Serii Fourier

Să se dezv în serie Fourier fcț per.

$x_k = 2k\pi$ pct de disc

a) $T = 2\pi$

$f : [0, 2\pi] \to \mathbb{R},\ f(x) = x^2$

Calc coef fourier

$$\boxed{a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega x)\,dx,\ n\in\mathbb{N}}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} x^2\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\left(\frac{\sin nx}{n}\right)' dx$$

$$= \frac{1}{\pi}\left[x^2\cdot\frac{\sin nx}{n}\Big|_0^{2\pi} - \int_0^{2\pi} 2x\,\frac{\sin nx}{n}\,dx\right]$$

$$= 0 - \frac{2}{\pi n}\int_0^{2\pi} x\sin(nx)\,dx$$

$$= -\frac{2}{\pi n}\int_0^{2\pi} x\left(-\frac{\cos nx}{n}\right)' dx$$

$$= -\frac{2}{\pi n}\left(-x\cdot\frac{\cos nx}{n}\Big|_0^{2\pi} + \int_0^{2\pi}\frac{\cos nx}{n}\,dx\right)$$

$$= +\frac{2}{\pi n}\cdot\frac{2\pi}{n}\cdot\underbrace{\cos 2n\pi}_{1} - \underbrace{\frac{2}{\pi n^2}\cdot\frac{\sin nx}{n}\Big|_0^{2\pi}}_{0}$$

$$= \frac{4}{n^2},\ n\in\mathbb{N}^*$$

$$\boxed{b_n = \frac{2}{T}\int_0^T f(x)\sin(n\omega x)\,dx},\ n\in\mathbb{N}^*$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\left(-\frac{\cos nx}{n}\right)' dx$$

$$= \frac{1}{\pi}\left[x^2\cdot\left(-\frac{\cos nx}{n}\right)\Big|_0^{2\pi} + \int_0^{2\pi} 2x\,\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi n}\,4\pi^2\cdot(\underbrace{-\cos 2\pi n}_{-1}) + \frac{2}{\pi n}\int_0^{2\pi}\left(\frac{\sin nx}{n}\right)' dx$$

$$= -\frac{4\pi}{n} + \underbrace{\frac{2}{\pi n}\,\frac{\sin nx}{n}\Big|_0^{2\pi}}_{0}$$

$$= -\frac{4\pi}{n},\ n\in\mathbb{N}^*$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega x) + b_n\sin(n\omega x)\right)$$

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \frac{2}{2\pi}\int_0^{2\pi} x^2\,dx =$$

$$= \frac{1}{\pi}\,\frac{x^3}{3}\Big|_0^{2\pi} = \frac{8\pi^3}{3\pi} = \frac{8\pi^2}{3}$$

$$\frac{a_0}{2} = \frac{8\pi^2}{3}\cdot\frac{1}{2} = \frac{4\pi^2}{3}$$

$$f(x) \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx)\right]$$

T Dirichlet $\Rightarrow$

$$S(x) = \begin{cases} f(x), & x \neq 2k\pi \\ \dfrac{f(0)+f(2\pi)}{2} = \dfrac{(2\pi)^2}{2} = \dfrac{4\pi^2}{2} = 2\pi^2, & x = 2k\pi,\ k\in\mathbb{Z}\end{cases}$$

$4\pi^2$

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $\quad f(x) = x^2$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx, \; x \in (0, 2\pi) \quad (*)$$

$$x = \pi \text{ în } (*) \Rightarrow \pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(n\pi) - 0$$

$$\frac{3\pi^2 - 4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \; \Big| : \left(-\frac{1}{4}\right)$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Să se dezv în serie F. de sinusuri fct periodică cu $T = 4$

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}$$

Prelung. prin imparitate $\quad l = 2$

$$f_i(x) = \begin{cases} f(x), & x \in [0, 2) \\ -f(-x), & x \in [-2, 0] \end{cases} \qquad f_i : [-2, 2]$$

$$f_i(x) = \begin{cases} -(-2+x) & , x \in [-2, -1] \\ -(-x) & , x \in [-1, 0] \\ x & , x \in [0, 1] \\ 2 - x & , x \in [1, 2] \end{cases}$$

$$b_n = \frac{4}{T}\int_0^2 f(x)\sin(n\omega x)\,dx = \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$a_n = 0, \; n \in \mathbb{N}$

$$b_n = \int_0^1 x \cdot \left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)' dx + \int_1^2 (2-x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)' dx$$

$$= \frac{-2x}{n\pi}\cdot\left(+\cos\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi x}{2}\,dx - \frac{(2-x)\cdot 2}{n\pi}\cdot\cos\frac{n\pi x}{2}\Big|_1^2 -$$

$$-\frac{2}{n\pi}\int_1^2 (-1)\cdot\left(-\cos\frac{n\pi x}{2}\right)dx = -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n\pi}\cdot\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\Big|_0^1 - (2-x)\cdot\frac{2}{n\pi}\cdot\cos\frac{n\pi x}{2}\Big|_1^2 +$$

$$+\frac{2}{n\pi}\int \cos\frac{n\pi x}{2}\,dx = \cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^1 + \cancel{\frac{2}{n\pi}\cos\frac{n\pi}{2}} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^1 =$$

$$= \frac{8}{\dots}\sin\frac{n\pi}{2}$$

) $f:\mathbb{R}\to\mathbb{R}$

$T=2$

$f(x)=|x|$

a) suma pe int $[-1,1]$

dom de divizibilitate

$f(-x)=|-x|=|x|=f(x) \Rightarrow$ fct. pară $\qquad \omega=\frac{2\pi}{T}=\pi$

$\Rightarrow b_n=0,\ n\in\mathbb{N}^*$

$$a_n=\frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos(n\omega x)\,dx=2\int_0^1 x\cdot\cos(n\pi x)\,dx,\ n\in\mathbb{N}$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos\frac{2n\pi x}{b-a}+\underbrace{b_n\sin\frac{2n\pi x}{b-a}}_{0}\right)$$

$$a_0=2\int_0^1 x\cdot\cos 0\,dx=2\int_0^1 x\,dx=x^2\Big|_0^1=1$$

$$a_n=2\int_0^1 x\cos(n\pi x)\,dx=2\int_0^1 x\left(\frac{\sin n\pi x}{n\pi}\right)'dx=2\cdot x\cdot\frac{\sin n\pi x}{n\pi}\Big|_0^1-$$

$$-\frac{2}{n\pi}\int_0^1\frac{\sin n\pi x}{1}\,dx=\underbrace{2\cdot 1\cdot\frac{\sin\pi n}{n\pi}-2\cdot 0}_{0}-\frac{2}{n\pi}\int_0^1\left(-\frac{\cos n\pi x}{n\pi}\right)'dx=$$

