# The comonadic smell of spreadsheets

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D2 =B2*C2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Fruit | Weight (kg) | Price / kg | Total price | % price |
| 2 | Apples | 1.5 | 1.75 | 2.625 | 0.4738267148 |
| 3 | Bananas | 0.7 | 1.29 | 0.903 | 0.1629963899 |
| 4 | Tangerines | 0.6 | 2.5 | 1.5 | 0.2707581227 |
| 5 | | | | | |
| 6 | | | | | |
| 7 | | | | | |
| 8 | | Total | 5.54 | | |
| 9 | | | | | |
| 10 | | | | | |

**Figure 1:** Example of a 2D spreadsheet

H8 | fx

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | Fruit | Weight (kg) | Price / kg | Total price | % price | |
| 2 | Apples | 1.5 | 1.75 | =B2*C2 | =D2/$C$8 | |
| 3 | Bananas | 0.7 | 1.29 | =B3*C3 | =D3/$C$8 | |
| 4 | Tangerines | 0.6 | 2.5 | =B4*C4 | =D4/$C$8 | |
| 5 | | | | | | |
| 6 | | | | | | |
| 7 | | | | | | |
| 8 | | Total | =SUM($C$2:$C$ | | | |
| 9 | | | | | | |
| 10 | | | | | | |

