THEORETICAL COMPUTER SCIENCE TUTORING (4)

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Problem 3.1 from the exam held on June 18, 2018

Let $\boldsymbol{L_1} \subseteq \boldsymbol{\Sigma}^*$ be an **acceptable** but undecidable language and let $\boldsymbol{L_2} \subseteq \boldsymbol{\Sigma}^*$ be a decidable language. Consider the following function $\boldsymbol{f}: \sigma^* \to \mathbb{N} : \ \forall x \in \Sigma^*$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } x \in L_1 \\ |x| & \text{if } x \notin L_1 \wedge x \in L_2 \\ 0 & \text{otherwise} \end{cases}$$

Show whether $\boldsymbol{f}$ is a computable function

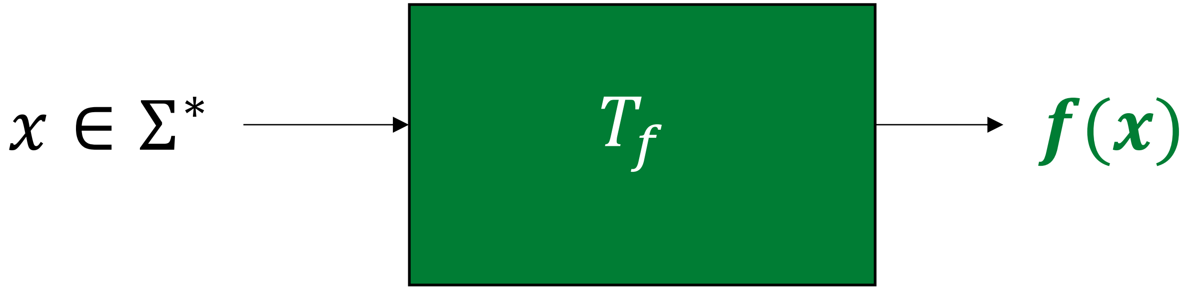
Problem 3.1 from the exam held on June 18, 2018

Claim

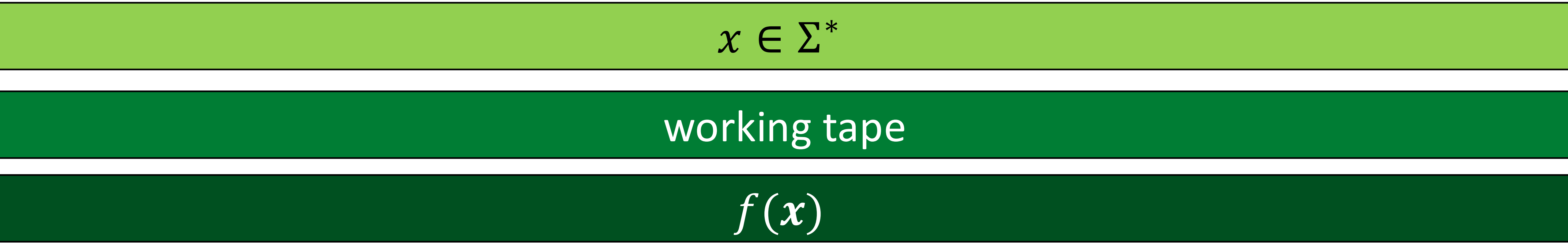
f is **not** computable

Proof

Let's assume for contradiction that f is computable. We could then construct a Turing machine T_1 that decides L_1



suppose that T_f has three tapes



Problem 3.1 from the exam held on June 18, 2018

Let's build a recognizer T_1

T_1

$x \in \Sigma^*$

working tape

$f(x)$

Calculate $f(x)$ on the third tape

if $f(x) = 1$ ✔ T_1 accepts

else if $f(x) = |x|$ or $f(x) = 0$ ✘ T_1 rejects

T_1 **decides** L_1 ⚡ contradiction

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]

Prove that for every constant $k \in \mathbb{N}$, $\mathbf{2^{n^k}}$ is a **time-constructible** function.

