INFORMATION AND COMPUTATION 117, 12–18 (1995)

Resolution for Quantified Boolean Formulas

HANS KLEINE BÜNING

FB 17—Mathematik/Informatik, Universität Paderborn, 33098 Paderborn, Germany

MAREK KARPINSKI

Institut für Informatik, Universität Bonn, 53012 Bonn, Germany

AND

ANDREAS FLÖGEL

FB 17—Mathematik/Informatik, Universität Paderborn, 33098 Paderborn, Germany

A complete and sound resolution operation directly applicable to the quantified Boolean formulas is presented. If we restrict the resolution to unit resolution, then the completeness and soundness for extended quantified Horn formulas is shown. We prove that the truth of a quantified Horn formula can be decided in $O(rn)$ time, where n is the length of the formula and r is the number of universal variables, whereas in contrast the evaluation problem for extended quantified Horn formulas is coNP-complete for formulas with prefix $\forall\exists$. Further, we show that the resolution is exponential for extended quantified Horn formulas.

1. INTRODUCTION

We are interested in the evaluation problem for quantified Boolean formulas, i.e., the problem of determining whether a formula $Q_1 z_1 \cdots Q_n z_n(\alpha_1 \wedge \cdots \wedge \alpha_m)$ is true, where Q_i is either $\exists$ or $\forall$ and α_i is a propositional clause. It is well known that the evaluation problem is PSPACE-complete (Stockmeyer and Meyer, 1973) and that subclasses are decidable in polynomial time. A linear time algorithm for quantified Boolean formulas with clauses of at most two variables can be found in (Aspvall *et al.*, 1979). Motivated by the possibility of several natural query-like extensions of standard Prolog, in (Karpinski *et al.*, 1987) a cubic time algorithm for quantified Horn formulas has been established based on a generalized unit resolution for such formulas. We adapt this approach and introduce a generalized resolution, called the Q-resolution. A Q-resolution is an ordinary resolution where only existential variables can be matched and, additionally, each universal variable of a clause for which no existential variable occurs after the universal variable in the prefix is omitted. We prove that Q-resolution is *complete* and *sound* for quantified Boolean formulas.

If we restrict Q-resolution to Q-unit resolution, where one of the clauses contains one existential variable and arbitrarily many universal variables analogously to Horn formulas, the completeness and soundness for quantified Horn formulas is shown.

A natural extension of the method applies to a class of formulas where the universal literals of a clause can be arbitrary, but the existential literals are in the Horn form. We will see that for this class, called the extended quantified Horn formulas, the evaluation problem is coNP-complete, but strangely the Q-unit resolution remains complete and sound.

As mentioned above, the evaluation problem for quantified Horn formulas is solvable in polynomial time. Based on simplification rules which transform quantified Horn formulas in linear time to Horn formulas with prefix $\forall\exists$, we improve the time upper bound of the evaluation problem.

In the last chapter we will investigate sets of Horn formulas denoted as multi Horn formulas. We will show that the satisfiability problem of multi Horn formulas, i.e., the problem of whether all Horn formulas are satisfiable, is closely related to the evaluation problem for quantified Horn formulas.

2. GENERALIZED RESOLUTION

In this paper, we shall demand for technical reasons that quantified formulas be of the form $\forall x_1 \exists y_1 \cdots \forall x_{k-1} \exists y_{k-1} \forall x_k (\alpha_1 \wedge \cdots \wedge \alpha_n)$, where $\forall x_i$ is an abbreviation for $\forall x_{n_{i-1}+1} \cdots \forall x_{n_i}$ with $n_0 = 0$ and $\exists y_i$ stands for $\exists y_{m_{i-1}+1} \cdots \exists y_{m_i}$ with $m_0 = 0$ and $\alpha_1, \ldots, \alpha_n$ clauses. Further, we assume that all variables occurring in $\alpha_1, \ldots, \alpha_n$ are bound and no variable appears positive and negative in some α_i.

0890-5401/95 $6.00

We say that a literal L_1 is before a literal L_2, if the variable of L_1 occurs in the order of the prefix before the variable of L_2 and we write clauses in the form $(L_1 \vee \cdots \vee L_t)$, where L_{i-1} is before L_i for $1 < i \leqslant t$. A $\forall$-literal (resp., $\exists$-literal) is a universal literal of the form x_l or $\overline{x_l}$ (resp., existential literal y_l or $\overline{y_l}$). A *pure* $\forall$-clause is a non-tautological clause consisting exclusively of $\forall$-literals. In particular, the empty clause $\sqcup$ is a pure $\forall$-clause.

Looking for the desired resolution operation we can observe the following:

If a non-tautological clause with only universal variables appears in the quantified Boolean formula, then the formula is false. Thus, a pure $\forall$-clause can be replaced by the empty clause.

A formula $\exists y_1 \cdots y_n(\alpha_1 \wedge \cdots \wedge \alpha_m)$ is false iff ordinary resolution leads to the empty clause. That means existential variables will be matched. Therefore we will restrict the resolution in the case of clauses with existential and universal variables to existential literals only.

If a clause contains an existential variable and a universal variable x not before an existential variable, then the universal variable x can be removed without effect on the truth of the formula. Hence, we will remove all universal variables of a resolvent which do not precede an existential variable of the resolvent.