$$=\frac{2}{n\pi}\left(\frac{\cos n\pi x}{n\pi}\right)\Big|_0^1=\frac{2}{n\pi}\left(\frac{\cos\pi n}{\pi n}-\frac{1}{n\pi}\right)=\frac{2}{n^2\pi^2}(\cos n\pi-1)=\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$S(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]\cos(n\pi x)\Big]=\frac{1}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cdot\cos\pi x$$

f cont pe $(-1,1)\Rightarrow S(x)=f(x),\ \forall x\in(-1,1)$

$$S(1)=\frac{f(-1+0)+f(1-0)}{2}=\frac{1+1}{2}=1=S(-1)\Rightarrow f \text{ dezv în}$$

serie Fourier pe int $[-1,1]$

$\cdot\ b_n=\frac{8}{n^2\pi^2}\cdot\sin\frac{n\pi}{2}$

$$b_{2m-1}=\frac{8}{(2m-1)^2\pi^2}\cdot\sin$$

$$\begin{cases}0, & n=2m\\ \frac{8}{n^2\pi^2}\cdot(-1)^m, & n=2m-1,\ m\in\mathbb{N}\end{cases}$$

$$\xrightarrow{T.D} S(x)=\sum_{n=1}^{\infty}b_n\cos\frac{n\pi x}{2}=\sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi x}{2}=\sum_{m=1}^{\infty}\frac{8(-1)^{m-1}}{(2m-1)^2\pi^2}\frac{\sin(2m-1)}{2}$$

① Să se arate că urm. funcții nu au limită:

1) $f(x,y)=\frac{x^2-2y^2}{3x^2+y^2}$, $(x,y)\neq(0,0)$

Heine
lim relative
iterate

Heine

$\exists\, x_{n_1}, x_{n_2}$, $\lim_{n\to\infty} x_{n_2}=\lim_{n\to\infty} x_{n_1}=a$ a.î. $\begin{cases}\lim_{n\to\infty}(f(x_{n_1}))=l_1\\ \lim_{n\to\infty}(f(x_{n_2}))=l_2\end{cases}$

$\overset{\text{Heine}}{\Rightarrow}$ f nu are lim în a

Fie $(x_n,y_n)=\left(\frac{2}{n},\frac{1}{n}\right)\xrightarrow[n\to\infty]{}(0,0)$, $(x_n',y_n')=\left(\frac{3}{n},\frac{2}{n}\right)\xrightarrow[n\to\infty]{}(0,0)$

$$\left.\begin{array}{l}\lim_{n\to\infty} f\left(\frac{2}{n},\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\frac{4}{n^2}-\frac{2}{n^2}}{\frac{12}{n^2}+\frac{1}{n^2}}=\frac{2}{13}\\ \lim_{n\to\infty} f\left(\frac{3}{n},\frac{2}{n}\right)=\lim_{n\to\infty}\frac{\frac{9}{n^2}-\frac{8}{n^2}}{\frac{27}{n^2}+\frac{4}{n^2}}=\frac{1}{31}\end{array}\right\}\overset{C}{\underset{\text{Heine}}{\Rightarrow}}\nexists\lim_{(x,y)\to(0,0)} f(x,y)$$

Lim iterate

$$\left.\begin{array}{l}l_{12}=\lim_{y\to0}\left(\lim_{x\to0} f(x,y)\right)=\lim_{y\to0}\frac{-2y^2}{y^2}=-2\\ l_{21}=\lim_{x\to0}\left(\lim_{y\to0} f(x,y)\right)=\lim_{x\to0}\frac{x^2}{3x^2}=3\end{array}\right\}\Rightarrow l_{12}\neq l_{21}\Rightarrow\nexists\lim_{(x,y)\to(0,0)} f(x,y)$$

2) $f(x,y)=\frac{3x^3y^2}{2x^6+4y^4}$, $x,y\neq(0,0)$

Fie $(x_n,y_n)=\left(\frac{1}{n^2},\frac{1}{n^3}\right)\xrightarrow[n\to\infty]{}(0,0)$

$$\lim_{n\to\infty} f\left(\frac{1}{n^2},\frac{1}{n^3}\right)=\lim_{n\to\infty}\frac{\frac{3}{n^6}\cdot\frac{1}{n^6}}{\frac{2}{n^{12}}+\frac{4}{n^{12}}}=\frac{3}{6}=\frac{1}{2}$$

Fie $(x_n',y_n')=\left(\frac{1}{n^2},\frac{2}{n^3}\right)\xrightarrow[n\to\infty]{}(0,0)$

$$\lim_{n\to\infty} f\left(\frac{1}{n^2},\frac{2}{n^3}\right)=\lim_{n\to\infty}\frac{\frac{3}{n^6}\cdot\frac{4}{n^6}}{\frac{2}{n^{12}}+\frac{4\cdot16}{n^{12}}}=\frac{12}{2+64}=\frac{12}{66}=\frac{2}{11}$$

$\Rightarrow\nexists\lim_{(x,y)\to(0,0)} f(x,y)$

② Lim iterate și globale

i) $f(x,y) = \frac{xy^2}{x^2+y^4}, (x,y) \neq (0,0)$

$l_{12} = l_{21} \Rightarrow$ trebuie calc cu altă metodă ($\nexists$ lim it...

Fie $A_m = \{(x,y) \in \mathbb{R}^2 \mid x = my^2\}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_m}} f(x,y) = \lim_{y\to 0} \frac{my^2\cdot y^2}{m^2y^4+y^4} = \lim_{y\to 0} \frac{my^4}{y^4(m^2+1)} = \frac{m}{m^2+1} \Rightarrow f \text{ nu are}$$

lim în $(0,0)$

! (lim relat. depinde de m)

ii) $f(x,y) = x \sin\frac{1}{x} \cos\frac{1}{y}, (x,y) \neq (0,0)$

nu ne dă așa (sper)

crit maj pt l_{12}

c Heine l_{21}

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1}$$

---

③ a) $$\lim_{(x,y)\to(0,0)} \frac{y\cdot\sin^2(2x)}{x^2+3y^2} = \lim_{(x,y)\to(0,0)} \frac{y\cdot\overbrace{\sin^2(2x)}^{1}}{(x^2+3y^2)\cdot(4x^2)} \cdot 4x^2 = 1\cdot 0 = 0$$

$$\frac{4x^2y}{x^2+3y^2} = |4y|\cdot \underbrace{\frac{x^2}{x^2+3y^2}}_{<1} < |4y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\Rightarrow$ (C maj) lim... $=0$

b) $$\lim_{(x,y)\to(0,0)} \frac{\ln(1+x^4+y^4)}{2(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \frac{\overbrace{\ln(1+x^4+y^4)}^{1}}{2(x^2+y^2)(x^4+y^4)}(x^4+y^4) = \frac{1}{2}\lim_{(x,y)\to(0,0)} \underbrace{\frac{x^4+y^4}{x^2+y^2}}_{0}$$

$$\left|\frac{x^4+y^4}{x^2+y^2}\right| = \frac{x^4}{x^2+y^2} + \frac{y^4}{x^2+y^2} = x^2\underbrace{\frac{x^2}{x^2+y^2}}_{<1} + y^2\underbrace{\frac{y^2}{x^2+y^2}}_{<1} < x^2+y^2 \to 0$$

lim... $=0$ (c maj)

