Digital Communication Notes

Alessandro Vulcu

13 ottobre 2025

Indice

1   Transforms

1.1   Fourier Transforms

1.1.1   Continuous-Time Fourier Transform

A well-behaved continuous-time function $x(t)$ and its Fourier transform $X(f)$ are related by the *analysis* and *synthesis* equations:

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\, dt \qquad \text{(transform)} \qquad (A.1)$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}\, df \qquad \text{(inverse transform)} \qquad (A.2)$$

1.1.2   Continuous-Time Fourier Series

Periodic functions do not fall under the umbrella of well-behaved functions. Yet they are very important in the analysis of communication signals. We can side-step this problem by using the definition of the Fourier series and by invoking the *Dirac delta function* $\delta(t)$.

Consider a periodic signal $x(t)$ whose period $T > 0$ is the smallest real number such that $x(t) = x(t + T)$. The continuous-time Fourier series of such signal is defined as:

$$X[n] = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi nt}{T}}\, dt \qquad \text{(analysis)} \qquad (A.3)$$

$$x(t) = \sum_n X[n]e^{j\frac{2\pi nt}{T}} \qquad \text{(synthesis)} \qquad (A.4)$$

1.1.3   Table A.1: Continuous-Time Fourier Transform Properties

Property	Aperiodic Signal	Fourier Transform
Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Time shift	$x(t - t_0)$	$e^{-j2\pi ft_0}X(f)$
Conjugation	$x^*(t)$	$X^*(-f)$
Time reversal	$x(-t)$	$X(-f)$
Time scaling	$x(at)$	$\frac{1}{|a|}X\left(\frac{f}{a}\right)$
Convolution	$x(t) * y(t) = \int x(\tau)y(t - \tau)\, d\tau$	$X(f)Y(f)$
Autocorrelation	$x(t) * x^*(-t)$	$|X(f)|^2$
Modulation	$x(t)e^{j2\pi f_0 t}$	$X(f - f_0)$
Conjugate symmetry	$x(t)$ real	$X(f) = X^*(-f)$
Duality	$x(t) \longleftrightarrow X(f)$	$X(t) \longleftrightarrow x(-f)$
Parseval's theorem	$\int x(t)y^*(t)\, dt$	$\int X(f)Y^*(f)\, df$

1.1.4 Table A.2: Continuous-Time Fourier Transform Pairs

Function	Time-domain	Frequency-domain
Impulse	$\delta(t)$	1
Constant function	1	$\delta(f)$
Complex exponential	$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
Cosine	$\cos(2\pi f_0 t + \theta)$	$\frac{1}{2}\left[e^{j\theta}\delta(f - f_0) + e^{-j\theta}\delta(f + f_0)\right]$
Sine	$\sin(2\pi f_0 t + \theta)$	$\frac{1}{2j}\left[e^{j\theta}\delta(f - f_0) - e^{-j\theta}\delta(f + f_0)\right]$
Impulse train	$\sum_k \delta(t - kT)$	$\frac{1}{T}\sum_n \delta\left(f - \frac{n}{T}\right)$
Rectangular pulse	$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & \lvert t\rvert \le \frac{T}{2} \\ 0, & \text{else} \end{cases}$	$T\,\mathrm{sinc}(fT) = T\,\frac{\sin(\pi fT)}{\pi f}$
Bandlimited pulse	$W\,\mathrm{sinc}^2(fW)$	—
Sinc pulse	$\mathrm{sinc}(Wt) = \frac{\sin(\pi Wt)}{\pi Wt}$	$\frac{1}{W}\,\mathrm{rect}\left(\frac{f}{W}\right)$

The continuous-time Fourier series creates as an output a weighting (ponderazione, quanto peso) of the fundamental frequency of the signal $e^{j\frac{2\pi t}{T}}$ and its harmonics.

We can express the Fourier transform of a periodic signal as:

$$X(f) = \sum_n X[n]\,\delta\left(f - \frac{n}{T}\right) \tag{A.5}$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\,e^{j2\pi ft}\,df \tag{A.6}$$

where the unboundedness of (A.1) is circumvented by means of $\delta(f)$.

