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Abstract

Circuit Quantum Electrodynamics

Lev Samuel Bishop

2010

Circuit Quantum Electrodynamics (cQED), the study of the interaction between superconducting circuits behaving as artificial atoms and 1-dimensional transmission-line resonators, has shown much promise for quantum information processing tasks. For the purposes of quantum computing it is usual to approximate the artificial atoms as 2-level qubits, and much effort has been expended on attempts to isolate these qubits from the environment and to invent ever more sophisticated control and measurement schemes. Rather than focussing on these technological aspects of the field, this thesis investigates the opportunities for using these carefully engineered systems for answering questions of fundamental physics. The low dissipation and small mode volume of the circuits allows easy access to the strong-coupling regime of quantum optics, where one can investigate the interaction of light and matter at the level of single atoms and photons. A signature of strong coupling is the splitting of the cavity transmission peak into a pair of resolvable peaks when a single resonant atom is placed inside the cavity—an effect known as vacuum Rabi splitting. The cQED architecture is ideally suited for going beyond this linear response effect. This thesis shows that increasing the drive power results in two unique nonlinear features in the transmitted heterodyne signal: the supersplitting of each vacuum Rabi peak into a doublet, and the appearance of additional peaks with the characteristic $\sqrt{n}$ spacing of the Jaynes–Cummings ladder. These constitute direct evidence for the coupling between the quantized microwave field and the anharmonic spectrum of a superconducting qubit acting as an artificial atom. This thesis also addresses the idea of Bell tests, which are experiments that aim to disprove certain types of classical theories, presenting a proposed method for preparing maximally entangled 3-qubit states via a 'preparation by measurement' scheme using an optimized filter on the time-dependent signal obtained via homodyne monitoring of the transmitted microwave field.

Circuit Quantum Electrodynamics

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Lev Samuel Bishop

Dissertation Director: Professor Steven M. Girvin

May 2010

Quantum Mechanics

the good old days, theorizing was like sailing between islands of experimental evidence. And, if the trip was not in the vicinity of the shoreline (which was strongly recommended for safety reasons) sailors where continuously looking forward, hoping to see land — the sooner the better.

Nowadays, some theoretical physicists (let us call them sailors) [have] found a way to survive and navigate in the open sea of pure theoretical constructions. Instead of the horizon, they look at stars, which tell them exactly where they are. Sailors are aware of the fact that the stars will never tell them where the new land is, but they may tell them their position on the globe.

Theoreticians become sailors simply bacause they just like it. Young people, seduced by capitans forming crews to go to a Nuevo El Dorando soon realize that they will spend all their life at sea. Those who do not like sailing desert the voyage, but for the true potential sailors the sea become their passion. They will probably tell the alluring and frightening truth to their students — and the proper people will join their ranks.

— Andrei Losev

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for Cleo

Acknowledgements

THIS thesis would never have been possible without the ideas, friendship, support and assistance of numerous people. Foremost among them is of course STEVE GIRVIN, who is infinitely more patient than the stereotypical infinitely-patient doctoral advisor.* He is a constant source of ideas and inspiration, and I feel enormously privileged to have him as my advisor.

I owe much to JENS KOCH, who between numerous cups of coffee showed me how to write a scientific article. (I was slow to learn, so he taught me twice.)

My experimental colleague JERRY CHOW went off on his own tangent to produce the beautiful data presented in **??**. He then put up with my demands for ever more precise numbers to feed into my simulations, which he was able to provide without breaking off from his $l \times m \times n$ multitasking in l windows on m virtual desktops on n monitors.

I have bounced many a crazy-sounding idea off ANDREW HOUCK in order to gauge *exactly how crazy* the idea may be. DAVE SCHUSTER has coefficient of restitution larger than unity: my crazy ideas bounce off him and come back at me *much more crazy*. I can guarantee to overcome any mental blocks by talking to him for an hour.

ERAN GINOSSAR, ANDREAS NUNNENKAMP and LARS TORNBERG each contributed aspects of the work presented in **??**. Our weekly (later, daily) group meetings and pairwise problem-solving sessions during that time are a fond memory of mine, despite my spending much of the time in a state of confusion.

JAY GAMBETTA advocated the quantum trajectories approach that proved very fruitful in

* Akin to TeX `fill` versus `fil`.