Definition

A function $f: \mathbb{N} \to \mathbb{N}$ is **time-constructible** if there exists a Turing machine T of transducer type that, given an input integer n in unary (1^n), writes on the output tape $f(n)$ in unary $\left(1^{f(n)}\right)$ in $dtime(T, n) = O\big(f(n)\big)$

⏳ $O\big(f(n)\big)$

n

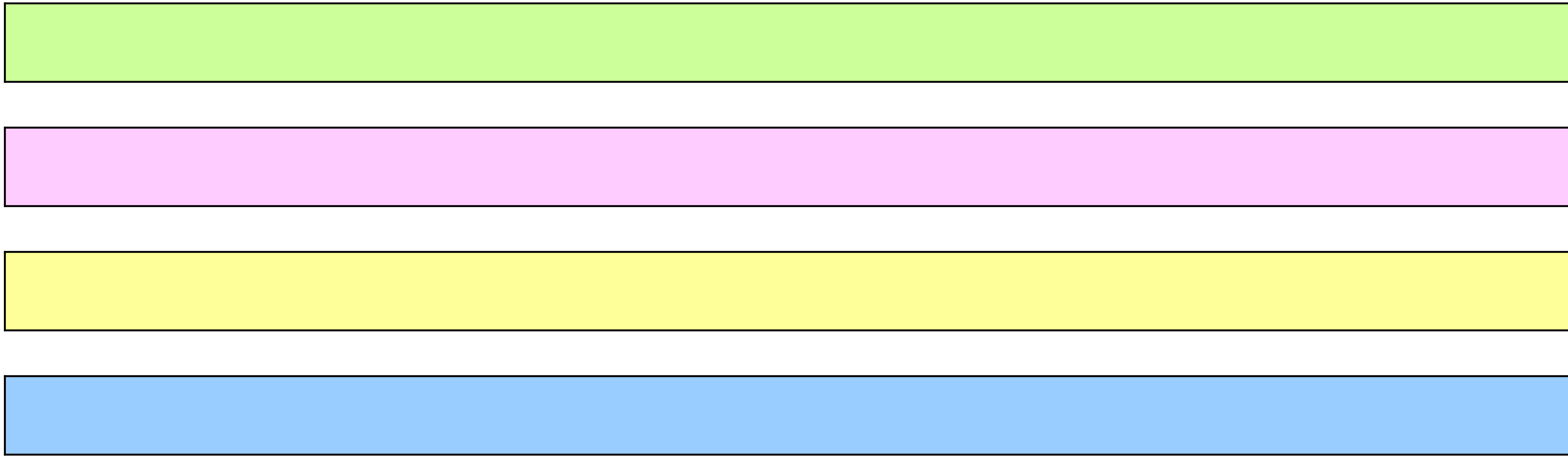
$f(n)$

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)

Claim

$\forall$ constant $k \in \mathbb{N}$, $\mathbf{2^{n^k}}$ is a **time-constructible**

Let's build a Turing machine that computes $\mathbf{2^{n^k}}$

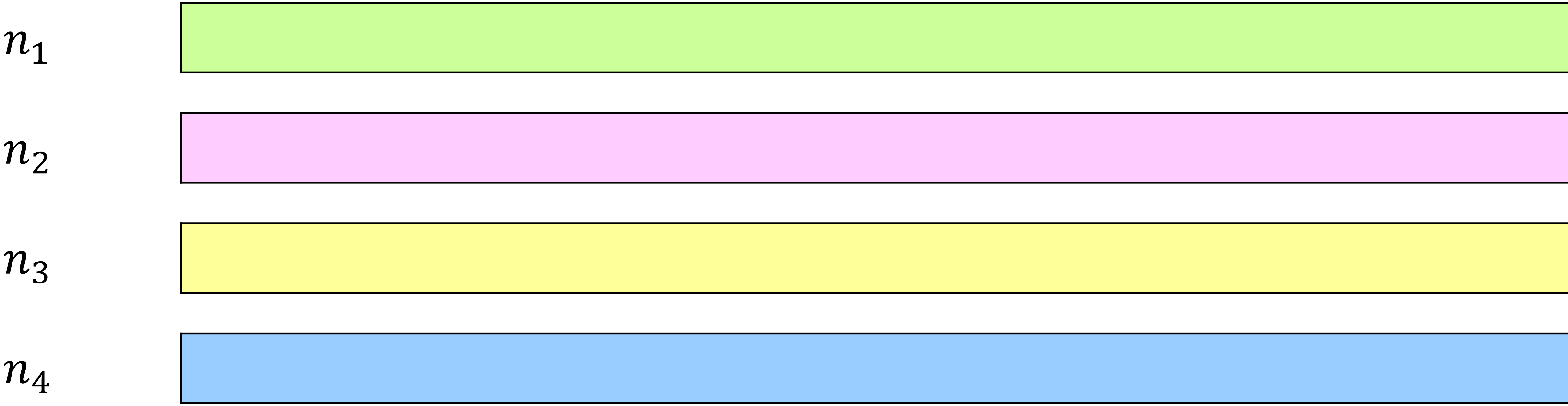


Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]

