Revenue Implications of Single Software Subscriptions

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Abstract

This thesis investigates the revenue potential of different pricing strategies for a single software distribution. The game-theoretic model presented is tailor-made to analyze software revenue in the domain of consumer software such as video games. A publisher is maximizing his expected revenue in a system with many different user types, each of them responding optimally to the publishers' pricing strategy. The publisher has three choices, to offer perpetual licenses only, subscription licenses only, or both. To attract more users, the publisher can adjust prices over time. The users' equilibrium strategies for this Markov Decision Process are found through backward induction. The publisher's revenue is searched via differential evolution. Numerical analysis finds that although many publishers in practice only provide perpetual licenses, by offering a subscription option they can increase revenue significantly. Revenue can be raised again if both options are offered in parallel. In this last setting, the majority of revenue is attributed to selling subscription licenses.

Keywords: Revenue maximization · Pricing · Consumer Software · Subscription · Price Discrimination · Backward Induction · Monopoly · Video game revenue

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1 Introduction

Two independent Swiss studies from 2021 show that more than 2 out of 5 Swiss people are playing video games more than once a week. The first study finds that 44% of all Swiss men and woman are playing video games regularly and extensively.[1] The second study concludes that more than 41% of the Swiss people are gaming more than once a week, a plus of almost 8% compared to the same survey in 2019. Despite an international boost in the gaming market due to the COVID-19 pandemic, the growth in Switzerland is considered to be sustainable and independent of the pandemic. (Hüttermann, 2021)

Globally, the video game industry is worth billions of dollars. In 2020, total revenue of the industry was over 150 billion U.S. dollars.[2] The huge economic possibilities for both selling in-game advertisements and licenses for the games itself are clearly recognisable. The majority of revenue goes to free-to-play (also called freemium) games via advertisement earnings. Nevertheless, also premium titles achieved a total revenue of almost 25 billion U.S. dollars with a rising trend.[3] [4] This thesis focuses on the revenue maximization of premium titles where the software publisher is selling licenses. These licences can be sold as perpetual or subscription licenses.

Perpetual licenses make the user an owner of the product for theoretically on infinite (unless the software is shut down). Subscribing works differently and permits the user to access the product for a finite amount of time until the subscription price has to be paid again if the user wants to have access for another time period. In contrast to a perpetual license where every upgrade has to be bought separately, subscription usually allows the user to always access the newest version of a product during the subscription period.

The model developed in this thesis will focus on the revenue maximization of consumer software such as but not restricted to video games. Consumer software means the direct selling of software to end users. This thesis focuses on products which experience a quality loss. That means, their attractiveness diminishes after the release over time if there are no upgrades or network effects available that keep the product current. This applies well to single video games and less to streaming services such as Spotify or Netflix. In practice, these software providers do not offer perpetual licenses since their upgrades are delivered continuously in the form of new content. The subscription model is therefore well accepted and best practice for content providers in the streaming sector. The situation for video games is fundamentally different where a publisher can rationally think about sales through perpetual licenses, subscription licenses or both at the same time.

Advantages of subscription licenses are a continuous revenue stream for the publisher and the lower barriers of entry for a single user. Obviously, for users interested in the product for a long time, revenue for the publisher can be far higher than the one-time payment of a perpetual license. If subscription and perpetual licenses are offered in parallel, there is the risk of market cannibalization between the two license models. Moreover, users only interested in the product for a short time probably stop subscribing quickly when the product is less novel and therefore, revenue gained via subscription will be lower than selling a perpetual license. Since subscription users always expect to access the newest version of a product, no direct additional revenue for upgrades is gained. (Dierks & Seuken, 2020)

In the sector of premium video games, perpetual licenses are still much more common than subscriptions. E.g. *Steam* is a big digital video game distributor with over 120 million active players per month in 2020.[5] On *Steam*, perpetual licenses are available for all products. Since 2013, a subscription option exists if the publisher wishes to offer this option.[6] *Steam* does not deliver statistics about how many games are offering subscription licenses. A not representative review of the 10 top selling games on August 11, 2021, showed that not a single one of these products was available through subscription.[7]

The model described in this thesis allows the comparison of revenue potential of perpetual licences, subscription licenses and offering both at the same time. Cloud-based synergies and network effects are excluded from the analysis in order to investigate the direct revenue potential of the product itself. In the model presented, a single software publisher (a monopolist) is publishing a product in a system with many different user types as potential customers. In the following, we refer to the publisher as *she* and to the user as *he*. We model the problem as a two-step game where a publisher first defines her pricing strategy and all corresponding prices for a discrete number of timesteps. In the second step, all user types respond optimally, i.e., maximize their welfare in respect of the given prices. This leads us to the first main research question of the thesis:

Research Question 1: *How can we effectively find optimal user strategies for any given publisher strategy?*

Since the user has full information about all future prices and his personal type characteristics, there is an optimal behaviour for every single situation at every timestep. These situations are called user states. The optimal behaviour is summarized in the so called optimal user strategy. The setting presented is a typical Markov Decision Process (MDP) where the user's welfare depends on a sequence of decisions which we call actions. The optimal solution to the MDP for every user type is found by an algorithm based on backward induction, a simple dynamic programming method. Starting at the last timestep and iterating back timestep by timestep, the optimal action for every possible user state can be determined. To make the algorithm tractable and as fast as possible, I exclude unreachable states and actions that can never be optimal from the set of possible states and actions respectively. Given the optimal user strategies, I investigate the second question.

Research Question 2: *Which type of license maximizes publisher revenue?*

I compare the revenue of the different licensing types, each optimally priced. The different types are: Offering only perpetual licenses, only subscription licenses or both in parallel. To find the optimal pricing for each license type, I am following the approach used by Dierks and Seuken (2020) in their paper *Revenue Maximization for Consumer Software: Subscription or Perpetual License?* To find the optimal prices, I perform an iterative search called differential evolution which is applicable for non-continuous problems. That is the case in this setting since I have to define a discrete number of different user valuations such that backward induction can calculate the optimal user strategies. The drawback of differential evolution is that it cannot guarantee to find the global optimum solution.

2 Literature Review

This section summarizes the related published work and at the same time positions the model introduced later together with the fact that – best of my knowledge – besides Dierks and Seuken (2020) no directly comparable research exists. My work is mainly related to two literature streams: Firstly, the revenue of video games, and secondly, the revenue of different pricing types with a focus on subscription licenses.

Many of the problems studied in earlier research have apparent similarities with my model, but differ from it in at least one of the following points: 1) The product type for which revenue shall be maximized. 2) The different pricing strategies investigated and their revenue impact. 3) Basic assumptions which have to be made to define the shape of the model. In this overview, I therefore collect the research structured according to these similarities but also identify the respective differences to understand the relationship to my research.

2.1 Video game revenue

Considering the topic of product type domain, this thesis develops and investigates an especially suitable model to investigate revenue for consumer software, i.e., selling software directly to customers (B2C) instead of software sold to businesses (B2B, e.g., enterprise software). Examples for consumer software are video games, news pages, messaging apps or entertainment apps such as music players or video streaming services. (Spacey, 2017). There has been recent research about revenue maximization focused on video games. Chen, Elmachtoub, Hamilton, and Lei (2020) showed how to maximize revenue from so called microtransactions, where real money is spent for virtual items to be used in games. For both free-to-play and premium games, these transactions are often handled via so called loot boxes, randomly allocating the virtual items and only revealing them after the microtransaction price has been paid. Burns, Roseboom, and Ross (2016) developed a model maximizing in-game advertisement revenue of freemium mobile games.

These types of investigated revenue opportunities clearly differ from my focus of research. I therefore refer to other product types to find earlier research about the revenue maximization for the pricing strategies explored in this thesis.

2.2 Subscription licenses

Already in 1998, Choudhary et al. analyzed the economics of a monopoly firm selling and renting software in a two-period game to users who differ in their reservation price, i.e., the maximum they are willing to pay. Some important foundations still hold for my model: Firstly, it is assumed that consumer software is a durable good, i.e., a good that is consumed over a long period of time. Secondly, price discrimination for the same product over different timesteps can increase revenue for the publisher. And, thirdly, users facing a multiple timestep game can delay their purchase (or rent) in expectation or knowledge of price drops in later periods.

In the model of Choudhary et al. (1998), in the first period, the base product is sold or rented at a lower price. In the second period, renters loose their use rights and an upgraded version of the product is sold and rented at the same price since no later timestep exists where renters could loose their use rights. The quality of the upgrade is affected by positive network externalities, i.e., the utility that a user's experience grows with an increasing number of agents using the product. (Katz & Shapiro, 1985) In this case, network externality is modeled through the quality increase of the upgraded product via bug reports or new features added due to feedback from users during the first period. The paper finds that in this setting, offering both a selling and renting option leads to higher revenue than only offering the selling option.

Later research has mostly developed related models to Choudhary et al. (1998). The similarities to my research are - among others - the monopoly setting and obviously the impact of a subscription option. Importantly, the models often investigated only two timesteps with network externalities present. To model the practice more realistically, often new variables have been added.

Zhang and Seidmann (2003) added a quality uncertainty to another two-period game with a monopolist and network externalities. The quality uncertainty of the product in the second timestep is modeled via a quality variance since users do not know how much research and development expenses are spent by the publisher. The author compares perpetual licenses to subscription licenses and a hybrid approach offering both. Offering the subscription option revenue dominates the perpetual option in this setting. With a strong network effect present, the hybrid approach generates a surplus in revenue compared to the pure subscription option. (Zhang & Seidmann, 2010)

To just name a few others, Jiang, Chen, and Mukhopadhyay (2007) introduced other factors like a piracy rate since piracy (i.e., copying and forbidden distribution of software) was a major problem for perpetually licensed software. Gilbert, Randhawa, and Sun (2014) differentiated users by their frequency of so called instances of need. The authors find that offering both perpetual and subscription licenses is often revenue optimal in this case. Balasubramanian, Bhattacharya, and Krishnan (2015) compared the revenue potential of subscription and perpetual licenses in a monopoly and duopoly setting.

In the domain of professional Software as a Service (SaaS), related research has been done. Normally, the setting there differs heavily from my assumptions. Subscription (which is often called SaaS in that context) does not stand for a rent for a defined time span but the scalable use of the software. I.e., a company as customer can either buy the perpetual license for unrestricted use of the product or can subscribe and pay dependent on how heavily the software is being used. The subscription price is then set for one unit of use instead of a timestep as in my case. (Rohitratana & Altmann, 2012; Sato & Nakashima, 2020) Therefore, perpetual license and subscription license have heterogeneous qualities. The same holds for packaged B2B software such as enterprise resource planning (ERP) or human resource management (HRM) software. Besides the subscription and perpetual option also the option of not consuming the software but developing the products in-house is researched by Postmus, Wijngaard, and Wortmann (2009). The research by Choudhary (2007) is again closer to my model environment: The author models subscription (SaaS) with homogeneous software quality under perpetual license and subscription license. I.e., subscription is no longer defined as scalable but allows for the unlimited use of the most recent version of the software. For a monopoly publisher who can decide on investments for quality increase in future timesteps, it is found that offering the subscription option leads to higher investments in software quality than the perpetual licensing.

Subscription was also researched in other areas such as ancillary services. These services are kind of an *extra* (can also be seen as upgrade) to an already owned or rented base product, e.g. a user renting a virtual machine from Amazon can also utilize its machine learning algorithms for an additional payment. (Wang, Dada, & Sahin, 2019) Moreover, Prasad, Mahajan, and Bronnenberg (2003) theoretically derived the optimal subscription price for websites such as online newspapers when there is a free usage of the product with (more) advertisement as a second option besides subscription for the customers.

The models presented in this review are diverse, nevertheless the overview shows that offering subscription licenses often acts as revenue booster.

2.3 Predecessor model

The model my thesis is based on has been defined by Dierks and Seuken (2020). The model omits network externalities but increases complexity by looking at user and publisher behaviour for an extended range of timesteps (instead of only two timesteps). Moreover, user types are not just defined by their valuation but by three more parameters presented in Section 3.2 addressing the user model. There is a base product offered over the whole lifetime of the product and an upgrade which is mostly offered during a later timestep. Four prices are defined as follows: A price for the base product before the upgrade is released, a price for the upgrade and a price for the base product from the timestep the upgrade is released. The subscription price is stable over all timesteps. The authors find that offering both license types is always strictly dominating offering only subscription or perpetual licenses. In most cases, offering only subscription licenses leads to higher revenue than only offering perpetual licenses.

I will refer to the model from Dierks and Seuken (2020) as *initial model* from now on while the model elaborated in this thesis is called *extended model*. There is no directly comparable predecessor model found in literature besides the initial model.

The model presented in this thesis will extend the initial model in the following way: Perpetual prices for the base price and upgrade can be set individually in each timestep. Normally, the subscription price is being held constant over the product lifetime. However, in section 5.2.3 I exhibit results if subscription price is also variable. It is the goal of the thesis to evaluate if relations between the different license types found for the initial model also hold for the extended model setting.

Thanks to rising computing power in the last decades, we can investigate revenue in a setting with an extended number of pricing variables, probably impossible in 1998 when Choudhary et al. (1998) presented their model over only two timesteps. With this thesis, I am therefore closing two major gaps in the literature about subscription licenses and revenue potential of video games: Firstly, I investigate an extended number of timesteps, and secondly, I allow variable pricing in each timestep.

3 Model

As in Dierks and Seuken (2020), I model the problem as a two stage game. In the first stage, the publisher commits to a pricing strategy. For every timestep between 1 and n_{max}, the publisher can set an individual price for the base product and the upgrade. For $n > n_{max}$, the prices are similar to the price in timestep n_{max}.

In the second stage of the game, over n_{max} timesteps users arrive to the system. The users have perfect information about the pricing strategy chosen by the publisher. For example, if a user arrives in timestep 3, he is in full knowledge of all prices for every n $\geq$ 3. After a user has arrived to the system, he faces a game with multiple timesteps. In the following, I model this as an infinite time horizon Markov Decision Process (MDP) where the decision maker – in this case the user – can choose any action available at the current state he is in. The user gains his reward depending on the actions he chooses and the states he is in. Note that once a state is reached, the previous history is not important any more which is called Markov property, a foundation of the definition of an MDP. (Russell & Norvig, 1995)

3.1 Publisher Model

The publisher is selling a digital product and wants to maximize her revenue. Table 1 delivers an overview about all variables defining the publisher model.

A base product is offered during all timesteps, while an optional upgrade to the product is offered from timestep m onward. The base product has quality q_b, the upgrade has quality q_u. Of course, quality can not just be set by the publisher but is also influenced by the perception of the users. Since I don't investigate the maximization of revenue against competitors but just for the product itself, only the two qualities in direct comparison are important.

Dierks and Seuken (2020) assume that the publisher has infinite supply and no marginal costs due to the digital nature of the good. This assumption is also maintained. Initial expenses for the production of the base product and the upgrade are omitted as well.

This thesis extends the initial model from Dierks and Seuken (2020) and allows the publisher to change the price of perpetual licenses in any time step. Her strategy space is defined through the price vector $p = (p_t,\, p_b,\, p_u,\, p_s)$. p_t defines the possible price strategy types the publisher can offer to the users. If $p_t =$ Buy, only perpetual prices are offered. If $p_t =$ Sub, only a subscription price is offered. Consequently, if $p_t =$ Both, perpetual and subscription prices are offered. The variable prices for the base product and the upgrade are defined through the price vectors $p_b = (p_{b_1}, p_{b_2}, ..., p_{b_{n_{max}}})$ and $p_u = (p_{u_m}, p_{u_{m+1}}, ..., p_{u_{n_{max}}})$ respectively. The constant subscription price is defined by p_s.

Once bought, the product and the upgrade can be used by the user for an infinite time horizon without any further payments. The upgrade can only be bought if the user already owns the base product or buys it during the same timestep. This is in clear contrast to subscription which gives access to all available upgrades immediately but expires if the subscription price is not paid any more. In the base case of the model, the subscription price is constant during all timesteps. See section 5.2.3 for results of a further model variation where the subscription price may sink in every timestep.

This thesis studies the maximization of the publishers utility which is defined as the expected revenue per user (see section 4.2 about Publisher Revenue for the details).

Variable		**Explanation**
$p = (p_t, p_b, p_u, p_s)$		price vector
	$p_t \in \{\text{Buy, Sub, Both}\}$	price strategy type
	$p_b = (p_{b_1}, p_{b_2}, ..., p_{b_{n_{max}}})$	base product price vector
	$p_u = (p_{u_m}, p_{u_{m+1}}, ..., p_{u_{n_{max}}})$	upgrade price vector
	p_s	subscription price
$q = (q_b, q_u)$		quality vector
	q_b	quality base product
	q_u	quality upgrade
m		timestep upgrade release

Table 1: Variables in the publisher model

3.2 User Model

Every user is described by it's type and state. The state is defined as tuple $\sigma = (d, o)$. When a user arrives to the system, he is interested in gaining access to the product, i.e., he has demand. The demand $d \in \{0, 1\}$ therefore is defined as Boolean variable, denoting if the user has interest to use the product. The variable is necessary since a user can loose and regain demand, trying to model a user in real life. $o \in \{0, 1\}^2$ denotes the ownership of the base product and the upgrade as Boolean variables. E.g., a user with $o_1 = 1$ and $o_2 = 0$ owns the base product but not the upgrade.

Four variables define a user's type, denoted in the tuple $\tau = (n_a, \delta, \gamma, v)$. The timestep when a user arrives to the system is denoted by $n_a \in \{1, ..., n_{max}\}$. The arrival time defines the first timestep the user can take an action, i.e., subscribing or buying the product. E.g., a user with $n_a = 1$ is in the system throughout the whole lifetime of the product. n_a is drawn from a distribution with the probability mass function f_a. $\delta \in (0, 1)$ and $\gamma \in (0, 1)$ denote the engagement factor and the quality decay factor and are drawn from the probability mass functions f_δ and f_γ respectively. The (long term) engagement factor δ is defined as the probability of not losing demand in the next timestep. In every timestep a user who uses the product has the probability $1-\delta$ that he looses demand, i.e., d is set from 1 to 0 and his utility in the future therefore will be 0. Note that when a user is not using the product, i.e., he doesn't subscribe and has not bought the product yet, this probability is 0. Once a user has lost demand, he won't regain demand except from the timestep of the upgrade release. When the upgrade is released, every user who lost demand has the one time complementary probability δ that he is interested again and d is set from 0 to 1. The quality decay factor γ on the other hand describes how fast the excellence of a product decreases according to the user, independently of the user's arrival time. This variable therefore describes the overall attractiveness of the product in the user's eyes since with time, curiosity for and freshness of the product fall continuously. The *realized quality* of a product is given by

$$\theta(o, \gamma, n, q) = o_1 (\sum_{i \in o} \gamma^{n-M_i} q_i) \tag{1}$$

where tuple $M = (1, m)$ is used to define the release timesteps, i.e., the base product is released in the first timestep and the upgrade in timestep m. Further, the sum above is multiplied by o_1 because ownership of the upgrade alone leads to a realized quality of 0. Lastly, $v \in [0, v_{max}]$ is drawn from a distribution with probability density function f_v. While having demand, v denotes the value a user has for a product of quality 1.

