THEORETICAL COMPUTER SCIENCE TUTORING (5)

Maurizio Fiusco

TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Problem 3 from the exam held on June 19, 2019

Given an undirected graph $G = (V, E)$ and an integer h, we say that a subset V' of the nodes of G is α-colorable with h colors if there exists a function $c : V' \to \{1, \dots, h\}$ such that, for every $u, v \in V'$, if $c(u) = c(v)$ then $(u, v) \in E$.

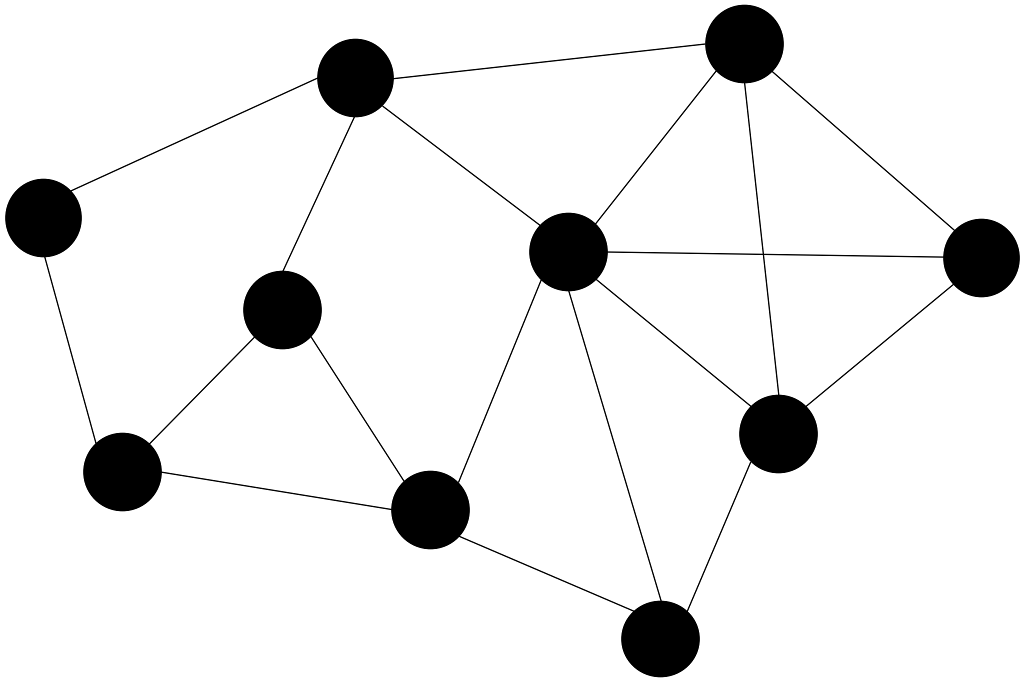
Consider the following decision problem: given an undirected graph $G = (V, E)$ and an integer k, decide whether there does not exist a subset $V' \subseteq V$ of cardinality at least k that is α-colorable with only one color.

After formalizing the problem using the triple $\langle I, S, \pi \rangle$, answer the following questions (in the order deemed appropriate), providing justification for each response.

- Is the problem in P?
- Is the problem in NP?
- Is the problem in coNP?

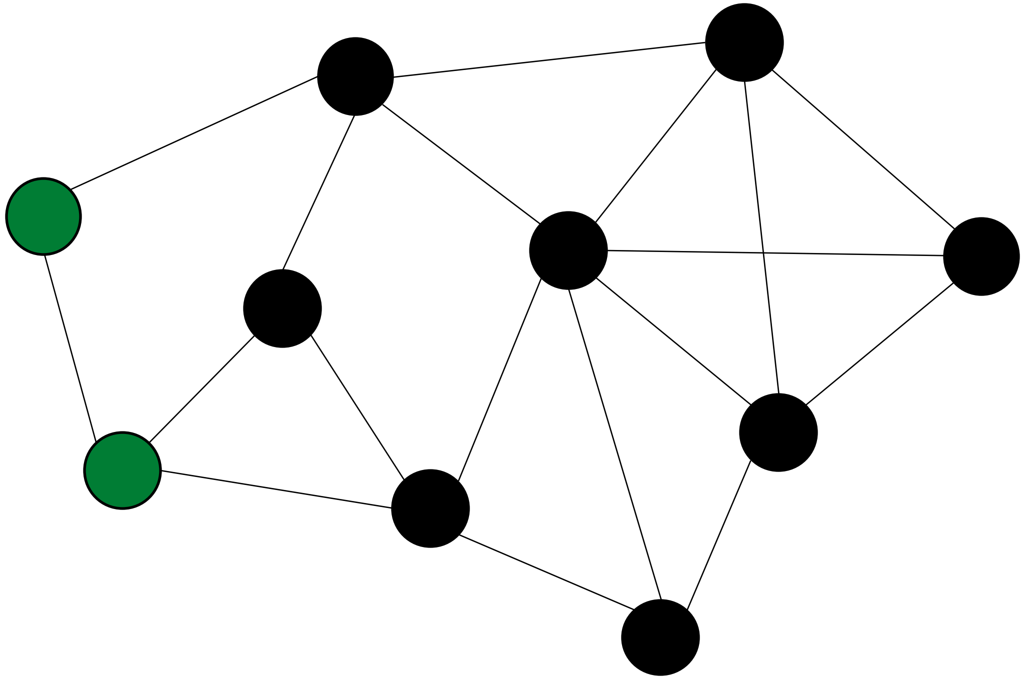
Problem 3 from the exam held on June 19, 2019

Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **α-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \to \{1, \dots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $\boldsymbol{(u, v) \in E}$.



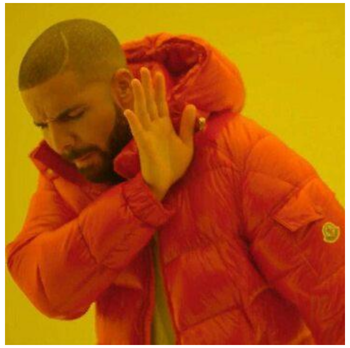
Problem 3 from the exam held on June 19, 2019

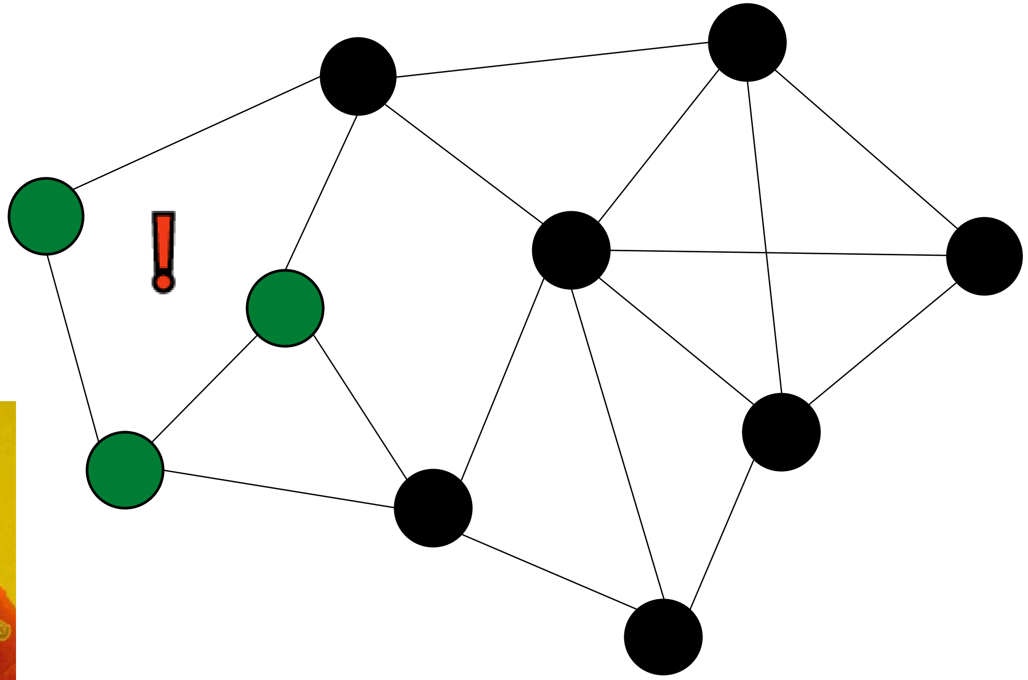
Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **α-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \to \{1, \ldots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}$.



