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Quantum Mechanics

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“ In the good old days, theorizing was like sailing between islands of experimental evidence. And, if the trip was not in the vicinity of the shoreline (which was strongly recommended for safety reasons) sailors where continuously looking forward, hoping to see land — the sooner the better.

Nowadays, some theoretical physicists (let us call them sailors) [have] found a way to survive and navigate in the open sea of pure theoretical constructions. Instead of the horizon, they look at stars, which tell them exactly where they are. Sailors are aware of the fact that the stars will never tell them where the new land is, but they may tell them their position on the globe.

Theoreticians become sailors simply bacause they just like it. Young people, seduced by capitans forming crews to go to a Nuevo El Dorando [...] soon realize that they will spend all their life at sea. Those who do not like sailing desert the voyage, but for the true potential sailors the sea become their passion. They will probably tell the alluring and frightening truth to their students — and the proper people will join their ranks. ”

— Andrei Losev

CONTENTS

PREFACE

To be written…

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Part I

THE BASIS

1 SCHWINGER'S APPROACH TO QUANTUM MECHANICS

> "I presume that all of you have already been exposed to some undergraduate course in Quantum Mechanics, one that leans heavily on de Broglie waves and the Schroedinger equation. I have never thought that this simple wave approach was acceptable as a general basis for the whole subject, and I intend to move immediately to replace it in your mind by a foundation that *is* perfectly general."
>
> J. Schwinger, *Quantum Mechanics. Symbolism of Atomic Measurements* Schwinger [2001].

1.1 INTRODUCTION

COMPARED with other traditional areas of physics, quantum mechanics is not easy. It often lacks physical intuitition and it relies on heavy mathematical background from the very beginning. As we shall see shortly, topics like uncertanty principle, the role of probability etc asks immediately for a theoretical framework/formalism... Phase space is not able to capture, it does not offer the tools. Naturally leading from the very beginning to adopt .

REFERENCES FOR CHAPTER 1

SCHWINGER, J.

2001 *Quantum Mechanics. Symbolism of Atomic Measurements*, Springer-Verlag, Berlin, ISBN: 3-540-41408-8. (Cited on p. 3.)

2 LINEAR OPERATORS IN HILBERT SPACES

Operator formulation of standard non-relativistic quantum mechanics heavily relies on the theory of linear operators in Hilber spaces. In particular, the spectral decomposition of self-adjoint operators (bounded and unbounded ones) is a key ingredient in formulating the basic rules of quantum mechanics. This chapter is aimed at providing the necessary mathematical background of functional analysis employed by non-relativistic quantum mechanics. It is a chapter on mathematics, not on quantum physics; for this reason care has been made to mathematical rigous more than what will be customary in later chapters. The Reader interested in how functional analysis applies to the formulation of quantum mechanics should jump to the next chapters. As we shall see, physicists are customary to employ Dirac's notation, a powerful mnemonic notation which naively speaking however lacks mathematical rigour; a dictionary is possible to translate Dirac notation to the rigorous theorems of functional analysis; more on this later on.

This chapter is mostly based on Reed and Simon [1980]. Some other books that have been helpful while writing this chapter includes: Berberian [1976]; Debnath and Mikusiński [2005]; Helmberg [1969]; Hutson and Pym [1980]; Teschl [2009].

2.1 BANACH AND HILBER SPACES

Unless stated otherwise, let $\mathbb{K}$ denote equivalently the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. (It is possible to develop the theory also for the skew-field of quaternions, but this case will not be discussed here to avoid dealing with the non-commutativity of quaternionic product.)

Norm

DEFINITION 2.1 (NORM): Let X be any vector space over $\mathbb{K}$. A "norm" on X is any application $X \to \mathbb{R}$ here denoted by $\|\cdot\|$ satisfying the following properties:

(a) $\|\varphi\| \geqslant 0, \quad \forall \varphi \in X$ (nonnegativity);

(b) $\|\varphi\| = 0$ if and only if $\varphi = 0$ (faithfulness)

(c) $\|\lambda\varphi\| = |\lambda| \, \|\varphi\|, \quad \forall \lambda \in \mathbb{K}$ and $\forall \varphi \in X$ (positive homogeneity);

(d) $\|\varphi + \psi\| \leqslant \|\varphi\| + \|\psi\|, \quad \forall (\varphi, \psi) \in X \times X$ (subadditivity).

Remark. Item (a) is not strictly necessary: it follows from items (b) to (d). In fact, from item (c) $\|-\varphi\| = \|\varphi\|$, from item (d) we have thus $\|\varphi + (-\varphi)\| \leqslant \|\varphi\| + \|\varphi\| = 2\,\|\varphi\|$, and from item (b) $\|\varphi - \varphi\| = 0$, thus $\|\varphi\| \leqslant 0$ for every $\varphi \in X$.

Remark. X does not need to be finite-dimensional. It can be infinite-dimensional. The dimensionality of the vector space is not relevant for this and the following definitions and properties.

A "normed vector space" is a vector space equipped with a norm, as formalized by the following definition.

Normed vector space

DEFINITION 2.2 (NORMED VECTOR SPACE): A "normed vector space" over $\mathbb{K}$ is a pair $(X, \|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is any norm on X.

If $(X, \|\cdot\|)$ is a normed vector space, the norm $\|\cdot\|$ *induces* a metric (i. e., a notion of distance) and thus a topology on X. This is proved by the following theorem.

Distance induced by a norm

THEOREM 2.1: Let $(X, \|\cdot\|)$ be any normed vector space over $\mathbb{K}$. Let $d\colon X \times X \to \mathbb{R}$ be the function defined by

$$\boxed{d(\varphi,\psi) = \|\varphi - \psi\|\,, \quad \forall(\varphi,\psi) \in X \times X\,.} \tag{2.1}$$

Then, (X, d) is a metric space. The metric in eq. (2.1) is called the "metric induced by the norm" on X.

Proof. Remember that, given a set X and an application $d\colon X \times X \to [0, +\infty[$, d is called a "metric" (or "distance") on X if it satisfies the following properties:

(a) $d(\varphi,\psi) \geqslant 0, \quad \forall(\varphi,\psi) \in X \times X$ (nonnegativity);

(b) $d(\varphi,\psi) = 0$ if and only if $\varphi = \psi$ (i. e., the only point of X at zero distance from ψ is ψ itself);

(c) $d(\varphi,\psi) = d(\psi,\varphi), \quad \forall(\varphi,\psi) \in X \times X$ (the distance is a symmetric function of its arguments);

(d) $d(\varphi,\psi) \leqslant d(\psi,\eta) + d(\eta,\psi), \quad \forall(\varphi,\psi,\eta) \in X \times X \times X$ (the so-called "triangle inequality").