**Figure 2:** Formulas

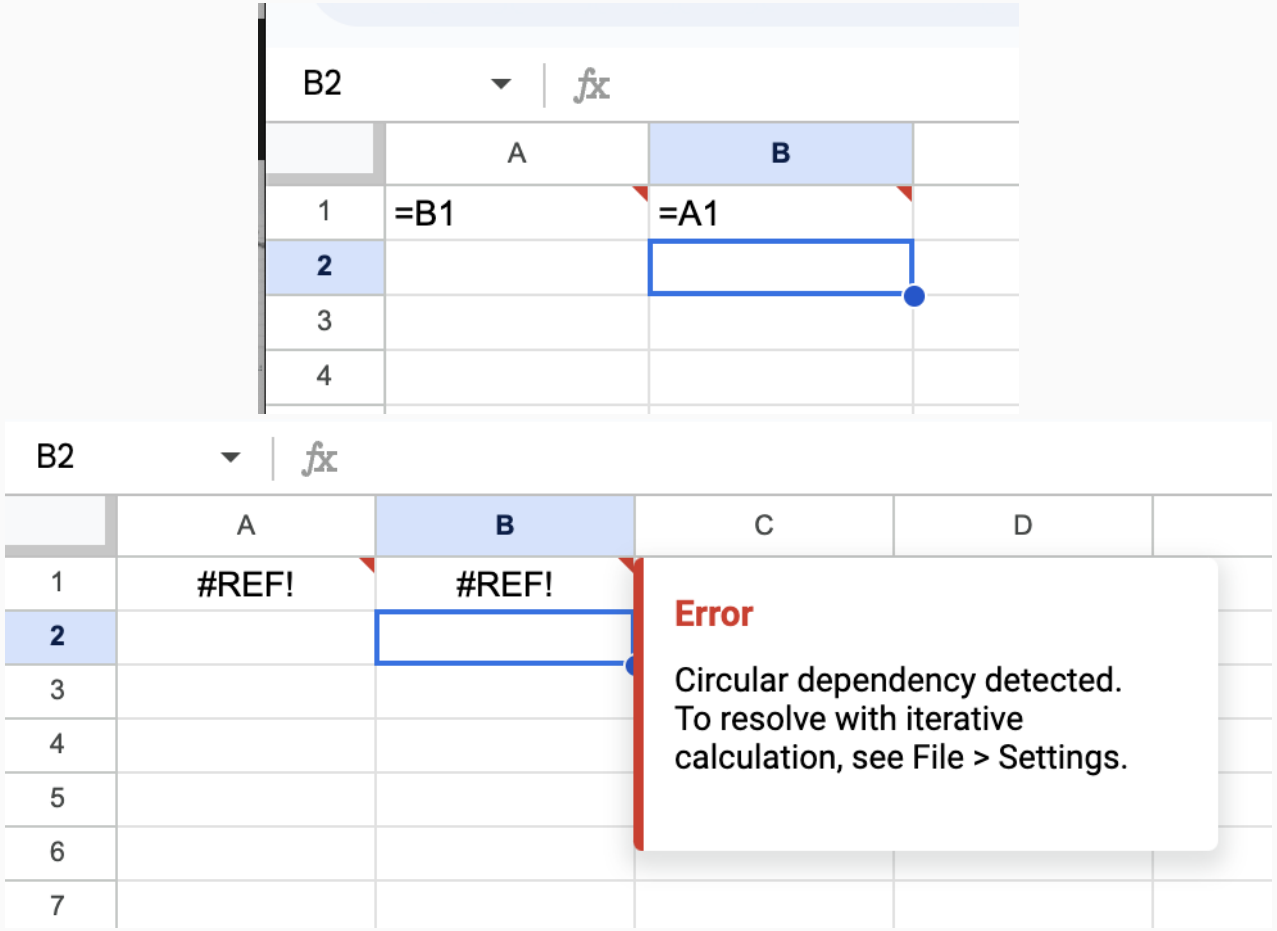


**Figure 3:** Circular dependencies

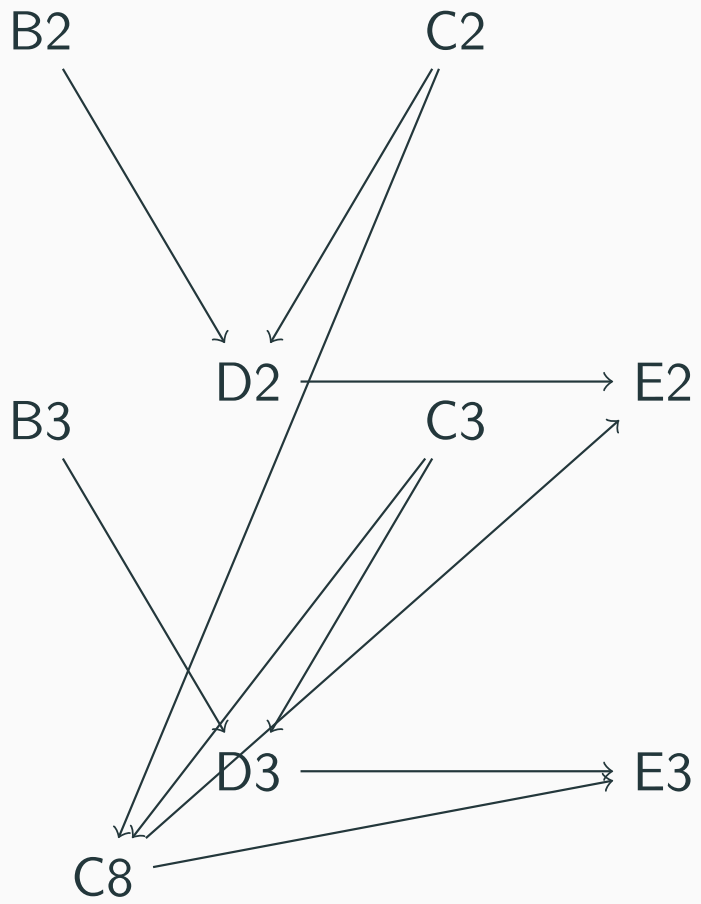


**Figure 4:** DAG

Associated to a spreadsheet is a DAG where

- nodes are cells
- there is a directed edge from $a$ to $b$ if and only if $b$ has a formula that depends on $a$ (i. e., $b$ depends on $a$)

Spreadsheet *evaluation* is analogous to *dependency resolution*

A modern spreadsheet application provides many additional capabilities, not in scope in this talk, such as

- Parsing of cell formulas
- Built-in functions
- Graphics (e. g., histograms, piecharts, etc)
- Formatting, exporting, drag-and-drop, copy-pasting, etc
- Programmable functions (e. g., in VBA)
- Data connectors
- Collaborative online spreadsheets
- And many more

C3 | fx 1.29

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | Fruit | Weight (kg) | Price / kg | Total price | % price | |
| 2 | Apples | 1.5 | 1.75 | 2.625 | 0.4738267148 | |
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| 5 | | | | | | |
| 6 | | | | | | |
| 7 | | | | | | |
| 8 | | Total | 5.54 | | | |
| 9 | | | | | | |
| 10 | | | | | | |

**Figure 5:** Recalculation and support graph

MVP / POC: recalculate the whole spreadsheet every time a cell changes

POST MVP: *incremental recalculation*

Traditional technique:

- Topological sorting of the support graph[1] (e. g., Kahn's algorithm)

[1] `https://en.wikipedia.org/wiki/Topological_sorting`

Production applications need to take into account more complicated challenges:[2]

- Performance tradeoffs (e. g., calculating the topological order also has an associated cost)
- Resource utilisation (e. g., compact representations)
- Volatile functions (e. g., `now()`, `rand()`)
- DAG caching
- Parallelisation algorithms, etc

[2] `https://learn.microsoft.com/en-us/office/client-developer/excel/excel-recalculation`