Problem 3 from the exam held on June 19, 2019

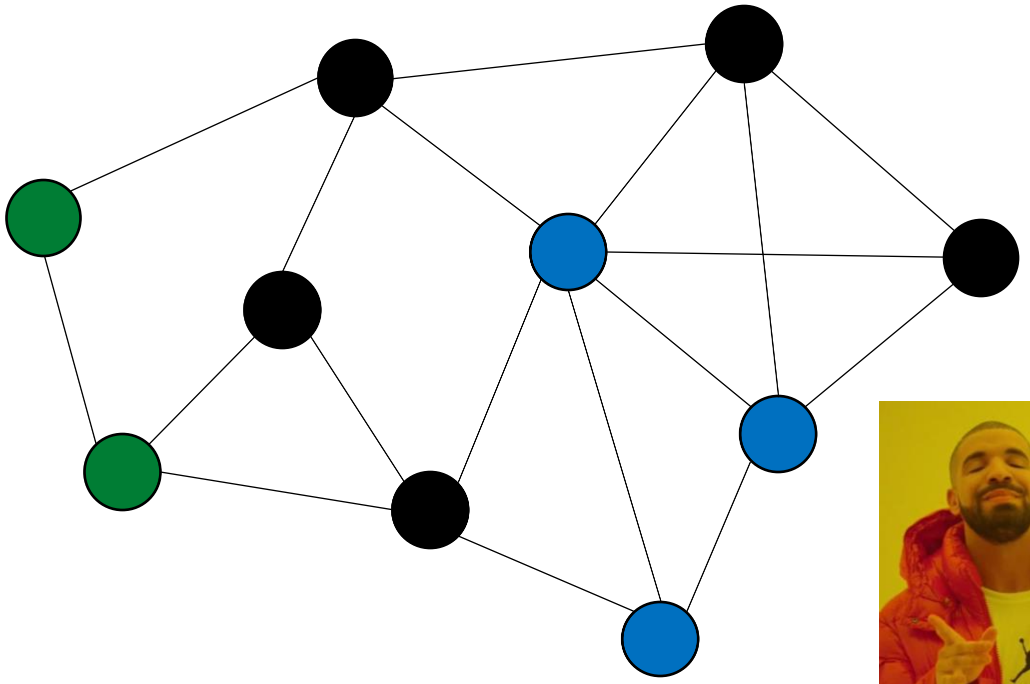
Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **α-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \to \{1, \dots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $\boldsymbol{(u, v) \in E}$.





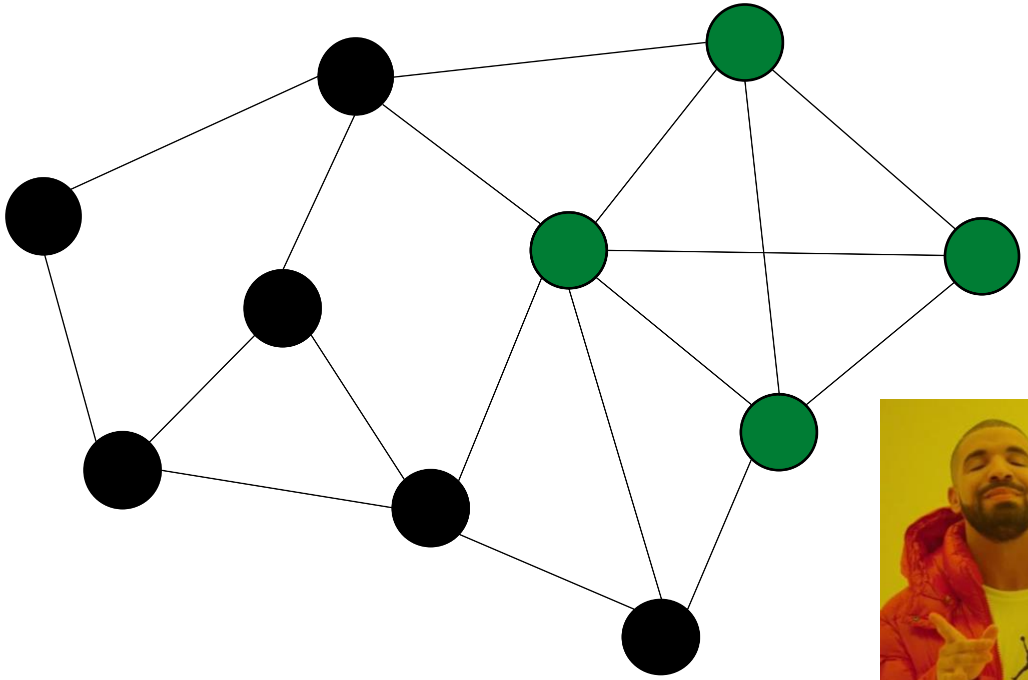
Problem 3 from the exam held on June 19, 2019

Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **α-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \rightarrow \{1, \dots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $\boldsymbol{(u, v) \in E}$.



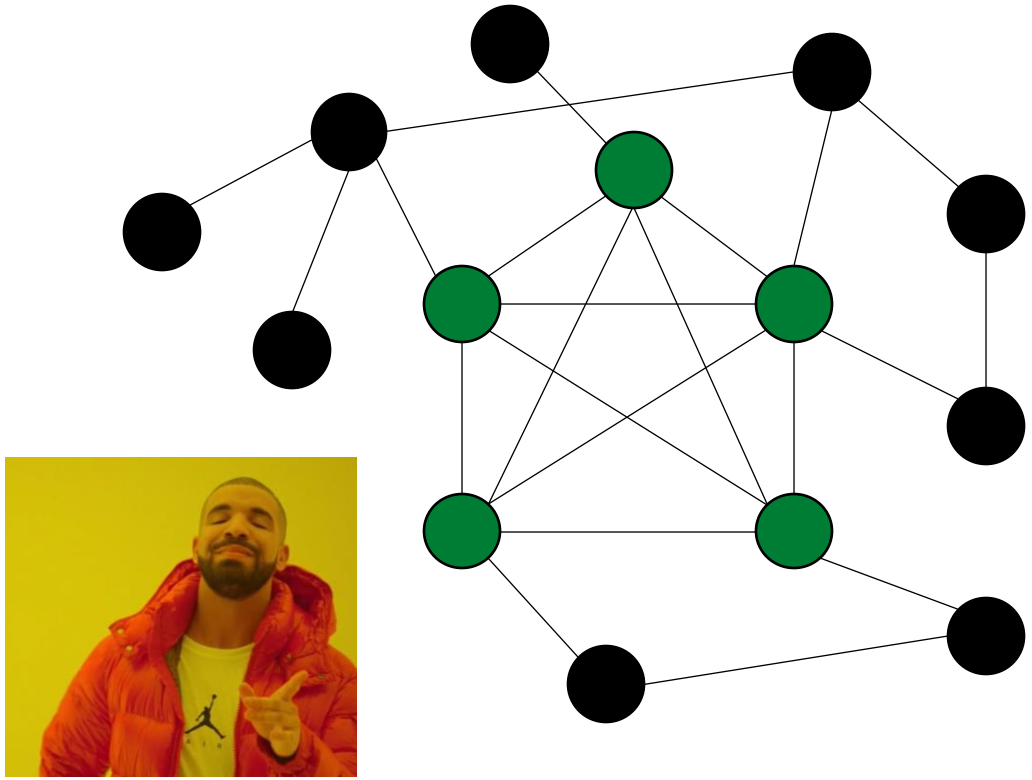
Problem 3 from the exam held on June 19, 2019

Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **$\boldsymbol{\alpha}$-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \to \{1, \dots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $\boldsymbol{(u, v) \in E}$.



Problem 3 from the exam held on June 19, 2019

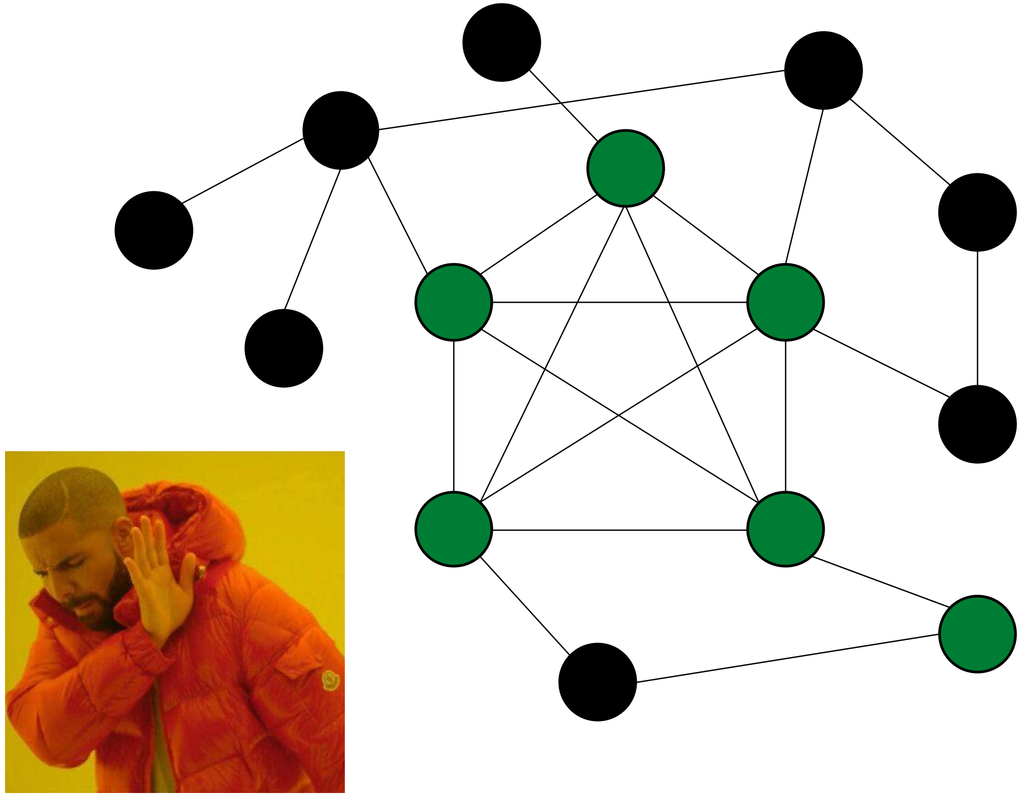
Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **α-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \rightarrow \{1, \ldots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $\boldsymbol{(u, v) \in E}$.