Thus, we need to show that eq. (2.1) defines a metric over X, i. e., we need to prova that such d satisfies items (a) to (d) above. Item (a) follows from property (a) of the norm. Item (b) follows from property (b) of the norm, since $\|\varphi - \psi\| = 0$ if and only if $\varphi - \psi = 0$, i. e., if and only if $\varphi = \psi$. Item (c) follows from property (c) of the norm, since

$$\begin{aligned} d(\varphi,\psi) &= \|\varphi - \psi\| = \|-(\psi - \varphi)\| = |-1|\,\|\psi - \varphi\| \\ &= \|\psi - \varphi\| = d(\psi,\varphi), \quad \forall(\varphi,\psi) \in X \times X\,. \end{aligned}$$

Item (d) follows from property (d) of the norm, since

$$\begin{aligned} d(\varphi,\psi) &= \|\varphi - \psi\| = \|\varphi - \eta + \eta - \psi\| \leqslant \|\varphi - \eta\| + \|\eta - \psi\| \\ &= d(\varphi,\eta) + d(\eta,\psi), \quad \forall(\varphi,\psi,\eta) \in X \times X \times X\,. \end{aligned}$$

This completes the proof. ∎

The metric induced by a norm fulfilles the following extra properties, the proof of which is straightforward and is left to the Reader as exercize:

1. $d(\varphi + \eta, \psi + \eta) = d(\varphi,\psi)$ (translation invariance), and
2. $d(\lambda\varphi, \lambda\psi) = |\lambda|\, d(\varphi,\psi)$ (homogeneity),

for all $(\varphi,\psi,\eta) \in X \times X$ and $\lambda \in \mathbb{K}$.

EXERCISE 2.1: Prove items 1 and 2 above.

Theorem 2.1 shows that any normed vector space is naturally endowed with a notion of distance. Please notice however that a normed vector space could be equipped also with distances other than the one induced by the norm; such distances are not necessary related to the norm; furthermore, eq. (2.1) is not the only one possible distance built from the norm (see **??** 2.2).

EXERCISE 2.2: Let $(X, \|\cdot\|)$ be a normed vector space. Let $d\colon X \times X \to \mathbb{R}$ be the function defined by

$$d(\varphi,\psi) = \frac{\|\varphi - \psi\|}{1 + \|\varphi - \psi\|}, \quad \forall(\varphi,\psi) \in X \times X$$

Show that d defines a *non*homogeneus, translation invariant metric on X.

Now that we have a "natural" notion of distance in normed vector spaces, i. e., eq. (2.1), all the concepts defined for metric spaces applies automatically, in particular, to normed linear spaces. Among these concepts are: continuity, limits, convergence, compactness, completeness, open sets etc.

Inverse triangle inequality

LEMMA 2.1: Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. The following holds:

$$\left|\|\psi\| - \|\varphi\|\right| \leqslant \|\psi - \varphi\| , \quad \forall(\psi, \varphi) \in X \times X . \tag{2.2}$$

Proof. The key ingredient is the triangle inequality of the norm. For all $(\varphi, \psi) \in X \times X$, we have

$$\|\varphi\| = \|\varphi - \psi + \psi\| \leqslant \|\varphi - \psi\| + \|\psi\|$$

and

$$\|\psi\| = \|\psi - \varphi + \varphi\| \leqslant \|\varphi - \psi\| + \|\psi\| .$$

Thus,

$$\begin{aligned} \|\varphi\| - \|\psi\| &\leqslant \|\varphi - \psi\| \\ \|\psi\| - \|\varphi\| &\leqslant \|\varphi - \psi\| . \end{aligned}$$

which is exactly the meaning of eq. (2.2). (To see this even more explicitly, it is enough to discuss the three cases $\|\psi\| < \|\varphi\|$, $\|\psi\| = \|\varphi\|$, $\|\psi\| > \|\varphi\|$, and remember that $\|\varphi - \psi\| \geqslant 0$.) ∎

Continuity of the norm

THEOREM 2.2: Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. The norm $\|\cdot\|$ is continuous on X (with respect to the metric induced by the norm).

Proof. We need to show that $\forall\psi \in X$ the following holds: for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $\varphi \in X$, if $d(\psi, \varphi) < \delta$ then $\tilde{d}(\|\psi\| , \|\varphi\|) < \varepsilon$, where $\tilde{d}$ is the euclidean distance in $\mathbb{R}$.

Notice that $d(\psi, \varphi) < \delta$ means

$$\|\psi - \varphi\| < \delta ,$$

and $\tilde{d}(\|\psi\| , \|\varphi\|) < \varepsilon$ means

$$\left|\|\psi\| - \|\varphi\|\right| < \epsilon .$$

Using eq. (2.2), it is enough to choose any $0 < \delta < \epsilon$. ∎

As a consequence of the continuity of the norm, if $(X, \|\cdot\|)$ is a normed vector space and $(\varphi_k)_{k\in\mathbb{N}}: \mathbb{N} \to X$ is a sequence in X convergent to some $\varphi \in X$, i. e.,

$$\lim_{k \to +\infty} \varphi_k = \varphi ,$$

then

$$\|\varphi\| = \left\|\lim_{k\to+\infty} \varphi_k\right\| = \lim_{k\to+\infty} \|\varphi_k\| . \tag{2.3}$$

EXERCISE 2.3: Justify all steps in eq. (2.3). (Hint: it is just a way to say that a function $f(x)$, in this case the norm, is continuous if and only if $\lim_{x\to x_0} f(x)f(x_0)$.)

DEFINITION 2.3 (EQUIVALENCE OF THE NORMS): Let X be a vector space over $\mathbb{K}$ and $\|\cdot\|_1 : X \to \mathbb{R}$ and $\|\cdot\|_2 : X \to \mathbb{R}$ two norms on X. The two norms are said to be "equivalent" if there exists a pair of strictly positive real numbers λ and μ such that

$$\begin{aligned} \alpha \|\varphi\|_1 &\leqslant \|\varphi\|_2 \\ &\leqslant \beta \|\varphi\|_1 , \end{aligned} \tag{2.4}$$

for all $\varphi \in X$.

Equivalent norms define the same notions of continuity and convergence and for many purposes do not need to be distinguished. As we shall prove later, for finite-dimensional (real or complex) vector space, all norms are equivalent. On the other hand, in the case of infinite-dimensional vector spaces, not all norms are equivalent and we need to specify which norm we are using.

Completeness of a metric space

Let us recall an important fact about Cauchy sequencies. Give any metric space (X, d), a convergent sequence in X is also a Cauchy sequence. The converse of this statement is not generally true however, i. e., there exists metric spaces where some Cauchy sequences do not need to converge.

