# Robótica Aérea
# Módulo III, Tema 2: Sensor fusion
# Tema 2.1: Bayes filter

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# Contenidos

# Probablistic robotics

- The standard MULTISENSOR DATA FUSION methods employed in robotics are based on **probabilistic methods**.
- PROBABILISTIC ROBOTICS intends to address the **uncertainties in robot perception and action**.
- Instead of relying on a single "best guess", probabilistic algorithms represent information by **probability distributions over a whole space of guesses**.
- By doing so, they can **represent ambiguity and degree of belief** in a mathematically sound way. **Estimation and control choices** are made **robustly by taking into account of the uncertainty**.
- Probabilistic methods used in robotics are generally based on **Bayes' rule** (Teorema de Bayes) for **combining prior information (información previa) and observation information**.
- We shall illustrate this concept with **mobile robot localization** as an example - the problem of *estimating a robot's coordinates relative to an external reference frame*.
- The **belief** $bel(x)$ (creencia) of its location is represented by a **probability density function** (función de densidad de probabilidad) over the space of all locations $x$.

# Mobile robot localization

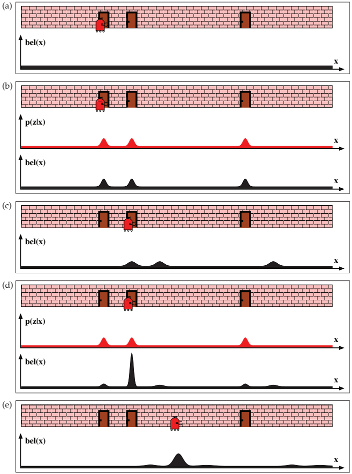


Figure: Mobile robot **localization** using probabilistic methods [Thrun, 2005].

- Prior to sensor measurements, the belief $bel(x)$ has a **uniform distribution** over all locations.
- After the first sensor measurement $z$, the robot realizes that it is **in front of a door**.
  - For a **given x position**, the **probability of the sensor detecting a door at** $x$ is given by $p(z|x)$, **three bell-shape distributions** in front of three indistinguishable doors.
  - The robot then **updates the belief** $bel(x)$ accordingly, including **three distinct hypotheses** which are equally plausible given the sensor data.
  - The robot also assigns **small but non-zero probabilities in front of walls**, to account for the **possibilities of errors in its assessment of seeing a door**. The ability to maintain **low-probability hypotheses** is essential for attaining **robustness**.

# Mobile robot localization

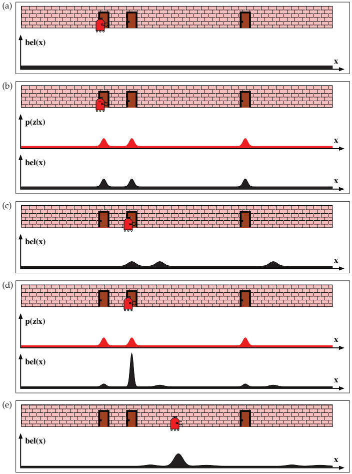


Figure: Mobile robot **localization** using probabilistic methods [Thrun, 2005].

- After **moving**, the **belief** has been **shifted** in the direction of motion. The **larger spread** of $bel(x)$ reflects the **uncertainty in robot motion**.
- The sensor detects a **second door**.
  - The probability of the sensor detecting a door, $p(z|x)$, is the **same as the first sensor measurement**.
  - However, for the belief $bel(x)$, based on prior information and current sensor measurements, the probabilistic algorithm now can place **most of the probability near one of the doors**. The robot is now **quite confident** about where it is.
- The probabilistic algorithm **updates the belief** $bel(x)$ **recursively** as the robot moves. At each iteration, there are two steps:
  - PREDICTION/PROPAGATION (propagación) by integrating equations of motion with control inputs
  - UPDATE/CORRECTION (corrección) of the prediction results with observation inputs

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# Discrete probability distribution

$$\sum_x p(x) = \sum_{i=1}^{n} P(X = x_i) = 1$$

- **Discrete random variable** (variable aleatoria discreta) $X$ can have a countable number of values in $\{x_1, x_2, \cdots, x_n\}$ .
- **Probability mass function** (PMF, función de masa de probabilidad) of $X$, $p(x) = P(X = x_i)$, denotes the **probability** of $X$ having a certain **value** $x_i$.
- For instance, a **fair coin** (moneda) gives $p(head, cara) = p(tail, cruz) = 1/2$.
- Discrete probability **sums up to one**.

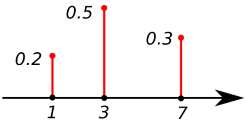


Figure: The probability mass function of a discrete probability distribution: $p(1) = 0.2$, $p(3) = 0.5$, $p(7) = 0.3$ (source, ©Oleg Alexandrov).