Problem 3 from the exam held on June 19, 2019

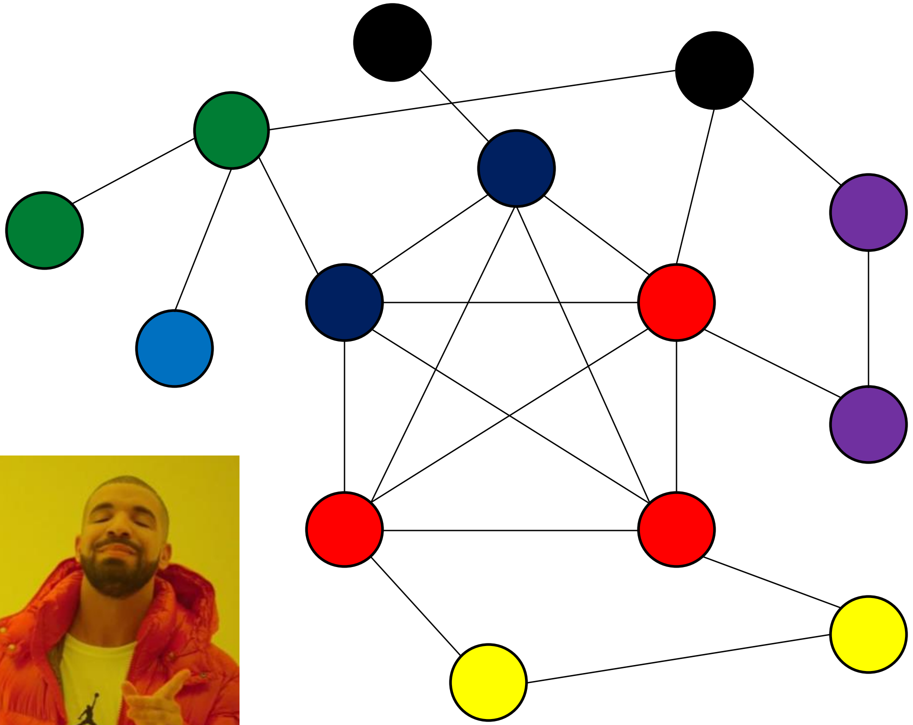
Given an undirected graph $G = (V, E)$ and an integer h, we say that a subset V' of the nodes of G is **α-colorable** with h colors if there exists a function $c : V' \rightarrow \{1, \dots, h\}$ such that, for every $u, v \in V'$, if $c(u) = c(v)$ then $(u, v) \in E$.

I can only color the cliques



Problem 3 from the exam held on June 19, 2019

Given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{h}$, we say that a subset $\boldsymbol{V'}$ of the nodes of G is **$\boldsymbol{\alpha}$-colorable** with $\boldsymbol{h}$ colors if there exists a function $c : V' \to \{1, \dots, h\}$ such that, for every $u, v \in \boldsymbol{V'}$, if $\boldsymbol{c(u) = c(v)}$ then $\boldsymbol{(u, v) \in E}$.



Problem 3 from the exam held on June 19, 2019

Given an undirected graph $G = (V, E)$ and an integer h, we say that a subset V' of the nodes of G is α-colorable with h colors if there exists a function $c : V' \rightarrow \{1, \dots, h\}$ such that, for every $u, v \in V'$, if $c(u) = c(v)$ then $(u, v) \in E$.

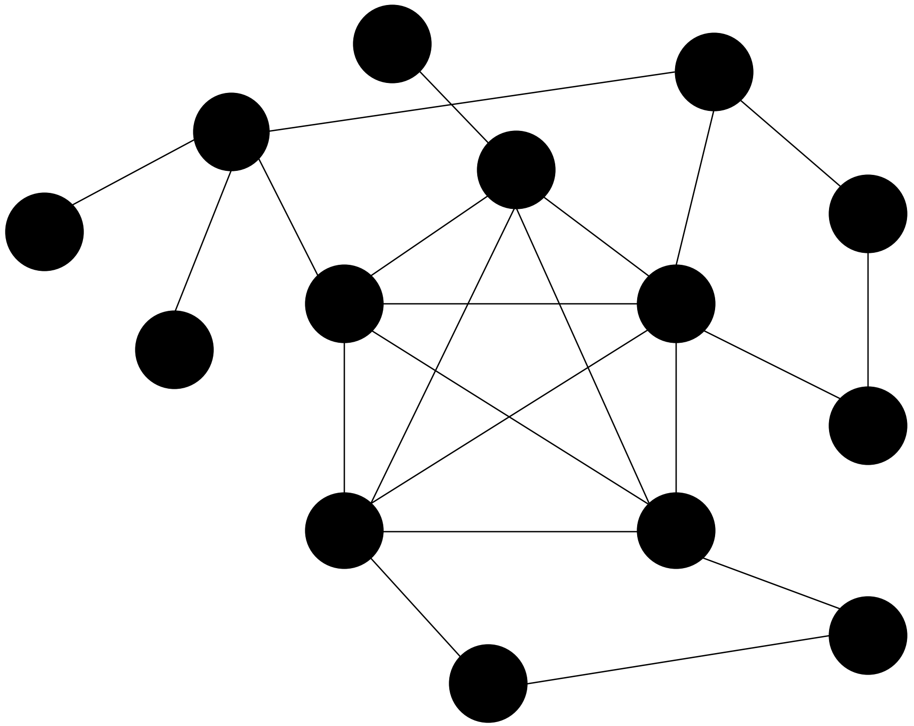
Consider the following decision problem: given an undirected graph $G = (V, E)$ and an integer k, decide whether there does not exist a subset $V' \subseteq V$ of cardinality at least k that is α-colorable with only one color.

After formalizing the problem using the triple $\langle I, S, \pi \rangle$, answer the following questions (in the order deemed appropriate), providing justification for each response.

- Is the problem in P?
- Is the problem in NP?
- Is the problem in $coNP$?

Problem 3 from the exam held on June 19, 2019

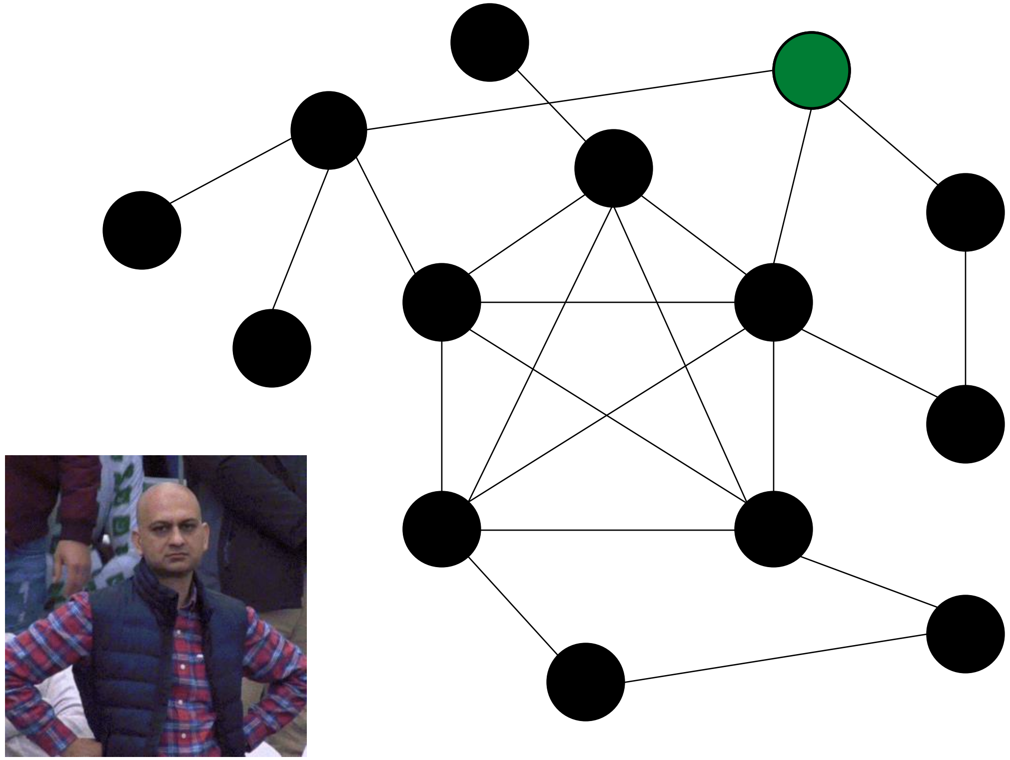
Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V}' \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only **one color**.