> A typical counter-example works as follow. Consider a sequence $(x_n)_n$ of rational numbers in $\mathbb{R}$ which is convergent to some irrational number. Thus, $(x_n)_n$ is a Cauchy sequence. Now, consider the same sequence as a sequence in $\mathbb{Q}$: of course, it is still a Cauchy sequence, but this time it is not convergent.

A metric space is "complete" if *all* Cauchy sequencies are convergent.

Banach space

Definition 2.4 (Banach space): Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. If (X, d) (where d is the metric induced by the norm) is complete, $(X, \|\cdot\|)$ is called a "Banach space".

We are almost ready to introduce the notion of Hilber space, which is the setting where we will develop the operator formulation of non-relativistic quantum mechanics. The first ingredient is the "inner product", defined below.

Complex numbers notation

In the following, for every $z \in \mathbb{C}$ we will denote the "complex conjugate" of z with z^* and the "modulus" of z with $|z|$. Remember that $z = \Re z + i\Im z$ (where $\Re z$ and $\Im z$ are the real and imaginary parts of z, respectively), $z^* = \Re z - i\Im z$ and $|z|^2 = zz^*$. The inverse of $z \neq 0$ is $z^{-1} = z^*/|z|^2$. We have $z \in \mathbb{R}$ if and only if $z^* = z$. The complex conjugation satisfies the "involution" property: $(z^*)^* = z$. Furthermore, $(z_1 z_2)^* = z_1^* z_2^*$, $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$, and $|z_1 z_2| = |z_1|\,|z_2|$.

Inner product

Definition 2.5 (inner product): Let X be a vector space over $\mathbb{K}$. A "inner product" on X is any application $X \times X \to \mathbb{K}$, hereafter denoted by $\langle\cdot|\cdot\rangle$, satisfying the following properties:

(a) $\langle\varphi|\psi + \eta\rangle = \langle\varphi|\psi\rangle + \langle\varphi|\eta\rangle\,, \quad \forall(\varphi, \psi, \eta) \in X \times X \times X;$

(b) $\langle\varphi|\lambda\psi\rangle = \lambda\,\langle\varphi|\psi\rangle\,, \quad \forall(\varphi, \psi) \in X \times X$ and $\lambda \in \mathbb{K}$;

(c) $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*\,, \quad \forall(\varphi, \psi) \in X \times X;$

(d) $\langle\varphi|\varphi\rangle \geqslant 0, \quad \forall\varphi \in X;$

(e) $\langle\varphi|\varphi\rangle = 0$ if and only if $\varphi = 0$.

Several remarks are in order.

Remark. Items (a) and (b) implies that the inner product is *linear* on the *second* component, namely:

$$\boxed{\langle\varphi|\lambda\psi + \mu\eta\rangle = \lambda\,\langle\varphi|\psi\rangle + \mu\,\langle\varphi|\eta\rangle\,,}$$

for all $(\varphi, \psi, \eta) \in X\times X\times X$ and for all $(\lambda, \mu) \in \mathbb{K}\times\mathbb{K}$. Item (c) together with items (a) and (b) implies that the inner product in general is *conjugate-linear* (or anti-linear) on the *first* component, namely

$$\boxed{\langle\lambda\varphi + \mu\eta|\psi\rangle = \lambda^*\,\langle\varphi|\psi\rangle + \mu^*\,\langle\varphi|\eta\rangle\,,}$$

for all $(\varphi, \psi, \eta) \in X \times X \times X$ and for all $(\lambda, \mu) \in \mathbb{K} \times \mathbb{K}$. Of course, if $\mathbb{K} = \mathbb{R}$, $\lambda^* = \lambda$, $\mu^* = \mu$ and the inner product becomes linear also on the first component (thus, it is *bi*linear); but this is not the case if $\mathbb{K} = \mathbb{C}$, where complex conjugation appears.

Remark. Item (b) is a matter of choice. Some authors prefer a different convention:

$$\langle \lambda\varphi|\psi\rangle = \lambda\,\langle\varphi|\psi\rangle\,, \quad \forall(\varphi,\psi)\in X\times X \text{ and } \lambda\in\mathbb{K}\,;$$

with this convention, the inner product would become linear on the first component and conjugate-linear on the second one. The convetion of having the inner product linear on the second component is the one most often employed by physicists, and the one used in this notes.

Remark. Regarding items (d) and (e), one may wonder what does mean $\langle\varphi|\varphi\rangle \geqslant 0$, since we expect $\langle\varphi|\varphi\rangle \in \mathbb{K}$, and if $\mathbb{K} = \mathbb{C}$ it might seem that the inequlity does not make sense. Actually, from item (c)

$$\langle\varphi|\varphi\rangle = \langle\varphi|\varphi\rangle^*\,, \quad \forall\varphi\in X\,,$$

Remark. Some authors prefer the notation (φ,ψ) instead of $\langle\varphi|\psi\rangle$; the notation $\langle\varphi|\psi\rangle$ is closer to the one uses by physicists and it is the first step towards the introduction of Dirac's notation. (Dirac's notation is more than simply writing the inner product this way; we will discuss this point in connection with the spectral theorem of linear operators.) In Dirac notation the vector ψ is denoted by $|\psi\rangle$, and it is called "ket"; there is a "kind of conjugation" (more on this later on) that converts the analogous "ket" $|\varphi\rangle$ to a so-called "bra" $\langle\varphi|$ and the inner product is considered as a product between a "bra" and a "ket" (resulting in a "braket"!). Of course, this is just a naming convention.

DEFINITION 2.6 (INNER PRODUCT SPACE): A "inner product space" over $\mathbb{K}$ is a pair $(X,\langle\cdot|\cdot\rangle)$, where X is a vector space over $\mathbb{K}$ and $\langle\cdot|\cdot\rangle : X\times X\to\mathbb{K}$ is an inner product on X.

Inner product space

The following lemma will be useful later on.

LEMMA 2.2: Let $(X,\langle\cdot|\cdot\rangle)$ be a inner product space over $\mathbb{K}$. If $\psi = 0$, then

$$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle = 0\,, \tag{2.5}$$

for every $\varphi\in X$.

The statement of the lemma itself looks rather obvious. However, a technical proof is given below. The linearity of the inner product is a key ingredient.