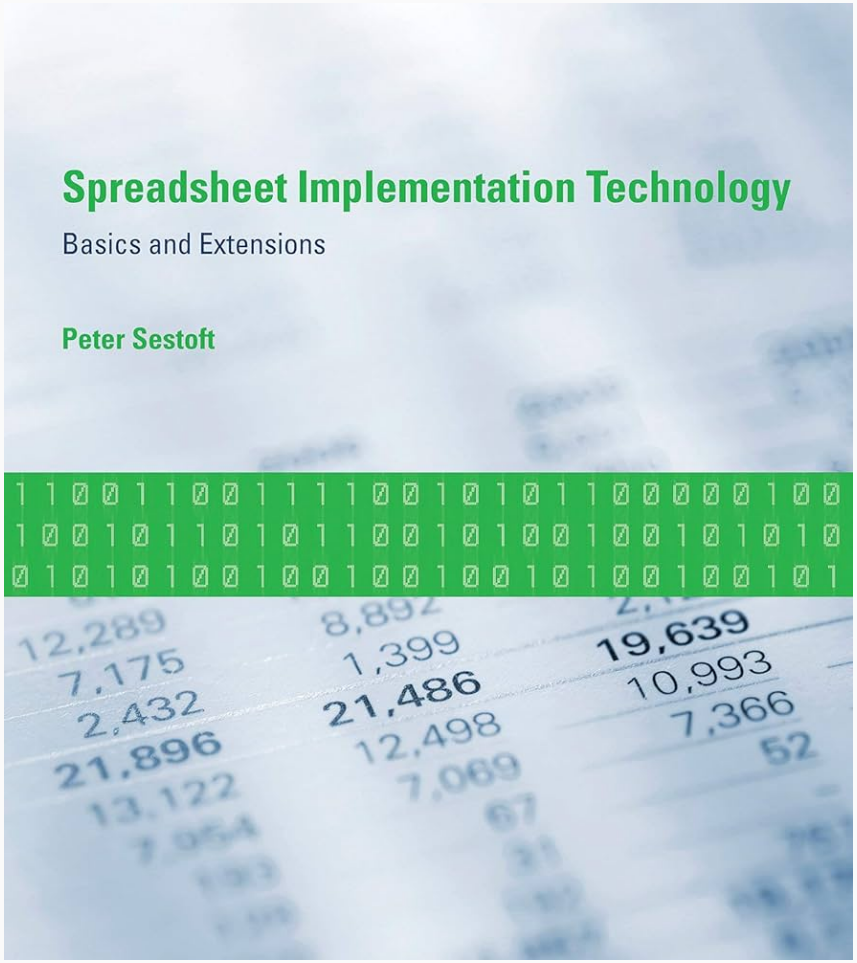



**Figure 6:** P. Sestoft, *Spreadsheet Implementation Technology*, MIT press (2014).[4]

[4] `https://direct.mit.edu/books/book/3071/Spreadsheet-Implementation-TechnologyBasics-and`

**Observation**

Spreadsheet-like evaluation can be expressed as a *fixed-point of higher dimensional comonads*

