A naive model for the vertical motion of an air bubble in water

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Introduction

I attended courses in free and SCUBA diving and they explained us the basic gas laws, during a lesson I was asking myself: what is the motion of an air bubble generated as some depth in the sea?

A naive model

The motion equation in one dimension is

$$m \frac{d^2 y}{dt^2} = F$$

where t is time, y is the depth in the water and F sums all the relevant forces applied to the bubble.
$y = 0$ is the water level and $y < 0$ is below the water.
F is the sum of just two component:

$$F = -m\,g + A$$

where m is the mass of the bubble, g is the gravity acceleration and A is the Archimede's force. The Archimede's force is the product of the density ρ of the water, the volume V of the bubble and the the gravity acceleration: $A = \rho\,V\,g$.
But the volume of the air bubble depends on the pressure p according to the perfect gas law:
$p\,V = n\,R\,T$ where n is the number of moles of air, R is the gas constant and T is the temperature.
The pressure p at depth y is due to the water column (so called hydrostatic pressure): $p = \rho\,g\,|\,y\,|$ (it's valid only when $y < 0$).
The number of moles n is related to the molar mass of the air M and the bubble mass m through $n = \frac{m}{M}$.
So

$$V = \frac{n\,R\,T}{p} = \frac{m}{M} \cdot \frac{R\,T}{\rho\,g\,|\,y\,|}$$

and

$$F = -m\,g + \rho\,V\,g$$

Substituting V in F I get

$$F = -m\,g + \rho\,g \cdot \frac{m}{M} \cdot \frac{R\,T}{\rho\,g\,|\,y\,|}$$

and I can simplify $\rho\,g$ (so this model is valid for any fluid? Uhmmm... first doubt about my naive model) to get

$$F = -m\,g + \frac{m}{M} \cdot \frac{R\,T}{|y|}$$

Now the sum of forces must be balanced by the product of the mass of the bubble and its acceleration:

$$m\,\frac{d^2 y}{dt^2} = -m\,g + \frac{m}{M} \cdot \frac{R\,T}{|y|}$$

I can simplify the mass of the bubble (uhm... second doubt about this model):

$$\frac{d^2 y}{dt^2} = -g + \frac{R\,T}{M} \cdot \frac{1}{|y|}$$

So let’s try to solve it symbolically!

Solution of the differential equation

In[1]:=
```
DSolve[y''[t] == -g + (R * T) / (M * Abs[y[t]]), y[t], t]
```

Out[1]=

$$\text{Solve}\left[\int_1^{y[t]} \frac{1}{\sqrt{c_1 + 2\int_1^{K[2]} \frac{R\,T - g\,M\,\text{Abs}[K[1]]}{M\,\text{Abs}[K[1]]}\,dK[1]}}\,dK[2]^2 == (t + c_2)^2,\ y[t]\right]$$

OK, it seems it cannot be solved... so I look for a numerical solution; first of all I set some values (in SI units):

In[2]:=
```
R := 8.31
T := 293
M := 28.97 / 1000
g := 9.8
h := -10
```

And then solve it numerically:

In[7]:=
```
ysol = NDSolveValue[{y''[t] == -g + (R * T) / (M * Abs[y[t]]), y[0] == h, y'[0] == 0}, y, {t, 0, 10}]
```

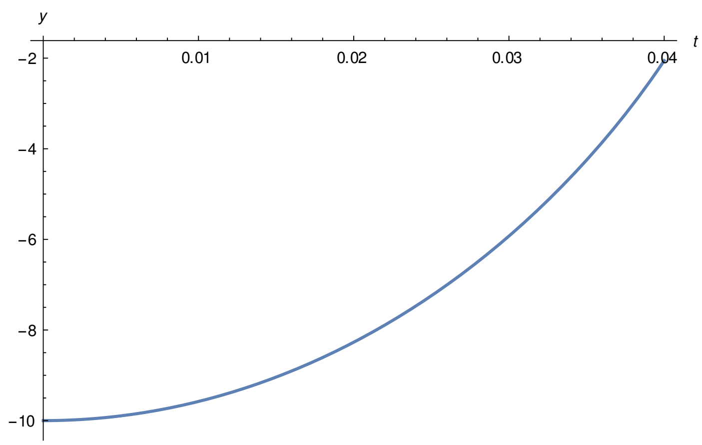
NDSolveValue : At t == 0.04325234866474409` , step size is effectively zero; singularity or stiff system suspected .

