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Non-Relativistic Quantum Mechanics

Quantum Field Theory

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“ In the good old days, theorizing was like sailing between islands of experimental evidence. And, if the trip was not in the vicinity of the shoreline (which was strongly recommended for safety reasons) sailors where continuously looking forward, hoping to see land — the sooner the better.

Nowadays, some theoretical physicists (let us call them sailors) [have] found a way to survive and navigate in the open sea of pure theoretical constructions. Instead of the horizon, they look at stars, which tell them exactly where they are. Sailors are aware of the fact that the stars will never tell them where the new land is, but they may tell them their position on the globe.

Theoreticians become sailors simply bacause they just like it. Young people, seduced by capitans forming crews to go to a Nuevo El Dorando [...] soon realize that they will spend all their life at sea. Those who do not like sailing desert the voyage, but for the true potential sailors the sea become their passion. They will probably tell the alluring and frightening truth to their students — and the proper people will join their ranks. ”

— Andrei Losev

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PREFACE

To be written…
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Part I

THE BASIS

1 SCHWINGER'S APPROACH TO QUANTUM MECHANICS

> "I presume that all of you have already been exposed to some undergraduate course in Quantum Mechanics, one that leans heavily on de Broglie waves and the Schroedinger equation. I have never thought that this simple wave approach was acceptable as a general basis for the whole subject, and I intend to move immediately to replace it in your mind by a foundation that *is* perfectly general."
>
> J. Schwinger, *Quantum Mechanics. Symbolism of Atomic Measurements* Schwinger [2001].

1.1 INTRODUCTION

COMPARED with other traditional areas of physics, quantum mechanics is not easy. It often lacks physical intuitition and it relies on heavy mathematical background from the very beginning. As we shall see shortly, topics like uncertanty principle, the role of probability etc asks immediately for a theoretical framework/formalism... Phase space is not able to capture, it does not offer the tools. Naturally leading from the very beginning to adopt .

REFERENCES FOR CHAPTER 1

SCHWINGER, JULIAN

2001 *Quantum Mechanics. Symbolism of Atomic Measurements*, Springer-Verlag, Berlin, ISBN: 3-540-41408-8. (Cited on p. 3.)

2 LINEAR OPERATORS IN HILBERT SPACES

Operator formulation of standard non-relativistic quantum mechanics heavily relies on the theory of linear operators in Hilber spaces. In particular, the spectral decomposition of self-adjoint operators (bounded and unbounded ones) is a key ingredient in formulating the basic rules of quantum mechanics. This chapter is aimed at providing the necessary mathematical background of functional analysis employed by non-relativistic quantum mechanics. It is a chapter on mathematics, not on quantum physics; for this reason care has been made to mathematical rigous more than what will be customary in later chapters. The Reader interested in how functional analysis applies to the formulation of quantum mechanics should jump to the next chapters. As we shall see, physicists are customary to employ Dirac's notation, a powerful mnemonic notation which naively speaking however lacks mathematical rigour; a dictionary is possible to translate Dirac notation to the rigorous theorems of functional analysis; more on this later on.

This chapter is mostly based on Reed and Simon [1980].

2.1 BANACH AND HILBER SPACES

Unless stated otherwise, let $\mathbb{K}$ denote equivalently the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. (It is possible to develop the theory also for the skew-field of quaternions, but this case will not be discussed here to avoid dealing with the non-commutativity of quaternionic product.)

Norm

DEFINITION 2.1 (NORM): Let X be any vector space over $\mathbb{K}$. A "norm" on X is any application $X \to \mathbb{R}$ denoted by $\|\cdot\|$ satisfying the following properties:

(a) $\|\varphi\| \geqslant 0, \quad \forall \varphi \in X$ (nonnegative);

(b) $\|\varphi\| = 0$ if and only if $\varphi = 0$ (strictly positive);

(c) $\|\lambda\varphi\| = |\lambda|\|\varphi\|, \quad \forall \lambda \in \mathbb{K}$ and $\forall \varphi \in X$ (positive homogeneity);

(d) $\|\varphi + \psi\| \leqslant \|\varphi\| + \|\psi\|, \quad \forall(\varphi, \psi) \in X \times X$ (the so-called "triangle inequality")

Item (a) is not strictly necessary: it follows from items (b) to (d). In fact, from item (c) $\|-\varphi\| = \|\varphi\|$, from item (d) we have thus $\|\varphi + (-\varphi)\| \leqslant \|\varphi\| + \|\varphi\| = 2\|\varphi\|$, and from item (b) $\|\varphi - \varphi\| = 0$.

