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Quantum Field Theory

Path Integral Methods and Renormalization

Quantum Field Theory

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“In the good old days, theorizing was like sailing between islands of experimental evidence. And, if the trip was not in the vicinity of the shoreline (which was strongly recommended for safety reasons) sailors where continuously looking forward, hoping to see land — the sooner the better.

Nowadays, some theoretical physicists (let us call them sailors) [have] found a way to survive and navigate in the open sea of pure theoretical constructions. Instead of the horizon, they look at stars, which tell them exactly where they are. Sailors are aware of the fact that the stars will never tell them where the new land is, but they may tell them their position on the globe.

Theoreticians become sailors simply bacause they just like it. Young people, seduced by capitans forming crews to go to a Nuevo El Dorando […] soon realize that they will spend all their life at sea. Those who do not like sailing desert the voyage, but for the true potential sailors the sea become their passion. They will probably tell the alluring and frightening truth to their students — and the proper people will join their ranks.”

— Andrei Losev

CONTENTS

PREFACE vii

APHORISMS ix

I FUNCTIONAL METHODS 1

1 FEYNMAN'S SPACE-TIME APPROACH TO QUANTUM MECHANICS 3
- 1.1 Introduction 3
- 1.2 Review of the quantum time-evolution operator 3
- 1.3 Path integral representation of the Feynman's kernel 6
- 1.4 Free particle 9
- 1.5 Path integral treatment of harmonic potential 10
- *1.6 Path integral treatment of Coulomb potential 10
- *1.7 Path integral treatment of finite square well potential 10
- 1.8 WKB from a path integral perspective 10
- 1.9 What's the use of all this? 10

2 FUNCTIONAL METHODS FOR SCALAR QUANTUM FIELDS 11
- 2.1 Path integral quantization of scalar fields 11
  - 2.1.1 Path integral quantization of interacting scalar field 11
  - 2.1.2 Review of canonical quantization of scalar fields 11
- 2.2 Perturbation expansion of Green functions of self-interacting scalar fields 12
- 2.3 Feynman diagrams 12
- 2.4 Troubles with perturbation theory 12
- 2.5 Symmetry factors of Feynman diagrams for scalar fields 12

II SYSTEMATICS OF RENORMALIZATION 13

III QUANTUM ELECTRODYNAMICS 15

3 SPINORS 17
- 3.1 The Lorentz group 17
  - 3.1.1 Notations and conventions 17

IV NON-ABELIAN GAUGE THEORIES AND THE STANDARD MODEL 19

4 FUNCTIONAL QUANTIZATION OF YANG-MILLS GAUGE FIELD THEORIES 21

5 QUANTUM CHROMODYNAMICS 23

6 INTRODUCTION TO LATTICE QCD 25

7 ULTRACOLD ATOM SIMULATIONS OF YANG-MILLS LATTICE GAUGE THEORIES 27

8 ANOMALIES 29

9 ELECTROWEAK SPONTANEOUSLY SYMMETRY BREAKING AND HIGGS MECHANISM 31

V INTRODUCTION TO SUSY 33

VI APPENDICES 35

A GAUSSIAN INTEGRALS 37
- A.1 Basic results 37
- A.2 Univariate Gaussian integrals 39
  - A.2.1 Real case and quadratic supplement trick 39
  - A.2.2 Fourier transform using Cauchy's residue theory 40

A.2.3 Fresnel integral 42
A.2.4 Complex case via differential equations and analytic continuation 43
A.2.5 Moment integrals 45
A.3 Multivariate Gaussian integrals 46
A.4 Wick theorem 48
B GROUP THEORETICAL METHODS 49
C CRASH COURSE OF GRASSMANN VARIABLES 51
D BAKER-CAMPBELL-HAUSDORFF FORMULA 53
E SADDLE-POINT METHOD AND WKB 55
E.1 Real case (alias, Laplace method) 55
E.2 Complex case 55
F RENORMALIZATION GROUP AND THE PRIME NUMBER GAS 57
G PATH INTEGRALS APPROACH TO CLASSICAL MECHANICS 59

PREFACE

Relativistic quantum theory of fields (QFT) is an extremely fashinating topic, full of far-reaching implications in several areas of modern Physics and which reveals a rich mathematical structure behind it. It is the language currently used to develop a major importance part of latest achievements in theoretical physics in several areas, ranging from high energy particle physics to statistical mechanics, solid state physics, cosmology, etc. It is a source of intriguing new physical questions and a furnace where to develop new challanging mathematical ideas.

QFT was originally developed in order to approach quantum mechanics in a way consistent with the requirements of Einstein's special relativity. Currently, it offers a comprehensive framework to deal with elementary excitations above the ground state of physical systems having an infinite number of degrees of freedom.

QFT is the theoretical framework currently employed to describe all foundamental interactions but gravitation in high-energy physics — in the Standard Model of particle physics, strong interactions are described by Quantum Chromodynamics (QCD), which is an unbroken non-abelian $\mathsf{SU}(3)$ color Yang-Mills gauge QFT, while the electroweak interactions are described by a non-abelian $\mathsf{SU}(2) \times \mathsf{U}(1)$ Yang-Mills gauge QFT spontaneously broken via the Higgs mechanism. Furthermore, QFT has let to a deeper understanding of the singular properties of a wide class of phase transitions at the critical point in statistical field theory and has allowed the development of field-theoretical descriptions of the statistical properties of certain geometrical models (e. g., self-avoiding random walks which are of major importance, for instance, in polymer physics).

The level of mathematical rigour will be suitable to an i

Sometimes we will run into lengthly calculations. Hopefully, this will be helpful. Learing QFT be doing it, this is the basic learning method behind this handout. Otherwise, it could become harder and dangerous to acquire familiarity with the tools and strategies of QFT without getting lost into the formalism.

Trieste, February 20, 2015 *Alessandro Candolini*

APHORISMS

The *lietemotiv* of these notes is *renormalization*. Let's sketch a summary to grab the key ideas. The section is mainly inspired by the introductory chapter of Zinn-Justin.

QFT in its most direct formulations run into a severe conceptual issue. QFT crashes due to the occurrence of infinities in the calculations of physical quantities.

To overcome this difficulty and avoid the occurrence of infinities, an empirical, somewhat *ad hoc* (but systematic) procedure, which goes under the name of renormalization, was eventually developed which allows extracting from (meaningless) divergent mathematical expressions (meaningfull) finite numerical predictions to be compared with experiments.

Often in mathematics one meets expressions which naively speaking are divergent but it is possible to append a meaning to them, e. g., via analytic continuation or by regularize them in a suitable way. Renormalization however is a different story. Experimental data are needed in order the renormalization algorithm to work. Renormalization recipe needs input from experiments. You have to provide *input* experimental parameters in order the renomalization machinary to work. The divergence is hidden behind some experimental parameters, it's not a purely mathematical fact like analytic continuation. Renormalization heals the divergences of the theory by sweep them under the carpet of few external parameters. In other words, infinities are absorbed into few experimental parameters.

The renormalization recipe works in this spirit: you have to carefully distinguish between *bare* quantities (e. g., mass, electric charge) and *effective* quantities. The latter ones are those which actually have to be related to experiments, while the former ones are additional auxiliary parameters of the physical model, which are necessary in order the renormalization recipe to work but whose physical meaning remains somewhat obscured. At this level, it might be said that renornalization is a way of organizing the calculations in order to get finite results from expressions which naively speaking would otherwise led to infinities.

From a pragmatic point of view, renormalization works: It has allowed and still allows calculations of increasing precision. In fact, the renormalization procedure would hardly have been convincing if the predictions were not confirmed with increasing precision by experiments.

However, this is not the whole story. A beautiful and deep idea comes at this point of the story, which goes at the heart of renormalization. The problem of infinities is related to an unexpected phenomenon: *Renormalization is related to the non-decoupling of very different lenght scales*. This idea was borned in the context of statistical mechanics of critical phenomena first, and then extended to particle physics. Today, those who are familiar with Kenneth Wilson's ideas and the renormalization group, will immediately say that actually there is no divergence. More or less, the story goes like this: Every QFT requires an ultraviole completion (thus being an effective theory). The only difference between renormalizable and non-renormalizable QFTs lies in the fact that the former are *insensitive* to the ultraviolet data (which can be absorbed in a few low-energy parameters) while the latter depend on the details of the ultraviolet completion. In other words, the fact whether the renormalization recipe is capable of absorbing the divergences or not for a given quantum field theory and how, tells us an important property of the quantum field theory.

Part I

FUNCTIONAL METHODS

Complements of non-relativistic quantum mechanics and application of functional methods and path integrals to scalar fields

1

FEYNMAN'S OVERALL SPACE-TIME APPROACH TO NON-RELATIVISTIC QUANTUM MECHANICS

> "I would also like to emphasize that by this time I was becoming used to a physical point of view different from the most costumary point of view. In the costumary view, things are discussed as a function of time in very great detail. For example, you have the field at this moment, a differential equation gives you the field at the next moment and so on; a method, which I shall call the Hamilton method, the time differential method. We have, instead [the action] a thing that describes the character of the path throughout all of space and time. The behavior of nature is determined by saying her whole space-time path has a certain character."
>
> R. F. Feynman
> *The development of the space-time view of quantum electrodynamics*,
> Nobel Lecture, December 11, 1965. Brown [2000].