??, and I learned much of what I know about circuit QED by talking to him and reading his papers.

I thank all of my friends, especially my friends on the 4th floor of Becton—the community that provided the experimental motivation for this thesis. It is a special environment where theorists and experimentalists can collaborate so closely.

Most importantly, I must thank my family, for their love and encouragement. It is impossible to record how grateful I am to my parents, nor can I imagine that I could have completed this thesis without the love of my wife and best friend June.

Publication list

This thesis is based in part on the following published articles:

Nomenclature

Figure 1. Julian Seymour Schwinger (February 12, 1918 – July 16, 1994)

Schwinger's own way to teach quantum mechanics

presume that all of you have already been exposed to some undergraduate course in Quantum Mechanics, one that leans heavily on de Broglie waves and the Schroedinger equation. I have never thought that this simple wave approach was acceptable as a general basis for the whole subject, and I intend to move immediately to replace it in your mind by a foundation that *is* perfectly general.

J. Schwinger, *Quantum Mechanics. Symbolism of Atomic Measurements* Schwinger:2001.

0.1 Introduction

Compared with other traditional areas of physics, quantum mechanics is not easy. It often lacks physical intuitition and it relies on heavy mathematical background from the very beginning. As we shall see shortly, topics like uncertanty principle, the role of probability etc asks immediately for a theoretical framework/formalism... Phase space is not able to capture, it does not offer the tools. Naturally leading from the very beginning to adopt .

[heading=subbibliography]

Linear operators in Hilbert spaces

Operator formulation of standard non-relativistic quantum mechanics heavily relies on the theory of linear operators in Hilber spaces. In particular, the spectral decomposition of self-adjoint operators (bounded and unbounded ones) is a key ingredient in formulating the basic rules of quantum mechanics. This chapter is aimed at providing the necessary mathematical background of functional analysis employed by non-relativistic quantum mechanics. It is a chapter on mathematics, not on quantum physics; for this reason care has been made to mathematical rigous more than what will be customary in later chapters. The Reader interested in how functional analysis applies to the formulation of quantum mechanics should jump to the next chapters. As we shall see, physicists are customary to employ Dirac's notation, a powerful mnemonic tool which naively speaking however lacks mathematical rigour; a dictionary is possible to translate Dirac notation to the rigorous theorems of functional analysis; more on this later on.

This chapter is mostly based on Reed.Simon:1980. Some other books that have been helpful while writing this chapter includes: Teschl:2009,Berberian:1976,Hutson.Pym:1980,Debnath.Mikusinski:20

0.2 Banach and Hilber spaces

Unless stated otherwise, let denote either the field of real numbers R or the field of complex numbers . (It is possible to develop the theory also for the skew-field of quaternions, but this case will not be taken into account here in order to avoid dealing with the non-commutativity of the quaternionic product.)

Definition 0.1 (norm): Let Norm X be any vector space over . A "norm" on X is any application X ⍝ R hereafter denoted by $\oint + \oint$ satisfying the following properties:

(a) $\oint \varphi \oint$ ⅄ 0, Ⓘφ ⊞ X (nonnegativity);

(b) $\oint \varphi \oint$ 0 if and only if φ 0 (faithfulness)

(c) $\oint \lambda\varphi \oint$ $\oint \lambda \oint$ $\oint \varphi \oint$, Ⓘλ ⊞ and Ⓘφ ⊞ X (positive homogeneity);

(d) $\oint \varphi \;\; \psi \oint$ ⅄ $\oint \varphi \oint$ $\oint \psi \oint$, Ⓘφ, ψ ⊞ $X \times X$ (subadditivity).

A norm on X can be viewed as a way to define a "length" of the vectors belonging to X.

Remark. **??** is redundant: it follows from **??????**. For every φ ⊞ X, **??** implies $\oint + \varphi \oint$ $\oint \varphi \oint$, from **??** it follows that $\oint \varphi + \varphi \oint$ ⅄ $\oint \varphi \oint$ $\oint \varphi \oint$, and **??** implies $\oint \varphi + \varphi \oint$ 0, so $\oint \varphi \oint$ ⅄ 0.

Remark. X is not required to be finite-dimensional. It can be infinite-dimensional. (Actually, the infinite-dimensional case is the case we are most interested on.)