④ Cont parț și cont fct:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$$\left.\begin{array}{l} \lim_{x\to 0} f(x,0) = 0 = f(0,0) \\ \lim_{y\to 0} f(0,y) = 0 = f(0,0) \end{array}\right\} \Rightarrow f \text{ continuă parțial în } (0,0) \text{ în rap. cu variab. } x,y$$

a) $f(x,y)=x\cdot \operatorname{arctg}\frac{x}{y}$, $y\neq 0$, $x_0=(-1,1)$, $\bar{h}=\bar{e}_1-2\bar{e}_2$

b) $f(x,y,z)=y\ln\frac{y^z}{z^x}$, $x,y,z>0$, $x_0=(2,1,3)$, $\bar{h}=3\bar{i}-2\bar{j}+\bar{k}$

Deriv part ord I în x_0 $\quad \frac{\partial f}{\partial x}(x_0), \frac{\partial f}{\partial y}(x_0), \frac{\partial f}{\partial z}(x_0)$

Deriv fct după dir h în x_0 $\quad \frac{\partial f}{\partial h}(x_0)$

Diferenț. ord I în x_0, expr dif totale ord I

$d_{x_0}f \qquad df$

Gradientul în x_0 ($\operatorname{grad}_{x_0} f$)

a) $\underline{\frac{\partial f}{\partial x}(x_0)}=\frac{\partial f}{\partial x}(-1,1)=\lim\limits_{x\to -1}\frac{f(x,1)-f(1,-1)}{x+1}=\lim\limits_{x\to -1}\frac{x\operatorname{arctg}x-\frac{\pi}{4}}{x+1}\overset{l'H}{\underset{0/0}{=}}$

$=\lim\limits_{x\to -1}\frac{\operatorname{arctg}x+\frac{x}{1+x^2}}{1}=-\frac{\pi}{4}-\frac{1}{2}$

$\underline{\frac{\partial f}{\partial y}(x_0)}=\frac{\partial f}{\partial y}(-1,1)=\lim\limits_{y\to 1}\frac{f(-1,y)-f(-1,1)}{y-1}=\lim\limits_{y\to 1}\frac{-\operatorname{arctg}(-\frac{1}{y})-\frac{\pi}{4}}{y-1}\underset{0/0}{=}$

$=\lim\limits_{y\to 1}\frac{(\operatorname{arctg}\frac{1}{y}-\frac{\pi}{4})'}{(y-1)'}=\lim\limits_{y\to 1}\frac{\frac{-1}{y^2}}{\frac{1+\frac{1}{y^2}}{1}}=\lim\limits_{y\to 1}-\cancel{\frac{1}{y^2}}\cdot\frac{\cancel{y^2}}{(y^2+1)}=-\frac{1}{2}$

$\underline{\frac{\partial f}{\partial h}(x_0)}=\frac{\partial f}{\partial x}(x_0)\cdot h_1+\frac{\partial f}{\partial y}(x_0)\cdot h_2=\left(-\frac{\pi}{4}-\frac{1}{2}\right)\cdot 1+\left(-\frac{1}{\cancel{2}}\right)\cdot(-\cancel{2})=$

$=-\frac{\pi}{4}-\frac{1}{2}+\overset{2}{\cancel{1}}=-\frac{\pi}{4}+\frac{1}{2}$

$\underline{d_{x_0}f}=\frac{\partial f}{\partial x}(x_0)\,dx+\frac{\partial f}{\partial y}(x_0)\,dy$

$=\left(-\frac{\pi}{4}-\frac{1}{2}\right)dx+\left(-\frac{1}{2}\right)dy$

$\underline{df}=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$

$\frac{\partial f}{\partial x}=\operatorname{arctg}\frac{x}{y}+\frac{\left(\frac{x}{y}\right)'}{1+\frac{x^2}{y^2}}=\operatorname{arctg}\frac{x}{y}+x\cdot\frac{\left(\frac{x}{y}\right)'}{1+\left(\frac{x}{y}\right)^2}=\operatorname{arctg}\frac{x}{y}+\frac{x\cdot\frac{1}{y}}{\frac{y^2}{1}+\frac{x^2}{y^2}}=$

$=\operatorname{arctg}\frac{x}{y}+\frac{x}{y}\cdot\frac{y^2}{y^2+x^2}=\operatorname{arctg}\frac{x}{y}+\frac{xy}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \cancel{x\,\text{arctg}\,\frac{x}{y}} + x\cdot\frac{\left(\frac{x}{y}\right)'}{1+\left(\frac{x}{y}\right)^2} = \cancel{x\,\text{arctg}\,\frac{x}{y}} + \frac{x\cdot\frac{-x}{y^2}}{\frac{y^2+x^2}{y^2}} = \cancel{x\,\text{arctg}\,\frac{x}{y}} + \ldots$$

$$= \cancel{x\,\text{arctg}\,\frac{x}{y}} - \frac{x^2}{y^2+x^2}$$

$$df = \left(\text{arctg}\,\frac{x}{y} + \frac{xy}{x^2+y^2}\right)dx - \frac{x^2}{x^2+y^2}\,dy$$

$$\text{grad}_{x_0} f = \frac{\partial f}{\partial x}(x_0)\,\bar{i} + \frac{\partial f}{\partial y}(x_0)\,\bar{j} = \left(-\frac{\pi}{4}-\frac{1}{2}\right)\bar{i} - \frac{1}{2}\bar{j}$$

b) $f(x,y,z) = yz\ln y - xy\ln z$

$$\frac{\partial f}{\partial x}(2,1,3) = \lim_{x\to 2}\frac{f(x,1,3)-f(2,1,3)}{x-2} = \lim_{x\to 2}\frac{-x\ln 3 + 2\ln 3}{x-2} = \lim_{x\to 2}\frac{-(x-2)\ln 3}{x-2}$$