1.1 INTRODUCTION

The *operator* formulation of non-relativistic quantum mechanics is based on the mathematical machinery of Hilbert spaces.* The (pure) physical states of a quantum system are represented by normalized vectors (referred to as state vectors) belonging to a separable complex Hilbert space, while the physical observables of the system are represented mathematically by (densely-defined) linear self-adjoint operators (often, unbounded ones) acting on the Hilbert space of state vectors. In this framework, the dynamics (in absence of measurement process) is ruled by a family of unitary linear evolution operators determined by the Hamiltonian of the quantum system (see section 1.2). The Hamiltonian encodes the dynamics of the system and specifies the evolution opererators through the Schrödinger equation. According to Feynman's words and for reasons which will become clearer in a while, this approach will be referred to as *time-differential*, *Hamiltonian* approach. On the contrary, Feynman's approach will happear to be a *global space-time* and *Lagrangian* approach. Furthermore, from the mathematical point of view, instead of von Neumann's spectral theory of linear self-adjoint operators (usually, unbounded) in Hilbert spaces one has to deal with Feynman's *path integrals* (i. e., integration in functional spaces).

1.2 REVIEW OF THE QUANTUM TIME-EVOLUTION OPERATOR

Time-differential approach means that the dynamics is ruled by a differential equation in time, which allows us to compute the state of the system at a given time starting from the knowledge of the state at a previous time (*initial* conditions).

* Originally, quantum mechanics was developed in two seemingly different ways, namely, the "matrix mechanics" of Heisenberg, Born and Jordan and the "wave mechanics" of Schrödinger. It was von Neumann, in its cornerstone book *Mathematical foundations of Quantum mechanics*, who revealed how Hilbert spaces provide the rigorous mathematical framework underlying both formulations. For an exposition of the tools of functional analysis involved in the operatorial approach of non-relativistic quantum mechanics refer to, e. g., the original book of von Neumann [1955] or Reed and Simon [1980] (and the other volumes). A recent textbook is that of Teschl [2009]. Although the use of Hilbert space is not generally questioned by newcomer students and is usually taken quite for granted, the reasons why from a physical perspective one eventually resolves in using the apparatus of linear operator in Hilbert space remains somewhat hidden. The use of Hilbert spaces can be *in*duced from physical insights in a way suggested by Schwinger in its lucid and deep lectures of quantum mechanics, see Schwinger [2001].

On the contrary, a space-time overall approach means that the time-evolution is determined by the whole space-time history (i. e., boundary conditions in time).

Unitary time-evolution operator

Let us work in the Schrödinger picture, where the time dependence is completely carried by state vectors. The evolution forward in time of every (pure) state of a (closed) quantum mechanical system, in absence of measurement processes, is *implemented* through the action of the (linear) quantum time-evolution operator $\hat{U}(t'', t')$ on the state vectors. $\hat{U}(t'', t')$ maps the state vector $|\psi(t')\rangle$ describing the state of a quantum system at the initial time t' to $|\psi(t'')\rangle$ which describes the state at the later time t'', assuming that no measurements have been performed on the system between t' and t''. In other words, $\hat{U}(t'', t')$ pushes the state forward in time:

$$|\psi(t'')\rangle = \hat{U}(t'', t')\,|\psi(t')\rangle\,, \quad t'' \geqslant t'\,. \tag{1.1}$$

Equation (1.1) tells that the time-evolution of a quantum states is fully *deterministic*: the knowledge of $|\psi(t')\rangle$ at the initial time t' implies the knowledge of $|\psi(t'')\rangle$ at any subsequent time t'' (if no measurements occurred in between t' to t'') The standard interpretation of state vectors in ordinary quantum mechanics is however a *probabilistic* one (Born interpretation), i. e., in general one is not able/allowed to extract from $|\psi(t'')\rangle$ deterministic predictions on the outcomes of a measurement process performed on this state. One is only allowed to predict the *probability* of a certain outcome. In the standard interpretation of quantum mechanics, this probability is not a matter of our ignorance and should instead be understood as intrinsic (i. e., not epistemic).

Two major remarks are in order: a) general physical considerations constrain the mathematical properties of $\hat{U}$, in other words physics implies restrictions on the properties of $\hat{U}$ for any system — as we sall see, $\hat{U}$ must be unitary, etc; b) the exact form of $\hat{U}$ must depend on the particular system we are investigating and the interactions which this system undergoes.

Schrödinger equation

The explicit form of $\hat{U}(t'', t')$ depends of course on the interactions that a particular system undergoes. Informations about the interactions is usually encoded in an observable which is called the quantum Hamiltonian operator $\hat{H}$. $\hat{H}$ determines $\hat{U}(t'', t')$ (and so the evolution forward in time of the quantum system) via the following time-differential (operator) equation

$$i\hbar \frac{\partial \hat{U}(t'', t')}{\partial t''} = \hat{H}(t'')\hat{U}(t'', t')\,, \tag{1.2}$$

which is the time-dependent Schrödinger equation for the time-evolution operator, together with the initial condition

$$\hat{U}(t', t') = \mathbb{1}\,. \tag{1.3}$$

In eq. (1.2), i denotes (as usual) the imaginary unit and $\hbar$ is a real parameter which has dimensions of an action and which will be later identified with the Plank's constant h divided by 2π.[*] $\hat{H}$ in general may depend explicitly on time (e. g., a time-dependent Hamiltonian appears when you study the interaction of a charge particle with an external electric or magnetic field which is not constant in time).

So, let us summarize the main steps to compute the time evolution of a quantum system:

(i) Provide the quantum Hamiltonian operator of the system;

(ii) once $\hat{H}$ is given, solve eqs. (1.2) and (1.3) for this $\hat{H}$ to get $\hat{U}(t'', t')$;

(iii) once you have calculated $\hat{U}(t'', t')$, you are able to compute the state vector $|\psi(t'')\rangle$ at every time $t'' \geqslant t'$ by just applying $\hat{U}(t'', t')$ to the initial state $|\psi(t')\rangle$ according to eq. (1.2).

Comparison with Hamiltonian dynamics

Brief comparison with the evolution in time of an Hamiltonian dynamical system in classical mechanics. At any time, the state of the system is fully specified by a point in the phase space of the system. Let $(\boldsymbol{q}', \boldsymbol{p}')$ the point representing the state at the initial time t'. The state $(\boldsymbol{q}'', \boldsymbol{p}'')$ describing the system at the later time $t'' \geqslant t'$ is obtained by applying the Hamiltonian flux to $(\boldsymbol{q}_0, \boldsymbol{p}_0)$. The Hamiltonian flux is obtained by solving the Hamilton's equations of motion. These equations involve the Hamiltonian function of the system.

How to write down the Hamiltonian of a quantum system; quantization

Ad item i. Strictly speaking, $\hat{H}$ should be inferred by experimental data in principle. Nevertheless, it might happen that classical mechanics provides guidelines to write down the quantum Hamiltonian operator. This is somewhat a strange procedure, which goes under the name of "quantization". There is a number of ways to start with a classical system and then to construct a quantized version of it. In this section, we will briefly recall the so-called "canonical quantization", which starts from the classical Hamiltonian. We shall see in the following sections that path integrals provide a different way to quantize a classical system, starting from the action functional. There are other quantization procedures.

Remark. Do not forget that the interplay between classical and quantum mechanics is a fashinating but still open topic. There are classical theories which still lack a quantum description (best example is quantum gravity), there are classical theories with multiple quantum versions (e. g., when operator ordering problems are encountered) and quantum theories without classical analogue.

us about quantum mechanics?

Explicit formulas for the quantum evolution operator

Ad. item ii. For time-independent quantum Hamiltonian operators, an explicitly formula for $\hat{U}(t'', t')$ is known, namely,

$$\hat{U}(t'', t') = e^{-\frac{i}{\hbar}\hat{H}(t''-t')}, \tag{1.4}$$

which can be obtained by formally integrating eq. (1.2) and which can be rigorously proven for all self-adjoint $\hat{H}$. In this case, $\hat{U}(t'', t')$ indeed depends only on the difference $t'' - t'$, so $\hat{U}(t'', t') = \hat{U}(t'' - t')$ and $\hat{U}$ becomes a (strongly-continuous) one-parameter group of (unitary) linear operators and the quantum Hamiltonian is the generator of such group.

In the general case of time-dependent Hamiltonians, one may still obtain a formal expression in terms of the Dyson's series:

$$\hat{U}(t'', t') = \mathrm{T}\, e^{-\frac{i}{\hbar}\int_{t'}^{t''} \hat{H}(\tau)\, d\tau}, \tag{1.5}$$

where T is Dyson's time-ordering symbol. Equation (1.5) reduces to eq. (1.4) when $\hat{H}$ does not depend explicitly on time. Equation (1.5) is explain in many textbooks on quantum mechanics; from a mathematical point of view, one uses the Neumann expansion, for a rigorous treatment see, e. g., Reed and Simon [1975, § X.12].

Main properties of the quantum evolution operator

Let us discuss some of the main properties of $\hat{U}$:

- From a mathematical point of view, these properties follow from eq. (1.2);
- From a physical point of view, it is natural to ascribe these properties to $\hat{U}$ from the very beginning; some authors in fact motivate eq. (1.2) starting from these properties.

Among these properties are:

UNITARITY: This means that

$$\hat{U}^\dagger(t'', t')\hat{U}(t'', t') = \hat{U}(t'', t')\hat{U}^\dagger(t'', t') = \mathbb{1}.$$

This is required for consistency with the probabilistic interpretation of $|\psi(t)\rangle$. In fact, it ensures probability conservation in time.

* $\hbar$ appears in the Heisenberg canonical commutation relations.

COMPOSITION PROPERTY: The evolution from t' to t'' should be equivalent to the evolution from t' to some intermediate time t plus the evolution from t to t'':

$$\hat{U}(t'', t') = \hat{U}(t'', t)\hat{U}(t, t'), \quad t'' > t > t', \tag{1.6}$$

in absence of measuremens from t' to t''. This property means that time-evolution has no memory: the evolution from time t to time t'' depends only on the state of the system at time t (in absence of measurement processes).