Proof. By linearity of the inner product,

$$\langle\varphi|\psi+\psi\rangle = \langle\varphi|\psi\rangle + \langle\varphi|\psi\rangle\,, \quad \forall(\varphi,\psi)\in X\times X\,.$$

In particular, if $\psi = 0$ we have $\psi+\psi=\psi$ and

$$\langle\varphi|\psi+\psi\rangle = \langle\varphi|\psi\rangle\,, \quad \forall\varphi\in X \text{ and } \psi = 0\,.$$

Thus

$$\langle\varphi|\psi\rangle + \langle\varphi|\psi\rangle = \langle\varphi|\psi\rangle\,, \quad \forall\varphi\in X \text{ and } \psi = 0\,,$$

which is an equation in $\mathbb{K}$ for the unknown $\langle\varphi|\psi\rangle$, whose only solution is $\langle\varphi|\psi\rangle = 0$. ■

Any inner product space is naturally endowed with a norm coming from the inner product. Let $(X,\langle\cdot|\cdot\rangle)$ be an inner product space over $\mathbb{K}$. Let $\|\cdot\| : X\to\mathbb{R}$ defined by

Norm induced by inner product

$$\boxed{\|\psi\| = \sqrt{\langle\psi|\psi\rangle}, \quad \forall\psi\in X\,.} \tag{2.6}$$

Observe that such $\|\cdot\|$ in eq. (2.6) is well-defined, since $\langle\psi|\psi\rangle \geqslant 0$ for every $\psi\in X$. The square root is not ambigous: it is not a square root of a complex number; it is the square root of a positive real number, we don't need to specify a branch for the square root function. We shall prove in a moment that $\|\cdot\|$ is actually a norm on X, this justify the notation $\|\cdot\|$. Such norm is the "norm induced by the inner product". Before proving this, we need a preliminary but extremely important result, which goes under the name of Cauchy-Schwarz inequality.

Cauchy-Schwarz inequality is of major importance. It is a key ingredient in several proofs of functional analysis. It has important implications also outside the realm of analysis. For example, the general formulation of the Heisenberg uncertainty principle in quantum mechanics (or the analogous time-bandwidth uncertainty principle for temporal signal transmission) is derived using the Cauchy-Schwarz inequality.

Cauchy-Schwarz inequality

THEOREM 2.3 (CAUCHY-SCHWARZ INEQUALITY): Let $(X, \langle\cdot|\cdot\rangle)$ be an inner product space. The following holds:

$$\boxed{|\langle\varphi|\psi\rangle| \leqslant \|\varphi\| \, \|\psi\| \,, \quad \forall(\varphi, \psi) \in X \times X \,.} \tag{2.7}$$

Remark. In eq. (2.7) we are using the definition eq. (2.6) but it is important to emphasize that we are *not* using (in both the statement and in the proof of Cauchy-Schwarz inequality) the fact that eq. (2.6) is a norm. We don't know at this point that eq. (2.6) defines a norm, we will prove that in the next theorem, using the Cauchy-Schwarz inequality.

Proof. We distinguish two cases. If $\psi = 0$, $\langle\varphi|\psi\rangle = 0$ (see eq. (2.5)) and $\langle\psi|\psi\rangle = \|\psi\|^2 = 0$ (by definition of the inner product), thus the inequality is satisfied.

Let us now consider $\psi \neq 0$. For every $\lambda \in \mathbb{K}$, we have

$$\langle\varphi + \lambda\psi|\varphi + \lambda\psi\rangle = \langle\varphi|\varphi\rangle + \lambda \langle\varphi|\psi\rangle + \lambda^* \langle\psi|\varphi\rangle + \lambda^*\lambda \langle\psi|\psi\rangle \,.$$

By item (d) in **??**, the left-hand side of this equation is $\langle\varphi + \lambda\psi|\varphi + \lambda\psi\rangle \geqslant 0$ and it is zero if and only if $\varphi + \lambda\psi = 0$. Thus

$$\langle\varphi|\varphi\rangle + \lambda \langle\varphi|\psi\rangle + \lambda^* \langle\psi|\varphi\rangle + \lambda^*\lambda \langle\psi|\psi\rangle \geqslant 0 \,,$$

for every $\varphi \in X$, $\psi \in X\backslash\{0\}$ and $\lambda \in \mathbb{K}$. Choose

$$\lambda = -\frac{\langle\varphi|\psi\rangle^*}{\langle\psi|\psi\rangle}, \quad \psi \neq 0 \,,$$

which makes sense since we are discussing the case $\psi \neq 0$. Plugin into the previosu equation yields

$$\langle\varphi|\varphi\rangle - \frac{\langle\varphi|\psi\rangle^*}{\langle\psi|\psi\rangle} - \frac{\langle\varphi|\psi\rangle}{\langle\psi|\psi\rangle} \langle\varphi|\psi\rangle^* + \frac{\langle\varphi|\psi\rangle^*}{\langle\psi|\psi\rangle} \frac{\langle\varphi|\psi\rangle}{\cancel{\langle\psi|\psi\rangle}} \cancel{\langle\psi|\psi\rangle} \geqslant 0 \,,$$

hence

$$\langle\varphi|\varphi\rangle - + \frac{\langle\varphi|\psi\rangle^* \langle\varphi|\psi\rangle}{\langle\psi|\psi\rangle} \geqslant 0 \,,$$

from which it follows

$$\langle\varphi|\varphi\rangle \langle\psi|\psi\rangle \geqslant |\langle\varphi|\psi\rangle|$$

(using the fact that $\langle\psi|\psi\rangle > 0$). Taking the square root of both sites (notice that both sides are surely positive) yields the expected result. ∎

THEOREM 2.4: Let $(X, \langle\cdot|\cdot\rangle)$ be an inner product space over $\mathbb{K}$. $(X, \|\cdot\|)$ with $\|\cdot\|$ defined by eq. (2.6) is a normed vector space over $\mathbb{K}$.

Proof. We need to check that eq. (2.6) makes sense and that it satisfies items (a) to (d) of the definition of the norm. ∎

Continuity of inner product

THEOREM 2.5: Let $(X, \langle\cdot|\cdot\rangle)$ be an inner product space over $\mathbb{K}$. $\langle\cdot\rangle\cdot$ is a continuous function of both arguments (with respect to the metric induced by the inner product).

Proof. Let us discuss the continuity on the second argument (the continuity on the first argument can be handled in a similar way).

We need to show that $\forall(\varphi, \psi) \in X \times X$ the following holds: for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $\eta \in X$, if $d(\psi, \eta) < \delta$ then $\tilde{d}(\langle\varphi|\psi\rangle, \langle\varphi|\eta\rangle) < \varepsilon$, where $\tilde{d}$ is the euclidean distance in $\mathbb{R}$.