In any timestep, a user has to make three decisions, collected in the user's action $a = (s, b)$ where $s \in \{0, 1\}$ denotes if he chooses to subscribe and $b \in \{0, 1\}^2$ denotes if he buys the base product and upgrade respectively. Subscription gives immediate access to the newest version of the product during the corresponding timestep, i.e., $o_s^n = [1, 0]$ if $n < m$ and $o_s^n = [1, 1]$ otherwise. Buying gives everlasting ownership of the bought product(s), i.e. changes the ownership vector in every later timestep from o to $o' = max(o, b)$.

Variable		Explanation
$\tau = (n_a, \delta, \gamma, v)$		user type
	$n_a \in \{1, ..., n_{max}\}$	timestep arrival
	$\delta \in (0, 1)$	engagement factor
	$\gamma \in (0, 1)$	quality decay factor
	$v \in [0, v_{max}]$	user value
$\sigma = (d, o)$		user state
	$d \in \{0, 1\}$	demand
	$o \in \{0, 1\}^2$	ownership
$a = (s, b)$		user action
	$s \in \{0, 1\}$	subscription action
	$b \in \{0, 1\}^2$	buy action

Table 2: Variables in the user model

Some more intermediate calculations and definitions lead us to the final formula for the *expected utility* of a user:

The *normalized immediate reward* w_n of a user with type τ in timestep n and state σ paying action a is given by

$$w_n(a, \tau, \sigma) = d((1 - s)\theta(max(o, b), \gamma, n, q) + s\theta(o_s^n, \gamma, n, q)). \tag{2}$$

.

His *immediate payment* in timestep n is given by

$$\rho_n(a, p) = p_s s + p_{b_n} b_1 + p_{u_n} b_2. \tag{3}$$

His overall *immediate utility* in timestep n is therefore given by

$$u_n(a, \tau, \sigma, p) = v w_n(a, \tau, \sigma) - \rho_n(a, p). \tag{4}$$

A strategy $\alpha(n, \tau) : \mathbb{N} \times \{0, 1\} \times \{0, 1\}^2 \to \{0, 1\} \times \{0, 1\}^2$ maps timesteps and user states to actions. The *normalized expected reward* for playing strategy α is given by

$$w(\alpha, \tau) = \sum_{n=n_a}^{\infty} \sum_{\sigma'} P(\sigma_n = \sigma' | \alpha, \tau, \sigma) w_n(a, \tau, \sigma), \tag{5}$$

where $P(\sigma_n = \sigma' | \alpha, \tau, \sigma)$ stands for the probability of the user being in state σ' during timestep n given α, τ, σ. Similarly, the *expected payment* is given by

$$\rho(\alpha, \tau, p) = \sum_{n=n_a}^{\infty} \sum_{\sigma'} P(\sigma_n = \sigma' | \alpha, \tau, \sigma) \rho_n(a, p). \tag{6}$$

The user's overall *expected utility* with strategy α is consequently given by

$$u(\alpha, \tau, p) = vw(\alpha, \tau) - \rho(\alpha, \tau, p). \tag{7}$$

4 Equilibrium Strategies

In this section, I want to find the optimal strategies for the publisher and the users in the model environment. It is important to note that the users act independently of each other and therefore only react to the publishers pricing strategy. Moreover, due to the simplicity of the model, I am also neglecting other offers or publishers, making it relatively easy to deterministically calculate the optimal user behaviour through backward induction. However, the publisher has to consider many user types as possible customers when he sets the prices for the product. This is highly complex for two or more user types already and therefore this section only gives the formula for the publisher revenue which is later maximized via differential evolution.

4.1 User Equilibrium Strategies

All users have full information about the prices set by the publisher. For any given user type τ it is therefore possible to exactly calculate his equilibrium strategy, i.e., maximizing his expected utility through the optimal strategy.

In the initial model proposed by Dierks and Seuken (2020), the authors proved that the optimal strategies for each user type can only come from a small set of possible strategies, namely two potentially optimal strategies before timestep m and three potentially optimal strategies for $n \geq m$. Over a continuous space of user types it was therefore possible to calculate optimal user strategies. This is a striking difference from the extended model where the equilibrium strategy for every user type has to be found individually and the set of potentially optimal strategies cannot be restricted.

However, since I do not model user interaction, I can solve the MDP for any user type in isolation through backward induction. Russell and Norvig (1995) proposed this relatively simple dynamic programming algorithm for solving MDPs, at that time under the name *Value Iteration*. Instead of calculating the expected utility for every possible strategy α, backward induction uses the Markov property (memoryless property) as an abbreviation for its solution path with large runtime improvements. Since the past does not matter when a state is reached, there is exactly one optimal action for every state. See also algorithm 1 to note that the best action is the action maximizing the sum of the immediate utility with the action plus the expected utilities of the states in the next timestep, weighted with its probabilities reached with the best action. (Russell & Norvig, 1995) It is very unlikely that two actions are exactly equally good, i.e., lead to the same expected utility. Should this nevertheless happen, I define that buying or subscribing is preferred to not buying or not subscribing. Moreover, buying is preferred to subscribing.

In the following, I present the preliminaries used to conduct the backward induction algorithm applied to my model, displayed in Algorithm 1.

To find the best action, for every timestep and for every state, I iterate over the set of possible actions A. A is independent of the state and timestep but depends on the publishers pricing strategy. The three boolean variables $s \in \{0, 1\}$ and $b \in \{0, 1\}^2$ theoretically allow eight different actions. For runtime improvements of the algorithm, I can exclude impossible or actions played with probability 0 for all user types.

For p_t = Buy, the four actions where $s = 1$ can be removed since subscription is not possible. For p_t = Sub, I can exclude the six actions where $b_1 = 1$ or $b_2 = 1$ for the same reason. For p_t = Both, $a = (1, [1, 1])$ can be removed from the set of possible actions since it cannot be optimal for any user to subscribe and buy the base product and upgrade at the same time. This leaves us with seven possible actions for p_t = Both, four actions for p_t = Buy and two actions for p_t = Sub which results in a reduction of the necessary computing power.

S defines the set of reachable user states. $d \in \{0, 1\}$ and $o \in \{0, 1\}^2$ define eight possible user states. I remove states which are reached with probability 0. It is not possible to own the upgrade without owning or buying the base product and therefore I can remove the two states where $o_1 = 0$ and $o_2 = 1$. For $p_t \in$ {Buy, Both}, the six remaining states define S. For p_t = Sub, S can be reduced to the two states where $o_1 = 0$ and $o_2 = 0$ since ownership is not possible when only the subscription option is offered.

Lastly, I define the Probability Transition Matrix between states as P. As an important part of the MDP definition, these transitions are partly influenced by the action taken by the user and partly random. The ownership vector of the state is deterministic and changes if $b_1 = 1$ or $b_2 = 1$. On the other hand, the change of d is random and depends on the engagement factor δ. The transition probabilities are implicitly stated in Section 3.2 about the User Model.

While the backward induction is similar for all other timesteps, states in n_{max} have to be treated specially. Russell and Norvig (1995) called these states terminal states. Since n_{max} is the last timestep I am modeling, the utility, reward and payment calculations for this timestep have to include all timesteps until infinity. In my model, in timesteps with $n > n_{max}$, the prices are the same as in n_{max}. In the following, I show how the action decision in timestep n_{max} implicitly defines the optimal actions until $d = 0$. I distinguish two main cases for the actions in timestep n_{max}, namely $s = 0$ and $s = 1$. For any action with $s = 0$:

Lemma 1 *Case 1: If a user type $\tau = (n_a, \delta, \gamma, v)$ buys the base product or the upgrade in timestep n_{max} or later, the optimal strategy is to buy the product(s) in timestep n_{max}.*

Proof. The payments in timesteps $n > n_{max}$ don't decrease, whereas the realized quality of the product(s) do, modeled through the quality decay factor γ. Therefore, if the user waits with buying the product, his reward is lower which cannot be optimal and he will therefore buy in timestep n_{max} already. ∎

For any action with $s = 1$:

Lemma 2 *Case 2. If max(b, o) = [0, 0], a user type τ will subscribe in timesteps $n > n_{max}$ as long as his immediate utility is non-negative, i.e., $u_n(a, \tau, \sigma, p) \geq 0$.*
Case 3. If max(b, o) = [1, 0], a user type τ will subscribe in timesteps $n > n_{max}$ as long as his immediate reward for the upgrade is larger or equal to the subscription price.
Case 4. If max(b, o) = [1, 1], a user type τ won't subscribe.

Proof. Case 2. When subscribing without ownership of any of the product, the user obtains immediate utility $v\theta([1,1], \gamma, n, q) - p_s$ when the base product and the upgrade are released which is the case for all timesteps $n \geq n_{max}$. This utility decreases in n. It follows directly that as long this immediate utility is positive, it's optimal to subscribe for the user.
Case 3. Subscription increases reward for the user by the reward for the upgrade whereas the reward for the base product comes through the ownership of the base product without subscription. Therefore, as long as the immediate reward $v\gamma^{n-M_2}q_u$ for the upgrade is bigger or equal to p_s, the user can increase his utility and $s = 1$ dominates $s = 0$.
Case 4. This follows directly from the fact that through subscription, no additional reward can be gained if the user has ownership of both base product and upgrade but has to pay p_s. The action is strictly dominated by the corresponding action with $s = 0$. ∎

This leads to the following utility and payment calculations for the cases presented above. For Case 1 (with $s = 0$), the utility for the action played in timestep n_{max} itself can be calculated as usual. For all $n > n_{max}$, no further payments will be due since all products have been bought already. Therefore, the immediate reward for the timesteps until infinity can be calculated as geometric series taking into account the probability of loosing demand (engagement factor δ) and the quality decay factor γ. The reward for the timestep n_{max} has to be substracted again since it is already included in the utility for the timestep n_{max} itself.

$$u^{\infty}_{n_{max}}(a, \tau, \sigma_n, p|s = 0) = u_n(a, \tau, \sigma, p) + \frac{vw_n(a, \tau, \sigma)}{(1 - \delta\gamma)} - vw_n(a, \tau, \sigma) \tag{8}$$

Since there are no further payments after n_{max}, it holds that $p^{\infty}_{n_{max}} = p_n(a, p)$ as usual.

If $s = 1$, for Case 2 with max(b, o) = [0, 0], (i.e., the action played must be $a = [1, [0, 0]]$), I define the last timestep the user still subscribes if he has demand as $n_s^{last_1}$. I.e., $n_s^{last_1}$ is the largest $n \geq n_{max}$ where it holds that:

$$p_s \leq v\theta([1,1], \gamma, n, q). \tag{9}$$

Therefore, the expected utility for timestep n_{max} can be calculated as:

$$u^{\infty}_{n_{max}}(a, \tau, \sigma_n, p | s = 1, max(b, o) = [0, 0]) = \sum_{n=n_{max}}^{n_s^{last_1}} (u_n(a, \tau, \sigma, p))\delta^{n-n_{max}} \tag{10}$$

The according immediate payment therefore is:

$$\rho^{\infty}_{n_{max}}(a, p | s = 1, max(b, o) = [0, 0]) = \sum_{n=n_{max}}^{n_s^{last_1}} (p_s s)\delta^{n-n_{max}}. \tag{11}$$

For Case 3 with max(b, o) = [1, 0], there exists $n_s^{last_2}$, defined as the last timestep the user still subscribes if he has demand. I.e., the immediate reward for the upgrade itself is higher than the subscription price, hence $n_s^{last_2}$ is the largest $n \geq n_{max}$ where it holds that:

$$p_s \leq v\gamma^{n-M_2} q_u. \tag{12}$$

The formula for the immediate utility for timestep n_{max} is therefore given by the geometric series for the utility for the ownership of the base product and the utility for the upgrade thanks to subscriptions:

$$\begin{aligned} &u^{\infty}_{n_{max}}(a, \tau, \sigma_n, p | s = 1, max(b, o) = [1, 0]) = \\ &\quad \frac{vw_n(a, \tau, \sigma)}{(1 - \delta\gamma)} + \sum_{n=n_{max}}^{n_s^{last_2}} (v\gamma^{n-M_2} q_u - p_s)\delta^{n-n_{max}} \end{aligned} \tag{13}$$

The according expected payment therefore is:

$$\rho^{\infty}_{n_{max}}(a, p | s = 1, max(b, o) = [1, 0]) = \sum_{n=n_{max}}^{n_s^{last_2}} (p_s s)\delta^{n-n_{max}}. \tag{14}$$

Lastly, for Case 4 with max(b, 0) = [1, 1], the utility for the user can be calculated similarly to Equation 8 where $s = 0$. Clearly, the utility in timestep n_{max} for the case with $s = 1$ will be lower since the subscription price has to be paid. As noted above, this can never be optimal, therefore the action will never be played and no formula for the expected payment is necessary.

$$\begin{aligned} &u^{\infty}_{n_{max}}(a, \tau, \sigma_n, p | s = 1, max(b, o) = [1, 1]) = \\ &\quad u^{\infty}_{n_{max}}(a, \tau, \sigma_n, p | s = 0) - p_s \end{aligned} \tag{15}$$

Combining the four cases leads to:

$$
\begin{aligned}
&u^{\infty}_{n_{max}}(a,\tau,\sigma_n,p) = \\
&\qquad (1-s)(u^{\infty}_{n_{max}}(a,\tau,\sigma_n,p|s=0)) \\
&\qquad + s(u^{\infty}_{n_{max}}(a,\tau,\sigma_n,p|s=1,max(b,o)=[0,0]) \\
&\qquad + u^{\infty}_{n_{max}}(a,\tau,\sigma_n,p|s=1,max(b,o)=[1,0]) \\
&\qquad + u^{\infty}_{n_{max}}(a,\tau,\sigma_n,p|s=1,max(b,o)=[1,1]))
\end{aligned} \tag{16}
$$

Starting with the terminal states at n_{max}, backward induction until $n = 1$ is performed. Merging everything presented in this section until now leads to Algorithm 1. The output of this algorithm is the optimal user strategy α^*, containing the optimal action for every reachable state. I did not introduce a notation for the expected utility yet. The expected utility is denoted as $u^*_n(\tau,\sigma_n,p)$ (not to be confused with the immediate utility denoted as $(u_n(a,\tau,\sigma_n,p))$) and stands for the total utility a user can expect until infinity when he is playing the best action at every state, starting from the current state.

Algorithm 1: Get optimal user strategy

Input: σ, user type
p, price vector from publisher
A, set of available user actions
S, set of available user states
P, probability transition function between user states
Output: α^*, optimal user strategy
u^*, optimal user utilities

$n = n_{max}$
$u^*_n(\tau,\sigma_n,p) = \max_{a\in A}\left[u^{\infty}_n(a,\tau,\sigma_n,p)\right] \forall\sigma_n$
$\alpha^*_n(\tau,\sigma_n,p) = \arg\max_{a\in A}\left[u^{\infty}_n(a,\tau,\sigma_n,p)\right] \forall\sigma_n$
$n = n-1$
while $n > 0$ **do**
    $u^*_n(\tau,\sigma_n,p) = \max_{a\in A}\left[u_n(a,\tau,\sigma_n,p) + \sum_{\sigma_{n+1}\in S} P(\sigma_{n+1}|\sigma_n,a)u^*_{n+1}(\tau,\sigma_{n+1},p)\right] \forall\sigma_n$
    $\alpha^*_n(\tau,\sigma_n,p) =$
    $\arg\max_{a\in A}\left[u_n(a,\tau,\sigma_n,p) + \sum_{\sigma_{n+1}\in S} P(\sigma_{n+1}|\sigma_n,a)u^*_{n+1}(\tau,\sigma_{n+1},p)\right] \forall\sigma_n$
    $n = n-1$
end

Taking a closer look at the algorithm shows that backward induction has to be performed separately for every user type τ with different δ, γ and v. But due to the memoryless property of the MDP, for differing n_{max} we can reuse the best actions calculated for every user state. This allows to significantly improve performance, i.e., instead of n_{max} backward inductions for ever combination of δ, γ and v, backward induction has to be performed just once. Note that these user types of course differ in their starting states $\sigma_{n_a} = [1, [0, 0]]$ (i.e., the user has demand but no ownership) which is reached with probability $P = 1$.

4.2 Publisher Revenue

Given the Users Equilibrium Strategies, the formula for the *publisher's expected revenue per user* for a strategy $p = (p_t, p_b, p_u, p_s)$ follows directly:

$$\pi(p) = \sum_{n_a} \sum_{\delta} \sum_{\gamma} \int_{v} \rho(\alpha^*(n_a, \delta, \gamma, v), p) f_a(n_a) f_\delta(\delta) f_\gamma(\gamma) f_v(v) dv \tag{17}$$

Since $f_v(v)$ has to be approximated with the help of a discrete number of user values (defined through x_u, see section 5.1), the valuations can not be found as integral but have to be given as sum as well:

$$\pi(p) = \sum_{n_a} \sum_{\delta} \sum_{\gamma} \sum_{v} \rho(\alpha^*(n_a, \delta, \gamma, v), p) f_a(n_a) f_\delta(\delta) f_\gamma(\gamma) f_v(v) \tag{18}$$

I.e., the formula weighs the expected payments for every user type acting optimally with the probability of the user type. Finding the equilibrium strategy for the publisher is more difficult than for the user. Due to the very diverse user types, the maximization problem for the publisher is highly non-convex. I do not deliver an extended theoretical analysis, instead I will perform differential evolution searches to find the best publisher strategies (see section 5 about the numerical analysis). Some starting considerations for a theoretical approach can be found in the appendix section B.

5 Numerical Analysis

After outlining the setup and describing the results, this section delineates the revenue potential of different pricing strategies for the base case, some adaptions and comparative statics. Analogously to Dierks and Seuken (2020) I call these optimal prices Opt(Buy) when $p_t = \text{Buy}$, Opt(Sub) when $p_t = \text{Sub}$, Opt(Both) when $p_t = \text{Both}$ and Opt(Both|Opt(Buy)) when $p_t = \text{Both}$ and the perpetual prices are taken from Opt(Buy). Note that even though the base case is equivalent, the results presented in the following cannot be directly compared in absolute numbers to the results presented in Dierks and Seuken (2020) since a discrete number of user valuations is investigated while Dierks and Seuken (2020) looked at a continuous space of user valuations.

