Digital Communication Notes

Alessandro Vulcu

6 ottobre 2025

Indice

1 Transforms

1.1 Fourier Transforms

1.1.1 Continuous-Time Fourier Transform

A well-behaved continuous-time function $x(t)$ and its Fourier transform $X(f)$ are related by the *analysis* and *synthesis* equations:

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,dt \qquad \text{(transform)} \tag{A.1}$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}\,df \qquad \text{(inverse transform)} \tag{A.2}$$

1.1.2 Continuous-Time Fourier Series

Periodic functions do not fall under the umbrella of well-behaved functions. Yet they are very important in the analysis of communication signals. We can side-step this problem by using the definition of the Fourier series and by invoking the *Dirac delta function* $\delta(t)$.

Consider a periodic signal $x(t)$ whose period $T > 0$ is the smallest real number such that $x(t) = x(t+T)$. The continuous-time Fourier series of such signal is defined as:

$$X[n] = \frac{1}{T}\int_{0}^{T} x(t)e^{-j\frac{2\pi nt}{T}}\,dt \qquad \text{(analysis)} \tag{A.3}$$

$$x(t) = \sum_{n} X[n]e^{j\frac{2\pi nt}{T}} \qquad \text{(synthesis)} \tag{A.4}$$

1.1.3 Table A.1: Continuous-Time Fourier Transform Properties

Property	**Aperiodic Signal**	**Fourier Transform**
Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Time shift	$x(t - t_0)$	$e^{-j2\pi ft_0}X(f)$
Conjugation	$x^*(t)$	$X^*(-f)$
Time reversal	$x(-t)$	$X(-f)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Convolution	$x(t) * y(t) = \int x(\tau)y(t-\tau)\,d\tau$	$X(f)Y(f)$
Autocorrelation	$x(t) * x^*(-t)$	$\lvert X(f)\rvert^2$
Modulation	$x(t)e^{j2\pi f_0 t}$	$X(f - f_0)$
Conjugate symmetry	$x(t)$ real	$X(f) = X^*(-f)$
Duality	$x(t) \longleftrightarrow X(f)$	$X(t) \longleftrightarrow x(-f)$
Parseval's theorem	$\int x(t)y^*(t)\,dt$	$\int X(f)Y^*(f)\,df$

1.1.4 Table A.2: Continuous-Time Fourier Transform Pairs

Function	Time-domain	Frequency-domain
Impulse	$\delta(t)$	1
Constant function	1	$\delta(f)$
Complex exponential	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
Cosine	$\cos(2\pi f_0 t+\theta)$	$\frac{1}{2}\big[e^{j\theta}\delta(f-f_0)+e^{-j\theta}\delta(f+f_0)\big]$
Sine	$\sin(2\pi f_0 t+\theta)$	$\frac{1}{2j}\big[e^{j\theta}\delta(f-f_0)-e^{-j\theta}\delta(f+f_0)\big]$
Impulse train	$\sum_k \delta(t-kT)$	$\frac{1}{T}\sum_n \delta(f-\frac{n}{T})$
Rectangular pulse	$\mathrm{rect}\big(\frac{t}{T}\big)=\begin{cases}1, & \lvert t\rvert\le\frac{T}{2}\\ 0, & \text{else}\end{cases}$	$T\ \mathrm{sinc}(fT)=T\,\frac{\sin(\pi fT)}{\pi f}$
Bandlimited pulse	$W\ \mathrm{sinc}^2(fW)$	—
Sinc pulse	$\mathrm{sinc}(Wt)=\frac{\sin(\pi Wt)}{\pi Wt}$	$\frac{1}{W}\,\mathrm{rect}\big(\frac{f}{W}\big)$

The continuous-time Fourier series creates as an output a weighting (ponderazione, quanto peso) of the fundamental frequency of the signal $e^{j\frac{2\pi t}{T}}$ and its harmonics.

We can express the Fourier transform of a periodic signal as:

$$X(f)=\sum_n X[n]\,\delta\Big(f-\frac{n}{T}\Big) \tag{A.5}$$

$$x(t)=\int_{-\infty}^{+\infty} X(f)\,e^{j2\pi ft}\,df \tag{A.6}$$

where the unboundedness of (A.1) is circumvented by means of $\delta(f)$.

Dato che un segnale periodico ha durata infinita, le sue trasformate non sono funzioni ordinarie, ma distribuzioni formate da impulsi di Dirac.

1.2 Discrete-Time Fourier Transform

Consider now a well-behaved discrete-time signal $x[n]$. Its discrete-time Fourier transform analysis and synthesis relationships are:

$$X(\nu)=\sum_n x[n]\,e^{-j2\pi n\nu} \tag{A.7}$$

$$x[n]=\int_{-1/2}^{1/2} X(\nu)\,e^{j2\pi n\nu}\,d\nu \tag{A.8}$$

where the frequency ν is defined on any finite interval of unit length, typically $[-\frac{1}{2},\frac{1}{2}]$.

1.3 Discrete Fourier Transform

While the discrete-time Fourier transform offers a sound analytical framework for discrete-time signals, it is inconvenient due to its continuous frequency.

So an alternative is the **DFT** and the **Fast Fourier Transform (FFT)**.

The DFT is extremely important in digital signal processing and communications.

