Numerics (Part IV)

[Lectures 2 & 3] Sp-Line Interpolation (Quadratic and Cubic)

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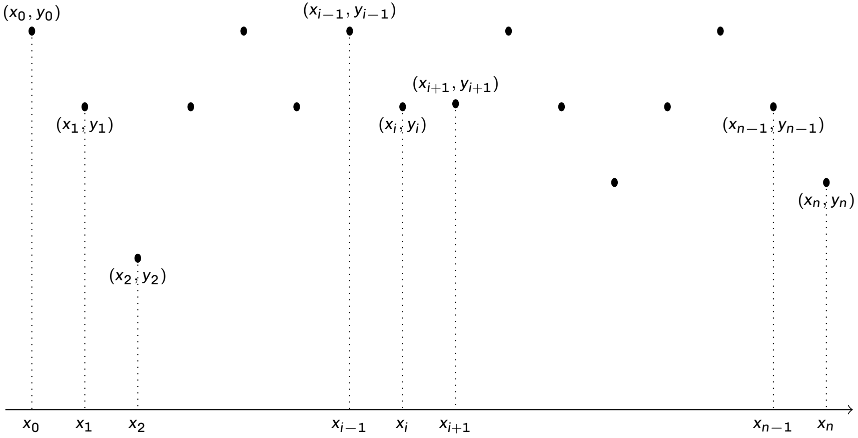
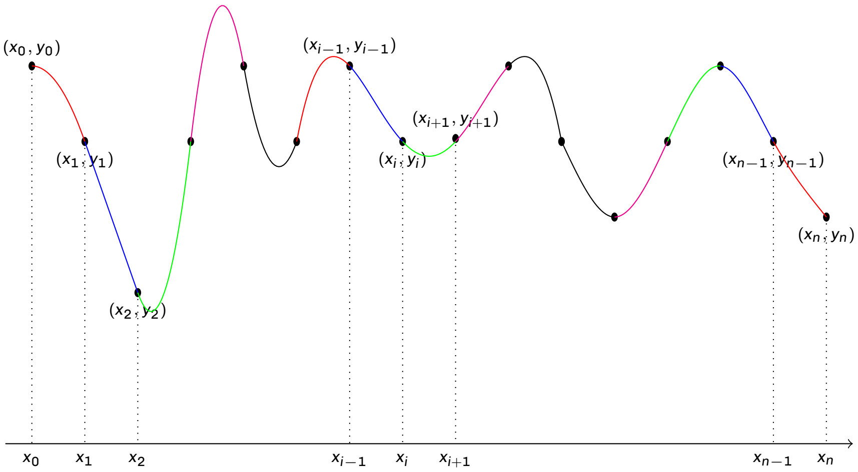
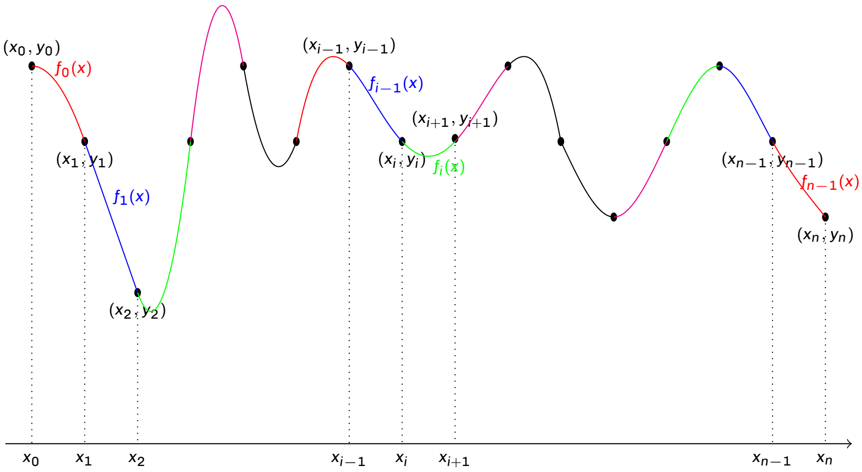
Sp-line Approach



Sp-line Approach



Sp-line Approach



Sp-line Approach

Input

- Given $n + 1$ points $\{(x_i, y_i)\}_{i=0}^{n}$
- such that $x_{i+1} > x_i \; \forall i$ (sorted from left to right)

Output

- Find a function $f(x)$ defined between x_0 and x_n
- Such that $f(x_i) = y_i \; \forall i = 0, 1, \ldots, n$ (f crosses the points)
- f continuous (optionally first k derivatives)
- For $x_i \leq x < x_{i+1}, f(x) = f_i(x)$

  (with f_i polynomial of given degree)

Three (kinds of) Sp-line

Connecting consecutive points by:

- Line segments (linear sp-line)
- Parabolic arcs (quadratic sp-line)
- Cubic arcs (cubic sp-line)

- **Linear** sp-line: a single segment for each two points, no degrees of freedom (might have sharp points)
- **Quadratic** sp-line: infinite parabolic curves crossing two points (asking continuous derivative)
- **Cubic** sp-line: infinite cubic curves crossing two points (asking continuous first and second derivative)