Dato che un segnale periodico ha durata infinita, le sue trasformate non sono funzioni ordinarie, ma distribuzioni formate da impulsi di Dirac.

1.2 Discrete-Time Fourier Transform

Consider now a well-behaved discrete-time signal $x[n]$. Its discrete-time Fourier transform analysis and synthesis relationships are:

$$X(\nu) = \sum_n x[n]\,e^{-j2\pi n\nu} \tag{A.7}$$

$$x[n] = \int_{-1/2}^{1/2} X(\nu)\,e^{j2\pi n\nu}\,d\nu \tag{A.8}$$

where the frequency ν is defined on any finite interval of unit length, typically $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

1.3 Discrete Fourier Transform

While the discrete-time Fourier transform offers a sound analytical framework for discrete-time signals, it is inconvenient due to its continuous frequency.

So an alternative is the **DFT** and the **Fast Fourier Transform (FFT)**.

The DFT is extremely important in digital signal processing and communications.

It applies to finite-length discrete-time signals and, since any finite-length signal can be repeated to form a periodic discrete-time signal, the DFT can also be interpreted as applying to periodic signals.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad\qquad k = 0, \ldots, N-1 \qquad\qquad (A.9)$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \qquad\qquad n = 0, \ldots, N-1 \qquad\qquad (A.10)$$

Length-N sequence	N-point DFT
$x[n]$	$X[k]$
$ax[n] + by[n]$	$aX[k] + bY[k]$
$x[n]$	$NX[((-k))_N]$
$x[(n-m))_N]$	$e^{j2\pi km/N} X[k]$
$\sum_m x[m]y[((n-m))_N]$	$X[k]Y[k]$
$x^*[n]$	$X^*[((-k))_N]$
$x[n]$ real	$X[k] = X^*[((-k))_N]$

1.4 Z-Transform

The Z-transform converts a function of a discrete real variable to a function of a complex variable z. This converts difference equations into algebraic equations and convolution into products.

The Z-transform of a causal function $x[n]$ is:

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \qquad\qquad (A.13)$$

while the inversion of $X(z)$ back onto $x[n]$ requires an integration on the complex plane.

2 Matrix Algebra

2.1 Column Space, Row Space, Null Spaces

The **column space** of an $N \times M$ matrix A is the set of all linear combinations of its column vectors. It is therefore a subspace (whose dimension is at most M) of the N-dimensional vector space.

We can write the linear combination as the product of A with the vector $\mathbf{x} = [x_0, \ldots, x_{M-1}]^T$:

$$A \begin{bmatrix} x_0 \\ \vdots \\ x_{M-1} \end{bmatrix} = \mathbf{y} = A\mathbf{x} \qquad\qquad (B.1)$$

2.1.1 Example B.1

The column space of

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.2}$$

is the set of vectors $\mathbf{y} = [y_0\ y_1\ y_2]^T$ having the form

$$\mathbf{y} = A\mathbf{x} \tag{B.3}$$

$$= \begin{bmatrix} 3x_1 \\ 2x_0 \\ x_1 \end{bmatrix}. \tag{B.4}$$

These vectors satisfy $y_0 = 3y_2$, which defines a subspace of dimension $M = 2$ (that is, a plane) on a vector space of dimension $N = 3$.

2.1.2 Example B.2

The **row space** of A in (B.2), in turn, is the set of vectors $\mathbf{y}$ having the form

$$\mathbf{y} = A^T\mathbf{x} \tag{B.5}$$

$$= \begin{bmatrix} 2x_1 \\ 3x_0 + x_2 \end{bmatrix}, \tag{B.6}$$

which defines the entire vector space of dimension $M = 2$.

The column rank and row rank correspond to the dimensions of the column space and row space, respectively.

The fact that the column and row ranks coincide in this case is not a coincidence. The row and column ranks always coincide, giving the **rank of a matrix**.