There is a more technical problem with the tautological clauses. The ordinary resolution may lead to tautological clauses, but in order to generate the empty clause they are not helpful. For the sake of simplicity we assume that quantified Boolean formulas do not contain tautological clauses and we will forbid tautological resolvents.

Now we introduce our generalized resolution operation, called Q-resolution:

DEFINITION 1. Assume that the formula does not contain tautological clauses and that a clause is seen as a set of literals, so there are no multiple occurrences of the same literal in one clause.

(a) Let α be a pure $\forall$-clause; then we replace α by the empty clause. In every other clause we delete all $\forall$-literals that do not precede a $\exists$-literal.

(b) Let α_1 be a clause with $\exists$-literal y_l and let α_2 be a clause with $\exists$-literal $\overline{y_l}$. In both clauses all occurrences of $\forall$-literals that do not precede any $\exists$-literal are removed. Then the *Q-resolvent* α of α_1 and α_2 is obtained as follows:

1. Remove all occurrences of y_l and $\overline{y_l}$ in $\alpha_1 \vee \alpha_2$.

2. If the resulting clause contains complementary literals then no resolvent exists. Otherwise the resolvent is obtained by removing all occurrences of $\forall$-literals that do not precede any $\exists$-literal occurring in the resulting clause.

Comparing ordinary resolution with Q-resolution, we see that only literals bound by existential quantifiers can be matched and universal variables that do not precede a $\exists$-literal will be eliminated.

EXAMPLE. $\forall x_1 x_2 \exists y_1 y_2 \forall x_3((x_1 \vee \bar{x}_2 \vee \bar{y}_1 \vee \bar{y}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{y}_1) \wedge (\bar{x}_2 \vee y_2) \wedge (\bar{x}_1 \vee \bar{y}_2))$.

The following Q-resolution steps can be performed:

$$(x_1 \vee \bar{x}_2 \vee \bar{y}_1 \vee \bar{y}_2 \vee x_3), (\bar{x}_2 \vee y_2) \vdash_{\overline{Q-\mathrm{Res}}} (x_1 \vee \bar{x}_2 \vee \bar{y}_1)$$

$$(\bar{x}_1 \vee \bar{y}_2), (\bar{x}_2 \vee y_2) \vdash_{\overline{Q-\mathrm{Res}}} \sqcup,$$

where $\vdash_{\overline{Q-\mathrm{Res}}}$ denotes the application of Q-resolution steps.

It is well known that resolution is complete and sound for propositional formulas in CNF; i.e., a formula $\exists y_1 \cdots y_n (\alpha_1 \wedge \cdots \wedge \alpha_m)$ is false iff the empty clause can be derived by resolution applied to $\alpha_1 \wedge \cdots \wedge \alpha_m$. A similar result will be shown for Q-resolution and quantified Boolean formulas.

THEOREM 2.1. *A quantified Boolean formula Φ is false iff $\Phi \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$.*

Proof. At first we assume Φ is false and show by induction over k (the number of quantifiers) that the empty clause can be generated by Q-resolution.

1. $k = 1$ and $\Phi = \exists y_1(\phi_1 \wedge \cdots \wedge \phi_m)$. Since in that case Q-resolution equals ordinary resolution and resolution is complete, the empty clause can be derived.

2. $k = 1$ and $\Phi = \forall x_1(\phi_1 \wedge \cdots \wedge \phi_m)$. Since Φ is false at least one ϕ_i is non-tautological and for a pure $\forall$-clause we obtain the empty clause.

3. $k > 1$ and $\Phi = \exists y_1 \cdots y_{m_1} \forall \cdots (\phi_1 \wedge \cdots \wedge \phi_m)$. Let Φ_0, resp. Φ_1, be the formula which we obtain by the assignment $y_1 = 0$, resp. $y_1 = 1$. By the induction hypothesis this yields $\Phi_0 \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$ and $\Phi_1 \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$.

Hence, we see that $\Phi \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$ or $\Phi \vdash_{\overline{Q-\mathrm{Res}}} y_1$ and $\Phi \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$ or $\Phi \vdash_{\overline{Q-\mathrm{Res}}} \bar{y}_1$. Therefore we can conclude $\Phi \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$.

4. $k > 1$ and $\Phi = \forall x_1 \cdots x_{n_1} \exists \cdots (\phi_1 \wedge \cdots \wedge \phi_m)$. Now let Φ_0, resp. Φ_1, be the formula obtained by removing all clauses with literals x_1, resp. $\bar{x}_1$, and omitting all literals $\bar{x}_1$, resp. x_1. That means Φ_0 is the formula with assignment $x_1 = 1$ and Φ_1 the formula with assignment $x_1 = 0$.

Since Φ is false at least one of Φ_0 and Φ_1 is false. Without loss of generality let Φ_0 be false. By the induction hypothesis the empty clause is derivable by Q-resolution starting with Φ_0. If we add the removed $\bar{x}_1$-occurrences in clauses of Φ_0 then no literal x_1 occurs. Then we have generated the empty clause.

Altogether we have shown that the empty clause is derivable if the formula Φ is false.

To prove the converse direction it suffices to show the following: Let $\Phi = \forall x_1 \exists y_2 \cdots \exists y_{k-1} \forall x_k (\phi_1 \wedge \cdots \wedge \phi_m)$ be a quantified formula and ϕ a clause derivable by the Q-resolution of ϕ_i and ϕ_j; then $\forall x_1 \exists y_2 \cdots \forall x_k (\phi_1 \wedge \cdots \wedge \phi_m \wedge \phi)$ is true if Φ is true. This can be seen directly from the definition of Q-resolution. Q.E.D.