$$= -\ln 3$$

$$\frac{\partial f}{\partial y}(2,1,3) = \lim_{y\to 1}\frac{f(2,y,3)-f(2,1,3)}{y-1} = \lim_{y\to 1}\frac{3y\ln y - 2y\ln 3 + 2\ln 3}{y-1} \overset{l'H}{\underset{\frac{0}{0}}{=}}$$

$$= \lim_{y\to 1}\frac{3\ln y + 3 - 2\ln 3}{1} = 3 - 2\ln 3$$

$$\frac{\partial f}{\partial z}(2,1,3) = \lim_{z\to 3}\frac{f(2,1,z)-f(2,1,3)}{z-3} = \lim_{z\to 3}\frac{-2\ln z + 2\ln 3}{z-3} \overset{\frac{0}{0}}{=} \lim_{z\to 3} -\frac{2}{z} = -\frac{2}{3}$$

$$\frac{\partial f}{\partial h}(2,1,3) = \frac{\partial f}{\partial x}(2,1,3)\cdot 3 + \frac{\partial f}{\partial y}(2,1,3)\cdot(-2) + \frac{\partial f}{\partial z}(2,1,3)\cdot 1 =$$

$$= -\ln 3\cdot 3 - 2\cdot(3-2\ln 3) - \frac{2}{3} = -3\ln 3 - 6 + 4\ln 3 - \frac{2}{3} = \ln 3 - \frac{20}{3}$$

$$d_{x_0} f = \frac{\partial f}{\partial x}(x_0)\,dx + \frac{\partial f}{\partial y}(x_0)\,dy + \frac{\partial f}{\partial y}(x_0)\,dz = -\ln 3\,dx + (3-2\ln 3)\,dy - \frac{2}{3}\,dz$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\frac{\partial f}{\partial x} = -y\ln z$$

$$\frac{\partial f}{\partial y} = z\ln y + z - x\ln z = z + \ln\frac{y^z}{z^x}$$

$$\frac{\partial f}{\partial z} = y\ln y - \frac{xy}{z}$$

$$df = (-y\ln z)\,dx + \left(z + \ln\frac{y^z}{z^x}\right)dy + \left(y\ln y - \frac{xy}{z}\right)dz$$

$$\text{grad}_{x_0} f = \frac{\partial f}{\partial x}(x_0)\,\bar{i} + \frac{\partial f}{\partial y}(x_0)\,\bar{j} + \frac{\partial f}{\partial z}(x_0)\,\bar{k} = -\ln 3\,\bar{i} + (3-2\ln 3)\,\bar{j} - \frac{2}{3}\bar{k}$$

fct vect $F : D \subset \mathbb{R}^3 \to \mathbb{R}^2$

$F(x,y,z) = \frac{y}{2z} 2^{xy}, \frac{z+2x}{y^2+3xz}$, $x_0 = (-1,1,2)$ $J_F(x_0)$

$$\begin{cases} f_1(x,y,z) = \frac{y}{2z} \cdot 2^{xy} \\ f_2(x,y,z) = \frac{z+2x}{y^2+3xz} \end{cases}$$

$$JF(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x_0) & \frac{\partial f_1}{\partial y}(x_0) & \frac{\partial f_1}{\partial z}(x_0) \\ \frac{\partial f_2}{\partial x}(x_0) & \frac{\partial f_2}{\partial y}(x_0) & \frac{\partial f_2}{\partial z}(x_0) \end{pmatrix}$$

$$\cancel{\frac{\partial f_1}{\partial x}(-1,1,2) = \lim_{x \to -1} \frac{f_1(x,1,2) - f_1(-1,1,2)}{x+1}} = \lim_{x \to -1} \frac{\frac{1}{4} \cdot 2^x - \frac{1}{4} \cdot 2^{-2}}{x+1} =$$

$$\cancel{= \lim_{x \to -1} \frac{2^{x-2} - 2^{-4}}{x+1}}$$

$$(2^a)' = \ln 2 \cdot 2^a$$

$$\frac{\partial f_1}{\partial x} = \left(\frac{y}{2z} \cdot 2^{xy}\right)'_x = \frac{y}{2z} \cdot \ln 2 \cdot 2^{xy} \cdot (xy)' = \frac{y}{2z} \cdot \ln 2 \cdot 2^{xy} \cdot y$$

$$\frac{\partial f_1}{\partial x}(-1,1,2) = \frac{1}{4} \cdot \ln 2 \cdot 2^{-1} \cdot 1 = \frac{1}{8} \ln 2$$

$$\frac{\partial f_1}{\partial y} = \left(\frac{y}{2z} \cdot 2^{xy}\right)'_y = \frac{1}{2z} \cdot 2^{xy} + \frac{y}{2z} \cdot \ln 2 \cdot 2^{xy} \cdot x$$

$$\frac{\partial f_1}{\partial y}(-1,1,2) = \frac{1}{4} \cdot 2^{-1} + \frac{1}{4} \cdot \ln 2 \cdot 2^{-1}(-1) = \frac{1}{8} - \frac{\ln 2}{8} = \frac{1}{8}(1 - \ln 2)$$

$$\frac{\partial f_1}{\partial z} = \left(\frac{y}{2z} \cdot 2^{xy}\right)'_z = -\frac{y}{2z^2} \cdot 2^{xy}$$

$$\frac{\partial f_1}{\partial z}(-1,1,2) = -\frac{1}{8} \cdot 2^{-1} = -\frac{1}{16}$$

$$\frac{\partial f_2}{\partial x} = \left(\frac{z+2x}{y^2+3xz}\right)'_x = \frac{2(y^2+3xz) - (z+2x)(3z)}{(y^2+3xz)^2}$$

$$\frac{\partial f_2}{\partial x}(-1,1,2) = \frac{2(1-3\cdot 2) - (2-2)}{(1-3\cdot 2)^2} = -\frac{2}{5}$$

$$\frac{\partial f_2}{\partial y} = \left(\frac{z+2x}{y^2+3xz}\right)'_y = \frac{-(z+2x)(y^2+3xz)'}{(y^2+3xz)^2} = \frac{-2y(z+2x)}{(y^2+3xz)^2}$$

$$\frac{\partial f_2}{\partial y}(-1,1,2) = \frac{-2(2-2)}{\dots} = 0$$

$$\frac{\partial f_2}{\partial z} = \left(\frac{z+2x}{y^2+3xz}\right)'_z = \frac{(y^2+3xz)+(z+2x)\cdot 3x}{(y^2+3xz)^2}$$

grad – c

$\left.\begin{array}{l}\text{div}\\ \text{rot}\end{array}\right\}$ c vect

$$\frac{\partial f_2}{\partial z}(-1,1,2) = \frac{(1-6)+(2-2)}{(1-6)^2} = -\frac{5}{-5^2} = -\frac{1}{5}$$

$$J\bar{F}(x_0) = \begin{pmatrix} \frac{1}{8}\ln 2 & \frac{1}{8}(1-\ln 2) & -\frac{1}{16} \\ -\frac{2}{5} & 0 & -\frac{1}{5} \end{pmatrix}$$

