Exercise 2.7 Modeling a car[1]

Let us consider the car as shown in Figure 1. The driver of the car (on the left hand side on the figure) has two controls: the acceleration of the front wheels (assumed to be motorized) and the rotation speed of the steering wheel. The brakes here represent a negative acceleration. We will denote by δ the angle between the front wheels and the axis of the car, by θ the angle made by the car with respect to the horizontal axis and by (x, y) the coordinates of the middle of the rear axle.

The state variables of our system are composed of

- the position coordinates, in other words all the knowledge necessary to draw the car, more specifically the x, y coordinates of the center of the rear axle, the orientation θ of the car, and the angle δ of the front wheels
- the kinetic coordinate v representing the speed of the center of the front axle (indeed, the sole knowledge of this value and the position coordinates allows to calculate all the speeds of all the other elements of the car).

Calculate the state equations of the system. We will assume that the two wheels have the same speed v (even though in reality, the inner wheel during a turn is slower than the outer one). Thus, as illustrated on the right hand side figure, everything happens as if there were only two virtual wheels situated at the middle of the axles.

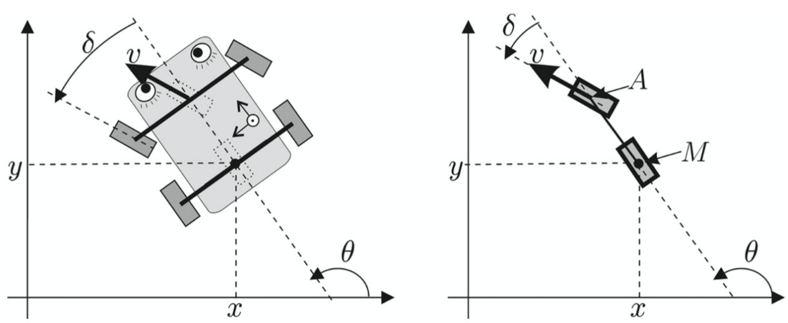

Figure 1: Car moving on a plane (view from above).

Solution of the Exercise 2.7

Calculate the state equations of the system. We will assume that the two wheels have the same speed v (even though in reality, the inner wheel during a turn is slower than the outer one). Thus, as illustrated on the right hand side figure, everything happens as if there were only two virtual wheels situated at the middle of the axles.

[1]Adapted from https://www.ensta-bretagne.fr/jaulin/automooc.pdf

The state vector is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \theta \\ v \\ \delta \end{pmatrix}.$$

As usual, the model we want to obtain is of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

The control inputs, u_1 and u_2, correspond to the acceleration of the front wheels, $\dot{v}$, and the angular velocity of the steering wheels, $\dot{\delta}$, respectively. It is easy to see that

$$\begin{aligned} \dot{x} &= v_M \cos\theta, \\ \dot{y} &= v_M \sin\theta. \end{aligned}$$

However, v_M the speed of point M is not a state variable. To eliminate v_M from these equation, we have to consider the velocity composition rule, the formula of Varignon,

$$\mathbf{v} = \mathbf{v}_A = \mathbf{v}_M + \overrightarrow{AM} \times \overrightarrow{\omega},$$

where the vector $\overrightarrow{\omega}$ represents the angular velocity of the car and it is a vector perpendicular to the plane having the same direction as unit vector $\mathbf{k}$. Let us express this vectorial relation in the frame of the car, which is represented in the figure:

$$\begin{pmatrix} v\cos\delta \\ v\sin\delta \\ 0 \end{pmatrix} = \begin{pmatrix} v_M \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -L \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix},$$

where L is the distance between the front and rear axes. After computing the cross product we get

$$\begin{pmatrix} v\cos\delta \\ v\sin\delta \\ 0 \end{pmatrix} = \begin{pmatrix} v_M \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ L\dot{\theta} \\ 0 \end{pmatrix}.$$

From these expressions we can deduce that,

$$\begin{aligned} v\cos\delta &= v_M, \\ v\sin\delta &= L\dot{\theta}. \end{aligned}$$

We have obtained the following expressions:

$$\begin{aligned} v_M &= v\cos\delta, \\ \dot{\theta} &= \frac{v\sin\delta}{L}. \end{aligned}$$

Then,

$$\begin{aligned}\dot{x} &= v_M \cos\theta = v\cos\delta\cos\theta,\\ \dot{y} &= v_M \sin\theta = v\cos\delta\sin\theta,\end{aligned}$$

and the state equations of the car are

$$\begin{aligned}\dot{x} &= v\cos\delta\cos\theta,\\ \dot{y} &= v\cos\delta\sin\theta,\\ \dot{\theta} &= \frac{v\sin\delta}{L}\\ \dot{v} &= u_1,\\ \dot{\delta} &= u_2.\end{aligned}$$