Robust Finite-Time Control of Robot Manipulators via Discontinous Integral Action

A Lyapunov-Based Approach with Output Feedback

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Contents

Section 1

Introduction

Motivation and Context

- **Why is robot control still a relevant problem?**
- **Classic challenges:** nonlinearity, uncertainty, and disturbances
- **The key problem:** linear control shows limitations against strong nonlinearities and lacks robustness

Objectives

Main Objective

To design a robust finite-time controller for robot manipulators that ensures stability and performance in the presence of uncertainties and disturbances.

Specific Objectives:

- Develop a controller featuring **discontinuous integral action** for disturbance rejection and zero steady-state error.
- Ensure the stability of the closed-loop system in the presence of **quadratic Coriolis terms**.
- Generalize the methodology for the **MIMO case** (n-DOF manipulators).
- Eliminate the need for **velocity measurements** by employing an output feedback approach using continuous homogeneous observers.

Section 2

Theoretical Background

Differential Inclusions

Why Differential Inclusions?

Uncertain or discontinuous systems are more appropriately described by Differential Inclusions (DI)

$$\dot{x} \in F(t, x)$$

than by Differential Equations (DE).

Solution of a DI:

A solution of the DI $\dot{x} \in F(t, x)$ is any function $x(t)$, defined in some interval $I \subseteq [0, \infty)$, which is:

- Absolutely continuous on each compact subinterval of I
- Satisfies $\dot{x}(t) \in F(t, x(t))$ almost everywhere on I

For a discontinuous DE $\dot{x} = f(t, x)$, the function $x(t)$ is a generalized solution if and only if it is a solution of the associated DI $\dot{x} \in F(t, x)$.

Filippov Differential Inclusions

We consider the DI $\dot{x} \in F(t,x)$ associated to $\dot{x} = f(t,x)$ as given by **A.F. Filippov's approach** - the Filippov DI with Filippov solutions. Filippov (1988)

Standard Assumptions:

The multivalued map $F(t,x)$ satisfies the standard assumptions if:

(H1) $F(t,x)$ is nonempty, compact, convex subset of $\mathbb{R}^n$

(H2) $F(t,x)$ is upper semi-continuous as a set valued map of x

(H3) $F(t,x)$ is Lebesgue measurable as a set valued map of t

(H4) $F(t,x)$ is locally bounded

Existence Theorem

If $F(t,x)$ satisfies (H1)-(H4), then for each pair $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^n$ there exists at least one solution $x(t)$ with $x(t_0) = x_0$.

Homogeneity - Dilation and Functions

Dilation Operator:

For $x = [x_1, \ldots, x_n]^\top$ and $\lambda > 0$:

$$\Lambda_r^\lambda x := [\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n]^\top$$

where $r = [r_1, \ldots, r_n]^\top$ with $r_i > 0$ are the **weights** of the coordinates.

***r*-Homogeneous Function:**

A function $V : \mathbb{R}^n \to \mathbb{R}$ is called r-homogeneous of degree $l \in \mathbb{R}$ if:

$$V(\Lambda_r^\lambda x) = \lambda^l V(x) \quad \text{for every } \lambda > 0$$

***r*-Homogeneous Vector Field:**

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called r-homogeneous of degree l if:

$$f(\Lambda_r^\lambda x) = \lambda^l \Lambda_r^\lambda f(x)$$

Homogeneity - Norm and Systems

Homogeneous Norm:
Given a vector r and dilation $\Lambda_r^\lambda x$, the homogeneous norm is defined by:

$$\|x\|_{r,p} := \left(\sum_{i=1}^{n} |x_i|^{p/r_i} \right)^{1/p}, \quad \forall x \in \mathbb{R}^n$$

for any $p \geq 1$.

Homogeneous System:
A system is called homogeneous if its vector field (or vector-set field) is r-homogeneous of some degree.

A vector-set field $F(x) \subset \mathbb{R}^n$ is r-homogeneous of degree l if:

$$F(\Lambda_r^\lambda x) = \lambda^l \Lambda_r^\lambda F(x)$$

Homogeneous Differential Inclusions

Homogeneous DI:

A DI $\dot{x} \in F(x)$ is r-homogeneous of degree l if the vector-set field $F(x)$ satisfies:

$$F(\Lambda_r^\lambda x) = \lambda^l \Lambda_r^\lambda F(x), \quad \forall \lambda > 0$$

Key Property - Local to Global:

For homogeneous systems of degree $l < 0$, local stability implies global stability. This remarkable property allows homogeneity to "extend" local properties to global ones.

Lyapunov Analysis:

For homogeneous DI, if there exists a homogeneous Lyapunov function $V(x)$ such that:

$$\frac{\partial V}{\partial x} \nu \leq -c(\|x\|) \quad \text{for all } \nu \in F(x)$$

then the origin is Uniformly Globally Asymptotically Stable (UGAS).

Finite-Time Stability

Global Uniform Finite-Time Stability (GUFTS):

A DI $\dot{x} \in F(x)$ is GUFTS at 0 if:

- $x(t) = 0$ is a Lyapunov-stable solution
- For any $R > 0$ there exists $T > 0$ such that any trajectory starting within $\|x\| < R$ reaches zero in time T

Fundamental Result: For r-homogeneous DI's of degree $l < 0$:

$$\text{Local asymptotic stability} \Rightarrow \text{Global finite-time stability}$$

Application: Homogeneous discontinuous controllers achieve:

- Finite-time convergence
- Robustness against perturbations
- Global stability from local analysis

References

Filippov, A. F. (1988). *Differential Equations with Discontinuous Righthand Side*. Kluwer Academic Publishers, Dordrecht, The Netherlands.