3. Q-UNIT-RESOLUTION

Analogously with the case of a unit resolution for ordinary resolution we can define a Q-unit-resolution. A clause ϕ is called a $\exists$-*unit clause* if ϕ contains exactly one $\exists$-literal and arbitrarily many $\forall$-literals. A *positive* $\exists$-unit clause is a unit clause with a positive $\exists$-literal.

DEFINITION 2. The Q-unit-resolution (Q-U-Res) is a Q-resolution where one of the clauses is a positive $\exists$-unit clause.

The Q-unit-resolution is useful not only for quantified Horn formulas, but for an extension of quantified Horn formulas too. As we will see the $\forall$-part of the clauses can be arbitrarily given, whereas the $\exists$-part must be in Horn form.

DEFINITION 3. A formula $\Phi = \forall x_1 \exists y_1 \cdots \forall x_k (\phi_1 \wedge \cdots \wedge \phi_m)$ is an extended quantified Horn formula, if for each ϕ_i the $\exists$-part is a Horn clause, i.e., the clause without all $\forall$-literals is a Horn clause.

THEOREM 3.1. *The Q-unit-resolution is complete and sound for extended quantified Horn formulas.*

Proof. We show that a formula Φ is false iff the empty clause can be derived from Φ by Q-unit-resolution. Since Q-unit-resolution is a restriction of Q-resolution, it suffices to prove that the empty clause is derivable if Φ is false.

Let $\mathscr{U} := \{\sigma \vee w : w$ is a $\exists$-literal, σ disjunction of $\forall$-literals, $\Phi \vdash_{\overline{Q-U-Res}} (\sigma \vee w)\}$ be the set of derivable $\exists$-unit clauses, i.e., clauses containing exactly one $\exists$-literal.

Now we construct the truth assignments by means of $\mathscr{U}$.

Let $\Phi = \forall x_1 \exists y_1 \cdots \forall x_k (\phi_1 \wedge \cdots \wedge \phi_m)$ be given with $x_1 = x_1 \cdots x_{n_1}, x_{i+1} = x_{n_i+1} \cdots x_{n_{i+1}}$ for $1 \leqslant i \leqslant k-1$. A truth assignment $\mathfrak{I}_i = (a_1, ..., a_{n_i}) \in \{0, 1\}^{n_i}$ is an arbitrary assignment to the $\forall$-variables $x_1, ..., x_{n_i}$. We define for $y_{m_{i-1}+t}$ for $1 \leqslant t \leqslant m_i - m_{i-1}$ the following assignment:

> If a $\exists$-unit clause $(\sigma \vee y_{m_{i-1}+t})$ is in $\mathscr{U}$ and σ is false for the assignment $\mathfrak{I}_i$ then $\mathfrak{I}_i(y_{m_{i-1}+t}) = 1$.
> In other cases we define $\mathfrak{I}_i(y_{m_{i-1}+t}) = 0$.

By the definition we have determined a scheme of truth assignments for Φ that is denoted as $\mathfrak{I}$.

Now we will show that Φ is true under the truth assignments $\mathfrak{I}$ if the empty clause is not derivable.

For each clause ϕ in Φ, let t be the number of $\exists$-literals occurring in ϕ. By induction on t we show that ϕ is true for $\mathfrak{I}$.

If $t=1$ then ϕ is in $\mathscr{U}$: If the $\exists$-literal is positive then the clause is true under $\mathfrak{I}$. The case where the $\exists$-literal is negative is not possible: Assuming $\phi = (\sigma \vee \bar{y}_{m_{i-1}+t})$ it follows that $\mathfrak{I}_i(\sigma \vee \bar{y}_{m_{i-1}+t}) = 0$ and therefore $\mathfrak{I}_i(\sigma) = 0$ and $\mathfrak{I}_i(\bar{y}_{m_{i-1}+t}) = 1$. Hence there is some clause $(\sigma' \vee y_{m_{i-1}+t}) \in \mathscr{U}$. Since $\mathfrak{I}_i(\sigma') = 0$ the disjunctions σ and σ' do not contain complementary literals. The Q-resolution applied to $(\sigma \vee y_{m_{i-1}+t})$ and $(\sigma' \vee \bar{y}_{m_{i-1}+t})$ yields the empty clause. This is a contradiction.

For $t > 1$ the $\exists$-part of ϕ contains a negative $\exists$-literal $\bar{y}_{m_{i-1}+t}$ because of the Horn property. Let σ be the $\forall$-part of the clause ϕ.

We assume that ϕ is false under $\mathfrak{I}$. Furthermore, let σ' be the subclause of σ such that the $\forall$-literals in σ are less than $y_{m_{i-1}+t}$ in the prefix order.

Then we obtain $\mathfrak{I}_i(\sigma) = 0$ and $\mathfrak{I}_i(\bar{y}_{m_{i-1}+t}) = 0$. Hence there is some $\exists$-unit clause $(\sigma'' \vee y_{m_{i-1}+t}) \in \mathscr{U}$ and therefore $\Phi \vdash_{\overline{Q-U-Res}} (\sigma'' \vee y_{m_{i-1}+t})$ and $\mathfrak{I}_i(\sigma'') = 0$. Since $\mathfrak{I}_i(\sigma') = 0$ and $\mathfrak{I}_i(\sigma'') = 0$ there is no complementary pair of literals in $\sigma' \vee \sigma''$. Hence we can apply Q-unit-resolution to $(\sigma'' \vee y_{m_{i-1}+t})$ and ϕ, obtaining a clause ϕ^* with at most $t-1$ $\exists$-literals.