5.1 Setup

In the following, the respective parameter choices are explained. Dierks and Seuken (2020) have developed a numerical base case, built on the analysis of comprehensive sales data for video games. These were accessible through *Steam Spy*[8], a platform collecting user numbers and other publicly available data from the online game store *Steam*. Defining the setup includes the various distributions that describe the totality of users and parameters chosen as base case. I explicitly retain the base case to make my results comparable to Dierks and Seuken (2020). There, reasoning for the chosen parameters can be found, while in the following, the base case is presented without deeper justification for the chosen values.

The arrival distribution f_a accounts for the fact that typically more users arrive in the first timestep than in later timesteps. x_a defines the arrivals during the first timestep compared to later timesteps, while arrivals in $n > 1$ remain constant until n_{max}. $x_a = 5$ is chosen as base case.

$$f_a(n_a) = \begin{cases} \frac{x_a}{x_a + n_{max} - 1} & \text{if } n_a = 1 \\ \frac{1}{x_a + n_{max} - 1} & \text{if } 1 < n_a \leq \text{n}_{max} \end{cases} \tag{19}$$

The distribution of quality decay factors is discrete as well, defining $\gamma \in \{0.85, 0.9, 0.95\}$, i.e., the quality of a game decays at a rate of 5%, 10% or 15%, depending on the users type. x_γ defines the probability that $\gamma = 0.9$, while $\gamma = 0.85$ and $\gamma = 0.95$ are equally common. $x_\gamma = 0.8$ is defined as base case.

$$f_\gamma(\gamma) = \begin{cases} x_\gamma & \text{if } \gamma = 0.9 \\ \frac{1 - x_\gamma}{2} & \text{if } \gamma \in \{0.85, 0.95\} \end{cases} \tag{20}$$

The engagement factors are set up as a two-type distribution, distinguishing a short-term and a long-term user where the long-term user has a lower probability of loosing demand in each timestep. I therefore define x_{δ_s} as the probability of loosing demand for the short term user and $P(x_{\delta_s})$ for the corresponding probability of the user type. x_{δ_l} stands for the probability of loosing demand for the long term user.

$$f_\delta(\delta) = \begin{cases} P(x_{\delta s}) & \text{if } \delta = \mathrm{x}_{\delta_s} \\ 1 - P(x_{\delta_s}) & \text{if } \delta = \mathrm{x}_{\delta_l} \end{cases} \tag{21}$$

For the base case it is expected that only 20% of the users are long term users, having a 90% probability to keep demand in every timestep. The remaining 80% of the users are short term users who loose demand with a probability of 50% in every timestep. In summary: $x_{\delta s} = 0.5$, $x_{\delta l} = 0.9$ and $P(x_{\delta s}) = 0.8$.

The user valuations are assumed to follow a truncated normal distribution with mean $\mu = 25$ in the interval [0, 50]. The standard deviation $\sigma = 10$ is defined as base case for f_v.

In contrast to Dierks and Seuken (2020), my simulation suite only allows the numerical analysis of a discrete number of user types. Therefore, I define x_u as the discrete number of user valuations, chosen as the means of evenly spaced intervals over the whole valuation interval. The probability of these valuations are defined as the probability of the interval, found with the help of the cumulative distribution function (CDF) of SciPy[9]. E.g., for $x_u = 2$, there are only two intervals to look at, namely [0, 25] and [25, 50]. This delivers the two user values $v_1 = 12.5$ and $v_2 = 37.5$, found as the means of the intervals. Both user types are weighted as $f_v(v_1) = f_v(v_2) = 0.5$, since their frequency of occurrence is exactly the same, probabilistically underpinned with the help of the CDF since $CDF(0) = 0$, $CDF(25) = 0.5$ and $CDF(50) = 1$. Figure 9 in section 5.2.4 about comparative statics explains, why $x_u = 10$ seems as a good approach for the continuous user valuation space. To illustrate the approach, Table 3 shows the different values v and weights $f_v(v)$ (summing up to 1 if not rounded) for the base case $x_u = 10$.

Interval	**User value (v)**	CDF(lower)	CDF(upper)	**Weight ($\mathbf{f_v(v)}$)**
[0, 5]	2.5	0	0.0167	0.0167
[5, 10]	7.5	0.0167	0.0614	0.0446
[10, 15]	12.5	0.0614	0.1544	0.093
[15, 20]	17.5	0.1544	0.3061	0.1518
[20, 25]	22.5	0.3061	0.5	0.1939
[25, 30]	27.5	0.5	0.6939	0.1939
[30, 35]	32.5	0.6939	0.8456	0.1518
[35, 40]	37.5	0.8456	0.9386	0.093
[40, 45]	42.5	0.9386	0.9833	0.0446
[45, 50]	47.5	0.9833	1	0.0167

Table 3: User values and associated weights to discretely approximate the continuous user valuation space for $x_u = 10$. All values rounded to 4 digits.

For every user type with valuation v, there exist 3 (number of different quality decay factors γ) times 2 (number of different engagement factors δ) times n_{max} (number of different arrival times n_a) versions.

For the numerical analysis, the upgrade is defined as half as valuable as the base product: $q_b = 1$, $q_u = 0.5$. Lastly, the base case investigates a setting with $n_{max} = 12$ and $m = 7$.

5.2 Results

This section presents the revenue potential for different pricing strategies of the publisher. Firstly, the base case results are outlined, followed by some model modifications and comparative statics. The results are found via differential evolution (DE) search. DE is an iterative approach which searches many different combinations of its input variables, trying to improve its solution. This genetic algorithm was invented by Storn and Price (1997) and has some positive properties which I benefit from. Since DE is simple and it is not using the gradient of a function, it can solve non-continuous problems such as the one presented in this thesis. DE is known for its robustness and good global convergence. (Das & Suganthan, 2010) In my case, the variables are all prices p_b, p_u and p_s. The number of variables is dependent on m, n_{max} and p_t. The function to be maximized is $\pi(p)$. One disadvantage of DE is the possibility of thousands of evaluations (and therefore long runtimes if a single evaluation is complex). As mentioned, note that for one evaluation (i.e., every p which is analyzed by DE), backward induction for every user type has to be performed to get the optimal strategy α^* from which $\pi(p)$ is calculated, weighting the expected payments for every user type with the probability of the user type. Therefore, reducing the runtime of a single evaluation with shortcuts presented for the backward induction algorithm is crucial.

Another disadvantage is that DE cannot guarantee finding the global optimum. For the highly non-convex problem presented, I therefore work with the best DE result for $\pi(p)$ out of 10 DE searches. Of course, this increased number of DE runs increases runtime but improves quality and comparability of the results since it is possible that DE falls into a local optimum occasionally but it's unlikely that this happens 10 times. DE in *SciPy* has many different parameters which can be set. After some experimentation, I stayed with the standard values since other parameter choices could not improve the result significantly and often led to increased runtimes. Nevertheless, in the simulation suite (see Appendix A), the parameters *popsize* and *strategy* can be controlled through the config-file. DE demands a lower and upper bound between which the iterative search is performed. I set the lower bound for all prices to 0. For all perpetual prices, the upper bound is set to 300 and for the subscription price it is set to 100. These values are chosen to improve runtime (compared to a very high upper bound such as 1000) and are high enough such that it is not optimal to buy or subscribe at this upper bound price for any user type.

All pricing vectors and more details about the results presented in the following can be found in the extended results summarized in section C of the appendix.

5.2.1 Base case

This section discusses the results for the base case defined in the setup section. Note that $x_u = 10$ is defined as base case, i.e., the publisher maximizes his revenue by playing against 10 (valuations) $\cdot$ 12 (arrival times) $\cdot$ 3 (quality decay factors) $\cdot$ 2 (engagement factors) = 720 different user types. Table 4 summarizes the optimal results for each publisher strategy. From now on, to keep it short, *Revenue* stands for the publisher's expected revenue per user $\pi(p)$ and *Welfare* stands for the user's expected utility $u(\alpha^*, \tau, p)$. *Overall Welfare* stands for the sum of *Revenue* and *Welfare*.

	Revenue	**Welfare**	**Overall Welfare**	**Revenue ($\mathbf{p_b}$)**	**Revenue ($\mathbf{p_u}$)**	**Revenue ($\mathbf{p_s}$)**
Opt(Buy)	34.4	31.53	65.93	27.76	6.63	0
Opt(Sub)	36.87	21.12	57.99	0	0	36.87
Opt(Both)	40.28	23.61	63.88	10.57	1.13	28.58
Opt(Both|Opt(Buy))	34.42	31.54	65.96	27.53	6.59	0.31

Table 4: Base case results for the four different publisher strategies p_t. Extended results in Table 10.

For p_t = Buy (i.e., the publisher only offers perpetual licences), the numerical analysis conducted shows that $\pi(p) = 34.4$. The chosen price vectors p_b and p_u show a relatively steady discount with growing n. The base price is almost halved over the entire period of time, as it falls from $p_{b_1} = 53.3$ to $p_{b_{12}} = 28.22$. The price drop for the upgrade is even more extreme, from $p_{u_7} = 15.26$ to $p_{u_{12}} = 2.17$. For this optimal setting of p, about 80% of the total revenue is gained through the base product. Doing an extended backward induction analysis over all 720 different user types shows that 69.42% of all users will buy the base product. Note that this doesn't mean that 69.42% of the 720 discrete user types will buy the base product since this would not include the probability of a user type. 69.42% though is calculated with respect to the probability of the user types. Interestingly, 100% of these users buying the base product also have an incentive to buy the upgrade, at latest in the last timestep due to the heavy price drop. Still, the revenue of the upgrade makes up only 20 per cent of the total revenue. This is 1) due to the lower prices in general and 2) due to the fact that users can loose demand before buying the upgrade while all interested users will for sure buy the base product. Also if they wait multiple timesteps after their arrival until buying, they will buy because demand can only be lost when the product is used according to my model.

For p_t = Sub, the revenue increases to $\pi(p) = 36.87$, which corresponds to a rise of 7% compared to the optimal setting with p_t = Buy. This revenue is achieved with a constant and relatively moderate subscription price of $p_s = 16.24$, attracting 77.5% of the users to subscribe at least once, compared to 69.42% of the users buying the product when p_t = Buy. This is thanks to lower entry barriers for short-term users. Nevertheless, the user welfare drops significantly by almost 50%. The additional revenue for the publisher cannot compensate for this loss of user welfare and therefore the overall welfare is also significantly lower (around 14%) than for case with p_t = Buy. The drop in welfare and increase in revenue can be explained as follows: Firstly, low-valuation long-term users (i.e., all immediate rewards lower than subscription price) never subscribe while they buy the base product even though their immediate utility is highly negative, anticipating future utilities. For middle-valuation users there is no clear pattern. But, secondly and even more importantly, high-valuation users experience less welfare but pay more, especially if they are long-term users. To illustrate the second point, I compare revenue and welfare for a user type with $n_a = 1$, $\delta = 0.9$, $\gamma = 0.9$ and $v = 27.5$. In the setting with p_t = Buy, publisher revenue is 67.84 and user welfare is 178.31. For p_t = Sub, revenue is 139.51 and welfare is 57.49. In summary, even though more users play the game for at least one timestep in p_t = Sub, the welfare decreases because all users stop playing for timesteps where the immediate reward is lower than the immediate payment. If the product can be bought such as in p_t = Buy, users will use the product significantly longer, namely as long as the immediate reward is positive regardless of the current price (since the product has already been bought in an earlier timestep).

If the publisher offers both a perpetual and a subscription price (i.e, p_t = Both), revenue rises once more, to $\pi(p) = 40.28$ for optimal price vectors. This comes with the positive side effect of an increased user welfare compared to Opt(Sub). Opt(Both) comes with very high base prices (e.g., $p_{b_1} = 288.92, p_{b_2} = 144.93, p_{b_{12}} = 67.71$) such that the publisher skims a high one-time revenue from long-term engagement users for the sale of perpetual licences. At the same time, upgrade price ($p_{u_7} = 21.65$) and subscription price ($p_s = 16.18$) stay relatively moderate. In this setting, 78% of all users play the game for at least one timestep, compared to 77.5% in Opt(Sub) and 69.42% in Opt(Buy). The overall welfare is only decreased by around 3% compared to Opt(Buy) while it was much lower in Opt(Sub), showing that most of the lost user welfare is compensated through the increase in revenue. The major part (70.95%) of the revenue is gained through the sale of subscription licences in this setting. Interestingly, the sale of both a subscription and perpetual price to the same user type is the case for 11% of all users (if the user has demand at the corresponding timesteps). This can have two reasons: 1) The user subscribes in a timestep before buying and pay the perpetual price later when price dropped. 2) The user subscribes even though the base product is already owned because the perpetual price for the upgrade is too high but the subscription price is below the immediate reward for the upgrade. In this setting, only the first case occurs, especially there are some user types subscribing in the first timestep and buying in the second timestep since the perpetual price for the base product is already halved at $n = 2$.

Finally, I analyze the introduction of a subscription price to a setting with the optimal perpetual prices from Opt(Buy), calling this Opt(Both|Opt(Buy)). This has the effect that no user is worse off since he can still pay the old perpetual prices if the newly introduced subscription price doesn't allow a higher welfare. Subscription price $p_s = 24.31$ is significantly higher than in Opt(Sub) and Opt(Both). If it was lower, too many users would deviate to subscription, lowering revenue for the publisher. In this setting, I observe a Pareto improvement since both user and publisher cannot be worse off than before. The improvement is vanishingly small, even clearly below 1% for revenue, welfare and total welfare. Therefore, it can be stated that without higher perpetual prices such as in Opt(Both), the introduction of a subscription price has not the desired revenue improvement effect.

I conclude this subsection with further takeaways from the base case. Since I assume that most publishers are experienced with the price setting of perpetual licences, I want to evaluate the identified equilibria regarding their stability, dependent on the subscription price. Namely, I investigate the risk of a subscription price that is not chosen optimally, for example due to lack of experience of the publisher. With the help of multiple backward inductions, revenue for different subscription prices is summarized in Figure 1. Obviously, changing p_s has no effect on Opt(Buy). I observe that the revenue for Opt(Sub) and

Opt(Both) lies above the one for Opt(Buy) for a fairly large range of p_s around the optimal subscription price. My main statements about revenue in this section are therefore not highly dependent on the absolutely perfect pricing of p_s but also hold if the publisher sets this price a little off. This fact reinforces the generality of my statements. On the other hand, for almost all cases of Opt(Both|Opt(Buy)), adding a subscription option with p_s results in a revenue decrease. Revenue is only equally high if p_s is high enough such that no user type subscribes which in fact corresponds to the setting of Opt(Buy). Offering a subscription option additionally to perpetual prices can therefore almost exclusively lead to a revenue increase if the perpetual prices are raised as is the case for Opt(Both).

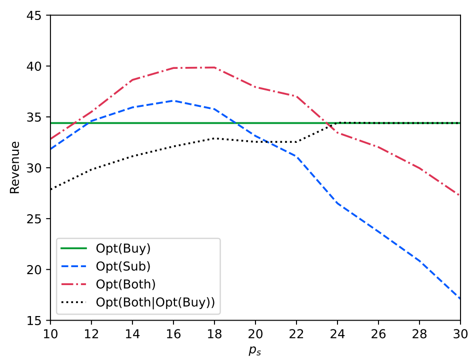

Figure 1: Revenue for different subscription prices p_s holding p_b and p_u constant

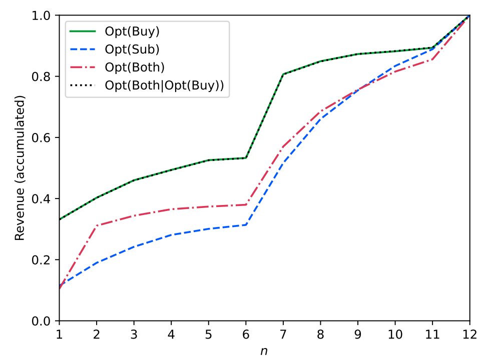

Figure 2: Accumulated revenue over time span n_{max} for optimal pricing of the four different p_t

I want to further investigate the revenue generation over time, depicted in Figure 2. (Depending on the print quality or the computer display the reader opens this thesis, possibly one of the lines of Opt(Buy) or Opt(Both|Opt(Buy)) is not visible since the lines overlap exactly.) Note that the tick up in all curves at the final timestep $n = 12$ won't be observable in practice because in the model presented, n_{max} includes all revenues up to infinity. For Opt(Buy), two main facts are observable: Although prices are further reduced later, the main revenue is generated at $n = 1$ (with the sale of the base product) and $n = 7$ (with a rise in sales of the base product and the sale of the newly introduced upgrade) where users experience the highest immediate rewards thanks to the freshness of the product and the upgrade release respectively. With decreasing prices, also part of the short-term engagement users and newly arrived users can be convinced to buy. The price decrease must not be too strong since otherwise, the high payments from timesteps $n = 1$ and $n = 7$ would cease and also long-term engagement users would wait longer until buying.

Revenues for Opt(Sub) are more evenly distributed. The flattening curve before m can be explained by 1) users loosing demand after subscribing in an earlier timestep and 2) users anticipating the upgrade release. E.g., if a user still has demand after $n = 4$, it can be optimal to not subscribe until m since he then still has demand for the product for sure. For the flattening curve after m, only argument 1) is decisive, i.e., the arriving users who start with subscription immediately cannot compensate for the user types who are loosing demand. An interesting effect can be observed for Opt(Both) where no single user buys the product in $n = 1$, explaining the similar behaviour of Opt(Sub) and Opt(Both) in the first timestep, since p_s is almost identical. Already in the second timestep, thanks to a reduced base price, Opt(Both) earns almost the total revenue until m. The very high base price in $n = 1$ can therefore be seen as a tactical instrument from the publisher. Opt(Both) then observes an up tick in revenue at m as well, followed by a flattened curve with a last uptick because perpetual prices are discounted heavily in $n = 12$. Opt(Both|Opt(Buy)) has not to be discussed further since it behaves the same as Opt(Buy) since almost no users choose the subscription option.

Comparison with Dierks and Seuken (2020)

I summarize the best results for every pricing strategy p_t in Table 5 for the model introduced by Dierks and Seuken (2020) which I am calling the *initial model*. Afterwards, I can compare the results with the *extended model* presented in this thesis. As mentioned before, the initial model differs from the extended model by only allowing four prices: a base product price $p_b^{<m}$ before the upgrade release, a base product price $p_b^{\geq m}$ after upgrade release, an upgrade price p_u and a subscription price p_s. Therefore, as soon as not only subscription licenses are offered, the extended model differs from the initial model.

