Exercise. Control of a planar RR planar robot manipulator in the operational space (with $\mathbf{J}^T$ method).[1]

The planar horizontal *RR* robot manipulator represented in Figure 1 is a paradigmatic model which appears in most robotics textbooks. Moreover, it qualitatively corresponds to the model of the first two links of a Selective Compliant Assembly Robot Arm (SCARA) robot without taking into account the vertical link (see Figure 2).

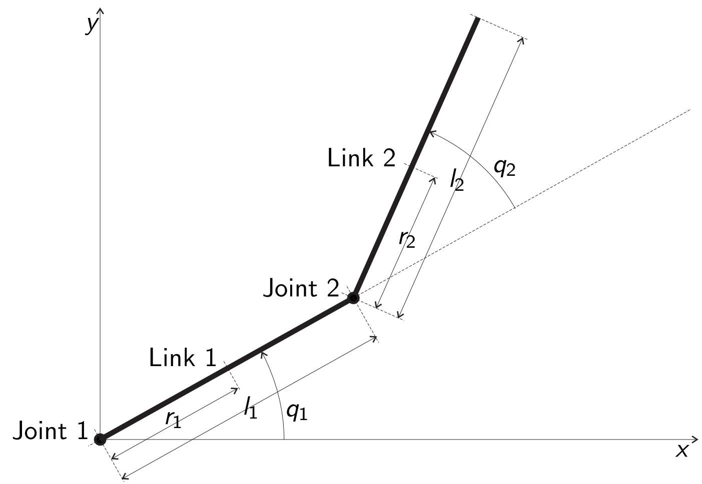

Figure 1: RR robot manipulator that moves in a horizontal plane.

However, this simple robot manipulator has very complex nonlinear dynamics which makes it attractive in control engineering and it is also interesting in the sense that it comprises most of the kinematical and dynamical properties a typical industrial robot has and that makes it suitable for testing control techniques.

[1]From B. Siciliano, L. Sciavicco, L. Villani, G. Oriolo, Robotics: Modelling, Planning and Control, Springer, 2009.

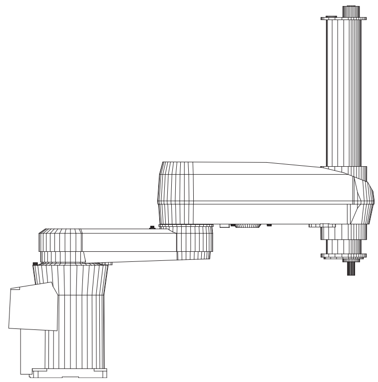

Figure 2: Selective Compliant Assembly Robot Arm (SCARA) robot manipulator.

The planar *RR* manipulator is composed of two homogeneous links and two actuated joints moving in a horizontal plane $\{x, y\}$, as shown in Figure 1, where l_i is the length of link i, r_i is the distance between joint i and the mass center of link i, m_i is the mass of link i, and I_{z_i} is the barycentric inertia with respect to a vertical axis z of link i, for $i = 1, 2$. Since the robot manipulator moves in a horizontal plane, the dynamic model of this robotic system is represented by the second order differential equation

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{u},$$

where the two matrices $\mathbf{B}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ have the following expressions

$$\begin{aligned} \mathbf{B}(\mathbf{q}) &= \begin{bmatrix} \alpha + 2\beta\cos(q_2) & \delta + \beta\cos(q_2) \\ \delta + \beta\cos(q_2) & \delta \end{bmatrix}, \\ \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) &= \begin{bmatrix} -\beta\sin(q_2)\dot{q}_2 & -\beta\sin(q_2)(\dot{q}_1 + \dot{q}_2) \\ \beta\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}. \end{aligned}$$

$\mathbf{q} = (q_1, q_2)^T$ is the vector of configuration variables, where q_1 is the angular position of link 1 with respect to the x axis of the reference frame $\{x, y\}$ and q_2 is the angular position of link 2 with respect to link 1 as illustrated in Figure 1. The vector $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2)^T$ is the vector of angular velocities, where $\dot{q}_1$ and $\dot{q}_2$ are the angular velocities at joint 1 and joint 2, respectively. The vector $\ddot{\mathbf{q}} = (\ddot{q}_1, \ddot{q}_2)^T$ is the vector of accelerations, where $\ddot{q}_1$ and $\ddot{q}_2$ are the accelerations at joint 1 and joint 2, respectively. The control inputs of the system are $\mathbf{u} = (u_1, u_2)$, where u_1 is the torque applied by the actuator at joint 1, and u_2 is the torque applied by the actuator at joint 2. The parameters α, β and δ have the following expressions

$$\begin{aligned} &\alpha = I_{z_1} + I_{z_2} + m_1 r_1^2 + m_2(l_1^2 + r_2^2), \\ &\beta = m_2 l_1 r_2, \\ &\delta = I_{z_2} + m_2 r_2^2. \end{aligned}$$