Out[7]=
```
InterpolatingFunction[ Domain: {{0., 0.0433}} Output: scalar ]
```

I would say the problem is the *singularity* (even because I do not know what a *stiff system* is ☺).
This is the plot of the depth y vs time t:

In[8]:=
```
Plot[ysol[t], {t, 0, 0.04}, AxesLabel → {t, y}]
```

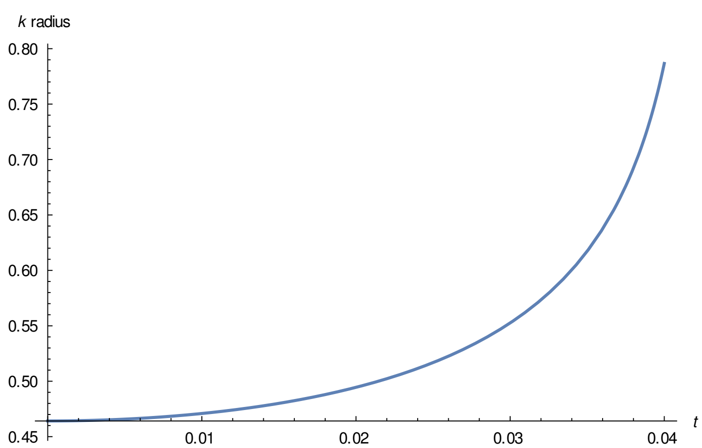
Out[8]=



And this is the plot of the bubble radius multiplied by a constant *k*:

In[9]:=
```
kradius[t_] := CubeRoot[-1 / ysol[t]]
Plot[kradius[t], {t, 0, 0.04}, AxesLabel → {t, k radius}]
```

Out[10]=

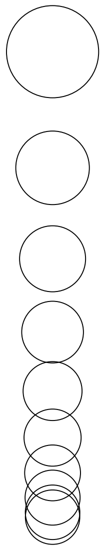


Now I would like to draw the bubble in ten equispaced time instants (copied from https://mathematica.stackexchange.com/a/207586)

In[11]:=
```
bubble[t_] := Circle[{0, ysol[t]}, kradius[t]]
```

In[12]:=
```
Graphics[{bubble /@ Array[# &, 10, {0, 0.04}]}, ImageSize → 50]
```

Out[12]=

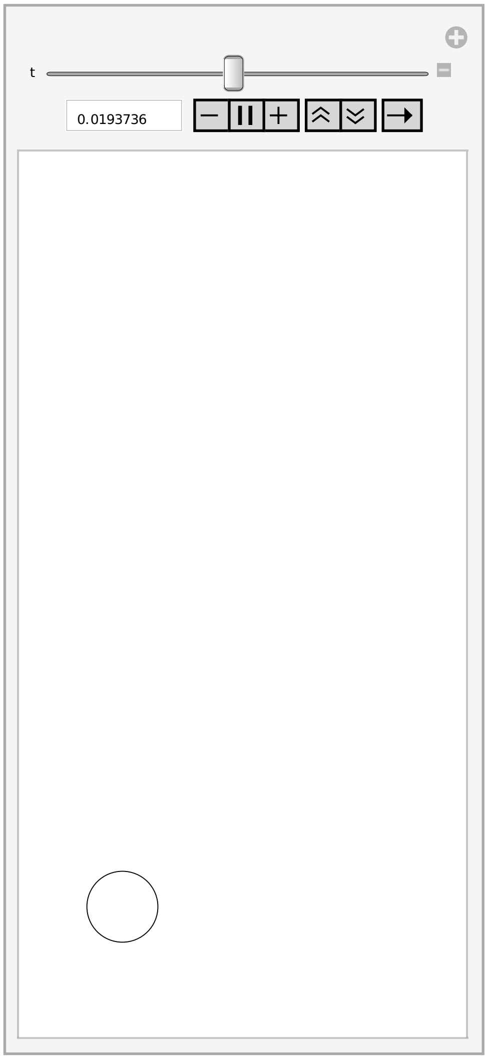


An animation

An animation (copied from https://mathematica.stackexchange.com/a/17205):

In[13]:=
```
Manipulate [
 Graphics [{Circle [{0, ysol [t]}, kradius [t]]}, PlotRange → {{-1, 1}, {-11, 0}}], {t, 0, 0.04}]
```

Out[13]=



What’s wrong with this model?

The main problem I can see, based on my limited experience in observing bubbles, is that it neglects the “friction” of the water acting on the moving bubble: I remember that I saw bubbles starting as sphere but then deforming into mushroom-like shapes.
Then I have doubts about the mass of my bubble.

I think this model will make physicists laughing (and numerical analysts too)... but I am not able to grasp the 1917 formulation by Lord Rayleigh and my Mathematica trial license is soon expiring! If I had had more time I would have tried to solve the Lord Rayleigh’s equation.

Further readings

Lord Rayleigh O.M. F.R.S. (1917) VIII. *On the pressure developed in a liquid during the collapse of a spherical cavity*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 34:200, 94-98, DOI: 10.1080/14786440808635681