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V'} \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only one color.

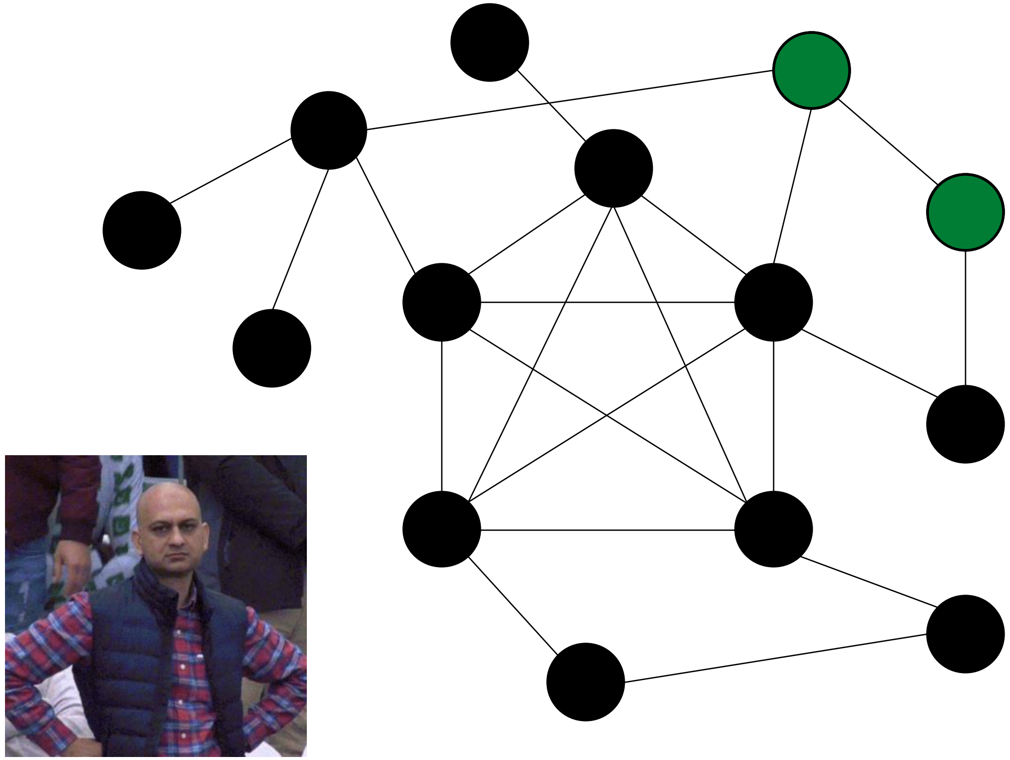
$k = 1$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V'} \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only **one color**.

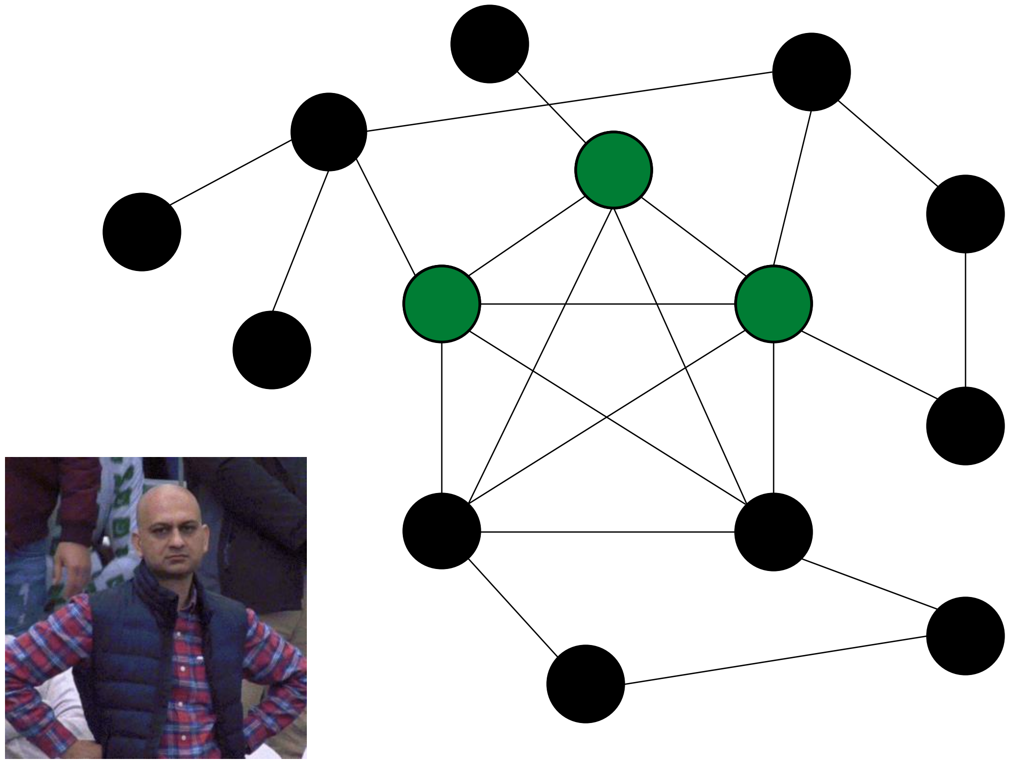
$k = 2$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V'} \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only **one color**.

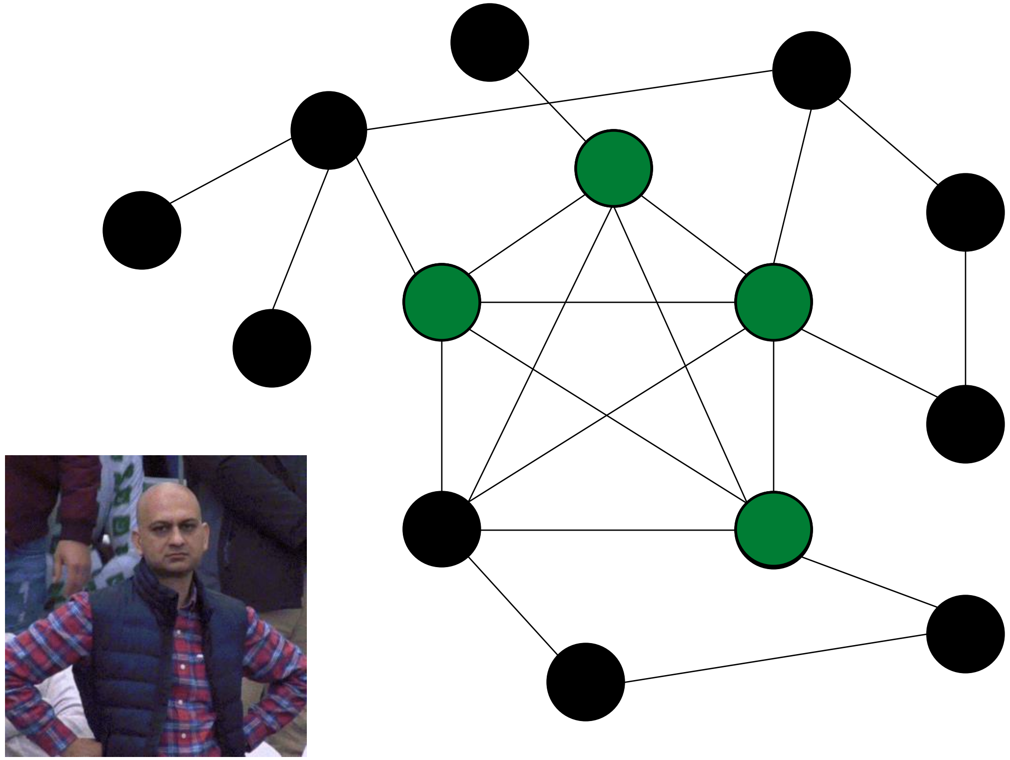
$k = 3$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V'} \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only **one color**.