Since adding the clause ϕ^* to Φ does not change the truth of Φ we can apply the induction hypothesis with ϕ^* and obtain our desired result: ϕ is true. Q.E.D.

THEOREM 3.2. *For every t ($t \geqslant 1$) there exists a quantified extended Horn formula Φ_t of length $18t+1$ which is false, and the refutation to the empty clause requires at least 2^t Q-resolution steps.*

Proof. The formula Φ_t is defined as follows: $\exists y_0 y_1 y'_1 \forall x_1 \exists y_2 y'_2 \forall x_2 \exists y_3 y'_3 \cdots \forall x_{t-1} \exists y_t y'_t \forall x_t \exists y_{t+1} \cdots y_{t+t}$

$$
\begin{aligned}
&(\bar{y}_0) \wedge \\
&(y_0 \leftarrow y_1 y'_1) \wedge \\
&(y_1 \leftarrow x_1 y_2 y'_2) \wedge (y'_1 \leftarrow \bar{x}_1 y_2 y'_2) \wedge \\
&(y_2 \leftarrow x_2 y_3 y'_3) \wedge (y'_2 \leftarrow \bar{x}_2 y_3 y'_3) \wedge \\
&\vdots \qquad\qquad\qquad \vdots \\
&\vdots \qquad\qquad\qquad \vdots \\
&(y_{t-1} \leftarrow x_{t-1} y_t y'_t) \wedge (y'_{t-1} \leftarrow \bar{x}_{t-1} y_t y'_t) \wedge \\
&(y_t \leftarrow x_t y_{t+1} \cdots y_{t+t}) \wedge (y'_t \leftarrow \bar{x}_t y_{t+1} \cdots y_{t+t}) \wedge \\
&(x_1 \vee y_{t+1}) \wedge \cdots \wedge (x_t \vee y_{t+t}) \wedge \\
&(\bar{x}_1 \vee y_{t+1}) \wedge \cdots \wedge (\bar{x}_t \vee y_{t+t}).
\end{aligned}
$$

As can easily be seen, each positive $\exists$-unit clause with $\exists$-literal y_i (y'_i) ($i = 1, ..., t$) is of the form $\overset{(-)}{x_1} \vee \overset{(-)}{x_2} \vee \cdots \vee \overset{(-)}{x_{i-1}} \vee y_i^{(\prime)}$, and is obtained by Q-unit-resolution in at least 2^{t-i} steps. So each of the units y_1 and y'_1 is obtained in at least 2^{t-1} steps leading to the empty clause in at least 2^t Q-unit-resolution steps. (This also shows the falsity of Φ_t.)

A gain in complexity by using unrestricted Q-resolution would result from the merging of literals from the two parents of a resolution step. But each clause with head y_i (y'_i) that is not an $\exists$-unit contains x_i ($\bar{x}_i$) and at least one y_j or y'_j with $j > i$ in the body.

So, if one has resolved a clause containing $y_i y'_i$ in the body with a clause with the head y_i (y'_i) that is not an $\exists$-unit, the resolvent contains at least $x_i y'_i$ ($\bar{x}_i y_i$) and all y'_j of the clause with the head y_i (y'_i) with $j > i$. There is no possibility of using this resolvent in a resolution step with a clause with the head y'_i (y_i) before all $y_j^{(\prime)}$ with $j > i$ are eliminated by resolution, so that the blocking x_i ($\bar{x}_i$) is deleted. Even if the clause with the head y'_i has the same $y_j^{(\prime)}$ ($j > i$) in the body as the clause with the head y_i no merging is possible and the same $y_j^{(\prime)}$ has to be eliminated by resolution for a second time.

Since $(\bar{y}_0)$ is the only negative clause in Φ_t, one has to resolve with $(y_0 \leftarrow y_1 y'_1)$. So one has to eliminate y_1 and y'_1. This means we have to resolve upon y_2 and y'_2 when $x_1 y_2 y'_2$ is introduced in the resolution tree and again when $\bar{x}_1 y_2 y'_2$ is introduced; one has to resolve upon $y_3 y'_3$ four times (when $x_1 x_2$, $x_1 \bar{x}_2$, $\bar{x}_1 x_2$, and $\bar{x}_1 \bar{x}_2$ are introduced) and so on.

This results in at least 2^t Q-resolution steps. Q.E.D.

THEOREM 3.3. *The evaluation problem for extended quantified Horn formulas with prefix $\forall\exists$ is coNP-complete.*

Proof. That the falsity of an extended quantified Horn formula with prefix $\forall\exists$ can be decided nondeterministically in polynomial time is obvious. Thus the evaluation problem belongs to coNP and the completeness remains to be shown.

We will associate with each formula $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_n$ in 3-CNF an extended quantified Horn formula, such that the propositional formula α is satisfiable iff the extended quantified Horn formula is false.

To each clause $\alpha_i = (L_{i,1} \vee L_{i,2} \vee L_{i,3})$ with literals $L_{i,j}$ we associate three clauses $\phi_{i,1} = (L_{i,1} \vee y_i)$, $\phi_{i,2} = (L_{i,2} \vee y_i)$, and $\phi_{i,3} = (L_{i,3} \vee y_i)$.