Se cons. câmpul – scalar $\boxed{f(x,y,z) = xyz}$

– vectorial $\boxed{\bar{v}(x,y,z) = \underset{P}{\frac{y}{z}}\bar{i} + \underset{Q}{\frac{z}{x}}\bar{j} + \underset{R}{\frac{x}{y}}\bar{k}}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ v. de poz

$$\text{grad } f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k} = yz\bar{i} + xz\bar{j} + xy\bar{k}$$

$$\text{div } \bar{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 \Rightarrow \text{câmp } \underline{\text{solenoidal}}$$

$$\text{rot } \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{z} & \frac{z}{x} & \frac{x}{y} \end{vmatrix} = \frac{\partial}{\partial y}\frac{x}{y}\bar{i} + \frac{\partial}{\partial x}\frac{z}{x}\bar{k} + \frac{\partial}{\partial z}\frac{y}{z}\bar{j} - \frac{\partial}{\partial y}\frac{y}{z}\bar{k} - \frac{\partial}{\partial z}\frac{z}{x}\bar{i} - \frac{\partial}{\partial x}\frac{x}{y}\bar{j} =$$

$$= -\frac{x}{y^2}\bar{i} - \frac{z}{x^2}\bar{k} - \frac{y}{z^2}\bar{j} - \frac{1}{z}\bar{k} - \frac{1}{x}\bar{i} - \frac{1}{y}\bar{j} =$$

$$= \bar{i}\left(-\frac{x}{y^2} - \frac{1}{x}\right) - \bar{j}\left(\frac{y}{z^2} + \frac{1}{y}\right) + \bar{k}\left(-\frac{z}{x^2} - \frac{1}{z}\right)$$

$$\bar{v}\cdot\bar{r} = \frac{xy}{z} + \frac{yz}{x} + \frac{xz}{y}$$

$$\bar{v}\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{y}{z} & \frac{z}{x} & \frac{x}{y} \\ x & y & z \end{vmatrix} = \left(\frac{z^2}{x} - x\right)\bar{i} + \left(\frac{x^2}{y} - y\right)\bar{j} + \left(\frac{y^2}{z} - z\right)\bar{k}$$

$$\nabla(\bar{v}\cdot\bar{r}) = \left(\frac{y}{z} - \frac{yz}{x^2} + \frac{z}{y}\right)\bar{i} + \left(\frac{x}{z} + \frac{z}{x} - \frac{xz}{y^2}\right)\bar{j} + \left(-\frac{xy}{z^2} + \frac{y}{x} + \frac{x}{y}\right)\bar{k}$$

$$\nabla(\bar{v}\times\bar{r}) = -\frac{z^2}{x^2} - 1 - \frac{x^2}{y^2} - 1 - \frac{y^2}{z^2} - 1 = -3 - \left(\frac{z^2}{x^2} - \frac{x^2}{y^2} - \frac{y^2}{z^2}\right)$$

$$\nabla(\bar{v}\times\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z^2}{x}-x & \frac{x^2}{y}-y & \frac{y^2}{z}-z \end{vmatrix} = \frac{2y}{z}\bar{i} + \frac{2x}{y}\bar{k} + \frac{2z}{x}\bar{j} - \bar{k}\cdot 0 - \bar{i}\cdot 0 - \bar{j}\cdot 0 = \frac{2y}{z}\bar{i} + \frac{2x}{y}\bar{k} + \frac{2z}{x}\bar{j}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$(\sin a)' = \cos a$$

$f(x,y,z) = x^2 \sin\frac{y}{z}$, $z \neq 0$, $X_0 = (1, \pi, 2)$

a) $d_{x_0}^2 f$ b) $H_f(x_0)$

a) $$d_{x_0}^2 f = \frac{\partial^2 f}{\partial x^2}(x_0)dx^2 + \frac{\partial^2 f}{\partial y^2}(x_0)dy^2 + \frac{\partial^2 f}{\partial z^2}(x_0)dz^2 + \frac{2\partial^2 f}{\partial x\partial y}(x_0)dxdy + \frac{2\partial^2 f}{\partial y\partial z}(x_0)dydz + \frac{2\partial^2 f}{\partial z\partial x}(x_0)dzdx$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\sin\frac{y}{z}\right) = 2\sin\frac{y}{z} \Rightarrow \frac{\partial^2 f}{\partial x^2}(x_0) = 2\sin\frac{\pi}{2} = 2$$

$$\frac{\partial f}{\partial x} = \left(x^2\sin\frac{y}{z}\right)'_x = 2x\sin\frac{y}{z}$$

$$\frac{\partial^2 f}{\partial y^2}(x_0) = -\frac{1}{4}\cdot\sin\frac{\pi}{2} = -\frac{1}{4}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(x^2\cos\frac{y}{z}\cdot\frac{1}{z}\right)'_y = \frac{x^2}{z^2}\cdot\left(-\sin\frac{y}{z}\right)$$

$X_0 = (1, \pi, 2)$

$$\frac{\partial^2 f}{\partial z^2}(x_0) = \underbrace{\frac{2\cdot 1\cdot\pi}{8}\cos\frac{\pi}{2}}_{0} - \frac{\pi^2}{16}\sin\frac{\pi}{2} = -\frac{\pi^2}{16}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \left(-\frac{x^2y}{z^2}\cos\frac{y}{z}\right)'_z = \frac{2x^2y}{z^3}\cos\frac{y}{z} - \frac{x^2y}{z^2}\sin\frac{y}{z}\cdot\frac{y}{z^2}$$

$$\frac{\partial^2 f}{\partial x\partial y}(x_0) = \cos\frac{\pi}{2} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \left(\frac{x^2}{z}\cos\frac{y}{z}\right)'_x = \frac{2x}{z}\cos\frac{y}{z}$$

$$\frac{\partial^2 f}{\partial y\partial z}(x_0) = -\frac{1}{4}\cdot 0 + \frac{\pi}{8}\sin\frac{\pi}{2} = \frac{\pi}{8}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \left(-\frac{yx^2}{z^2}\cos\frac{y}{z}\right)'_y = \frac{-x^2\cdot z^2}{z^4}\cos\frac{y}{z} + \frac{yx^2}{z^3}\sin\frac{y}{z}$$

$$\frac{\partial^2 f}{\partial z\partial x}(x_0) = -\frac{2\pi}{4}\cos\frac{\pi}{2} = 0$$