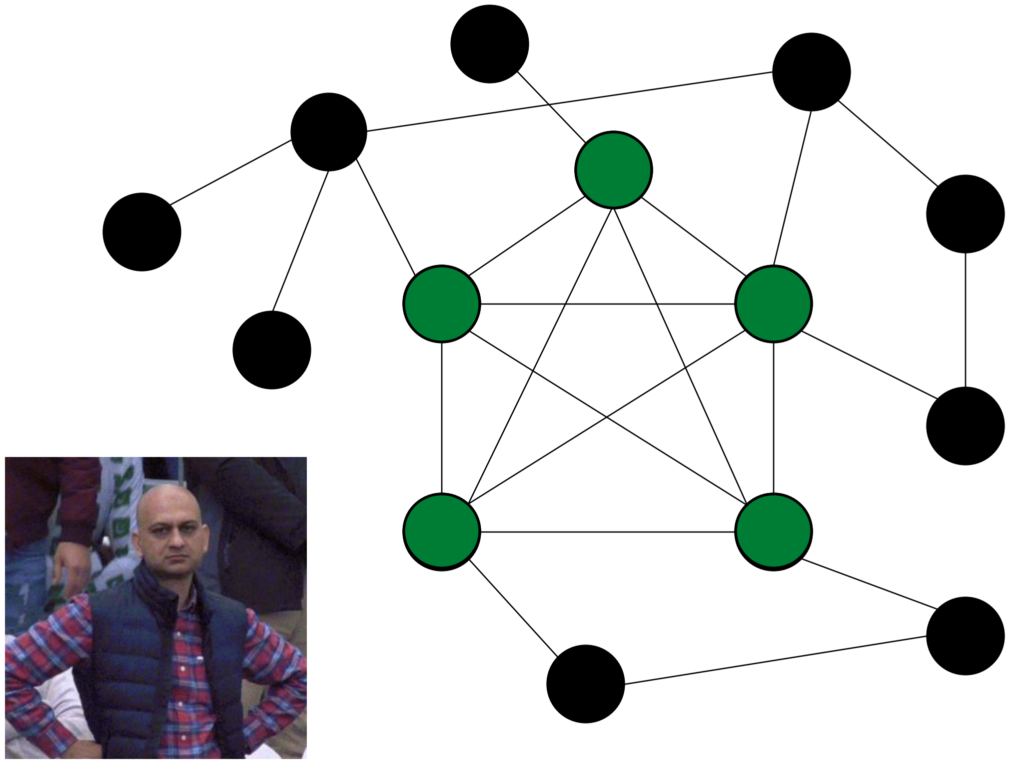
$k = 4$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V'} \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **$\boldsymbol{\alpha}$-colorable** with only **one color**.

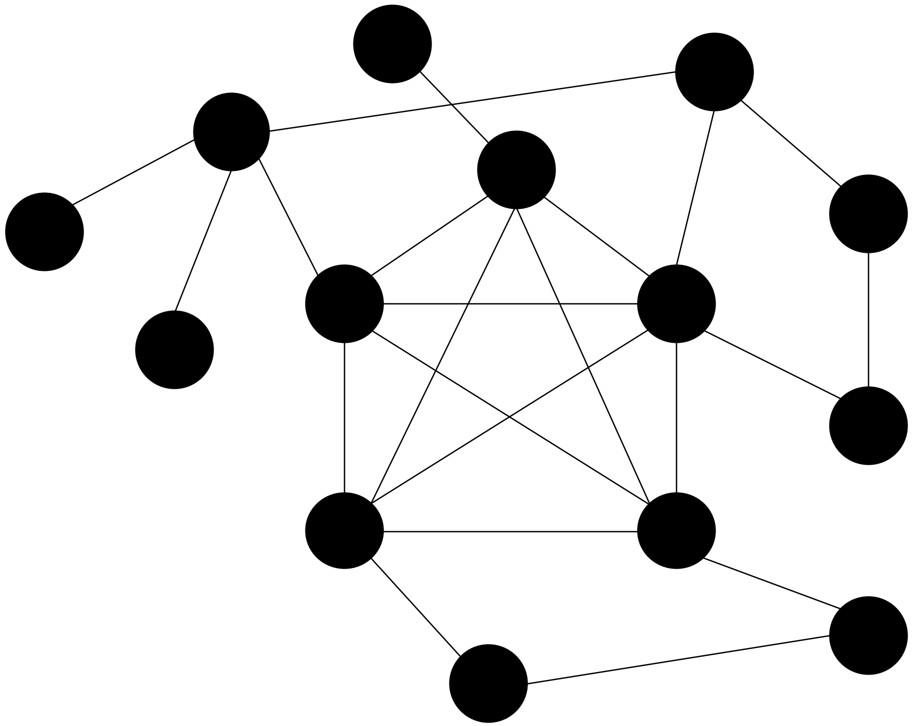
$k = 5$



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $G = (V, E)$ and an integer k, decide whether there **does not exist** a subset $V' \subseteq V$ of cardinality at least k that is **α-colorable** with only one color.

$k \geq 6$ ❓



Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V}' \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only one color.

- $I_{exam} = \{\langle G = (V, E), k\rangle : G \text{ is an undirected graph} \wedge k \in \mathbb{N}\}$
- $S_{exam}(G, k) = \{V' \subseteq V\}$
- $\pi_{exam}\big(G, k, S_{exam}(G, k)\big) = \forall\, V' \in S_{exam}(G, k), \mathrm{c}\colon \mathrm{V}' \to \{1\}\big[|\boldsymbol{V}'| < \boldsymbol{k} \vee \exists\, u, v \in \boldsymbol{V}'\ [\boldsymbol{c}(\boldsymbol{u}) = \boldsymbol{c}(\boldsymbol{v}) \wedge (\boldsymbol{u}, \boldsymbol{v}) \notin \boldsymbol{E}]\big]$

Problem 3 from the exam held on June 19, 2019

Consider the following decision problem: given an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ and an integer $\boldsymbol{k}$, decide whether there **does not exist** a subset $\boldsymbol{V}' \subseteq \boldsymbol{V}$ of cardinality at least $\boldsymbol{k}$ that is **α-colorable** with only **one color**.

- $I_{exam} = \{\langle G = (V, E), k\rangle : G$ is an undirected graph $\land\, k \in \mathbb{N}\}$
- $S_{exam}(G, k) = \{V' \subseteq V\}$
- $\pi_{exam}\big(G, k, S_{exam}(G, k)\big) = \forall\, V' \in S_{exam}(G, k)\, \big[|\mathbf{V}'| < \mathbf{k} \lor \exists\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}'\, [(\boldsymbol{u}, \boldsymbol{v}) \notin E]\big]$

Problem 3 from the exam held on June 19, 2019

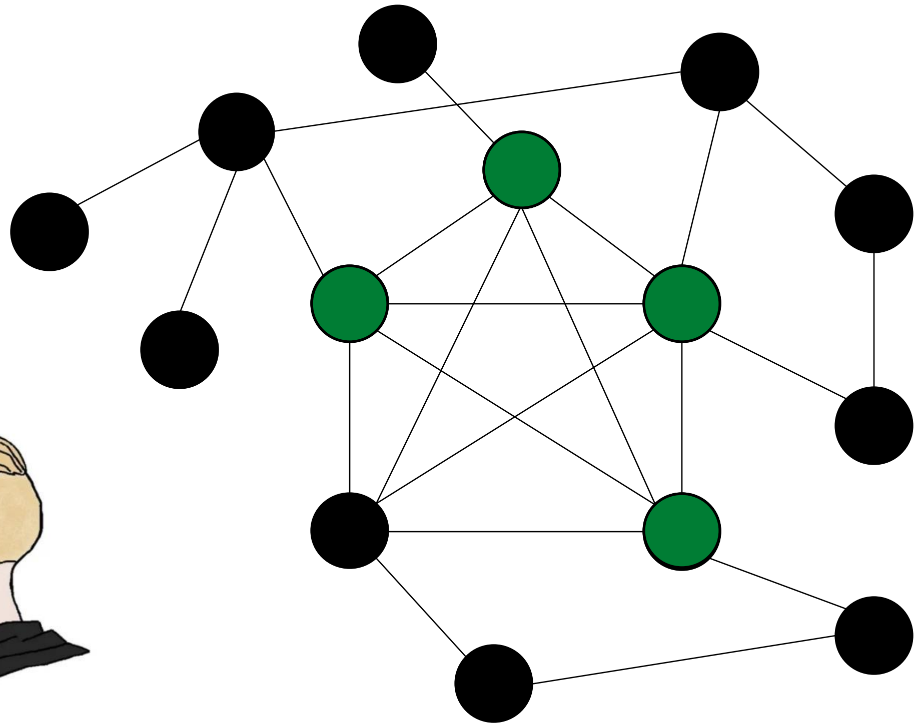
- $I_{\neg exam} = \{\langle G = (V, E), k \rangle : G$ is an undirected graph $\wedge\, k \in \mathbb{N}\}$
- $S_{\neg exam}(G, k) = \{V' \subseteq V\}$
- $\pi_{\neg exam}\big(G, k, S_{\neg exam}(G, k)\big) = \exists\, V' \in S_{\neg exam}(G, k)\, \big[|\mathbf{V}'| \geq \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}'\, [(\boldsymbol{u}, \boldsymbol{v}) \in E]\big]$

