Optimal Equivariant Matchings on the 6-Cube

With an Application to the King Wen Sequence

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Abstract

We characterize perfect matchings on the Boolean hypercube $\{0,1\}^n$ that are equivariant under the Klein four-group $K_4 = \langle \mathrm{comp}, \mathrm{rev} \rangle$ generated by bitwise complement and reversal. For $n = 6$, we prove there exists a *unique* K_4-equivariant matching minimizing total Hamming cost among matchings using only comp or rev pairings, achieving cost 120 versus 192 for the complement-only matching. The optimal matching is determined by a simple "reverse-priority rule": pair each element with its reversal unless it is a palindrome, in which case pair with its complement. We verify that the historically significant King Wen sequence of the I Ching is isomorphic to this optimal matching. Notably, allowing $\mathrm{comp} \circ \mathrm{rev}$ pairings yields lower cost (96), but the King Wen sequence follows the structurally simpler rule. All results are formally verified in Lean 4 with the Mathlib library.

1 Introduction

The Boolean hypercube $\{0,1\}^n = \{0,1\}^n$ admits a natural action by the Klein four-group $K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by bitwise complement comp and bit reversal rev. A *K_4-equivariant perfect matching* is a partition of $\{0,1\}^n$ into pairs such that paired elements lie in the same K_4-orbit.

This paper addresses the following optimization problem:

> *Among K_4-equivariant perfect matchings on $\{0,1\}^n$ where each pair uses either* comp *or* rev*, which minimize total Hamming distance?*

For $n = 6$, we prove uniqueness: there is exactly one optimal matching of this form, determined by a simple greedy rule. As an application, we verify that the King Wen sequence—a traditional ordering of the 64 hexagrams of the I Ching into 32 pairs—is isomorphic to this unique optimal matching. We also note that relaxing the constraint to allow $\mathrm{comp} \circ \mathrm{rev}$ pairings yields a lower-cost matching (96 vs 120), but one without the structural elegance of the reverse-priority rule.

1.1 Main Results

Theorem 1.1 (Uniqueness of Optimal comp/rev Matching)**.** *For $n = 6$, there exists a unique K_4-equivariant perfect matching on $\{0,1\}^6$ minimizing total Hamming distance among matchings where each pair is either $\{h, \mathrm{comp}(h)\}$ or $\{h, \mathrm{rev}(h)\}$. This matching is given by the* reverse-priority rule*:*

$$\mathrm{partner}(h) = \begin{cases} \mathrm{rev}(h) & \text{if } h \neq \mathrm{rev}(h) \\ \mathrm{comp}(h) & \text{if } h = \mathrm{rev}(h) \end{cases}$$

The total Hamming cost is 120, compared to 192 for the complement-only matching.

Theorem 1.2 (Isomorphism with King Wen)**.** *The King Wen sequence of the I Ching defines a perfect matching on* $\{0,1\}^6$ *that is isomorphic to the unique optimal* K_4*-equivariant matching of Theorem* **??***.*

1.2 Formal Verification

All theorems are formally verified in Lean 4 with the Mathlib library. Key modules:

- `IChing/Hexagram.lean`: $\{0,1\}^6$ representation, Hamming distance
- `IChing/Symmetry.lean`: K_4-action, orbit structure
- `IChing/KingWenOptimality.lean`: Optimality and uniqueness proofs

2 Preliminaries

2.1 Hexagrams as Binary Vectors

Definition 2.1 (Hexagram)**.** A *hexagram* is an element $h \in \{0,1\}^6 = \{0,1\}^6$. We index positions 0 (bottom) through 5 (top), following the traditional convention that hexagrams are read from bottom to top. The correspondence between hexagrams and binary numbers was noted by Leibniz [**?**].

There are $2^6 = 64$ hexagrams, corresponding to the vertices of the 6-dimensional hypercube.

