horizontal component of its velocity. Since the carts moves horizontally, the vertical component of its velocity is zero. The term

$$\frac{1}{2}m_r\left(\frac{d}{dt}(s - l\sin\theta)\right)^2$$

represents the kinetic energy of the rod due to the horizontal component of its velocity, and the term

$$\frac{1}{2}m_r\left(\frac{d}{dt}(l\cos\theta)\right)^2$$

represents the kinetic energy of the rod due to the vertical component of its velocity.

The total potential energy of the system, using the horizontal position of the pendulum as the reference position at which the potential energy is zero, can be written as:

$$V = m_r g l\cos\theta.$$

Therefore, the Lagrangean is given by:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_c + m_r)\dot{s}^2 - m_r l\dot{s}\dot{\theta}\cos\theta + \frac{1}{2}m_r l^2\dot{\theta}^2 - m_r g l\cos\theta. \end{aligned}$$

The Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} &= u, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0. \end{aligned}$$

From

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = u,$$

since

$$\begin{aligned}
\frac{\partial L}{\partial \dot{s}} &= (m_c + m_r)\dot{s} - m_r l\dot{\theta}\cos\theta, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} &= (m_c + m_r)\ddot{s} - m_r l\ddot{\theta}\cos\theta + m_r l\dot{\theta}^2\sin\theta, \\
\frac{\partial L}{\partial s} &= 0.
\end{aligned}$$

the first Lagrange equation is

$$(m_c + m_r)\,\ddot{s} - m_r l\ddot{\theta}\cos\theta + m_r l\dot{\theta}^2\sin\theta = u.$$

Likewise, from,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0.$$

since

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= -m_r l\dot{s}\cos\theta + m_r l^2\dot{\theta}, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= -m_r l\ddot{s}\cos\theta + m_r l\dot{s}\sin\theta\;\dot{\theta} + m_r l^2\ddot{\theta}, \\
\frac{\partial L}{\partial \theta} &= m_r l\dot{s}\dot{\theta}\sin\theta + m_r g l\sin\theta.
\end{aligned}$$

the second Lagrange equation is

$$m_r l\left(-g\sin\theta + l\ddot{\theta} - \ddot{s}\cos\theta\right) = 0.$$

Thus, in this case the Lagrange equations are

$$\begin{aligned}
&(m_c + m_r)\,\ddot{s} - m_r l\ddot{\theta}\cos\theta + m_r l\dot{\theta}^2\sin\theta - u = 0, \\
&m_r l\left(-g\sin\theta + l\ddot{\theta} - \ddot{s}\cos\theta\right) = 0.
\end{aligned}$$

Solve this set of equations for $\ddot{s}$ and $\ddot{\theta}$ using the `Matlab` `Symbolic Toolbox`. Compare the result with that obtained with the Newton approach. Upload the `Matlab` code to Aula Virtual.