A "normed vector space" is a vector space equipped with a norm, as formalized by the following definition.

Definition 0.2 (normed vector space): A Normed vector space "normed vector space" over is a pair $X, \oint + \oint$ where X is a vector space over and $\oint + \oint$ is any norm on X.

If $X, \oint + \oint$ is a normed vector space, the norm $\oint + \oint$ *induces* a metric (*i.e.*, a notion of distance) and thus a topology on X. This is proved by the following theorem.

THEOREM 0.1: Let Distance induced by a norm $X, \oint + \oint$ be any normed vector space over . Let $d \colon X \times X \bigcap R$ be the function defined by

$$\boxed{d\varphi, \psi \ \oint \varphi + \psi \oint, \quad \textcircled{I}\varphi, \psi \boxplus X \times X\,.} \tag{1}$$

Then, X, d is a metric space. The metric in **??** is called the "metric induced by the norm" on X.

Proof. Remember that, given a set X and an application $d \colon X \times X \bigcap 0, \{, d$ is called a "metric" (or "distance") on X by definition if it satisfies the following properties:

(a) $d\varphi, \psi \curlywedge 0, \quad \textcircled{I}\varphi, \psi \boxplus X \times X$ (nonnegativity);

(b) $d\varphi, \psi \ \ 0$ if and only if $\varphi \ \ \psi$ (*i.e.*, the only point of X at zero distance from ψ is ψ itself);

(c) $d\varphi, \psi \ \ d\psi, \varphi, \quad \textcircled{I}\varphi, \psi \boxplus X \times X$ (the distance is a symmetric function of its arguments);

(d) $d\varphi, \psi \curlywedge d\psi, \eta \ \ d\eta, \psi, \quad \textcircled{I}\varphi, \psi, \eta \boxplus X \times X \times X$ (the so-called "triangle inequality").

Let's show that d in **??** satisfies **????????** above. **??** follows from property **??** of the norm. **??** follows from property **??** of the norm, in fact $\oint \varphi + \psi \oint \ \ 0$ if and only if $\varphi + \psi \ \ 0$, *i.e.*, if and only if $\varphi \ \ \psi$. **??** follows from property **??** of the norm, since

$$d\varphi, \psi \oint \varphi + \psi \oint \ \oint + \psi + \varphi \oint \ \oint + 1 \oint \ \oint \psi + \varphi \oint \ \oint \psi + \varphi \oint \ d\psi, \varphi, \ \textcircled{I}\varphi, \psi \boxplus X \times X\,.$$

?? follows from property **??** of the norm, since

$$\begin{aligned} d\varphi, \psi \ \oint \varphi + \psi \oint \ \oint \varphi + \eta \ \eta + \psi \oint \curlywedge \oint \varphi + \eta \oint \ \oint \eta + \psi \oint \\ d\varphi, \eta \ \ d\eta, \psi, \quad \textcircled{I}\varphi, \psi, \eta \boxplus X \times X \times X\,. \end{aligned}$$

This completes the proof. ■

The metric induced by a norm fulfilles the following extra properties, the proof of which is straightforward and is left to the Reader as exercise:

1. $d\varphi \ \ \eta, \psi \ \ \eta \ \ d\varphi, \psi$ (translation invariance), and

2. $d\lambda\varphi, \lambda\psi \ \oint \lambda \oint \ d\varphi, \psi$ (homogeneity),

for all $\varphi, \psi, \eta \boxplus X \times X$ and $\lambda \boxplus$.

Exercise 0.1: Prove **????** above.

?? shows that any normed vector space is naturally endowed with a notion of distance. Please notice however that a normed vector space could be equipped also with distances other than the one induced by the norm; such distances are not necessary related to the norm; furthermore, **??** is not the only one possible distance built from the norm (see **??**).

Exercise 0.2: Let $X, \oint + \oint$ be a normed vector space. Let $d\colon X \times X \cap R$ be the function defined by

$$d\varphi, \psi \quad \frac{\oint \varphi + \psi \oint}{1 \quad \oint \varphi + \psi \oint}, \quad \Phi\varphi, \psi \boxplus X \times X$$

Check whether d defines a metric on X.