Definition 2.2 (Hamming Distance)**.** For $h_1, h_2 \in \{0,1\}^6$, the *Hamming distance* is

$$d_H(h_1, h_2) = \#\{i : h_1(i) \neq h_2(i)\} = \sum_{i=0}^{5} |h_1(i) - h_2(i)|$$

2.2 The Klein Four-Group Action

Definition 2.3 (Complement and Reversal)**.** Define operations on hexagrams:

$$\begin{aligned} \mathrm{comp}(h)(i) &= 1 - h(i) \quad \text{(bitwise complement)} \\ \mathrm{rev}(h)(i) &= h(5-i) \quad \text{(bit reversal)} \end{aligned}$$

Proposition 2.4. *The operations* comp *and* rev *satisfy:*

1. $\mathrm{comp} \circ \mathrm{comp} = \mathrm{id}$ *(complement is an involution)*
2. $\mathrm{rev} \circ \mathrm{rev} = \mathrm{id}$ *(reversal is an involution)*
3. $\mathrm{comp} \circ \mathrm{rev} = \mathrm{rev} \circ \mathrm{comp}$ *(they commute)*

Thus $\{\mathrm{id}, \mathrm{comp}, \mathrm{rev}, \mathrm{comp} \circ \mathrm{rev}\}$ *forms the Klein four-group* $\mathbb{Z}_2 \times \mathbb{Z}_2$*.*

Definition 2.5 (Orbit)**.** The *orbit* of a hexagram h under the Klein four-group is

$$\mathrm{orbit}(h) = \{h, \mathrm{comp}(h), \mathrm{rev}(h), \mathrm{comp}(\mathrm{rev}(h))\}$$

Proposition 2.6 (Orbit Sizes)**.** *For any hexagram* h*,* $|\mathrm{orbit}(h)| \in \{2, 4\}$*.*

- *Size 4: generic hexagrams (48 total, forming 12 orbits)*

- *Size 2: palindromes with* $h = \mathrm{rev}(h)$ *(8 total, forming 4 orbits)*
- *Size 2: anti-symmetric hexagrams with* $\mathrm{rev}(h) = \mathrm{comp}(h)$ *but* $h \neq \mathrm{rev}(h)$ *(8 total, forming 4 orbits)*

Definition 2.7 (Anti-Symmetric Hexagram)**.** A hexagram h is *anti-symmetric* if $h(i) \neq h(5-i)$ for all $i \in \{0,1,2\}$. Equivalently, $\mathrm{rev}(h) = \mathrm{comp}(h)$.

Definition 2.8 (Palindrome)**.** A hexagram h is a *palindrome* if $\mathrm{rev}(h) = h$.

2.3 Hamming Distances of Group Actions

Proposition 2.9. *For any hexagram* h*:*

1. $d_H(h, \mathrm{comp}(h)) = 6$ *(complement flips all bits)*
2. $d_H(h, \mathrm{rev}(h)) \leq 6$*, with equality iff* $h(i) \neq h(5-i)$ *for all* i

Lemma 2.10 (Key Lemma)**.** *If* $d_H(h, \mathrm{rev}(h)) = 6$*, then* $\mathrm{rev}(h) = \mathrm{comp}(h)$*.*

Proof. If all 6 positions differ between h and $\mathrm{rev}(h)$, then for each i: $\mathrm{rev}(h)(i) = h(5-i) \neq h(i)$. Since bits are in $\{0,1\}$, this means $h(5-i) = 1 - h(i) = \mathrm{comp}(h)(i)$. Thus $\mathrm{rev}(h) = \mathrm{comp}(h)$. □

This lemma is crucial: when reversal achieves maximum distance 6, it coincides with complement, so there is no choice to make.

3 The King Wen Sequence

3.1 Definition

The King Wen sequence assigns each hexagram a number 1–64 according to tradition [?]. We represent it via a mapping from King Wen numbers to binary:

Definition 3.1 (King Wen Binary Mapping)**.** Define $\mathrm{KW} : \{0, \ldots, 63\} \to \{0,1\}^6$ as follows (index 0 corresponds to King Wen number 1):

KW	1	2	3	4	5	6	7	8
Binary	63	0	17	34	23	58	2	16
KW	9	10	11	12	13	14	15	16
Binary	55	59	7	56	61	47	4	8

(Full table in Appendix.)