	Revenue	**Welfare**	**Overall Welfare**	**Revenue ($\mathbf{p_b}$)**	**Revenue ($\mathbf{p_u}$)**	**Revenue ($\mathbf{p_s}$)**	$\mathbf{p_b^{<m}}$	$\mathbf{p_b^{\geq m}}$	$\mathbf{p_u}$	$\mathbf{p_s}$
Opt(Buy)	33.19	30.51	63.7	23.14	10.05	0	51.19	22.96	18.79	∞
Opt(Sub)	36.87	21.12	57.99	0	0	36.87	∞	∞	∞	16.24
Opt(Both)	39.88	25.2	65.08	12.82	3.88	23.18	124.46	64.89	27.53	16.24
Opt(Both|Opt(Buy))	33.2	30.51	63.71	23.14	10.05	0.01	51.19	22.96	18.79	25.45

Table 5: Base case results for the initial model presented in Dierks and Seuken (2020)

Note that these results for the base case differ in absolute numbers from the paper of Dierks and Seuken (2020) because the user valuations are discrete instead of continuous in my setting. But, importantly, the relations in revenue and welfare between the different licensing options are mostly unchanged. The exception is the revenue and welfare for Opt(Both|Opt(Buy)). In the continuous user value setting presented in Dierks and Seuken (2020), both revenue and welfare are about 1% higher than for Opt(Buy). Since revenue and welfare in Opt(Both|Opt(Buy)) are not better off than Opt(Buy) in this thesis for both the initial and extended model it can be concluded that the main reason for this effect is the discrete number of user values and not variable perpetual prices. Nevertheless, it's important to note that this finding cannot be confirmed in this thesis.

One can observe two main conspicuities when comparing Table 4 and 5: 1) As expected due to less freedom in pricing, the publisher in the initial model has lower revenue for Opt(Buy), Opt(Both) and Opt(Both|Opt(Buy)) than for the extended model presented in Table 4. 2) The relations between the different pricing strategies stay the same, i.e., offering only perpetual prices results in lower revenue than offering subscription licences or both types of licences together.

I want to further investigate point 1) by opposing the optimal prices from the initial and extended model. It is noticeable that the extended model cannot boost revenue tremendously. Revenue in Opt(Both) can only be increased by 1%. One can explain this fact by the subscription licenses which make up for the biggest part of the revenue. In Opt(Buy), the extended model with 18 price variables can increase revenue by 4% compared to the initial model with 3 price variables. Figure 3 shows the perpetual prices p_b and p_u over time.

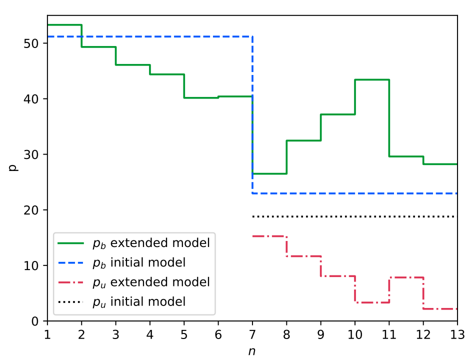

Figure 3: Comparing perpetual prices in Opt(Buy) for the initial and extended model over time.

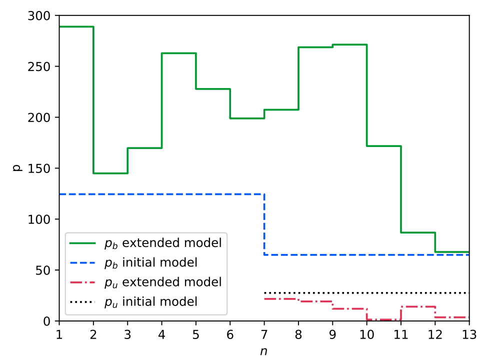

Figure 4: Comparing perpetual prices in Opt(Both) for the initial and extended model over time.

It is striking that the optimal perpetual prices are pretty close to each other for both models at timesteps 1 and m, i.e., the timesteps of price changes in the initial model. These timesteps are also responsible for the greatest part of the revenue, already seen in Figure 2. By lowering p_b steadily before m and by lowering p_u after m, 4% of additional revenue can be gained. This number is not higher due to the high non-convexity of the maximization problem, caused by the very different user types. Therefore, a price change often increases revenue for some of the users but at the same time lowers it for other user types.

5.2.2 Discounted vs. Freely Chosen Perpetual Prices

Until now, without explaining the decision in detail, it was assumed that perpetual prices of future timesteps cannot be higher than the perpetual price in the first timestep or the upgrade release for p_b and p_u respectively. In reality, perpetual prices often won't ever get higher than in the timestep of a release because this normally annoys customers and sales numbers will decrease automatically which cannot be the goal of the publisher. Actually, in the version of discounted perpetual prices, I worked with a slightly customized Publisher Model compared to section 3.1: $p_b = (p_{b_1}, p_B)$ with $p_B \in (0,1]^{n_{max}-1}$ where the requested price is p_{b_1} for the first timestep and for every $n > 1$ is calculated as $p_{b_n} = p_{b_1} p_{B_{n-1}}$. Analogously, $p_u = (p_{u_m}, p_U)$ and $p_U \in (0,1]^{n_{max}-m}$ lead to $p_{u_n} = p_{u_m}$ if $n = m$, $p_{u_n} = p_{u_m} p_{U_{n-m}}$ if $n > m$ and ∞ otherwise.

Let's investigate the option for freely chosen perpetual prices in the following. There are three meaningful reasons why a publisher could set a lower price in the first timestep than in a later one. 1) The access to the product is cheaper in the first timestep because the product is not yet fully completed but is already sold in form of a prototype version. 2) The publisher increases the base price together with an upgrade release since the quality of the whole product has been improved. 3) The publisher wants to push more users from the perpetual option towards the subscription option.

Reasons 1) and 2) have not been modeled in this thesis. Reason 3) will be investigated in the following. Table 6 displays the results for freely chosen perpetual prices.

	Revenue	**Welfare**	**Overall Welfare**	**Revenue ($\mathbf{p_b}$)**	**Revenue ($\mathbf{p_u}$)**	**Revenue ($\mathbf{p_s}$)**
Opt(Buy)	34.12	30.99	65.11	24.95	9.17	0
Opt(Sub)	36.87	21.12	57.99	0	0	36.87
Opt(Both)	40.5	23.42	63.92	8.55	3.19	28.77
Opt(Both|Opt(Buy))	34.36	31.07	65.43	24.54	9.02	0.8

Table 6: Results for freely chosen perpetual prices. Extended results in Table 11.

Even though the publisher has more freedom in choosing the perpetual prices, revenue cannot be increased compared to the base case displayed in Table 4. Even the pricing of p_b and p_u resembles the discounted case as is shown in Figure 5 where I compare the prices for Opt(Buy).

Figure 6 displays the high non-convexity of the problem where perpetual licenses prices are compared for Opt(Both) for the base case with discounted prices and the freely chosen perpetual prices. Even though the prices are rather different, Revenue, User Welfare and Overall Welfare are very similar. This is due to the highly non-convex problem: If prices are discounted at different timesteps, the publisher does not make the highest profit at the same timesteps. But this revenue shift between timesteps has no effect on the Total Welfare over all timesteps. Most importantly, it also cannot be observed that subscription licenses would generate a larger portion of revenue (Revenue(p_s)) = 28.77 for the freely chosen prices versus Revenue(p_s)) = 28.58 for the base case with discounted prices). Reason 3) explained above therefore does not apply.

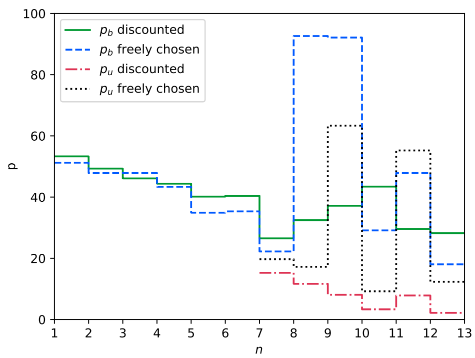

Figure 5: Comparing perpetual prices in Opt(Buy) for discounted and freely chosen prices.

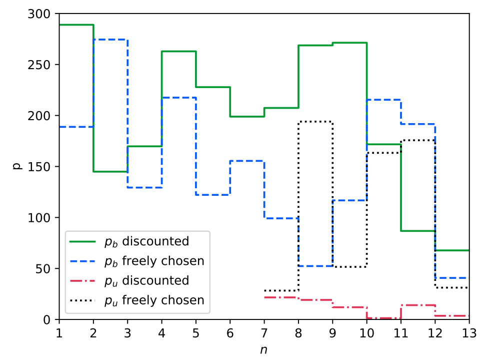

Figure 6: Comparing perpetual prices in Opt(Both) for discounted and freely chosen prices.

5.2.3 Variable Subscription Price

Another extension of the model is possible by setting a unique subscription price for every timestep. Normally, subscription licenses are renewed automatically if the user does not terminate the contract. Therefore, it should not be possible for the publisher to set a higher subscription price in any later timestep since users would be negatively surprised by a higher payment than their first contract payment. If the subscription price would increase, it theoretically is possible that personalised subscription prices are defined where users who started subscribing earlier could keep their lower subscription prices from the timestep of the contract start. This is not modeled here and therefore, subscription price must stay the same or decrease in any later timestep.

The publisher model from Section 3.1 therefore changes accordingly: The subscription price p_s is now also modeled as tuple $p_s = (p_{s_1}, p_S)$ with $p_S \in (0,1]^{n_{max}-1}$. Therefore, the subscription price for any timestep with $n > 1$ is calculated as multiplication of the subscription price in the timestep before and the according element of p_S, i.e., $p_{s_n} = p_{s_{n-1}} p_{S_{n-1}}$. In the User Model from section 3.2, the *immediate payment* is now given by:

$$\rho_n(a,p) = p_{s_n} s + p_{b_n} b_1 + p_{u_n} b_2. \tag{22}$$

The best-of-10 DE search delivers the following results:

	Revenue	Welfare	Overall Welfare	Revenue (p_b)	Revenue (p_u)	Revenue (p_s)
Opt(Buy)	34.4	31.53	65.93	27.76	6.63	0
Opt(Sub)	39.17	23.4	62.57	0	0	39.17
Opt(Both)	41.01	20.98	62	11.23	0.64	29.14
Opt(Both|Opt(Buy))	34.89	31.98	66.87	25.23	6.07	3.59

Table 7: Results for variable subscription price. Extended results in Table 12.

A variable subscription price increases revenue for the publisher again compared to the stable subscription price shown in Table 4. Opt(Sub) grows by 6%, Opt(Both) by 2% and Opt(Both|Opt(Buy)) by 1 %. Note that Opt(Both|Opt(Buy)) finally has some small positive effect on revenue and welfare compared to Opt(Buy). The revenue is achieved by more or less steadily decreasing subscription prices, starting higher than the optimal subscription price in the stable case and decreasing below for both the Opt(Buy) and Opt(Both) case (see Figures 7 and 8 respectively).

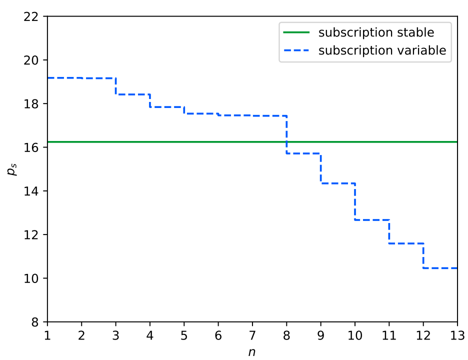

Figure 7: Comparing p_s in Opt(Buy) for stable and variable case

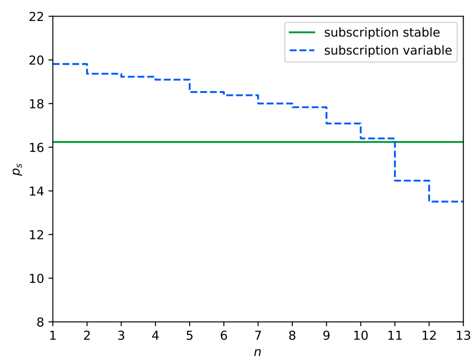

Figure 8: Comparing p_s in Opt(Both) for stable and variable case

The very moderate p_s towards the last timesteps in Opt(Buy) lets 83.5% of all users subscribe at least once while with p_s stable this number was 77.5%. In Opt(Both), 78.4% of the users in the system will play the game for at least one timestep compared to 78% with p_s stable.

5.2.4 Comparative Statics

This section examines the robustness of the results above with the help of comparative statics, motivating the topic as follows: *This is a typical problem confronting the economist: in the absence of precise quantitative data he must infer analytically the qualitative direction of movement of a complex system.* (Samuelson, 1941, p. 98) This is the case for my model and to test the stability of my results, I will vary the parameters x_u, x_{δ_s}, x_a, x_γ and σ to compare the different equilibrium states for the four pricing strategies p_t. I.e., if a small variation in a parameter value would completely change the relation between the different strategies p_t, my results would loose meaningfulness since they would be only found for a very specific parameter combination which is defined as base case. Further checks would include correlations between multiple parameters. Notwithstanding, I only focus on single parameter changes, comparing the different pricing strategies p_t, normalized to Opt(Buy). For Figures 9 to 15, the ticks on the x axis show the different values for which best-of-5 DE runs have been conducted (around 10 values per parameter). Opt(Buy) is normalized to 1 such that the other price strategy types can be analyzed in relation to Opt(Buy).

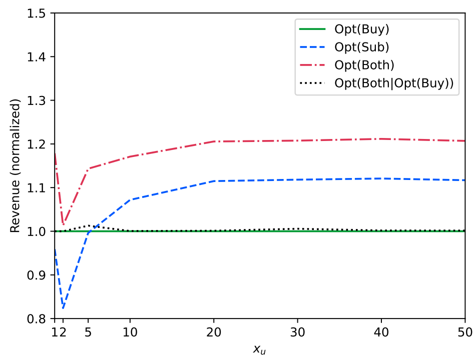

Figure 9: Revenue for different number of user valuations (varying x_u). Extended results in Table 13.

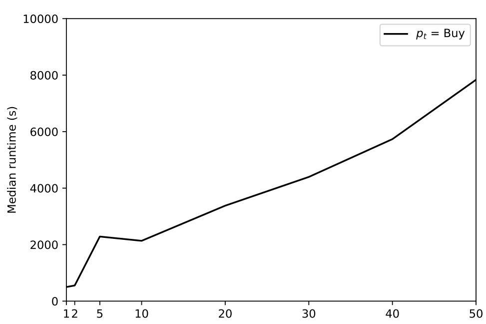

Figure 10: Median DE runtime in seconds for different number of user valuations (varying x_u)

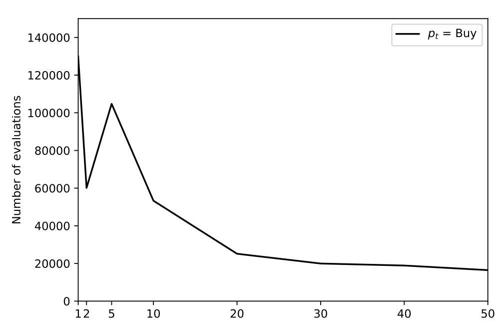

Figure 11: Median DE number of evaluations for different number of user valuations (varying x_u)

Firstly, I explain why $x_u = 10$ is a good base case of the model, already used in the whole results section. The higher the number of different valuation users (x_u), the better we can approximate the normal distribution for the valuations. Figure 9 shows that for $x_u \geq 10$, the relations between Opt(Buy), Opt(Sub), Opt(Both) and Opt(Both|Opt(Buy)) are relatively similar. This insight is useful to support $x_u = 10$ as a good choice. Figures 10 and 11 show why $x_u = 10$ is the best choice. The Figures display the median runtime and number of evaluations of DE respectively for the base case with $p_t =$ Buy, executed on a single node of the Kraken Cluster from the Departments of Informatics at the University of Zurich. $p_t =$ Buy runtimes are compared since it has the longest runtimes of all pricing types. In most cases, DE for $p_t =$ Buy takes a little longer than for $p_t =$ Both even though $p_t =$ Buy has one variable less (no subscription price). But the subscription price seems to work as a regulator for DE to find the optimum faster. Note that for growing x_u, one evaluation of DE takes longer since backward induction has to be performed more often. The runtime for one single evaluation therefore increases linearly with increasing x_u. Even if DE needs less evaluations for larger x_u until it outputs a solution (see Figure 11), it is important to compare the effective runtimes of DE until the solution is found: Figure 10 shows that DE for $x_u = 10$ is four times faster than for $x_u = 50$. With limited computational power as it was the case for my thesis, $x_u = 10$ is therefore a good choice with reasonable runtime.

Next, the comparative statics for the parameters x_{δ_s}, x_a, x_γ and σ are analyzed:

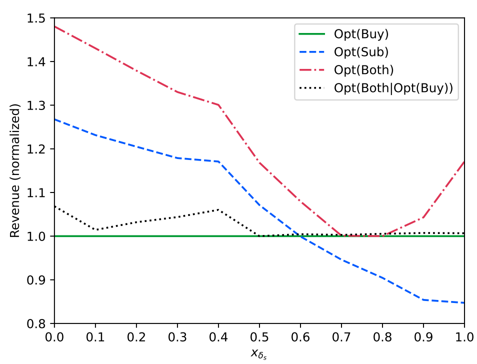

Figure 12: Revenue for different engagement distributions (varying x_{δ_s}). Extended results in Table 14.

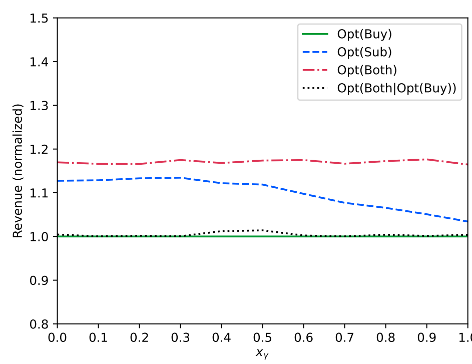

Figure 13: Revenue for different quality decay distributions (varying x_γ). Extended results in Table 15.

Figure 12 shows that the more different the short-term engagement user is from the long-term engagement user (with $x_{\delta_l} = 0.9$), the higher is the potential revenue improvement of $p_t =$ Sub and $p_t =$ Both compared to $p_t =$ Buy.

Figure 13 shows that if the quality decay factors are distributed more unbalanced, i.e. if x_γ increases and user types with $\gamma = 0.9$ are most common, the potential revenue improvement of p_t = Sub compared to p_t = Buy decreases.