concatenation

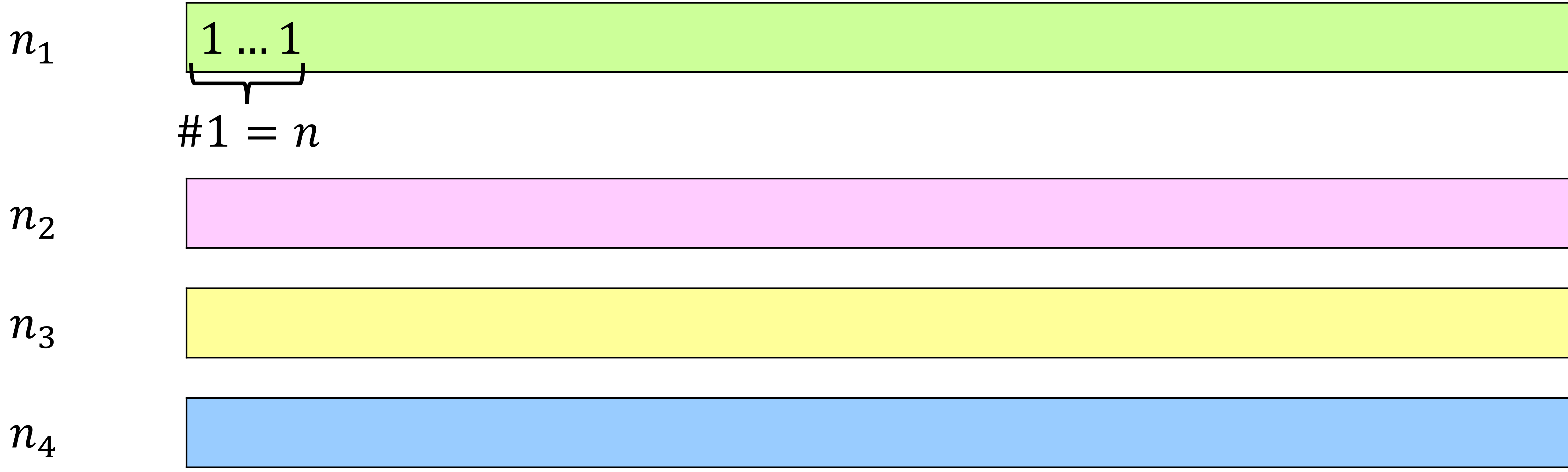
$$\underbrace{1 \dots 1}_{\#1 = 2^j} + \underbrace{1 \dots 1}_{\#1 = 2^j} = \underbrace{1 \dots 1}_{\#1 = 2^{j+1}}$$

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]



$\boldsymbol{n_1 \leftarrow n}$

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)



$\boldsymbol{n_2 \leftarrow k^{th}}$**-power(**$\boldsymbol{n_1}$**)**

We know how to compute the k^{th} power in $O(n^k)$ steps

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]

n_1 $\quad \underbrace{1 \dots 1}_{\#1 = n}$

n_2 $\quad \underbrace{1 \dots 1}_{\#1 = n^k}$

n_3

n_4

$\boldsymbol{n_3 \leftarrow 2}$

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]()

n_1 $\quad \underbrace{1 \dots 1}_{\#1 = n}$

n_2 $\quad \underbrace{1 \dots 1}_{\#1 = n^k}$

n_3 $\quad \underbrace{11}_{\#1 = 2^1}$

n_4

Let $\boldsymbol{i_2}$ be the position of the head on the second tape

$\boldsymbol{i_2 \leftarrow 2}$

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)

n_1	$1 \ldots 1$
n_2	$1\mathbf{1} \ldots 1$
n_3	11
n_4	

while$(\boldsymbol{i_2 \leq n_2})$ **do**

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]()