Definition 3.2 (King Wen Pairs)**.** The *King Wen pairs* are (h_{2k}, h_{2k+1}) for $k = 0, \ldots, 31$, where $h_i = \mathrm{KW}(i)$.

3.2 The Equivariance Theorem

Theorem 3.3 (100% Structural Regularity)**.** *Every King Wen pair* (h_1, h_2) *satisfies:*

$$h_2 = \mathrm{comp}(h_1) \quad \text{or} \quad h_2 = \mathrm{rev}(h_1)$$

The 32 pairs partition as:

- *4 pairs: palindromes, paired by complement ($h_2 = \mathrm{comp}(h_1)$, distance 6)*
- *4 pairs: anti-symmetric, paired by reversal = complement ($h_2 = \mathrm{rev}(h_1) = \mathrm{comp}(h_1)$, distance 6)*
- *24 pairs: generic, paired by reversal ($h_2 = \mathrm{rev}(h_1)$, distance 2 or 4)*

Proof. Verified computationally over all 32 pairs. In Lean 4: `decide`. □

Corollary 3.4. *The King Wen pairing respects the Klein four-group: paired hexagrams always lie in the same orbit.*

4 Optimality of the King Wen Matching

4.1 The Reverse-Priority Rule

Definition 4.1 (Reverse-Priority Matching)**.** Define the *priority partner* function:

$$\mathrm{partner}(h) = \begin{cases} \mathrm{comp}(h) & \text{if } h = \mathrm{rev}(h) \text{ (palindrome)} \\ \mathrm{rev}(h) & \text{otherwise} \end{cases}$$

The intuition: prefer reversal (which has distance ≤ 6) unless reversal is trivial (palindrome), in which case use complement.

Theorem 4.2 (Involution)**.** *The priority partner function is an involution:* $\mathrm{partner}(\mathrm{partner}(h)) = h$.

Proof. Two cases:

1. If h is a palindrome, then $\mathrm{comp}(h)$ is also a palindrome (complement preserves palindrome structure), so $\mathrm{partner}(\mathrm{comp}(h)) = \mathrm{comp}(\mathrm{comp}(h)) = h$.
2. If h is not a palindrome, then $\mathrm{rev}(h)$ is also not a palindrome, so $\mathrm{partner}(\mathrm{rev}(h)) = \mathrm{rev}(\mathrm{rev}(h)) = h$.

□

4.2 Cost Minimization

Definition 4.3 (Total Hamming Cost)**.** For a perfect matching M on $\{0,1\}^6$, the *total Hamming cost* is

$$\mathrm{Cost}(M) = \sum_{\{h_1,h_2\} \in M} d_H(h_1, h_2)$$

Theorem 4.4 (Optimality among comp/rev Matchings)**.** *Among K_4-equivariant perfect matchings on $\{0,1\}^6$ where each pair uses* comp *or* rev*, the reverse-priority matching uniquely minimizes total Hamming cost.*

Proof. Restricting to comp/rev pairings, the options for each hexagram h are $\mathrm{comp}(h)$ or $\mathrm{rev}(h)$.

- By Proposition **??**, $d_H(h, \mathrm{rev}(h)) \leq 6 = d_H(h, \mathrm{comp}(h))$.
- Equality holds iff $\mathrm{rev}(h) = \mathrm{comp}(h)$ (Lemma **??**).
- For palindromes, $\mathrm{rev}(h) = h$, so the only non-trivial option is $\mathrm{comp}(h)$.

Thus the reverse-priority rule chooses the minimum-distance option at each step. Since there is no actual choice when distances are equal (the options coincide), the matching is uniquely determined. □

Remark 4.5 (The comp $\circ$ rev Alternative)**.** On a size-4 orbit, a third equivariant pairing exists: $\{h, \mathrm{comp}(\mathrm{rev}(h))\}$. For orbits where $d_H(h, \mathrm{rev}(h)) = 4$, we have $d_H(h, \mathrm{comp}(\mathrm{rev}(h))) = 2$, making the comp $\circ$ rev pairing strictly cheaper. A mixed strategy using comp $\circ$ rev on such orbits achieves total cost $96 < 120$. However, this requires case-by-case analysis of each orbit, whereas the reverse-priority rule is a uniform structural principle. The King Wen sequence follows the elegant rule rather than the raw optimum.

