

# Numerics (Part IV)

[Lecture 1] Polynomial Interpolation

Alessandro Antonucci

`alessandro.antonucci@supsi.ch`

# A Better Algorithm for Polynomial Interpolation

- An alternative form for the polynomial, ex.  $p_3(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + c_3(x-x_0)(x-x_1)(x-x_2)$
- $p_n(x) = \sum_{i=0}^n c_i \left[ \prod_{j=0}^{i-1} (x - x_j) \right]$
- Requiring the passage through the  $n + 1$  points

1	0	0	...	0	$y_0$
1	$x_1 - x_0$	0	...	0	$y_1$
		...	...		
1	$x_n - x_0$	$(x_n - x_0)(x_n - x_1)$	...	$\prod_{j=0}^{n-1} (x_n - x_j)$	$y_n$

- Triangular: quadratic (instead of cubic) complexity

# Finite Differences Method

- Data  $\{(x_i, y_i)\}_{i=0}^n$
- Order-1 Differences  $f[x_i] := f(x_i) \quad \forall i = 0, \dots, n$
- Order-2  $f[x_i, x_{i+1}] := \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \quad \forall i = 0, \dots, n-1$
- Order-3  $f[x_i, x_{i+1}, x_{i+2}] := \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} \quad \forall i = 0, \dots, n-2$
- Order-4  $f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] := \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i} \quad \forall i = 0, \dots, n-3$
- Order- $k$  = Difference between last and first  
Order- $(k-1)$  divided by difference between last and first  $x$
- Order- $(n+1)$   $f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$

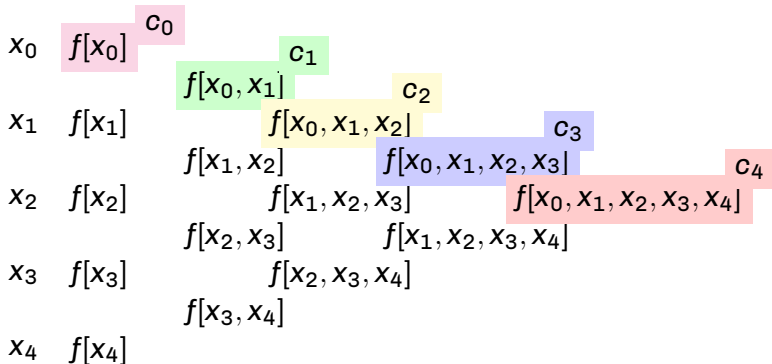
## Finite Differences Method (ii)

- Calculations on a pyramid-shaped structure (ex. 5 points,  $n = 4$  )

$x_0$	$f[x_0]$				
		$f[x_0, x_1]$			
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		$f[x_3, x_4]$			
$x_4$	$f[x_4]$				

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- Calculations on a pyramid-shaped structure (ex. 5 points,  $n = 4$ )



## Space Complexity from $O(n^2)$ to $O(n)$

$x_0$	$f[x_0]$				
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- Storing all the computations in a list of size 5!