Remark. Equation (1.6) will play a key role in section 1.3.

1.3 PATH INTEGRAL REPRESENTATION OF THE FEYNMAN'S KERNEL

In this section, the Feynman's formulation of one-particle non-relativistic quantum mechanics will be developed starting from the matrix elements in configuration space of the quantum time-evolution operator (we will construct a path integral representation of it, for a particular class of Hamiltonian operators). Historically, Feynman's came up with the path integral formulation in a completely different way, then he proved the equivalence with the usual formulation. We strongly invite to read Feynman's original paper.*

The setting

We shall focus on the simplest quantum system: one *spinless* particle, having no other internal degrees of freedom, moving in one dimension without constrains on the motion. Several generalizations are possible: two and three-dimensional cases, spherical coordinates, etc.†

> Notice that this leaves out even some simple quantum mechanical systems such as the infinite square well (i. e., a particle in a box). The reader might be wonder of that, because the Schrödinger equation is easily solvable in this case. Its path integral treatment instead is difficult to handle. We shall show a trick to approach infinite potential well and other quantum systems in a restricted domain by advocating the image-point method. A proof of the applicability of the method relies on supersymmetric quantum mechanics, so this point will be discussed later. The solution of a spinless particle in a central Coulomb potentials by means of path integrals is also possible, and it will be discussed, but also in this case special care is needed. The extension to spin poses some problems (eventually, spin is a relativistic effect) and we will discuss further the role of spin in a later chapter.

Remember that the algebra of the quantum operators is (almost, see remark below) fixed by the so-called Born, Heisenberg and Jordan canonical commutation relations, namely

$$[\hat{q}, \hat{p}] = i\hbar\,, \tag{1.7}$$

where $\hbar = h/2\pi$ and h is the Planck's constant and $\hat{q}$ and $\hat{p}$ are the quantum mechanical position and momentum operators.

It is costumary to introduce "eigenstates" of position and momentum operators:

$$\hat{q}\,|q\rangle = q\,|q\rangle$$

and

$$\hat{p}\,|p\rangle = p\,|p\rangle\,.$$

* Feynman, 1948.

† For a comprehensive treatment refer to, e. g., Grosche and Steiner, 1998; Kleinert, 2009.

Brief mathematical interlude. The operators $\hat{q}$ and $\hat{p}$ have continuous spectrum and cannot be bounded. It can be shown that this is necessary in order eq. (1.7) to hold. To avoid such (rather technical) difficulties coming from working with unbounded operators, at a more rigorous mathematical level would be that of replacing eq. (1.7) with teh so-called Weyl's commutation relations; refer, e. g., to Zeidler, 2009. The eigenvectors of $\hat{q}$ and $\hat{p}$ should be understood in a generalized (improper) sense. von Neumann's spectral theory of (possibly unbounded) densily-defined linear operators in Hilbert space is a mathematically rigorous approach.

Definition of the Feynman's kernel/propagator

The main character of the story is

$$K(t'', q''; t', q') = \left\langle q'' \middle| \hat{U}(t'', t') \middle| q' \right\rangle, \quad t'' \geqslant t', \tag{1.8}$$

which is called the "Feynman's kernel" or "propagator". (Do not confuse it with the propagator appearing in quantum field theories.) Often, a slightly modified version of eq. (1.8) is used:

$$K(t'', q''; t', q') = \left\langle q'' \middle| \hat{U}(t'', t') \middle| q' \right\rangle \vartheta(t'' - t'), \tag{1.9}$$

where ϑ is the Heaviside step function defined by

$$\vartheta(t) = \begin{cases} 1 & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0 \end{cases}$$

for all $t \in \mathbb{R}$. Equation (1.9) ensures that $K(t'', q''; t', q')$ is zero for every $t'' < t'$. We shall use this latter definition of the propagator.

Meaning of the propagator

The meaning of the $K(t'', q''; t', q')$ is:

- Mathematical interpretation: Matrix elements of the time-evolution operator in the basis of eigenstates of the position operator;* remember that any operator is fixed once you have prescribed its matrix elements with respect to some basis.
- Physical interpretation: Probability *amplitude* between the localized states (eigenstates of the position operator) $|q'\rangle$ and $|q''\rangle$, i. e., probability amplitude for a particle located at q' at initial time t' to be found at q'' at a later time t''; see fig. 1.1.

From a mathematical point of view, $K(t'', q''; t', q')$ is also a *Green's function* of the Schrödinger equation in configuration space. Let us discuss this point.

The Green function is not unique, because you can also add to it a solution of the homogeneus differential equation.

This implies that our approch is nothing but a traditional responsive approach to partial differential equation. The propagator can be used as a kernel to construct by convolution solutions of inhomogeneus differential problems.

Computing the kernel in the standard way

The main point is that it is possible to construct an integral representation of the Feynman's kernel in term of path integrals. The computation of the kernel using the operatorial formulation of quantum mechanics is as follows:

- You need $\hat{H}$ and you have first to solve eq. (1.2);
- Once you have found explicitly the time-evolution operator, compute its matrix elements. Often, it might become easier to compute this matrix elements by fixing a particular realization of the Hilbert space.

In this section, we shall see that we can compute the same kernel in a completely different way, without having to solve any differential equation, and even without knowledge of $\hat{H}$! We can compute it through a representation in terms of path integrals which involves the action functional of the corresponding classical system.

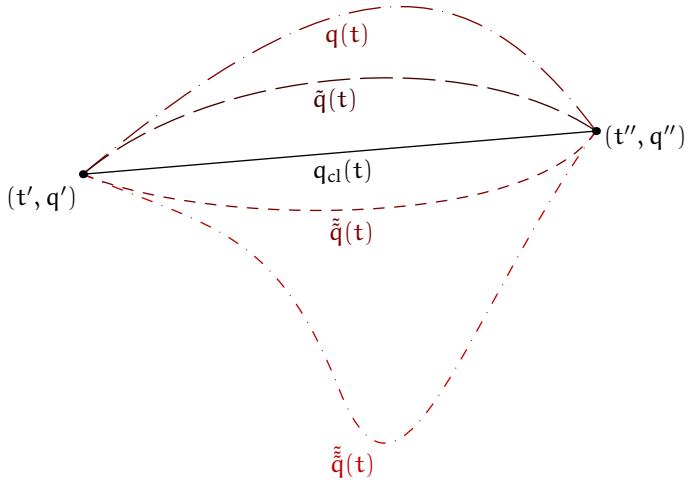


FIGURE 1.1. From a physical point of view, the Feynman's kernel is the answer to the following simple question: if a particle is measured at a position q' at time t' what is the probability *amplitude* that it will be found at some other position q'' at a later time t''? It is the position representation of the Schrödinger time-evolution operator. We shall see that according to Feynman's formulation of quantu mechanics the kernel can be computed by summing contributions coming from all possible paths in the confguration space joining (t', q') with (t'', q'').

First step: Discretization

The first step to construct the path integral representation of the kernel is to discretize the the time interval between t' and t'' into a "large" number, say N, of steps, which for simplicity are chosen to be equidistant, i. e., each of them have the same length Δt given by

$$\Delta t = \frac{t'' - t'}{N},$$

so that we can define the intermediate discrete times (i. e., an equidistant time lattice)

$$t_k = t' + k\Delta t, \quad k = 1, \dots, N-1.$$

It is useful to introduce also the notations $t_0 = t'$ and $t_N = t''$.

From finite-time kernel to small-time kernel

Using the foundamental composition law eq. (1.6), we can write

$$\hat{U}(t'', t') = \hat{U}(t'', t_{N-1})\hat{U}(t_{N-1}, t_{N-2}) \cdots \hat{U}(t_2, t_1)\hat{U}(t_1, t'),$$

or

$$\hat{U}(t'', t') = \prod_{k=1}^{N} \hat{U}(t_k, t_{k-1}).$$

Thus,

$$\langle q''|\hat{U}(t'', t')|q'\rangle = \left\langle q'' \middle| \prod_{k=1}^{N} \hat{U}(t_k, t_{k-1}) \middle| q' \right\rangle.$$

Inserting $N-1$ resolutions of the identity, which formally read

$$\int_{\mathbb{R}} |q\rangle \langle q| \, dq = \mathbb{1},$$

between each of the n terms in the product above, we get

$$\boxed{\langle q''|\hat{U}(t'', t')|q'\rangle = \int \prod_{k=1}^{N-1} dq_k \prod_{k=1}^{N} \langle q_k|\hat{U}(t_k, t_{k-1})|q_{k-1}\rangle} \tag{1.10}$$

* According to the previous remark, this "basis" has to be understood in a "generalized" sense. The rigorous approach would be that of introducing the spectral family of projectors, etc

with the additional notations $|q_0\rangle = |q'\rangle$ and $|q_N\rangle = |q''\rangle$. Notice that the initial and finale positions $q_0 = q'$ and $q_N = q''$ are *not* integrated over.