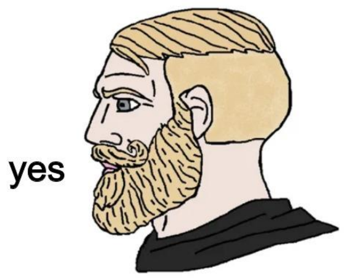
- $I_{clique} = \{\langle G = (V, E), k \rangle : G$ is an undirected graph $\wedge\, k \in \mathbb{N}\}$
- $S_{clique}(G, k) = \{V' \subseteq V\}$
- $\pi_{clique}\big(G, k, S_{clique}(G, k)\big) = \exists\, V' \in S_{clique}(G, k)\, \big[|\mathbf{V}'| \geq \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}'\, [(\boldsymbol{u}, \boldsymbol{v}) \in E]\big]$

Problem 3 from the exam held on June 19, 2019

- $I_{clique} = \{\langle G = (V, E), k\rangle : G \text{ is an undirected graph} \wedge k \in \mathbb{N}\}$
- $S_{clique}(G, k) = \{V' \subseteq V\}$
- $\pi_{clique}\left(G, k, S_{clique}(G, k)\right) = \exists\, V' \in S_{clique}(G, k)\left[{\color{red}|\mathbf{V'}| \geq \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V'}[(\boldsymbol{u}, \boldsymbol{v}) \in E]}\right]$

$k = 4$





Problem 3 from the exam held on June 19, 2019

Clique is **NP-complete** $\Rightarrow$ **¬exam** is **NP-complete** $\Rightarrow$ **exam** is **coNP-complete**

⚠ To be more formal, you should formally specify the reduction between **Clique** and **¬exam**, but the problems are practically identical

Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $G = (V, E, w)$ with $w: E \to \mathbb{N}$ and two integers h and k, decide whether G has a clique of k nodes such that the sum of the weights of its edges is at most h.

After formalizing the problem using the triple $\langle I, S, \pi \rangle$, answer the following questions (in the order deemed appropriate), providing justification for each response.

- Is the problem in P?
- Is the problem in NP?
- Is the problem in coNP?

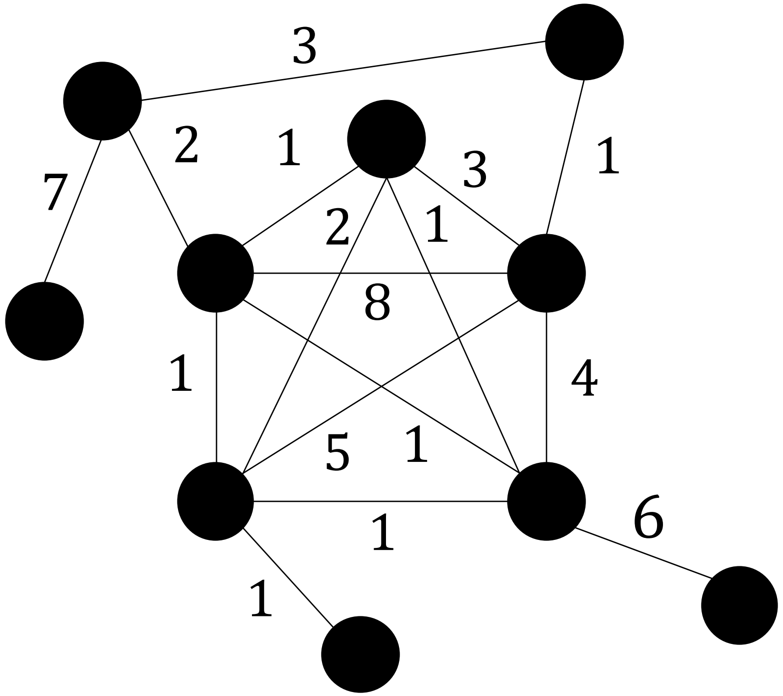
Note well: Deciding whether a graph contains a clique of exactly k nodes is an NP-complete problem.

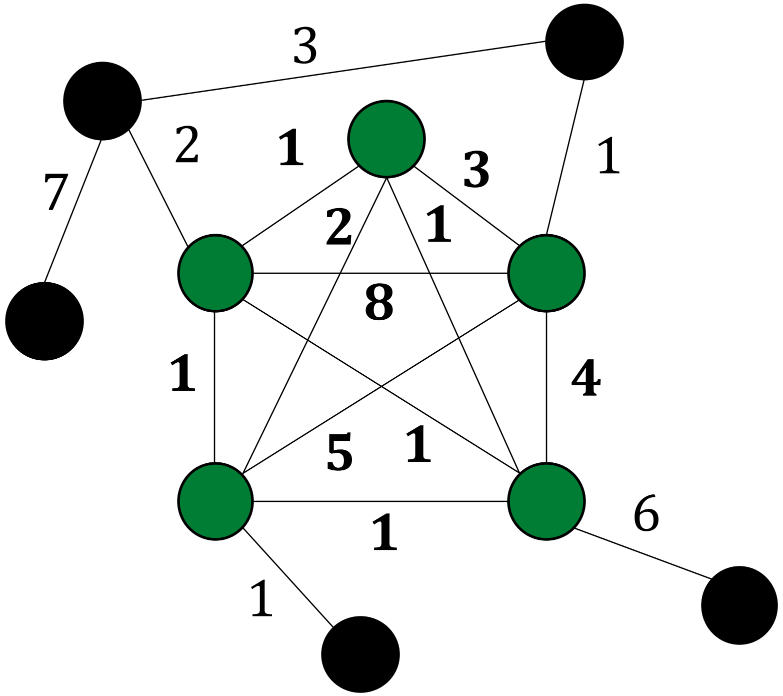
Suggestion: Remember the reduction that demonstrates that TSP is NP-complete.

Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $G = (V, E, w)$ with $w: E \to \mathbb{N}$ and two integers h and k, decide whether G has a **clique** of **k nodes** such that the **sum of the weights** of its edges is at most h.

$k = 5$
$h = 100$

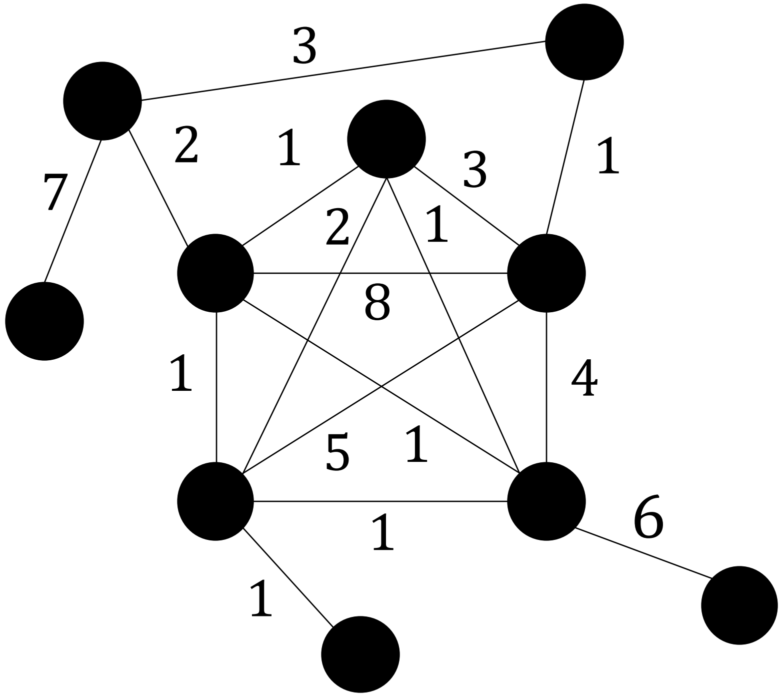




Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{w})$ with $\boldsymbol{w}: \boldsymbol{E} \to \mathbb{N}$ and two integers $\boldsymbol{h}$ and $\boldsymbol{k}$, decide whether G has a **clique** of $\boldsymbol{k}$ **nodes** such that the **sum of the weights** of its edges is at most $\boldsymbol{h}$.