# Continuous probability distribution

$$P(X \in [a, b]) = \int_a^b p(x)\,\mathrm{d}x, \quad \int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$$

- CONTINUOUS RANDOM VARIABLE (variable aleatoria continua) $X$ can have a **continuum of values**.
- PROBABILITY DENSITY FUNCTION (PDF, función de densidad de probabilidad) $p(x)$ is the probability "per unit length".
    - It is also called PROBABILITY DISTRIBUTION FUNCTION.
    - We can **NOT specify** the probability for $X$ to have a **particular value** $x$.
    - $P(X \in [x, x + \mathrm{d}x]) = p(x)\,\mathrm{d}x$ is the probability of $X$ falling within the **infinitesimal interval** (intervalo infinitesimal) $[x, x + \mathrm{d}x]$.
    - $P(X \in [a, b]) = \int_a^b p(x)\,\mathrm{d}x$ is the probability of $X$ falling within the **interval** $[a, b]$.
- **Integration** of the PDF over the **whole continuous space**, i.e., zeroth moment $\int_{-\infty}^{\infty} x^0 p(x)\,\mathrm{d}x$, equals to **one**.
- There can be $p(x) > 1$ **for a continuous PDF**, but **NOT for a discrete PMF**. For instance, Dirac Delta function with infinite value at origin and zero elsewhere, $\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$.
- An example in thermodynamics: **1D Maxwellian velocity distribution function**.

# 1D Maxwellian velocity distribution function

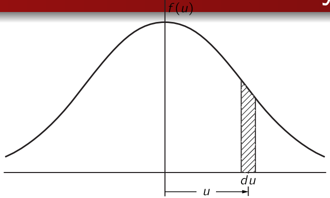


Figure: A **velocity distribution function**. The shaded area is $f(u)\,\mathrm{d}u$. (source, ©R. Feynman).

$$f_M(u) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{mu^2/2}{k_B T}\right)$$

- A gas in **thermal equilibrium at rest** has the mean velocity $\bar{u} = 0$.
- The temperature $T$ (in K) measures the **random jiggling motion** of the atoms. With $k_B$ the Boltzmann constant, $k_B T$ has the unit of energy J.
- The **possibility (fraction) of particles having velocities within a range** $\mathrm{d}u$ about $u$, $[u, u + \mathrm{d}u]$, is $f_M(u)\,\mathrm{d}u$.

- For the zeroth moment, you can prove $\int_{-\infty}^{\infty} u^0 f_M(u)\,\mathrm{d}u = 1$ by using Gaussin integral $\int_{-\infty}^{\infty} exp(-x^2)\,\mathrm{d}x = \sqrt{\pi}$.
- If the number density of this gas is $n$ (in $\mathrm{m}^{-3}$), the **number density of particles having velocities within a range** $\mathrm{d}u$ about $u$, $[u, u + \mathrm{d}u]$, is $nf_M(u)\,\mathrm{d}u$ and we have $\int_{-\infty}^{\infty} nf_M(u)\,\mathrm{d}u = n\int_{-\infty}^{\infty} f_M(u)\,\mathrm{d}u = n$.
- The **zero mean velocity** can be demonstrated by the first moment $\int_{-\infty}^{\infty} u^1 f_M(u)\,\mathrm{d}u = \bar{u} = 0$.
- The temperature measures the average energy of the particle, which can be derived from the second moment as $\int_{-\infty}^{\infty} \frac{mu^2}{2} f_M(u)\,\mathrm{d}u = k_B T$.

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# Joint probability

The JOINT PROBABILITY (probabilidad conjunta) of two random variables $X$ and $Y$ describes the probability that both events $X = x$ and $Y = y$ occur together.

$$p(x, y) = P(X = x, Y = y)$$

| | $X = 0$ | $X = 1$ |
|---|---|---|
| $Y = 0$ | 30 | 20 |
| $Y = 1$ | 10 | 40 |

$$P(X = 0, Y = 0) = 30\%$$
$$P(X = 0, Y = 1) = 10\%$$
$$P(X = 1, Y = 0) = 20\%$$
$$P(X = 1, Y = 1) = 40\%$$

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# Conditional probability

The CONDITIONAL PROBABILITY (probabilidad condicionada) of $X = x$ given $Y = y$ ($x$ dado $y$) describes the probability that the event $X = x$ occurs **given the occurrence of the other event** $Y = y$ as a condition. In other words, it is the probability of $X = x$ conditioned on $Y = y$.

$$p(x|y) = P(X = x|Y = y)$$

| | $X = 0$ | $X = 1$ |
|---|---|---|
| $Y = 0$ | 30 | 20 |
| $Y = 1$ | 10 | 40 |

$$P(X = 0, Y = 0) = 30\% \qquad P(X = 0|Y = 0) = \frac{30}{30 + 20} = 60\%$$

$$P(X = 0, Y = 1) = 10\% \qquad P(X = 0|Y = 1) = \frac{10}{10 + 40} = 20\%$$

$$P(X = 1, Y = 0) = 20\% \qquad P(X = 1|Y = 0) = \frac{20}{30 + 20} = 40\%$$

$$P(X = 1, Y = 1) = 40\% \qquad P(X = 1|Y = 1) = \frac{40}{10 + 40} = 80\%$$

# Total probability

| | $X=0$ | $X=1$ |
|---|---|---|
| $Y=0$ | 30 | 20 |
| $Y=1$ | 10 | 40 |

$$P(X=0,Y=0)=30\% \qquad P(X=0|Y=0)=\frac{30}{30+20}=60\% \qquad P(X=0)=\frac{30+10}{100}=40\%$$

