

The comonadic smell of spreadsheets

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D2				$\sum x = B2 * C2$		
	A	B	C	D	E	
1	Fruit	Weight (kg)	Price / kg	Total price	% price	
2	Apples	1.5	1.75	2.625	0.4738267148	
3	Bananas	0.7	1.29	0.903	0.1629963899	
4	Tangerines	0.6	2.5	1.5	0.2707581227	
5						
6						
7						
8		Total	5.54			
9						
10						

Figure 1: Example of a 2D spreadsheet

H8						
	A	B	C	D	E	F
1	Fruit	Weight (kg)	Price / kg	Total price	% price	
2	Apples	1.5	1.75	=B2*C2	=D2/\$C\$8	
3	Bananas	0.7	1.29	=B3*C3	=D3/\$C\$8	
4	Tangerines	0.6	2.5	=B4*C4	=D4/\$C\$8	
5						
6						
7						
8		Total	=SUM(\$C\$2:\$C\$4)			
9						
10						

Figure 2: Formulas

The image shows two views of a spreadsheet. The top view shows the initial state with a circular dependency between B1 and A1. The bottom view shows the resulting error state after the dependency is detected.

	A	B
1	=B1	=A1
2		
3		
4		

	A	B	C	D
1	#REF!	#REF!		
2				
3				
4				
5				
6				
7				

Error
Circular dependency detected.
To resolve with iterative calculation, see File > Settings.

Figure 3: Circular dependencies

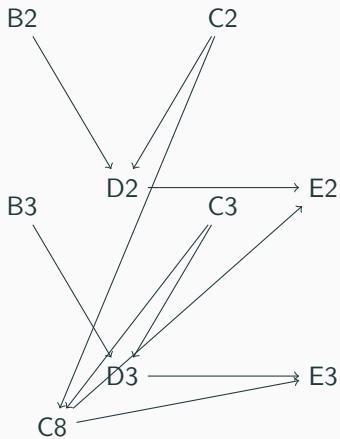


Figure 4: DAG

Associated to a spreadsheet is a DAG where

- nodes are cells
- there is a directed edge from a to b if and only if b has a formula that depends on a (i. e., b depends on a)

Spreadsheet *evaluation* is analogous to *dependency resolution*

A modern spreadsheet application provides many additional capabilities, not in scope in this talk, such as

- Parsing of cell formulas
- Built-in functions
- Graphics (e. g., histograms, piecharts, etc)
- Formatting, exporting, drag-and-drop, copy-pasting, etc
- Programmable functions (e. g., in VBA)
- Data connectors
- Collaborative online spreadsheets
- And many more

C3	▼	fx	1.29			
	A	B	C	D	E	F
1	Fruit	Weight (kg)	Price / kg	Total price	% price	
2	Apples	1.5	1.75	2.625	0.4738267148	
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5						
6						
7						
8		Total	5.54			
9						
10						
11						

Figure 5: Recalculation and support graph

MVP / POC: recalculate the whole spreadsheet every time a cell changes

POST MVP: *incremental recalculation*

Traditional technique:

- Topological sorting of the support graph¹ (e. g., Kahn's algorithm)

¹https://en.wikipedia.org/wiki/Topological_sorting

Production applications need to take into account more complicated challenges:²

- Performance tradeoffs (e. g., calculating the topological order also has an associated cost)
- Resource utilisation (e. g., compact representations)
- Volatile functions (e. g., `now()`, `rand()`)
- DAG caching
- Parallelisation algorithms, etc

²<https://learn.microsoft.com/en-us/office/client-developer/excel/excel-recalculation>

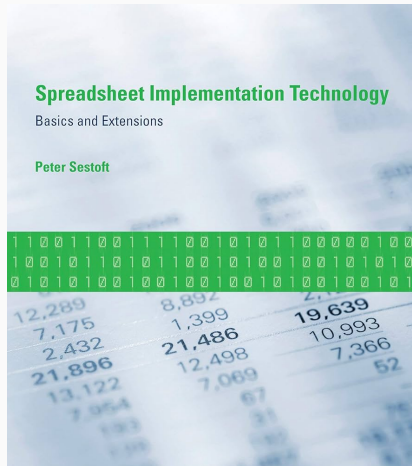


Figure 6: P. Sestoft, *Spreadsheet Implementation Technology*, MIT press (2014).⁴