$k = 6$
$h = 100$

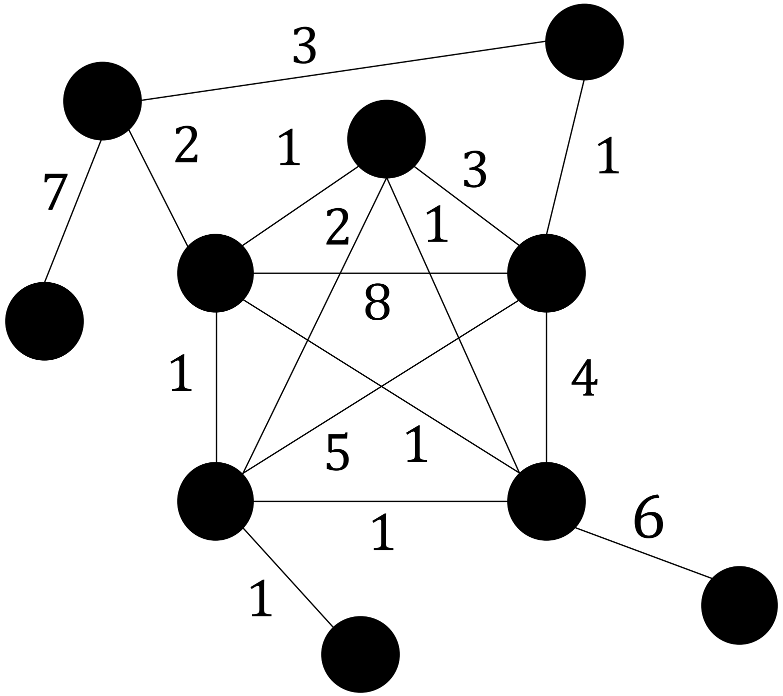


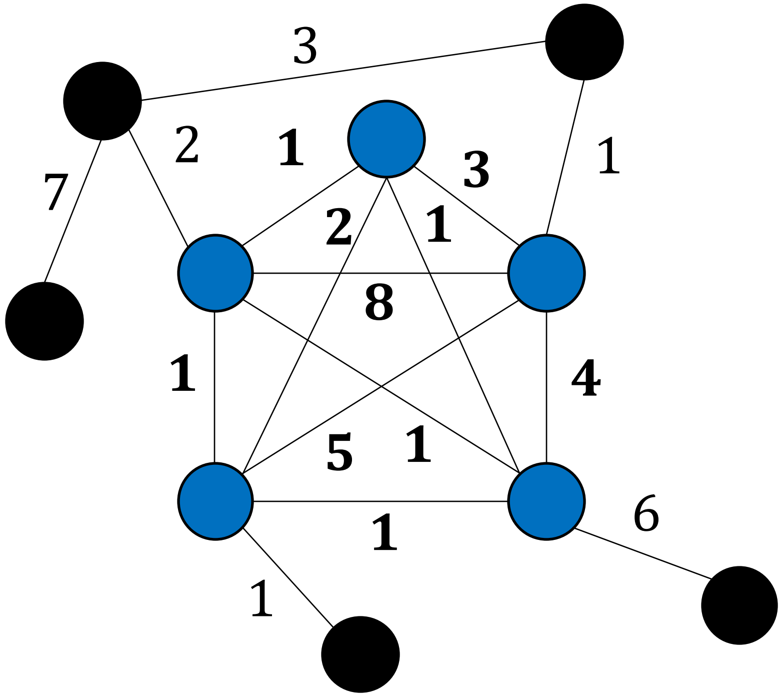
Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{w})$ with $\boldsymbol{w}: \boldsymbol{E} \rightarrow \mathbb{N}$ and two integers $\boldsymbol{h}$ and $\boldsymbol{k}$, decide whether G has a **clique** of **k nodes** such that the **sum of the weights** of its edges is at most $\boldsymbol{h}$.

$k = 5$

$h = 15$ ❌





Problem 3 from the exam held on June 16, 2022

Consider the following decision problem: given an undirected and weighted graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{w})$ with $\boldsymbol{w}: \boldsymbol{E} \rightarrow \mathbb{N}$ and two integers $\boldsymbol{h}$ and $\boldsymbol{k}$, decide whether G has a **clique** of $\boldsymbol{k}$ **nodes** such that the **sum of the weights** of its edges is at most $\boldsymbol{h}$.

- $I_{ex} = \{\langle G = (V, E, w), k, h\rangle : G$ is an undirected and weighted graph $\wedge\, w: E \rightarrow \mathbb{N} \wedge k, h \in \mathbb{N}\}$
- $S_{ex}(G, k, h) = \{V' \subseteq V\}$
- $\pi_{ex}\big(G, k, S_{ex}(G, k)\big) = \exists\, V' \in S_{ex}(G, k) \left[|\mathbf{V}'| = \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}' \left[(\boldsymbol{u}, \boldsymbol{v}) \in E\right] \wedge \sum_{u,v \in V'} w(\{u, v\}) \leq h\right]$

Problem 3 from the exam held on June 16, 2022

- $I_{ex} = \{\langle G = (V, E, \cancel{w}), k, \cancel{h} \rangle$: G is an undirected and weightedgraph $\wedge\ w: E \rightarrow \mathbb{N} \wedge k, h \in \mathbb{N}\}$
- $S_{ex}(G, k, \cancel{h}) = \{V' \subseteq V\}$
- $\pi_{ex}\big(G, k, S_{ex}(G, k, h)\big) = \exists\, V' \in S_{ex}(G, k, h)\ \big[|\mathbf{V}'| = \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}'\ [(\boldsymbol{u}, \boldsymbol{v}) \in E] \wedge \sum_{u,v \in V'} \cancel{w}(\{u, v\}) \leq h\big]$

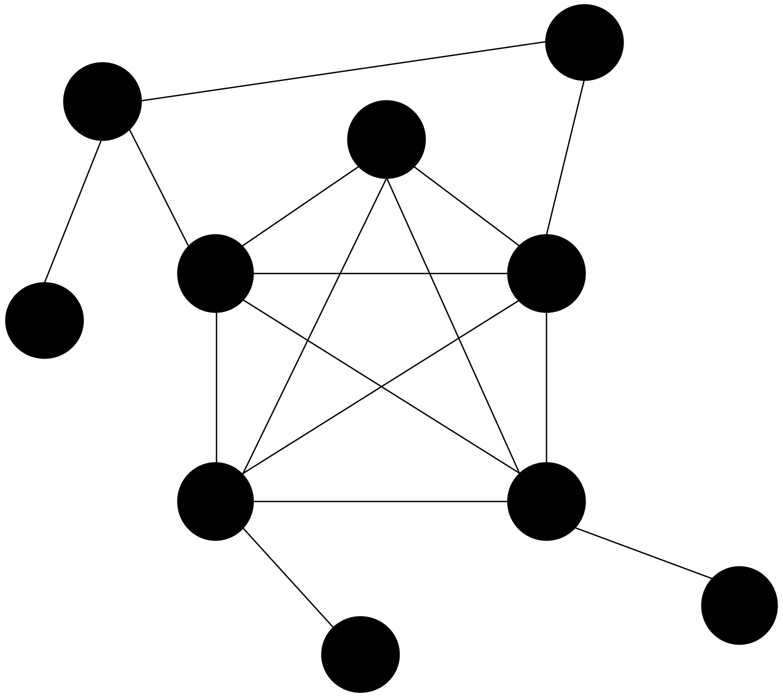
Note well: Deciding whether a graph contains a clique of exactly k nodes is an NP-complete problem.

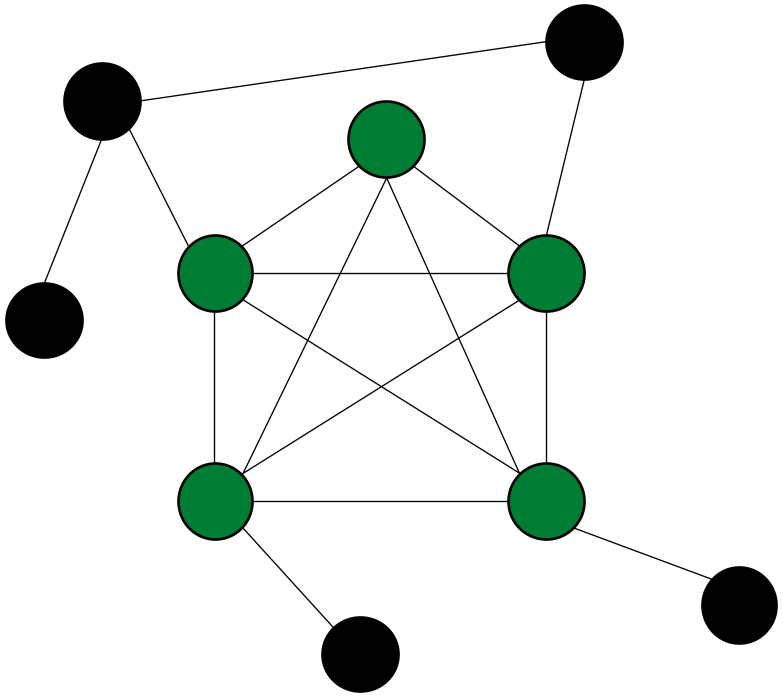
Problem 3 from the exam held on June 16, 2022