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \left(2x\sin\frac{y}{z}\right)'_z = -\frac{2xy}{z^2}\cos\frac{y}{z}$$

$$d_{x_0}^2 f = 2dx^2 - \frac{1}{4}dy^2 - \frac{\pi^2}{16}dz^2 + \frac{\pi}{4}dy\,dz$$

$$H_f(x_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{\pi}{8} \\ 0 & \frac{\pi}{8} & -\frac{\pi^2}{16} \end{pmatrix}$$

$T_2(x,y,z) =$ formula ...

$$f(x,y) = (x+2y^2)\cdot e^{2x-y}$$

$$\frac{\partial^{56} f}{\partial x^{13} \partial y^{43}} = \frac{\partial^{13}}{\partial x^{13}}\left(\frac{\partial^{43} f}{\partial y^{43}}\right) \quad (*)$$

Leibniz

$$\boxed{(u\cdot v)^{(n)} = u^{(n)}\cdot v + C_n^1 u^{(n-1)} v' + \ldots + C_n^{n-1}\cdot u'\cdot v^{(n-1)} + C_n^n \cdot u \cdot v^{(n)}}$$

Fie $u(y) = x + 2y^2$

$v(y) = e^{2x-y}$

$$(u\cdot v)^{(43)} = \underbrace{u^{(43)}\cdot v + C_{43}^1 \cdot u^{(42)}\cdot v' + \ldots}_{0} + C_{43}^{41}\cdot u''\cdot v^{(41)} + C_{43}^{42}\cdot u'\cdot v^{(42)} + C_{43}^{43}\cdot u \cdot v^{(43)}$$

$u'(y) = 4y$

$u''(y) = 4$

$u^{(3)}(y) = 0$

...

$u^{(43)}(y) = 0$

$v'(y) = -e^{2x-y}$

$v''(y) = e^{2x-y}$

....

$v^{(43)}(y) = -e^{2x-y}$

$$\begin{array}{r} 21 \\ 43 \\ \hline 63 \\ 84 \\ \hline 903 \cdot \\ 4 \\ \hline 3612 \end{array}$$

$$\Rightarrow (u\cdot v)^{43} = \frac{43!}{2!\cdot 41!}\cdot 4\cdot(-e^{2x-y}) + 43\cdot 4y\cdot e^{2x-y} + (x+2y^2)\cdot(-e^{2x-y})$$

$$= e^{2x-y}(-x - 2y^2 + 172y - 3612)$$

$$(*) \Rightarrow \frac{\partial^{13}(g)}{\partial x^{13}} = (**)$$

Fie $u(x) = e^{2x-y}$

$v(x) = -x-2y^2+172y-3612$

$$(u\cdot v)^{(13)} = u^{(13)}\cdot v + C_{13}^{1}\cdot u^{(12)}\cdot v' + \underbrace{\ldots + C_{13}^{12}\cdot u'\cdot v^{(12)} + C_{13}^{13}\cdot u\cdot v^{(13)}}_{0}$$

$u'(x) = 2e^{2x-y}$ $\qquad$ $v'(x) = -1$

$u''(x) = 4e^{2x-y}$ $\qquad$ $v''(x) = 0$

$\ldots$ $\qquad$ $\ldots$

$u^{(13)}(x) = 2^{13}\cdot e^{2x-y}$ $\qquad$ $v^{(13)}(x) = 0$

$$(u\cdot v)^{(13)} = 2^{13}\cdot e^{2x-y}\cdot(-x-2y^2+172y-3612) - 13\cdot 2^{12}\cdot e^{2x-y} =$$

$$= 2^{12}e^{2x-y}(-2x-4y^2+344y-7224-13) \quad (**)$$

$f(x,y) = \sqrt[13]{x^2y^{24}+y^{17}x^9}$

Arătați $\underbrace{\frac{\partial^2 f}{\partial x^2}(1,1) + 2\frac{\partial^2 f}{\partial x\partial y}(1,1) + \frac{\partial^2 f}{\partial y^2}(1,1)}_{(*)} = 2\sqrt[13]{2}$

$f(tx,ty) = \sqrt[13]{t^{26}x^2y^{24}+t^{26}y^{17}x^9} = t^2 f(x,y) \Rightarrow f$ omog în sens Euler

$$\overset{f}{\underset{\text{Euler}}{\Rightarrow}} x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = \underline{n(n-1)}\cdot f(x,y) \Rightarrow \boxed{n=2}$$

$$\Rightarrow (*) = 2(2-1)\cdot f(1,1) = 2\cdot\sqrt[13]{1\cdot 1+1\cdot 1} = 2\sqrt[13]{2}, \text{ adev.}$$

eterm pct de extr local

$f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}, \quad x,y,z \neq 0$

$$\begin{cases} \frac{\partial x}{\partial f} = 0 \\ \frac{\partial y}{\partial f} = 0 \\ \frac{\partial z}{\partial f} = 0 \end{cases} \Rightarrow \begin{cases} \overset{4x^2}{1} - \frac{y^2}{4x^2} = 0 & \frac{4x^2 - y^2}{4x^2} = 0 \Rightarrow y^2 = 4x^2 \,/\sqrt{\ } \\ & y = 2x \\ \frac{2y}{4x} - \frac{z^2}{y^2} = 0 & \Leftarrow \;\Rightarrow \frac{4x}{4x} = \frac{z^2}{y^2} \Rightarrow z = y \\ \frac{2z}{y} - \frac{2}{z^2} = 0 \Rightarrow & \frac{2z}{z} - \frac{2}{z^2} = 0 \Rightarrow 1 = \frac{1}{z^2} \Rightarrow z = \pm 1 \end{cases}$$

Pt $z = y = 1 \Rightarrow \frac{2}{4x} - 1 = 0 \Rightarrow x = \frac{1}{2}$ $\qquad M_1 = \left(\frac{1}{2}, 1, 1\right)$