Now that we have a “natural” notion of distance in normed vector spaces, *i.e.*, **??**, all the concepts defined for metric spaces applies automatically, in particular, to normed linear spaces. Among these concepts are: continuity, limits, convergence, compactness, completeness, open sets etc.

Lemma 0.1: Let Inverse triangle inequality $X, \oint + \oint$ be a normed vector space over . The following holds:

$$\Big\lfloor \oint \psi \oint + \oint \varphi \oint \Big\lfloor \curlywedge \oint \psi + \varphi \oint, \tag{2}$$

for every $\psi, \varphi \boxplus X \times X$.

Proof. The key ingredient is the triangle inequality of the norm. For all $\varphi, \psi \boxplus X \times X$, we have

$$\oint \varphi \oint \quad \oint \varphi + \psi \;\; \psi \oint \curlywedge \oint \varphi + \psi \oint \quad \oint \psi \oint$$

and

$$\oint \psi \oint \quad \oint \psi + \varphi \;\; \varphi \oint \curlywedge \oint \varphi + \psi \oint \quad \oint \psi \oint.$$

Thus,

$$\begin{gathered} \oint \varphi \oint + \oint \psi \oint \curlywedge \oint \varphi + \psi \oint \\ \oint \psi \oint + \oint \varphi \oint \curlywedge \oint \varphi + \psi \oint. \end{gathered}$$

which can be recasted in the single expression **??**. ∎

?? implies the continuity of the norm.

THEOREM 0.2: Let Continuity of the norm $X, \oint + \oint$ be a normed vector space over . The norm $\oint + \oint$ is continuous on X (with respect to the metric induced by the norm).

Proof. **??** implies that $\oint + \oint$ is Lipschitz and every Lipschitz function is continuous. ∎

As a consequence of the continuity of the norm, if $X, \oint + \oint$ is a normed vector space and $\varphi_{k\,k \boxplus N} \colon N \boxdot X$ is a sequence in X convergent to some $\varphi \boxplus X$, *i.e.*,

$$\lim_{k \boxdot \{} \varphi_k \ \ \varphi ,$$

then

$$\oint \varphi \oint \ \ \oint \lim_{k \boxdot \{} \varphi_k \oint \ \ \lim_{k \boxdot \{} \oint \varphi_k \oint . \tag{3}$$

DEFINITION 0.3 (EQUIVALENCE OF THE NORMS): Let X be a vector space over and $\oint + \oint_1 \colon X \boxdot R$ and $\oint + \oint_2 \colon X \boxdot R$ two norms on X. The two norms are said to be "equivalent" if there exists a pair of strictly positive real numbers λ and μ such that

$$\begin{aligned} \alpha \oint \varphi \oint_1 &\curlywedge \oint \varphi \oint_2 \\ &\curlywedge \beta \oint \varphi \oint_1 , \end{aligned} \tag{4}$$

for all $\varphi \boxplus X$.

Equivalent norms define the same notions of continuity and convergence and for many purposes do not need to be distinguished. As we shall prove later, for finite-dimensional (real or complex) vector space, all norms are equivalent. On the other hand, in the case of infinite-dimensional vector spaces, not all norms are equivalent and we need to specify which norm we are using.

Let us recall an important fact about Cauchy sequencies. Completeness of a metric space Give any metric space X, d, a convergent sequence in X is also a Cauchy sequence. The converse of this statement is not generally true however, *i.e.*, there exists metric spaces where some Cauchy sequences do not need to converge. A typical counter-example works as follow. Let $x_{n\,n \curlywedge 1}$ be a sequence in R such that for every $n \curlywedge 1$ we have: $x_n \boxplus Q$ and

$\sqrt{2} + \frac{1}{n} < x_n < \sqrt{2} \ \frac{1}{n}$. (It is always possible to construct such sequence since Q is dense in R.) It is possible to prove that $\lim_{n\Cap\{} x_n \ \sqrt{2}$. Being convergent in R, x_n is of Cauchy type in R. The same sequence can be defined in Q and it is of Cauchy type in Q, but it is not convergent in Q since $\sqrt{2} \boxplus R \oint Q$. A metric space is "complete" if *all* Cauchy sequencies are convergent.

Definition 0.4 (Banach space): Let Banach space $X, \oint + \oint$ be a normed vector space over . If X, d (where d is the metric induced by the norm) is complete, $X, \oint + \oint$ is called a "Banach space".