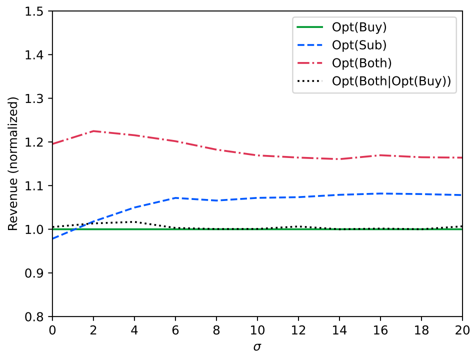

Figure 14: Revenue for different value distributions (varying σ). Extended results in Table 16.

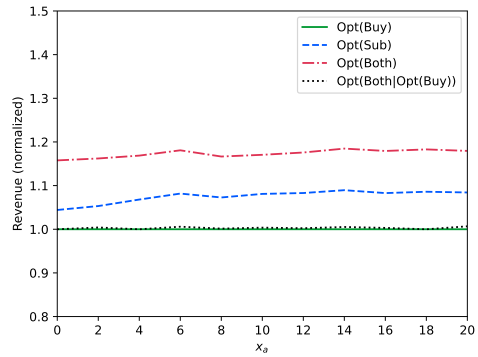

Figure 15: Revenue for different quality decay distributions (varying x_a). Extended results in Table 17

Figure 14 shows that the variance of the user valuation (i.e., σ) has low impacts on the relative revenue potential if $\sigma > 6$. For very small variance (i.e., $\sigma \leq 6$), p_t = Sub and p_t = Both are worse off relatively compared to p_t = Buy compared to higher variances. In fact, the user types are more similar with small variance and given the base case parameter choices, many users will buy the product even if they are short-term engagement users (i.e., they have a small δ). Therefore, the relative revenue potential of p_t = Buy compared to p_t = Sub is higher than for larger σ.

Figure 15 shows that x_a has no big impact on the relative revenue potential of different strategies. If less users arrive in the first timestep (i.e., x_a is small), the potential revenue improvement of p_t = Sub and p_t = Both compared to p_t = Buy is less strong.

I conclude that the more diverse the user types are, the higher is the potential revenue improvement of p_t = Sub and p_t = Both compared to p_t = Buy.

6 Discussion

This thesis investigated the revenue potential of different pricing strategies. A monopoly publisher maximizes his revenue in a system with many different user types who are interested in buying his product. Since prices change over time, the problem is modeled as a Markov Decision Process and optimal user strategies have been found via backward induction. The model used is tailor-made to analyze the selling of consumer software such as video games. This research closes a gap in the revenue maximization of premium games. The focus of our numerical analysis was the effect of introducing subscription licenses besides or as a substitute to perpetual licenses. Therefore, I closed a gap in the research about the revenue maximization through subscription licenses over multiple timesteps by introducing variable prices in every timestep. The optimal pricing vectors for the publisher have been found via differential evolution search.

The most important findings from the numerical analysis for publishers of consumer software in the real world are the following:

If the publisher offers subscription licenses instead of perpetual licenses, he can increase revenue in most cases. This holds mostly also if we vary single setting parameters as it was done in the section 5.2.4 about comparative statics. Subscription licensing has a potential revenue improvement compared to perpetual licensing also if the subscription price is not chosen optimally which I showed for the base case in section 5.2.1. Publishers in the real world should note that one disadvantage of offering subscription licenses is a significant decrease in user welfare.

User welfare can be increased again if both perpetual licenses and subscription licenses are offered in parallel. In this setting, higher perpetual prices are optimal for the publisher. Since the publisher has the highest expected revenue in this setting, I propose to publishers to offer both licensing types at the same time.

Furthermore, it can be stated that setting the highest perpetual price at the timestep of the release is optimal for a publisher. Discounting the prices in later timesteps more or less steadily is often maximizing the revenue. This findings are outlined in section 5.2.2.

I also investigated a setting with variable subscription prices in section 5.2.3. Even though this pricing strategy is not common in practice, I find that revenue can be increased again by offering variable subscription prices alone or in parallel with perpetual prices.

Since I extended the initial model from Dierks and Seuken (2020), I compared my results to theirs. I was able to confirm the most important results, explained above. It is found that my extended model can increase revenue for every pricing strategy compared to the initial model as it allows the publisher to set a unique price in every timestep. Since the effects of the introduced variables are not tremendous, I reason that if computational power is very scarce or the model shall be as simple as possible, the initial model by Dierks and Seuken (2020) can approximate the extended model reasonably well with much fewer pricing variables.

Although the extended model presented in this thesis includes many details, it cannot completely represent reality and may also be used as a starting point for later research. In the following, I present the limitations combined with future work.

Limitations and Future Work

To the best of my knowledge, no model exists comparing pricing strategy types that comes closer to reality. This is mainly due to the highly differentiated user types and the variable prices in each timestep. Still, one of the main limitation of this thesis is that the conclusions so far only hold for a very specific setting of parameter choices. Therefore, further validations are a first step to give more weight to my findings. Smart model extensions in a second step could further approximate the model to the real world as could the incorporation of other fundamental considerations such as network externalities. The most important limitations and inspirations for future work are summarized in the following. Firstly, I focus on further research without model adaption.

Model validation with real world data. The chosen parameters for the base case have been derived by Dierks and Seuken (2020) with respect to sales data from SteamSpy. Nevertheless, the model could be calibrated to a single video game regarding its historic performance. Then, resulting strategy and revenue of the data compared to the model could be analyzed. SteamSpy just counts owners of the product but cannot identify how many of these owners are permanent owners. I.e., a distinction between perpetual sales and subscriptions cannot be made, which also applies to games which offer both, such as FIFA 21.[10] A useful validation is dependent on this distinction, otherwise only the case p_t = Buy could be validated but the main research focus and the strength of my model is targeted toward offering subscription licenses as substitute or in addition to perpetual licenses.

Wide parameter choices analysis. The comparative statics displayed in section 5.2.4 still only display a small subset of possible parameter choices and take some given assumptions as quite untested. It could be of interest to investigate the relations between Opt(Buy), Opt(Sub), Opt(Both) and Opt(Both|Opt(Buy)) without a model extension but with an extensive number of different parameter choices. Some ideas are: The quality ratio between the base product (q_b) and the upgrade (q_u) could be varied since these values are just strict assumptions made for the base case. Further, a range of different upgrade release timesteps m and the effect of a changing n_{max} could be tested.

Theoretical analysis. User types are discrete in this thesis, not allowing for continuous user values since backward induction only works for discrete user types. It could be investigated if for a given p, a continuous user type distribution could be classified into a discrete number of optimal user strategies as in Dierks and Seuken (2020). This would better approximate reality since user values will not be discrete in reality. Furthermore, this thesis did not develop an extended theoretical approach to find the global optimum for publisher pricing since DE search was performed instead. Further theoretical analysis could deliver new insights, see section B in the appendix about optimal pricing for $p_t =$ Buy as possible starting point.

Secondly, I outline possible further research with model adaption.

Network externalities. This thesis excluded network externalities explicitly, assuming that no user interaction exists in this pure two-player-game between publisher and user. It seems plausible that for a game where the publisher has already developed an upgrade and has no multiplayer mode, single users do not influence each other. Since a major research stream (see section 2 about related literature) focuses on settings with network externalities, it would be of interest to verify the observed revenue advantage of Opt(Sub) against Opt(Buy) with these externaltities. Similar to e.g. Zhang and Seidmann (2010), this could be implemented with a quality uncertainty factor. To the best of my knowledge, this would be the first research comparing perpetual and subscription licenses over more than two timesteps with network externalities included.

Duopoly instead of monopoly. Another setting known from literature would be a duopoly instead of a monopoly, where two publishers are selling a qualitatively identical product and are competing against each other. (Balasubramanian et al., 2015)

User with partial information. The full information assumption is not realistic in practice. An interesting consideration would be the uncertainty about the own user type, i.e. a user will not know his valuation, engagement factor and quality decay factor before playing the game and could learn about his preferences from timestep to timestep. A machine learning model (typically Reinforcement Learning can be used to solve MDPs (Russell & Norvig, 1995)) could be deployed, where users would learn about their optimal actions instead of calculating them deterministically through backward induction as in this thesis.

Multiple upgrades. Many games on Steam allow for multiple upgrades, e.g. games such as Stellaris which are further developed over time.[11] This possible model extension could also help or even be necessary to do an exact model calibration if sales data would be available from a game with multiple upgrades.

7 Conclusion

This thesis analyzed the maximization of revenue for a publisher selling perpetual, or subscription licenses, or both at the same time. I confirmed the main findings from Dierks and Seuken (2020) in an extended model environment with a publisher being allowed to set different perpetual prices in every timestep. I examined a wide range of strategy options, i.e., allowing the publisher to set higher prices than in the first timestep or to additionally set variable subscription prices.

Assuming optimal pricing, I showed that offering a subscription option increases revenue for the publisher: Offering only subscription licences already delivers significantly higher revenue compared to offering only perpetual licences. The publisher can further increase her expected earnings by offering both options in combination. A revenue increase of 10% to 20% results in a lower expected user welfare of around 30%. The more different and diverse the user types are, the bigger is the revenue improvement of offering a subscription option. When both options are available, perpetual prices are much higher than in the single case, while the subscription price stays fairly stable. For this reason, in the "both" option subscription licenses generate the biggest portion of revenue.

The optimal pricing of perpetual licenses follows a staircase shape (by lowering price from timestep to timestep) when offered alone but also when offered alongside subscription licenses.

An interesting additional option is the introduction of a variable subscription price. Lowering the subscription price in every timestep raises the expected user welfare, since licenses are bought by more user types, and publisher revenue increases due to the raised number of users subscribing at least once.

Endnotes

[1]See https://www.persoenlich.com/digital/gamen-ist-kein-nischenmarkt-mehr (accessed August 22, 2021).

[2]See https://newzoo.com/insights/trend-reports/newzoo-global-games-market-report-2020-light-version/ (accessed August 22, 2021). Note that name and e-mail have to be given to access the report.

[3]See https://www.marketingcharts.com/cross-media-and-traditional/videogames-traditional-and-cross-channel-115988 (accessed August 33, 2021).

[4]See https://tagn.wordpress.com/2021/01/10/superdata-reviews-2020-digital-game-revenue/ (accessed August 22, 2021).

[5]See https://steamcommunity.com/groups/steamworks/announcements/detail/2961646623386540827 (accessed August 22, 2021)

[6]See https://store.steampowered.com/oldnews/10463 (accessed August 22, 2021).

[7]See https://store.steampowered.com/search/?filter=topsellers (accessed August 11, 2021).

[8]See https://steamspy.com/(accessed August 22, 2021))

[9]https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html (accessed August 22, 2021)

[10]See https://store.steampowered.com/app/1313860/EA_SPORTS_FIFA_21/ for the current subscription and buying option for FIFA 21 at Steam (accessed August 22, 2021)]

[11]See https://store.steampowered.com/app/281990/Stellaris/ for the whole range of Stellaris products. (accessed August 22, 2021)

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Appendices

A Simulation Suite

This part describes the simulation framework used to perform an extensive amount of computational experiments to investigate the revenue potential of different pricing strategies. The simulation suite is written in Python and independently from the work of Dierks and Seuken (2020). There is no graphical user interface implemented since code can be run from command line. There is also no database implemented since small adaptions to the model or simulation variables make it impossible to reuse or compare results and it would be complex to store and compare in a database environment; i.e., changing n_{max} opens a completely different model environment. Therefore, the input for each run can be controlled via a *config.ini* file, holding all necessary model and simulation variables. The outputs, i.e., the simulation results, are stored locally under the desired path as *.csv* files containing all relevant parameter sets.

There are four main functions implemented, the config-file defines which one is being run: (1) Differential evolution to find optimal price vectors, (2) Backward induction to evaluate revenue and welfare for defined price vectors, (3) Search for optimal price vectors for case $p_t = Buy$ for single user types, used only in appendix B.1, (4) Hardcoded version of the theoretical search for multiple user types in appendix B.2.

Information about all parameters and which values they accept can be found in the *README.md* file.

B Theoretical Publisher Equilibrium

In the following, I show the difficulties of finding a publisher equilibrium theoretically with the help of some notes about the optimal publisher strategy for a single user type. Since this is never the case in reality, the second part of the chapter describes equilibrium analysis for the publisher with multiple user types. Since in this thesis, optimal prices have been found with DE searches, I compare the theoretically optimal price considerations with the corresponding DE results. The whole section focuses on p_t = Buy since this case is complicated enough and the difficulties are already coming to light.

B.1 Publisher Strategies for Single User Type

This section describes the theoretical approach to find the best publisher strategy for p_t = Buy for a single user type τ and compares its results with a differential evolution search.

If no subscription option is offered, the optimal prices for the base product and the upgrade are equal to the user's expected utility for buying the corresponding product, i.e., all of the welfare is gained by the publisher, but it is still optimal to buy for the user with expected utility $= 0$. The following formulas therefore also calculate the value for the user if he owns the product from this timestep on without paying anything for it. Hence, proven by Dierks and Seuken (2020), the highest price paid for the base product by user type τ (with $n_a \leq n$) at timestep n with the help of geometric series and taking to account probability δ of regaining demand in timestep m if demand was lost before can be rewritten and adopted as follows:

$$p_b^*(\tau, n | n < m, p_t = Buy) = \Big(\frac{\gamma^{n-1} q_b (1 - (\gamma\delta)^{m-n})}{1 - \gamma\delta} + (\delta(1 - \delta^{m-n}) + \delta^{m-n}) \frac{\gamma^{m-1} q_b}{1 - \gamma\delta}\Big) v, \quad (23)$$

and for $n \geq m$:

$$p_b^*(\tau, n | n \geq m, p_t = Buy) = \frac{\gamma^{n-1} q_b}{1 - \gamma\delta} v \quad (24)$$

The highest price paid for the upgrade by user type τ (with $n_a \leq n$) at timestep $n \geq m$ (i.e., upgrade is already released) is:

$$p_u^*(\tau, n, p_t = Buy) = \frac{\gamma^{n-1} q_u}{1 - \gamma\delta} v \quad (25)$$

To maximize her revenue, the publisher can now set the base price to $p_b^*(\tau, n_a)$ for $n \geq n_a$. The upgrade price can be set to $p_u^*(\tau, max(n_a, m))$ for $n \geq max(n_a, m)$. This forces the user to buy both base product and upgrade as early as possible, i.e. in the timestep with his highest willingness to pay, maximizing the publisher's revenue while the user's revenue is still 0.

Best-of-5 differential evolution is performed here for two single user types to (1) illustrate that differential evolution has no problem to find the global optimum with the same revenue and welfare value as the theoretical approach and (2) show that different prices can lead to the same revenue, i.e., revenue can be split between base product and upgrade by lowering one of the prices and increasing the other price at the same time. I choose $q_1 = 1$, $q_2 = 0.5$, $n = 4$, $m = 3$.

	Revenue	Welfare	Overall Welfare	$\mathbf{p_{b_1}}$	$\mathbf{p_{b_2}}$	$\mathbf{p_{b_3}}$	$\mathbf{p_{b_4}}$	$\mathbf{p_{u_3}}$	$\mathbf{p_{u_4}}$
$\tau = (\mathbf{1}, \mathbf{0.5}, \mathbf{0.9}, \mathbf{18})$									
Theoretical Approach	52.9	0	52.9	42.67	42.67	42.67	42.67	16.36	16.36
Differential Evolution	52.9	0	52.9	50.26	47.3	48.37	43.69	4.21	2.83
$\tau = (\mathbf{2}, \mathbf{0.9}, \mathbf{0.95}, \mathbf{37})$									
Theoretical Approach	389.45	0	389.45	∞	263.14	263.14	263.14	127.59	127.59
Differential Evolution	389.44	0.01	389.45	∞	375.02	409.48	374.44	14.56	7.61

Table 8: Comparing the theoretical approach with DE results for two single user types.

The exact maximized revenue for a single user type for the cases p_t = Sub and p_t = Both are not further investigated in this thesis since the desired illustration of increased complexity for multiple user types gets clear with case p_t = Buy already.

B.2 Publisher Strategies for Multiple User Types

To maximize the publishers expected revenue for p_t = Buy for multiple user types, theoretical analysis gets more complicated. What follows is a theoretical approach for which it is shown that the approach cannot deliver an optimal solution since DE finds a price vector which leads to higher revenue. This shows the non-convexity of the optimization problem for the publisher.

In the following, I choose $q_1 = 1$, $q_2 = 0.5$, $n = 2$, $m = 2$ with two valuation users, i.e., $x_u = 2$. The idea for the theoretical maximization with multiple user types can be summarized as follows: As explained in section B.1, for a single user type revenue can be theoretically maximized for p_t = Buy. In a setting with different user types, we can therefore find the optimal prices for every single user type. These optimal prices can be gathered in a list for every single price which has to be chosen by the publisher. I.e., this approach delivers a list for potentially optimal prices for p_{b_1}, p_{b_2} and p_{u_2}. Therefore, searching over all possible price combinations given by these lists, it could be possible to maximize revenue for the publisher, since every price is at least maximizing revenue for one single user. Table 9 shows that this consideration does not hold.

	Revenue	Welfare	Overall Welfare	$\mathbf{p_{b_1}}$	$\mathbf{p_{b_2}}$	$\mathbf{p_{u_2}}$
Theoretical Approach	58.7	22.59	81.3	79.08	59.21	32.61
Differential Evolution	59.1	22.58	81.68	79.47	59.98	32.04

Table 9: Comparing the theoretical approach with DE for multiple user types

Since DE finds a higher revenue than the theoretical approach, we conclude that this approach is not finding the global optimum, mainly due to its high non-convex character where user types can deviate and e.g. buy in later timesteps. Secondly, also due to its horrible runtime for increasing n, I reject this theoretical approach.

Note that I still cannot state that DE finds the global optimum since we do not know this value in the optimum. I only showed that the theoretical approach cannot be optimal since DE finds price vectors which lead to higher revenue.