In addition, we have also two additional subspaces:

- **Orthogonal complement of row space** (null space of A): all vectors satisfying $A\mathbf{x} = 0$, which has dimension $M - \text{rank}(A)$.

- **Orthogonal complement of column space** (null space of A^T): with dimension $N - \text{rank}(A)$.

3 Special Matrices

3.1 Hermitian Matrices

A complex matrix A is said to be **Hermitian** if $A^* = A$.

- They are quadratic.

- They have real elements on the diagonal.

- Off-diagonal elements are complex conjugates of each other.

- Eigenvalues are always real.

$$A = \begin{bmatrix} 2 & 1+i & 4 \\ 1-i & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

3.2 Unitary Matrices

A complex matrix U is said to be **unitary** if $U^*U = UU^* = I$.

- U is nonsingular, and $U^* = U^{-1}$.

- The columns of U form an orthonormal set, as do the rows of U.

- For any complex vector $\mathbf{x}$, the vector $\mathbf{y} = U\mathbf{x}$ satisfies $|\mathbf{y}| = |\mathbf{x}|$. Thus, $\mathbf{y}$ is a rotated version of $\mathbf{x}$, and U embodies that rotation.

3.3 Fourier Matrices

An $N \times N$ **Fourier matrix** U is a unitary matrix whose (i, j)th entry equals $e^{j2\pi ij/N}$. It follows that the jth column, for $j = 0, \ldots, N-1$, is given by

$$U_j = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi j/N} \\ \vdots \\ e^{j2\pi(N-1)j/N} \end{bmatrix} \tag{B.7}$$

The **DFT** of a vector $\mathbf{x}$ is

$$\mathbf{X} = \sqrt{N}\, U^*\mathbf{x} \tag{B.8}$$

and the **IDFT** is

$$\mathbf{x} = \frac{1}{\sqrt{N}} U\mathbf{X} \tag{B.9}$$

Indeed, by interpreting the entries of $\mathbf{x}$ and $\mathbf{X}$ as sequences, (B.8) and (B.9) are scaled versions of the $\text{DFT}_N\{\cdot\}$ and $\text{IDFT}_N\{\cdot\}$ transforms in (A.9) and (A.10).

3.4 Toeplitz and Circulant Matrices

A **Toeplitz matrix** is constant along each of its diagonals.

A **Toeplitz circular matrix** is completely described by any of its rows, of which the other rows are just circular shifts with offsets to the row indices:

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 5 \\ 5 & 1 & 2 \end{bmatrix}. \tag{B.10}$$

If A is an $N \times N$ **circulant matrix**, then the following holds:

- The eigenvectors of A equal the columns of the Fourier matrix U in (B.7).

- The eigenvalues of A equal the entries of $U^* \mathbf{a}$, where $\mathbf{a}$ is any column of A.

Hence, the eigenvalues of a circulant matrix are directly the DFT of any o

4 Matrix Decompositions

4.1 Eigenvalue Decomposition

Any $N \times N$ Hermitian matrix A can be decomposed as:

$$A = U \Lambda U^* = \sum_{i=0}^{N-1} \lambda_i(A) \, \mathbf{u}_i \mathbf{u}_i^*, \qquad \text{(B.12, B.13)}$$

where:

- $U = [\, \mathbf{u}_0 \; \cdots \; \mathbf{u}_{N-1} \,]$ is a unitary matrix whose columns are the eigenvectors of A.

- $\Lambda = \mathrm{diag}(\lambda_0(A), \dots, \lambda_{N-1}(A))$ is a diagonal matrix containing the eigenvalues of A.

Each eigenvector $\mathbf{u}_i$ satisfies:

$$A\mathbf{u}_i = \lambda_i(A)\mathbf{u}_i, \quad i = 0, \dots, N-1. \qquad \text{(B.14)}$$

The eigenvalues of a Hermitian matrix are always real. A Hermitian matrix A is:

- **Positive semidefinite** if, for every nonzero complex vector $\mathbf{x}$,

$$\mathbf{x}^* A \mathbf{x} \geq 0. \qquad \text{(B.15)}$$

- **Positive definite** if the inequality is strict.