n_1	$1 \ldots 1$
n_2	$1\mathbf{1} \ldots 1$
n_3	11
n_4	

while$(\boldsymbol{i_2 \leq n_2})$ **do**

$\boldsymbol{n_4 \leftarrow n_3}$

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)

n_1	$1 \ldots 1$
n_2	$1\mathbf{1} \ldots 1$
n_3	11
n_4	11

while$(\boldsymbol{i_2} \leq \boldsymbol{n_2})$ **do**

$\boldsymbol{n_4} \leftarrow \boldsymbol{n_3}$

$\boldsymbol{n_3} \leftarrow \boldsymbol{n_3} + \boldsymbol{n_4}$

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)

n_1	$1 \ldots 1$
n_2	$1\mathbf{1} \ldots 1$
n_3	1111
n_4	11

while$(\boldsymbol{i_2 \leq n_2})$ **do**

$$\boldsymbol{n_4 \leftarrow n_3}$$
$$\boldsymbol{n_3 \leftarrow n_3 + n_4}$$
$$\boldsymbol{i_2 \leftarrow \mathbf{i}_2 + 1}$$

Problem 6.1 from Es-LinguaggiECompplessita.pdf (uniroma2.it)

n_1	$1 \ldots 1$
n_2	$11\mathbf{1} \ldots 1$
n_3	1111
n_4	11

while$(\boldsymbol{i_2 \leq n_2})$ **do**

$$\boldsymbol{n_4 \leftarrow n_3}$$
$$\boldsymbol{n_3 \leftarrow n_3 + n_4}$$
$$\boldsymbol{i_2 \leftarrow \mathrm{i}_2 + 1}$$

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]

n_1 $1 \dots 1$

n_2 $11\mathbf{1} \dots 1$

n_3 $\underbrace{1111}_{\#1 = 2^{i_2}}$

n_4 11

After each iteration

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)

n_1	$1 \ldots 1$
n_2	$111 \ldots \mathbf{1}$
n_3	$\underbrace{1 \ldots 1}_{\#1 = 2^{n_2}}$
n_4	11

$n_2 = n^k$

output n_3

Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]

$$
\begin{array}{lll}
n_1 \longleftarrow n & & \\
n_2 \leftarrow k^{th}\text{-power}(n_1) & O(n^k) & \\
n_3 \longleftarrow 2 & & \\
i_2 \leftarrow 2 & & \\
\text{while}(i_2 \leq n_2) \text{ do} & 2 \leq i_2 \leq n^k & \\
\quad n_4 \leftarrow n_3 & \#1 \ = \ 2^{i_2-1} & \displaystyle\sum_{i_2=2}^{n^k} \left[2^{i_2-1} + 2^{i_2}\right] \\
\quad n_3 \leftarrow n_3 + n_4 & \#1 \ = \ 2^{i_2} & \\
\quad i_2 \leftarrow \mathrm{i}_2 + 1 & & \\
\text{output } n_3 & &
\end{array}
$$

Problem 6.1 from Es-LinguaggiEComplessita.pdf (uniroma2.it)

$$\sum_{i_2=2}^{n^k}\left[\mathbf{2}^{i_2-1}+\mathbf{2}^{i_2}\right]=\sum_{i_2=2}^{n^k}\mathbf{2}^{i_2-1}+\sum_{i_2=2}^{n^k}\mathbf{2}^{i_2}=$$

$$=\frac{\mathbf{1}}{\mathbf{2}}\sum_{i_2=2}^{n^k}\mathbf{2}^{i_2}+\sum_{i_2=2}^{n^k}\mathbf{2}^{i_2}=\left(\frac{\mathbf{1}}{\mathbf{2}}+\mathbf{1}\right)\sum_{i_2=2}^{n^k}\mathbf{2}^{i_2}=$$

$$=\left(\frac{\mathbf{3}}{\mathbf{2}}\right)\sum_{i_2=2}^{n^k}\mathbf{2}^{i_2}=\left(\frac{3}{2}\right)\left(2^{n^k+1}-1-1-2\right)<$$