Balancing Parameters vs. Constraints

- Connecting $n + 1$ points with n segments/arcs
- Linear sp-line
  - two parameters for each line segment: $2n$ parameters
  - each segment connects two points: $2n$ constraints
  - $2n - 2n = 0$ (balanced)
- Quadratic sp-line
  - every arc is parabolic (3 parameters): $3n$ parameters
  - every arc crosses two points: $2n$ constraints
  - derivative continuous in the junction points: $n - 1$ constraints
  - $3n - 2n - (n - 1) = 1$ (a free parameter)
- Cubic sp-line
  - every arc is cubic (4 parameters) $4n$ parameters
  - every arc crosses two points: $2n$ constraints
  - der. I and II continuous in the junction points: $2(n - 1)$ constraints
  - $4n - 2n - 2(n - 1) = 2$ (two free parameters)

QUADRATIC SP-LINE

Quadratic Sp-Line

- Derivative of a quadratic sp-line is a linear sp-line
- For each $i = 0, 1, \ldots, n$, let us define $z_i := f'(x_i)$
- Rewriting the derivative of $f_i(x_i)$

$$f_i'(x) = \frac{z_{i+1} - z_i}{x_{i+1} - x_i}(x - x_i) + z_i$$

- Integrating (from the derivative to the function)

$$f_i(x) = \frac{z_{i+1} - z_i}{x_{i+1} - x_i}\frac{(x - x_i)^2}{2} + z_i(x - x_i) + \alpha_i$$

- $f_i(x_i) = y_i$ implies $\alpha_i = y_i$
- $f_i(x_{i+1}) = y_{i+1}$ implies

$$y_{i+1} = \frac{z_{i+1} - z_i}{x_{i+1} - x_i}\frac{(x_{i+1} - x_i)^2}{2} + z_i(x_{i+1} - x_i) + y_i$$

Quadratic Sp-Line (Recap)

- z_0 given
- recursively getting $z_1, z_2, \ldots, z_n$

$$z_{i+1} = -z_i + 2\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

- these parameters are used to obtain the parabolic curves

$$f_i(x) = \frac{z_{i+1} - z_i}{2(x_{i+1} - x_i)}(x - x_i)^2 + z_i(x - x_i) + y_i$$

with $x_i \leq x \leq x_{i+1}$

CUBIC SP-LINE

Cubic Sp-line

- Second derivative of a cubic sp-line is a linear sp-line
- For each $i = 0, 1, \ldots, n$, let $z_i := f''(x_i)$
- Let $h_i := x_{i+1} - x_i$ denote the distance between two points on the x-axis
- Rewriting the second derivative of $f_i(x_i)$

$$f_i''(x) = \frac{z_{i+1}}{h_i}(x - x_i) - \frac{z_i}{h_i}(x - x_{i+1})$$

- Integrating two times

$$f_i'(x) = \frac{z_{i+1}}{2h_i}(x - x_i)^2 - \frac{z_i}{2h_i}(x - x_{i+1})^2 + \alpha_i$$

$$f_i(x) = \frac{z_{i+1}}{6h_i}(x - x_i)^3 - \frac{z_i}{6h_i}(x - x_{i+1})^3 + \alpha_i(x - x_i) + \beta_i$$

Cubic Sp-Line

$$f_i(x) = \frac{z_{i+1}}{6h_i}(x - x_i)^3 - \frac{z_i}{6h_i}(x - x_{i+1})^3 + \alpha_i(x - x_i) + \beta_i$$

Finding α_i and β_i by setting $f(x_i) = y_i$ and $f(x_{i+1}) = y_{i+1}$

$$\alpha_i = \frac{y_{i+1} - y_i}{h_i} - \frac{h_i}{6}(z_{i+1} - z_i) \quad \beta_i = y_i - z_i\frac{h_i^2}{6}$$

Getting relation between z_{i+1}, z_i, z_{i-1} where $f'_{i-1}(x_i) = f'_i(x_i)$

Cubic Sp-Line (recap)

- z_0 and z_n are given (typically $z_0 = 0$ e $z_n = 0$)
- getting $z_1, z_2, \ldots, z_n$ by the linear system

  $$h_{i-1}z_{i-1}+2(h_{i-1}+h_i)z_i+h_iz_{i+1} = 6\left(\frac{y_{i+1}-y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right)$$

  $\forall i = 1, \ldots, n-1$
- with these parameters we write the cubic curves as: $f_i(x) =$

  $$\frac{z_{i+1}(x-x_i)^3 + z_i(x_{i+1}-x)^3}{6h_i}+\left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6}z_{i+1}\right)(x-x_i)+\left(\frac{y_i}{h_i} - \frac{h_i}{6}z_i\right)(x_{i+1}-x)$$

  $\forall x_i \leq x \leq x_{i+1}$

Cubic Sp-Line (recap, ii)

- If the x of the points are at the same distance $h_i = h$ for each i, and $h = 1$, formulae rewrite as:

$$z_{i-1} + 4z_i + z_{i+1} = 6\left(y_{i+1} - 2y_i + y_{i-1}\right)$$

$$f_i(x) = \frac{z_{i+1}}{6}(x - x_i)^3 + \frac{z_i}{6}(x_{i+1} - x)^3 + \\ + \left(y_{i+1} - \frac{z_{i+1}}{6}\right)(x - x_i) + \left(y_i - \frac{z_i}{6}\right)(x_{i+1} - x)$$