If A has full rank ($\mathrm{rank}(A) = N$), it is invertible, and the eigenvectors of A and A^{-1} coincide. The inverse is given by:

$$A^{-1} = U \Lambda^{-1} U^* = U \, \mathrm{diag}\!\left(\frac{1}{\lambda_0(A)}, \dots, \frac{1}{\lambda_{N-1}(A)} \right) U^*. \qquad \text{(B.16, B.17)}$$

Eigenvectors are defined up to phase rotations; that is, $e^{j\phi}\mathbf{u}_i$ is also a valid eigenvector corresponding to the same eigenvalue.

4.2 Singular-Value Decomposition (SVD)

For any matrix $A \in \mathbb{C}^{N \times M}$ (or $\mathbb{R}^{N \times M}$), there exists a factorization:

$$A = U \, \Sigma \, V^*, \qquad \text{(B.18)}$$

where $U \in \mathbb{C}^{N \times N}$ and $V \in \mathbb{C}^{M \times M}$ are unitary (orthogonal if real), and $\Sigma \in \mathbb{R}^{N \times M}$ is a diagonal matrix with non-negative entries $\sigma_0 \geq \sigma_1 \geq \cdots \geq 0$, called the **singular values** of A.

Singular vectors. Let u_j and v_j denote the j-th left and right singular vectors, respectively. Then:

$$A v_j = \sigma_j u_j, \qquad A^* u_j = \sigma_j v_j, \qquad j = 0, \dots, \min(N, M) - 1. \qquad \text{(B.19, B.20)}$$

Relationship with eigenvalue decomposition.

- The *squared singular values* of A are the nonzero eigenvalues of A^*A and AA^*:

$$A^*A = V\,\Sigma^*\Sigma\,V^*, \qquad AA^* = U\,\Sigma\Sigma^*\,U^*, \qquad \lambda_j(A^*A) = \sigma_j^2.$$

- The right singular vectors v_j are the eigenvectors of A^*A.

- The left singular vectors u_j are the eigenvectors of AA^*.

Rank and fundamental subspaces. If $r = \text{rank}(A)$ is the number of nonzero singular values, then:

$$\dim\mathcal{R}(A) = \dim\mathcal{R}(A^*) = r, \qquad \dim\mathcal{N}(A) = M - r, \qquad \dim\mathcal{N}(A^*) = N - r.$$

Table B.1 — Bases of the column, row, and null spaces via SVD

Subspace	Dimension	Basis (from SVD)
Column space $\mathcal{R}(A)$	r	First r columns of U
Row space $\mathcal{R}(A^*)$	r	First r columns of V
Null space of A^*	$N - r$	Last $N - r$ columns of U
Null space of A	$M - r$	Last $M - r$ columns of V

Notes.

- The SVD provides orthonormal bases for the column, row, and null spaces of A.

- Each singular value σ_j measures how much A stretches the vector v_j along direction u_j.

4.3 QR Decomposition

Given $N \geq M$, any $N \times M$ matrix A can be factored as

$$A = QR, \tag{B.21}$$

where Q is an $N \times N$ unitary (orthogonal if real) matrix, and R is an $N \times M$ upper-triangular matrix whose last $(N - M)$ rows are zero.

This factorization is particularly useful for solving least-squares problems and for numerical stability in matrix computations.

4.4 Trace and Determinant

The **trace** of a square matrix equals the sum of the entries on its main diagonal, and also the sum of its eigenvalues:

$$\text{tr}(A) = \sum_i a_{ii} = \sum_i \lambda_i.$$

It is a linear operation invariant under change of basis and under pre- or post-multiplication by unitary matrices.

