Numerics (Part IV)

[Lecture 1] Polynomial Interpolation

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A Better Algorithm for Polynomial Interpolation

- An alternative form for the polynomial, ex.$p_3(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2)$
- $p_n(x) = \sum_{i=0}^{n} c_i \left[\prod_{j=0}^{i-1}(x - x_j)\right]$
- Requiring the passage through the $n + 1$ points

$$\left[\begin{array}{ccccc|c}
1 & 0 & 0 & \dots & 0 & y_0 \\
1 & x_1 - x_0 & 0 & \dots & 0 & y_1 \\
 & & \dots & \dots & & \\
1 & x_n - x_0 & (x_n - x_0)(x_n - x_1) & \dots & \prod_{j=0}^{n-1}(x_n - x_j) & y_n
\end{array}\right]$$

- Triangular: quadratic (instead of cubic) complexity

Finite Differences Method

- Data $\{(x_i, y_i)\}_{i=0}^{n}$
- Order-1 Differences $f[x_i] := f(x_i)\ \forall i = 0, \ldots, n$
- Order-2 $f[x_i, x_{i+1}] := \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}\ \forall i = 0, \ldots, n-1$
- Order-3 $f[x_i, x_{i+1}, x_{i+2}] := \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$ $\forall i = 0, \ldots, n-2$
- Order-4 $f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] := \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}$ $\forall i = 0, \ldots, n-3$
- Order-k = Difference between last and fiirst Order-$(k-1)$divided by difference between last and first x
- Order-$(n+1)$ $f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$

Finite Differences Method (ii)

- Calculations on a pyramid-shaped structure (ex. 5 points, $n = 4$)

$$
\begin{array}{llllll}
x_0 & f[x_0] & & & & \\
 & & f[x_0,x_1] & & & \\
x_1 & f[x_1] & & f[x_0,x_1,x_2] & & \\
 & & f[x_1,x_2] & & f[x_0,x_1,x_2,x_3] & \\
x_2 & f[x_2] & & f[x_1,x_2,x_3] & & f[x_0,x_1,x_2,x_3,x_4] \\
 & & f[x_2,x_3] & & f[x_1,x_2,x_3,x_4] & \\
x_3 & f[x_3] & & f[x_2,x_3,x_4] & & \\
 & & f[x_3,x_4] & & & \\
x_4 & f[x_4] & & & &
\end{array}
$$

Finite Differences Method (ii)

- Calculations on a pyramid-shaped structure (ex. 5 points, $n = 4$)

$$
\begin{array}{llllll}
 & & c_0 & & & \\
x_0 & f[x_0] & & & & \\
 & & c_1 & & & \\
 & & f[x_0, x_1] & c_2 & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] & c_3 & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] & c_4 \\
x_2 & f[x_2] & & f[x_1, x_2, x_3] & & f[x_0, x_1, x_2, x_3, x_4] \\
 & & f[x_2, x_3] & & f[x_1, x_2, x_3, x_4] & \\
x_3 & f[x_3] & & f[x_2, x_3, x_4] & & \\
 & & f[x_3, x_4] & & & \\
x_4 & f[x_4] & & & &
\end{array}
$$

Space Complexity from $O(n^2)$ to $O(n)$

$$
\begin{array}{llllll}
x_0 & f[x_0] & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] & \\
x_2 & f[x_2] & & f[x_1, x_2, x_3] & & f[x_0, x_1, x_2, x_3, x_4] \\
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x_2 & f[x_2] & & f[x_1, x_2, x_3] & & f[x_0, x_1, x_2, x_3, x_4] \\
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$$
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x_0 & \boxed{f[x_0]} & & & & \\
 & & \boxed{f[x_0, x_1]} & & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] & & \\
 & & \boxed{f[x_1, x_2]} & & f[x_0, x_1, x_2, x_3] & \\
x_2 & f[x_2] & & f[x_1, x_2, x_3] & & f[x_0, x_1, x_2, x_3, x_4] \\
 & & \boxed{f[x_2, x_3]} & & f[x_1, x_2, x_3, x_4] & \\
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x_4 & f[x_4] & & & &
\end{array}
$$

- Storing all the computations in a list of size 5!