# Agenda

1. A taste of comonads
2. Comonadic vibes meet spreadsheets
3. Down the rabbit hole

# A taste of comonads

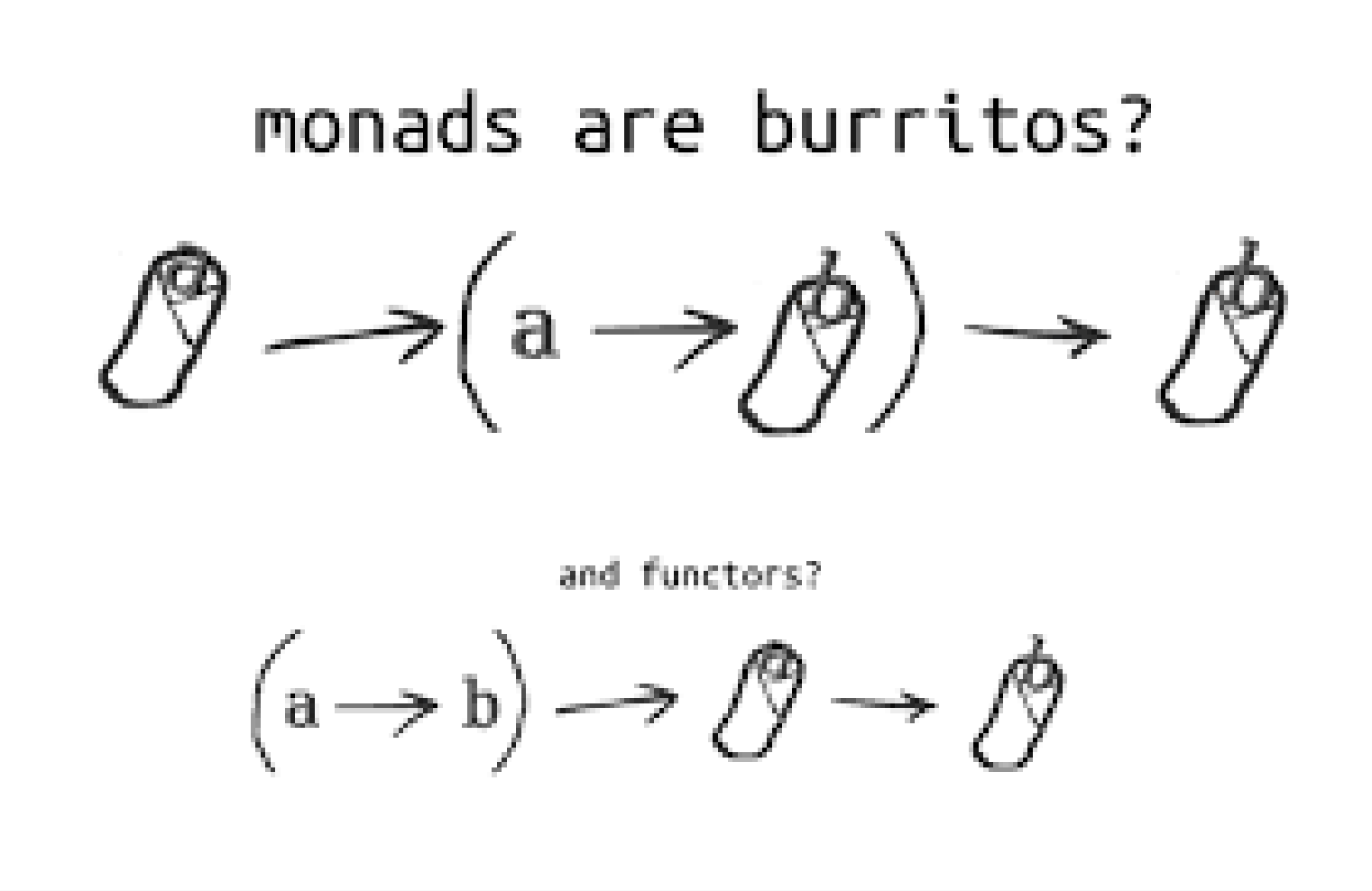



**Figure 7:** Monads are burritos

**Definition**

Comonads are co-burritos

Functor hierarchy:

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

together with some laws.