- $I_{nw} = \{\langle G = (V, E, w), k\rangle : G \text{ is an undirected graph} \land w: E \rightarrow \mathbb{N} \land k \in \mathbb{N}\}$
- $S_{nw}(G, k) = \{V' \subseteq V\}$
- $\pi_{nw}\big(G, k, S_{nw}(G,k)\big) = \exists\, V' \in S_{nw}(G,k) \left[|\mathbf{V}'| = \mathbf{k} \land \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}' \left[(\boldsymbol{u}, \boldsymbol{v}) \in E\right]\right]$

$\geq$ (it's **Clique**)

$k = 5$



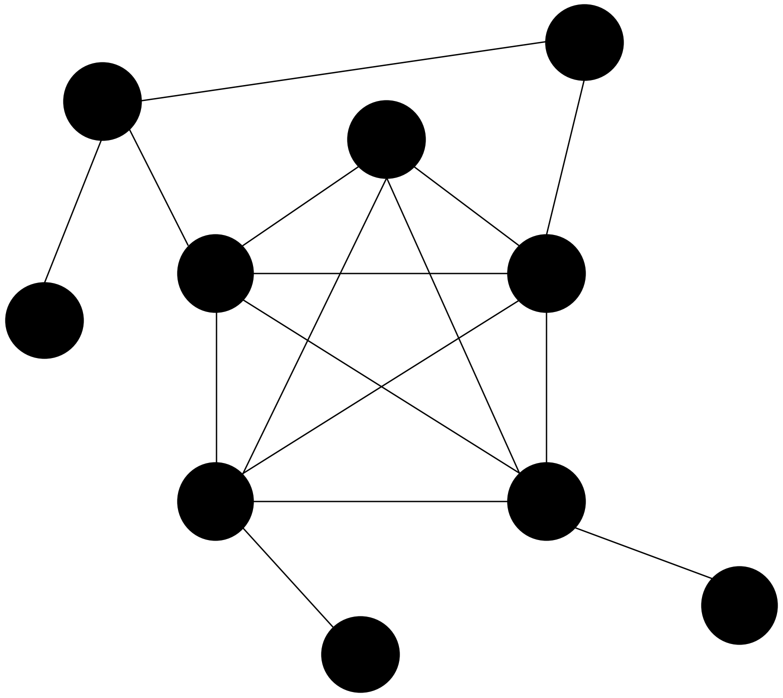


Problem 3 from the exam held on June 16, 2022

- $I_{nw} = \{\langle G = (V, E, w), k\rangle :$ G is an undirected graph $\wedge\ w: E \rightarrow \mathbb{N} \wedge k \in \mathbb{N}\}$
- $S_{nw}(G, k) = \{V' \subseteq V\}$
- $\pi_{nw}\big(G, k, S_{nw}(G, k)\big) = \exists\, V' \in S_{nw}(G, k)\ \Big[|\mathbf{V}'| = \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}'\ [(\boldsymbol{u}, \boldsymbol{v}) \in E]\Big]$

$\geq$ (it's Clique)

$k = 6$



Problem 3 from the exam held on June 16, 2022

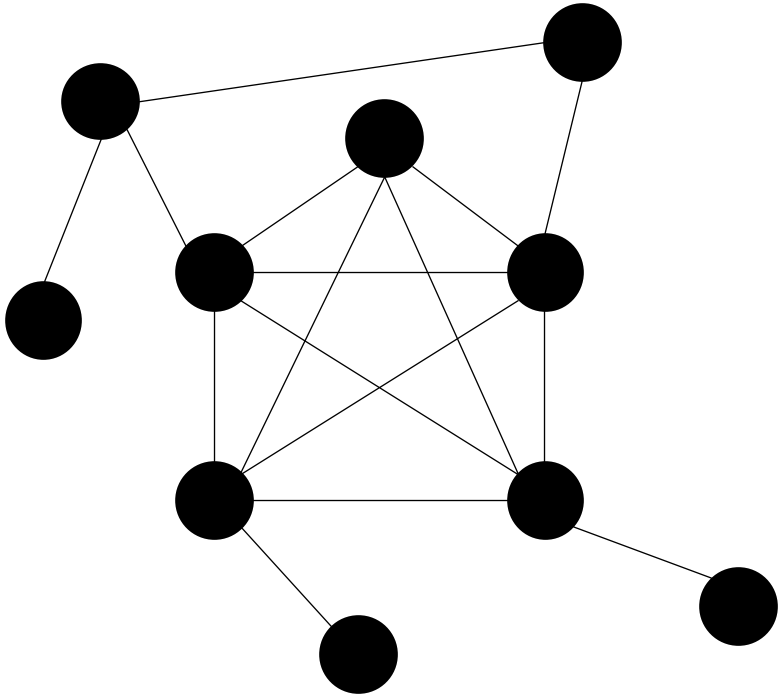
- $I_{ex} = \{\langle G = (V, E, w), k, h \rangle : G$ is an undirected and weighted graph $\wedge\, w: E \rightarrow \mathbb{N} \wedge k, h \in \mathbb{N}\}$
- $S_{ex}(G, k, h) = \{V' \subseteq V\}$
- $\pi_{ex}\big(G, k, S_{ex}(G, k, h)\big) = \exists\, V' \in S_{ex}(G, k, h) \left[|\mathbf{V}'| = \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}' \,[(\boldsymbol{u}, \boldsymbol{v}) \in E] \wedge \sum_{u,v \in V'} w(\{u, v\}) \leq h \right]$

Let's reduce from **Clique**

- $I_{Clique} = \{\langle G = (V, E, w), k \rangle : G$ is an undirected graph $\wedge\, w: E \rightarrow \mathbb{N} \wedge k \in \mathbb{N}\}$
- $S_{Clique}(G, k) = \{V' \subseteq V\}$
- $\pi_{Clique}\left(G, k, S_{Clique}(G, k)\right) = \exists\, V' \in S_{Clique}(G, k) \left[|\mathbf{V}'| = \mathbf{k} \wedge \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}' \,[(\boldsymbol{u}, \boldsymbol{v}) \in E] \right]$

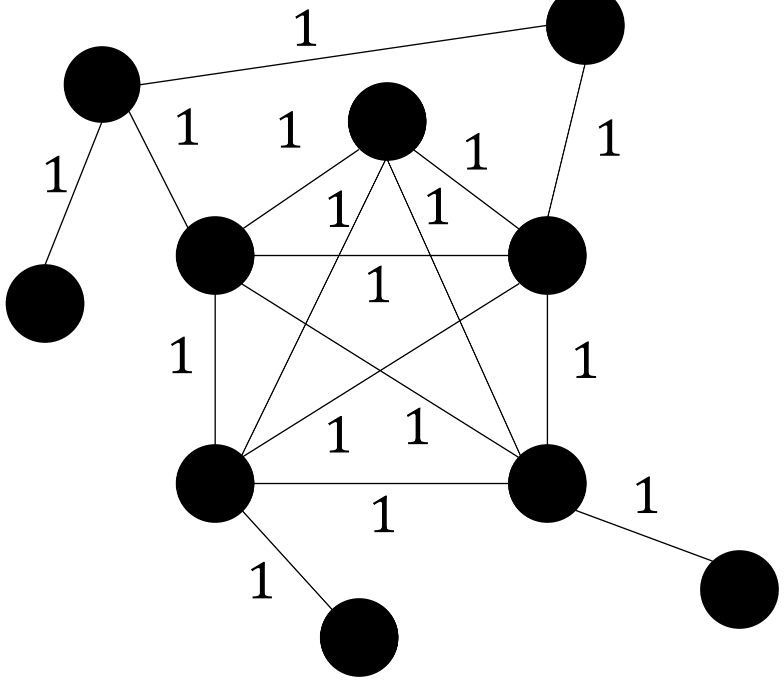
Problem 3 from the exam held on June 16, 2022

k



Instance of **Clique**

$k, \boldsymbol{h = |E|}$



Instance of **our problem**

Problem 3 from the exam held on June 16, 2022

Instance of **Clique**

$I_{Clique} = \{\langle G = (V, E), k\rangle : G$ is an undirected graph $\wedge\, k \in \mathbb{N}\}$

Instance of **our problem**

$I_{ex} = \{\langle G_{ex} = (V, E, w), k, h = |E|\rangle$

$: G_{ex}$ is an undirected and weighted graph $\wedge\, w: E \rightarrow \mathbb{N} \wedge k \in \mathbb{N}\}$

If G has a **clique** of $\boldsymbol{k}$ vertices, G_{ex} has a **clique** of $\boldsymbol{k}$ vertices, such that the **sum of the weights** of its edges is at most $\boldsymbol{h}$, I_{ex} is a “yes” instance

If G doesn’t have a **clique** of $\boldsymbol{k}$ vertices, G_{ex} doesn’t have a **clique** of $\boldsymbol{k}$ vertices, I_{ex} is a “no” instance