$$P(X=0,Y=1)=10\% \qquad P(X=0|Y=1)=\frac{10}{10+40}=20\% \qquad P(X=1)=\frac{20+40}{100}=60\%$$

$$P(X=1,Y=0)=20\% \qquad P(X=1|Y=0)=\frac{20}{30+20}=40\% \qquad P(Y=0)=\frac{30+20}{100}=50\%$$

$$P(X=1,Y=1)=40\% \qquad P(X=1|Y=1)=\frac{40}{10+40}=80\% \qquad P(Y=1)=\frac{10+40}{100}=50\%$$

Joint probability $\quad p(x,y)=p(x|y)p(y)$

Total probability $\quad p(x)=\int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}y=\int_{-\infty}^{\infty} p(x|y)p(y)\,\mathrm{d}y$

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# Joint, conditional, total probability

| $p(x,y)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $p(y)$ |
|---|---|---|---|---|---|
| $y_1$ | $1/8$ | $1/16$ | $1/32$ | $1/32$ | $1/4$ |
| $y_2$ | $1/16$ | $1/8$ | $1/32$ | $1/32$ | $1/4$ |
| $y_3$ | $1/16$ | $1/16$ | $1/16$ | $1/16$ | $1/4$ |
| $y_2$ | $1/4$ | $0$ | $0$ | $0$ | $1/4$ |
| $p(x)$ | $1/2$ | $1/4$ | $1/8$ | $1/8$ | $\sum = 1$ |

- Joint probability $p(x_2, y_2) = 1/8$.
- Total probability $p(y_2)$?

$$p(y_2) = \sum_{x_i} p(y_2, x_i) = 1/16 + 1/8 + 1/32 + 1/32 = 1/4$$

- Conditional probability $p(x_2|y_2)$? $p(x_2|y_2) = \frac{1/8}{1/4} = 1/2$.
- $\sum_{x_i} p(x_i) = \sum_{y_i} p(y_i)$? $\sum_x p(x) = \sum_y p(y) = 1$.
- $\sum_{x_i} p(x_i|y_2)$? $\sum_{x_i} p(x_i|y_2) = 1$.

# Bayes' rule

$$p(x,y) = p(x|y)p(y) = p(y|x)p(x) \quad \Longrightarrow \quad \boxed{p(H|D) = \frac{p(D|H)p(H)}{p(D)}}$$

- In Bayesian probability theory, there are two events, being **hypothesis** $H$ (e.g., position $x$) and **observation data** $D$ (e.g., measurement $z$ from the sensor to detect a door).
- Our **goal** is to obtain the **probability of the hypothesis** after **consideration of the data**. This conditional probability is called POSTERIOR PROBABILITY DISTRIBUTION (probabilidad a posteriori), $p(H|D)$ (e.g., $p(x|z)$, *the probability of position $x$ for given sensor measurement $z$*).
- The PRIOR PROBABILITY DISTRIBUTION (probabilidad a priori), $p(H)$, summarizes the knowledge we have regarding the hypothesis $H$ **prior to considering data** $D$.
- The LIKEHOOD FUNCTION (función de verosimilitud), $p(D|H)$, assesses the **probability of the observed data** given that the **hypothesis is true** (e.g., $p(z|x)$, *the probability of sensor measurement $z$* for given position $x$).
- Typically, what we **need** is the posterior probability $p(H|D)$. However, what is **easier** to obtain is the likehood function $p(D|H)$.
- Bayes' rule provides a **convenient way** to compute the posterior probability $p(H|D)$ using the likehood function $p(D|H)$.

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# Bayes' rule

$$p(H|D) = \frac{p(D|H)p(H)}{p(D)}$$

- The total probability, $p(D)$, describes **how likely it is to make a observation**. It can be obtained by **integrating** (or summing) $p(D|H)p(H)$ over all possible hypotheses $H$.

$$\boxed{p(D) \overset{\text{total prob.}}{=\!=\!=} \sum_{H} p(D,H) \overset{\text{joint prob.}}{=\!=\!=} \sum_{H} p(D|H)p(H)}$$

- To calculate the posterior probability distribution $p(H|D)$ for **any value** of $H$, the value of $p(D)$ is the **same**. For this reason, a **constant** normalizer coefficient $\eta$ is normally used in Bayes' rule, which guarantees $\Sigma_x p(x|y) = 1$.

$$\boxed{p(H|D) = \eta p(D|H)p(H), \ \eta = \frac{1}{p(D)}}$$

- Bayes rule can be used for **multiple conditions**.

$$\boxed{p(x|y,z)} = \frac{p(x,y,z)}{p(y,z)} = \frac{p(y|x,z)p(x,z)}{p(y,z)} \boxed{= \frac{p(y|x,z)p(x|z)}{p(y|z)}}$$

- Bayes rule plays a **predominant role** in probabilistic robotics.