A "normed vector space" is a vector space equipped with a norm, as formalized by the following definition.

Normed vector space

DEFINITION 2.2 (NORMED VECTOR SPACE): A "normed vector space" over $\mathbb{K}$ is a pair $(X, \|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is any norm on X.

If $(X, \|\cdot\|)$ is a normed vector space, the norm $\|\cdot\|$ *induces* a metric (i. e., a notion of distance) and thus a topology on X. This is proved by the following theorem.

Distance induced by a norm

THEOREM 2.1: Let $(X, \|\cdot\|)$ be any normed vector space over $\mathbb{K}$. Let $d\colon X \times X \to \mathbb{R}$ be the function defined by

$$\boxed{d(\varphi, \psi) = \|\varphi - \psi\|, \quad \forall(\varphi, \psi) \in X \times X\,.} \tag{2.1}$$

Then, (X, d) is a metric space. The metric in eq. (2.1) is called the "metric induced by the norm" on X.

Proof. We need to prove that eq. (2.1) defines a metric over X, i. e., we need to show that d satisfies:

(a) $d(\varphi, \psi) \geqslant 0, \quad \forall(\varphi, \psi) \in X \times X$ (nonnegativity);

(b) $d(\varphi, \psi) = 0$ if and only if $\varphi = \psi$ (i. e., the only point of X at zero distance from ψ is ψ itself);

(c) $d(\varphi, \psi) = d(\psi, \varphi), \quad \forall(\varphi, \psi) \in X \times X$ (the distance is a symmetric function of its arguments);

(d) $d(\varphi, \psi) \leqslant d(\psi, \eta) + d(\eta, \psi), \quad \forall(\varphi, \psi, \eta) \in X \times X \times X$ (the so-called "triangle inequality").

Item (a) follows from property (a) of the norm. Item (b) follows from property (b) of the norm, since $\|\varphi - \psi\| = 0$ if and only if $\varphi - \psi = 0$, i. e., if and only if $\varphi = \psi$. Item (c) follows from property (c) of the norm, since

$$\begin{aligned} d(\varphi, \psi) &= \|\varphi - \psi\| = \|-(\psi - \varphi)\| = |-1|\|\psi - \varphi\| \\ &= \|\psi - \varphi\| = d(\psi, \varphi), \quad \forall(\varphi, \psi) \in X \times X. \end{aligned}$$

Item (d) follows from property item (d) of the norm, since

$$\begin{aligned} d(\varphi, \psi) &= \|\varphi - \psi\| = \|\varphi - \eta + \eta - \psi\| \leqslant \|\varphi - \eta\| + \|\eta - \psi\| \\ &= d(\varphi, \eta) + d(\eta, \psi), \quad \forall(\varphi, \psi, \eta) \in X \times X \times X. \end{aligned}$$

This completes the proof. ∎

The metric induced by a norm fulfilles the following extra properties, the proof of which is straightforward and is left to the Reader:

1. $d(\varphi + \eta, \psi + \eta) = d(\varphi, \psi), \quad \forall(\varphi, \psi, \eta) \in X \times X \times X$ (translation invariance);
2. $d(\lambda\varphi, \lambda\psi) = |\lambda| d(\varphi, \psi), \quad \forall(\varphi, \psi) \in X \times X$ and $\lambda \in \mathbb{K}$ (homogeneity);

Theorem 2.1 shows that any normed vector space is naturally endowed with a notion of distance. Please notice however that a normed vector space could be equipped also with distances other than the one induced by the norm; such distances are not necessary related to the norm; furthermore, eq. (2.1) is not the only one possible distance built from the norm.