The robotic problem we study is a regulation problem: find the control inputs $u_1(t)$, $u_2(t)$ that steer the tip of the manipulator from an initial position $\mathbf{p}_I = (p_{1_I}, p_{2_I})^T$ to the final position $\mathbf{p}_F = (p_{1_F}, p_{2_F})^T$. Suppose that it is a rest-to-rest motion, that is, both the initial and final velocities of the tip of the manipulator are assumed to be zero.

The state space representation of the dynamics of the manipulator in which $\mathbf{x} = (x_1, x_2, x_3, x_4)^T = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$ can be calculated as follows. From

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{u},$$

since the matrix $\mathbf{B}(\mathbf{q})$ is always invertible, we have

$$\ddot{\mathbf{q}} = -\mathbf{B}^{-1}(\mathbf{q})\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{B}^{-1}(\mathbf{q})\mathbf{u}.$$

From the definition of the state space vector, we have $\dot{x}_1 = x_3$, $\dot{x}_2 = x_4$. Since $\ddot{q}_1 = \dot{x}_3$ and $\ddot{q}_2 = \dot{x}_4$ we can write

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \\ \begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} &= -\mathbf{B}^{-1}(x_1, x_2)\mathbf{C}(\mathbf{x}) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mathbf{B}^{-1}(x_1, x_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

The relation between the configuration variables $\mathbf{q} = (q_1, q_2)^T$ and the position of the tip $\mathbf{p} = (p_1, p_2)^T$ is

$$\begin{aligned} p_1 &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2), \\ p_2 &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2), \end{aligned}$$

which, defining the output variables as $y_1 = p_1$ and $y_2 = p_2$ and using the state variables, can be rewritten as

$$\begin{aligned} y_1 &= l_1 \cos x_1 + l_2 \cos(x_1 + x_2), \\ y_2 &= l_1 \sin x_1 + l_2 \sin(x_1 + x_2). \end{aligned}$$

The Jacobian matrix of this transformation is

```
syms l1 l2 q1 q2

p1 = (l1*cos(q1) +l2*cos(q1 + q2));
p2 = (l1*sin(q1) +l2*sin(q1 + q2));

J = jacobian([p1, p2], [q1, q2])
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```
J =

[- l2*sin(q1 + q2) - l1*sin(q1), -l2*sin(q1 + q2)]
[  l2*cos(q1 + q2) + l1*cos(q1),  l2*cos(q1 + q2)]
```

$$\mathbf{J}(\mathbf{q}) = \begin{pmatrix} -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) & l_2 \cos(q_1 + q_2) \end{pmatrix}.$$

It is clear that

$$\dot{\mathbf{p}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}.$$

Using the state variables, the Jacobian matrix can be rewritten as

$$\mathbf{J}(x_1, x_2) = \begin{pmatrix} -l_2 \sin(x_1 + x_2) - l_1 \sin(x_1) & -l_2 \sin(x_1 + x_2) \\ l_2 \cos(x_1 + x_2) + l_1 \cos(x_1) & l_2 \cos(x_1 + x_2) \end{pmatrix}.$$

Assuming that $\mathbf{p}_d = (p_{1_d}, p_{2_d})^T$, a first regulation law in the operational space is

$$\mathbf{u} = \mathbf{J}^T(\mathbf{q})\mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \mathbf{K}_D\dot{\mathbf{q}}$$

which using the state space variables can be rewritten as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{J}^T(x_1, x_2)\mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \mathbf{K}_D \begin{pmatrix} x_3 \\ x_4 \end{pmatrix},$$

where $\mathbf{K}_P$ and $\mathbf{K}_D$ are positive definite matrices which are symmetric and can be chosen as diagonal matrices.

