

Tarea 1

Control por Modos Deslizantes

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Solve exercises 1.2, 1.7, 2.1, 2.2, 2.3, 2.5

Problem 1: 1.1

A simplified m longitudinal motion of an underwater vehicle can be described by

$$m\ddot{x} + k\dot{x}|\dot{x}| = u \quad (1)$$

where x is the position of the vehicle, u is the control input (a force that is provided by a propeller), m is the mass of the vehicle, and $k > 0$ is the drag coefficient. Assuming the value of m is known exactly, the drag coefficient is bounded $k_1 \leq k \leq k_2$, and the position and its derivative (velocity) x, \dot{x} are measured:

- Obtain a state system model of the vehicle using $x_1 = x$, $x_2 = \dot{x}$ as the state variables.
- Design a conventional sliding mode control law u that drives $x_1, x_2 \rightarrow 0$ as time increases.
- Simulate the control system for $x_1(0) = 2$ m, $x_2(0) = 0.5$ m/s, $m = 4$ kg, and $k = 1.5 + 0.4 \sin(2t)$ [kg/ms]. Plot the time histories of the sliding variable, the control function u , the position x_1 , and the velocity x_2 .
- Identify the quantities that reach zero in finite time and the ones that approach zero asymptotically.

Hint: The function $k\dot{x}|\dot{x}|$ can be bounded as

$$|k\dot{x}|\dot{x}|| \leq k_2\dot{x}^2 = 1.9\dot{x}^2.$$

Solution

- Using the state variables $x_1 = x$ and $x_2 = \dot{x}$, the dynamic model of the vehicle can be expressed as:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{m} (-kx_2|x_2| + u). \end{aligned}$$

- Define the sliding surface as:

$$\sigma = x_2 + cx_1, \quad c > 0,$$

with its time derivative given by:

$$\dot{\sigma} = c\dot{x}_1 + \dot{x}_2 = cx_2 + \frac{1}{m}(-kx_2|x_2| + u).$$

Consider the following Lyapunov candidate function:

$$V = \frac{1}{2}\sigma^2,$$

which has the time derivative:

$$\dot{V} = \sigma\dot{\sigma} = \sigma \left(cx_2 + \frac{1}{m}(-kx_2|x_2| + u) \right).$$

Choose the control input as:

$$u = m(-cx_2 + v),$$

so that:

$$\dot{V} = \sigma \left(-\frac{k}{m}x_2|x_2| + v \right).$$

Using the provided bound on the nonlinear term from the hint:

$$|kx_2|x_2|| \leq k_2x_2^2 = 1.9x_2^2,$$

we can bound the derivative of V as:

$$\dot{V} \leq |\sigma| \frac{1.9}{m}x_2^2 + \sigma v.$$

To dominate the first term and ensure convergence, choose:

$$v = - \left(\rho + \frac{1.9}{m}x_2^2 \right) \text{sign}(\sigma),$$

where $\rho > 0$ is a design parameter.

- c) The simulation of the control system using $c = 2$ and $\rho = 2$ yields the following results:

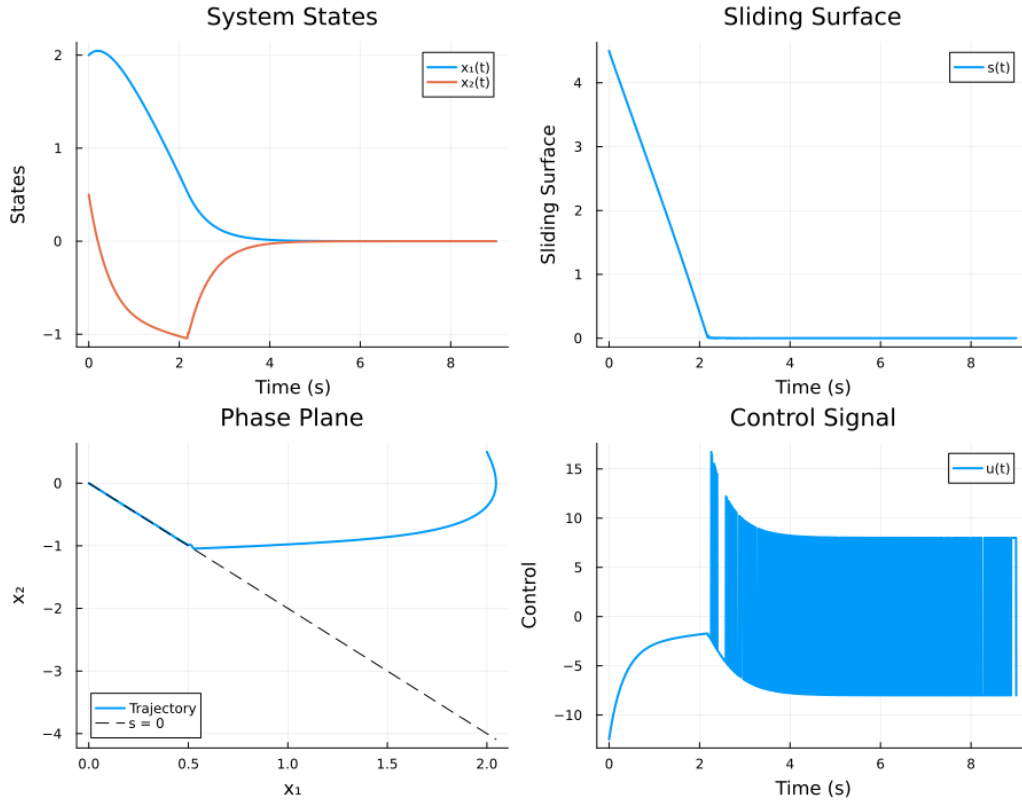


Figure 1: Simulation results for the sliding mode control system.

- d) We can observe that the sliding variable σ reaches zero in finite time, indicating that the system enters the sliding mode, while the states x_1 and x_2 approach zero asymptotically.

□

Problem 2: 1.2

Repeat Exercise 1.1 approximating the *sign* function in the control law by the sigmoid function $\text{sign}(\sigma) \approx \frac{\sigma}{|\sigma| + \varepsilon}$ and separately by the saturation function

$$\text{sign}(\sigma) \approx \begin{cases} 1 & \text{if } \sigma > \varepsilon, \\ \frac{\sigma}{\varepsilon} & \text{if } |\sigma| \leq \varepsilon, \\ -1 & \text{if } \sigma < -\varepsilon. \end{cases}$$

Solution

As the analysis is practically the same as in Exercise 1.1, we will only show the results for the sigmoid and saturation approximations, and a discussion of the results.

c) i) Using the saturation approximation, we have:

$$u = m(-cx_2 + v) = m\left(-cx_2 - \left(\rho + \frac{1.9}{m}x_2^2\right)\text{sat}(\sigma)\right).$$