Exercise 2.1: Let $(X, \|\cdot\|)$ be a normed vector space. Let $d\colon X \times X \to \mathbb{R}$ be the function defined by

$$d(\varphi, \psi) = \frac{\|\varphi - \psi\|}{1 + \|\varphi - \psi\|}, \quad \forall(\varphi, \psi) \in X \times X$$

Show that d defines a nonhomogeneus, translation invariant metric on X.

Now that we have a "natural" notion of distance in normed vector spaces, i. e., eq. (2.1), all the concepts defined for metric spaces applies automatically, in particular, to normed linear spaces. Among these concepts are: continuity, limits, completeness, open sets etc.

Lemma 2.1: Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. The following holds:

$$\big|\|\psi\| - \|\varphi\|\big| \leqslant \|\psi - \varphi\|, \quad \forall(\psi, \varphi) \in X \times X. \tag{2.2}$$

Proof. The key ingredient is the triangle inequality of the norm. For all $(\varphi, \psi) \in X \times X$, we have

$$\|\varphi\| = \|\varphi - \psi + \psi\| \leqslant \|\varphi - \psi\| + \|\psi\|$$

and

$$\|\psi\| = \|\psi - \varphi + \varphi\| \leqslant \|\varphi - \psi\| + \|\psi\| .$$

Thus,

$$\begin{aligned} \|\varphi\| - \|\psi\| &\leqslant \|\varphi - \psi\| \\ \|\psi\| - \|\varphi\| &\leqslant \|\varphi - \psi\| . \end{aligned}$$

which is exactly the meaning of eq. (2.2). (To see this even more explicitly, it is enough to discuss the three cases $\|\psi\| < \|\varphi\|$, $\|\psi\| = \|\varphi\|$, $\|\psi\| > \|\varphi\|$, and remember that $\|\varphi - \psi\| \geqslant 0$.) ∎

Continuity of the norm

THEOREM 2.2: Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. The norm $\|\cdot\|$ is continuous on X (with respect to the metric induced by the norm).

Proof. We need to show that $\forall \psi \in X$ the following holds: for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $\varphi \in X$, if $d(\psi, \varphi) < \delta$ then $\tilde{d}(\|\psi\|, \|\varphi\|) < \varepsilon$, where $\tilde{d}$ is the euclidean distance in $\mathbb{R}$.

Notice that $d(\psi, \varphi) < \delta$ means

$$\|\psi - \varphi\| < \delta ,$$

and $\tilde{d}(\|\psi\|, \|\varphi\|) < \varepsilon$ means

$$|\|\psi\| - \|\varphi\|| < \epsilon .$$

Using eq. (2.2), it is enough to choose any $0 < \delta < \epsilon$. ∎

Completeness of a metric space

Let us recall an important fact about Cauchy sequencies. Give any metric space (X, d), a convergent sequence in X is also a Cauchy sequence. The converse of this statement is not generally true however, i. e., there exists metric spaces where some Cauchy sequences do not need to converge.

> A typical counter-example works as follow. Consider a sequence $(x_n)_n$ of rational numbers in $\mathbb{R}$ which is convergent to some irrational number. Thus, $(x_n)_n$ is a Cauchy sequence. Now, consider the same sequence as a sequence in $\mathbb{Q}$: of course, it is still a Cauchy sequence, but this time it is not convergent.

A metric space is "complete" if *all* Cauchy sequencies are convergent.

Banach space

DEFINITION 2.3 (BANACH SPACE): Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. If (X, d) (where d is the metric induced by the norm) is complete, $(X, \|\cdot\|)$ is called a "Banach space".

We are almost ready to introduce the notion of Hilber space, which is the setting where we will develop the operator formulation of non-relativistic quantum mechanics. The first ingredient is the "inner product", defined below.

Complex numbers notation

In the following, for every $z \in \mathbb{C}$ we will denote the "complex conjugate" of z with z^* and the "modulus" of z with $|z|$. Remember that $z = \Re z + i\Im z$ (where $\Re z$ and $\Im z$ are the real and imaginary parts of z, respectively), $z^* = \Re z - i\Im z$ and $|z|^2 = zz^*$. The inverse of $z \neq 0$ is $z^{-1} = z^*/|z|^2$. We have $z \in \mathbb{R}$ if and only if $z^* = z$. The complex conjugation satisfies the "involution" property: $(z^*)^* = z$. Furthermore, $(z_1 z_2)^* = z_1^* z_2^*$, $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$, and $|z_1 z_2| = |z_1||z_2|$.