Equation (1.10) holds in general. Now, we consider the special case in which the Hamiltonian operator takes the standard form

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{q}), \tag{1.11}$$

where m is the mass of the particle and $\hat{q}$ and $\hat{p}$ are the quantum mechanical position and momentum operators as before. $\hat{V}(\hat{q})$ is some time-*in*dependent local* interaction potential. More general classes of Hamiltonian operators will be not consider in this note. In the case of eq. (1.11), the propagator of the k^{th} sub-interval (the short-time kernel), using eq. (1.4), is

$$\begin{aligned}\left\langle q_k \left| \hat{U}(t_k, t_{k-1}) \right| q_{k-1} \right\rangle &= \left\langle q_k \left| e^{\frac{i}{\hbar}\hat{H}(t_k - t_{k-1})} \right| q_{k-1} \right\rangle \\ &= \left\langle q_k \left| e^{\frac{i}{\hbar}\Delta t\left(\frac{\hat{p}^2}{2m} + \hat{V}(\hat{q})\right)} \right| q_{k-1} \right\rangle.\end{aligned}$$

At this point, it would be nice if it were possible to write something like

$$e^{-\frac{i}{\hbar}\Delta t\left(\frac{\hat{p}^2}{2m} + \hat{V}(\hat{q})\right)} = e^{-\frac{i}{\hbar}\Delta t\frac{\hat{p}^2}{2m}} e^{-\frac{i}{\hbar}\Delta t\hat{V}(\hat{q})}.$$

Unfortunately, in general this is not the case. Remember that in general

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}}$$

does *not* hold when $\hat{A}$ and $\hat{B}$ does *not* commute. An explicit expression for $\exp\left(\left(\hat{A} + \hat{B}\right)\right)$ is known in the form of Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{\hat{C}},$$

where the first terms of $\hat{C}$ are given by

1.4 FREE PARTICLE

The Lagrangian of the free particle is

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2$$

and the action functional is

$$S[q(\tau)] = \int_{t'}^{t''} \frac{1}{2} m \dot{q}^2(\tau)\, d\tau$$

with *non*-homogeneous Dirichlet boundary conditions on the continuous paths $q(t)$ fixed:

$$\begin{cases} q(t') = q' \\ q(t'') = q'' \end{cases}$$

The path integral representation of the kernel is given by

$$K(x'', t''|x', t') = \int_{x(t')=x'}^{x(t'')=x''} e^{\frac{i}{\hbar}\int_{t'}^{t''} \frac{1}{2} m \dot{x}^2\, \mathcal{D}t}\, \mathcal{D}x(t) = \lim \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}} \int_{\mathbb{R}^{N-1}} \prod_{k=1}^{N-1} x_k e^{\frac{i}{\hbar}\sum_{k=1}^{N} \frac{1}{2} m \left(\frac{x_k - x_{k-1}}{\varepsilon}\right)^2 \varepsilon} \tag{1.12}$$

The multiple integral is

* Local means that $\langle q_1|\hat{V}(\hat{q})|q_2\rangle$ has support restricted to $q_1 = q_2$. For example, spin dependent potentials are not of this type.

1.5 PATH INTEGRAL TREATMENT OF HARMONIC POTENTIAL

*1.6 PATH INTEGRAL TREATMENT OF COULOMB POTENTIAL

*1.7 PATH INTEGRAL TREATMENT OF FINITE SQUARE WELL POTENTIAL

1.8 WKB FROM A PATH INTEGRAL PERSPECTIVE

1.9 WHAT'S THE USE OF ALL THIS?

2 FUNCTIONAL METHODS FOR SCALAR QUANTUM FIELDS

The general strategy behind functional methods in quantum field theory is similar to that a statistical mechanics:

- build up the key object of the theory, which is the *generating* functional

$$Z[J] = \int e^{-\frac{i}{\hbar} S[\varphi]} \mathcal{D}\varphi \tag{2.1}$$

  in which the dynamics of the fields is *encoded*;

- extract (i. e., *decode*) the physical quantities from the generating functional.

$Z(J)$ plays the role of the partition function in statistical mechanics. In fact, physicists are used to employ the same symbol Z to denote both of them. Primary interest is on physical quantities like a) scattering cross-sections, b) decay rates of unstable particles, c) magnetic moments, etc. However, the theoretical calculations of these quantities usually proceed through the following two steps:

- first, more abstract quantities, i. e., the n-point correlation function, are computed from the generating functional,

- then the S matrix which determined scattering cross-sections and decay rates is related to correlation functions.

2.1 PATH INTEGRAL QUANTIZATION OF SCALAR FIELDS

The Lagrangian density L_{free} of the non-interacting real neutral massive scalar field $\varphi(x)$ in the $(d+1)$-dimensional Minkowski space is[*]

$$L_{\text{free}}\left(\varphi(x), \partial_\mu \varphi(x)\right) = \frac{1}{2}\left(\partial_\mu \varphi(x)\right)\left(\partial^\mu \varphi(x)\right) - \frac{1}{2} m^2 \varphi^2(x)\,. \tag{2.2}$$

It is a *function* of the field and its first-order space-time derivatives at one space-time point x. The parameter m will later be identified with the bare mass of the bosons.

The corresponding action functional $S[\varphi(x)]$ is defined as usual as the space-time integral of the Lagrangian density, i. e.,

$$S[\varphi(x)] = \int L\, d^4x = \int \left(\frac{1}{2}\left(\partial_\mu \varphi(x)\right)\left(\partial^\mu \varphi(x)\right) - \frac{1}{2} m^2 \varphi^2(x)\right) d^4x\,. \tag{2.3}$$

S is a real-valued *functional* of the field configuration φ.

2.1.1 *Path integral quantization of interacting scalar field*

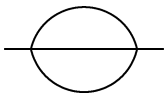
2.1.2 *Review of canonical quantization of scalar fields*

In order to make some comparisons between path integral and operator approaches to the quantum theory of scalar fields, we briefly sketch a summary of canonical quantization of scalar fields.

[*] Our conventions are summarized in **??**.

2.2 PERTURBATION EXPANSION OF GREEN FUNCTIONS OF SELF-INTERACTING SCALAR FIELDS

2.3 FEYNMAN DIAGRAMS



(2.4)

2.4 TROUBLES WITH PERTURBATION THEORY

2.5 SYMMETRY FACTORS OF FEYNMAN DIAGRAMS FOR SCALAR FIELDS

Part II

SYSTEMATICS OF RENORMALIZATION

Renormalization theory is developed systematically for scalar fields

Part III

QUANTUM ELECTRODYNAMICS

Illustration of the methods of quantum field theory to QED

3 SPINORS

SPINORS are fashinating objects. They were first invented by mathematician Cartan to classify Lie algebras. In mathematics they are related to

- classification of Lie algebras;
- quaternions;
- representations of groups, in particular: the rotation group, and the Lorentz group;
- Clifford algebras;

In physics, spinor fields describe fermions. The link between physics and mathematics is related

This chapter provide a review and summary of the Lorentz group, its algebra and its representations, with focus on the spinor representations. This section is mostly based on **Dreiner.Haber.ea:2010**.

3.1 THE LORENTZ GROUP

This section is mostly based on **Barone:2004**.

3.1.1 *Notations and conventions*

The metric tensor of the flat Minkowsky space (i. e., special relativity) is taken to be with signature $(+,-,-,-)$, i. e.,

$$g^{\mu\nu} = g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)\,. \tag{3.1}$$

The homogeneus Lorentz transformations are *linear*, *homogeneus* transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{3.2}$$

that leave invariant the interval $x^{\nu}x_{\nu}$.

Part IV

NON-ABELIAN GAUGE THEORIES AND THE STANDARD MODEL

Quantum chromodynamics and electroweak theory

4

FUNCTIONAL QUANTIZATION OF YANG-MILLS GAUGE FIELD THEORIES

5 QUANTUM CHROMODYNAMICS

6 INTRODUCTION TO LATTICE QCD

7

ULTRACOLD ATOM SIMULATIONS OF YANG-MILLS LATTICE GAUGE THEORIES

8

ANOMALIES

9

ELECTROWEAK SPONTANEOUSLY SYMMETRY BREAKING AND HIGGS MECHANISM

Part V

INTRODUCTION TO SUSY

Part VI

APPENDICES

A GAUSSIAN INTEGRALS

GAUSSIAN integrals are of major practical importance in several fields (mathematics, probability and statistics, quantum mechanics and field theory, etc.). The starting point is $\int_{\mathbb{R}} e^{x^2}\, dx = \sqrt{\pi}$. This appendix is concerned with generalizations of this formula. First, univariate and multivariate real and complex Gaussian integration is refreshed, including the calculation of moment integrals (it will be shown how the latter ones can be efficiently computed in a systematic way by invoking Wick's theorem; in connection with this, the technique of generating functions of moments and comulants is discussed). The aim of this part is three-fold: (a) recollect well-known techniques belonging to calculus courses for evaluating in closed-form finite-dimensional Gaussian integrals, (b) with the aim of eventually apply generalizations of these methods to the infinite-dimensional case and (c) introducing notations and terminology borrowed from quantum field theory. From the field-theoretical point of view, finite-dimensional Gaussian integrals can be understood as a *zero-dimensional* counterpart of a *free bosonic* field theory. Moment integrals are *correlation functions* (i. e., Green's functions in quamtum field theory). Corresponding to *interacting* (bosonic) field theories are non-Gaussian integrals to be treated perturbatively; each contribution in the perturbation expansion takes the form of a moment integral which can be computed by using Wick's theorem; it is also possible to introduce some diagrammatics (the analogous of Feynman's diagrams in quantum fields) to manage the terms of the perturbation expansion. Finally, the mathematical theory of infinite-dimensional Gaussian integration is discussed and applied to calculate the path integrals of free particle and harmonic oscillator in quantum mechanics. Non-perturbative treatment of certain non-Gaussian integrals via saddle-point method is discussed in **??** and corresponds to *semiclassical* approximation. Gaussian integration over Grassmannian variables (useful in the path integral quantization of *fermionic* fields) is discussed in appendix C. This appendix is mostly inspired by **Zeidler:2009b**, in particular chapters from 7.8 to 7.9, and **Zinn-Justin:2005**, expecially chapter 1.

