Robust Finite-Time Control of Robot Manipulators via Discontinous Integral Action

A Lyapunov-Based Approach with Output Feedback

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Section 1

Introduction

Motivation and Context

- **Why is robot control still a relevant problem?**
- **Classic challenges:** nonlinearity, uncertainty, and disturbances
- **The key problem:** linear control shows limitations against strong nonlinearities and lacks robustness

Objectives

Main Objective

To design a robust finite-time controller for robot manipulators that ensures stability and performance in the presence of uncertainties and disturbances.

Specific Objectives:

- Develop a controller featuring **discontinuous integral action** for disturbance rejection and zero steady-state error.
- Ensure the stability of the closed-loop system in the presence of **quadratic Coriolis terms**.
- Generalize the methodology for the **MIMO case** (n-DOF manipulators).
- Eliminate the need for **velocity measurements** by employing an output feedback approach using continuous homogeneous observers.

Section 2

Problem Statement

Mathematical Formulation

Manipulator Dynamics

Nominal

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

Uncertainties + Disturbances

$$\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta G(q) + d(q,\dot{q},\ddot{q}) = \tau$$

Control Objectives:

- Design τ to achieve:
  - Finite-Time Regulation (FTR)
  - Finite-Time Tracking (FTT)

FTR: If $q_d \in \mathbb{R}^n$ is constant, then the joint position error $\tilde{q} = q - q_d$ and the joint velocity $\dot{q}$ satisfy $\lim_{t\to T(\tilde{q}_0,\dot{q}_0)} \tilde{q}(t) = 0$ and $\lim_{t\to T(\tilde{q}_0,\dot{q}_0)} \dot{q}(t) = 0$, for all IC $\tilde{q}_0, \dot{q}_0 \in \mathbb{R}^n$.

FTT: If $\|q_d\| < \alpha_\nu$ and $\alpha_\nu \in \mathbb{R}_{>0}$ is known, then $\tilde{q}$ and the joint velocity error $\dot{\tilde{q}} = \dot{q} - \dot{q}_d$ satisfy $\lim_{t\to T(\tilde{q}_0,\dot{q}_0)} \tilde{q}(t) = 0$ and $\lim_{t\to T(\tilde{q}_0,\dot{q}_0)} \dot{\tilde{q}}(t) = 0$, for all IC $\tilde{q}_0, \dot{\tilde{q}}_0 \in \mathbb{R}^n$.

Mathematical Formulation

Manipulator Dynamics

Nominal

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

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$$\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta G(q) + d(q,\dot{q},\ddot{q}) = \tau$$

Control Objectives:

- Design τ to achieve:
  - Finite-Time Regulation (FTR)
  - Finite-Time Tracking (FTT)
- Handle uncertainties

Mathematical Formulation

Manipulator Dynamics

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$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

Uncertainties + Disturbances

$$\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta G(q) + d(q,\dot{q},\ddot{q}) = \tau$$

Control Objectives:

- Design τ to achieve:
  - Finite-Time Regulation (FTR)
  - Finite-Time Tracking (FTT)
- Handle uncertainties
- Implement Output Feedback Controller

Technical Challenges

Main Challenges:

- Quadratic Coriolis terms $C(q, \dot{q})\dot{q}$

Coriolis Challenge

$$C(q, \dot{q})\dot{q} = \sum_{i,j,k} C_{ijk}(q)\dot{q}_j\dot{q}_k$$

- Quadratic in velocity $\rightarrow$ strong nonlinearity
- Cross-coupling between all joints
- Velocity-dependent uncertainty amplification

Technical Challenges

Main Challenges:

- Quadratic Coriolis terms $C(q, \dot{q})\dot{q}$
- Discontinuous dynamics analysis

Discontinuous Analysis

Problem: τ contains $\text{sign}(e)$ terms

- Classical derivatives don't exist
- Need differential inclusions: $\dot{x} \in F(x)$
- Prove finite-time convergence

Technical Challenges

Main Challenges:

- Quadratic Coriolis terms $C(q, \dot{q})\dot{q}$
- Discontinuous dynamics analysis
- MIMO coupling effects

MIMO Complexity

Each joint affects others through:

- Inertial coupling
- Centrifugal forces
- Gravitational terms

Technical Challenges

Main Challenges:

- Quadratic Coriolis terms $C(q,\dot{q})\dot{q}$
- Discontinuous dynamics analysis
- MIMO coupling effects
- Output feedback design

Observer Design and Analysis

- Estimate $\dot{q}$ from position measurements only:
$$\dot{\hat{q}} = f(q, \hat{q})$$
- Analyse how does this affect to closed-loop stability.

Section 3

Literature Review

Literature Review: Finite-Time Control for Manipulators

Focus Areas

- Discontinuous PID control for finite-time convergence
- Output feedback design using homogeneous observers
- Finite-time stability analysis for robot manipulators
- Robustness to uncertainties and quadratic nonlinearities

Papers Analyzed:

1. **Moreno (2020)** Discontinuous PID (SISO baseline)
2. **Mercado-Uribe and Moreno (2020)** Advanced theory and proofs
3. **Cruz-Zavala et al. (2021)** Finite-time for manipulators
4. **Cruz-Zavala and Moreno (2017)** Homogeneous Lyapunov tools

Key Question

How to extend discontinuous PID from SISO to MIMO manipulators with quadratic Coriolis terms?

Research Gap No existing work combines: disc-PID + MIMO (manipulators) + quadratic terms handling + output feedback

Paper 1: Moreno 2020 - Discontinuous PID for SISO Systems

Core Idea: Discontinuous integral action $\dot{\sigma} = \text{sign}(e_1)$ achieves finite-time tracking and disturbance rejection for SISO systems with relative degree 2.

Mathematical Contribution:

- Controller: $u = -k_1 \left\lceil e_1 + k_2 \lceil e_2 \rfloor^{3/2} \right\rfloor^{1/3} - k_3 \sigma$
- Finite-time stability proof using homogeneous LF
- Continuous homogeneous observer for velocity estimation:

$$\dot{\hat{e}}_1 = -\lambda l_1 \lceil \hat{e}_1 - e_1 \rfloor^{2/3} + \hat{e}_2$$

$$\dot{\hat{e}}_2 = -\lambda^2 l_2 \lceil \hat{e}_1 - e_1 \rfloor^{1/3} - k_1 a_N \left\lceil e_1 + k_2 \lceil \hat{e}_2 \rfloor^{3/2} \right\rfloor^{1/3}$$

- Output feedback implementation (position-only measurements)

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = b_0 + a_0 u \\ \dot{\sigma} = \text{sign}(e_1) \end{cases}$$

SISO system with Relative Degree=2

Paper 1: Moreno 2020 - Discontinuous PID for SISO Systems

Core Idea: Discontinuous integral action $\dot{\sigma} = \text{sign}(e_1)$ achieves finite-time tracking and disturbance rejection for SISO systems with relative degree 2.

Advantages:

+ Finite-time convergence with zero steady-state error
+ Universal internal model for Lipschitz references/disturbances
+ Simple low-dimensional controller structure
+ Relaxed minimum-phase assumption (only BIBS required)
+ Continuous control signal despite discontinuous integral
+ Robust to parameter uncertainties

Key Innovation

Discontinuous integral acts as "universal servomechanism" for large class of signals

Assumptions:

- $0 < a_m \leq a_0 \leq a_M$
- $|\dot{\rho}| \leq L$ (Lipschitz)
- BIBS zero dynamics

Paper 1: Moreno 2020 - Discontinuous PID for SISO Systems

Core Idea: Discontinuous integral action $\dot{\sigma} = \text{sign}(e_1)$ achieves finite-time tracking and disturbance rejection for SISO systems with relative degree 2.