$$\sum_{i=0}^{n}\mathbf{2}^{i}=\mathbf{2}^{n+1}-\mathbf{1}$$

$$<3\cdot 2^{n^k}\in O\left(2^{n^k}\right)$$

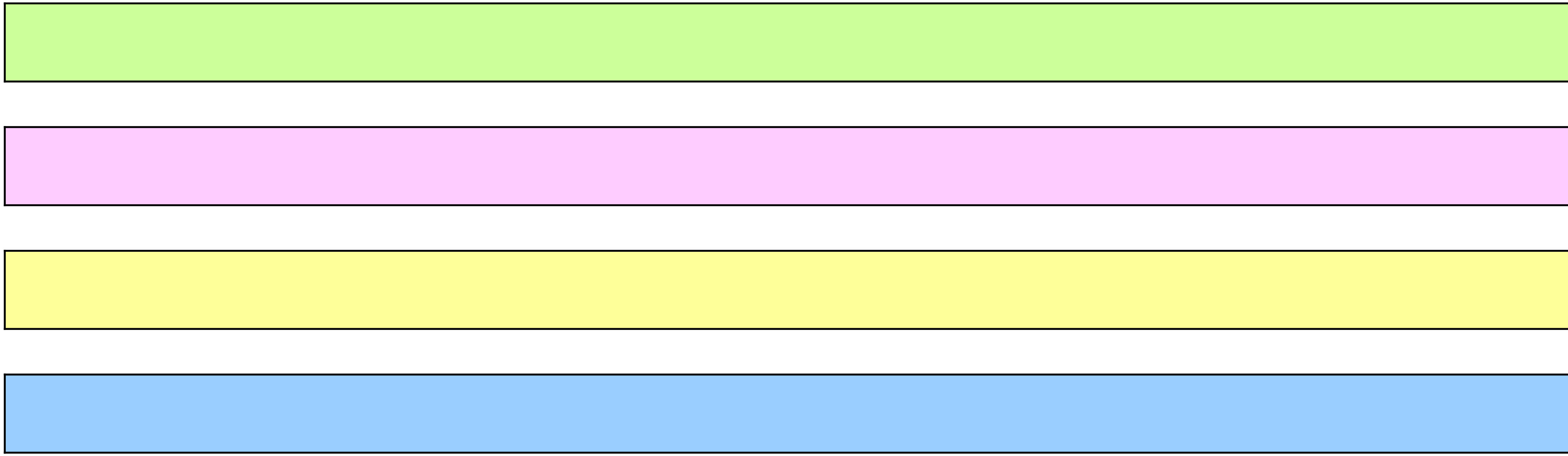
Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

Prove that $\boldsymbol{f(n) = n^n}$ is a **time-constructible** function

Claim

$\boldsymbol{f(n) = n^n}$ is a **time-constructible** function

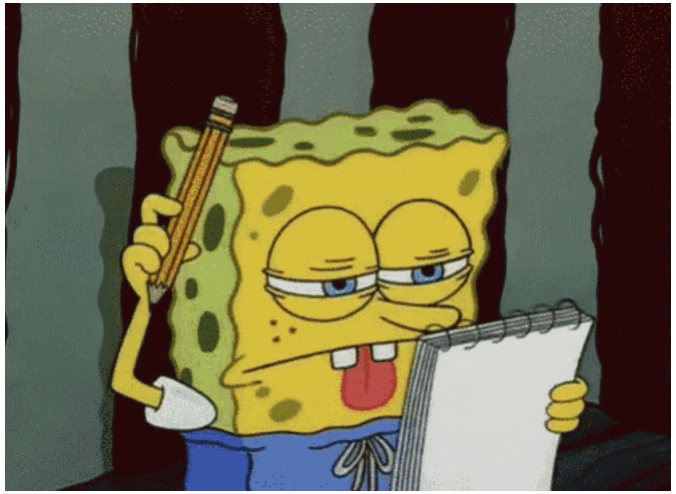
Let's build a Turing machine that computes $\boldsymbol{n^n}$



Problem 6.1 from [Es-LinguaggiEComplessita.pdf (uniroma2.it)]()