# Independent, conditionally independent

- Two random variables $X$ and $Y$ are INDEPENDENT if *the occurrence of event* $X = x$ *does not affect the probability of occurrence of event* $Y = y$. The **joint probability** writes
$$p(x, y) = p(x)p(y) \ .$$
**Equivalently**, two random variables $X$ and $Y$ are **independent** if *knowing the value of* $Y$ *does not change the probabilities of* $X$. The **conditional probability** writes
$$p(x|y) = \frac{p(x, y)}{p(y)} = \frac{p(x)p(y)}{p(y)} = p(x) \ .$$

- Two random variables $X$ and $Y$ are CONDITIONALLY INDEPENDENT given a third **conditioning random variable** $Z$ if and only if they are **independent** in their **conditional probability** distribution given $Z$.
$$p(x, y|z) = p(x|z)p(y|z) \ .$$
**Equivalently**, **conditional independence** implies that *a variable* $Y$ *carries no information about a variable* $X$ *if the value of conditioning variable* $Z$ *is known*.
$$p(x|z) = \frac{p(x, y|z)}{p(y|z)} = \frac{p(x, y, z)}{p(y, z)} = p(x|y, z), \quad p(y|z) = p(y|x, z) \ .$$

- Generally, for conditional independence and (absolute) independence, **one condition does not imply the other.**
$$p(x, y|z) = p(x|z)p(y|z) \quad \not\Leftrightarrow \quad p(x, y) = p(x)p(y) \ .$$

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# Glossary

- Probability mass function, $\sum_x p(x) = 1$.
- Probability density function or probability distribution function, $\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$.
- Joint probability, $p(x, y)$.
- Conditional probability, $p(x|y) = p(x, y)/p(y)$.
- Total probability, $p(x) = \sum_y p(x, y) = \sum_y p(x|y)p(y)$.
- Bayes' rule

$$p(H|D) = \frac{p(D|H)p(H)}{p(D)} = \frac{p(D|H)p(H)}{\sum_H p(D|H)p(H)} = \eta p(D|H)p(H)$$

hypothesis $H$, observation data $D$, prior probability distribution $p(H)$, posterior probability distribution $p(H|D)$, likehood function $p(D|H)$, normalizer $\eta$.

- Bayes' rule with multiple conditions

$$p(x|y, z) = \frac{p(y|x, z)p(x|z)}{p(y|z)} = \frac{p(y|x, z)p(x|z)}{\sum_x p(y|x, z)p(x|z)} = \eta p(y|x, z)p(x|z)$$

- Totally independent, $p(x, y) = p(x)p(y)$ and $p(x|y) = p(x)$.
- Conditionally independent, $p(x, y|z) = p(x|z)p(y|z)$ and $p(x|y, z) = p(x|z)$.

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# State, measurement, and control

- STATE is denoted by $\boldsymbol{x}$, which can be generally defined as the collection of all aspects of **the robot and its environment** that can impact the future (e.g., *pose, velocity, angular velocity, location of obstacle or landmark*).
- ENVIRONMENT MEASUREMENT DATA is denoted by $\boldsymbol{z}$, which **provides information** about a momentary **state of the environment** (e.g., *camera images, range scans, GPS, star tracker data*). Environment OBSERVATION tends to **increase the robot's knowledge**.
- CONTROL DATA is denoted by $\boldsymbol{u}$, which carries information about the **change of sate** in the environment caused by control actions (e.g., *pull or push action, acceleration data, odometer that measures the evolution of a robot's wheels, thruster thrust and reaction wheel torque on a spacecraft*). Motion tends to **induce a loss of** knowledge due to the inherent noise in robot actions.
- The specific variables included in $\boldsymbol{x}$, $\boldsymbol{u}$, and $\boldsymbol{z}$ **depend on the context**.
- TIME is generally defined **discretely** at instant $t_k$, $k = 0, 1, 2 \cdots$ (0 for the initial instant).

# Dynamic Bayes network

- At the instant $t_k$, the following quantities are defined
  - $x_k$, the **state** vector to be estimated at $t_k$;
  - $x_{0:k} = \{x_0, x_1 \cdots x_k\}$, the **history** of states;
  - $u_k$, the **control** inputs applied at time $t_{k-1}$ to change the state from $t_{k-1}$ to $t_k$.
  - $u_{1:k} = \{u_1, u_2 \cdots u_k\}$, the **history** of control inputs.
  - $z_k$, the **measurements** (i.e., observations) made at $t_k$ to estimate $x_k$;
  - $z_{1:k} = \{z_1, z_2 \cdots z_k\}$, the **history** of observations;

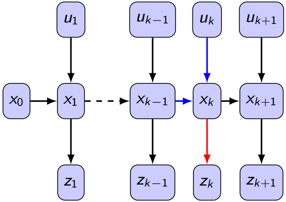


Figure: The **dynamic Bayes network** characterizes the evolution of controls, states, and measurements.