Inner product

DEFINITION 2.4 (INNER PRODUCT): Let X be a vector space over $\mathbb{K}$. A "inner product" on X is any application $X \times X \to \mathbb{K}$, hereafter denoted by $\langle \cdot | \cdot \rangle$, satisfying the following properties:

(a) $\langle \varphi | \psi + \eta \rangle = \langle \varphi | \psi \rangle + \langle \varphi | \eta \rangle , \quad \forall (\varphi, \psi, \eta) \in X \times X \times X;$

(b) $\langle \varphi | \lambda\psi \rangle = \lambda \langle \varphi | \psi \rangle , \quad \forall (\varphi, \psi) \in X \times X$ and $\lambda \in \mathbb{K};$

(c) $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$, $\quad\forall(\varphi,\psi) \in X \times X$;

(d) $\langle\varphi|\varphi\rangle \geqslant 0$, $\quad\forall\varphi \in X$;

(e) $\langle\varphi|\varphi\rangle = 0$ if and only if $\varphi = 0$.

Several remarks are in order.

Remark. Items (a) and (b) implies that the inner product is *linear* on the *second* component, namely:

$$\boxed{\langle\varphi|\lambda\psi + \mu\eta\rangle = \lambda\,\langle\varphi|\psi\rangle + \mu\,\langle\varphi|\eta\rangle\,,}$$

for all $(\varphi,\psi,\eta) \in X\times X\times X$ and for all $(\lambda,\mu) \in \mathbb{K}\times\mathbb{K}$. Item (c) together with items (a) and (b) implies that the inner product in general is *conjugate-linear* (or anti-linear) on the *first* component, namely

$$\boxed{\langle\lambda\varphi + \mu\eta|\psi\rangle = \lambda^*\,\langle\varphi|\psi\rangle + \mu^*\,\langle\varphi|\eta\rangle\,,}$$

for all $(\varphi,\psi,\eta) \in X \times X \times X$ and for all $(\lambda,\mu) \in \mathbb{K} \times \mathbb{K}$. Of course, if $\mathbb{K} = \mathbb{R}$, $\lambda^* = \lambda$, $\mu^* = \mu$ and the inner product becomes linear also on the first component (thus, it is *bi*linear); but this is not the case if $\mathbb{K} = \mathbb{C}$, where complex conjugation appears.

Remark. Item (b) is a matter of choice. Some authors prefer a different convention:

$$\langle\lambda\varphi|\psi\rangle = \lambda\,\langle\varphi|\psi\rangle\,, \quad \forall(\varphi,\psi) \in X \times X \text{ and } \lambda \in \mathbb{K}\,;$$

with this convention, the inner product would become linear on the first component and conjugate-linear on the second one. The convetion of having the inner product linear on the second component is the one most often employed by physicists, and the one used in this notes.

Remark. Regarding items (d) and (e), one may wonder what does mean $\langle\varphi|\varphi\rangle \geqslant 0$, since we expect $\langle\varphi|\varphi\rangle \in \mathbb{K}$, and if $\mathbb{K} = \mathbb{C}$ it might seem that the inequlity does not make sense. Actually, from item (c)

$$\langle\varphi|\varphi\rangle = \langle\varphi|\varphi\rangle^*\,, \quad \forall\varphi \in X\,,$$

Remark. Some authors prefer the notation (φ,ψ) instead of $\langle\varphi|\psi\rangle$; the notation $\langle\varphi|\psi\rangle$ is closer to the one uses by physicists and it is the first step towards the introduction of Dirac's notation. (Dirac's notation is more than simply writing the inner product this way; we will discuss this point in connection with the spectral theorem of linear operators.) In Dirac notation the vector ψ is denoted by $|\psi\rangle$, and it is called "ket"; there is a "kind of conjugation" (more on this later on) that converts the analogous "ket" $|\varphi\rangle$ to a so-called "bra" $\langle\varphi|$ and the inner product is considered as a product between a "bra" and a "ket" (resulting in a "braket"!). Of course, this is just a naming convention.