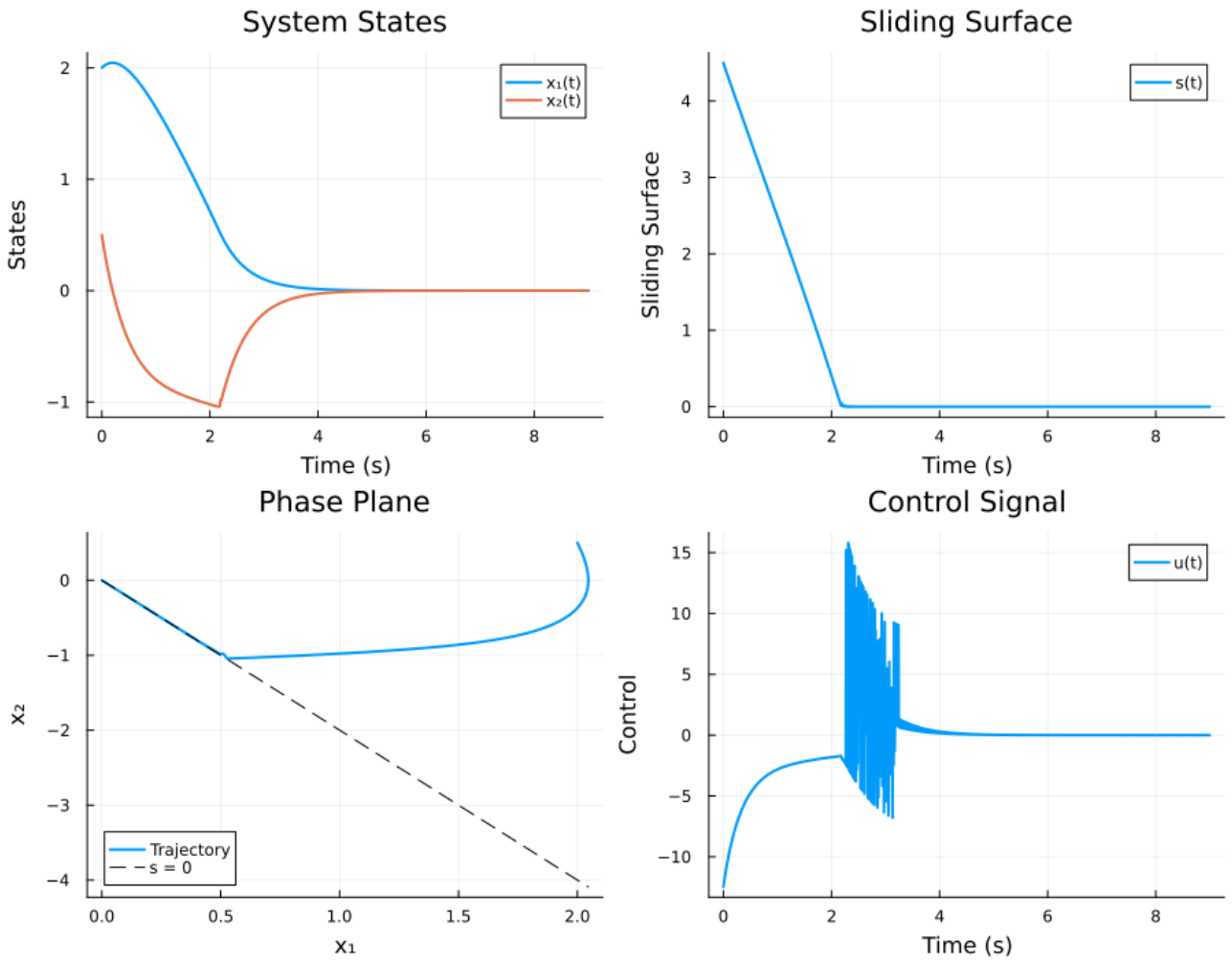


Figure 2: Simulation results for the sliding mode control system using saturation approximation.

ii) Using the sigmoid approximation, we have:

$$u = m(-cx_2 + v) = m \left(-cx_2 - \left(\rho + \frac{1.9}{m} x_2^2 \right) \frac{\sigma}{|\sigma| + \varepsilon} \right).$$

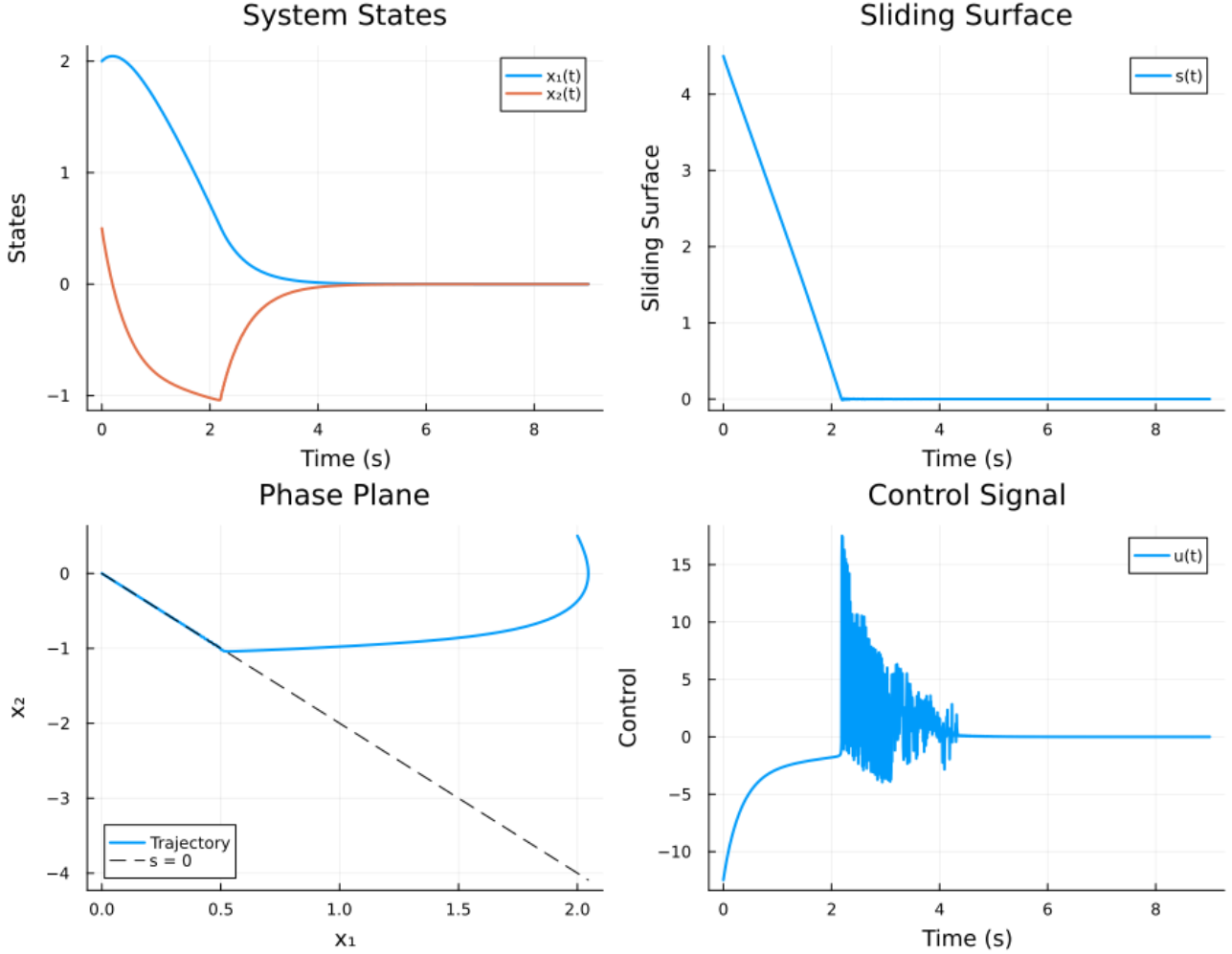


Figure 3: Simulation results for the sliding mode control system using sigmoid approximation.