- The **evolution** of states and measurements is governed by **probabilistic distributions**.
  - Since the emergence of state $x_t$ might be **conditioned on all past states, measurements, and controls**, the **evolution of state** is characterized by

$$p(x_k | x_{0:k-1}, z_{1:k-1}, u_{1:k}) \ .$$

  - The **evolution of measurement** $z_k$ is characterized by

$$p(z_k | x_{0:k}, z_{1:k-1}, u_{1:k}) \ .$$

# Markov assumption

The MARKOV ASSUMPTION postulates that **past and future data are independent if one knows the current state $x_k$**.

- Assuming that the state $x_{k-1}$ is a **sufficient summary of all previous** states, controls and measurements, the state transition probability (probabilidad de transición de estado) characterizes the **state evolution** as

$$p(x_k | x_{0:k-1}, z_{1:k-1}, u_{1:k}) \approx p(x_k | x_{k-1}, u_k) \; .$$

In other words, **given** the conditioning variables $\{x_{k-1}, u_k\}$, there is **conditional independence** between $x_k$ and $\{x_{0:k-2}, z_{1:k-1}, u_{1:k-1}\}$.

- Assuming that $x_k$ is a **sufficient summary of all previous** states, controls and measurements, the measurement probability (probabilidad de medición) characterizes the **measurement evolution** as

$$p(z_k | x_{0:k}, z_{1:k-1}, u_{1:k}) \approx p(z_k | x_k) \; .$$

In other words, **given** the conditioning variable $\{x_k\}$, there is **conditional independence** between $z_k$ and $\{x_{0:k-1}, z_{1:k-1}, u_{1:k}\}$.

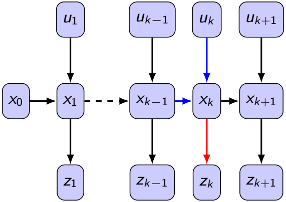


Figure: The **dynamic Bayes network** characterizes the evolution of controls, states, and measurements.

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# Probabilistic filtering

- FILTERING (técnicas de filtrado) is concerned with the sequential process of maintaining a probabilistic model for a **state** that evolves over time and is periodically observed by a sensor. It forms the basis for many problems in **tracking and navigation**.
- The general filtering problem can be formulated in **Bayesian form**.
- The robot has an internal belief of its **true state**. BELIEFS are represented by conditional probabilities of state ($\boldsymbol{x}$) **conditioned** on the available data ($\boldsymbol{z}$ and $\boldsymbol{u}$).

- PREDICTION (predicción) $\overline{bel}(x_k)$: the probability distribution of the state $x_k$ at time $t_k$, just **after executing control** $u_k$ and **before incorporating measurement** $z_k$, conditioned on all past measurements $z_{1:k-1}$ and all past controls $u_{1:k}$.

$$\overline{bel}(x_k) = p(x_k | z_{1:k-1}, u_{1:k})$$

- BELIEF (creencia) $bel(x_k)$: the probability distribution of the state $x_k$ at time $t_k$, **after incorporating measurement** $z_k$, conditioned on all past measurements $z_{1:k}$ and all past controls $u_{1:k}$.

$$bel(x_k) = p(x_k | z_{1:k}, u_{1:k}) \ .$$

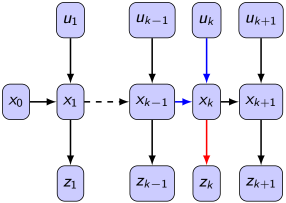


Figure: The **dynamic Bayes network** characterizes the evolution of controls, states, and measurements.

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# Bayes filter: inputs and outputs

**Recursive_Bayes_filter(** $bel(x_{k-1}), u_k, z_k$ **)**

1: **for** all possible values of the state $x_k$ **do**
2: PREDICTION: $\overline{bel}(x_k) = \int p(x_k|u_k, x_{k-1}) bel(x_{k-1}) \, \mathrm{d}x_{k-1}$
3: UPDATE: $bel(x_k) = \eta p(z_k|x_k) \overline{bel}(x_k)$
4: **end for**
5: **return** $bel(x_k)$

- The above **pseudo-algorithm** depicts one **iteration** of the **recursive Bayes Filter**. At each iteration, the **current belief** $bel(x_k)$ is calculated from the belief $bel(x_{k-1})$ obtained in the **previous iteration**.
- Inputs
  - $bel(x_{k-1})$, the **belief** at $t_{k-1}$;
  - $u_k$, the **control** at $t_k$;
  - $z_k$, the **measurement** at $t_k$.
- Outputs (return):
  - $bel(x_k)$, the **updated belief** at $t_k$.