Then we define $\Phi := \forall x_1 \cdots x_s \exists y_1 \cdots y_n ((\bar{y}_1 \vee \cdots \vee \bar{y}_n) \wedge \phi_{1,1} \wedge \phi_{1,2} \wedge \phi_{1,3} \wedge \phi_{2,1} \cdots \wedge \phi_{n,3})$, where $x_1, \ldots, x_s$ are the variables of the propositional formula α.

If Φ is false, then it yields $\Phi \vdash_{\overline{Q-\mathrm{Res}}} \sqcup$. Since for each i one of the clauses $\phi_{i,1}$ or $\phi_{i,2}$ or $\phi_{i,3}$ must be matched, there is some L_{i,j_i} in α_i such that $\{L_{1,j_1}, \ldots, L_{n,j_n}\}$ does not contain complementary literals. Thus we can define the truth assignment I for α by choosing $I(L_{i,j_i}) = 1$.

In the converse direction let α be satisfiable. Then there is some truth assignment I such that $I(\alpha) = 1$. Hence, for each i there is some literal L_{i,j_i} with $I(L_{i,j_i}) = 1$. Now we can apply the Q-resolution to Φ, obtaining the empty clause as follows:

$$(\bar{y}_1 \vee \cdots \vee \bar{y}_n), (L_{1,j_1} \vee y_1)$$
$$\vdash_{\overline{Q-\mathrm{Res}}} (L_{1,j_1} \vee \bar{y}_2 \vee \cdots \vee \bar{y}_n), (L_{2,j_2} \vee \bar{y}_2) \vdash_{\overline{Q-\mathrm{Res}}} \cdots$$
$$\vdash_{\overline{Q-\mathrm{Res}}} (L_{1,j_1} \vee \cdots \vee L_{n,j_{n-1}} \vee \bar{y}_n), (L_{n,j_n} \vee y_n) \vdash_{\overline{Q-\mathrm{Res}}} \sqcup.$$

Since the satisfiability problem for 3-CNF is NP-complete and the transformation can be performed in polynomial time, we have proved our claim. Q.E.D.

Using Q-unit-resolution we can improve the upper time bound of the evaluation problem for quantified Horn formulas.

THEOREM 3.4. *The evaluation problem for quantified Horn formulas can be decided in $O(rn)$ time, where n is the length of the formula and r is the number of $\forall$-variables occurring positive in the formula.*

Proof. Let Φ be a quantified Horn formula of the form $\forall x_1 \exists y_1 \cdots \exists y_{k-1} \forall x_k (\phi_1 \wedge \cdots \wedge \phi_m)$ without pure $\forall$-clauses and without tautological clauses.

Each clause of Φ belongs to one of three classes. P_Φ is the conjunction of clauses with a positive $\exists$-literal, N_Φ^+ is the conjunction of clauses with a positive $\forall$-literal, and N_Φ are the remaining clauses.

Then Φ is false if and only if there exists a clause $\phi \in N_\Phi^+$ or $\phi \in N_\Phi$ such that $\forall x_1 \exists y_1 \cdots \exists y_{k-1} \forall x_k (P_\Phi \wedge \phi)$ is false.

Since N_Φ and P_Φ contain only negative $\forall$-literals no difficulties with the $\forall$-literals can occur when building Q-resolvents. So if $\phi \in N_\Phi$ then $\forall x_1 \exists y_1 \cdots \exists y_{k-1} \forall x_k (P_\Phi \wedge \phi)$ is false if and only if unit-resolution yields the empty clause when applied to the formula obtained by removing all $\forall$-literals in $(P_\Phi \wedge \phi)$.

In the other case, if $\phi \in N_\Phi^+$ then let x_i denote the positive $\forall$-literal in Φ. Since x_i is the only $\forall$-literal occurring positive in $\forall x_1 \exists y_1 \cdots \exists y_{k-1} \forall x_k (P_\Phi \wedge \phi)$, all other $\forall$-literals can be removed for this test. Then for each $\exists$-variable y_j at most two different $\exists$-unit clauses can occur, y_j and $y_j \vee \neg x_i$. Then $\forall x_1 \exists y_1 \cdots \exists y_{k-1} \forall x_k (P_\Phi \wedge \phi)$ is false if and only if the set of $\exists$-unit clauses of the form y_j that are derivable by means of Q-resolution contains all $\exists$-variables occurring in ϕ. Since ϕ contains a positive $\forall$-literal x_i, no Q-resolvent of ϕ or clause of the form $y_j \vee \neg x_i$ exists.

If y_j is before x_i in the prefix of Φ, then, due to the elimination of $\forall$-literals not before an $\exists$-literal, only the $\exists$-unit clause y_j occurs. So we can use propositional unit resolution in order to get all positive $\exists$-unit clauses y_j with y_j before x_i that are derivable from P_Φ without taking the occurrences of the negative $\forall$-literal x_i into consideration.

To obtain the derivable $\exists$-unit clauses y_j with y_j not before x_i we use the newly derived unit clauses together with the clauses from P_Φ, that do not contain x_i. The use of a clause that contains x_i would result in $\exists$-unit clauses of the form $y_j \vee \neg x_i$, if y_j is not before x_i.

From this we get the following algorithm for the evaluation problem for quantified Horn formulas.