The **determinant** of a square matrix equals the product of its eigenvalues, each counted with its algebraic multiplicity. If the determinant is nonzero, the matrix is invertible; otherwise, it is singular. For an invertible matrix:

$$\det(A^{-1}) = \frac{1}{\det(A)}. \tag{B.22}$$

Useful determinant properties. For properly dimensioned matrices A and B:

$$\det(I + AB) = \det(I + BA), \tag{B.23}$$

where I denotes the identity matrix of suitable size. Also,

$$\det(AB) = \det(BA) = \det(A)\det(B). \tag{B.24, B.25}$$

Trace properties. The trace satisfies the cyclic property:

$$\text{tr}(AB) = \text{tr}(BA), \tag{B.26}$$

and, more generally, is invariant under cyclic permutations of factors.

Summary.

- $\text{tr}(A)$ is the sum of eigenvalues of A.

- $\det(A)$ is the product of eigenvalues of A.

- Both quantities are invariant under unitary transformations.

4.5 Frobenius Norm

The Frobenius norm of an $N \times M$ matrix A equals

$$\|A\|_F = \sqrt{\text{tr}(AA^*)}, \tag{B.27}$$

which can also be written as

$$\|A\|_F = \sqrt{\text{tr}(A^*A)}. \tag{B.28}$$

Expanding the argument of the square root, one obtains

$$\|A\|_F = \sqrt{\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} |A_{i,j}|^2}. \tag{B.29}$$

In the special case where A is a vector, this reduces to the standard Euclidean norm. Unless otherwise stated, this is the norm definition used throughout the text. The distance between two vectors is the norm of their difference, unless otherwise stated as Euclidean distance.

4.6 Moore–Penrose Pseudoinverse

The pseudoinverse generalizes the concept of a matrix inverse. For an $N \times M$ rectangular matrix A, the **Moore–Penrose pseudoinverse** $A^\dagger$ is the unique matrix satisfying:

$$AA^\dagger A = A, \tag{B.30}$$
$$A^\dagger AA^\dagger = A^\dagger, \tag{B.31}$$

and such that both $AA^\dagger$ and $A^\dagger A$ are Hermitian.

Special cases.

- If A^*A is invertible, it is easily verified that

$$A^\dagger = (A^*A)^{-1}A^*, \tag{B.32}$$

which satisfies $A^\dagger A = I$. This may be the case when $N \geq M$.

- If AA^* is invertible, then

$$A^\dagger = A^*(AA^*)^{-1}, \tag{B.33}$$

which satisfies $AA^\dagger = I$. This may occur when $N \leq M$.

- If A is square and invertible, both formulas coincide, and the pseudoinverse reduces to the regular inverse:

$$A^\dagger = A^{-1}.$$

4.7 Matrix Inversion Lemma

An identity that is often useful in many derivations is the **matrix inversion lemma**, which states that:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1}DA^{-1}. \tag{B.34}$$

This formula, also known as the **Woodbury matrix identity**, allows computing the inverse of the linear operator on the left-hand side whenever the inverses of its two main components, A and C, are known.

4.8 Kronecker Product

The **Kronecker product** extends the outer product of vectors to matrices. Given an $N_A \times M_A$ matrix A and an $N_B \times M_B$ matrix B, their Kronecker product is the $(N_A N_B) \times (M_A M_B)$ matrix defined as:

$$A \otimes B = \begin{bmatrix} [A]_{0,0}B & \cdots & [A]_{0,M_A-1}B \\ \vdots & \ddots & \vdots \\ [A]_{N_A-1,0}B & \cdots & [A]_{N_A-1,M_A-1}B \end{bmatrix}. \tag{B.35}$$

Like the regular matrix product, the Kronecker product is noncommutative, linear, and associative. In addition, the following properties hold:

$$(A \otimes B)^* = A^* \otimes B^*, \tag{B.36}$$

and, if both A and B are invertible,

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{B.37}$$

Finally, if A and B have r_A and r_B nonzero singular values, respectively, then $A \otimes B$ has $r_A r_B$ nonzero singular values given by all the cross-products of those singular values.