Notice that $d(\psi, \eta) < \delta$ means

$$\|\psi - \eta\| < \delta\,,$$

and $\tilde{d}(\langle\varphi|\psi\rangle, \langle\varphi|\eta\rangle) < \varepsilon$ means

$$|\langle\varphi|\psi\rangle - \langle\varphi|\eta\rangle| < \varepsilon\,.$$

Using linearity of the inner product and Cauchy-Schwarz inequality yields

$$|\langle\varphi|\psi\rangle - \langle\varphi|\eta\rangle| = |\langle\varphi|\psi - \eta\rangle| \leqslant \|\varphi\|\,\|\psi - \eta\| \leqslant \delta\,\|\varphi\|\,.$$

If $\varphi = 0$, the result is 0 and we are done. Otherwise, if $\varphi \neq 0$, it is enough to choose $0 < \delta < \epsilon/\|\varphi\|$. ∎

As a consequence of the continuity of the inner product, if $(X, \langle\cdot|\cdot\rangle)$ is a inner product space and $(\psi_k)_{k\in\mathbb{N}}\colon \mathbb{N} \to X$ is a sequence in X such that $\sum_{k\in\mathbb{N}} \psi_k$ is convergent to some $\psi \in X$, i. e.,

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \psi_k = \psi\,,$$

then

$$\langle\varphi|\psi\rangle = \langle\varphi|\lim_{n\to+\infty} \sum_{k=1}^{n} \psi_k\rangle = \lim_{n\to+\infty} \langle\varphi|\sum_{k=1}^{n} \psi_k\rangle = \lim_{n\to+\infty} \sum_{k=1}^{n} \langle\varphi|\psi_k\rangle\,. \tag{2.8}$$

EXERCISE 2.4: Justify all steps in eq. (2.8).

Hilbert space

DEFINITION 2.7 (HILBERT SPACE): Let $(X, \langle\cdot|\cdot\rangle)$ be a inner product space over $\mathbb{K}$. If the metric space (X, d) (with the distance arising from the inner product) is complete, $(X, \langle\cdot|\cdot\rangle)$ is called an "Hilbert space".

In short: Banach spaces are complete normed vector spaces and Hilbert spaces are complete inner product spaces. Hilbert spaces are a special case of Banach space, where the norm comes from an inner product. The underlying iner product inducing the norm introduces extra features (in particular, some related to the notion of orthogonality) which are not present in general Banach spaces (see next section for details on this).

In the next sections, we will focus on Hilbert spaces only, and we will not consider incomplete inner product space. On reason for this is that any incomplete metric space admit a completion (more on this later, when discussing BLT theorem).

Hereafter, if $(X, \langle\cdot|\cdot\rangle)$ is an inner product space, if not explicitly stated, we will also intend that $\|\cdot\|$ denotes the norm induced by the inner product, and that the convergence, etc refers to the distance induced by that norm.

Is it possible to distinguish if in a normed vector space, the norm is coming from some underlying inner space? The answer is yes, and we are going to prove immediately an interesting criterion to perform this check.

The following result is basic to establish the later theorem. It will also be useful later, when discussing positive operators.

Lemma 2.3 (polarization identity): Let $(X, \langle\cdot|\cdot\rangle)$ be a inner product space. In the case $\mathbb{K}=\mathbb{R}$,

$$\langle\varphi|\psi\rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right), \quad \forall(\varphi,\psi)\in X\times X. \tag{2.9}$$

In the case $\mathbb{K}=\mathbb{C}$,

$$\langle\varphi|\psi\rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right) + \frac{i}{4}\left(\|\varphi+i\psi\|^2 + \|\varphi-i\psi\|^2\right), \quad \forall(\varphi,\psi)\in X\times X. \tag{2.10}$$

Remark. The polarization identity can be stated in the following concise way:

$$\langle\varphi|\psi\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \left\|\varphi + i^k\psi\right\|^2. \tag{2.11}$$

Proof. ■

The following result is called Jordan-von Neumann theorem.*

Jordan-von Neumann theorem

Parallelogram law

Theorem 2.6 (Jordan-von Neumann theorem): Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. The norm $\|\cdot\|$ comes from an inner product *if and only if* the following identity holds:

$$\|\varphi+\psi\|^2 + \|\varphi-\psi\|^2 = 2\|\varphi\|^2 + \|\psi\|^2, \quad \forall(\varphi,\psi)\in X\times X. \tag{2.12}$$

Equation (2.12) is known as “parallelogram law”.

Proof. Let’s prove first that if $(X, \langle\cdot|\cdot\rangle)$ is a Hilbert space, then the norm induced by the inner product fullfills eq. (2.12). This is the straightforward part of the proof. By definition of the norm induced by the inner product,

$$\begin{aligned}
\|\varphi+\psi\| &= \langle\varphi+\psi|\varphi+\psi\rangle \\
&= \langle\varphi|\varphi\rangle + \langle\varphi|\psi\rangle + \langle\psi|\varphi\rangle + \langle\psi|\psi\rangle \\
&= \left\|\varphi^2\right\| + 2\Re\langle\varphi|\psi\rangle + \|\psi\|^2 \\
\|\varphi-\psi\| &= \langle\varphi-\psi|\varphi-\psi\rangle \\
&= \langle\varphi|\varphi\rangle - \langle\varphi|\psi\rangle - \langle\psi|\varphi\rangle + \langle\psi|\psi\rangle \\
&= \left\|\varphi^2\right\| - 2\Re\langle\varphi|\psi\rangle + \|\psi\|^2
\end{aligned}$$

for all $(\varphi,\psi)\in X\times X$; summing the two we have eq. (2.12).

Now, let’s prove that if $(X, \|\cdot\|)$ is a Banach space whose norm satisfies eq. (2.12) then the norm comes from an inner product. It is possible to recover the underlying inner product by means of the polarization identity. Since the polarization identity takes two differen forms, depending on whether $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, we should distinguish two cases. We will discuss the case $\mathbb{K}=\mathbb{C}$, the case $\mathbb{C}=\mathbb{R}$ is analogous and the details are left to the Reader.

In the Banach space $(X, \|\cdot\|)$ it is always possible to define an application $\langle\cdot|\cdot\rangle : X\times X \to \mathbb{K}$ by letting

$$\langle\varphi|\psi\rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right) + \frac{i}{4}\left(\|\varphi+i\psi\|^2 - \|\varphi-i\psi\|^2\right), \quad \forall(\varphi,\psi)\in X\times X.$$

We need to prove that this application indeed is a inner product on X, i. e., we need to prove that it satisfies items (a) to (e) in **??**.

Ad (a).
Ad (b).
Ad (b).
Ad (c).
Ad (d).
Ad (e). ■

* See Jordan and von Neumann, 1935.