⁴<https://direct.mit.edu/books/book/3071/Spreadsheet-Implementation-TechnologyBasics-and>

Observation

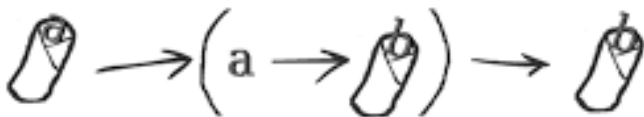
Spreadsheet-like evaluation can be expressed as a *fixed-point of higher dimensional comonads*

Agenda

1. A taste of comonads
2. Comonadic vibes meet spreadsheets
3. Down the rabbit hole

A taste of comonads

monads are burritos?



and functors?

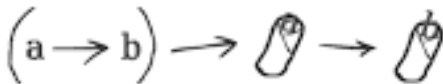


Figure 7: Monads are burritos

Definition

Comonads are co-burritos

Functor hierarchy:

```
class Functor f where  
  fmap :: (a -> b) -> f a -> f b
```

together with some laws.

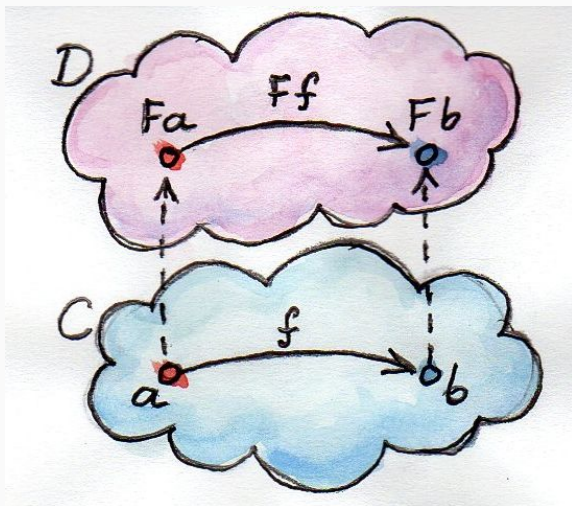


Figure 8: Functors: *Lifting* of 1-ary pure functions to the f -world

```
f1 :: [Int] -> [String]
```

```
-- vs
```

```
f2 :: Int -> String
```

```
f3 :: [Int] -> [String]
```

```
f3 = fmap f2
```

```
class (Functor f) => Applicative f where
  pure :: Applicative f => a -> f a
  ap :: Applicative f => f (a -> b) -> f a -> f b
```

together with laws governing properties of `fmap`, `ap`, and `pure`.

Applicatives provide lifting of N -ary pure functions in the f -context

```
class (Applicative f) => Monad f where
  return :: Monad f => a -> f a
  join :: Monad f => f (f a) -> f a

-- flatMap
bind :: Monad f => m a -> (a -> m b) -> m b
```

In short:

- Functor \sim `map`
- Applicative \sim `mapN` (or equivalent)
- Monad \sim `flatMap` (or equivalent)

(and the caveat that the implementation must satisfies certain “reasonable” laws that rule out “unexpected” behaviours)

This seems very abstract but from a software engineer perspective lifting provides

- Separation of concerns
- Reusability
- Composability

`flatMap` ensures no “callback hell” of nested effects

$$f \ (f \ (f \ (\dots (f \ a) \)) \)) = f \ a$$

Duality:

```
class Functor f => Comonad f where
  coreturn :: f a -> a
  cojoin  :: f a -> f ( f a )
```

(and, as usual, there are important laws)

```
class Functor w => Comonad w where
    extract :: f a -> a
    duplicate :: w a -> w ( w a )

-- coKleisli composition / extract
coFlatMap :: Comonad w => (w a -> b) -> w a -> w b
coFlatMap f = fmap f . duplicate
```

(and, as usual, there are important laws)

- Monadic values are typically *produced* in effectful computations

$a \rightarrow m\ b$

- Comonadic values are typically *consumed* in context-sensitive computations (e.g., “queries”)