# Bayes filter: prediction/propagation step

**Recursive_Bayes_filter(** $bel(x_{k-1}), u_k, z_k$ **)**

1: **for** all possible values of state $x_k$ **do**
2: PREDICTION: $\overline{bel}(x_k) = \int p(x_k|u_k, x_{k-1})bel(x_{k-1})\,\mathrm{d}x_{k-1}$
3: UPDATE: $bel(x_k) = \eta p(z_k|x_k)\overline{bel}(x_k)$
4: **end for**
5: **return** $bel(x_k)$

PREDICTION/PROPAGATION step (predicción/propogación): the prediction, $\overline{bel}(x_k)$, is given by the **integration** of the multiplication of state transition probability, $p(x_k|u_k, x_{k-1})$, and belief at $t_{k-1}$, $bel(x_{k-1})$, over all possible state at $t_{k-1}$, $x_{k-1}$.

$$
\begin{aligned}
\overline{bel}(x_k) &\xlongequal{\text{prediction def.}} p(x_k|z_{1:k-1}, u_{1:k}) \\
&\xlongequal{\text{total probability}} \int p(x_k, x_{k-1}|z_{1:k-1}, u_{1:k})\,\mathrm{d}x_{k-1} \\
&\xlongequal{\text{conditional probability}} \int p(x_k|x_{k-1}, z_{1:k-1}, u_{1:k})p(x_{k-1}|z_{1:k-1}, u_{1:k})\,\mathrm{d}x_{k-1} \\
&\xlongequal{\text{Markov assumption}} \int p(x_k|x_{k-1}, u_k)p(x_{k-1}|z_{1:k-1}, u_{1:k})\,\mathrm{d}x_{k-1} \\
&\xlongequal{\text{conditional independence } u_k, x_{k-1}} \int p(x_k|x_{k-1}, u_k)p(x_{k-1}|z_{1:k-1}, u_{1:k-1})\,\mathrm{d}x_{k-1} \\
&\xlongequal{\text{belief def.}} \int p(x_k|u_k, x_{k-1})bel(x_{k-1})\,\mathrm{d}x_{k-1}
\end{aligned}
$$

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# Bayes filter: update/correction step

**Recursive_Bayes_filter(** $bel(x_{k-1}), u_k, z_k$ **)**

1: **for** all possible values of state $x_k$ **do**
2: PREDICTION: $\overline{bel}(x_k) = \int p(x_k|u_k, x_{k-1})bel(x_{k-1})\,\mathrm{d}x_{k-1}$
3: UPDATE: $bel(x_k) = \eta p(z_k|x_k)\overline{bel}(x_k)$
4: **end for**
5: **return** $bel(x_k)$

MEASUREMENT UPDATE/CORRECTION step (corrección): the belief, $bel(x_k)$, is given by the **multiplication** of normalizer, $\eta$, measurement probability, $p(z_k|x_k)$, and prediction, $\overline{bel}(x_k)$.

$$
\begin{aligned}
bel(x_k) &\overset{\text{belief def.}}{=\!=\!=} p(x_k|z_{1:k}, u_{1:k}) \\
&\overset{\text{Bayes' rule}}{=\!=\!=} \frac{p(z_k|x_k, z_{1:k-1}, u_{1:k})p(x_k|z_{1:k-1}, u_{1:k})}{p(z_k|z_{1:k-1}, u_{1:k})} \\
&\overset{\text{Normalizer}}{=\!=\!=} \eta p(z_k|x_k, z_{1:k-1}, u_{1:k})p(x_k|z_{1:k-1}, u_{1:k}) \\
&\overset{\text{Markov assumption}}{=\!=\!=} \eta p(z_k|x_k)p(x_k|z_{1:k-1}, u_{1:k}) \\
&\overset{\text{prediction def.}}{=\!=\!=} \eta p(z_k|x_k)\overline{bel}(x_k)
\end{aligned}
$$

$$
\eta = \frac{1}{p(z_k|z_{1:k-1}, u_{1:k})} = \frac{1}{\int p(z_k|x_k, z_{1:k-1}, u_{1:k})p(x_k|z_{1:k-1}, u_{1:k})\,\mathrm{d}x_k}
$$

$$
= \frac{1}{\int p(z_k|x_k)\overline{bel}(x_k)\,\mathrm{d}x_k}
$$

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# Robot and door state: model

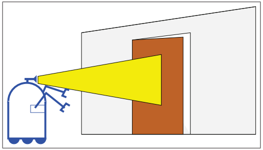

Figure: A mobile robot estimating the state of a door [Thrun, 2005].

- We use the example of a **robot estimating the state of a door** to illustrate Bayes filter algorithm.
- **State**

$$X_k = \{\text{open}, \text{closed}\}$$

- **Measurement**

$$Z_k = \{\text{s\_open}, \text{s\_closed}\}$$

- **Control**

$$U_k = \{\text{push}, \text{nothing}\}$$

- The robot is assumed to have the following initial beliefs,

$$bel(X_0 = \text{open}) = 0.5 \ ,$$
$$bel(X_0 = \text{closed}) = 0.5 \ .$$

The robot **does not know** the state of the door, thus **assigning equal prior probability** for the two possible door state values.

- The **noise** of the sensor is assumed to be characterized by the following measurement probabilities,

$$P(Z_k = \text{s\_open} | X_k = \text{open}) = 0.6 \ ,$$
$$P(Z_k = \text{s\_closed} | X_k = \text{open}) = 0.4 \ ,$$
$$P(Z_k = \text{s\_open} | X_k = \text{closed}) = 0.2 \ ,$$
$$P(Z_k = \text{s\_closed} | X_k = \text{closed}) = 0.8 \ .$$

The sensor is **relatively reliable** in detecting a **closed** door (error probability 0.2), but not an open door (error probability 0.4).