A.1 BASIC RESULTS

key identity

THE starting point is

$$I = \int_{\mathbb{R}} e^{-\varphi^2}\, d\varphi = \sqrt{\pi}\,. \tag{A.1}$$

A theorem by Liouville states that (roughly speaking) the anti-derivatives of the Gaussian function $e^{-\varphi^2}$ cannot be written in terms of elementary functions;[*] thus the foundamental theorem of calculus does not provide help in evaluating the integral in eq. (A.1). So, where does eq. (A.1) come from? Let us briefly refresh some arguments which leads to eq. (A.1), skipping technical mathematical details.

Proof by squaring

One of the standard ways to prove eq. (A.1) is the following remarkable trick of squaring, attributed to Poisson.[†] Consider the square the original integral

$$I^2 = \left(\int_{\mathbb{R}} e^{-\varphi^2}\, d\varphi\right)\left(\int_{\mathbb{R}} e^{-\psi^2}\, d\psi\right),$$

[*] See, e. g., **Boros.Moll:2004** and references therein for a more precise statement and for a proof of this theorem.

[†] It may be just surprising to learn that this ingenious trick is so fruitful "only" in this case and there is essentially no wider applicability of it as an integration method. The proof of this fact can be found in R. Dowson, *On a "Singular" Integration Technique of Poisson*, Am. Math. Monthly, 112 (2005), pp. 270–272, and *Poisson's remarkable calculation — a method or a trick?*, Elem. Math. 65 (2010), pp. 1–8.

where, in order to avoid confusion, a different dummy variable ψ has been introduced in place of φ in the second integral. Fubini's theorem applies, allowing to convert the original product of two one-dimensional integrals along the real φ- and ψ-lines into only one double integral over the two-dimensional (φ, ψ)-plane:

$$I^2 = \int_{\mathbb{R}^2} e^{-(\varphi^2+\psi^2)}\, d\varphi\, d\psi\,.$$

This integral can be computed by going over to standard plane polar coordinates

$$\begin{cases} \phi = \rho\cos(\vartheta) \\ \psi = \rho\sin(\vartheta) \end{cases}, \quad \vartheta \in]0, 2\pi[\text{ and } \rho \in]0, +\infty[\,.$$

We get

$$I^2 = \int_0^{2\pi} d\vartheta \int_0^{+\infty} d\rho\, \rho e^{-\rho^2}\,,$$

where the factor ρ comes from the Jacobian of the polar transformation. The last integral becomes the product of two one-dimensional integrals, which can be integrated immediately. We obtain in this way

$$I^2 = \left(\int_0^{2\pi} d\vartheta\right)\left(\int_0^{+\infty} \rho e^{-\rho^2}\, d\rho\right) = 2\pi \lim_{b\to+\infty}\left[-\frac{e^{-\rho^2}}{2}\right]_{\rho=0}^{\rho=b} = \pi\,,$$

hence $I = \sqrt{\pi}$.* Rigorously speaking, one should take care of the manipulatations of double integrals over the *unbounded* domain $\mathbb{R}^2$, eventually however Gaussian integrals are absolutely convergent and no problem occurs in this case.†

Proof by differentiating under the integral sign

Here is another (even if a bit technical) way to prove eq. (A.1), which involves only one-dimensional integrals along the real line. It is useful in a calculus course where double integrals have not been covered yet. But it is also interesting by its own for the following reason: the underlying strategy is the trick of differentiating under the integral sign, a general strategy we shall use often (see appendix A.2.4). The argument goes as follows, skipping again technical details. The hint was suggested, e. g., in the book *Mathematical Analysis* by Tom M. Apostol (problem 9-17, p. 246). Let $A:]0, +\infty[\to \mathbb{R}$ be the real-valued function over the positive reals defined by

$$a(t) = \left(\int_0^t e^{-\varphi^2}\, d\varphi\right)^2 + \int_0^1 \frac{e^{-t^2(1+\varphi^2)}}{1+\varphi^2}\, d\varphi\,,$$

for every $t \in]0, +\infty[$. A straight calculation shows that

$$\begin{aligned} \frac{dA(t)}{dt} &= 2e^{-t^2}\int_0^t e^{-\varphi^2}\, d\varphi - 2t\int_0^1 e^{-t^2(1+\varphi^2)}\, d\varphi \\ &= 2te^{-t^2}\int_0^1 e^{-t^2\xi^2}\, d\xi - 2te^{-t^2}\int_0^1 e^{-t^2\varphi^2}\, d\varphi = 0, \quad \text{where } \xi = \varphi/t\,. \end{aligned}$$

(Leibniz rule has been applied to interchange derivative and integration symbols.) Therefore, $A(t)$ is constant over $]0, +\infty[$ and its value is found by taking the limit $t \downarrow 0$:

$$\lim_{t\downarrow 0} A(t) = \int_0^1 \frac{1}{1+\varphi^2}\, d\varphi = \arctan(\varphi)\Big|_{\varphi=0}^{\varphi=1} = \frac{\pi}{4}\,.$$

* The Gaussian function is always strictly positive; by the monotony of integrals, $I > 0$ and so the solution $I = -\sqrt{\pi}$ is not acceptable.

† A careful derivation if all integrals are understood as Henstock-Kurzweil integrals can be found, e. g., in: A. Fonda, *Lezioni sulla teoria dell'integrale*, Edizioni Goliardiche, Trieste (2003), chapter 2.

(Of course, you should be able to explain why it is possible to move the limit inside the integration.) Finally, by taking the limit $t \to +\infty$, the integral over $[0, 1]$ vanishes and we get

$$\left(\int_0^{+\infty} e^{-\varphi^2}\, d\varphi\right)^2 = \frac{\pi}{4},$$

and eq. (A.1) is proven.*

A.2 UNIVARIATE GAUSSIAN INTEGRALS

In this section, the result of the previous section is used to calculate the value of one-dimensional Gaussian integrals with general quadratic exponent, i. e., of the form

$$I(a, J) = \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 + J\varphi}\, d\varphi\,, \tag{A.2}$$

where $a, J \in \mathbb{C}$. The factor ½and the signs are conventional. With this conventions, the integral is (absolutely) convergent *if and only* $\Re a > 0$. The integral is convergent (not absolutely) also in the case $\Re a = 0$ (i. e., Fresnel integrals), but the calculation in this case needs special treatment and it will be discussed separately (appendix A.2.3).

Several strategies are possible to evaluate $I(a, J)$; some are suitable for certain cases, some others can be applied to all circumstances. In the following, we shall exploit the following strategies:

- quadratic supplement trick;
- contour integration in the complex plane via Cauchy's residue theory;
- Schwinger-Dyson like differential equation.

The quadratic supplement trick is suitable for the *real* case (appendix A.2.1). Naively applied to the general complex case, the quadratic supplement trick remarkably yields the correct result, even if rigorously speaking the method is no longer valid (but of course it can still serve for mnemonic purposes). Switching to contour integration in the complex domain (where one can use the powerful tool of Cauchy's residue theory) is useful to deal with some particular cases (e. g., Fresnel integral and the Fourier transform of the Gaussian function, see appendices A.2.2 and A.2.3). The general case can be established by writing and solving a differential equation for $I(a, J)$ (appendix A.2.4).

A.2.1 *Real case and quadratic supplement trick*

One-dimensional real Gaussian integral with no source term

In this section, we calculate $I(a, J)$ when $a, J \in \mathbb{R}$, $a > 0$. First, consider $J = 0$. By a change of variables,

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2}\, d\varphi = \sqrt{\frac{2}{a}} \int_{\mathbb{R}} e^{-\xi^2}\, d\xi, \quad \text{where } \xi = \sqrt{\frac{a}{2}}\varphi\,,$$

and applying eq. (A.1) to the integral on the right-hand side we get

$$I(a, 0) = \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2}\, d\varphi = \sqrt{\frac{2\pi}{a}}, \quad a \in \mathbb{R},\, a > 0\,. \tag{A.3}$$

Quadratic supplement trick

Now, consider $J \neq 0$ (source term). The trick is to convert the integral to an integral

* There are other proofs of eq. (A.1) based, for instance, on contour integration in the complex plane, different changes of variables, etc. See, e. g., **Boros.Moll:2004**.

of the form eq. (A.3) by completing the square in the exponent, i. e.,

$$-\frac{1}{2}a\varphi^2 + J\varphi = -\frac{1}{2}a\left(\varphi^2 - 2\frac{J}{a}\varphi\right) = -\frac{1}{2}a\left[\left(\varphi - \frac{J}{a}\right)^2 - \left(\frac{J}{a}\right)^2\right]$$
$$= -\frac{1}{2}a\left(\varphi - \frac{J}{a}\right)^2 + \frac{J^2}{2a}\,.$$

By shifting the integration variable and using the translation invariance of the integration measure and of the endpoints of the integration range, we finally get

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 + J\varphi}\,d\varphi = e^{\frac{J^2}{2a}}\int_{\mathbb{R}} e^{-\frac{1}{2}a\left(\varphi - \frac{J}{a}\right)^2}\,d\varphi = e^{\frac{J^2}{2a}}\sqrt{\frac{2\pi}{a}}\,.$$

One-dimensional real Gaussian integral with source term

Therefore,

$$I(a, J) = \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 + J\varphi}\,d\varphi = \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}}\,, \tag{A.4}$$

for every $a, J \in \mathbb{R}$, $a > 0$. As we shall see in appendix A.2.4, eq. (A.4) holds also if $a, J \in \mathbb{C}$, $\Re a > 0$, in that case the square root of a complex number should be understood in the sense of principal value.

Gaussian probability distribution

Equation (A.4) in the real case is important in the discussion of the Gaussian probability distribution, which is obiquitous in probability and statistics. The Gaussian probability density function $p : \mathbb{R} \to \mathbb{R}$ of a real variable x is defined as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,,$$

where μ and σ are real parameters ($\sigma > 0$). Using eq. (A.4), it is easy to prove that such $p(x)$ is normalized:

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1\,.$$

We shall see later that the mean value of this distribution is μ and the variance is σ^2.