$w\ a \rightarrow b$

Maybe and `[]` are not comonads (why?)

```
class Copointed f where
  extract :: f a -> a
```

```
instance Comonad NonEmpty where
    extract = head
    duplicate = tails1
```

```
{-# LANGUAGE OverloadedLists #-}  
-- return all suffices including itself  
  
tails [1,2,3] 'shouldBe'  
      [[1,2,3], [2, 3], [3]]
```

```
{-# LANGUAGE OverloadedLists #-}  
coFlatMap f [1,2,3] 'shouldBe'  
    [f [1,2,3], f [2, 3], f [3]]  
  
-- in this case  
-- coFlatMap f = fmap f . tails
```


Simple moving average of a time series $[\phi_1, \dots, \phi_N]$ with window size k at the time t is

$$SMA_k(t) = \frac{1}{k} \sum_{j=t-k+1}^t \phi_j \quad (1)$$

```
{-# LANGUAGE OverloadedLists #-}
```

```
sma :: Window -> NonEmpty Int -> [Double]
```

```
sma 3 [1,2,3,4,5,6,7,8,9,10] 'shouldBe '  
    [2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0]
```

because

- $(1 + 2 + 3)/3 = 6/3 = 2$
- $(2 + 3 + 4)/3 = 9/3 = 3$
- $(3 + 4 + 5)/3 = 12/3 = 4$

```

smaLocal :: Window -> NonEmpty Double
    -> Maybe Double
smaLocal k s = (/ (fromIntegral k)) <$>
    sumK k s

sumFirstK :: Num a =>
    Window -> NonEmpty a -> Maybe a
sumFirstK (Window k) _ | k <= 0 = Nothing
sumFirstK (Window 1) (MyNonEmpty a _) = Just a
sumFirstK k (MyNonEmpty a as) =
    case nonEmpty as of
        Just as2 -> (+a) <$> sumFirstK (k-1) as
        Nothing -> Nothing

```

```
sma :: Window -> MyNonEmpty Double
      -> MyNonEmpty (Maybe Double)
sma size = coFlatMap (smaLocal size)
```

Folkloristic understanding of some comonads usages

- the function passed to `coFlatMap` represents a *local* computation (can explore the “neighbourhood” of the focus point, but produces a single value)
- `cojoin` produces a (lazy) view of the structure from all perspectives
- `coFlatMap` applies the local computation to all perspectives

Similar examples:

- Numerical derivation
- Reconciliation in a list of events
- etc

Interesting usages:

- Cellular automaton (e. g., Conway's game of life)
- Store, Moore machine and comonadic UIs

Comonadic vibes meet spreadsheets

```
data Stream a = Stream a (Stream a)
    deriving (Eq, Show, Functor)

instance Comonad Stream where
    extract (Stream a _) = a
    duplicate s'@(Stream _ as) =
        Stream s' (duplicate as)
```

```
iterate :: (a -> a) -> a
         -> Stream a
iterate f seed =
    Stream seed (iterate f (f seed))

takeN :: Int -> Stream a -> [a]
takeN 0 _ = []
takeN n (Stream a as) = a : takeN n as
```

```
naturals :: Stream Int  
naturals = iterate (+1) 0
```

```
takeN 10 naturals 'shouldBe' [0,1,2,3,4,5,6,7,8,9]
```

One-dimensional, bidirectional, homogeneous spreadsheet of type *a*

```
data Sheet1 = Sheet1 (Stream a) a (Stream a)
  deriving (Eq, Show, Functor)
```

```
focus :: Sheet1 a -> a
focus (Sheet1 _ a _ ) = a
```

```
moveL :: Sheet1 a -> Sheet1 a
moveL (Sheet1 (Stream a as) focus s)
    = Sheet1 as a (Stream focus s)
```

```
moveR :: Sheet1 a -> Sheet1 a
moveR (Sheet1 s focus (Stream a as))
    = Sheet1 (Stream focus s) a as
```

```
instance Comonad Sheet1 where
  extract = focus
  duplicate s =
    Sheet1 (allLeft s) s (allRight s) where
      allLeft = iterate moveL
      allRight = iterate moveR
```

Duplicate is “diagonalisation” (as with other comonads derived from zippers)

```
naturalsSheet1 :: Sheet1 Int
naturalsSheet1 = Sheet1 (constS 0) 0 naturals
```

We are looking for sheets of functions:

```
sheet :: Num a => Sheet1 (Sheet1 a -> a)
```

and the evaluate function should look like

```
evaluate :: Num a => Sheet1 (Sheet1 a -> a)  
        -> Sheet1 a
```