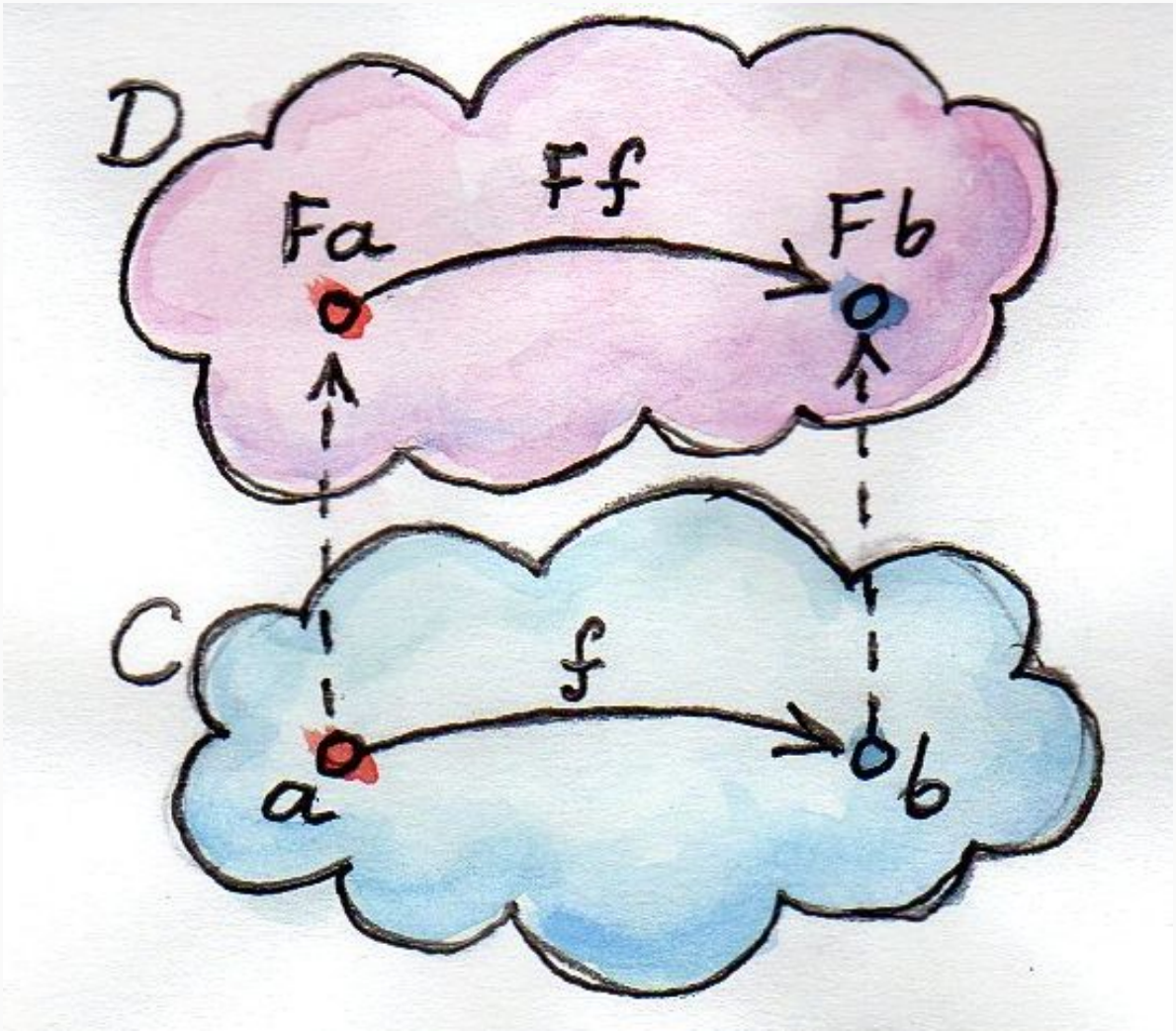



**Figure 8:** Functors: *Lifting* of 1-ary pure functions to the $f$-world

```
f1 :: [Int] -> [String]

-- vs

f2 :: Int -> String

f3 :: [Int] -> [String]
f3 = fmap f2
```

```
class (Functor f) => Applicative f where
  pure :: Applicative f => a -> f a
  ap :: Applicative f => f (a ->b) -> f a -> f b
```

together with laws governing properties of `fmap`, `ap`, and `pure`.

Applicatives provide lifting of $N$-ary pure functions in the $f$-context

```
class (Applicative f) => Monad f where
  return :: Monad f => a -> f a
  join :: Monad f => f (f a) -> f a

-- flatMap
bind :: Monad f => m a -> (a -> m b) -> m b
```

In short:

- Functor $\sim$ `map`
- Applicative $\sim$ `mapN` (or equivalent)
- Monad $\sim$ `flatMap` (or equivalent)

(and the caveat that the implementation must satisfies certain "reasonable" laws that rule out "unexpected" behaviours)

This seems very abstract but from a software engineer perspective lifting provides

- Separation of concerns
- Reusability
- Composability

`flatMap` ensures no "callback hell" of nested effects

```
f ( f ( f ( ... (f a)  ) )) = f a
```

Duality:

```
class Functor f => Comonad f where
    coreturn :: f a -> a
    cojoin :: f a -> f ( f a )
```

(and, as usual, there are important laws)

```
class Functor w => Comonad w where
   extract :: f a -> a
   duplicate :: w a -> w ( w a )

-- coKleisli composition / extract
coFlatMap :: Comonad w => (w a -> b) -> w a -> w b
coFlatMap f = fmap f . duplicate
```

(and, as usual, there are important laws)

- Monadic values are typically *produced* in effectful computations

```
a -> m b
```

- Comonadic values are typically *consumed* in context-sensitive computations (e. g., “queries”)

```
w a -> b
```

Maybe and [] are not comonads (why?)

```
class Copointed f  where
   extract :: f a -> a
```

```
instance Comonad NonEmpty where
    extract = head
    duplicate = tails1
```

```
{-# LANGUAGE OverloadedLists #-}
-- return all suffices including itself

tails [1,2,3] `shouldBe`
        [[1,2,3], [2, 3], [3]]
```

```haskell
{-# LANGUAGE OverloadedLists #-}
coFlatMap f [1,2,3] `shouldBe`
        [f [1,2,3], f [2, 3], f [3]]

-- in this case
-- coFlatMap f = fmap f . tails
```