C Extended Experimental Results

p_t	Rev.	Welf.	Tot. Welf.	p_b												p_u						p_s												Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
				1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	34.4	31.53	65.93	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	27.76	6.63	0
Sub	36.87	21.12	57.99	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	36.87
Both	40.28	23.61	63.88	288.92	144.93	169.75	262.88	227.77	198.88	207.38	268.73	271.31	171.72	86.77	67.71	21.65	19.18	11.99	1.32	14.04	3.59	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	10.57	1.13	28.58
Both|Opt(Buy)	34.42	31.54	65.96	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	24.3	24.3	24.3	24.3	24.3	24.3	24.3	24.3	24.3	24.3	24.3	24.3	27.53	6.59	0.31

Table 10: Extended Results for Table 4 about the base case

p_t	Rev.	Welf.	Tot. Welf.	p_b												p_u						p_s												Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
				1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	34.12	30.99	65.11	51.25	47.86	47.89	43.38	34.9	35.31	22.22	92.62	92.14	29.1	47.92	18.02	19.71	17.22	63.36	9.21	55.23	12.32	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.95	9.17	0
Sub	36.87	21.12	57.99	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	36.87
Both	40.5	23.42	63.92	188.82	274.45	129.32	217.5	122.15	155.47	99.16	52.35	116.76	215.42	191.59	40.73	28.3	194.01	51.58	163.35	175.72	31.2	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	8.55	3.19	28.77
Both|Opt(Buy)	34.36	31.07	65.43	51.25	47.86	47.89	43.38	34.9	35.31	22.22	92.62	92.14	29.1	47.92	18.02	19.71	17.22	63.36	9.21	55.23	12.32	24.22	24.22	24.22	24.22	24.22	24.22	24.22	24.22	24.22	24.22	24.22	24.22	24.54	9.02	0.8

Table 11: Extended Results for Table 6 about freely chosen perpetual prices

p_t	Rev.	Welf.	Tot. Welf.	p_b												p_u						p_s												Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
				1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	34.4	31.53	65.93	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	27.76	6.63	0
Sub	39.17	23.4	62.57	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	19.18	19.16	18.42	17.84	17.54	17.46	17.44	15.71	14.34	12.67	11.59	10.46	0	0	39.17
Both	41.01	20.98	62	283.83	159.44	184.04	242.27	225.75	121.81	196.88	159.12	84.74	207.07	85.31	67.94	13.59	10.07	10.17	12.55	4.95	1.2	19.81	19.37	19.23	19.1	18.53	18.38	18	17.83	17.09	16.4	14.47	13.51	11.23	0.65	29.14
Both|Opt(Buy)	34.89	31.98	66.87	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	72.82	67.17	55.25	38.28	36.24	30.81	25.51	20.65	18.55	18.16	17.69	16.85	25.23	6.07	3.59

Table 12: Extended Results for Table 7 about variable subscription prices

p_t	x_u	Rev.	Welf.	Tot. Welf.	p_b												p_u						p_s												Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
					1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	1	46.58	25.87	72.45	52.49	50.36	43.44	51.59	39.62	34.12	38.84	50.12	28.28	32	25.97	21.75	19.27	18.37	9.29	12.04	8.32	5.59	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	36.52	10.07	0
Buy	2	40.66	21.34	62	83.33	77.31	77.54	64.96	60.32	63.14	46.06	41.43	49.58	50.95	47.26	26.67	19.45	17.33	14.75	14.98	9.82	13.27	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	32.69	7.97	0
Buy	5	36.69	28.55	65.24	68.13	53.2	53.07	54.34	46.12	34.11	29.37	24.36	48.22	23.77	45.04	19.94	20.02	17.2	14.8	9.83	14.09	7.36	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	27.39	9.29	0
Buy	10	34.4	31.53	65.93	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	27.76	6.63	0
Buy	20	32.95	31.61	64.56	48.95	46.07	44.51	46.04	38.6	44.36	24.77	25.28	26.26	23.25	27.92	26.08	19.65	18.15	15.03	12.44	6.02	8.12	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.53	8.42	0
Buy	30	32.8	33.5	66.3	49.45	46.55	41.29	38.5	33.8	34.39	34.48	27.86	26.01	26.5	23.44	19.17	16.78	12.44	10.98	9.54	6.12	8.97	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.66	7.15	0
Buy	40	32.55	33.52	66.06	49.39	48	40.72	42.36	36.02	34.52	23.89	28.47	24.67	23.83	23.36	25.02	16.83	14.55	9.73	9.47	9.49	6.55	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.79	7.76	0
Buy	50	32.68	33.49	66.16	48.28	46.86	39.64	38.16	33.52	30.26	24.21	24.42	21.23	21.15	24.86	16.48	19.51	16.93	14.59	10.75	11.07	10.59	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	23.83	8.85	0
Sub	1	44.67	19.29	63.96	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	0	0	44.67
Sub	2	33.5	14.47	47.97	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	22.84	22.84	22.84	22.84	22.84	22.84	22.84	22.84	22.84	22.84	22.84	22.84	0	0	33.5
Sub	5	36.55	24.56	61.1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	14.83	14.83	14.83	14.83	14.83	14.83	14.83	14.83	14.83	14.83	14.83	14.83	0	0	36.55
Sub	10	36.87	21.12	57.99	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	36.87
Sub	20	36.73	22.37	59.1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.66	15.66	15.66	15.66	15.66	15.66	15.66	15.66	15.66	15.66	15.66	15.66	0	0	36.73
Sub	30	36.68	22.43	59.11	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.63	15.63	15.63	15.63	15.63	15.63	15.63	15.63	15.63	15.63	15.63	15.63	0	0	36.68
Sub	40	36.48	22.72	59.2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.51	15.51	15.51	15.51	15.51	15.51	15.51	15.51	15.51	15.51	15.51	15.51	0	0	36.48
Sub	50	36.5	23.17	59.67	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.32	15.32	15.32	15.32	15.32	15.32	15.32	15.32	15.32	15.32	15.32	15.32	0	0	36.5
Both	1	54.9	8.34	63.24	284.64	189.3	167.25	283.66	138.14	125.6	151.76	129.62	97.57	110.72	80.27	75.86	13.37	4.23	9.02	12.72	7.41	2.39	20.78	20.78	20.78	20.78	20.78	20.78	20.78	20.78	20.78	20.78	20.78	20.78	22.02	1.71	31.16
Both	2	41.2	15.35	56.54	295.99	189.75	181.86	222.69	139.13	294.26	98.74	121.47	89.05	60.6	94.87	25.4	17.71	4.88	10.36	4.11	7.15	14.45	29.64	29.64	29.64	29.64	29.64	29.64	29.64	29.64	29.64	29.64	29.64	29.64	17.33	4.39	19.47
Both	5	41.95	15.51	57.45	292.8	269.95	160.34	209.25	132	140.59	121.18	239.93	196.24	85.4	113	82.32	22.73	19.91	10.22	12.84	12.34	12.81	20.8	20.8	20.8	20.8	20.8	20.8	20.8	20.8	20.8	20.8	20.8	20.8	13.9	2.03	26.02
Both	10	40.28	23.61	63.88	288.92	144.93	169.75	262.88	227.77	198.88	207.38	268.73	271.31	171.72	86.77	67.71	21.65	19.18	11.99	1.32	14.04	3.59	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	16.18	10.57	1.13	28.58
Both	20	39.72	22.84	62.56	255.92	135.64	216.93	219.92	111.96	107.06	255.79	142.39	138.47	149.33	68.17	55.9	17.33	16.34	0.36	12.56	5.87	10.9	17.33	17.33	17.33	17.33	17.33	17.33	17.33	17.33	17.33	17.33	17.33	17.33	13.19	1.58	24.95
Both	30	39.61	22.98	62.6	226.62	134.87	161.2	150.85	221.91	96.69	85.33	133.52	85.02	64.06	54.69	51.82	20.39	18.65	14.49	16.72	18.97	17.8	17.19	17.19	17.19	17.19	17.19	17.19	17.19	17.19	17.19	17.19	17.19	17.19	12.39	2.42	24.81
Both	40	39.44	22.41	61.85	298.62	155.47	133.8	281.03	202.2	222.62	146.42	146.46	153.65	78.89	150.28	79.34	10.51	10.08	8.13	5.91	5.61	6.77	17.23	17.23	17.23	17.23	17.23	17.23	17.23	17.23	17.23	17.23	17.23	17.23	11.89	0.91	26.64
Both	50	39.44	22.69	62.13	253.49	205.25	134.87	127.68	122.86	227.87	95.03	252.8	250.68	216.32	251.94	74.57	12.56	1.21	6.13	9.84	2.49	0.4	17	17	17	17	17	17	17	17	17	17	17	17	13.41	0.94	25.09
Both|Opt(Buy)	1	46.58	25.87	72.45	52.49	50.36	43.44	51.59	39.62	34.12	38.84	50.12	28.28	32	25.97	21.75	19.27	18.37	9.29	12.04	8.32	5.59	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	36.52	10.07	0
Both|Opt(Buy)	2	40.66	21.34	62	83.33	77.31	77.54	64.96	60.32	63.14	46.06	41.43	49.58	50.95	47.26	26.67	19.45	17.33	14.75	14.98	9.82	13.27	53.57	53.57	53.57	53.57	53.57	53.57	53.57	53.57	53.57	53.57	53.57	53.57	32.69	7.97	0
Both|Opt(Buy)	5	37.16	28.63	65.79	68.13	53.2	53.07	54.34	46.12	34.11	29.37	24.36	48.22	23.77	45.04	19.94	20.02	17.2	14.8	9.83	14.09	7.36	26.2	26.2	26.2	26.2	26.2	26.2	26.2	26.2	26.2	26.2	26.2	26.2	25.58	8.96	2.62
Both|Opt(Buy)	10	34.42	31.57	65.99	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	22.83	22.83	22.83	22.83	22.83	22.83	22.83	22.83	22.83	22.83	22.83	22.83	26.85	6.34	1.22
Both|Opt(Buy)	20	32.98	31.61	64.6	48.95	46.07	44.51	46.04	38.6	44.36	24.77	25.28	26.26	23.25	27.92	26.08	19.65	18.15	15.03	12.44	6.02	8.12	24.41	24.41	24.41	24.41	24.41	24.41	24.41	24.41	24.41	24.41	24.41	24.41	24.52	8.41	0.05
Both|Opt(Buy)	30	32.99	33.59	66.57	49.45	46.55	41.29	38.5	33.8	34.39	34.48	27.86	26.01	26.5	23.44	19.17	16.78	12.44	10.98	9.54	6.12	8.97	23.17	23.17	23.17	23.17	23.17	23.17	23.17	23.17	23.17	23.17	23.17	23.17	25.05	6.91	1.02
Both|Opt(Buy)	40	32.61	33.52	66.13	49.39	48	40.72	42.36	36.02	34.52	23.89	28.47	24.67	23.83	23.36	25.02	16.83	14.55	9.73	9.47	9.49	6.55	22.86	22.86	22.86	22.86	22.86	22.86	22.86	22.86	22.86	22.86	22.86	22.86	24.65	7.7	0.25
Both|Opt(Buy)	50	32.73	33.56	66.29	48.28	46.86	39.64	38.16	33.52	30.26	24.21	24.42	21.23	21.15	24.86	16.48	19.51	16.93	14.59	10.75	11.07	10.59	22.07	22.07	22.07	22.07	22.07	22.07	22.07	22.07	22.07	22.07	22.07	22.07	22.89	8.18	1.66

Table 13: Extended Results for Figure 9 about comparative statics for x_u

p_t	x_δ	Rev.	Welf.	Tot. Welf.	p_b																														Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
					1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	0	19.66	24.73	44.4	88.46	78.63	60.1	48.64	41.21	38.38	20.05	23.76	61.52	60.84	75.82	68.78	3.02	1.48	2.33	1.85	3.02	2.21	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	18	1.66	0
Buy	0.1	21.28	27.37	48.65	82.83	79.53	70.27	56.46	46.44	39.52	23.02	38.86	23.95	23.69	17.78	20.98	2.42	1.07	1.33	0.19	1.52	0.33	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	19.85	1.43	0
Buy	0.2	23.15	27.36	50.51	87.81	80.15	76.48	67.54	52.05	66.16	27.8	25.13	43.05	33	72.53	20.11	0.16	0.02	0.16	0.11	0.12	0.16	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	23.05	0.1	0
Buy	0.3	25.49	27.25	52.73	90.72	83.26	75.26	64.98	52.32	53.23	30.4	35.99	65.39	27.96	35.67	21.96	1.08	0.92	0.79	0.18	0.39	0.86	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.83	0.65	0
Buy	0.4	28.14	26.43	54.58	99.56	92.95	73.25	66.5	54.84	53.15	23.49	65.58	30.34	21.03	26.17	19.07	12.65	7.49	4.66	8.22	6.68	6.62	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	20.5	7.64	0
Buy	0.5	34.4	31.53	65.93	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	27.76	6.63	0
Buy	0.6	42.24	31.76	74	63.05	59.54	54.02	47.59	41.84	40.88	26.45	26.35	35.14	29.52	24.69	24	23.68	22.6	20.78	17.29	15.67	13.37	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	30.16	12.09	0
Buy	0.7	52.79	32.4	85.19	92.46	91.14	79.84	76.2	81.57	61.38	53.08	50.72	70.6	50.07	45.1	46.02	11.58	4.75	4.51	2.45	9.39	6.65	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	47.49	5.3	0
Buy	0.8	66.94	43.59	110.53	99.89	93.29	89.96	71.52	80.68	82.69	66.27	84.73	63.47	62.18	91.65	45.86	16.14	11.28	8.86	1.07	4.19	3.01	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	57.98	8.96	0
Buy	0.9	89.78	55.12	144.9	99.83	94.31	91.16	77.96	64.28	74.33	64.05	77.38	53.61	64.84	88.78	40.91	55.82	52.57	49.52	50.91	38.87	30.71	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	57.29	32.49	0
Buy	1	125.28	130.56	255.84	99.69	96.69	88.63	94.45	78.16	98.11	87.99	88.98	72.53	88.45	76.27	70.25	85.52	79.49	82.25	76.78	83.32	77.48	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	71.24	54.04	0
Sub	0	24.92	14.66	39.58	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	24.92
Sub	0.1	26.2	15.42	41.62	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	0	0	26.2
Sub	0.2	27.9	16.42	44.32	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	0	0	27.9
Sub	0.3	30.04	17.65	47.69	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	0	0	30.04
Sub	0.4	32.96	19.15	52.1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	32.96
Sub	0.5	36.84	21.15	57.99	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	0	0	36.84
Sub	0.6	42.21	24.29	66.5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.02	16.02	16.02	16.02	16.02	16.02	16.02	16.02	16.02	16.02	16.02	16.02	0	0	42.21
Sub	0.7	49.94	30.76	80.7	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	0	0	49.94
Sub	0.8	60.54	38.08	98.63	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	14.56	14.56	14.56	14.56	14.56	14.56	14.56	14.56	14.56	14.56	14.56	14.56	0	0	60.54
Sub	0.9	76.66	54.98	131.64	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	0	0	76.66
Sub	1	106.15	74.82	180.97	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	12.12	12.12	12.12	12.12	12.12	12.12	12.12	12.12	12.12	12.12	12.12	12.12	0	0	106.15
Both	0	29.11	16.86	45.97	263.03	138.82	144.25	190.42	98.76	184.36	124.09	116.65	231.34	158.1	191.75	62.75	12.91	8.39	2.78	7.51	3.97	8.71	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	17.94	13.53	1.52	14.07
Both	0.1	30.43	17.15	47.58	287.16	128.26	167.88	131.81	224.52	133.96	116.63	178.75	134.68	196.51	65.23	63.76	18.61	8.89	9.31	12.07	14.77	18.33	17.79	17.79	17.79	17.79	17.79	17.79	17.79	17.79	17.79	17.79	17.79	17.79	12.92	2.03	15.48
Both	0.2	31.93	18.06	49.99	218.97	131.47	140.89	150.59	142.21	112.46	214.78	167.84	195.08	78.48	127.4	62.15	12.6	3.83	5.86	12.17	4.31	9.39	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	14.61	1.51	15.82
Both	0.3	33.9	19.6	53.5	193.74	128.47	177.7	101.77	146.13	79.62	80.84	100.27	89.58	103.14	92.63	65.28	24.69	19.92	20.04	14.84	8.13	4.96	17.89	17.89	17.89	17.89	17.89	17.89	17.89	17.89	17.89	17.89	17.89	17.89	13.7	2.79	17.4
Both	0.4	36.61	22.1	58.71	293.09	158.12	115	205.41	119.64	285.47	137.2	114.85	107.14	74.38	128.57	55.82	16.63	9.41	1.78	14.19	8.28	15.44	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	12.26	2.03	22.31
Both	0.5	40.18	23.3	63.49	243.5	218.06	215.33	107.56	172.89	120.8	163.67	129.22	96.74	100.34	107.22	59.69	244.34	88.21	93.68	67.74	65.72	12.16	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	5.29	1.08	33.81
Both	0.6	45.61	25.89	71.49	228.23	186.17	164.69	168.61	226.49	196.74	203.14	177.53	93.78	99.45	76.97	59.27	26.61	10.03	16.38	12.17	25.98	12.67	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	5.26	1.12	39.23
Both	0.7	52.84	33	85.84	287.2	147.65	133.3	241.65	113.56	211.28	186.81	191.02	218.18	85.36	238.89	63.68	14.21	8.49	10.68	5.13	7.35	7.46	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	15.07	11.26	1.24	40.34
Both	0.8	66.94	40.31	107.25	275.95	185.09	158.39	104.36	202.88	252.67	229.12	214.05	158.36	227.18	152.66	45.16	9.47	4.7	7.99	6.95	4	3.52	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	21.26	1.73	43.95
Both	0.9	93.64	53.52	147.16	267.2	204.38	124.71	115.65	135.36	86.65	258.79	185.79	191.52	151.48	72.8	70.89	17.63	12.07	11.38	13.3	4.45	2.07	18.01	18.01	18.01	18.01	18.01	18.01	18.01	18.01	18.01	18.01	18.01	18.01	67.69	8.95	17
Both	1	146.72	86.05	232.77	288.11	253	196.78	228.8	217.32	252.57	208.08	273.17	184.29	186.62	88.93	82.37	204.67	140.11	150.81	77.45	63.28	80.54	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	45.79	32.92	68.01
Both|Opt(Buy)	0	21.01	27.78	48.79	88.46	78.63	60.1	48.64	41.21	38.38	20.05	23.76	61.52	60.84	75.82	68.78	3.02	1.48	2.33	1.85	3.02	2.21	17.91	17.91	17.91	17.91	17.91	17.91	17.91	17.91	17.91	17.91	17.91	17.91	8.31	0.5	12.2
Both|Opt(Buy)	0.1	21.58	27.5	49.09	82.83	79.53	70.27	56.46	46.44	39.52	23.02	38.86	23.95	23.69	17.78	20.98	2.42	1.07	1.33	0.19	1.52	0.33	23.24	23.24	23.24	23.24	23.24	23.24	23.24	23.24	23.24	23.24	23.24	23.24	18.07	1.25	2.26
Both|Opt(Buy)	0.2	23.89	27.56	51.45	87.81	80.15	76.48	67.54	52.05	66.16	27.8	25.13	43.05	33	72.53	20.11	0.16	0.02	0.16	0.11	0.12	0.16	23.26	23.26	23.26	23.26	23.26	23.26	23.26	23.26	23.26	23.26	23.26	23.26	20.78	0.09	3.03
Both|Opt(Buy)	0.3	26.6	27.55	54.15	90.72	83.26	75.26	64.98	52.32	53.23	30.4	35.99	65.39	27.96	35.67	21.96	1.08	0.92	0.79	0.18	0.39	0.86	23.88	23.88	23.88	23.88	23.88	23.88	23.88	23.88	23.88	23.88	23.88	23.88	22.36	0.58	3.66
Both|Opt(Buy)	0.4	29.84	26.97	56.81	99.56	92.95	73.25	66.5	54.84	53.15	23.49	65.58	30.34	21.03	26.17	19.07	12.65	7.49	4.66	8.22	6.68	6.62	23.39	23.39	23.39	23.39	23.39	23.39	23.39	23.39	23.39	23.39	23.39	23.39	18.58	6.7	4.57
Both|Opt(Buy)	0.5	34.4	31.53	65.93	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	29.03	29.03	29.03	29.03	29.03	29.03	29.03	29.03	29.03	29.03	29.03	29.03	27.76	6.63	0
Both|Opt(Buy)	0.6	42.43	31.77	74.21	63.05	59.54	54.02	47.59	41.84	40.88	26.45	26.35	35.14	29.52	24.69	24	23.68	22.6	20.78	17.29	15.67	13.37	24.06	24.06	24.06	24.06	24.06	24.06	24.06	24.06	24.06	24.06	24.06	24.06	30.03	12.01	0.4
Both|Opt(Buy)	0.7	52.94	32.45	85.38	92.46	91.14	79.84	76.2	81.57	61.38	53.08	50.72	70.6	50.07	45.1	46.02	11.58	4.75	4.51	2.45	9.39	6.65	23.86	23.86	23.86	23.86	23.86	23.86	23.86	23.86	23.86	23.86	23.86	23.86	46.68	5.16	1.09
Both|Opt(Buy)	0.8	67.3	43.65	110.95	99.89	93.29	89.96	71.52	80.68	82.69	66.27	84.73	63.47	62.18	91.65	45.86	16.14	11.28	8.86	1.07	4.19	3.01	23.11	23.11	23.11	23.11	23.11	23.11	23.11	23.11	23.11	23.11	23.11	23.11	57.71	8.93	0.66
Both|Opt(Buy)	0.9	90.43	55.4	145.84	99.83	94.31	91.16	77.96	64.28	74.33	64.05	77.38	53.61	64.84	88.78	40.91	55.82	52.57	49.52	50.91	38.87	30.71	20.32	20.32	20.32	20.32	20.32	20.32	20.32	20.32	20.32	20.32	20.32	20.32	56.5	31.78	2.15
Both|Opt(Buy)	1	126.12	131.7	257.81	99.69	96.69	88.63	94.45	78.16	98.11	87.99	88.98	72.53	88.45	76.27	70.25	85.52	79.49	82.25	76.78	83.32	77.48	16.56	16.56	16.56	16.56	16.56	16.56	16.56	16.56	16.56	16.56	16.56	16.56	68.05	52.49	5.57