Let Φ be the formula $\forall x_1 \exists y_1 \cdots \exists y_{k-1} \forall x_k (\phi_1 \wedge \cdots \wedge \phi_m)$. We assume that Φ contains no pure $\forall$-clause. In this case Φ would be false. Furthermore we assume that Φ does not contain tautological clauses.

The following algorithm stops with the result "*false*" if Φ is false and with the result "*true*" otherwise.

The algorithm uses a procedure UNITRES. If the propositional Horn formula HF is satisfiable then UNITRES(HF) yields true and false otherwise. Furthermore, if UNITRES(HF) yields true, then all variables are marked for which the positive literals are consequences of HF. UNITRES is an adaptation of the linear time algorithm for the satisfiability problem for Horn formulas, cf. Dowling and Gallier (1984). Note that UNITRES is only used for propositional Horn formulas.

If ϕ is a clause then $\exists$-part(ϕ) denotes the clause obtained by removing all $\forall$-literals in ϕ.

begin
  let P_Φ be the set of clauses in Φ that contain a positive $\exists$-literal.
  let N_Φ^+ be the set of clauses in Φ that contain a positive $\forall$-literal.
  let N_Φ be the set of clauses in Φ that contain only negative literals.
  HF := $\{\exists\text{-part}(\phi) \mid \phi \in P_\Phi \cup N_\Phi\}$
  if UNITRES(HF) = *false* **then stop** with result *false*
  foreach $\forall$-variable x_i occurring positive in Φ **do**
    HF := $\{\exists\text{-part}(\phi) \mid \phi \in P_\Phi\}$ and additionally all clauses in HF are marked, that were obtained from a clause in P_Φ containing the variable x_i.
    Use UNITRES(HF) to mark all $\exists$-variables y_j that can be derived as positive unit clauses in HF.
    Unmark all variables y_j in HF that are not before x_i in the prefix of Φ. {Now all positive $\exists$-unit clauses y_j that may be derived in HF with y_j before x_i in the prefix of Φ are marked.}
    Remove in HF all clauses containing a positive $\exists$-literal y with y before x_i in the prefix of Φ.
    HF := HF $\cup \{y_j \mid y_j$ is marked variable in HF$\}$
    Unmark all variables in HF.
    Remove in HF all marked clauses.
    HF := HF $\cup \{\exists\text{-part}(\phi) \mid \phi \in N_\Phi^+$ and ϕ contains positive $\forall$-literal $x_i\}$
    if UNITRES(HF) = *false* **then stop** with result *false*
  endforeach
  stop with result *true*
end

Let be r the number of $\forall$-variables occurring positive in Φ and n the length of Φ. Then all steps in the loop and preceding the loop can be done in $O(n)$ time. Altogether the algorithm requires $O(rn)$ time. Q.E.D.

4. MULTI HORN FORMULAS

In this section we present a problem which is in a certain sense equivalent to the evaluation problem for quantified Horn formulas. This problem is concerned with sets of formulas. When we deal with a set of formulas, sometimes some formulas have some clauses in common and therefore we can store these clauses only once and add an index to the clause indicating to which formula the clause belongs.

A *multi Horn formula* Φ consists of a set of index-Horn clauses and an *index set* $\text{Ind}(\Phi) = \{1, ..., m\}$. An *index-Horn clause* is a tuple $([i_1, ..., i_m], \alpha)$, where α is a propositional Horn clause and $[i_1, ..., i_m]$ is a list of natural numbers with $\{i_1, ..., i_m\} \in \text{Ind}(\Phi)$. The Horn formula $\Phi(i)$ of a multi Horn formula Φ is defined as follows:

$$\Phi(i) := \{\alpha_j \mid (I_j, \alpha_j) \text{ is an index-Horn clause of } \Phi, i \in I_j\}.$$

EXAMPLE.

$$\Phi = \{([1], \bar{A}_1), ([2], \bar{A}_2), ([2,3], A_1 \vee \bar{A}_2), ([1,2], A_1 \vee \bar{A}_3), ([3], A_2 \vee \bar{A}_4), ([2,3], A_3)\}$$

is a multi Horn formula with $\text{Ind}(\Phi) = \{1, 2, 3\}$ and with formulas

$$\Phi(1) = \{\bar{A}_1, A_1 \vee \bar{A}_3\}$$
$$\Phi(2) = \{\bar{A}_2, A_1 \vee \bar{A}_2, A_1 \vee \bar{A}_3, A_3\}$$
$$\Phi(3) = \{A_1 \vee \bar{A}_2, A_2 \vee \bar{A}_4, A_3\}.$$

One typical problem is the satisfiability problem for multi Horn formulas:

$$\text{SAT(Multi Horn)} := \{\Phi \text{ multi Horn} \mid \forall i \in \text{Ind}(\Phi) : \Phi(i) \in \text{SAT}\}$$

It is easy to see that the formula given above is in SAT(Multi Horn). Since the satisfiability problem for Horn formulas can be solved in linear time, obviously the satisfiability of a multi Horn formula Φ can be decided in $O(mn)$ time, where n is the length of Φ and $m = |\text{Ind}(\Phi)|$.

Since a quantified Horn formula can be transformed in linear time into a formula with 3-clauses only, we restrict ourselves in the following to multi Horn formulas with 3-clauses only. Then we can improve the above mentioned upper time bound for SAT(Multi Horn).