# Robot and door state: model

- The following state transition probabilities are assumed for two types of controls.
  - The robot uses its manipulator to **push** the door open:

$$P(X_k = \text{open} | U_k = \text{push}, X_{k-1} = \text{open}) = 1 \ ,$$
$$P(X_k = \text{closed} | U_k = \text{push}, X_{k-1} = \text{open}) = 0 \ ,$$
$$P(X_k = \text{open} | U_k = \text{push}, X_{k-1} = \text{closed}) = 0.8 \ ,$$
$$P(X_k = \text{closed} | U_k = \text{push}, X_{k-1} = \text{closed}) = 0.2 \ .$$

    If the door is **closed**, the robot has 80% *chance of success to open it*. If the door is already **open**, it remains *open*.
  - The robot **does nothing**:

$$P(X_k = \text{open} | U_k = \text{nothing}, X_{k-1} = \text{open}) = 1 \ ,$$
$$P(X_k = \text{closed} | U_k = \text{nothing}, X_{k-1} = \text{open}) = 0 \ ,$$
$$P(X_k = \text{open} | U_k = \text{nothing}, X_{k-1} = \text{closed}) = 0 \ ,$$
$$P(X_k = \text{closed} | U_k = \text{nothing}, X_{k-1} = \text{closed}) = 1 \ .$$

    The door **remains** its original state.

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# Robot and door state: Bayes filter

**Recursive_Bayes_filter(** $bel(x_{k-1}), u_k, z_k$ **)**

1: **for** all possible values of the state $x_k$ **do**
2: $\quad \overline{bel}(x_k) = \int p(x_k|u_k, x_{k-1}) bel(x_{k-1})\, \mathrm{d}x_{k-1}$
3: $\quad bel(x_k) = \eta p(z_k|x_k)\overline{bel}(x_k)$
4: **end for**
5: **return** $bel(x_k)$

- At $k = 1$, the robot **does nothing** ($u_1$ = nothing) and it **senses an open door** ($z_1$ = s_open).
- **Prediction**: $\overline{bel}(x_1) = \sum_{x_0} p(x_1|u_1, x_0) bel(x_0)$.

$$\begin{aligned}
\overline{bel}(X_1 = \text{open}) &= P(X_1 = \text{open}|U_1 = \text{nothing}, X_0 = \text{open}) && \times bel(X_0 = \text{open}) \\
&+ P(X_1 = \text{open}|U_1 = \text{nothing}, X_0 = \text{closed}) && \times bel(X_0 = \text{closed}) \\
&= 1 \times 0.5 + 0 \times 0.5 = 0.5
\end{aligned}$$

$$\begin{aligned}
\overline{bel}(X_1 = \text{closed}) &= P(X_1 = \text{closed}|U_1 = \text{nothing}, X_0 = \text{open}) && \times bel(X_0 = \text{open}) \\
&+ P(X_1 = \text{closed}|U_1 = \text{nothing}, X_0 = \text{closed}) && \times bel(X_0 = \text{closed}) \\
&= 0 \times 0.5 + 1 \times 0.5 = 0.5
\end{aligned}$$

- Since the robot does nothing and we are **sure** about the consequences, the prediction is the same as previous belief, $\overline{bel}(x_1) = bel(x_0)$.

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# Robot and door state: Bayes filter

**Recursive_Bayes_filter(** $bel(x_{k-1}), u_k, z_k$ **)**

1: **for** all possible values of the state $x_k$ **do**
2: $\quad \overline{bel}(x_k) = \int p(x_k|u_k, x_{k-1}) bel(x_{k-1})\,\mathrm{d}x_{k-1}$
3: $\quad bel(x_k) = \eta p(z_k|x_k)\overline{bel}(x_k)$
4: **end for**
5: **return** $bel(x_k)$

- **Update**: $bel(x_1) = \eta P(Z_1 = \text{s\_open}|x_1)\overline{bel}(x_1)$.

$$P(Z_1 = \text{s\_open}|X_1 = \text{open}) \times \overline{bel}(X_1 = \text{open}) = 0.6 \times 0.5 = 0.3$$
$$P(Z_1 = \text{s\_open}|X_1 = \text{closed}) \times \overline{bel}(X_1 = \text{closed}) = 0.2 \times 0.5 = 0.1$$
$$\eta = 1/(0.3 + 0.1) = 2.5$$
$$bel(X_1 = \text{open}) = \eta P(Z_1 = \text{s\_open}|X_1 = \text{open}) \times \overline{bel}(X_1 = \text{open}) = 0.75$$
$$bel(X_1 = \text{closed}) = \eta P(Z_1 = \text{s\_open}|X_1 = \text{closed}) \times \overline{bel}(X_1 = \text{closed}) = 0.25$$

- The goal of normalization: $bel(X_1 = \text{open}) + bel(X_1 = \text{closed}) = 1$.