Simple moving average of a time series $[\phi_1, \ldots, \phi_N]$ with window size $k$ at the time $t$ is

$$SMA_k(t) = \frac{1}{k} \sum_{j=t-k+1}^{t} \phi_j \tag{1}$$

```
{-# LANGUAGE OverloadedLists #-}

sma :: Window -> NonEmpty Int -> [Double]

sma 3 [1,2,3,4,5,6,7,8,9,10] `shouldBe`
    [2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0]
```

because

- $(1 + 2 + 3)/3 = 6/3 = 2$
- $(2 + 3 + 4)/3 = 9/3 = 3$
- $(3 + 4 + 5)/3 = 12/3 = 4$

```
smaLocal :: Window -> NonEmpty Double
   -> Maybe Double
smaLocal k s = (/ (fromIntegral k)) <$>
         sumK k s

sumFirstK :: Num a =>
   Window -> NonEmpty a -> Maybe a
sumFirstK (Window k) _ | k <= 0 = Nothing
sumFirstK (Window 1) (MyNonEmpty a _) = Just a
sumFirstK k (MyNonEmpty a as) =
   case nonEmpty as of
     Just as2 -> (+a) <$> sumFirstK (k-1) as
     Nothing -> Nothing
```

```
sma :: Window -> MyNonEmpty Double
          -> MyNonEmpty (Maybe Double)
sma size = coFlatMap (smaLocal size)
```

## Folkloristic understanding of some comonads usages

- the function passed to `coFlatMap` represents a *local* computation (can explore the "neighbourhood" of the focus point, but produces a single value)
- `cojoin` produces a (lazy) view of the structure from all perspectives
- `coFlatMap` applies the local computation to all perspectives

Similar examples:

- Numerical derivation
- Reconciliation in a list of events
- etc

Interesting usages:

- Cellular automaton (e. g., Conway's game of life)
- Store, Moore machine and comonadic UIs

# Comonadic vibes meet spreadsheets

```
data Stream a = Stream a (Stream a)
  deriving (Eq, Show,Functor)

instance Comonad Stream where
   extract (Stream a _) = a
   duplicate s'@(Stream _ as) =
     Stream s' (duplicate as)
```

```
iterate :: (a -> a) -> a
  -> Stream a
iterate f seed =
  Stream seed (iterate f (f seed))

takeN :: Int -> Stream a -> [a]
takeN 0 _ = []
takeN n (Stream a as) = a : takeN n as
```

```
naturals :: Stream Imt
naturals = iterate (+1) 0

takeN 10 naturals `shouldBe` [0,1,2,3,4,5,6,7,8,9]
```

One-dimensional, bidirectional, homogeneous spreadsheet of type *a*

```
data Sheet1 = Sheet1 (Stream a) a (Stream a)
    deriving (Eq,Show, Functor)
```

```
focus :: Sheet1 a -> a
focus (Sheet1 _ a _ ) = a

moveL :: Sheet1 a -> Sheet1 a
moveL (Sheet1 (Stream a as) focus s)
   = Sheet1 as a (Stream focus s)

moveR :: Sheet1 a -> Sheet1 a
moveR (Sheet1 s focus (Stream a as))
   = Sheet1 (Stream focus s) a as
```

```
instance Comonad Sheet1 where
  extract = focus
  duplicate s =
    Sheet1 (allLeft s) s (allRight s) where
      allLeft = iterate moveL
      allRight = iterate moveR
```

Duplicate is “diagonalisation” (as with other comonads derived from zippers)

```
naturalsSheet1 :: Sheet1 Int
naturalsSheet1 = Sheet1 (constS 0) 0 naturals
```

We are looking for sheets of functions:

```
sheet :: Num a => Sheet1 (Sheet1 a -> a)
```

and the evaluate function should look like

```
evaluate :: Num a => Sheet1 (Sheet1 a -> a)
  -> Sheet1 a
```

Comonadic fixed-points at rescue[5]

```
kfix :: ComonadApply w => w ( w a -> a ) -> w a
```