LEMMA 4.3. *3-SAT(Multi Horn) is decidable in linear time for multi Horn formulas with the index-clauses.*

Proof. Let a multi Horn formula Φ be given. We replace each index-clause $([i_1, ... i_r], \alpha_i)$ by $([i_1], \alpha_i) \wedge \cdots \wedge ([i_r], \alpha_i)$. Then the resulting formula Φ^* has a length $O(\text{length}(\Phi))$ and each clause belongs to exactly one $\Phi^*(i)$. Thus we have separated Φ^*, obtaining $|\text{Ind}(\Phi)|$ Horn formulas, each of them decidable in linear time. Q.E.D.

Instead of index lists containing the indices to which a clause belongs, we can list the complement; that means that for $\mathrm{Ind}(\Phi) = \{1, ..., m\}$ and a index-clause (I, α) we can write (I^c, α), where $I^c := [1, ..., m] - I$. Such clauses are called *index-complement clauses*, and in order to obtain short expressions in the following we allow both representations.

EXAMPLE. $\Phi = \{([1,2], \alpha_1), ([2]^c, \alpha_2), ([1,3], \alpha_3), ([1], \alpha_4), ([2]^c, \alpha_5)\}$ is a multi Horn formula with $\mathrm{Ind}(\Phi) = \{1, 2, 3, 4\}$ and with formulas

$$\Phi(1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$
$$\Phi(2) = \{\alpha_1\}$$
$$\Phi(3) = \{\alpha_2, \alpha_3, \alpha_5\}$$
$$\Phi(4) = \{\alpha_2, \alpha_5\}.$$

The following transformation associates each quantified Horn formula Φ of the form $\Phi = \forall x \exists y(\phi_1 \wedge \cdots \wedge \phi_m)$ with a multi Horn formula multi(Φ) such that Φ is true iff multi(Φ) $\in$ SAT. Again we assume that Φ does not contain pure $\forall$-clauses and tautological clauses.

Transformation 1. Let $\Phi = \forall x_1 \cdots x_s \exists y_1 \cdots y_t(\phi_1 \wedge \cdots \wedge \phi_m)$ be given; then the associated multi Horn formula Φ is constructed as follows:

If a pure $\exists$-clause ϕ_i exists **then** $\mathrm{Ind}(\Phi) := \{0, ..., s\}$
else $\mathrm{Ind}(\Phi) := \{1, ..., s\}$

and

if $\phi_i = (\bar{x}_{i_1} \vee \cdots \vee \bar{x}_{i_{k-1}} \vee x_{i_k} \vee \bar{x}_{i_{k+1}} \vee \cdots \vee \bar{x}_{i_q} \vee \bar{y}_{j_1} \vee \cdots \vee \bar{y}_{j_p})$ **then** $\mathrm{multi}(\phi_i) := ([i_k], \bar{y}_{j_1} \vee \cdots \vee \bar{y}_{j_p})$

if $\phi_i = (\bar{x}_{i_1} \vee \cdots \vee \bar{x}_{i_q} \vee \bar{y}_{j_1} \vee \cdots \vee \bar{y}_{j_p}(\vee y_{j_r}))$ **then** $\mathrm{multi}(\phi_i) := ([i_1, ..., i_q]^c, \bar{y}_{j_1} \vee \cdots \vee \bar{y}_{j_p}(\vee y_{j_r}))$

if ϕ_i contains no $\forall$-literals **then** $\mathrm{multi}(\phi_i) := ([\ ]^c, \phi_i)$

and $\mathrm{multi}(\Phi) := (\mathrm{multi}(\phi_1) \wedge \cdots \wedge \mathrm{multi}(\phi_m))$.

THEOREM 4.2. *Each quantified Horn formula $\Phi = \forall x \exists y(\phi_1 \wedge \cdots \wedge \phi_m)$ is true iff the associated multi Horn formula multi(Φ) is in SAT(Multi Horn).*

Proof. Let Φ be the formula $\forall x_1 \cdots x_s \exists y_1 \cdots y_t(\phi_1 \wedge \cdots \phi_m)$. We abbreviate

$\mathrm{Pos}(0) := \{\phi \mid \phi$ is clause in Φ without $\forall$-literals and negative $\exists$-literals only$\}$,

$\mathrm{Neg}(0) := \{\phi \mid \phi$ is clause in Φ with one positive $\exists$-literal and without $\forall$-literals$\}$ and for each $i > 0$:

$\mathrm{Neg}(i) := \{\sigma \mid \sigma$ is clause in Φ, $\bar{x}_i$ in $\sigma\}$

$\mathrm{Pos}(i) := \{\sigma \mid \sigma$ is clause in Φ, x_i in $\sigma\}$.

Then Φ is false iff $\Phi \vdash_{Q-U-\mathrm{Res}} \sqcup$ iff there is some i with

$K(i) := \mathrm{Pos}(i) \cup \mathrm{Pos}(0) \cup \mathrm{Neg}(0) \cup_{j \neq i} \mathrm{Neg}(j) \vdash_{Q-U-\mathrm{Res}} \sqcup$.