Subsequent authors after Jordan and von Neumann have found norm conditions weaker than eq. (2.12) which characterize inner product spaces amongst normed vector spaces. See, e. g., Reznick [1978] and references therein; a recent paper is Scholtes [2010], which also contains references to the relevant literature on this subject.

2.2 CANONICAL PROTOTYPES OF BANACH AND HILBERT SPACES

2.2.1 *The Hilbert spaces* $\mathbb{R}^N$ *and* $\mathbb{C}^N$

Let $N \in \mathbb{N}\setminus\{0\}$ be any positive integer number. Consider the vector space $\mathbb{C}^N$, whose elements are N-ple of complex numbers, endowed with the usual operations of elementwise addition and elementwise multiplication of an N-ple by a complex number. For every

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}, \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix},$$

in $\mathbb{C}^N$, define an application $\langle\cdot|\cdot\rangle : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ as follows:

$$\boxed{\langle \boldsymbol{\varphi}|\boldsymbol{\psi}\rangle = \sum_{k=1}^{N} \varphi_k^* \psi_k\,.} \tag{2.13}$$

In a similar manner one can define the usual (Euclidean) inner product on $\mathbb{R}^N$, which takes the same form of eq. (2.13) without complex conjugation (because in that case all numbers involved are real numbers).

THEOREM 2.7: $(\mathbb{C}^N, \langle\cdot|\cdot\rangle)$, where $\langle\cdot|\cdot\rangle$ is defined by eq. (2.13), is an inner product space.

Proof. It is straightforward to show that $\langle\cdot|\cdot\rangle$ defined in eq. (2.13) satifies Items (a) to (e) of definition 2.5. ∎

THEOREM 2.8: $(\mathbb{C}^N, \langle\cdot|\cdot\rangle)$, where $\langle\cdot|\cdot\rangle$ is defined by eq. (2.13), is an Hilbert space.

Proof. We need to check completeness. ∎

2.2.2 *The Hilbert space* l^2

Let l^2 be the space of all and only the sequences $(\varphi_k)_{k\geqslant 1}$ in $\mathbb{K}$ such that the series $\sum_{k=1}^{+\infty} |\varphi_k|^2$ is convergent in $\mathbb{R}$.

It is easy to show that, with the usual addition and multiplication for an element of $\mathbb{K}$, l^2 is a vector space over $\mathbb{K}$. Define

$$\boxed{\langle \varphi|\psi\rangle = \sum_{k=1}^{+\infty} \varphi_k^* \psi_k\,,} \tag{2.14}$$

for every $\varphi = (\varphi_k)_{k\geqslant 1}$ and $\psi = (\psi_k)_{k\geqslant 1}$ in l^2.

THEOREM 2.9: $(l^2, \langle\cdot|\cdot\rangle)$, where $\langle\cdot|\cdot\rangle$ is defined by eq. (2.14), is an inner product space.

Proof. Similar to the proof that $\mathbb{C}^N$ is an inner product space, with some care to handle the limits of the various sums. ∎

THEOREM 2.10: $(l^2, \langle\cdot|\cdot\rangle)$, where $\langle\cdot|\cdot\rangle$ is defined by eq. (2.14), is an Hilbert space.

2.2.3 *The Hilbert* $L^2(\Omega)$

Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary non-empty subset of $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and let $\tilde{L}^2(\Omega)$ be the set consisting of all and only the functions defined on Ω and taking values on $\mathbb{K}$, i. e., $\psi\colon \Omega \to \mathbb{K}$, such that their square modulus is Lebesgue-integrable over Ω, i. e., the following integral exists and it is finite (in the sense of Lebesgue):

$$\int_\Omega |\psi|^2 .$$

Remark. Even if $\mathbb{K} = \mathbb{C}$, the modulus is real-valued so the integral above is always an integral of a real function.

It is not difficult to make $\tilde{L}^2(\Omega)$ a vector space. Our goal would be to endow $\tilde{L}^2(\Omega)$ also with an inner product. A problem would arise however, due to the fact that there are non-zero functions in $\tilde{L}^2(\Omega)$ whose square modulus integral is zero, and this would ultimately make not possible to satisfy property (e) of the inner product.

To overcome this difficulty, a slightly more techical construction is needed. The formal construction involves working with equivalence classes and proceeds as follows. First, we will equip $\tilde{L}^2(\Omega)$ with an equivalence relation, which allows to identify two functions whenever they are equal almost everywhere (i. e., when they differ only on a set of Lebesgue zero measure). Then, the quotient space with respect to this equivalence relation can be made a vector space over $\mathbb{K}$ (with suitable definitions of addition and multiplication by a element of $\mathbb{K}$) and can be equipped with an inner product. We will see that, with this inner product, the quotient space becomes an Hilbert space.

As a preliminary step, consider the subset $M \subseteq \tilde{L}^2(\Omega)$ defined as follows:

$$M = \left\{ \psi \in \tilde{L}^2(\Omega) \;\middle|\; \int_\Omega |\psi|^2 = 0 \right\}$$

The following lemma is not strictly necessary to be mentioned, but it makes the next proofs clearer.

Lemma 2.4: For every $(\varphi, \psi) \in M \times M$ and for every $\lambda \in \mathbb{C}$, $\varphi + \psi$ and $\lambda\psi$ belong to M.

In words: any linear combination of vectors of M is an element of M itself.

Proof. By linearity of the integral,

$$\int_\Omega |\lambda\psi|^2 = \int_\Omega |\lambda|^2\, |\psi|^2 = |\lambda|^2 \int_\Omega |\psi|^2 ,$$

thus $\lambda\psi \in M$. Using the monotonicity of the integral,

$$0 \leqslant \int_\Omega |\varphi + \psi|^2 \leqslant \int_\Omega |\psi|^2 + \int_\Omega |\varphi|^2 = 0 ,$$

thus $\psi + \varphi \in M$. ■

Let $\sim$ be the relation on $\tilde{L}^2(\Omega)$ defined in this way: for every $(\varphi, \psi) \in \tilde{L}^2(\Omega) \times \tilde{L}^2(\Omega)$, $\varphi \sim \psi$ if $\varphi - \psi \in M$. In words: $\varphi \sim \psi$ if the two functions agree outside a set of (Lebesgue) zero measure.

Lemma 2.5: $\sim$ is an equivalence relation over $\tilde{L}^2(\Omega)$.