[5] Why `ComonadApply`? We need some "zippiness", see references at the end

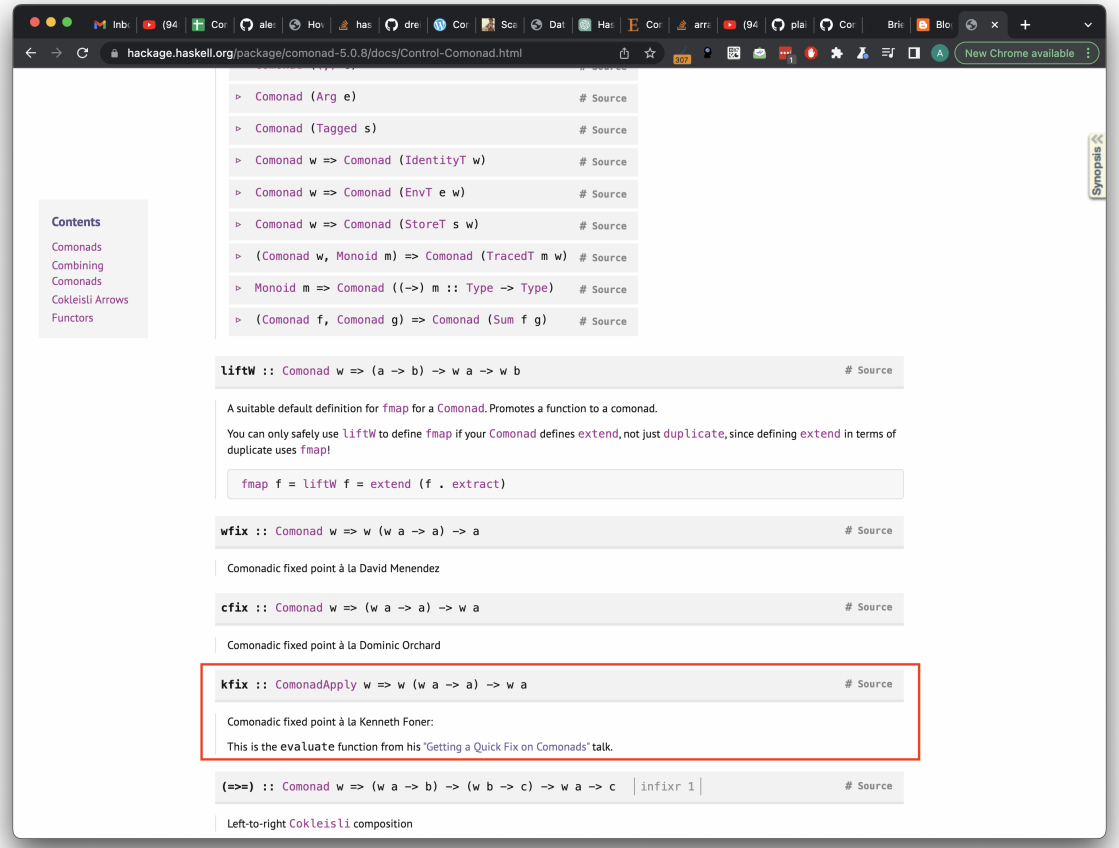



**Figure 9:** Kenneth fixed point for `ComonadApply`

# Down the rabbit hole

Next steps

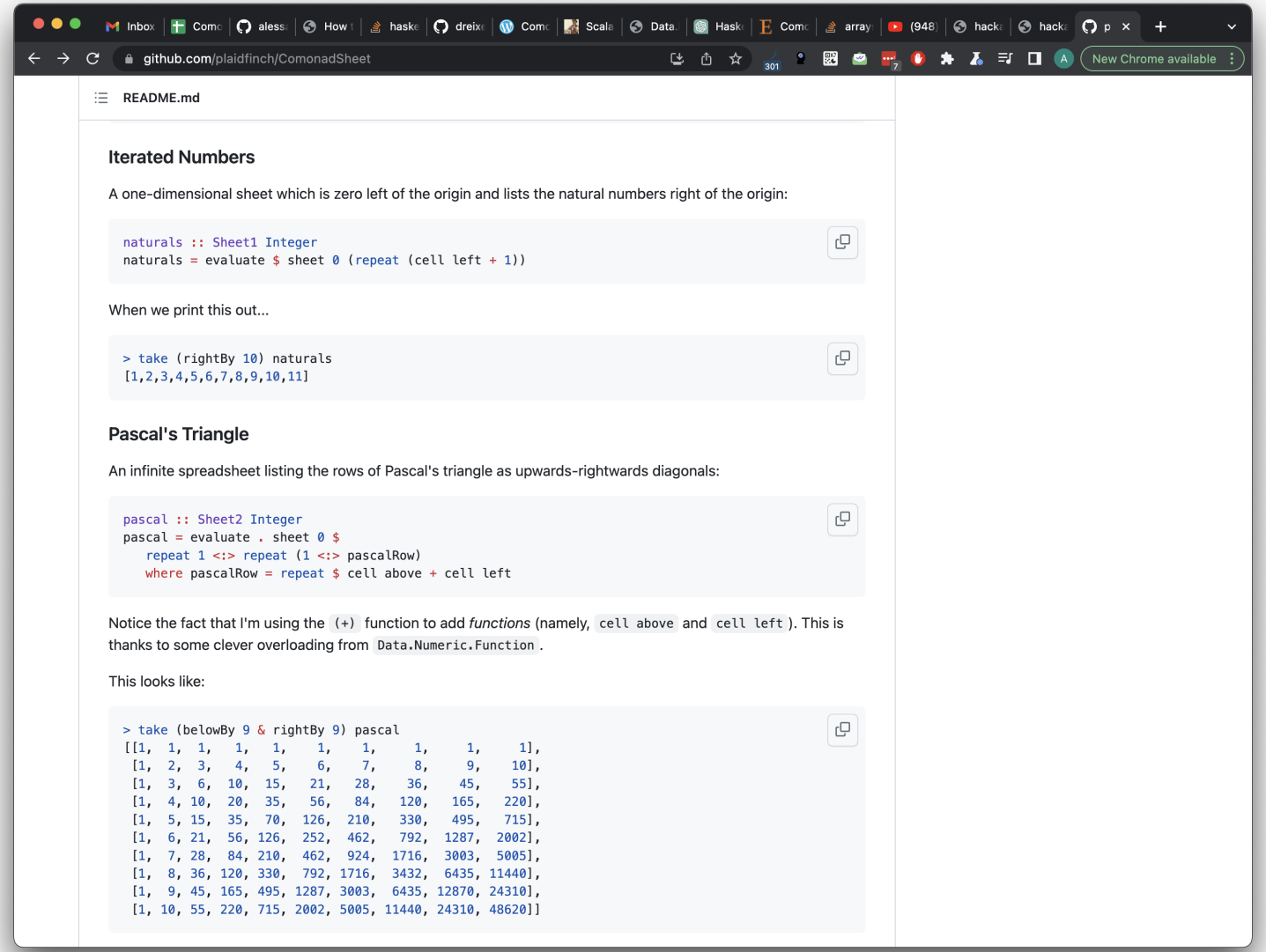
- build helper functions (aka, DSL) to build a spreadsheet
- generalise to multiple dimensions