Table 14: Extended Results for Figure 12 about comparative statics for x_δ

p_t	x_γ	Rev.	Welf.	Tot. Welf.	p_b																														Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
					1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	0	33.78	36.53	70.31	48.78	47.59	44.39	45.41	38.99	39.61	26.35	32.35	35.52	38.24	35.02	31.61	14.71	14.06	10.29	10.01	13.39	9.17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	26.44	7.34	0
Buy	0.1	33.53	36.78	70.31	48.33	48.05	48.25	41.12	37.52	34	26.65	33.75	32.73	33.57	30.39	26.96	14.44	11.77	8.58	10.25	9.68	11.28	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	26.75	6.78	0
Buy	0.2	33.27	36.51	69.78	49.57	49.37	45.48	47.24	41.33	38.68	27.9	31.43	43.81	35.82	23.49	29.36	12.78	9.97	2.05	10.39	10.04	6.65	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	26.8	6.47	0
Buy	0.3	33.04	33.47	66.51	58.37	54.23	44.96	54.16	44.62	33.87	25.68	32.5	30.16	27.17	31.91	27.34	16.06	10.72	6.77	12.8	10.03	13.07	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.76	7.28	0
Buy	0.4	33.3	32.44	65.74	62.64	52.64	48.1	44.67	60.42	33.93	26.03	31.93	39.1	33.86	20.5	20.06	15.91	11.86	7.21	6.27	12.25	15.08	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.94	8.36	0
Buy	0.5	33.2	31.77	64.97	65.37	51.56	54.45	44.49	40.52	39.47	25.8	30.6	30.15	35.3	26.77	34.38	15.92	11.29	10.77	7.72	5.68	7.79	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.56	7.64	0
Buy	0.6	33.61	32.79	66.4	56.34	51.75	51.79	51.95	41.93	38.75	30.76	40.27	39.09	41.37	37.25	22.24	10.95	5.72	0.65	3.23	3.71	8.21	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	28.41	5.21	0
Buy	0.7	34.28	32.17	66.45	52.36	52.22	47.57	42.05	37.97	34.9	23.73	27.44	37.42	30.1	22.7	19.43	18.08	14.49	14.68	4.82	9.67	10.46	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.73	8.55	0
Buy	0.8	34.44	32.07	66.51	52.83	48.16	45.47	42.8	36.54	40.29	27.74	25.63	31.7	29.11	22.08	22.25	16.11	12.15	7.77	8.36	11.25	12.86	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	26.78	7.66	0
Buy	0.9	34.85	31.95	66.8	55.8	51.65	48.4	46.13	41.05	37.92	30.37	33.01	33.65	25.52	25.78	19.34	11.63	7.92	8.81	5.13	6.62	8.85	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	29.16	5.69	0
Buy	1	35.39	31.12	66.51	54.13	52.32	49.04	43.74	40.49	35.24	40.85	27.01	27.77	25.46	26.49	20.21	14.73	11.22	6.15	8.61	9.26	4.56	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	29.03	6.36	0
Sub	0	38.09	23.89	61.98	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	0	0	38.09
Sub	0.1	37.84	23.48	61.32	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	0	0	37.84
Sub	0.2	37.7	22.96	60.66	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	0	0	37.7
Sub	0.3	37.48	22.51	60	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	0	0	37.48
Sub	0.4	37.36	21.98	59.34	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	0	0	37.36
Sub	0.5	37.14	21.53	58.67	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	16.71	0	0	37.14
Sub	0.6	36.89	21.12	58.01	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.68	16.68	16.68	16.68	16.68	16.68	16.68	16.68	16.68	16.68	16.68	16.68	0	0	36.89
Sub	0.7	36.92	21.68	58.6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	0	0	36.92
Sub	0.8	36.69	21.32	58.01	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.16	16.16	16.16	16.16	16.16	16.16	16.16	16.16	16.16	16.16	16.16	16.16	0	0	36.69
Sub	0.9	36.62	20.81	57.43	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.17	16.17	16.17	16.17	16.17	16.17	16.17	16.17	16.17	16.17	16.17	16.17	0	0	36.62
Sub	1	36.6	20.24	56.84	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	0	0	36.6
Both	0	39.51	25.43	64.94	293.16	203.69	128.5	223.26	167.85	260.97	247.06	222.45	216.65	172.48	110.49	79.73	299.55	265.59	208.91	106.6	261.61	35.07	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	4.89	1.92	32.7
Both	0.1	39.1	24.84	63.94	272.97	178.16	236.18	254.28	193.99	151.86	143.03	234.81	176.88	145.21	164.91	112.35	146.25	46.78	127.88	30.36	41.47	0.26	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	5.65	0.01	33.43
Both	0.2	38.79	24.12	62.91	236.31	227.88	173.25	215.38	191.75	225.75	173.08	137.78	120.27	152.16	146.68	72.2	107.52	92.07	22.63	24.8	75.63	41.65	16.74	16.74	16.74	16.74	16.74	16.74	16.74	16.74	16.74	16.74	16.74	16.74	3.29	1.9	33.6
Both	0.3	38.82	26.72	65.54	244.03	137.81	149.9	181.32	205.93	155.95	226.83	182.69	91.48	86	147.97	115.74	5.93	2.87	2.95	4.6	2.95	4.36	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	16.69	13.01	0.51	25.31
Both	0.4	38.9	24.97	63.87	294.95	159.2	269.24	230.69	130.59	109.74	238.69	263.47	88.83	257.08	86.56	119.91	14.44	7.08	7.21	0.05	7.22	5.1	16.61	16.61	16.61	16.61	16.61	16.61	16.61	16.61	16.61	16.61	16.61	16.61	9.54	0.93	28.42
Both	0.5	38.97	23.39	62.36	222.4	162.47	133.42	196.08	162.93	196.1	125.2	99.87	154.18	65.89	80.38	61.95	133.97	69.25	60.76	22.05	41.18	15.84	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	6.01	1.93	31.02
Both	0.6	39.49	25.62	65.1	193.45	119.85	191.49	115.5	129.49	123.42	101.79	131.07	99.73	67.54	89.5	61.01	22.48	17.15	14.59	19.88	19.3	20.06	16.5	16.5	16.5	16.5	16.5	16.5	16.5	16.5	16.5	16.5	16.5	16.5	11.88	2.57	25.03
Both	0.7	39.99	24.9	64.89	221.2	117.92	158.83	198.01	220.44	109.19	185.08	83.99	146.7	114.42	69.42	59.69	55.14	19.28	16.59	34.06	10	26.72	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	10.75	1.64	27.6
Both	0.8	40.37	25.23	65.6	258.98	160.15	126.72	213.45	103.68	120.37	172.74	140.16	91.48	179.56	73.3	66.41	8.29	3.03	2.24	5.31	4.82	5.67	16.11	16.11	16.11	16.11	16.11	16.11	16.11	16.11	16.11	16.11	16.11	16.11	14.55	0.87	24.96
Both	0.9	41	19.81	60.81	252.9	190.31	141.44	212.19	116.13	145.61	174.43	128.95	94.9	202.63	139.21	68.03	19.36	17.94	5.18	7.82	14.54	4	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	13.72	1.51	25.77
Both	1	41.22	23.12	64.34	269.6	104.57	116.06	166.44	178.19	83.04	210.05	110.89	126.44	121.5	152.25	46.26	46.26	40.72	32.2	29.39	19.29	22.74	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	9.08	3.72	28.43
Both|Opt(Buy)	0	33.93	36.56	70.49	48.78	47.59	44.39	45.41	38.99	39.61	26.35	32.35	35.52	38.24	35.02	31.61	14.71	14.06	10.29	10.01	13.39	9.17	23.69	23.69	23.69	23.69	23.69	23.69	23.69	23.69	23.69	23.69	23.69	23.69	25.88	7.17	0.88
Both|Opt(Buy)	0.1	33.54	36.79	70.33	48.33	48.05	48.25	41.12	37.52	34	26.65	33.75	32.73	33.57	30.39	26.96	14.44	11.77	8.58	10.25	9.68	11.28	24.24	24.24	24.24	24.24	24.24	24.24	24.24	24.24	24.24	24.24	24.24	24.24	26.52	6.7	0.32
Both|Opt(Buy)	0.2	33.33	36.51	69.84	49.57	49.37	45.48	47.24	41.33	38.68	27.9	31.43	43.81	35.82	23.49	29.36	12.78	9.97	2.05	10.39	10.04	6.65	23.77	23.77	23.77	23.77	23.77	23.77	23.77	23.77	23.77	23.77	23.77	23.77	26.72	6.46	0.15
Both|Opt(Buy)	0.3	33.06	33.58	66.64	58.37	54.23	44.96	54.16	44.62	33.87	25.68	32.5	30.16	27.17	31.91	27.34	16.06	10.72	6.77	12.8	10.03	13.07	24.15	24.15	24.15	24.15	24.15	24.15	24.15	24.15	24.15	24.15	24.15	24.15	23.79	7.24	2.02
Both|Opt(Buy)	0.4	33.7	32.65	66.35	62.64	52.64	48.1	44.67	60.42	33.93	26.03	31.93	39.1	33.86	20.5	20.06	15.91	11.86	7.21	6.27	12.25	15.08	24.61	24.61	24.61	24.61	24.61	24.61	24.61	24.61	24.61	24.61	24.61	24.61	23.08	8.33	2.3
Both|Opt(Buy)	0.5	33.66	31.95	65.61	65.37	51.56	54.45	44.49	40.52	39.47	25.8	30.6	30.15	35.3	26.77	34.38	15.92	11.29	10.77	7.72	5.68	7.79	24.37	24.37	24.37	24.37	24.37	24.37	24.37	24.37	24.37	24.37	24.37	24.37	23.74	7.63	2.29
Both|Opt(Buy)	0.6	33.69	32.81	66.5	56.34	51.75	51.79	51.95	41.93	38.75	30.76	40.27	39.09	41.37	37.25	22.24	10.95	5.72	0.65	3.23	3.71	8.21	24.11	24.11	24.11	24.11	24.11	24.11	24.11	24.11	24.11	24.11	24.11	24.11	28.06	5.17	0.47
Both|Opt(Buy)	0.7	34.28	32.17	66.46	52.36	52.22	47.57	42.05	37.97	34.9	23.73	27.44	37.42	30.1	22.7	19.43	18.08	14.49	14.68	4.82	9.67	10.46	24.48	24.48	24.48	24.48	24.48	24.48	24.48	24.48	24.48	24.48	24.48	24.48	25.71	8.55	0.03
Both|Opt(Buy)	0.8	34.57	32.18	66.76	52.83	48.16	45.47	42.8	36.54	40.29	27.74	25.63	31.7	29.11	22.08	22.25	16.11	12.15	7.77	8.36	11.25	12.86	22.47	22.47	22.47	22.47	22.47	22.47	22.47	22.47	22.47	22.47	22.47	22.47	25.72	7.14	1.72
Both|Opt(Buy)	0.9	34.9	31.95	66.84	55.8	51.65	48.4	46.13	41.05	37.92	30.37	33.01	33.65	25.52	25.78	19.34	11.63	7.92	8.81	5.13	6.62	8.85	24.12	24.12	24.12	24.12	24.12	24.12	24.12	24.12	24.12	24.12	24.12	24.12	29.16	5.69	0.05
Both|Opt(Buy)	1	35.52	31.15	66.67	54.13	52.32	49.04	43.74	40.49	35.24	40.85	27.01	27.77	25.46	26.49	20.21	14.73	11.22	6.15	8.61	9.26	4.56	24.27	24.27	24.27	24.27	24.27	24.27	24.27	24.27	24.27	24.27	24.27	24.27	28.7	6.22	0.61