Proof. Let's check the equivalence relation properties:

REFLEXIVITY : for every $\psi \in \tilde{L}^2(\Omega)$, $\psi - \psi = 0$ (where 0 denotes the identically zero function, i. e., the function which takes value zero everywhere on Ω) and thus $\psi - \psi \in M$;

SYMMETRY : for every ψ and φ in $\tilde{L}^2(\Omega)$, if $\psi \sim \varphi$ also $\varphi \sim \psi$; in fact, $\psi \sim \varphi$ means $\psi - \varphi \in M$ and, for lemma 2.4, also $\varphi - \psi = (-1)(\psi - \varphi) \in M$;

TRANSITIVITY : for every ψ, φ and η in $\tilde{L}^2(\Omega)$, if $\psi \sim \eta$ and $\eta \sim \varphi$, then $\psi \sim \varphi$; in fact, if $\psi - \eta \in M$ and $\eta - \psi \in M$, then also $(\psi - \eta) + (\eta - \varphi) \in M$ by lemma 2.4.

■

We introduce the following notation: for every ψ in $\tilde{L}^2(\Omega)$, let $[\psi]$ denote the equivalence class of ψ under $\sim$; furthermore, let $L^2(\Omega)$ denote the quotient space (i. e., the space of all possibile equivalence classes) of $\tilde{L}^2(\Omega)$ by $\sim$.

Define addition and multiplication by a scalar constant in $L^2(\Omega)$ in the following way. For every $[\psi]$ and $[\varphi]$ in $L^2(\Omega)$ and $\lambda \in \mathbb{K}$, put

$$\begin{aligned} [\psi] + [\varphi] &= [\psi + \varphi] \,, \\ \lambda[\psi] &= [\lambda\psi] \,. \end{aligned}$$

First of all, let us check that these operations are well-defined.

LEMMA 2.6: For every ψ and $\tilde{\psi}$ in $[\psi]$ and for every φ, $\tilde{\varphi}$ in $[\varphi]$,

$$\begin{aligned} [\psi] + [\varphi] &= [\tilde{\psi}] + [\tilde{\psi}] \,, \\ \lambda[\psi] &= \lambda[\tilde{\psi}] \,. \end{aligned}$$

Proof. There exists η and ξ in M such that $\tilde{\psi} = \psi + \eta$ and $\tilde{\varphi} = \varphi + \xi$; then, $\tilde{\psi} + \tilde{\varphi} = (\psi + \eta) + (\varphi + \xi) = (\psi + \varphi) + (\eta + \xi)$, where $\eta + \xi \in M$. Thus $(\tilde{\psi} + \tilde{\varphi}) - (\psi + \varphi) = \eta + \xi \in M$, $(\tilde{\psi} + \tilde{\varphi}) \sim (\psi + \varphi)$. In the same way one proves that $\lambda\tilde{\psi} \sim \lambda\psi$. ■

THEOREM 2.11: $(L^2(\Omega), +, \cdot)$ is a vector space over $\mathbb{K}$, where the addition and multiplication are those defined above.

It is a standard result of linear algebra that the quotient space with the above definitions of addition and multiplication is a vector space. More generally, this result applies to every quotient space, no matter what is the underlying set and what is the specific equivalence relation. We leave the proof of theorem 2.11 as exercise.

EXERCISE 2.5: Prove theorem 2.11. Generalize the proof to arbitrary quotient spaces under a generic equivalence relation.

In $L^2(\Omega)$, define an application $\langle\cdot|\cdot\rangle$ by letting

$$\boxed{\langle[\varphi]\,|\,[\psi]\rangle = \int_\Omega \varphi^*\psi \,,} \tag{2.15}$$

where on the right hand side φ is any function belonging to $[\varphi]$ and ψ is any function belonging to $[\psi]$.

Remark. In order to simplify the notation, we will denote the equivalence class $[\psi]$ containing ψ by ψ itself.

Remark. Given $\varphi\colon \Omega \to \mathbb{K}$, the function $\varphi^*\colon \Omega \to \mathbb{K}$ is defined by $\varphi^*(x) = (\varphi(x))^*$ for all $x \in \Omega$.

We need to show that such $\langle\cdot|\cdot\rangle$ is well-defined, i. e.: (a) show that the integral exists and is convergent, and (b) show that the integral is independent from the choice of $\psi \in [\psi]$ and $\varphi \in [\varphi]$.

Let us recall the following theorem from the theory of Lebesgue integration.

THEOREM 2.12: Let $\varphi\colon\Omega\to\mathbb{K}$ be Lebesgue integrable on Ω and let $\psi\colon\Omega\to\mathbb{K}$ be any function satisfying

$$|\psi(x)|\leqslant|\varphi(x)|\,,$$

for all $x\in\Omega$. Then, ψ is Lebesgue integrable on Ω.

To show that the integral exists, notice that for every $(z,w)\in\mathbb{C}\times\mathbb{C}$, we have

$$|zw|\leqslant\frac{1}{2}|z|^2+\frac{1}{2}|w|^2\,; \tag{2.16}$$

in fact,

$$0\leqslant(|z|-|w|)^2=|z|^2+|w|^2-2\,|zw|\,.$$

Thus,

$$|\varphi^*(x)\psi(x)|\leqslant\frac{1}{2}|\varphi(x)|^2+\frac{1}{2}|\psi(x)|^2\,,\quad\forall x\in\Omega\,,$$

thus the integral of $|\varphi^*(x)\psi(x)|$ exists and is absolutely convergent, and this ensures the convergence of the integral of $\varphi^*(x)\psi(x)$.

EXERCISE 2.6: Show that the integral in eq. (2.15) is independent from the choice of $\psi\in[\psi]$ and $\varphi\in[\varphi]$.

THEOREM 2.13: $(L^2(\Omega),\langle\cdot|\cdot\rangle)$, where $\langle\cdot|\cdot\rangle$ is defined by eq. (2.15), is a inner product space.

Proof. We need to show that $\langle\cdot|\cdot\rangle$ defined by eq. (2.15) satisfies items (a) to (e) in **??**.
Linearity of $\langle\cdot|\cdot\rangle$ follows immediately from the linearity of the integral:

$$\begin{aligned}\langle\varphi|\psi+\eta\rangle&=\int_\Omega\varphi^*\,(\psi+\eta)=\int_\Omega\varphi^*\psi+\int_\Omega\varphi^*\eta=\langle\varphi|\psi\rangle+\langle\varphi|\eta\rangle\\ \langle\varphi|\lambda\psi\rangle&=\int_\Omega\varphi^*\,(\lambda\psi)=\lambda\int_\Omega\varphi^*\psi=\lambda\,\langle\varphi|\psi\rangle\end{aligned}$$

Furthrmore,

$$\langle\psi|\varphi\rangle=\int_\Omega\psi^*\varphi=\int_\Omega(\psi\varphi^*)^*=\left(\int_\Omega\varphi^*\psi\right)^*=\langle\varphi|\psi\rangle^*$$