A.2.2 *Fourier transform using Cauchy's residue theory*

Connection with the Fourier transform

Consider $I(a, J)$ with complex $a, J \in \mathbb{C}$, $\Re a > 0$. Let us first consider the case

- $a \in \mathbb{R}$, $a > 0$;
- $\Re J = 0$.

We may write $J = -i\omega$ for some $\omega \in \mathbb{R}$ (the minus sign is conventional). So, we have to compute

$$\tilde{I}(a, \omega) = \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 - i\omega\varphi}\,d\varphi, \quad a, \omega \in \mathbb{R}, a > 0\,. \tag{A.5}$$

Of course, the case $\omega = 0$ fits the results of appendix A.2.1, so we are interested in $\omega \neq 0$. Notice that $\tilde{I}(a, \omega)$ is nothing but the Fourier transform of the (real) Gaussian function $e^{-\frac{1}{2}a\varphi^2}$.

Naively speaking, the quadratic supplement trick is not rigorous in the complex case

At first sight, one could think to use the quadratic supplement trick as in the real case. Completing the square

$$-\frac{1}{2}a\varphi^2 - i\omega\varphi = -\frac{1}{2}a\left[\left(\varphi + \frac{i\omega}{a}\right)^2 - \left(\frac{i\omega}{a}\right)^2\right] = -\frac{1}{2}a\left(\varphi + \frac{i\omega}{a}\right)^2 - \frac{\omega^2}{2a}\,,$$

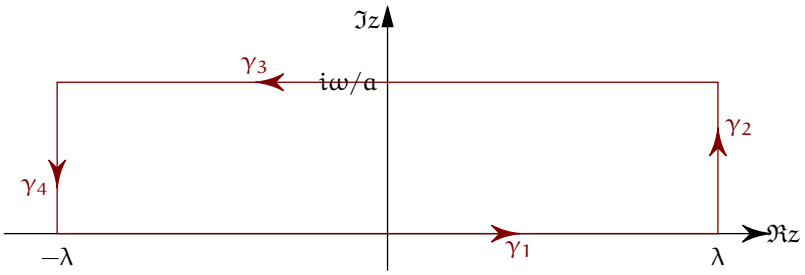


FIGURE A.1. Simple closed positively-oriented contour γ in the complex z-plane obtained by joining γ_1, γ_2, γ_3 and γ_4. λ is a positive real number which eventually will be sent to infinity (we are interested in taking the limit $\lambda \to +\infty$). The parameters ω and a are the real numbers appearing in eq. (A.5). The illustration corresponds to the case $\omega > 0$, but you can analogously consider the picture corresponding to the case $\omega < 0$.

we get

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 - i\omega\varphi}\, d\varphi = e^{-\frac{\omega^2}{2a}} \int_{\mathbb{R}} e^{-\frac{1}{2}a\left(\varphi + \frac{i\omega}{a}\right)^2}\, d\varphi\,.$$

To get the previous expression, all we have used are algebraic manipulations, so there is nothing wrong in doing so. Now, we would like to change variables in the integral

$$\xi = \sqrt{\frac{a}{2}}\left(\varphi + \frac{i\omega}{a}\right), \tag{A.6}$$

so that

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 - i\omega\varphi}\, d\varphi = e^{-\frac{\omega^2}{2a}} \sqrt{\frac{2\pi}{a}}\,. \tag{A.7}$$

Remarkably enough, eq. (A.7) holds true even if, naively speaking, the argument we have used to prove it is *not* correct. Why is the trick not rigorous in this case? The reason is that eq. (A.6), naively speaking, implies converting the original integral with respect to the real variable φ into a new integral with respect to the new *complex* variable ξ (ξ in general carries also an imaginary part). In other words, we are changing the path of integration in the complex plane from the real line to a line which is parallel to the real line.

There are many ways to establish eq. (A.7) on a rigorous ground. In this section, we give an argument using Cauchy's residue theory and contour integration in the complex plane. This approach will help use to understand better why, although naively speaking not correct, the quadratic supplement trick gives the right value; the reason is that the other contributions coming for changing the integration path in the complex plane are vanishing. In appendix A.2.4, a more general strategy based on the use of a differential equation will allow us to fully rederive the results of this section and also to establish the value of $I(a, J)$ in the more general case $a, J \in \mathbb{C}$, $\Re a > 0$. Since the results of this section will be rederived as a particular case in appendix A.2.4, if you prefer you can skip the rest of this section.

Proof by Cauchy's residue theory

Consider the contour integral of the complex Gaussian function

$$\oint_\gamma e^{-\frac{1}{2}az^2}\, dz, \quad a \in \mathbb{R}, a > 0\,,$$

where γ is the simple closed positively-oriented rectangular contour drawn in fig. A.1. Since $e^{-\frac{1}{2}az^2}$ is always an entire function of the complex variable z (i. e., it is analytic

in the whole complex z-plane) then Cauchy-Goursat theorem applies,*

$$\oint_\gamma e^{-\frac{1}{2}az^2}\,dz = 0\,, \tag{A.8}$$

in particular this is true for all λ. Of course, this integral can be computed in another way, namely

$$\oint_\gamma e^{-\frac{1}{2}az^2}\,dz = \sum_{i=1}^{4}\int_{\gamma_i} e^{-\frac{1}{2}az^2}\,dz \tag{A.9}$$

where the paths $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are that drawn in Figure A.1. These integrals can be computed once a parametric representation of γ_i's is given. A parametric reprentation of γ_1 is the function $z_1\colon [-\lambda,\lambda] \to \mathbb{C}$ defined by $z_1(\varphi) = \varphi$ for every $\varphi \in [-\lambda,\lambda]$; we have that $\frac{dz_1(\varphi)}{d\varphi} = 1$ and

$$\int_{\gamma_1} e^{-\frac{1}{2}az^2}\,dz = \int_{-\lambda}^{\lambda} e^{-\frac{1}{2}a\varphi^2}\,d\varphi\,.$$

As $\lambda \to +\infty$, $\int_{\gamma_1} e^{-\frac{1}{2}az^2}\,dz \to \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2}\,d\varphi = \sqrt{\frac{2\pi}{a}}$. A parametric reprentation of γ_2 is the function $z_2\colon [0,1] \to \mathbb{C}$ defined by $z_2(\varphi) = \lambda + \frac{i\varphi\omega}{a}$ for every $\varphi \in [0,1]$; we have that $\frac{dz_2(\varphi)}{d\varphi} = \frac{i\omega}{a}$ and

$$\int_{\gamma_2} e^{-\frac{1}{2}az^2}\,dz = \frac{i\omega}{a}\int_0^1 e^{-\frac{1}{2}a\left(\lambda+\frac{i\varphi\omega}{a}\right)^2}\,d\varphi\,.$$

Since the modulus of this last integral is bounded by

$$\left|\int_0^1 e^{-\frac{1}{2}a\left(\lambda+\frac{i\varphi\omega}{a}\right)^2}\,d\varphi\right| \leqslant e^{-\frac{1}{2}a\lambda^2}\int_0^1 e^{\varphi^2\frac{\omega^2}{2a}}\,d\varphi\,,$$

which goes to zero as $\lambda \to +\infty$, the integral along γ_2 gives a vanishing contribution to the integral over γ. In the same way, it is not difficult to confirm that also the integral along γ_4 does not contribute to $\oint_\gamma e^{-\frac{1}{2}az^2}\,dz$ in the limit $\lambda \to +\infty$. Finally, a parametric reprentation of γ_3 is the function $z_3\colon [-\lambda,\lambda] \to \mathbb{C}$ defined by $z_3(\varphi) = -\varphi + \frac{i\omega}{a}$ for every $\varphi \in [-\lambda,\lambda]$ (the minus sing accounts of the direction of the path); we have that $\frac{dz_3(\varphi)}{d\varphi} = -1$ and

$$\int_{\gamma_3} e^{-\frac{1}{2}az^2}\,dz = -\int_{-\lambda}^{\lambda} e^{-\frac{1}{2}a\left(-\varphi+\frac{i\omega}{a}\right)^2}\,d\varphi = -e^{\frac{\omega^2}{2a}}\int_{-\lambda}^{\lambda} e^{-\frac{1}{2}a\varphi^2-i\omega\varphi}\,d\varphi\,.$$

As $\lambda \to +\infty$, $\int_{\gamma_3} e^{-\frac{1}{2}az^2}\,dz \to -e^{\frac{\omega^2}{2a}}\tilde{I}(a,\omega)$. Equations (A.8) and (A.9) then implies

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2-i\omega\varphi}\,d\varphi = e^{-\frac{\omega^2}{2a}}\sqrt{\frac{2\pi}{a}}\,.$$

A.2.3 *Fresnel integral*

In this section we shall prove that

$$\int_{\mathbb{R}} e^{\pm i\varphi^2}\,d\varphi = e^{\pm i\frac{\pi}{4}}\sqrt{\pi}\,, \tag{A.10}$$

* Cauchy-Goursat theorem can be stated as follows: For any single-valued analytic function $f(z)$ in a simply-connected domain Ω in the complex plane the complex contour integral of f along any simple closed positively-oriented contour Γ in Ω vanishes, i. e., $\oint_\gamma f(z)\,dz = 0$. Stated otherwise: If $f(z)$ is analytic everywhere interior to and on a simple closed positively-oriented contour Γ in the complex plane, then $\oint_\gamma f(z)\,dz = 0$.