Pt $z = y = -1 \Rightarrow -\frac{2}{4x} - 1 = 0 \Rightarrow x = -\frac{1}{2}$ $\qquad M_2\left(-\frac{1}{2}, -1, -1\right)$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(1 - \frac{y^2}{4x^2}\right)'_x = +\frac{2y^2}{4x^3}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(\frac{y}{2x} - \frac{z^2}{y^2}\right)'_y = \frac{2x}{4x^2} + \frac{2z^2}{y^3}$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \left(\frac{2z}{y} - \frac{2}{z^2}\right)'_z = \frac{2}{y} + \frac{4}{z^3}$

$$H = \begin{pmatrix} \frac{2y^2}{4x^3} & -\frac{2y}{4x^2} & 0 \\ -\frac{2y}{4x^2} & \frac{1}{2x} + \frac{2z^2}{y^3} & -\frac{2z}{y^2} \\ 0 & -\frac{2z}{y^2} & \frac{2}{y} + \frac{4}{z^3} \end{pmatrix}$$

$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 0$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\frac{2y}{4x^2}$

$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = -\frac{2z}{y^2}$

$$Hf\left(\tfrac{1}{2}, 1, 1\right) = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix} \qquad \begin{matrix} \Delta_1 > 0 \\ \Delta_2 > 0 \\ \Delta_3 > 0 \end{matrix} \Rightarrow \left(\tfrac{1}{2}, 1, 1\right) \text{ pct min local}$$

$Hf\left(-\frac{1}{2}, -1, -1\right) = \ldots$

1) $\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{t}} dt = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$x^2 = t$

$t^5 \cdot t^{-\frac{1}{2}}$

$x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}} dt$

$x = 0, t = 0$

$x = \infty, t = \infty$

2) $\int_0^\infty x^{10} \cdot e^{-x^2+3} dx = e^3 \int_0^\infty (\sqrt{t})^{10} \cdot e^{-t^2} \frac{dt}{2\sqrt{t}} = \frac{e^3}{2}\int_0^\infty t^{\frac{9}{2}} e^{-t^2} dt =$

$x^2 = t$

$= \frac{e^3}{2}\Gamma\left(\frac{11}{2}\right) = \frac{e^3}{2}\Gamma\left(5+\frac{1}{2}\right) =$

$x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}} dt$

$= \frac{e^3}{2} \frac{9!!}{2^5}\sqrt{\pi}$

3) $\int_0^1 \sqrt{x^{15} - x^{16}} dx = \int_0^1 \sqrt{x^{15}(1-x)} dx = \int_0^1 x^{\frac{15}{2}} \cdot (1-x)^{\frac{1}{2}} dx = \beta\left(\frac{17}{2}, \frac{3}{2}\right)$

$\Rightarrow \beta\left(\frac{17}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{17}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{17}{2}+\frac{3}{2}\right)} = \frac{\Gamma\left(8+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right)}{\Gamma(10)} = \frac{15!!\sqrt{\pi}}{2^8} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{9!} =$

$= \frac{15!!\,\pi}{2^9 \cdot 9!}$

4) $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx \cdot \cos^{10} x\,dx$

$= \frac{1}{2}\beta\left(3, \frac{11}{2}\right) = \frac{1}{2}\frac{\Gamma(3)\cdot\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(3+\frac{11}{2}\right)} = \frac{1}{2}\frac{2!\cdot\Gamma\left(5+\frac{1}{2}\right)}{\Gamma\left(8+\frac{1}{2}\right)} =$

$\frac{2}{3} + \frac{11}{2} = \frac{1}{\cdot}$

$8 + \frac{1}{2}$

$= \frac{9!!\cdot\cancel{\sqrt{\pi}}}{2^5} \cdot \frac{2^{\cancel{8}\,3}}{15!!\cancel{\sqrt{\pi}}} = \frac{8\cdot 9!!}{15!!} = \frac{8}{11\cdot 13\cdot 15} = 2145$

Calc., folosind fct. Γ:

1) $\int_0^\infty x^3 e^{-x}\,dx = I_1$

2) $\int_0^\infty \sqrt{x}\, e^{-x}\,dx = I_2$

3) $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

4) $\int_1^\infty x\sqrt[3]{x^2-1}\cdot e^{-x^2}\,dx$

5) $\int_{-\infty}^0 x^4\cdot e^x\,dx$

1) $I_1 = \Gamma(4) = ((4-1)!) = 6$

2) $I_2 = \int_0^\infty x^{\frac{1}{2}} e^{-x}\,dx = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

3) $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \int_0^\infty x^{-\frac{1}{2}}\cdot e^{-x}\,dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4) $\int_1^\infty x\sqrt[3]{x^2-1}\cdot e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty \sqrt[3]{t}\cdot e^{-t-1}\,dt = \frac{1}{2e}\int_0^\infty t^{\frac{1}{3}} e^{-t}\,dt =$

$x^2-1=t$

$x^2=t+1$

$2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$

$x=1, t=0$

$= \frac{1}{2e}\Gamma\left(\frac{1}{3}+1\right) = \frac{1}{6e}\Gamma\left(\frac{1}{3}\right)$

5) $\int_{-\infty}^0 x^4\cdot e^x\,dx$

$x \to -x \Rightarrow \int_0^\infty x^4 e^{-x}\,dx = \Gamma(5) = 4! = 24$

b) pct extrem local

$f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$, $x, y, z \neq 0$

$$\begin{cases} \frac{\partial f}{\partial x} = 1 - \frac{y^2}{4x^2} \\ \frac{\partial f}{\partial y} = \frac{2y}{4x} - \frac{z^2}{y^2} \\ \frac{\partial f}{\partial z} = \frac{2z}{y} - \frac{2}{z^2} \end{cases} \Rightarrow \begin{cases} 1 - \frac{y^2}{4x^2} = 0 \\ \frac{y}{2x} - \frac{z^2}{y^2} = 0 \\ \frac{2z}{y} - \frac{2}{z^2} = 0 \end{cases}$$

$y^2 = 4x^2 \,|\sqrt{\ } \Rightarrow y = 2x$

$\frac{2x}{2x} - \frac{z^2}{y^2} = 0 \Rightarrow z^2 = y^2 \,|\sqrt{\ }$
$\Rightarrow z = y = 2x$

$y/\ \frac{2y}{y} - \frac{2}{y^2} = 0$

$2y^2 - 2 = 0$
$2y^2 = 2 \,|:2$
$\Rightarrow y = \pm 1$

Pt $\begin{cases} y = 1 \quad z = 1 \quad x = \frac{1}{2} \\ y = -1 \quad z = -1 \quad x = -\frac{1}{2} \end{cases}$