The first term is a control action proportional to the position error $(\mathbf{p}_d - \mathbf{p})$ in the operational space which can be interpreted as a force/moment applied to the end effector of the manipulator. The transpose Jacobian matrix maps end-effector forces/moments into joint forces/moments $\mathbf{u} = \mathbf{J}^T\mathbf{F}$. The second term is a damping action proportional to the joint velocities $\dot{\mathbf{q}}$. A mechanical interpretation of this regulation law is given in Figure 3.

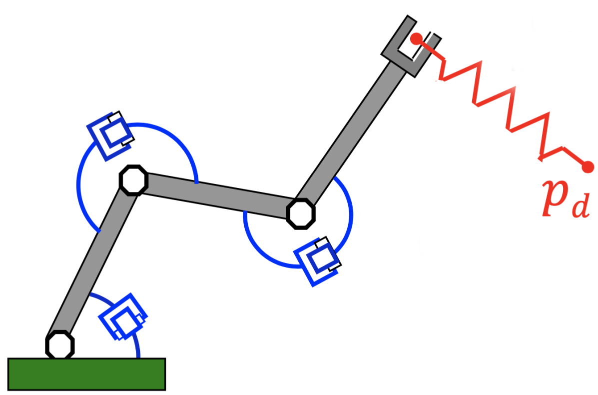

Figure 3: Mechanical interpretation of the control law $\mathbf{u} = \mathbf{J}^T(\mathbf{q})\mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \mathbf{K}_D\dot{\mathbf{q}}$.

A second regulation law in the operational space is

$$\mathbf{u} = \mathbf{J}^T(\mathbf{q})\left[\mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \mathbf{K}_D\dot{\mathbf{p}}\right]$$

in which the only difference with respect to the previous one is that damping action proportional to the end effector velocity $\dot{\mathbf{p}}$. Since

$$\dot{\mathbf{p}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{J}(x_1, x_2)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

and using the state space variables, this regulation law can be rewritten as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{J}^T(x_1, x_2)\mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \mathbf{J}^T(x_1, x_2)\mathbf{K}_D\mathbf{J}(x_1, x_2)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}.$$

A mechanical interpretation of this regulation law is given in Figure 4.

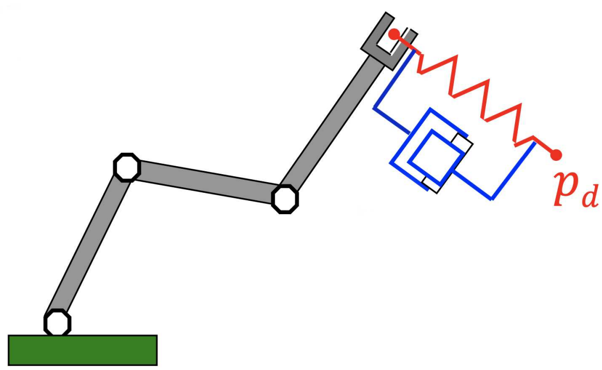

Figure 4: Mechanical interpretation of the law $\mathbf{u} = \mathbf{J}^T(\mathbf{q})\left[\mathbf{K}_P(\mathbf{p}_d - \mathbf{p}) - \mathbf{K}_D\dot{\mathbf{p}}\right]$.

The parameters of the dyanamic model of the robot are $I_{z_1} = 1$ kg m^2, $I_{z_2} = 1$ kg m^2, $m_1 = 1$ kg, $m_2 = 1$ kg, $l_1 = 1$ m, $l_2 = 1$ m, $r_1 = \frac{l_1}{2}$ m, $r_2 = \frac{l_2}{2}$ m.

1) Assuming that $\mathbf{p}_I = (p_{1_I}, p_{2_I})^T = (0, 1)^T$ m and $\mathbf{p}_F = (p_{1_F}, p_{2_F})^T = (1, 1)^T$ m, implement both controllers in `Matlab` to steer the tip of the robot from $\mathbf{p}_I$ to $\mathbf{p}_F$. Show, plotting the relevant variables and an animation, that the controllers satisfies the specifications.

2) Assuming that $\mathbf{p}_A = (p_{1_A}, p_{2_A})^T = (0.5, 1)^T$ m and $\mathbf{p}_B = (p_{1_B}, p_{2_B})^T = (0, 0.5)^T$ m are, respectively, the pick and place positions of the tip of the robot manipulator, implement the second controller in `Matlab` to iteratively move the tip of the robot from $\mathbf{p}_A$ to $\mathbf{p}_B$ and vice versa. Show, plotting the relevant variables and an animation, that the controller satisfies the specifications.

Solve