5 Random Variables and Processes

5.1 Random Variables

The random variables in this section are regarded as continuous unless otherwise indicated, yet most of the concepts extend to discrete distributions with a probability mass function (PMF) in place of the probability density function (PDF), and with a proper replacement of integrals by summations.

5.1.1 Bayes' Theorem

Given two random variables x and y with joint PDF $f_{xy}(\cdot, \cdot)$ and with marginals $f_x(\cdot)$ and $f_y(\cdot)$, the respective conditional distributions are obtained as:

$$f_{y|x}(y|x) = \frac{f_{xy}(x, y)}{f_x(x)}, \tag{C.1}$$

$$f_{x|y}(x|y) = \frac{f_{xy}(x, y)}{f_y(y)}, \tag{C.2}$$

respectively for $f_x(x) > 0$ and $f_y(y) > 0$.

Bayes' theorem then states:

$$f_{x|y}(x|y) = \frac{f_{y|x}(y|x) \, f_x(x)}{f_y(y)}. \tag{C.3}$$

Bayes' theorem has been said to play a role similar to that of Pythagoras' theorem in geometry — a comparison that seems appropriate given its importance and the triangular relationship it establishes between the joint distribution and the two conditionals.

5.2 Expectation

Given a real scalar x with PDF $f_x(\cdot)$, the expected value (also called the **mean** or **average**) of x can be obtained directly from $f_x(x)$ as:

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} x \, f_x(x) \, dx. \tag{C.4}$$

It can also be expressed in terms of the cumulative distribution function (CDF) $F_x(\cdot)$, where

$$F_x(x) = \int_{-\infty}^{x} f_x(\lambda) \, d\lambda, \quad \text{and} \quad f_x(x) = \frac{dF_x(x)}{dx},$$

through the relationship:

$$\mathbb{E}[x] = \int_{0}^{\infty} \left(1 - F_x(x)\right) dx - \int_{-\infty}^{0} F_x(x) \, dx. \tag{C.5}$$

This latter form is often useful when working with the CDF directly, since it avoids differentiating $F_x(x)$. If x is complex, the expression in (C.4) applies with integration over the complex plane. When integration limits are not explicitly indicated, they are to be understood as covering the support of the random variable — the region where its probability is nonzero.

5.3 Correlation

The covariance between two random scalars x and y, with respective means $\mu_x = \mathbb{E}[x]$ and $\mu_y = \mathbb{E}[y]$, is given by:

$$R_{xy} = \mathbb{E}\big[(x - \mu_x)(y - \mu_y)^*\big]. \tag{C.6}$$

When $x = y$, this reduces to the variance:

$$\text{var}[x] = \sigma_x^2 = R_{xx}.$$

In signal processing and communications, the term *correlation* is often used interchangeably with *covariance*. In statistics, however, the term *correlation* usually refers to the *correlation coefficient*, obtained by scaling:

$$\rho_{xy} = \frac{R_{xy}}{\sigma_x \sigma_y},$$

which ensures $\rho_{xy} \in [-1, 1]$.

For random vectors $\mathbf{x}$ and $\mathbf{y}$, the covariance (or correlation) matrix is defined as:

$$R_{xy} = \mathbb{E}\big[(\mathbf{x} - \mu_x)(\mathbf{y} - \mu_y)^*\big]. \tag{C.7}$$

If $\mathbf{x} = \mathbf{y}$, this reduces to the covariance (or correlation) matrix of $\mathbf{x}$, denoted R_{xx}. Properly scaling the entries of R_{xx} yields a matrix of correlation coefficients.