Corollary 4.6 (Cost Values)**.** *The total Hamming cost of the reverse-priority matching:*

- *Palindrome pairs:* $4 \times 6 = 24$
- *Anti-symmetric pairs:* $4 \times 6 = 24$
- *Generic pairs:* $12 \times 2 + 12 \times 4 = 72$

Total: $24 + 24 + 72 = 120$*.*

For comparison, the complement-only matching has cost $32 \times 6 = 192$*.*

4.3 Uniqueness

Theorem 4.7 (Uniqueness)**.** *Any equivariant matching satisfying the reverse-priority rule equals the priority partner function.*

Proof. The reverse-priority rule uniquely specifies the partner at each hexagram. By Theorem **??**, this defines a valid involution, hence a perfect matching. □

Corollary 4.8 (King Wen is Canonical)**.** *The King Wen sequence encodes the unique cost-minimizing* comp/rev *equivariant matching on* $\{0,1\}^6$*.*

5 Discussion

5.1 Orbit Structure

The K_4-action on $\{0,1\}^6$ partitions the 64 elements into orbits of size 2 or 4:

- **Size-4 orbits**: 12 orbits containing 48 generic elements
- **Size-2 orbits**: 8 orbits containing 16 palindromes (elements fixed by rev)

Within each orbit, the equivariant matching must pair elements. For size-4 orbits, there are three equivariant pairings induced by the non-identity involutions comp, rev, and comp $\circ$ rev. The reverse-priority rule uses rev (falling back to comp for palindromes), which minimizes cost among comp/rev matchings. A mixed strategy using comp $\circ$ rev on orbits where $d_H(h, \mathrm{rev}(h)) = 4$ achieves the global minimum cost of 96.

5.2 Uniqueness Mechanism

The key to uniqueness is Lemma **??**: when $d_H(h, \mathrm{rev}(h)) = 6$, we have $\mathrm{rev}(h) = \mathrm{comp}(h)$, so the two candidate partners coincide. This eliminates all apparent degrees of freedom in the optimization.

5.3 Formal Verification

All results are machine-checked in Lean 4 using the Mathlib library:

- `decide` tactic for computational verification of all 32 pairs
- Constructive proof that the reverse-priority function is an involution
- Proof of optimality via case analysis on orbit structure

5.4 Extensions

The K_4-action generalizes to $\{0,1\}^n$ for any n. The orbit structure and optimal matching problem remain well-defined; we conjecture the reverse-priority rule remains optimal for all n.

References

[1] Richard Wilhelm and Cary F. Baynes. *The I Ching or Book of Changes.* Princeton University Press, 1967.

[2] Gottfried Wilhelm Leibniz. Explication de l'arithmétique binaire. *Mémoires de l'Académie Royale des Sciences*, 1703.

A Full King Wen Binary Table

KW	Bin	KW	Bin	KW	Bin	KW	Bin
1	63	17	25	33	60	49	29
2	0	18	38	34	15	50	46
3	17	19	3	35	40	51	9
4	34	20	48	36	5	52	36
5	23	21	41	37	53	53	52
6	58	22	37	38	43	54	11
7	2	23	32	39	20	55	13
8	16	24	1	40	10	56	44
9	55	25	57	41	35	57	54
10	59	26	39	42	49	58	27
11	7	27	33	43	31	59	50
12	56	28	30	44	62	60	19
13	61	29	18	45	24	61	51
14	47	30	45	46	6	62	12
15	4	31	28	47	26	63	21
16	8	32	14	48	22	64	42

Binary values represent the hexagram as a 6-bit integer, with bit 0 as the bottom line.