$M\left(\frac{1}{2}, 1, 1\right)$ $\qquad M\left(-\frac{1}{2}, -1, -1\right)$

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{2y^2}{4x^3} \\ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{1}{2x} + \frac{2z^2}{y^3} \\ \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{2}{y} + \frac{4}{z^3}. \\ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\frac{y}{2x^2} \\ \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 0 \\ \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = -\frac{2z}{y^2} \end{cases}$$

$$H = \begin{pmatrix} \frac{y^2}{2x^3} & -\frac{y}{2x^2} & 0 \\ -\frac{y}{2x^2} & \frac{1}{2x} + \frac{2z^2}{y^3} & -\frac{2z}{y^2} \\ 0 & -\frac{2z}{y^2} & \frac{2}{y} + \frac{4}{z^3} \end{pmatrix}$$

$$H_f\left(\frac{1}{2}, 1, 1\right) = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix} \qquad H_f\left(-\frac{1}{2}, -1, -1\right) = \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -6 \end{pmatrix}$$

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 16 > 0$

$\Delta_3 = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{vmatrix} = 72 + 0 + 0 - 0 - 16 - 24 > 0$

$\Rightarrow \left(\frac{1}{2}, 1, 1\right)$ pct min

$\Delta_1 = -4 < 0$

$\Delta_2 = 8 > 0$

$\Delta_3 = -72 + 16 + 24 < 0$

$\Rightarrow \left(-\frac{1}{2}, -1, -1\right)$ pct max local

Examen Analiză ①

I. a) serie Fourier de sinusuri fct periodică

$T = 2\pi$

$f(x) = x^2, \quad 0 \le x \le \pi$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega x) + b_n \sin(n\omega x))$

$\omega = \frac{2\pi}{T} = 1$

Serie de sin $\Rightarrow a_n = 0$

$a_0 = 0$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$b_n = \frac{2}{T} \int_0^T f(x) \sin(nx)\,dx = \frac{1}{\pi} \int_0^{2\pi} (x+1) \sin(nx)\,dx$

$= \frac{1}{\pi} \int_0^{2\pi} x \sin(nx)\,dx + \frac{1}{\pi} \int_0^{2\pi} \sin(nx)\,dx$

$\frac{1}{\pi} \int_0^{2\pi} x \sin(nx)\,dx$

$= \frac{1}{\pi} \int_0^{2\pi} x \cdot \left(-\frac{\cos nx}{n}\right)' dx$

$= \frac{1}{\pi} \left(-x \cdot \frac{\cos nx}{n} \Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos nx}{n}\,dx \right)$

$= -\frac{1}{\pi} \cdot \frac{2\pi}{n} \cdot 1 + \frac{1}{\pi n} \cdot \frac{\sin nx}{n} \Big|_0^{2\pi}$

$= -\frac{2}{n} + \cancel{\frac{1}{\pi n}} \cdot 0$

$\frac{1}{\pi} \int_0^{2\pi} \sin(nx)\,dx$

$= -\frac{1}{\pi} \frac{\cos nx}{n} \Big|_0^{2\pi}$

$= \cancel{-\frac{1}{\pi} \frac{1}{n} + \frac{1}{\pi \cdot n}}$

$= 0$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} -\frac{2}{n} \sin(nx)$

$$f(x,y) = g\left(x^2y^2, \frac{y}{x}\right), \qquad g \in C^2(D),\ D \subset \mathbb{R}^2$$

Fie $u(x,y) = x^2y^2$

$v(x,y) = \frac{y}{x}$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} \end{aligned}} \begin{aligned} &= 2xy^2 \frac{\partial g}{\partial u} - \frac{y}{x^2} \frac{\partial g}{\partial v} \\ &= 2x^2y \frac{\partial g}{\partial u} + \frac{1}{x} \frac{\partial g}{\partial v} \end{aligned}$$

Considerăm operatorii de derivare:

$$\delta_x \square = 2xy^2 \frac{\partial \square}{\partial u} - \frac{y}{x^2} \frac{\partial \square}{\partial v}$$

$$\delta_y \square = 2x^2y \frac{\partial \square}{\partial u} + \frac{1}{x} \frac{\partial \square}{\partial v}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2x^2y \frac{\partial g}{\partial u} + \frac{1}{x} \frac{\partial g}{\partial v}\right) =$$

$$= 2x^2 \frac{\partial g}{\partial u} + 2x^2y\, \delta_y\left(\frac{\partial g}{\partial u}\right) + \frac{1}{x} \delta_y\left(\frac{\partial g}{\partial v}\right)$$

$$= 2x^2 \frac{\partial g}{\partial u} + 2x^2y\left[2x^2y \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right) + \frac{1}{x} \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial u}\right)\right] + \frac{1}{x}\left[2x^2y \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v}\right) + \right.$$

$$\left. + \frac{1}{x} \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right)\right]$$

$$= 2x^2 \frac{\partial g}{\partial u} + 4x^4y^2 \frac{\partial^2 g}{\partial u^2} + \underbrace{2xy \frac{\partial^2 g}{\partial v \partial u} + 2xy \frac{\partial^2 g}{\partial u \partial v}} + \frac{1}{x^2} \frac{\partial^2 g}{\partial v^2}$$

$$= 2x^2 \frac{\partial g}{\partial u} + 4x^4y^2 \frac{\partial^2 g}{\partial u^2} + 4xy \frac{\partial^2 g}{\partial u \partial v} + \frac{1}{x^2} \frac{\partial^2 g}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy^2 \frac{\partial g}{\partial u} - \frac{y}{x^2} \frac{\partial g}{\partial v}\right) =$$

$$= 2y^2 \frac{\partial g}{\partial u} + 2xy^2\, \delta_x\left(\frac{\partial g}{\partial u}\right) + \frac{2y}{x^3} \frac{\partial g}{\partial v} - \frac{y}{x^2} \delta_x\left(\frac{\partial g}{\partial v}\right) =$$

$$= 2y^2 \frac{\partial g}{\partial u} + 2xy^2\left[2x^2 \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right) - \frac{y}{x^2} \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u}\right)\right] + \frac{2y}{x^3} \frac{\partial g}{\partial v} - \frac{y}{x^2}\left[2xy^2 \frac{\partial}{\partial y} \cdot \right.$$

$$\left. \cdot \left(\frac{\partial g}{\partial v}\right) - \frac{y}{x^2} \cdot \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right)\right] =$$

$$= 2y^2 \frac{\partial g}{\partial u} + 2x^3y^2 \frac{\partial^2 g}{\partial u^2} - \frac{2y^3}{x} \frac{\partial^2 y}{\partial u \partial v} + \frac{2y}{x^3} \frac{\partial g}{\partial u} - \frac{2y^3}{x} \frac{\partial^2 g}{\partial u \partial v} + \frac{y^2}{x^4} \frac{\partial^2 g}{\partial v^2}$$