In both cases, we can observe that the system seems to have finite time convergence in the sliding variable, and the chattering attenuation in the control input is evident. The saturation approximation leads to a more abrupt control action at the beginning, while the sigmoid approximation provides a smoother transition. Despite this, both approximations yield similar results in terms of system behavior and performance. It is clear that both approximations are effective in reducing chattering, but the theory suggests that both approximations only drive the sliding variable asymptotically to zero, so these are not ideal sliding modes, even though, in this case, and with that very small ε , the system behaves as if it were a sliding mode in both cases. \square

Problem 3: 1.6

For the DC motor modeled by

$$\begin{aligned} J \frac{d\omega}{dt} &= k_m i - T_L, \\ L \frac{di}{dt} &= -iR - k_b \omega + u, \end{aligned} \quad (2)$$

where J is the moment of inertia, i is the armature current, L and R are the armature inductance and resistance respectively, ω is the motor angular speed, k_b is a constant of back electromotive force, k_m is a motor torque constant, T_L is an unknown load torque which is bounded and has a bounded derivative, and u is a control function defined by the armature voltage.

Design a sliding mode control u , steering the angular speed ω to zero assuming both i and ω are measurable and all parameters are known. Simulate the control system with the following parameters:

$$\begin{aligned} R &= 1 \Omega, \\ L &= 0.5 \text{ H}, \\ k_m &= 5 \times 10^{-2} \text{ Nm/A}, \\ k_b &= k_m, \\ J &= 10^{-3} \text{ Nms}^2/\text{rad}, \\ T_L &= 0.1 \sin(t) \text{ Nm}, \\ \omega(0) &= 1 \text{ rad/s}, \\ i(0) &= 0. \end{aligned}$$

Plot the time histories of the sliding variable, the control function u , the current i , and the angular speed ω .

Solution

The system (??) can be represented in state-space form as:

$$\begin{aligned} \dot{x}_1 &= \frac{k_m}{J} x_2 - \frac{T_L}{J}, \\ \dot{x}_2 &= -\frac{k_b}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u, \end{aligned}$$

where $x_1 = \omega$ (angular speed) and $x_2 = i$ (armature current).

The control objective is to drive $\omega = x_1 \rightarrow 0$ as $t \rightarrow \infty$. To this end, we define the sliding variable as

$$\begin{aligned} s &= \dot{x}_1 + \lambda x_1 \\ &= \frac{k_m}{J} x_2 - \frac{T_L}{J} + \lambda x_1. \end{aligned}$$

The term $-\frac{T_L}{J}$ is considered as an unmatched disturbance, let's redefine the sliding variable as:

$$s = \frac{k_m}{J} x_2 + \lambda x_1.$$

On the sliding surface $s = 0$, the dynamics reduce to:

$$x_2 = -\frac{\lambda J}{k_m} x_1.$$

Substituting into the first state equation yields:

$$\dot{x}_1 = -\lambda x_1 - \frac{T_L}{J}.$$

Thus, the disturbance that enters in the first equation (where control is absent) will prevent the state variable from converging to zero in the sliding mode. Nevertheless, let's continue with the typical approach to see this phenomena.

Consider the Lyapunov candidate:

$$V = \frac{1}{2}s^2 \quad \Rightarrow \quad \dot{V} = s\dot{s}.$$

Computing \dot{s} , we have:

$$\dot{s} = \frac{k_m}{J}\dot{x}_2 + \lambda\dot{x}_1 = \frac{k_m}{J}\left(-\frac{k_b}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u\right) + \lambda\left(\frac{k_m}{J}x_2 - \frac{T_L}{J}\right).$$

Simplifying, we can express \dot{V} as:

$$\dot{V} = s \left[k_m \left(-\left(\frac{R}{L} + \lambda \right) x_2 - \frac{k_b}{L} x_1 + \frac{1}{L} u \right) - \lambda T_L \right].$$

To enforce $\dot{s} = -\rho \text{sign}(s)$, with $\rho > 0$, we choose the control input:

$$u = (R - \lambda L)x_2 + k_b x_1 + \frac{L}{k_m}v,$$

where $v = -\rho \text{sign}(s)$.