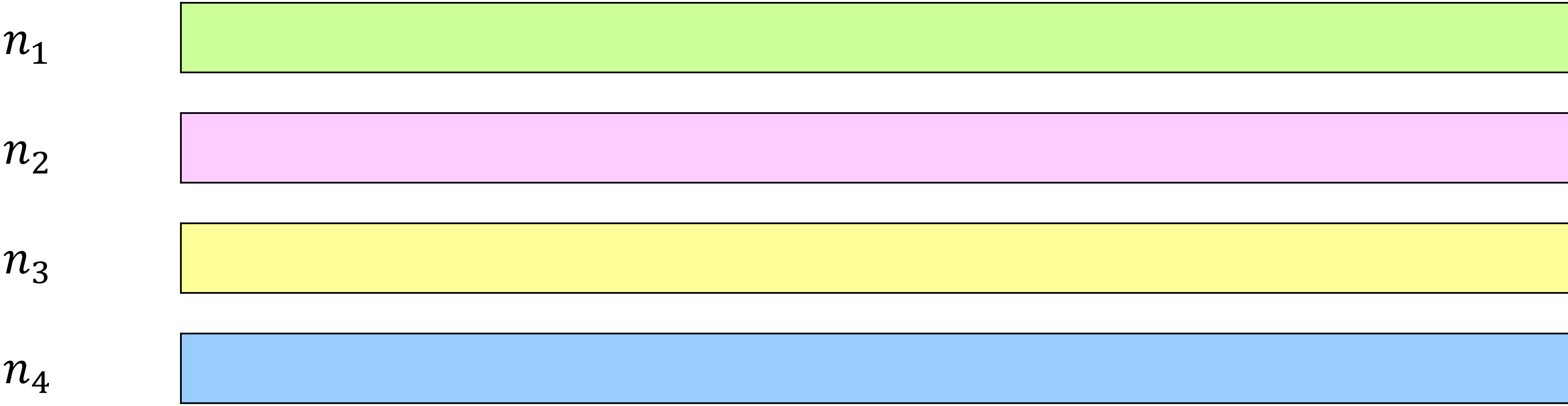
concatenation

$$\underbrace{\underbrace{1 \dots 1}_{\#1 = n^j} + \dots + \underbrace{1 \dots 1}_{\#1 = n^j}}_{n \text{ blocks}} = \underbrace{1 \dots 1}_{\#1 = n^{j+1}}$$



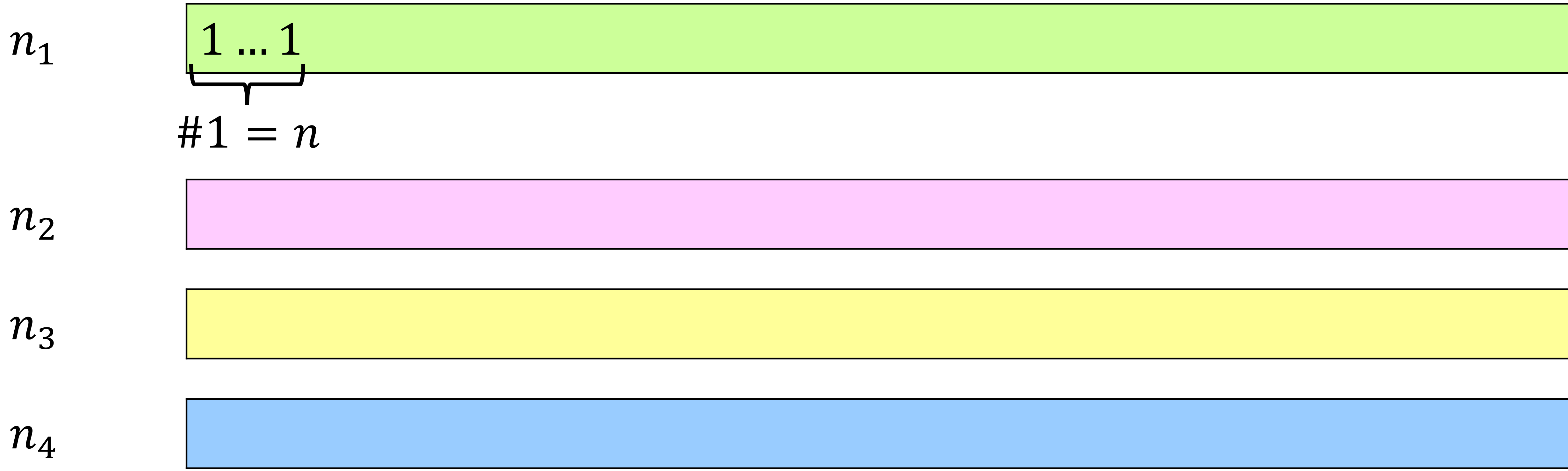
💡 2 nested cycles

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)