It applies to finite-length discrete-time signals and, since any finite-length signal can be repeated to form a periodic discrete-time signal, the DFT can also be interpreted as applying to periodic signals.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, \ldots, N-1 \tag{A.9}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \qquad n = 0, \ldots, N-1 \tag{A.10}$$

Length-N sequence	**N-point DFT**
$x[n]$	$X[k]$
$ax[n] + by[n]$	$aX[k] + bY[k]$
$x[n]$	$NX[((-k))_N]$
$x[((n-m))_N]$	$e^{j2\pi km/N}X[k]$
$\sum_m x[m]y[((n-m))_N]$	$X[k]Y[k]$
$x^*[n]$	$X^*[((-k))_N]$
$x[n]$ real	$X[k] = X^*[((-k))_N]$

1.4 Z-Transform

The Z-transform converts a function of a discrete real variable to a function of a complex variable z. This converts difference equations into algebraic equations and convolution into products.

The Z-transform of a causal function $x[n]$ is:

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \tag{A.13}$$

while the inversion of $X(z)$ back onto $x[n]$ requires an integration on the complex plane.

2 Matrix Algebra

2.1 Column Space, Row Space, Null Spaces

The **column space** of an $N \times M$ matrix A is the set of all linear combinations of its column vectors. It is therefore a subspace (whose dimension is at most M) of the N-dimensional vector space.

We can write the linear combination as the product of A with the vector $\mathbf{x} = [x_0, \ldots, x_{M-1}]^T$:

$$A\begin{bmatrix} x_0 \\ \vdots \\ x_{M-1} \end{bmatrix} = \mathbf{y} = A\mathbf{x} \tag{B.1}$$

2.1.1 Example B.1

The column space of

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.2}$$

is the set of vectors $\mathbf{y} = [y_0\ y_1\ y_2]^T$ having the form

$$\begin{aligned} \mathbf{y} &= A\mathbf{x} && \text{(B.3)} \\ &= \begin{bmatrix} 3x_1 \\ 2x_0 \\ x_1 \end{bmatrix}. && \text{(B.4)} \end{aligned}$$

These vectors satisfy $y_0 = 3y_2$, which defines a subspace of dimension $M = 2$ (that is, a plane) on a vector space of dimension $N = 3$.

2.1.2 Example B.2

The **row space** of A in (B.2), in turn, is the set of vectors **y** having the form

$$\begin{aligned} \mathbf{y} &= A^T\mathbf{x} && \text{(B.5)} \\ &= \begin{bmatrix} 2x_1 \\ 3x_0 + x_2 \end{bmatrix}, && \text{(B.6)} \end{aligned}$$

which defines the entire vector space of dimension $M = 2$.

The column rank and row rank correspond to the dimensions of the column space and row space, respectively.

The fact that the column and row ranks coincide in this case is not a coincidence. The row and column ranks always coincide, giving the **rank of a matrix**.

In addition, we have also two additional subspaces:

- **Orthogonal complement of row space** (null space of A): all vectors satisfying $A\mathbf{x} = 0$, which has dimension $M - \text{rank}(A)$.
- **Orthogonal complement of column space** (null space of A^T): with dimension $N - \text{rank}(A)$.

3 Special Matrices

3.1 Hermitian Matrices

A complex matrix A is said to be **Hermitian** if $A^* = A$.

- They are quadratic.
- They have real elements on the diagonal.
- Off-diagonal elements are complex conjugates of each other.

- Eigenvalues are always real.

$$A = \begin{bmatrix} 2 & 1+i & 4 \\ 1-i & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

3.2 Unitary Matrices

A complex matrix U is said to be **unitary** if $U^*U = UU^* = I$.

- U is nonsingular, and $U^* = U^{-1}$.
- The columns of U form an orthonormal set, as do the rows of U.
- For any complex vector **x**, the vector $\mathbf{y} = U\mathbf{x}$ satisfies $|\mathbf{y}| = |\mathbf{x}|$. Thus, **y** is a rotated version of **x**, and U embodies that rotation.

3.3 Fourier Matrices

An $N \times N$ **Fourier matrix** U is a unitary matrix whose (i, j)th entry equals $e^{j2\pi ij/N}$. It follows that the jth column, for $j = 0, \ldots, N-1$, is given by

$$U_j = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi j/N} \\ \vdots \\ e^{j2\pi(N-1)j/N} \end{bmatrix} \tag{B.7}$$

The **DFT** of a vector **x** is

$$\mathbf{X} = \sqrt{N}\, U^*\mathbf{x} \tag{B.8}$$

and the **IDFT** is

$$\mathbf{x} = \frac{1}{\sqrt{N}} U\mathbf{X} \tag{B.9}$$

Indeed, by interpreting the entries of **x** and **X** as sequences, (B.8) and (B.9) are scaled versions of the $\text{DFT}_N\{\cdot\}$ and $\text{IDFT}_N\{\cdot\}$ transforms in (A.9) and (A.10).

3.4 Toeplitz and Circulant Matrices

A **Toeplitz matrix** is constant along each of its diagonals.

A **Toeplitz circular matrix** is completely described by any of its rows, of which the other rows are just circular shifts with offsets to the row indices:

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 5 \\ 5 & 1 & 2 \end{bmatrix}. \tag{B.10}$$

If A is an $N \times N$ **circulant matrix**, then the following holds:

- The eigenvectors of A equal the columns of the Fourier matrix U in (B.7).

- The eigenvalues of A equal the entries of $U^*\mathbf{a}$, where **a** is any column of A.

Hence, the eigenvalues of a circulant matrix are directly the DFT of any o