Substituting into the Lyapunov derivative:

$$\dot{V} = s [-\lambda T_L + v].$$

To ensure $\dot{V} < 0$, we bound the disturbance: $|T_L| \leq 0.1$. Then:

$$\dot{V} \leq \lambda \cdot 0.1|s| + sv = \lambda \cdot 0.1|s| - \rho|s| = -|s|(\rho - 0.1\lambda).$$

Hence, choosing $\lambda = 1$ and $\rho = 0.2$, the condition $\dot{V} < 0$ is guaranteed.

Simulation results with this control strategy are presented below.

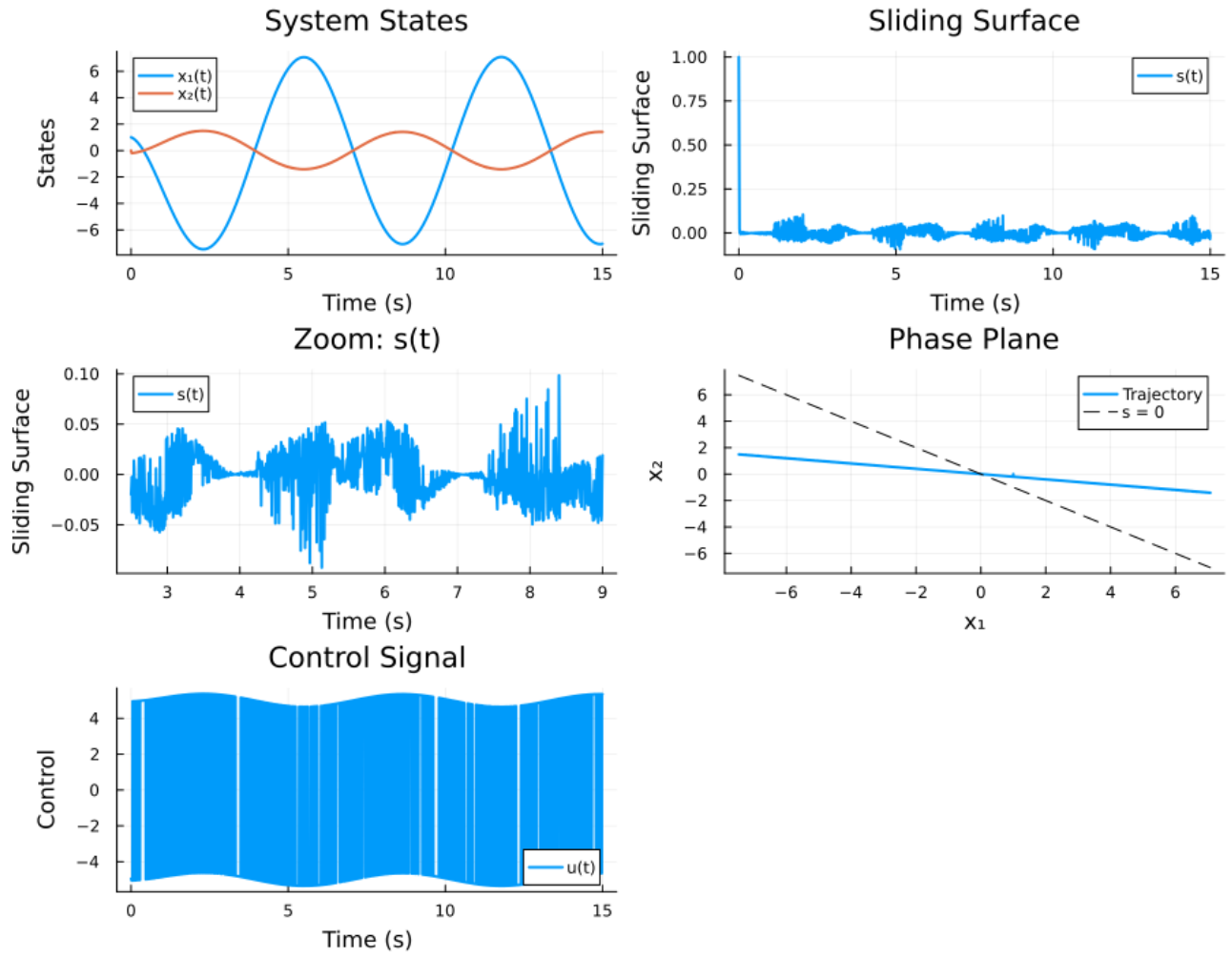


Figure 4: Simulation results for the sliding mode control system.

The simulation shows that the system converges to the sliding mode, but the disturbance T_L prevents the $x_1 = \omega$ state variable from converging asymptotically to zero.

Problem 4: 1.7

Repeat Exercise 1.6 assuming that only the angular speed $\omega(t)$ is measured. Design a sliding mode observer to estimate the armature current, ensuring that $\hat{i}(t) \rightarrow i(t)$ as $t \rightarrow \infty$. Simulate the closed-loop control system with this observer. Plot the time histories of the sliding variable, the control input $u(t)$, the angular speed $\omega(t)$, the actual current $i(t)$, and the estimated current $\hat{i}(t)$.