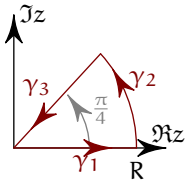


FIGURE A.2. Contour used to calculate Fresnel integral

This is called "Fresnel integral". (Sometimes, the name "Fresnel" integral refers to the real and imaginary parts of the integral above.) It arises in many situations (e. g., the description of near field Fresnel diffraction in optics); we have already met it in chapter 1.

At first sight one might wonder why the integral in eq. (A.10) is convergent. After all, by definition

$$\begin{aligned}\int_{\mathbb{R}} e^{\pm i\varphi^2}\, d\varphi &= \int_{\mathbb{R}} \Re e^{\pm i\varphi^2}\, d\varphi + i\int_{\mathbb{R}} \Im e^{\pm i\varphi^2}\, d\varphi \\ &= \int_{\mathbb{R}} \cos\left(\varphi^2\right)\, d\varphi \pm i\int_{\mathbb{R}} \sin\left(\varphi^2\right)\, d\varphi\,,\end{aligned}$$

and the sine and cosine functions do *not* even decay to zero at infinity! In fact, the Fresnel integral is *not* absolutely convergent, bacause the modulus of the integral function is always equal to one. However, Fresnel integral do converge. Qualitatively speaking, this is due to cancellation of the rapid oscillations of $\cos(\varphi)^2$ and $\sin(\varphi)^2$ as $\varphi \to \infty$. Notice that this intuitive idea, togerther with eq. (A.10), are the underlying ingredients of the stationary phase method for asymptotic evaluation of certain rapidly oscillating parametric integrals—a method which can also be generalized even to path integrals (WKB emerge in the path integral formulation of non-relativistic quantum mechanics in this way). The convergence can be proven by integrating by parts. This strategy however does not help us in evaluating the integral. One method to establish eq. (A.10) is to use complex integration and Cauchy's residue theory.

Proof of convergence

To prove the convergence of the Fresnel integral,

Once we have proven that the integral is convergent, in order to calculate it we can use the tools of complex analysis.

Proof via Laplace transform

Other proof involves making use of the Laplace transform and Feynman's trick of differentiating under the integral sign.

A.2.4 *Complex case via differential equations and analytic continuation*

> "One thing I never did learn was contour integration. I had learned to do integrals by various methods shown in a book that my high school physics teacher Mr. Bader had given me. One day he told me to stay after class. «Feynman,» he said, «you talk too much and you make too much noise. I know why. You're bored. So I'm going to give you a book. You go up there in the back, in the corner, and study this book, and when you know everything that's in this book, you can talk again.» So every physics class, I paid no attention to what was going on with Pascal's Law, or whatever they were doing. […] I was up in the back with this book: Advanced Calculus, by Woods. Bader knew I had studied Calculus for the Practical Man a little bit, so he gave me the real works—it was for a junior or senior course in college. It had Fourier series, Bessel functions, determinants, elliptic functions—all kinds of wonderful stuff that I didn't know anything about. That book also showed how to differentiate parameters under the integral sign—it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So

> because I was self-taught using that book, I had peculiar methods of doing integrals. [...] The result was, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. [...] So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me. ❞
>
> R. F. Feynman
> *Surely You're Joking, Mr. Feynman!*
> pp. 71-72, Bantam Books (1986).

In this section, we employ the trick of differentiating under the integral sign (the method Feynman is referring to in the passage above) to evaluate exactly $I(a, J)$ in the general complex case when $a, J \in \mathbb{C}$, $\Re a > 0$. This trick is very powerful also elsewhere. Furthermore, the same trick will allow us to compute integrals of the moments in a very efficient manner (expecially in the multi-dimensional case) and will lead us to establish Wick's theorem. Furthermore, its functional generalization forms the basis on which the Feynman path integral is used in practise in order to compute correlation functions in quantum field theory.

Complex Gaussian integral with source term

THEOREM A.1: Let $a, J \in \mathbb{C}$ and $\Re a > 0$. Then,

$$\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 + J\varphi}\, d\varphi = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}, \tag{A.11}$$

where the square root should be understood as principal value.

Principal value of the square root of a complex number

Remark. Let $z \in \mathbb{C}\backslash\{0\}$ by any non-zero complex number of the form

$$z = |z|\, e^{i\vartheta}, \quad -\pi < \vartheta < \pi.$$

(Notice that we exclude the negative real axis.) The principal value of the complex square-root of z is

$$\sqrt{z} = \sqrt{|z|} e^{i\frac{\vartheta}{2}}.$$

In this way, $\sqrt{z}$ defines a function which is holomorphic on the whole complex z-plane but the negative real axis, and which provides the analytic continuation of $\sqrt{x}$ for positive real values x.

First step: analytic continuation when no source is present

Proof. Let us consider $I(a, 0)$, with $a \in \mathbb{C}$, $\Re a > 0$. We already know that

$$I(a, 0) = \sqrt{\frac{2\pi}{a}}, \quad a \in \mathbb{R}, a > 0.$$

For complex a, $I(a, 0)$ is holomorphic in the half complex a-plane $\Re a > 0$. Consider the function $\sqrt{2\pi/a}$, where the square root is understood in the senso of principal value. Then, this function is also analytic in the half plane $\Re a > 0$ (it is analytic everewhere but the negative real axis) and its value coincide with the value of $I(a, 0)$ along the positive real axis. By analytic continuation,

$$I(a, 0) = \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2}\, d\varphi = \sqrt{\frac{2\pi}{a}}, \quad a \in \mathbb{C}, \Re a > 0. \tag{A.12}$$

Second step: The Schwinger-Dyson differential equation

To calculate $I(a, J)$, $J \in \mathbb{C}$, take the derivative with respect to J:

$$\frac{\partial I(a, J)}{\partial J} = \frac{\partial}{\partial J} \int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2 + J\varphi}\, d\varphi = \int_{\mathbb{R}} \varphi e^{-\frac{1}{2}a\varphi^2 + J\varphi}\, d\varphi,$$

using Leibniz rule to interchange derivative and integral symbols. Integrating by parts

$$\int_{\mathbb{R}} \varphi e^{-\frac{1}{2}a\varphi^2+J\varphi}\,d\varphi = -\frac{1}{a}\int_{\mathbb{R}} \underbrace{(-a\varphi)\,e^{-\frac{1}{2}a\varphi^2}}_{\frac{\partial}{\partial\varphi}\exp\left(-\frac{1}{2}a\varphi^2\right)} e^{J\varphi}\,d\varphi$$
$$= \frac{J}{a}\int_{\mathbb{R}} e^{-\frac{1}{2}a\varphi^2+J\varphi}\,d\varphi = \frac{J}{a}I(a,J)\,.$$

So, we have to solve the following differential equation:

$$\frac{\partial I(a,J)}{\partial J} - \frac{J}{a}I(a,J) = 0\,, \tag{A.13}$$

which can be somewhat consider the analogous of Schwinger-Dyson differential equations between in quantum field theory. Equation (A.13) is a first-order homogeneus linear partial diffential equation which be solved, for instance, by multiplying both sides by $e^{-\frac{J^2}{2a}}$ (the technique of integrating factors). In this way, we get

$$e^{-\frac{J^2}{2a}}\frac{\partial I(a,J)}{\partial J} - \frac{J}{a}e^{-\frac{J^2}{2a}}I(a,J) = 0\,,$$

i. e.,

$$\frac{\partial}{\partial J}\left\{e^{-\frac{J^2}{2a}}I(a,J)\right\} = 0\,,$$

which means that $e^{-\frac{J^2}{2a}}I(a,J)$ is a function of a only (i. e., it does not depend on J). Since we know that for $J=0$, this function is $I(a,0)=\sqrt{2\pi/a}$, we must have

$$I(a,J) = \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}}, \quad a, J \in \mathbb{C}, \Re a > 0\,.$$

This completes the proof. ∎

A.2.5 *Moment integrals*

In this section, we focus on calculating integrals of the form

$$\int_{-\infty}^{+\infty} x^m e^{-ax^2+Jx}\,dx\,,$$

where $m \in \mathbb{N}$ and $a, J \in \mathbb{C}$, $\Re a > 0$.

In the real case (i. e., $a, J \in \mathbb{R}$, $a > 0$), integrals of this form arise in connection with the computation of moments of the Gaussian probability distribution. Let us discuss briefly this point, in the context of univariate probability density functions on $\mathbb{R}$.

Moments of a probability distribution

The n^{th}-moment of a probability density function $p\colon \Omega \to \mathbb{R}$ on some (measurable) subset $\Omega \subseteq \mathbb{R}$ is (by definition)

$$\int_{\Omega} x^n p(x)\,dx\,,$$

provided that this integral exists.* In particular, the first moment (if it exists) is called the "mean value" of the probability distribution. It is easily verified that the mean value of the Gaussian distribution $p_{\mu,\sigma^2}\colon \mathbb{R} \to \mathbb{R}$ be the Gaussian probability density function

$$p_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,.$$

* A canonical example of a "pathological" probability distribution which does does not have finite moments is the Cauchy distribution (also known as Breit-Wigner distribution in high-energy physics, or Lorenzian function), whose standard form is $p(x) = \frac{1}{\pi(1+x^2)}$ for every $x \in \mathbb{R}$; its first order moment does not exist and the higher order moments are divergent.

is equal to the parameter μ. In analogy, it is also useful to define the n^{th}-moment of p. with respect to its mean value (the so-called "central moments"):

$$\int_\Omega (x-\mu)^n \, p(x)\, dx\,.$$

The first moment with respect to the mean value is of course equal to zero. The second moment with respect to the mean value is called the variance. It is easily verified that the variance of the Gaussian distribution p_{μ,σ^2} is equal to σ^2.