Inner product space

Definition 2.5 (inner product space): A "inner product space" over $\mathbb{K}$ is a pair $(X, \langle\cdot|\cdot\rangle)$, where X is a vector space over $\mathbb{K}$ and $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{K}$ is an inner product on X.

The following lemma will be useful later on.

Lemma 2.2: Let $(X, \langle\cdot|\cdot\rangle)$ be a inner product space over $\mathbb{K}$. If $\psi = 0$, then

$$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle = 0\,, \tag{2.3}$$

for every $\varphi \in X$.

The statement of the lemma itself looks rather obvious. However, a technical proof is given below. The linearity of the inner product is a key ingredient.

Proof. By linearity of the inner product,

$$\langle \varphi | \psi + \psi \rangle = \langle \varphi | \psi \rangle + \langle \varphi | \psi \rangle \,, \quad \forall (\varphi, \psi) \in X \times X \,.$$

In particular, if $\psi = 0$ we have $\psi + \psi = \psi$ and

$$\langle \varphi | \psi + \psi \rangle = \langle \varphi | \psi \rangle \,, \quad \forall \varphi \in X \text{ and } \psi = 0 \,.$$

Thus

$$\langle \varphi | \psi \rangle + \langle \varphi | \psi \rangle = \langle \varphi | \psi \rangle \,, \quad \forall \varphi \in X \text{ and } \psi = 0 \,,$$

which is an equation in $\mathbb{K}$ for the unknown $\langle \varphi | \psi \rangle$, whose only solution is $\langle \varphi | \psi \rangle = 0$. ∎

Any inner product space is naturally endowed with a norm coming from the inner product. Let $(X, \langle \cdot | \cdot \rangle)$ be an inner product space over $\mathbb{K}$. Let $\| \cdot \| \colon X \to \mathbb{R}$ defined by

Norm induced by inner product

$$\boxed{\| \psi \| = \sqrt{\langle \psi | \psi \rangle}, \quad \forall \psi \in X \,.} \tag{2.4}$$

Observe that such $\| \cdot \|$ in eq. (2.4) is well-defined, since $\langle \psi | \psi \rangle \geqslant 0$ for every $\psi \in X$. The square root is not ambigous: it is not a square root of a complex number; it is the square root of a positive real number, we don't need to specify a branch for the square root function. We shall prove in a moment that $\| \cdot \|$ is actually a norm on X, this justify the notation $\| \cdot \|$. Such norm is the "norm induced by the inner product". Before proving this, we need a preliminary but extremely important result, which goes under the name of Cauchy-Schwarz inequality.

Cauchy-Schwarz inequality is of major importance. It is a key ingredient in several proofs of functional analysis. It has important implications also outside the realm of analysis. For example, the general formulation of the Heisenberg uncertainty principle in quantum mechanics (or the analogous time-bandwidth uncertainty principle for temporal signal transmission) is derived using the Cauchy-Schwarz inequality.

Theorem 2.3 (Cauchy-Schwarz inequality): Let $(X, \langle \cdot | \cdot \rangle)$ be an inner product space. The following holds:

Cauchy-Schwarz inequality

$$\boxed{|\langle \varphi | \psi \rangle| \leqslant \| \varphi \| \| \psi \|, \quad \forall (\varphi, \psi) \in X \times X \,.} \tag{2.5}$$

Remark. In eq. (2.5) we are using the definition eq. (2.4) but it is important to emphasize that we are *not* using (in both the statement and in the proof of Cauchy-Schwarz inequality) the fact that eq. (2.4) is a norm. We don't know at this point that eq. (2.4) defines a norm, we will prove that in the next theorem, using the Cauchy-Schwarz inequality.

Proof. We distinguish two cases. If $\psi = 0$, $\langle \varphi | \psi \rangle = 0$ (see eq. (2.3)) and $\langle \psi | \psi \rangle = \| \psi \|^2 = 0$ (by definition of the inner product), thus the inequality is satified.

