Exercise 2.1 First and second order system[1]

We consider an integrator and a second-order system described by

(i) $\dot{y} = u$,

(ii) $\ddot{y} + a_1\dot{y} + a_0 y = bu$

where, u is the input y the output. Find a state representation in matrix form and give the characteristic polynomial for the two systems.

Solution of Exercise 2.1

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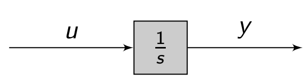
(i) The first system is represented by a first order differential equation

$$\dot{y} = u$$

This system is an integrator in which the input is u and the output is y. The integrator stores the state x which, in this case, is equal to y.



To find the state representation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned}$$

in which $\mathbf{A}$ is the evolution matrix, $\mathbf{B}$ is the control matrix, $\mathbf{C}$ is the observation matrix, and $\mathbf{D}$ is the direct matrix. $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector, and $\mathbf{y}$ is the output vector.

From the equation

$$\dot{y} = u$$

introducing the state variable x we obtain

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= x(t), \end{aligned}$$

[1] Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

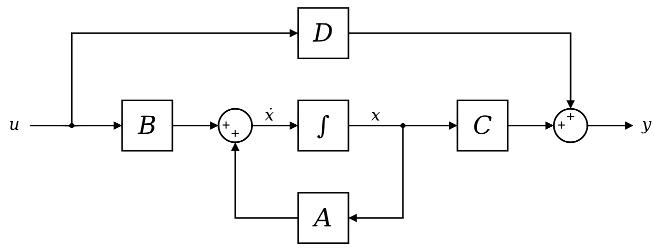


Figure 1: State space model.

which corresponds to

$$\begin{aligned}\dot{x}(t) &= 0\,x(t) + 1\,u(t),\\ y(t) &= 1\,x(t) + 0\,u(t).\end{aligned}$$

It is easy to see that the matrices of the state space representation are in this case the following scalars $A = 0$, $B = 1$, $C = 1$, and $D = 0$.

The characteristic polynomial is $\det(\lambda\mathbf{I} - \mathbf{A}) = \det(\lambda \cdot 1 - 0) = \lambda$.

The eigenvalues of matrix $\mathbf{A}$ are the roots of the characteristic polynomial. In this case the only root of the characteristic polynomial is $\lambda = 0$.

Moreover, the eigenvalues of matrix $\mathbf{A}$ correspond to the poles of the transfer function $\mathbf{G}(s)$ of the system, which can be computed from matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ as follows:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

In this case we get

$$G(s) = 1(sI - 0)^{-1}1 + 0 = \frac{1}{s},$$

which, as expected, is the transfer function of an integrator. The only pole of this transfer function is $s = 0$.

(ii) The second system is is represented by a second order differential equation

$$\ddot{y} + a_1\dot{y} + a_0 y = bu.$$

In this system, the input is u and the output is y. In this case the state vector is

$$\mathbf{x} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since $\ddot{y} = \dot{x}_2$, the state space representation of the system is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -a_1 x_2(t) - a_0 x_1(t) + bu(t),\\ y(t) &= x_1(t),\end{aligned}$$

which in matrix form corresponds to

$$\begin{aligned}\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} &= \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0\\ b\end{pmatrix} u(t),\\ y(t) &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + 0\ u(t),\end{aligned}$$

or

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\ b\end{pmatrix} u(t),\\ y(t) &= \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 0\ u(t).\end{aligned}$$

It is easy to see that the matrices of the state space representation are in this case the following

$$\mathbf{A} = \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\ b\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1 & 0\end{pmatrix}, D = 0.$$

The characteristic polynomial is

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \det\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} - \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}\right) = \det\begin{pmatrix}\lambda & -1\\ a_0 & \lambda + a_1\end{pmatrix} = \lambda^2 + a_1\lambda + a_0.$$

The matrix **A** is the companion matrix of the characteristic polynomial. The coefficients of the characteristic polynomial coincide with the elements of last row of the matrix **A** with the sign changed.