Numerics (Part III)

[Lecture 0] Linear Algebra Basics

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Vectors

- Vector (array): ordered sequence of (real) numbers
- Vector size: number of elements in the sequence
- We distinguish "row" vectors (horizontal)

  from "col" vectors (vertical)
- Transpose: swap rows and cols (and vice versa)

$$\vec{x} = [1 \quad 4 \quad 0] \quad \vec{x}^t = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

$$len(\vec{x}) = 3 \quad (\vec{x})_1 = 1 \quad (\vec{x})_2 = 4$$

Vector Calculus

- Scalar: single-element array
- Multiplying an array by a scalar (elementwise operation)

$$(\alpha\vec{x})_i = \alpha\,(\vec{x})_i$$

- Summing two arrays (also elementwise)

$$(\vec{x}+\vec{y})_i = (\vec{x})_i + (\vec{y})_i$$

- Product of two arrays (row by col) not elementwise

$$\vec{x}\cdot\vec{y} = \sum_i (\vec{x})_i(\vec{y})_i$$

Vector Calculus (exe)

- $\alpha = 3, \vec{x} = [1 \quad 4 \quad 0], \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$
- $\alpha\vec{y} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} \quad \vec{y} + \vec{z} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$
- $\vec{x} \cdot \vec{y} = 6 \quad \vec{x} \cdot \vec{z} = 1$
- $\vec{y} \cdot \vec{z} = ?$ strictly speaking not possible

Matrices

- Matrix = array of arrays
- Dimensions: number of rows and cools

  $\hat{A}$ matrix with n rows and m cols: $dim(\hat{A}) = [n, m]$
- $dim(\hat{A}) = [n, n]$? $\quad \hat{A}$ square matrix

$$\hat{A} = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 5 & 2 & 0 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 5 & 0 \end{bmatrix} \quad \hat{C} = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

$$dim(\hat{A}) = [3, 3] \quad dim(\hat{B}) = [3, 2] \quad dim(\hat{C}) = [2, 3]$$

$$(\hat{A})_{1,2} = 4 \quad (\hat{B})_{3,1} = 5 \quad (\hat{C})_{2,3} = 1$$

Kind of Matrices

- Diagonal (of a square matrix): array of elements with same row and col index
- Upper triangular matrix (ex. $\hat{A}$): elements under the diagonal are all zero
- Lower triangular matrix (ex. $\hat{C}$): elements over the diagonal are all zero
- Diagonal matrix (ex. $\hat{B}$): matrix at the same time upper and lower triangular

$$\hat{A} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \hat{C} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

Matrix Calculus

- Matrix (elementwise) multiplication by a scalar

$$(\alpha\hat{A})_{ij} = \alpha\,(\hat{A})_{ij}$$

- (Elementwise) Sum of two matrices

$$(\hat{A} + \hat{B})_{ij} = (\hat{A})_{ij} + (\hat{B})_{ij}$$

- Matrix multiplication (not elementwise!)

$$(\hat{A} \cdot \hat{B})_{ij} = \sum_k (\hat{A})_{ik}(\hat{B})_{kj}$$

- $dim(\hat{A}) = [n_A, m_A]$, $dim(\hat{B}) = [n_B, m_B]$, $dim(\hat{A} \cdot \hat{B}) = [n_A, m_B]$

  matrices can be multiplied iff $m_A = n_B$

Matrix Calculus (exe)

$$\alpha = 3 \quad \hat{A} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\alpha \cdot \hat{A} = \begin{bmatrix} 3 & -3 & 6 \\ 3 & 0 & 6 \\ 3 & 6 & 0 \end{bmatrix} \quad \hat{A} \cdot \hat{B} = \begin{bmatrix} 1 & -2 & 7 \\ 3 & -1 & 7 \\ 5 & 3 & 3 \end{bmatrix}$$

- Exercise: Check $\hat{A} \cdot \hat{B} \neq \hat{B} \cdot \hat{A}$

Identity Matrix

- Identity matrix $\hat{I}$: diagonal matrix, ones on the diagonal
- Identity as neutral element $\hat{I} \cdot \hat{A} = \hat{A} \cdot \hat{I} = \hat{A}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Inverse Matrix

- Given scalar k, its *reciprocal* k^{-1} is a number such that

$$k \cdot k^{-1} = k^{-1} \cdot k = 1$$

- Given matrix $\hat{A}$, its *inverse* $\hat{A}^{-1}$ is such that

$$\hat{A} \cdot \hat{A}^{-1} = \hat{A}^{-1} \cdot \hat{A} = \hat{I}$$

- Matrix inversion is not trivial, the algorithms we will study are basically intended to do that

$$\hat{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \hat{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Matrix Representation of a Linear System

$$\hat{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\hat{A} \cdot \vec{x} = \vec{b}$$

$$\begin{cases} x_1 - x_2 + 2x_3 = 2 \\ 2x_1 - x_2 - x_3 = 0 \\ x_1 + x_2 + x_3 = 3 \end{cases} \qquad \vec{x} = \hat{A}^{-1} \cdot \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$