The fact that complex conjugation can be moved outside the integration symbol is rigorously justified as follows. Let $\psi\colon\Omega\to\mathbb{C}$ be a complex-valued function defined on Ω, and set $\psi_1,\psi_2\colon\Omega\to\mathbb{R}$ by letting $\psi_1=\Re\psi$ and $\psi_2=\Im\psi$. By definition, ψ is Lebesgue integrable on Ω if and only if ψ_1 and ψ_2 are Lebesgue integrable on Ω and in that case we put

$$\int_\Omega\psi=\int_\Omega\psi_1+i\int_\Omega\psi_2$$

Clearly, if ψ is Lebesgue integrable, ψ^* is Lebesgue integrable and

$$\begin{aligned}\int_\Omega\psi^*&=\int_\Omega\psi_1-i\int_\Omega\psi_2\\ &=\left(\int_\Omega\psi\right)^*.\end{aligned}$$

Finally, the monotonicity of the integral ensures that

$$\langle\psi|\psi\rangle=\int_\Omega\psi^*\psi=\int_\Omega|\psi|^2\geqslant0\,,$$

and $\int_\omega|\psi|^2=0$ if and only if ψ belongs to the class of equivalence of the function which is identically zero on Ω, i. e., $\psi=0$. ∎

THEOREM 2.14 (RIESZ-FISCHER): $(L^2(\Omega), \langle\cdot|\cdot\rangle)$ is an Hilbert space.

Proof. We need to check completeness. ■

2.2.4 *The Banach spaces* l^p *and* $L^p(\Omega)$

2.3 ORTHOGONALITY

The notion of inner product allows to naturally equip an inner product space with a notion of orthogonality between vectors Some of the following results (e. g., Pitagorean theorem) can be used to some extend to export the notion of orthogonality to general Banach spaces, but here we will restrict ourselves to Hilbert spaces.

DEFINITION 2.8: Let $(X, \langle\cdot,\cdot\rangle)$ be any inner product space. For every $(\varphi, \psi) \in X \times X$, φ and ψ are said to be mutually "orthogonal" if

Orthogonality of two vectors

$$\langle\varphi|\psi\rangle = 0\,. \tag{2.17}$$

The extension to a (possibly not-countable) set of vectors is trivial and it is formalized by the following definition.

DEFINITION 2.9: Let $(X, \langle\cdot,\cdot\rangle)$ be any inner product space and $W \subseteq X$ a non-empty subset of X. W is a set of mutually orthogonal vectors if

Orthogonality of a set of vectors

$$\langle\varphi|\psi\rangle = 0, \quad \forall(\varphi,\psi) \in W \times W \text{ with } \varphi \neq \psi\,. \tag{2.18}$$

If moreover

$$\langle\varphi\rangle = 1, \quad \forall\varphi \in W \tag{2.19}$$

the set W is said to be "orthonormal".

In particular, given a sequence $\mathbb{N} \to X$ of vectors of X, we say that it is "orthonormal" if the image set is an orthonormal set.

THEOREM 2.15 (PYTHAGOREAN THEOREM): Let $(X, \langle\cdot,\cdot\rangle)$.

1. for every orthogonal pair of vectors φ and ψ in X,
$$\|\varphi+\psi\|^2 = \|\varphi\|^2 + \|\psi\|^2\,;$$
2. for every orthogonal finite set of $N \in \mathbb{N}$ vectors $\varphi_1, \varphi_2, \ldots \varphi_N$ in X,
$$\left\|\sum_{k=1}^{N} \varphi_k\right\|^2 = \sum_{k=1}^{N} \|\varphi_k\|^2$$
3. if $(\varphi_k)_{k\in\mathbb{N}}$ is a sequence $\mathbb{N} \to X$ of mutually orthogonal vectors of X, then $\sum_{k=1}^{+\infty} \varphi_k$ is convergent in X *if and only if* $\sum_{k=1}^{+\infty} \|\varphi_k\|^2$ is convergent in $\mathbb{R}$ and in that case we have the following identity:
$$\left\|\sum_{k=1}^{+\infty} \varphi_k\right\|^2 = \sum_{k=1}^{+\infty} \|\varphi_k\|^2\,.$$

Proof. In the first case the proof is straightforward, we have already found the same expression in the proof of the parallelogram identity, we have by direct computation that

$$\begin{aligned}\|\varphi+\psi\|^2 &= \langle\varphi+\psi|\varphi+\psi\rangle \\ &= \langle\varphi|\varphi\rangle + \langle\varphi|\psi\rangle + \langle\psi|\varphi\rangle + \langle\psi|\psi\rangle \\ &= \|\varphi\|^2 + 2\underbrace{\Re\langle\varphi|\psi\rangle}_{=0} + \|\psi\|^2 \\ &= \|\varphi\|^2 + \|\psi\|^2\,.\end{aligned}$$

In the second case, the proof is by induction on N. The case $N = 2$ has been shown explicitly in the first part of the proof. Now, let's check that

$$\left\| \sum_{k=1}^{N} \varphi_k \right\|^2 = \sum_{k=1}^{N} \|\varphi_k\|^2$$

for a given N implies

$$\left\| \sum_{k=1}^{N+1} \varphi_k \right\|^2 = \sum_{k=1}^{N+1} \|\varphi_k\|^2$$

for $N + 1$. We have

The third case shows a classic trick of using the Cauchy criterion of convergence.

■

2.4 COMPLETE ORTHONORMAL SETS

DEFINITION 2.10: Let $(X, \langle \cdot | \cdot \rangle)$ a Hilbert space and $(\varphi_k)_k$ a family of vectors in X. $(\varphi_k)_k$ is called a "complete orthonormal set" if

1. $(\varphi_k)_k$ is an orthonormal set in X;
2. for every $\psi \in X$,

   $$\langle \varphi_k | \psi \rangle = 0, \quad \forall k$$

   if and only if $\psi = 0$.

2.5 THE FOURIES SERIES IN $L^2(\Omega)$

2.6 LINEAR OPERATORS: FIRST PROPERTIES

2.7 ISOMORPHISM

2.8 RIESZ THEOREM

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3 THE RULES OF THE GAME

4 ONE-DIMENSIONAL QUANTUM SYSTEMS

Part II

THE CORE

5 ANGULAR MOMENTUM

6 HYDROGEN ATOM

7 PERTURBATION THEORY

8 SCATTERING

Part III

ADVANCED TOPICS

9 PATH INTEGRALS

10 SEMICLASSICAL QUANTUM MECHANICS

11 SUPERSYMMETRIC QUANTUM MECHANICS

12 SECOND QUANTIZATION FORMALISM

Part IV

APPENDICES

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