Expectation values with respect to a probability dstribution

In general,

$$\langle f(x)\rangle = \int_\Omega f(x)p(x)\, dx$$

is the "expectation value" of the function $f\colon \Omega \to \mathbb{R}$ with respect to the probability distribution p (when this integral exists). So, moments of a distribution are expectation values of integer powers.

We shall see two different ways to evaluate those integrals. The first makes extensively use of Euler's integral representation of the Gamma function. This is very powerful to handle the one-dimensional case. However, it does not easily generalize to the case of multivariate Gaussian distributions. For this reason, it is helpful to recover the results using another method, which although being less elegant in my opinion (it involves differentiating multiple times the integral with respect to J, something that in the one-dimensional case may sound quite boring stuff) can be generalized without difficulties to cover the multi-dimensional case. This will lead us to the powerful tool of generating functions. Suitable adapted to quantum fields, it is the content of the famous and useful Wick's theorem, which we shal discuss more lenghtly in the setting of higher-dimensional Gaussian integrals.

The key result of this section is that such integrals are fully determined once ..., a powerful result which is well-known in probability and stochastic processes.

A.3 MULTIVARIATE GAUSSIAN INTEGRALS

Prototype of multi-dimensional (real) Gaussian integration

THE *prototype* of all N-dimensional (real) Gaussian integrals ($N \in \mathbb{N}$) is

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{k=1}^N \lambda_k \varphi_k^2} \prod_{k=1}^N d\varphi_k\,, \tag{A.14}$$

where $\lambda_k \in \mathbb{R}$, $\lambda_k > 0$ for $k = 1, \dots, N$. (The integral is absolutely convergent due to the fact that λ_k's are always strictly positive.) This integral factorizes into the product of N one-dimensional Gaussian integrals (Fubini's theorem) so

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{k=1}^N \lambda_k \varphi_k^2} \prod_{k=1}^N d\varphi_k = \prod_{k=1}^N \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_k \varphi_k^2}\, dx_k = \sqrt{\frac{(2\pi)^N}{\prod_{k=1}^N \lambda_k}}\,. \tag{A.15}$$

More generally, we consider the exponential of a *real* quadratic form in N real variables. Remember that any $N \times N$ real *symmetric* matrix determines a quadratic form in N variables and conservesely given any real quadratic form in N variables its coefficients can always be arranged into an $N \times N$ *symmetric* matrix. When the integral is convergent, one can decouple the variables in the exponent by diagonalizing the matrix and the integral factorizes again into a product of N one-dimensional Gaussian integrals as before. This is the content of the following theorem.

The determinant trick

THEOREM A.2: Let A be any a) symmetric and b) positive-definite (thus invertible) $N \times N$ *real* square matrix ($N \in \mathbb{N}$). Then, the following integral is *absolutely* convergent and its value is

$$\int_{\mathbb{R}^N} e^{\left(-\frac{1}{2}\sum_{k,l=1}^N \varphi_k A_{k,l} \varphi_l\right)} \prod_{k=1}^N d\varphi_k = \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}}. \tag{A.16}$$

Furthermore, for every $J \in \mathbb{R}^N$ the following integral is absolutely convergent and its value is

$$\begin{aligned}\int_{\mathbb{R}^N} e^{\left(-\frac{1}{2}\sum_{k,l=1}^N \varphi_k A_{k,l}\varphi_l + \sum_{k=1}^N J_k\varphi_k\right)} \prod_{k=1}^N d\varphi_k \\ = e^{\left(\frac{1}{2}\sum_{k,l=1}^N J_l \Delta_{k,l} J_k\right)} \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}},\end{aligned} \tag{A.17}$$

where $\Delta = A^{-1}$ denotes the *inverse* of the matrix A.

Matrix notation

First of all, it is convenient to rewrite the previous results using the more compact matrix notation; then eq. (A.16) reads

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle x|Ax\rangle} \prod_{k=1}^N dx_k = \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} \tag{A.18}$$

and eq. (A.17) reads

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle x|Ax\rangle + \langle J|x\rangle} \prod_{k=1}^N dx_k = e^{-\frac{1}{2}\langle J|\Delta J\rangle} \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}}, \tag{A.19}$$

where $\Delta = A^{-1}$ and $\langle\cdot|\cdot\rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^N$, i. e., in particular

$$\langle J|x\rangle = \sum_{k=1}^N J_k x_k$$

and

$$\langle x|Ax\rangle = \sum_{k,l=1}^N A_{k,l} x_k x_l \,.$$

To prove the theorem and to understand the role of the hypothesis, we need some remarks on real, symmetric and positive definite matrices.

Let A be any $N \times N$ square matric with *real* or *complex* entries.

DEFINITION A.1: A is said to be "symmetric" if

$A_{ij} = A_{ji}$

for all $(i, j) \in N \times N$.

The definition of positive-definite matrix requires to distinghuish between real and complex matrices. (Why? See explation later.)

DEFINITION A.2:

A. If A is a *real* matrix, then A is defined to be "positive-definite" if a) A is symmetric and b) the following holds:

$$\langle x|Ay\rangle \geqslant 0 \tag{A.20}$$

for all $(x, y) \in \mathbb{R}^N$.

B. If A has *complex* entries, the A is defined to be "positive-definite" if eq. (A.20).

THEOREM A.3: Let A be positive-definite *complex* matrix. Then, A is Hermitean.

Proof. The key tool here is the polarization identity. ∎

Proof. A is symetric, thus (spectral theorem in $\mathbb{R}^N$) there exists and orthogonal transformation which diagonalizes A. This means that there exists a real orthogonal $N \times N$ square matrix O such that $OAO^T = D$, where

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

is a diagonal matrix; its diagonal entries are nothing but the eigenvalues of A. The matrix O is orthogonal, thus $O^{-1} = O^T$. As a conseguence,

$$\det OO^T = \det OO^{-1} = \det \mathbb{1}_{N\times N} = 1$$

but also

$$\det OO^T = \det O \det O^T = (\det O)^2 ,$$

so $|\det O| = 1$.

Now, we change variables. Let

∎

A.4 WICK THEOREM

We define the "contraction"

$$\overbrace{\varphi_{k_1}\varphi_{k_2}} = \langle \varphi_{k_1}\varphi_{k_2}\rangle = \Delta_{k_1 k_2} \tag{A.21}$$

We introduce the following diagrammatic representation

$$\overbrace{\varphi_{k_1}\varphi_{k_2}} = \overset{k_1 \quad p \quad k_2}{\longrightarrow} \tag{A.22}$$

Exercise. A.4.1: F

or example

$$\overbrace{\hat{\varphi}_{k_1}\hat{\varphi}_{k_2}} = \overset{k_1 \quad p \quad k_2}{\longrightarrow} \tag{A.23}$$

$$\overbrace{\hat{\varphi}_{k_1}\hat{\varphi}_{k_2}} = \overset{k_1 \quad p \quad k_2}{\longrightarrow} \tag{A.24}$$

For example

$$\langle \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\rangle = \overbrace{\varphi_{k_1}\varphi_{k_2}}\,\overbrace{\varphi_{k_3}\varphi_{k_4}} + \overbrace{\varphi_{k_1}\overbrace{\varphi_{k_2}\varphi_{k_3}}\varphi_{k_4}} . \tag{A.25}$$

$$-i\Sigma_{ope} = \left[\longrightarrow + \quad \overset{\frown}{\longrightarrow} \quad + \cdots \right] \tag{A.26}$$

$$\overset{a \quad p \quad b}{\longrightarrow} = \frac{i}{\not{p} - m_0}\delta^{abc}{}_{c} . \tag{A.27}$$

$$\int e^{\frac{i}{\hbar}S[x(t)]}\,\mathcal{D}x(t) = \int e^{\frac{i}{\hbar}S[x(t)]}\,dx \tag{A.28}$$

$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy\,dz \tag{A.29}$$

B

GROUP THEORETICAL METHODS

This appendix provide a review and summary of group theory, in particular Lie groups, their algebras and representations.

C

CRASH COURSE OF GRASSMANN VARIABLES

D

E

SADDLE-POINT METHOD AND WKB

The stepeest descent method (also referred to as saddle point method) is a powerful tool to handle the problem of evaluating the asymptotic behavior (or at least its leading term) of parametric integrals of a certain form. For example, it might be useful in evaluating the asymptotic behavior of functions for which a suitable integral representation is available. An introductory discussion of these topics is given, e. g., by **king ablowitz** See also Zinn-Justin.

Let us begin with a special case of real integrals over one real variable (this is known also as Laplace's method) and then generalize to contour integrals on a complex domain. Generalization to an arbitrary number of variables is not covered here, even if such generalizations are often needed in the toolkit of every physicists.*

E.1 REAL CASE (ALIAS, LAPLACE METHOD)

E.2 COMPLEX CASE

Consider the complex integral along the real line:

$$I(\lambda) = \int_{\Omega} g(t) e^{\lambda f(t)} \, dt \,, \tag{E.1}$$

where $\Omega \subseteq \mathbb{R}$ is an interval of the real line, $f : \Omega \to \mathbb{R}$ a *real*-valued function on Ω, $g : \Omega \to \mathbb{C}$a *complex*-valued function on Ω and λ a *positive* real parameter. We will study the leading term in the asymptotic expansion of integrals of such form as $\lambda \to +\infty$. This is more properly referred to as Laplace method, then we will move to the more general case of contour integrals in the complex plane.

* Stepeest descent method generalizes also to functional integrations and path integrals, this is very important in dealing with non-perturbative corrections in quantum mechanics and quantum field theory.

FIGURE E.1. Saddle point

F

RENORMALIZATION GROUP AND THE PRIME NUMBER GAS

G

PATH INTEGRALS APPROACH TO CLASSICAL MECHANICS

INDEX

Kernel, 6

Time evolution operator, 3