Something along the lines of

```
naturalsSheet :: Sheet1 (Sheet1 Int -> Int)
naturalsSheet = sheet 0 (repeat (cell left + 1))

naturals :: Sheet1 Int
naturals = evaluate naturalSheet

evaluate = kfix
```


github.com/plaidfinch/ComonadSheet
New Chrome available
README.md
Iterated Numbers
A one-dimensional sheet which is zero left of the origin and lists the natural numbers right of the origin:
naturals :: Sheet1 Integer
naturals = evaluate $ sheet 0 (repeat (cell left + 1))
When we print this out...
> take (rightBy 10) naturals
[1,2,3,4,5,6,7,8,9,10,11]
Pascal's Triangle
An infinite spreadsheet listing the rows of Pascal's triangle as upwards-rightwards diagonals:
pascal :: Sheet2 Integer
pascal = evaluate . sheet 0 $
repeat 1 <:> repeat (1 <:> pascalRow)
where pascalRow = repeat $ cell above + cell left
Notice the fact that I'm using the (+) function to add functions (namely, cell above and cell left). This is thanks to some clever overloading from Data.Numeric.Function.
This looks like:
> take (belowBy 9 & rightBy 9) pascal
[[1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10],
[1, 3, 6, 10, 15, 21, 28, 36, 45, 55],
[1, 4, 10, 20, 35, 56, 84, 120, 165, 220],
[1, 5, 15, 35, 70, 126, 210, 330, 495, 715],
[1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002],
[1, 7, 28, 84, 210, 462, 924, 1716, 3003, 5005],
[1, 8, 36, 120, 330, 792, 1716, 3432, 6435, 11440],
[1, 9, 45, 165, 495, 1287, 3003, 6435, 12870, 24310],
[1, 10, 55, 220, 715, 2002, 5005, 11440, 24310, 48620]]

About comonads:

- https://bartoszmilewski.com/2017/01/02/comonads/
- https://blog.higher-order.com/blog/2015/10/04/scala-comonad-tutorial-part-2/
- https://reasonablypolymorphic.com/blog/cofree-comonads/
- https://reasonablypolymorphic.com/blog/comonadic-physics/index.html

About comonadic UIs:

- https://functorial.com/the-future-is-comonadic/main.pdf
- https://arthurxavierx.github.io/ComonadsForUIs.pdf

About cellular automata

- https://www.schoolofhaskell.com/user/edwardk/cellular-automata/part-1 and references therein

From *Löb's Theorem* in modal logic[6]

$$\Box(\Box\phi \to \phi) \to \Box\phi \tag{2}$$

to spreadsheet evaluation:

- Original post from Piponi (2006): `http://blog.sigfpe.com/2006/11/from-l-theorem-to-spreadsheet.html`
- Functional pearl paper from Kenneth Foner (2015): `https://dl.acm.org/doi/10.1145/2887747.2804310`
- ComonadSheet hackage package (unmaintained): `https://hackage.haskell.org/package/ComonadSheet`

[6] In this context, $\Box\phi$ is a formal provability predicate which reads "$\phi$ is provable"

# Questions?