Solution

Let the observer estimate the state \hat{x}_1 using:

$$\dot{\hat{x}}_1 = v,$$

and define the observation error:

$$z_1 = \hat{x}_1 - x_1.$$

Taking the derivative:

$$\dot{z}_1 = \dot{\hat{x}}_1 - \dot{x}_1 = v - \left(\frac{k_m}{J} x_2 - \frac{T_L}{J} \right).$$

Define the correction term as:

$$v = -\rho \operatorname{sign}(z_1),$$

with a gain satisfying:

$$\rho > \frac{k_m}{J} |x_2| + \frac{|T_L|_{\max}}{J} + \beta, \quad \beta > 0.$$

Thus, the observation error dynamics becomes:

$$\dot{z}_1 = -\frac{k_m}{J} x_2 + \frac{T_L}{J} - \rho \operatorname{sign}(z_1),$$

and the Lyapunov analysis gives:

$$z_1 \dot{z}_1 = z_1 \left(-\frac{k_m}{J} x_2 + \frac{T_L}{J} - \rho \operatorname{sign}(z_1) \right).$$

On the sliding surface $z_1 = 0$, the equivalent control satisfies:

$$\dot{z}_1 = 0 \quad \Rightarrow \quad -\frac{k_m}{J} x_2 + \frac{T_L}{J} + v_{\text{eq}} = 0.$$

Solving for x_2 , we get:

$$x_2 = \frac{T_L}{k_m} + \frac{J}{k_m} v_{\text{eq}}.$$

Since T_L is unknown, we estimate x_2 using only v_{eq} . To approximate v_{eq} , we use a *low-pass filter*:

$$\tau \dot{\hat{v}}_{\text{eq}} = -\hat{v}_{\text{eq}} - \rho \operatorname{sign}(z_1),$$

where τ is the filter time constant.

And this is a great achievement, as we have now an estimate of \dot{x}_1 , in which previously we had an unmatched disturbance. So we can redefine the sliding variable as:

$$s = \dot{\hat{x}}_1 + \lambda x_1$$

for the control purposes.

Additionally, after the reaching time t_r , we estimate:

$$\hat{x}_2(t) \approx \frac{T_L}{k_m} + \frac{J}{k_m} \hat{v}_{eq}(t).$$

Finally, simulating the same conditions and parameters as in Exercise 1.6, with the new sliding variable and with the observer gain $\rho_o = 20$, and the filter time constant $\tau = 0.01$, we get the following results:

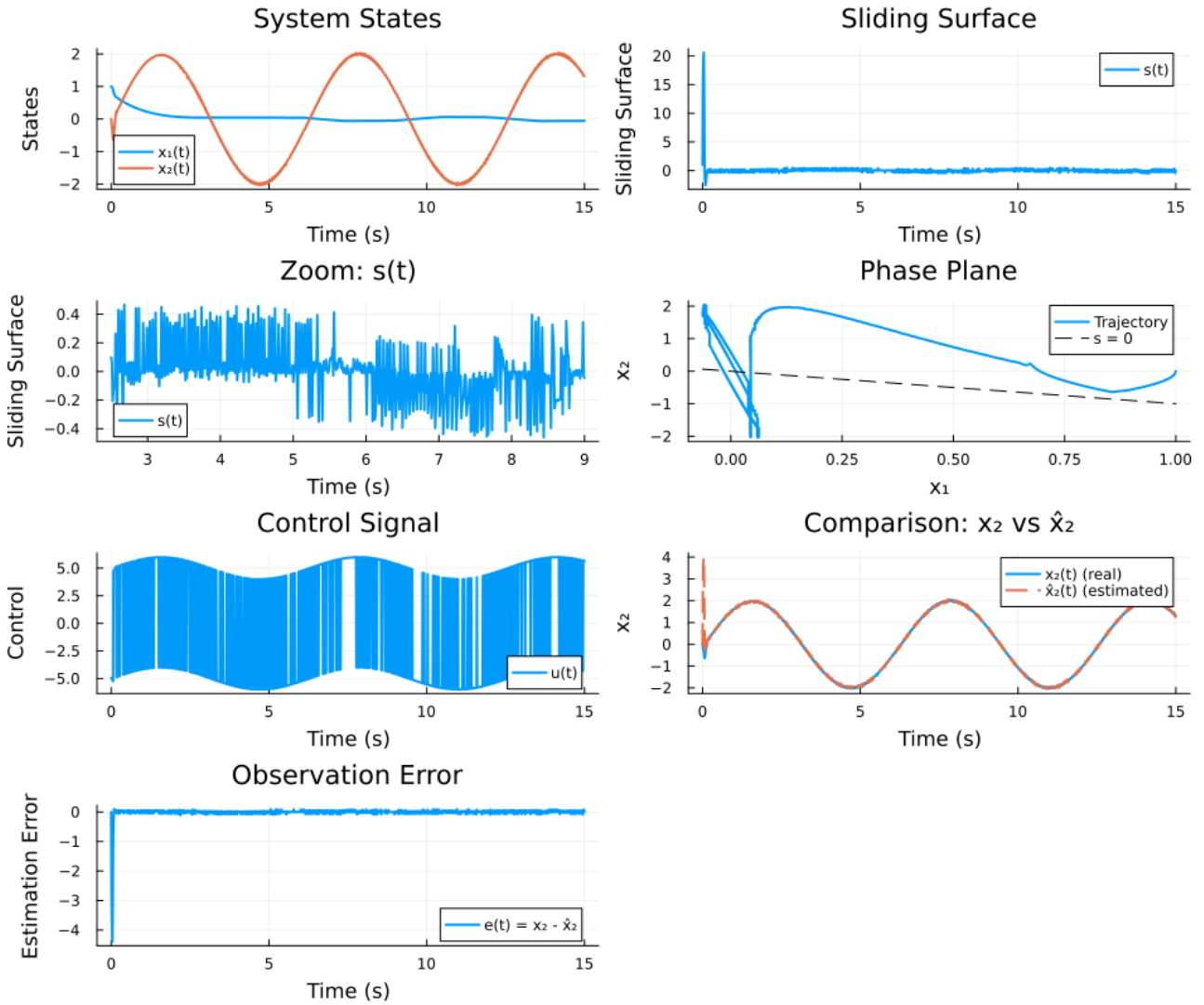


Figure 5: Simulation results for the sliding mode control system with observer.

In the simulation, we observe that the sliding mode observer successfully estimates the armature current $i(t)$, and the new sliding variable $s(t)$ converges to zero. A key advantage of this observer

is its ability to estimate the state variable \dot{x}_1 , which is crucial for defining the *new* sliding surface. This represents a significant improvement over the previous exercise, where we had to contend with an unmatched disturbance. Now, by effectively *observing* that disturbance through the observer, we are able to compensate for it. The simulation results confirm the convergence of the angular speed $x_1 = \omega$ to zero, fulfilling the original control objective.

Problem 5: 2.1

Consider the linear system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \sigma &= Gx\end{aligned}$$

with σ as the sliding variable. Find the equivalent control u_{eq} and the sliding mode equations when

1.

$$A = \begin{bmatrix} 2 & 19 \\ 3 & 29 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 9 & 12 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 9 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 1 & 29 & 0 \\ 1 & 12 & 0 \end{bmatrix}$$

Solution**Finding the equivalent control u_{eq} :**

The equivalent control u_{eq} is obtained by imposing that the sliding surface remains invariant, i.e., $\dot{\sigma} = 0$ on the sliding manifold.

1. First, differentiate σ :

$$\dot{\sigma} = G\dot{x}$$

where G defines the sliding surface. Substituting the system dynamics $\dot{x} = Ax + Bu$ yields:

$$\dot{\sigma} = G(Ax + Bu) = GAx + GBu$$

2. On the sliding surface, we require $\dot{\sigma} = 0$, which gives:

$$0 = GAx + GBu$$

3. Solving for u_{eq} leads to:

$$GBu_{eq} = -GAx \quad \Rightarrow \quad u_{eq} = -(GB)^{-1}GAx$$

assuming that GB is invertible.

4. Finally, substituting $u = u_{eq}$ into $\dot{x} = Ax + Bu$ provides the **sliding mode equations**.

Problem (a):

1. Compute GA , GB , and $(GB)^{-1}$:

$$GA = \begin{bmatrix} 9 & 12 \end{bmatrix} \begin{bmatrix} 3 & 19 \\ 3 & 29 \end{bmatrix} = \begin{bmatrix} 54 & 519 \end{bmatrix}$$

$$GB = \begin{bmatrix} 9 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 54 \quad \Rightarrow \quad (GB)^{-1} = \frac{1}{54}$$

2. Find u_{eq} :

$$\begin{aligned} u_{\text{eq}} &= -\frac{1}{54} \begin{bmatrix} 54 & 519 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= -x_1 - \frac{519}{54}x_2 \end{aligned}$$

3. Substitute u_{eq} into the system dynamics:

$$\begin{aligned} \dot{x} &= Ax + Bu_{\text{eq}} \\ &= \begin{bmatrix} 3 & 19 \\ 3 & 29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \left(-x_1 - \frac{519}{54}x_2 \right) \\ &= \begin{bmatrix} 2x_1 + 19x_2 - 2x_1 - 2\frac{519}{54}x_2 \\ 3x_1 + 29x_2 - 3x_1 - 3\frac{519}{54}x_2 \end{bmatrix} \end{aligned}$$