Comonadic fixed-points at rescue⁵

```
kfix :: ComonadApply w => w ( w a -> a ) -> w a
```

⁵Why ComonadApply? We need some “zippiness”, see references at the end

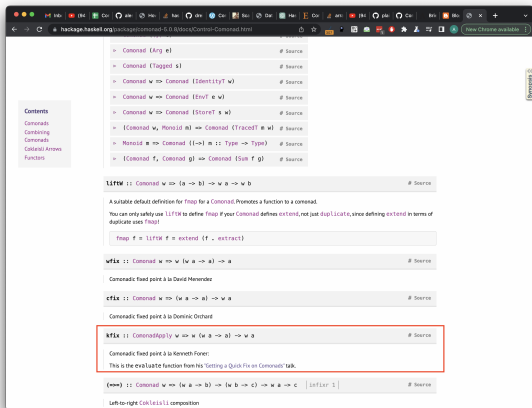


Figure 9: Kenneth fixed point for ComonadApply


```
naturals :: Sheet1 Integer
naturals = sheet 0 (repeat (cell left + 1))
```

Down the rabbit hole

The screenshot shows a web browser window with the address bar displaying `github.com/plaidfinch/ComonadSheet`. The page content is a README file titled `README.md`. It contains two main sections: **Iterated Numbers** and **Pascal's Triangle**.

Iterated Numbers

A one-dimensional sheet which is zero left of the origin and lists the natural numbers right of the origin:

```
naturals :: Sheet1 Integer
naturals = evaluate $ sheet 0 (repeat (cell left + 1))
```

When we print this out...

```
> take (rightBy 10) naturals
[1,2,3,4,5,6,7,8,9,10,11]
```

Pascal's Triangle

An infinite spreadsheet listing the rows of Pascal's triangle as upwards-rightwards diagonals:

```
pascal :: Sheet2 Integer
pascal = evaluate . sheet 0 $
  repeat 1 <|> repeat (1 <|> pascalRow)
  where pascalRow = repeat $ cell above + cell left
```

Notice the fact that I'm using the `(+)` function to add functions (namely, `cell above` and `cell left`). This is thanks to some clever overloading from `Data.Numeric.Function`.

This looks like:

```
> take (belowBy 9 & rightBy 9) pascal
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
 [1, 2, 3, 4, 5, 6, 7, 8, 9, 10],
 [1, 3, 6, 10, 15, 21, 28, 36, 45, 55],
 [1, 4, 10, 20, 35, 56, 84, 120, 165, 220],
 [1, 5, 15, 35, 70, 126, 210, 330, 495, 715],
 [1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002],
 [1, 7, 28, 84, 210, 462, 924, 1716, 3003, 5005],
 [1, 8, 36, 128, 330, 792, 1716, 3432, 6435, 11440],
 [1, 9, 45, 165, 495, 1287, 3003, 6435, 12870, 24310],
 [1, 10, 55, 228, 715, 2002, 5005, 11440, 24310, 48620]]
```

About comonads:

- <https://bartoszmilewski.com/2017/01/02/comonads/>
- <https://blog.higher-order.com/blog/2015/10/04/scala-comonad-tutorial-part-2/>
- <https://reasonablypolymorphic.com/blog/cofree-comonads/>
- <https://reasonablypolymorphic.com/blog/comonadic-physics/index.html>

About comonadic UIs:

- <https://functorial.com/the-future-is-comonadic/main.pdf>
- <https://arthurxavierx.github.io/ComonadsForUIs.pdf>

About cellular automata

- <https://www.schoolofhaskell.com/user/edwardk/cellular-automata/part-1> and references therein

From *Löb's Theorem* in modal logic⁶

$$\Box (\Box \phi \rightarrow \phi) \rightarrow \Box \phi \quad (2)$$

to spreadsheet evaluation:

- Original post from Piponi (2006): <http://blog.sigfpe.com/2006/11/from-l-theorem-to-spreadsheet.html>
- Functional pearl paper from Kenneth Foner (2015): <https://dl.acm.org/doi/10.1145/2887747.2804310>
- ComonadSheet hackage package (unmaintained): <https://hackage.haskell.org/package/ComonadSheet>

⁶In this context, $\Box \phi$ is a formal provability predicate which reads “ ϕ is provable”

Questions?