A concept frequently used in MIMO systems is that of **uncorrelatedness**: two variables x and y are said to be uncorrelated if $R_{xy} = 0$. If two random variables are independent, then they are also uncorrelated because:

$$\begin{aligned}
R_{xy} &= \mathbb{E}\big[(x - \mu_x)(y - \mu_y)^*\big] \\
&= \mathbb{E}[x - \mu_x]\,\mathbb{E}[y^* - \mu_y^*] \\
&= 0.
\end{aligned} \tag{C.8–C.10}$$

However, note that the converse is not necessarily true: two uncorrelated variables need not be independent.

$$\text{Independence:} \quad f_{xy}(x, y) = f_x(x)\, f_y(y).$$

5.4 Properness

Another important notion that distinguishes a relevant class of signals is **properness**. A complex random scalar x is said to be *proper complex* if

$$\mathbb{E}[x^2] = \mathbb{E}[|x|^2].$$

Properness requires that the real and imaginary parts of x, denoted $\Re\{x\}$ and $\Im\{x\}$, be **uncorrelated and have the same variance**.

This concept generalizes naturally to vectors: a complex random vector $\mathbf{x}$ is proper if

$$\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathsf{T}}] = \mathbb{E}[\mathbf{x}]\,\mathbb{E}[\mathbf{x}^{\mathsf{T}}].$$

In other words, $\Re\{\mathbf{x}\}$ and $\Im\{\mathbf{x}\}$ have identical covariance matrices, and their cross-covariance is zero.

Any subvector of a proper complex vector is also proper. For instance, if $\mathbf{x} = [x_0, x_1]^\mathsf{T}$ is proper, then both x_0 and x_1 are proper complex random variables. However, the converse is not necessarily true: two individually proper variables need not be jointly proper.

Properness is preserved under **affine transformations**. That is, if x is proper complex, then

$$y = Ax + b$$

is also proper complex for any constant matrix A and vector b.

5.5 Circular Symmetry

A random scalar x is said to be **circularly symmetric** if its distribution remains unchanged when x is rotated about its mean by an arbitrary angle ϕ, i.e.,

$$\text{the distribution of } (x - \mathbb{E}[x])e^{j\phi}$$

is identical to that of $(x - \mathbb{E}[x])$ for any arbitrary ϕ.

This property generalizes directly to random vectors and is preserved under affine transformations. For matrices, circular symmetry generalizes into the concept of **unitary invariance**, described below.

5.6 Unitary Invariance

A random matrix $\mathbf{X}$ is said to be **left unitarily invariant** if its distribution is identical to that of $\mathbf{UX}$ for any unitary matrix $\mathbf{U}$ independent of $\mathbf{X}$.

Similarly, $\mathbf{X}$ is **right unitarily invariant** if its distribution equals that of $\mathbf{XV}^*$ for any unitary matrix $\mathbf{V}$ independent of $\mathbf{X}$. If $\mathbf{X}$ is both left and right unitarily invariant, it is said to be **bi-unitarily invariant**.

5.7 Linear Transformations

If $\mathbf{x}$ is a complex random vector with PDF $f_{\mathbf{x}}(\cdot)$ and A is a nonsingular matrix, then the PDF of $\mathbf{y} = A\mathbf{x}$ is given by:

$$f_{\mathbf{y}}(y) = \frac{f_{\mathbf{x}}(A^{-1}y)}{|\det(A)|^2}. \tag{C.11}$$

5.8 Kurtosis

The relationship between the fourth- and second-order moments can be expressed in different ways. The definition adopted for the kurtosis of a real scalar x is:

$$\kappa(x) = \frac{\mathbb{E}[x^4]}{\mathbb{E}^2[x^2]}, \tag{C.12}$$

which satisfies $\kappa(x) \geq 1$, with equality holding if x is nonrandom.

If x is a zero-mean variable, this definition coincides with the common one used in statistics, where the fourth- and second-order moments are centered on the mean:

$$\kappa(x) = \frac{\mathbb{E}[(x - \mu)^4]}{\mathbb{E}^2[(x - \mu)^2]}.$$

This form is particularly suitable for analyzing transmit signals, which are typically zero-mean. However, for fading channels, the uncentered definition in (C.12) is often more convenient.

As an additional variant, the kurtosis can be adjusted so that it equals zero for a Gaussian variable. This adjusted version is known in signal processing as the **excess kurtosis**.