Thus we have sets of clauses K(i) for which the $\forall$-part does not play any role and therefore can be omitted. Then the associated index clauses to K(i) are exactly the Horn formulas multi($\Phi(i)$):

$(x_i \vee \bar{x}_{k_1} \vee \cdots \vee \bar{x}_{k_r} \vee \bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p})$
$\quad \in \mathrm{Pos}(i) : ([i], \bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p})$

$(\bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p})$
$\quad \in \mathrm{Pos}(0) : ([\ ]^c, \bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p})$

$(\bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p} \vee y_{l_q})$
$\quad \in \mathrm{Neg}(0) : ([\ ]^c, \bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p} \vee y_{l_q})$

$(\bar{x}_{k_1} \vee \cdots \vee \bar{x}_{k_r} \vee \bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p}(\vee y_{l_q}))$
$\quad \in \mathrm{Neg}(j)_{(j \neq i)} : ([k_1, ...k_r]^c, \bar{y}_{l_1} \vee \cdots \vee \bar{y}_{l_p}(\vee y_{l_q}))$
$\quad$ with $i \notin \{k_1, ..., k_r\}$.

Obviously multi($\Phi(i)$) is not satisfiable iff $K(i) \vdash_{Q-U-\mathrm{Res}} \sqcup$.

Hence we have proved Φ is true iff multi(Φ) is in SAT(Multi Horn). Q.E.D.

Each quantified Horn formula can be transformed in linear time into a formula with clauses having at most three literals. If the prefix is $\forall x \exists y$ then we can associate a multi Horn formula with 3-clauses too. By Theorem 4.2 the resulting Horn formula is in 3-SAT(Multi Horn) iff the quantified Horn formula is true.

COROLLARY 4.3. *One can compute in linear time for each quantified Horn formula Φ with prefix $\forall x \exists y$ a multi Horn formula multi(Φ), such that Φ is true iff multi(Φ) is in 3-SAT(Multi Horn).*

By Lemma 4.1 the problem 3-SAT(Multi Horn) is solvable in linear time, if the formula contains index-clauses only. In the case of index-complement-clauses used in Transformation 1, it is not known whether a linear time algorithm exists. Since this problem is directly related to the evaluation problem for quantified Horn formulas, a linear algorithm for multi Horn formulas with index and index-complement clauses would imply a linear algorithm for the evaluation problem for quantified Horn formulas with prefix $\forall x \exists y$.

In the other direction each multi Horn formula can be described in terms of quantified Horn formulas with prefix $\forall x \exists y$.

Transformation 2. Let Π be a multi Horn formula $\{(I_1^c, \alpha_1), ..., (I_s^c, \alpha_s), (I_{s+1}, \alpha_{s+1}), ..., (I_r, \alpha_r)\}$ with index set $\mathrm{Ind}(\Pi) = \{1, ..., m\}$, then we associate with Π the quantified Horn formula Φ as follows:

Without loss of generality we assume that $\alpha_1, ..., \alpha_r$ contain the variables $y_1, ..., y_p$.

For each clause α_i with some positive literal and representation (I_i, α_i) we replace I_i by $I_i^c := \mathrm{Ind}(\Pi) - I_i$, obtaining (I_i^c, α_i).

Then we define for each (I_i, α_i) (note that α_i contains negative literals only):

if $I_i = [i_1, ..., i_p]$ **then** $\phi_{i,1} := (x_{i_1} \vee \alpha_i), ..., \phi_{i,p} := (x_{i_p} \vee \alpha_i)$, and for each (I_i^c, α_i) (α_i contains some positive literal):
if $I_i^c = [i_1, ..., i_p]^c$ **then** $\phi_{i,1} = (\bar{x}_{i_1} \vee \cdots \vee \bar{x}_{i_p} \vee \alpha_i)$
and $\Phi := \forall x_1 \cdots x_m \exists y_1 \cdots \exists y_p (\phi_{1,1} \wedge \cdots \wedge \phi_{r,p})$.

Obviously this yields that Φ is true iff Π is in SAT(Multi Horn), because the previous transformation from quantified Horn formulas to multi Horn formulas applied to Φ leads to Π. Q.E.D.

5. CONCLUSION

We have introduced a complete and sound resolution operation directly applicable to quantified Boolean formulas. For propositional formulas in conjunctive normal form Haken (1985) has shown that ordinary resolution requires exponential time. Since Q-resolution is a generalization of ordinary resolution, obviously for a false formula at least exponentially many clauses must be generated. Because of the PSPACE-completeness of the evaluation problem it would be interesting to prove an essentially larger lower bound for our resolution operation.

Received September 5, 1990; in revised form October 16, 1992

REFERENCES

Aspvall, B., Plass, M. F., and Tarjan, R. E. (1979), A linear-time algorithm for testing the truth of certain quantified Boolean formulas, *Inform. Process. Lett.* **8**, 121–123.

Dowling, W. F., and Gallier, J. H. (1984), Linear-time algorithms for testing the satisfiability of propositional Horn formulae, *J. Logic Programming* **3**, 267–284.

Garey, M. R., and Johnson, D. S. (1979), "Computers and Intractability, A Guide to the Theory of NP-Completeness," Freeman, San Francisco.

Haken, A. (1985), The intractability of resolution, *Theoret. Comput. Sci.* **39**, 297–308.

Karpinski, M., Kleine Büning, H., and Schmitt, P. H. (1987), On the computational complexity of quantified Horn clauses, *in* "Proceedings, CSL '87," Lecture Notes in Computer Science, Vol. 329, pp. 129–137, Springer-Verlag, Berlin/New York.

Stockmeyer, L. J., and Meyer, A. R. (1973), Word problems requiring exponential time, *in* "Proceedings, 5th Annual ACM Symposium on Theory of Computing," pp. 1–9.