Limitations:

- SISO systems only (robots are MIMO)
- Assumes BIBS (invalid for quadratic Coriolis terms)
- Technical challenges for discontinuous integral action

Lipschitz violation

For manipulators: $\rho = \frac{C(q,\dot{q})\dot{q}}{M(q)}$

$$\dot{\rho} \propto \frac{d}{dt}[C(q,\dot{q})\dot{q}] \text{ unbounded}$$

Violates Assumption 3: $|\dot{\rho}| \leq L$

Paper 2: Mercado-Moreno 2020 - Generalized Discontinuous Integral Theory

Core Idea: Homogeneous discontinuous integral action for arbitrary relative degree (RD) systems, generalizing super-twisting algorithm to higher orders.

Mathematical Contribution:

- Controller: $u = -k_\rho \chi(\bar{x}_\rho) + z$, $\dot{z} \in -K_I \psi(\bar{x}_\rho)$
- Discontinuous integral $\psi(\bar{x}_\rho)$ can depend only on a portion of the state
- Smooth homogeneous Lyapunov function for arbitrary ρ
- Rigorous finite-time stability proof via modified backstepping
- Continuous control signal despite discontinuous integral

$$\begin{cases} \dot{x}_i = x_{i+1}, \;\; i = 1, \ldots, \rho - 1 \\ \dot{x}_\rho = g[u + \delta] \end{cases}$$

Arbitrary relative degree ρ

Paper 2: Mercado-Moreno 2020 - Generalized Discontinuous Integral Theory

Core Idea: Homogeneous discontinuous integral action for arbitrary relative degree (RD) systems, generalizing super-twisting algorithm to higher orders.

Advantages:

+ Generalizes super-twisting to arbitrary relative degree
+ Flexible integral structure (polynomial or rational forms)
+ Handles larger class of perturbations than classical methods
+ Smooth Lyapunov function (vs non-smooth in other works)
+ Rigorous mathematical framework with complete proofs
+ Scaling properties preserve stability

Key Innovation

Unified framework for discontinuous integral action across all relative degrees

Perturbation Class:

- $|\delta_1| \leq \Delta_1 \|x\|^{d+r_\rho}$
- $|\dot{\delta}_2| \leq \Delta_2$
- Much larger than constant perturbations

Paper 2: Mercado-Moreno 2020 - Generalized Discontinuous Integral Theory

Core Idea: Homogeneous discontinuous integral action for arbitrary relative degree (RD) systems, generalizing super-twisting algorithm to higher orders.

Limitations:

- SISO systems only (robots are MIMO)
- Arbitrary relative degree (unnecessary complexity for RD=2)
- Requires full state measurement $(\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})$
- Complex gain tuning procedure
- Needs robust exact differentiator for output feedback

Complexity Issue

For robots (RD=2):

- Arbitrary ρ adds unnecessary complexity
- Our approach: direct RD=2 design

Paper 3: Cruz-Nunez 2020 - Strict Lyapunov Functions for Robot FT Control

Core Idea: Energy shaping framework to construct strict Lyapunov functions for finite-time control of robot manipulators using continuous controllers.

Mathematical Contribution:

- Controller: $\tau = -\nabla_{\tilde{q}} U_c(\tilde{q}, q_d) - \nabla_{\dot{q}} F(\dot{q}) + M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d$
- Strict Lyapunov functions: $V_G = H^p + \gamma h(\tilde{q}) M \dot{q}$, $V_L = H^p + \gamma \tilde{q}^T M \dot{q}$
- Energy shaping conditions on potential U_c and dissipation F functions
- Unified framework for regulation and tracking problems
- Homogeneous controller design with $2r_2 > r_1 \geq r_2 > 0$

$$H = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U_d(\tilde{q}, q_d)$$

Total energy function

Key conditions:

- $\tilde{q}^T \nabla_{\tilde{q}} U_d \geq \beta |\tilde{q}|^{p_U+1}$
- $F(\dot{q}) = \frac{1}{p_F+1}\dot{q}^T D \lfloor \dot{q} \rceil^{p_F}$

Paper 3: Cruz-Nunez 2020 - Strict Lyapunov Functions for Robot FT Control

Core Idea: Energy shaping framework to construct strict Lyapunov functions for finite-time control of robot manipulators using continuous controllers.

