# Numerics (Part III)

[Lecture 3] LR Decomposition and Matrix Inversion

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# Idea (scalar case)

$$35 \cdot x = 8$$

$$7 \cdot 5x = 8$$

$$5x = y$$

$$7y = 8$$

$$\begin{cases} 5x = y \\ 7y = 8 \end{cases},$$

$$y = \frac{8}{7},$$

$$x = \frac{y}{5} = \frac{8}{35}$$

# LR Decomposition

- Given linear system $\hat{A} \cdot \vec{x} = \vec{b}$
- Decompose $\hat{A} = \hat{L} \cdot \hat{R}$

  ($\hat{L}$ lower triangular and $\hat{R}$ upper triangular)
- The system rewrites as $\hat{L} \cdot \hat{R} \cdot \vec{x} = \vec{b}$
- Equivalent to a pair of triangular systems $\begin{cases} \hat{R} \cdot \vec{x} = \vec{y} \\ \hat{L} \cdot \vec{y} = \vec{b} \end{cases}$
- Both systems are triangular!
- Solve the second and then the first (both by substitution)
- Problem: how to decompose $\hat{A}$ in the product $\hat{L} \cdot \hat{R}$?

# LR Decomposition (ii)

- For $\hat{L}\hat{R}$ decomposition just do Gauss on $\hat{A}$ only (not $\hat{b}$)
- Matrix $\hat{R}$ is the output of Gauss
- Matrix $\hat{L}$ is such that
  - all zeros over the diagonal
  - all ones on the diagonal
  - coeffs used to annihilate $\hat{A}$ elements

# $2 \times 2$ Example (Decomposition)

- $\hat{A} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$
- Gauss only on $\vec{A}$, $\texttt{pivot} = 2$, $\texttt{coeff} = -\frac{1}{2}$
- $\hat{L} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$, $\hat{R} = \begin{bmatrix} 2 & 1 \\ 0 & -\frac{3}{2} \end{bmatrix}$
- Exercise (check the decomposition)

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$$

# $2 \times 2$ Example (Solution)

- $\hat{A} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$
- Solving first $\hat{L} \cdot \vec{y} = \vec{b}$ and then $\hat{R} \cdot \vec{x} = \vec{y}$.

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & -\frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -\frac{3}{2} \end{bmatrix}$$

- $x_2 = 1, x_1 = 3$

# Complexity Analysis

- LR complexity same as Gauss $O(n^3)$
- After LR, two substitutions are needed $2O(n^2)$
- Same complexity as Gauss + substitution
- No benefits when solving a singly system
- Benefits when solving many systems,
  all with the same coefficients matrix

# Matrix Inversion (Application)

- $\hat{A} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \quad \hat{A}^{-1} = \begin{bmatrix} \color{red}{\bullet} & \color{blue}{\bullet} \\ \color{red}{\bullet} & \color{blue}{\bullet} \end{bmatrix}$
- $\hat{A} \cdot \hat{A}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} \color{red}{\bullet} & \color{blue}{\bullet} \\ \color{red}{\bullet} & \color{blue}{\bullet} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{I}$
- Solving (by LR) two systems with the same $\hat{A}$:
  - $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} \color{red}{\bullet} \\ \color{red}{\bullet} \end{bmatrix} = \begin{bmatrix} \color{red}{1} \\ \color{red}{0} \end{bmatrix}$
  - $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} \color{blue}{\bullet} \\ \color{blue}{\bullet} \end{bmatrix} = \begin{bmatrix} \color{blue}{0} \\ \color{blue}{1} \end{bmatrix}$