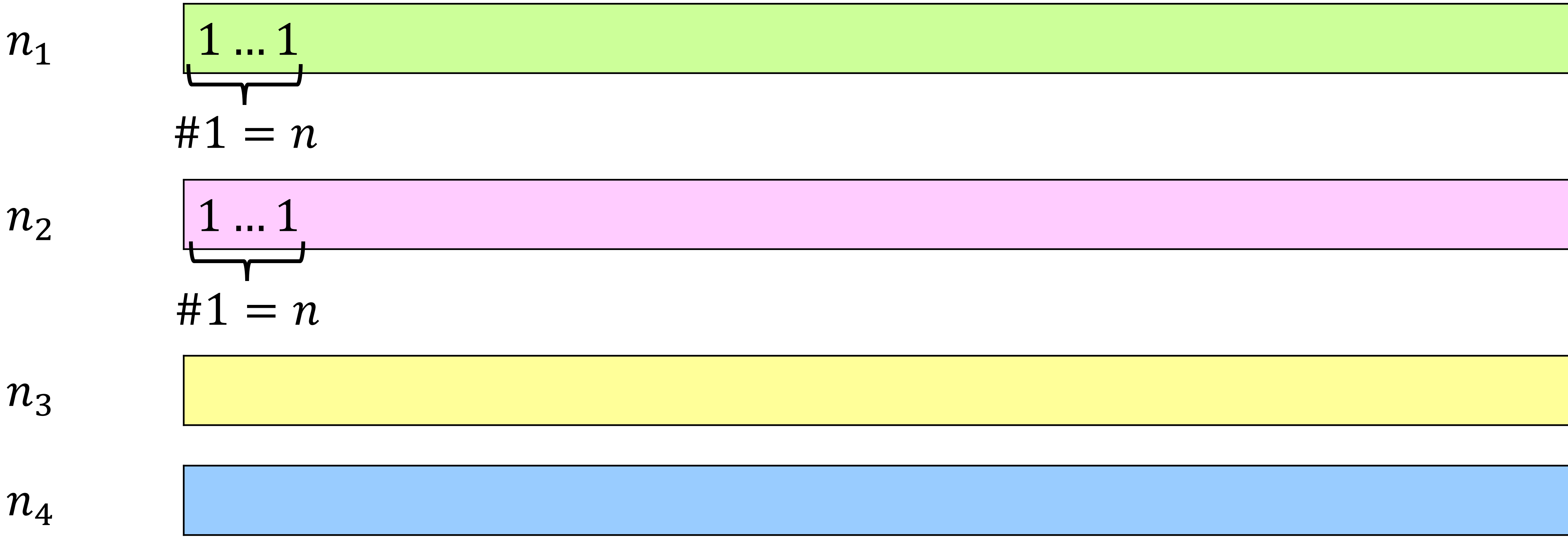
$\boldsymbol{n_1 \leftarrow n}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)



$\boldsymbol{n_2 \leftarrow n_1}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)



Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be the positions of the heads of the first two tapes

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

$\boldsymbol{n_3 \leftarrow n_1}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1 $\underbrace{1 \ldots 1}_{\#1 = n}$

n_2 $\underbrace{1 \ldots 1}_{\#1 = n}$

n_3 $\underbrace{1 \ldots 1}_{\#1 = n}$

n_4

$\boldsymbol{i \leftarrow 2}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1	$1\mathbf{1}\ldots 1$
n_2	$1\ldots 1$
n_3	$1\ldots 1$
n_4	

while$(\boldsymbol{i} \leq \boldsymbol{n})$ **do**

$\boldsymbol{n_4} \leftarrow \boldsymbol{n_3}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1	$1\mathbf{1} \dots 1$
n_2	$1 \dots 1$
n_3	$1 \dots 1$
n_4	$1 \dots 1$

while($\boldsymbol{i \le n}$) **do**

$\boldsymbol{n_4 \leftarrow n_3}$

$\boldsymbol{j \leftarrow 2}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1 | $1\mathbf{1} \dots 1$