Advantages:

+ Applies to robot manipulators (our target system)
+ Finite-time convergence with continuous control
+ No acceleration measurements required
+ Solves both regulation and tracking problems
+ Strict Lyapunov analysis ensures robust stability
+ Energy-based design is intuitive for mechanical systems

Key Achievement

First to solve finite-time tracking for robots with continuous controllers

Energy Shaping:

- Natural for mechanical systems
- Passivity-based design
- Proven stability framework

Paper 3: Cruz-Nunez 2020 - Strict Lyapunov Functions for Robot FT Control

Core Idea: Energy shaping framework to construct strict Lyapunov functions for finite-time control of robot manipulators using continuous controllers.

Limitations:

- No integral action (no disturbance rejection)
- No discontinuous terms for robustness
- Energy shaping approach different from PID structure
- Complex conditions on potential energy functions
- Requires full state feedback (position and velocity)

Missing Elements

For disturbance rejection:

- No integral action
- No discontinuous terms
- Limited robustness

Paper 4: Cruz-Moreno 2017 - Homogeneous HOSM Lyapunov Framework

Core Idea: Lyapunov-based design framework for homogeneous High-Order Sliding Mode controllers using modified backstepping approach.

Mathematical Contribution:

- General CLF framework: $V_\rho(x) = \gamma_{\rho-1} V_{\rho-1} + W_\rho(x)$
- Modified backstepping for discontinuous controllers
- Homogeneous CLF construction for arbitrary relative degree
- Controllers: $u_D = -k_\rho \lceil \sigma_\rho(x) \rfloor^0$, $u_Q = -k_\rho \frac{\sigma_\rho(x)}{M(x)}$
- Unified framework for nested and polynomial HOSM types

$$\begin{cases} \dot{x}_i = x_{i+1}, \; i = 1, \dots, \rho - 1 \\ \dot{x}_\rho \in [-C, C] + [K_m, K_M] u \end{cases}$$

Standard HOSM model

Homogeneity: $r_s = (\rho, \rho - 1, \dots, 1)$

Paper 4: Cruz-Moreno 2017 - Homogeneous HOSM Lyapunov Framework

Core Idea: Lyapunov-based design framework for homogeneous High-Order Sliding Mode controllers using modified backstepping approach.

Advantages:

- \+ Rigorous Lyapunov framework for HOSM design
- \+ Explicit gain calculation procedure
- \+ Covers large family of discontinuous controllers
- \+ Finite-time stability guarantees with convergence time bounds
- \+ Extension to variable-gain controllers
- \+ Foundation for subsequent discontinuous control work

Key Achievement

First explicit homogeneous CLF construction for arbitrary order HOSM

CLF Properties:

- r-homogeneous of degree m
- Smooth despite discontinuous control
- Recursive construction

Paper 4: Cruz-Moreno 2017 - Homogeneous HOSM Lyapunov Framework

Core Idea: Lyapunov-based design framework for homogeneous High-Order Sliding Mode controllers using modified backstepping approach.

Limitations:

- SISO systems only
- Pure sliding mode approach (no integral PID structure)
- Requires full state measurement up to $\sigma^{(\rho-1)}$
- Complex CLF construction for higher orders
- Conservative gain calculation in practice

Gap for disc-PID

Different paradigm:

- Sliding mode vs disc-PID structure
- No integral action for disturbances
- SISO limitation

Comparative Analysis: Methods vs. Requirements

Method	MIMO Systems	Integral Action	Coriolis Quadratic	Output Feedback
Moreno 2020	×	✓	×	✓
Mercado-Moreno 2020	×	✓	×	×
Cruz-Nunez 2020	✓	×	✓	✓
Cruz-Moreno 2017	×	×	×	×
Our Approach	✓	✓	✓	✓

Key Observations:

- No method combines ALL features
- SISO limitation in discontinuous integral approaches
- Missing quadratic terms handling

Trade-offs Identified:

- Discontinuous Integral vs. MIMO
- Integral action vs. manipulator application

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