Let us now consider $\psi \neq 0$. For every $\lambda \in \mathbb{K}$, we have

$$\langle \varphi + \lambda \psi | \varphi + \lambda \psi \rangle = \langle \varphi | \varphi \rangle + \lambda \langle \varphi | \psi \rangle + \lambda^* \langle \psi | \varphi \rangle + \lambda^* \lambda \langle \psi | \psi \rangle \,.$$

By item (d) in **??**, the left-hand side of this equation is $\langle \varphi + \lambda \psi | \varphi + \lambda \psi \rangle \geqslant 0$ and it is zero if and only if $\varphi + \lambda \psi = 0$. Thus

$$\langle \varphi | \varphi \rangle + \lambda \langle \varphi | \psi \rangle + \lambda^* \langle \psi | \varphi \rangle + \lambda^* \lambda \langle \psi | \psi \rangle \geqslant 0 \,,$$

for every $\varphi \in X$, $\psi \in X \backslash \{0\}$ and $\lambda \in \mathbb{K}$. Choose

$$\lambda = -\frac{\langle \varphi | \psi \rangle^*}{\langle \psi | \psi \rangle}, \quad \psi \neq 0 \,,$$

which makes sense since we are discussing the case $\psi \neq 0$. Plugin into the previosu equation yields

$$\langle\varphi|\varphi\rangle - \frac{\langle\varphi|\psi\rangle^*}{\langle\psi|\psi\rangle} - \frac{\langle\varphi|\psi\rangle}{\langle\psi|\psi\rangle}\langle\varphi|\psi\rangle^* + \frac{\langle\varphi|\psi\rangle^*}{\langle\psi|\psi\rangle}\frac{\langle\varphi|\psi\rangle}{\cancel{\langle\psi|\psi\rangle}}\cancel{\langle\psi|\psi\rangle} \geqslant 0\,,$$

hence

$$\langle\varphi|\varphi\rangle - + \frac{\langle\varphi|\psi\rangle^*\langle\varphi|\psi\rangle}{\langle\psi|\psi\rangle} \geqslant 0\,,$$

from which it follows

$$\langle\varphi|\varphi\rangle\,\langle\psi|\psi\rangle \geqslant |\langle\varphi|\psi\rangle|$$

(using the fact that $\langle\psi|\psi\rangle > 0$). Taking the square root of both sites (notice that both sides are surely positive) yields the expected result. ■

Theorem 2.4: Let $(X, \langle\cdot|\cdot\rangle)$ be an inner product space over $\mathbb{K}$. $(X, \|\cdot\|)$ with $\|\cdot\|$ defined by eq. (2.4) is a normed vector space over $\mathbb{K}$.

Proof. We need to check that eq. (2.4) makes sense and that it satisfies items (a) to (d) of the definition of the norm. ■

Continuity of inner product

Theorem 2.5: Let $(X, \langle\cdot|\cdot\rangle)$ be an inner product space over $\mathbb{K}$. $\langle\cdot\rangle\cdot$ is a continuous function of both arguments.

Hilbert space

Definition 2.6 (Hilbert space): Let $(X, \langle\cdot|\cdot\rangle)$ be a inner product space over $\mathbb{K}$. If the metric space (X, d) (with the distance arising from the inner product) is complete, $(X, \langle\cdot|\cdot\rangle)$ is called an "Hilbert space".

In short: Banach spaces are complete normed vector spaces and Hilbert spaces are complete inner product spaces. Hilbert spaces are a special case of Banach space, where the norm comes from an inner product. The underlying iner product inducing the norm introduces extra features (in particular, the notion of orthogonality) which are not present in general Banach spaces.

Hereafter, if $(X, \langle\cdot|\cdot\rangle)$ is an inner product space, if not explicitly stated, we will also intend that $\|\cdot\|$ denotes the norm induced by the inner product, and that the convergence, etc refers to the distance induced by that norm.

Is it possible to distinguish if in a normed vector space, the norm is coming from some underlying inner space? The answer is yes, and we are going to prove immediately an interesting criterion to perform this check.

The following result is basic to establish the later theorem. It will also be useful later, when discussing positive operators.