Table 15: Extended Results for Figure 13 about comparative statics for x_γ

p_t	σ	Rev.	Welf.	Tot. Welf.	p_b																														Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
					1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	0	45.65	26.1	71.75	62.46	48.23	51.43	45.06	49.56	38.62	30.72	34.4	27.15	30.39	42.77	17.45	15.16	13.99	12.79	3.35	3.26	10.09	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	36.46	9.19	0
Buy	2	42.63	27.23	69.86	68.4	55.96	44.97	41.74	51.49	56.09	25.81	29.38	26.44	34.05	28.28	24.76	15.81	13.63	7.46	3.35	3.45	0.14	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	34.37	8.25	0
Buy	4	39.86	28.8	68.65	59.25	48.08	44.64	45.99	36.53	31.57	26.02	36.76	31.87	26.59	34.33	14.09	16.99	12.87	12.77	3.96	12.27	10.69	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	31.03	8.83	0
Buy	6	37.25	29.06	66.31	51.54	48.31	44.23	42.19	36.87	37	22.58	29.28	24.71	34.2	37.08	18.63	19.2	17.91	14.19	6.16	13.83	11.65	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	26.8	10.45	0
Buy	8	35.71	30.83	66.54	52.97	49.02	47.37	40.99	37.01	33.76	25.18	29.24	32.29	31.42	24.94	23.43	16.71	13.89	5.77	5.93	8.98	6.25	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	28.26	7.45	0
Buy	10	34.4	31.53	65.93	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	27.76	6.63	0
Buy	12	33.53	31.94	65.47	55.04	49.3	47.94	43.73	46.07	33.83	26.09	37.56	31.38	25.55	22.63	24.26	16.08	11.74	10.36	13.02	10.91	8.88	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.92	7.62	0
Buy	14	32.76	32.42	65.17	59.24	54.81	50.45	48.9	43.34	45.8	28.97	32.6	36.99	24.86	24.2	24.7	12.69	8.79	6.25	11.78	9.05	8.7	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	26.11	6.65	0
Buy	16	32.31	31.98	64.29	57.68	55.38	54.02	48.82	39.12	46.28	25.52	29.6	35.64	29.59	30.28	30.57	16.65	12.54	11.56	11.77	7.19	5.39	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.62	7.69	0
Buy	18	32.12	32.46	64.58	56.5	52.49	49.99	42.89	39.64	34.64	24.87	27.68	22.54	26.85	43.41	30.85	20.03	16.31	11.28	14.64	18.33	5.05	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	24.13	7.99	0
Buy	20	31.95	33.73	65.68	53.94	47.98	47.08	41.28	42.12	40.99	36.52	32.3	35.57	33.45	39.2	18.47	16.72	12.6	11.8	3.15	6.1	11.91	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.41	6.55	0
Sub	0	44.66	19.29	63.96	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	15.23	0	0	44.66
Sub	2	43.39	15.52	58.91	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	16.75	0	0	43.39
Sub	4	41.84	20.44	62.28	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	15.06	15.06	15.06	15.06	15.06	15.06	15.06	15.06	15.06	15.06	15.06	15.06	0	0	41.84
Sub	6	39.92	22.68	62.6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	0	0	39.92
Sub	8	38.06	19.83	57.89	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.2	16.2	16.2	16.2	16.2	16.2	16.2	16.2	16.2	16.2	16.2	16.2	0	0	38.06
Sub	10	36.87	21.12	57.99	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	36.87
Sub	12	36	22.28	58.28	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	36
Sub	14	35.34	23.26	58.6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	0	0	35.34
Sub	16	34.96	23.88	58.83	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	0	0	34.96
Sub	18	34.71	24.33	59.04	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	34.71
Sub	20	34.45	23.71	58.16	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	16.72	0	0	34.45
Both	0	54.57	12.17	66.74	244.85	211.46	194.82	155.43	168.32	128.92	160.14	165.92	152.34	92.26	151.12	78.72	5.98	2.72	3.48	5.72	2.51	1.36	18.56	18.56	18.56	18.56	18.56	18.56	18.56	18.56	18.56	18.56	18.56	18.56	20.39	0.85	33.34
Both	2	52.21	14.47	66.69	235.92	158.22	132.23	158.49	110.07	94.15	130.91	129.84	81.57	228.7	74.8	55.35	30.55	24.08	14.53	9.3	2.88	16.1	18.47	18.47	18.47	18.47	18.47	18.47	18.47	18.47	18.47	18.47	18.47	18.47	17.37	3.75	31.09
Both	4	48.44	18.64	67.08	261.51	206.72	161.12	132.76	161.26	107.58	230.99	172.11	226.92	145.8	111.83	65.42	12.02	3.75	6.42	2.29	3.71	6.42	16.67	16.67	16.67	16.67	16.67	16.67	16.67	16.67	16.67	16.67	16.67	16.67	14.57	1.37	32.51
Both	6	44.76	21.7	66.46	271.44	259.94	186.38	104.1	97.33	87.09	189.21	103.82	143.29	177.06	143.81	55.98	15.36	10.38	7.06	6.06	12.91	15.33	16.73	16.73	16.73	16.73	16.73	16.73	16.73	16.73	16.73	16.73	16.73	16.73	12.97	2.29	29.5
Both	8	42.23	23.81	66.03	233.4	137.33	212.75	173.38	101.01	117.4	149.26	181.42	88.44	161.53	77.74	65.55	8.36	2.1	5.48	3.97	5.19	5.33	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	16.21	15.46	1.04	25.73
Both	10	40.21	23.27	63.49	279.57	251.26	191.06	199.27	147.03	123.51	152.99	113.37	151.78	169.15	222.31	43.55	75.81	13.78	34.58	72.69	47.28	28.45	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	3.86	2.52	33.83
Both	12	39.04	25.03	64.07	293.59	124.93	186.72	210.31	198.17	192.11	178.96	170.35	116.98	117.27	228.22	69.52	19.53	13.3	14.43	15.13	12.88	1.71	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	10.87	1.11	27.06
Both	14	38.02	25.54	63.56	263.07	160.23	121.7	142.26	235.68	221.83	130.57	191.48	183.87	156.08	87.55	59.01	129.4	76.4	52.04	64.15	105.16	12.85	16.19	16.19	16.19	16.19	16.19	16.19	16.19	16.19	16.19	16.19	16.19	16.19	4.91	1.07	32.04
Both	16	37.79	23.01	60.8	288.77	182.2	272.1	160.64	151.78	207.1	182.92	109.98	283.45	287.62	76.25	76.91	3.81	3.32	2.59	2.56	3.2	1.28	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	18.04	6.82	0.27	30.7
Both	18	37.42	23.22	60.64	298.44	272.46	262.3	216.34	262.3	122.79	143.6	188.78	179.17	109.92	278.39	81.72	3.07	1.3	2.56	2.95	2.5	2.27	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	9.83	0.25	27.33
Both	20	37.19	23.93	61.12	284.63	170.17	171.93	178.8	124.26	193.32	141.78	247.84	120.13	77.59	80.29	88.6	43.08	21.42	16.3	10.36	8.16	5.93	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	18.03	7.09	0.9	29.2
Both|Opt(Buy)	0	45.89	26.82	72.71	62.46	48.23	51.43	45.06	49.56	38.62	30.72	34.4	27.15	30.39	42.77	17.45	15.16	13.99	12.79	3.35	3.26	10.09	21.89	21.89	21.89	21.89	21.89	21.89	21.89	21.89	21.89	21.89	21.89	21.89	24.82	5.83	15.24
Both|Opt(Buy)	2	43.2	27.29	70.48	68.4	55.96	44.97	41.74	51.49	56.09	25.81	29.38	26.44	34.05	28.28	24.76	15.81	13.63	7.46	3.35	3.45	0.14	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	32.24	8.02	2.94
Both|Opt(Buy)	4	40.53	28.88	69.42	59.25	48.08	44.64	45.99	36.53	31.57	26.02	36.76	31.87	26.59	34.33	14.09	16.99	12.87	12.77	3.96	12.27	10.69	22.75	22.75	22.75	22.75	22.75	22.75	22.75	22.75	22.75	22.75	22.75	22.75	28.1	8.06	4.38
Both|Opt(Buy)	6	37.36	29.07	66.43	51.54	48.31	44.23	42.19	36.87	37	22.58	29.28	24.71	34.2	37.08	18.63	19.2	17.91	14.19	6.16	13.83	11.65	24.05	24.05	24.05	24.05	24.05	24.05	24.05	24.05	24.05	24.05	24.05	24.05	26.54	10.31	0.51
Both|Opt(Buy)	8	35.73	30.83	66.56	52.97	49.02	47.37	40.99	37.01	33.76	25.18	29.24	32.29	31.42	24.94	23.43	16.71	13.89	5.77	5.93	8.98	6.25	24.17	24.17	24.17	24.17	24.17	24.17	24.17	24.17	24.17	24.17	24.17	24.17	28.26	7.45	0.02
Both|Opt(Buy)	10	34.42	31.54	65.96	53.3	49.32	46.09	44.39	40.16	40.41	26.5	32.47	37.18	43.43	29.61	28.22	15.26	11.66	8.07	3.32	7.83	2.17	24.32	24.32	24.32	24.32	24.32	24.32	24.32	24.32	24.32	24.32	24.32	24.32	27.53	6.59	0.31
Both|Opt(Buy)	12	33.75	31.99	65.74	55.04	49.3	47.94	43.73	46.07	33.83	26.09	37.56	31.38	25.55	22.63	24.26	16.08	11.74	10.36	13.02	10.91	8.88	24.03	24.03	24.03	24.03	24.03	24.03	24.03	24.03	24.03	24.03	24.03	24.03	25.6	7.5	0.64
Both|Opt(Buy)	14	32.76	32.42	65.17	59.24	54.81	50.45	48.9	43.34	45.8	28.97	32.6	36.99	24.86	24.2	24.7	12.69	8.79	6.25	11.78	9.05	8.7	30.25	30.25	30.25	30.25	30.25	30.25	30.25	30.25	30.25	30.25	30.25	30.25	26.11	6.65	0
Both|Opt(Buy)	16	32.36	31.99	64.35	57.68	55.38	54.02	48.82	39.12	46.28	25.52	29.6	35.64	29.59	30.28	30.57	16.65	12.54	11.56	11.77	7.19	5.39	25.59	25.59	25.59	25.59	25.59	25.59	25.59	25.59	25.59	25.59	25.59	25.59	24.37	7.61	0.39
Both|Opt(Buy)	18	32.12	32.46	64.58	56.5	52.49	49.99	42.89	39.64	34.64	24.87	27.68	22.54	26.85	43.41	30.85	20.03	16.31	11.28	14.64	18.33	5.05	26.15	26.15	26.15	26.15	26.15	26.15	26.15	26.15	26.15	26.15	26.15	26.15	24.13	7.99	0.01
Both|Opt(Buy)	20	32.17	34.03	66.2	53.94	47.98	47.08	41.28	42.12	40.99	36.52	32.3	35.57	33.45	39.2	18.47	16.72	12.6	11.8	3.15	6.1	11.91	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	22.83	5.61	3.74

Table 16: Extended Results for Figure 14 about comparative statics for σ

p_t	x_a	Rev.	Welf.	Tot. Welf.	p_b																														Rev. (p_b)	Rev. (p_u)	Rev. (p_s)
					1	2	3	4	5	6	7	8	9	10	11	12	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12			
Buy	0	29.41	27.09	56.5	59.31	51.8	48.93	44.52	45.64	36.56	29.42	29.12	28.58	33.03	27.46	25.05	12.39	8.98	5.22	1.77	5.39	4.83	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	23.76	5.65	0
Buy	2	31.89	28.57	60.46	52.05	48.13	43.86	42.58	37.37	39.9	24.02	43.19	31.29	37.16	24.43	20.99	17.74	16.23	10.12	6.71	10.34	9.23	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	23.64	8.25	0
Buy	4	33.63	31.09	64.72	51.81	48.33	44.15	42.7	43.97	33.13	24.24	24.29	32.76	30.43	38.86	22.42	17.42	15.64	9.44	6.61	14.03	7.55	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	25.44	8.19	0
Buy	6	34.83	32.59	67.42	55.86	50.14	47.5	53.93	52.62	34.1	29.57	37.14	30.3	34.59	29.09	19.79	13.7	12.33	3.77	1.51	7.94	9.59	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	28.9	5.93	0
Buy	8	36.44	33.2	69.65	52.63	50.18	44.63	49.2	38.52	35.35	24.79	31.87	25.81	33.9	29.58	24.45	17.39	14.12	16.83	10.47	3.91	8.91	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	28.54	7.9	0
Buy	10	37.18	34.15	71.33	51.81	48.55	46.67	43.81	36.9	30.44	29.77	26.22	33.33	22.54	37.03	26.22	19.25	16.43	13.75	13.41	11.03	3.63	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	28.66	8.52	0
Buy	12	38	35.69	73.69	53.49	49.32	45.55	44.49	43.16	36.23	28.5	32.41	24.36	30.16	30.45	18.55	15.27	11.3	9.75	7.48	11.25	11.41	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	30.66	7.34	0
Buy	14	38.51	35.98	74.49	53.87	52.3	47.38	44.39	47.24	33.32	31.24	47.89	29.59	33.79	26.96	25.29	15.28	13.8	8.55	7.76	4.63	4.46	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	31.79	6.72	0
Buy	16	39.35	36.07	75.42	54.65	51.89	50.94	52.72	43.37	51.99	27.87	35.68	39.55	26.53	22.95	26.94	14.27	12.46	4.86	13.81	10.41	8.58	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	32.23	7.12	0
Buy	18	39.75	37.11	76.87	52.91	49.69	48.48	45.18	47.23	38.94	24.79	27.51	27.12	21.77	25.05	18.33	16.96	12.14	11.22	13.42	8.17	11.74	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	31.33	8.42	0
Buy	20	40.26	36.54	76.8	51.87	48.49	44.18	39.74	42.57	33.94	36.54	46.8	31.77	33.93	30.09	27.83	19.73	18.15	18.82	18.82	3.12	7.26	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	32.26	8	0
Sub	0	30.71	19.79	50.5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	14.55	0	0	30.71
Sub	2	33.59	22.43	56.01	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	14.54	14.54	14.54	14.54	14.54	14.54	14.54	14.54	14.54	14.54	14.54	14.54	0	0	33.59
Sub	4	35.92	20.45	56.37	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	35.92
Sub	6	37.68	21.75	59.42	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	0	0	37.68
Sub	8	39.1	22.74	61.84	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	39.1
Sub	10	40.19	23.63	63.82	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	0	0	40.19
Sub	12	41.15	24.26	65.4	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	41.15
Sub	14	41.95	24.81	66.76	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	41.95
Sub	16	42.61	25.31	67.92	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	0	0	42.61
Sub	18	43.17	25.74	68.91	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	0	0	43.17
Sub	20	43.65	26.13	69.78	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	16.22	0	0	43.65
Both	0	34.05	18.72	52.77	228.53	156.05	162.19	136.95	223.24	204.64	110.33	101.33	120.54	156.26	73.6	75.59	43.94	37.02	26.23	19.33	6.44	4.39	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	7.12	0.6	26.32
Both	2	37.06	20.97	58.03	225.89	167.36	200.46	195.81	148.03	153.44	86.93	78.12	118.82	207.97	87.64	61.54	197.5	171.96	33.22	163.91	48.62	10.38	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	16.24	5.61	0.95	30.51
Both	4	39.3	24.01	63.31	259.68	125.49	237.82	138.32	214.1	204.92	118.3	142.23	130.97	255.47	211.59	66.2	17.71	15.04	15.66	15.9	1.26	1.91	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	16.23	10.64	0.97	27.69
Both	6	41.13	24.88	66	289.66	129.5	174.31	191.95	173.49	105.33	175.93	128.36	111.39	120.97	266.18	69.05	13.29	6.76	2.79	10.22	5.89	1.33	16.6	16.6	16.6	16.6	16.6	16.6	16.6	16.6	16.6	16.6	16.6	16.6	13.75	1.19	26.19
Both	8	42.52	24.42	66.94	238.85	234.96	157.09	97.47	117.71	141.45	92.17	226.32	83.82	88.54	104.82	55.24	28.62	24.28	12.48	11.71	17.14	21.06	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	17.29	11.77	2.71	28.04
Both	10	43.52	23.92	67.44	297.22	163.18	275.57	132.58	188.36	204.99	151.8	249.3	123.37	82.29	80.09	191.3	11.27	6.53	7.1	4.44	8.79	5.98	17.31	17.31	17.31	17.31	17.31	17.31	17.31	17.31	17.31	17.31	17.31	17.31	12.18	0.91	30.43
Both	12	44.68	25.66	70.34	288.14	183.23	194.31	121.06	284.74	264.09	283.28	283.77	132.54	71.78	70.01	85.67	25.34	20.99	17.23	25.19	9.61	7.53	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	16.65	11.16	2.03	31.49
Both	14	45.62	26.72	72.35	252.29	190.62	218.81	110.47	130.75	227.27	212.35	73.86	105.72	118.43	131.43	73.52	20.98	11.45	18.12	9.88	9.98	14.52	17.22	17.22	17.22	17.22	17.22	17.22	17.22	17.22	17.22	17.22	17.22	17.22	12.83	2.25	30.54
Both	16	46.4	27.23	73.63	266.83	257.05	247.91	109.25	141.96	157.93	173.26	166.4	173.69	119.73	74.37	98.24	13.3	5.42	5.25	1.61	4.78	8.08	17.28	17.28	17.28	17.28	17.28	17.28	17.28	17.28	17.28	17.28	17.28	17.28	13.2	1.39	31.81
Both	18	47.02	27.83	74.85	270.73	236.64	124.17	170.61	164.87	209.72	99.14	181.24	115.03	178.42	180.72	72.62	8.81	8.08	6.91	2.1	1.08	7.03	17.3	17.3	17.3	17.3	17.3	17.3	17.3	17.3	17.3	17.3	17.3	17.3	16	1.19	29.83
Both	20	47.48	28.38	75.87	288.56	237.52	126.86	127.09	208.63	243.04	148.03	122.25	149.46	259.13	72.27	152.13	6.21	1.5	0.73	5.56	4.01	1.16	17.26	17.26	17.26	17.26	17.26	17.26	17.26	17.26	17.26	17.26	17.26	17.26	15.36	0.75	31.38
Both|Opt(Buy)	0	29.41	27.09	56.5	59.31	51.8	48.93	44.52	45.64	36.56	29.42	29.12	28.58	33.03	27.46	25.05	12.39	8.98	5.22	1.77	5.39	4.83	40.04	40.04	40.04	40.04	40.04	40.04	40.04	40.04	40.04	40.04	40.04	40.04	23.76	5.65	0
Both|Opt(Buy)	2	32.02	28.6	60.62	52.05	48.13	43.86	42.58	37.37	39.9	24.02	43.19	31.29	37.16	24.43	20.99	17.74	16.23	10.12	6.71	10.34	9.23	25.25	25.25	25.25	25.25	25.25	25.25	25.25	25.25	25.25	25.25	25.25	25.25	23.25	8.12	0.66
Both|Opt(Buy)	4	33.63	31.09	64.72	51.81	48.33	44.15	42.7	43.97	33.13	24.24	24.29	32.76	30.43	38.86	22.42	17.42	15.64	9.44	6.61	14.03	7.55	39.76	39.76	39.76	39.76	39.76	39.76	39.76	39.76	39.76	39.76	39.76	39.76	25.44	8.19	0
Both|Opt(Buy)	6	35.03	32.65	67.68	55.86	50.14	47.5	53.93	52.62	34.1	29.57	37.14	30.3	34.59	29.09	19.79	13.7	12.33	3.77	1.51	7.94	9.59	23.01	23.01	23.01	23.01	23.01	23.01	23.01	23.01	23.01	23.01	23.01	23.01	25	5.25	4.79
Both|Opt(Buy)	8	36.5	33.22	69.72	52.63	50.18	44.63	49.2	38.52	35.35	24.79	31.87	25.81	33.9	29.58	24.45	17.39	14.12	16.83	10.47	3.91	8.91	23.75	23.75	23.75	23.75	23.75	23.75	23.75	23.75	23.75	23.75	23.75	23.75	28.21	7.75	0.54
Both|Opt(Buy)	10	37.32	34.22	71.53	51.81	48.55	46.67	43.81	36.9	30.44	29.77	26.22	33.33	22.54	37.03	26.22	19.25	16.43	13.75	13.41	11.03	3.63	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	23.45	28.34	8.32	0.66
Both|Opt(Buy)	12	38.08	35.73	73.82	53.49	49.32	45.55	44.49	43.16	36.23	28.5	32.41	24.36	30.16	30.45	18.55	15.27	11.3	9.75	7.48	11.25	11.41	23.16	23.16	23.16	23.16	23.16	23.16	23.16	23.16	23.16	23.16	23.16	23.16	29.62	6.84	1.63
Both|Opt(Buy)	14	38.71	36.03	74.74	53.87	52.3	47.38	44.39	47.24	33.32	31.24	47.89	29.59	33.79	26.96	25.29	15.28	13.8	8.55	7.76	4.63	4.46	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	23.2	31.2	6.46	1.05
Both|Opt(Buy)	16	39.47	36.11	75.58	54.65	51.89	50.94	52.72	43.37	51.99	27.87	35.68	39.55	26.53	22.95	26.94	14.27	12.46	4.86	13.81	10.41	8.58	23.64	23.64	23.64	23.64	23.64	23.64	23.64	23.64	23.64	23.64	23.64	23.64	31.81	6.98	0.67
Both|Opt(Buy)	18	39.75	37.11	76.87	52.91	49.69	48.48	45.18	47.23	38.94	24.79	27.51	27.12	21.77	25.05	18.33	16.96	12.14	11.22	13.42	8.17	11.74	32.66	32.66	32.66	32.66	32.66	32.66	32.66	32.66	32.66	32.66	32.66	32.66	31.33	8.42	0
Both|Opt(Buy)	20	40.53	36.68	77.21	51.87	48.49	44.18	39.74	42.57	33.94	36.54	46.8	31.77	33.93	30.09	27.83	19.73	18.15	18.82	18.82	3.12	7.26	23.59	23.59	23.59	23.59	23.59	23.59	23.59	23.59	23.59	23.59	23.59	23.59	31.58	7.86	1.09

Table 17: Extended Results for Figure 15 about comparative statics for x_a