Therefore, the sliding mode dynamics are:

$$\dot{x} = \begin{bmatrix} -0.222x_2 \\ 0.166x_2 \end{bmatrix}$$

Problem (b):

1. Compute GA , GB , and $(GB)^{-1}$:

$$GA = \begin{bmatrix} 1 & 29 & 0 \\ 1 & 12 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 30 & 88 \\ 1 & 13 & 37 \end{bmatrix}$$

$$GB = \begin{bmatrix} 1 & 29 & 0 \\ 1 & 12 & 0 \end{bmatrix} \begin{bmatrix} 3 & 9 \\ 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 32 & -49 \\ 15 & -15 \end{bmatrix}$$

$$(GB)^{-1} = \frac{1}{255} \begin{bmatrix} -15 & 49 \\ -15 & 32 \end{bmatrix}$$

2. Find u_{eq} :

$$\begin{aligned} u_{\text{eq}} &= -(GB)^{-1}GAx \\ &= -\frac{1}{255} \begin{bmatrix} -15 & 49 \\ -15 & 32 \end{bmatrix} \begin{bmatrix} 1 & 30 & 88 \\ 1 & 13 & 37 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \frac{1}{15} \begin{bmatrix} -2 & -11 & -29 \\ -1 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

3. Substitute u_{eq} into the system dynamics:

$$\begin{aligned}\dot{x} &= Ax + Bu_{\text{eq}} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{15} \begin{bmatrix} 3 & 9 \\ 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 & -11 & -29 \\ -1 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \frac{1}{15} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 17 & 11 & 44 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\end{aligned}$$

Therefore, the sliding mode dynamics are:

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{17}{15}x_1 + \frac{11}{15}x_2 + \frac{44}{15}x_3 \end{bmatrix}$$

Problem 6: 2.1

Consider the system given by

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 + u_1 + 10u_2 \\ \dot{x}_2 &= x_2 + 3x_3 + u_1 - 2u_2 \\ \dot{x}_3 &= x_1 + x_3 - u_1 \\ \sigma_1 &= x_1 + 10x_2, \quad \sigma_2 = x_1 + 5x_2\end{aligned}$$

Find the system dynamics in the sliding mode $\sigma_1 = \sigma_2 = 0$.

Solution

1. First, we put the system in state-space form, so it can be solved as the previous problem:

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad \sigma = Gx \\ A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 10 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}\end{aligned}$$

2. Compute GA , GB , and $(GB)^{-1}$:

$$\begin{aligned}GA &= \begin{bmatrix} 1 & 11 & 31 \\ 1 & 6 & 16 \end{bmatrix} \\ GB &= \begin{bmatrix} 11 & -10 \\ 6 & 0 \end{bmatrix} \\ (GB)^{-1} &= \frac{1}{60} \begin{bmatrix} 0 & 10 \\ -6 & 11 \end{bmatrix}\end{aligned}$$

3. Find u_{eq} :

$$\begin{aligned}u_{eq} &= -(GB)^{-1}GAx \\ &= -\frac{1}{60} \begin{bmatrix} 0 & 10 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 11 & 31 \\ 1 & 6 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= -\frac{1}{60} \begin{bmatrix} 10 & 60 & 160 \\ 5 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\end{aligned}$$

4. Substitute u_{eq} into the system dynamics:

$$\begin{aligned}\dot{x} &= Ax + Bu_{eq} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \frac{1}{60} \begin{bmatrix} 1 & 10 \\ 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 60 & 160 \\ 5 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 7 & 1 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\end{aligned}$$

Therefore, the sliding mode dynamics are:

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{7}{6}x_1 + \frac{1}{6}x_2 + \frac{22}{6}x_3 \end{bmatrix}$$

Problem 7: 2.3

Consider the DC-DC buck converter in Fig. 2.19 which belongs to the class of attenuation circuits; the corresponding dynamic equations are given by:

$$\begin{aligned}L \frac{di}{dt} &= -v + uV_{in} \\ C \frac{dv}{dt} &= i - \frac{v}{R}\end{aligned}$$

where i is the current through the inductor L , v is the voltage across the capacitor C , V_{in} is the input voltage, and $u \in \{0, 1\}$ is the switching control signal. The goal is to stabilize the output voltage v at the desired level v_d . This goal is to be achieved via stabilization of the inductor current i at the desired level $i_d = \frac{v_d}{R}$ using sliding mode control, for this purpose: (a) Setting $\sigma = i - i_d$ and using the control input $u = \frac{1}{2}(1 - \text{sign}(\sigma))$, find the equivalent control and the sliding mode dynamics. (b) Considering $L = 20\text{mH}$, $C = 20\mu\text{ F}$, $R = 30\Omega$, $V_{in} = 15\text{ V}$, $v_d = 10\text{ V}$, and the initial conditions $i(0) = 0.1\text{ A}$ and $v(0) = 5\text{ V}$, confirm the efficacy of the controller by simulations.