# Robot and door state: Bayes filter

**Recursive_Bayes_filter(** $bel(x_{k-1}), u_k, z_k$ **)**

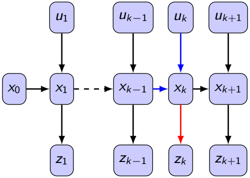
1: **for** all possible values of the state $x_k$ **do**
2: $\quad \overline{bel}(x_k) = \int p(x_k|u_k, x_{k-1}) bel(x_{k-1}) \, \mathrm{d}x_{k-1}$
3: $\quad bel(x_k) = \eta p(z_k|x_k) \overline{bel}(x_k)$
4: **end for**
5: **return** $bel(x_k)$

- At $k = 2$, the robot **pushes** the door ($u_2 = \text{push}$) and it **senses an open door** ($z_2 = \text{s\_open}$).
- **Prediction**: $\overline{bel}(X_2 = \text{open}) = 1 \times 0.75 + 0.8 \times 0.25 = 0.95$, $\overline{bel}(X_2 = \text{closed}) = 0 \times 0.75 + 0.2 \times 0.25 = 0.05$.
- **Normalizer**:

$$p(Z_2 = \text{s\_open}|X_2 = \text{open}) \times \overline{bel}(X_1 = \text{open}) = 0.6 \times 0.95 = 0.57$$
$$p(Z_2 = \text{s\_open}|X_2 = \text{closed}) \times \overline{bel}(X_1 = \text{closed}) = 0.2 \times 0.05 = 0.01$$
$$\eta = 1/0.58 \approx 1.724$$

- **Update**: $bel(X_2 = \text{open}) \approx 0.983$, $bel(X_2 = \text{closed}) \approx 0.017$.

# Glossary




- State, $\boldsymbol{x}$; Measurement, $\boldsymbol{z}$; Control, $\boldsymbol{u}$.
- Probabilistic distributions

| State evolution | State transition probability | Prediction |
|---|---|---|
| $p(x_k \mid x_{0:k-1}, u_{1:k}, z_{1:k-1})$ | $p(x_k \mid x_{k-1}, u_k)$ | $\overline{bel}(x_k) = p(x_k \mid z_{1:k-1}, u_{1:k})$ |
| Measurement evolution | Measurement probability | Belief |
| $p(z_k \mid x_{0:k}, u_{1:k}, z_{1:k-1})$ | $p(z_k \mid x_k)$. | $bel(x_k) = p(x_k \mid z_{1:k}, u_{1:k})$ |

- Markov assumption

$$p(x_k \mid x_{k-1}, u_k) \approx p(x_k \mid x_{0:k-1}, u_{1:k}, z_{1:k-1})$$
$$p(z_k \mid x_k) \approx p(z_k \mid x_{0:k}, u_{1:k}, z_{1:k-1})$$

- Bayes filter

$$\textbf{prediction}: \ \overline{bel}(x_k) = \int p(x_k \mid u_k, x_{k-1}) bel(x_{k-1}) \, \mathrm{d}x_{k-1}$$

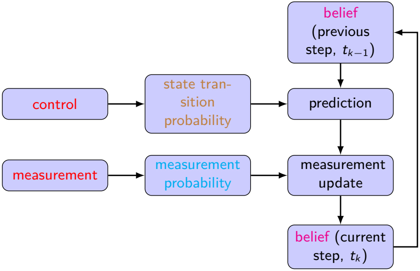
$$\textbf{update}: \ bel(x_k) = \eta p(z_k \mid x_k) \overline{bel}(x_k)$$

# Bayes filter algorithm

**Recursive_Bayes_filter(** $bel(x_{k-1})$, $u_k$, $z_k$ **)**

**for** all possible values of the state $x_k$ **do**
$\quad \overline{bel}(x_k) = \int p(x_k | u_k, x_{k-1}) bel(x_{k-1})\,\mathrm{d}x_{k-1}$
$\quad bel(x_k) = \eta p(z_k | x_k) \overline{bel}(x_k)$
**end for**
**return** $bel(x_k)$

- Define state $x_k$, control $u_k$, and measurement $z_k$.
- Assume initial belief $bel(x_0)$ at $t_0$.
- Provide control history $u_{1:k}$ and obtain measurement history $z_{1:k}$.
- Model state transition probability $p(x_k | u_k, x_{k-1})$ and measurement probability $p(z_k | x_k)$.
- Carry out the **recursive** Bayes filter algorithm to **update** the belief $bel(x_k)$ at each time step $t_k$.

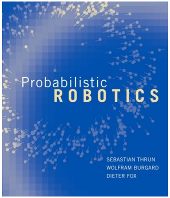


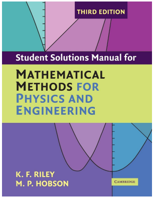

- Kalman filter is a kind of Gaussian filters that constitute the earliest tractable implementations of the Bayes filter for continuous spaces of random variables. All Gaussian filters share the basic idea that **beliefs are represented by normal distributions**.

# Acrónimos

| | |
|---|---|
| **KF** | Kalman Filter |
| **PDF** | Probability Distribution Function |
| **PMF** | Probability Mass Function |

# Bibliografía Recomendada

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