Lemma 2.3 (polarization identity): Let $(X, \langle\cdot|\cdot\rangle)$ be a inner product space. In the case $\mathbb{K} = \mathbb{R}$,

$$\langle\varphi|\psi\rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right), \quad \forall(\varphi,\psi) \in X \times X\,. \tag{2.6}$$

In the case $\mathbb{K} = \mathbb{C}$,

$$\langle\varphi|\psi\rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right) + \frac{i}{4}\left(\|\varphi+i\psi\|^2 + \|\varphi-i\psi\|^2\right), \quad \forall(\varphi,\psi) \in X \times X\,. \tag{2.7}$$

Proof. ■

The following result is called Jordan-von Neumann theorem.*

Jordan-von Neumann theorem

Parallelogram law

THEOREM 2.6 (JORDAN-VON NEUMANN THEOREM): Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$. The norm $\|\cdot\|$ comes from an inner product (i. e., the Banach space is also an Hilbert space) *if and only if* the following identity (known as "parallelogram law") holds

$$\|\varphi + \psi\|^2 + \|\varphi - \psi\|^2 = 2\|\varphi\|^2 + \|\psi\|^2, \quad \forall (\varphi, \psi) \in X \times X. \tag{2.8}$$

Proof. Let's prove first that if $(X, \langle\cdot|\cdot\rangle)$ is a Hilbert space, then the norm induced by the inner product fullfills eq. (2.8). This is the straightforward part of the proof. By definition of the norm induced by the inner product,

$$\begin{aligned}
\|\varphi + \psi\| &= \langle \varphi + \psi | \varphi + \psi \rangle \\
&= \langle \varphi|\varphi\rangle + \langle \varphi|\psi\rangle + \langle \psi|\varphi\rangle + \langle \psi|\psi\rangle \\
&= \|\varphi^2\| + 2\Re\langle \varphi|\psi\rangle + \|\psi\|^2 \\
\|\varphi - \psi\| &= \langle \varphi - \psi | \varphi - \psi \rangle \\
&= \langle \varphi|\varphi\rangle - \langle \varphi|\psi\rangle - \langle \psi|\varphi\rangle + \langle \psi|\psi\rangle \\
&= \|\varphi^2\| - 2\Re\langle \varphi|\psi\rangle + \|\psi\|^2
\end{aligned}$$

for all $(\varphi, \psi) \in X \times X$; summing the two we have eq. (2.8).

Now, let's prove that if $(X, \|\cdot\|)$ is a Banach space whose norm satisfies eq. (2.8) then the norm comes from an inner product. It is possible to recover the underlying inner product by means of the polarization identity. Since the polarization identity takes two differen forms, depending on whether $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, we should distinguish two cases. We will discuss the case $\mathbb{K} = \mathbb{C}$, the case $\mathbb{C} = \mathbb{R}$ is analogous and the details are left to the Reader.

In the Banach space $(X, \|\cdot\|)$ it is always possible to define an application $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{K}$ by letting

$$\langle \varphi|\psi\rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right) + \frac{i}{4}\left(\|\varphi+i\psi\|^2 - \|\varphi-i\psi\|^2\right), \; \forall (\varphi, \psi) \in X \times X.$$

We need to prove that this application indeed is a inner product on X, i. e., we need to prove that it satisfies items (a) to (e) in **??**.

Ad (a).

Ad (b).

Ad (b).

Ad (c).

Ad (d).

Ad (e). ∎

EXERCISE 2.2: Does theorem 2.6 apply to generic normed vector spaces over $\mathbb{K}$ which are not completed (i. e., which are not Banach spaces)?