n_2 | $1\mathbf{1} \dots 1$

n_3 | $1 \dots 1$

n_4 | $1 \dots 1$

while($i \leq n$) do

- $\boldsymbol{n_4 \leftarrow n_3}$
- $\boldsymbol{j \leftarrow 2}$
- **while(j $\leq n$) do**
  - $\boldsymbol{n_3 \leftarrow n_3 + n_4}$
  - $\boldsymbol{j \leftarrow j + 1}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1	$1\mathbf{1}\ldots1$
n_2	$11\mathbf{1}\ldots1$
n_3	$1\ldots11\ldots1$
n_4	$1\ldots1$

while($\boldsymbol{i \leq n}$) do

  $\boldsymbol{n_4 \leftarrow n_3}$

  $\boldsymbol{j \leftarrow 2}$

  while(j $\boldsymbol{\leq n}$) do

    $\boldsymbol{n_3 \leftarrow n_3 + n_4}$

    $\boldsymbol{j \leftarrow j + 1}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1	$1\mathbf{1}\ldots 1$
n_2	$1\ldots \mathbf{1}$
n_3	$1\ldots 1\ldots 1\ldots 1$
n_4	$1\ldots 1$

$\#\mathbf{1} = \boldsymbol{n_4 \cdot n}$

while($i \le n$) do

- $n_4 \leftarrow n_3$
- $j \leftarrow 2$
- **while(j $\le n$) do**
  - $n_3 \leftarrow n_3 + n_4$
  - $j \leftarrow j + 1$
- $i \leftarrow i + 1$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1 $11\mathbf{1} \ldots 1$

n_2 $1\mathbf{1} \ldots 1$

n_3 $1 \ldots 1 \ldots 1 \ldots 1$

n_4

while$(\boldsymbol{i} \leq \boldsymbol{n})$ **do**

- $\boldsymbol{n_4 \leftarrow n_3}$
- $\boldsymbol{j \leftarrow 2}$
- **while**$(\mathrm{j} \leq \boldsymbol{n})$ **do**
  - $\boldsymbol{n_3 \leftarrow n_3 + n_4}$
  - $\boldsymbol{j \leftarrow j + 1}$
- $\boldsymbol{i \leftarrow i + 1}$

...

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

n_1	$111 \ldots \mathbf{1}$
n_2	$11 \ldots \mathbf{1}$
n_3	$1 \ldots 1 \ldots 1 \ldots 1$
n_4	

while$(\boldsymbol{i} \leq \boldsymbol{n})$ **do**
- $\boldsymbol{n_4} \leftarrow \boldsymbol{n_3}$
- $\boldsymbol{j} \leftarrow \mathbf{2}$
- **while**$(\mathbf{j} \leq \boldsymbol{n})$ **do**
  - $\boldsymbol{n_3} \leftarrow \boldsymbol{n_3} + \boldsymbol{n_4}$
  - $\boldsymbol{j} \leftarrow \boldsymbol{j} + \mathbf{1}$
- $\boldsymbol{i} \leftarrow \boldsymbol{i} + \mathbf{1}$

output $\boldsymbol{n_3}$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

$\boldsymbol{n_1, n_2, n_3 \leftarrow n}$ — $\boldsymbol{O(n)}$

$\boldsymbol{i \leftarrow 2}$

while($\boldsymbol{i \leq n}$) do — $\boldsymbol{n}$ **times**

- $\boldsymbol{n_4 \leftarrow n_3}$ — $\boldsymbol{\#1 \; = n^{i-1}}$
- $\boldsymbol{j \leftarrow 2}$
- **while(j $\boldsymbol{\leq n}$) do** — $\boldsymbol{n}$ **times**
  - $\boldsymbol{n_3 \leftarrow n_3 + n_4}$ — **#1 added** $\boldsymbol{= n^{i-1}}$
  - $\boldsymbol{j \leftarrow j + 1}$
  - $\boldsymbol{O(n^i)}$
- $\boldsymbol{i \leftarrow i + 1}$

output $\boldsymbol{n_3}$

$$\sum_{i=2}^{n} n^i$$

Exercise 1 from Antonio Cruciani's tutoring (lesson 4)

$$\sum_{i=2}^{n} n^i = \frac{n^2 - n^{n+1}}{1-n} = \frac{n^{n+1} - n^2}{n-1} \in O(n^n)$$

$$\sum_{k=m}^{n} x^k = \frac{x^m - x^{n+1}}{1-x}$$