2.2 CANONICAL PROTOTYPES OF BANACH AND HILBERT SPACES

Let $N \in \mathbb{N}\backslash\{0\}$ be any positive integer number. Consider the vector space $\mathbb{C}^N$, whose elements are N-ple of complex numbers, endowed with the usual operations of elementwise addition and elementwise multiplication of an N-ple by a complex number. For every

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}, \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix},$$

* See Jordan and von Neumann, 1935.

in $\mathbb{C}^N$, define an application $\langle\cdot|\cdot\rangle : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ as follows:

$$\langle \boldsymbol{\varphi}|\boldsymbol{\psi}\rangle = \sum_{k=1}^{N} \varphi_k^* \psi_k \,. \tag{2.9}$$

It is possible to prove that this function indeed is a inner product on $\mathbb{C}^N$. We need to verify items (a) to (e) of definition 2.4. Linearity is traightforward:

$$\langle \boldsymbol{\varphi}|\boldsymbol{\psi}+\boldsymbol{\eta}\rangle = \sum_{k=1}^{N} \varphi_k^* \left(\psi_k + \eta_k\right) = \left(\sum_{k=1}^{N} \varphi_k^* \psi_k\right) + \left(\sum_{k=1}^{N} \varphi_k^* \eta_k\right) = \langle \boldsymbol{\varphi}|\boldsymbol{\psi}\rangle + \langle \boldsymbol{\varphi}|\boldsymbol{\eta}\rangle \,,$$
$$\forall (\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\eta}) \in C^N \times C^N \times \mathbb{C}^N \,,$$
$$\langle \boldsymbol{\varphi}|\lambda\boldsymbol{\psi}+\boldsymbol{\eta}\rangle = \sum_{k=1}^{N} \varphi_k^* \lambda \psi_k = \lambda \sum_{k=1}^{N} \varphi_k^* \psi_k = \lambda \langle \boldsymbol{\varphi}|\boldsymbol{\psi}\rangle \,, \ \forall (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in C^N \times C^N \text{ and } \lambda \in \mathbb{C} \,.$$

Item (c) follows from the fact that, for every $(z, w) \in \mathbb{C} \times \mathbb{C}$,

- $(z+w)^* = z^* + w^*$;
- $(zw)^* = z^* w^*$;
- $(z^*)^* = z$.

(It is easy to prove by induction on N that the first property is true also with a sum of N complex numbers.) Thus:

$$\langle \boldsymbol{\psi}|\boldsymbol{\varphi}\rangle = \sum_{k=1}^{N} \psi_k^* \varphi_k = \sum_{k=1}^{N} \left(\psi_k \varphi_k^*\right)^* = \left(\sum_{k=1}^{N} \varphi_k^* \psi_k\right)^* = \langle \boldsymbol{\varphi}|\boldsymbol{\psi}\rangle^*$$

2.3 ORTHOGONALITY

2.4 ORTHONORMAL BASIS

REFERENCES FOR CHAPTER 2

JORDAN, PASCUAL and JOHN VON NEUMANN

1935 "On inner products in linear, metric spaces", *Annals of Mathematics*, vol. 36, 3 (July 1935), pp. 719-723. (Cited on p. 11.)

REED, MARTIN and BERRY SIMON

1980 *Functional Analysis*, 2nd ed., Methods of Modern Mathematical Physics, Academic Press, London, vol. 1, ISBN: 0-12-585050-6. (Cited on p. 5.)

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APPENDICES

PESKIN, MICHAEL E. and DANIEL V. SCHROEDER

1995 *An introduction to Quantum Field Theory*, Perseus Books Publishing L.L.C., Reading, MA, ISBN: 0-201-50397-2.

ZEIDLER, EBERHARD

2009a *Gauge theory*, 1st ed., Quantum Field Theory. A Bridge between Mathematicians and Physicists, Springer, Berlin, vol. 3, ISBN: 978-3-540–85376-3 (hardcover).

2009b *Quantum Electrodynamics*, 1st ed., Quantum Field Theory. A Bridge between Mathematicians and Physicists, Springer, Berlin, vol. 2, ISBN: 978-3-540–85376-3 (hardcover).

ZINN-JUSTIN, JEAN

2002 *Quantum Field Theory and Critical Phenomena*, Oxford University Press, Oxford, ISBN: 0-19-850923-5.

2005 *Path Integrals in Quantum Mechanics*, Oxford University Press, Oxford, ISBN: 0-19-856674-3 (Hbk).

2007 *Phase Transition and Renormalization Group*, Oxford University Press, Oxford, ISBN: 978-0-19-922719-8 (Hbk).