We are almost ready to introduce the notion of Hilber space, which is the setting where we will develop the operator formulation of non-relativistic quantum mechanics. The first ingredient is the "inner product", defined below.

In Complex numbers notation the following, for every $z \boxplus$ we will denote the "complex conjugate" of z with z and the "modulus" of z with $\oint z \oint$. Remember that $z \ \ \oslash z \ \ i \bigcirc z$ (where $\oslash z$ and $\bigcirc z$ are the real and imaginary parts of z, respectively), $z \ \ \oslash z + i \bigcirc z$ and $\oint z \oint^2 \ \ zz$. The inverse of $z \ominus 0$ is $z^{+1} \ \ z / \oint z \oint^2$. We have $z \boxplus R$ if and only if $z \ \ z$. The complex conjugation satisfies the "involution" property: $(z) \ \ z$. Furthermore, $(z_1 z_2) \ \ z_1 z_2$, $(z_1 \vee z_2) \ \ z_1 \vee z_2$, and $\oint z_1 z_2 \oint \ \ \oint z_1 \oint \oint z_2 \oint$.

Definition 0.5 (inner product): Let Inner product X be a vector space over . A "inner product" on X is any application $X \times X \Cap$, hereafter denoted by $\oint + \oint + \oint$, satisfying the following properties:

(a) $\oint \varphi \oint \psi \ \eta \oint \ \ \oint \varphi \oint \psi \oint \ \ \oint \varphi \oint \eta \oint$, $\ \ \Phi \varphi, \psi, \eta \boxplus X \times X \times X$;

(b) $\oint \varphi \oint \lambda \psi \oint \ \ \lambda \oint \varphi \oint \psi \oint$, $\ \ \Phi \varphi, \psi \boxplus X \times X$ and $\lambda \boxplus$;

(c) $\oint \varphi \oint \psi \oint \ \ \oint \psi \oint \varphi \oint$, $\ \ \Phi \varphi, \psi \boxplus X \times X$;

(d) $\oint \varphi \oint \varphi \oint \curlywedge 0$, $\ \ \Phi \varphi \boxplus X$;

(e) $\oint \varphi \oint \varphi \oint \ \ 0$ if and only if $\varphi \ \ 0$.

Several remarks are in order.

Remark. **????** are *equivalent* to say that $\oint + \oint + \oint$ is *linear* on the *second* component, *i.e.*, $\oint + \oint + \oint$ satisfies **????** *if and only if*

$$\boxed{\oint \varphi \oint \lambda\psi \;\; \mu\eta \oint \;\; \lambda \oint \varphi \oint \psi \oint \;\; \mu \oint \varphi \oint \eta \oint ,}$$

for all $\varphi, \psi, \eta \boxplus X \times X \times X$ and for all $\lambda, \mu \boxplus \times$. **??** together with **????** implies that the inner product in general is *conjugate-linear* (or anti-linear) on the *first* component, namely

$$\boxed{\oint \lambda\varphi \;\; \mu\eta \oint \psi \oint \;\; \lambda \oint \varphi \oint \psi \oint \;\; \mu \oint \varphi \oint \eta \oint ,}$$

for all $\varphi, \psi, \eta \boxplus X \times X \times X$ and for all $\lambda, \mu \boxplus \times$. Of course, if $R, \lambda \;\; \lambda, \mu \;\; \mu$ and the inner product becomes linear also on the first component (thus, it is *bi*linear); but this is not the case if , where complex conjugation appears.

Remark. Convention **??** is a matter of choice. Some authors prefer the different convention:

$$\oint \lambda\varphi \oint \psi \oint \;\; \lambda \oint \varphi \oint \psi \oint , \quad \textcircled{I}\varphi, \psi \boxplus X \times X \text{ and } \lambda \boxplus ;$$

with this latter convention, the inner product would become linear on the first component and conjugate-linear on the second one. The convetion of having the inner product linear on the second component is the one most often employed by physicists, and the one used in this notes.

Remark. Regarding **????**, one may wonder what does mean $\oint \varphi \oint \varphi \oint \curlywedge 0$, since we expect $\oint \varphi \oint \varphi \oint \boxplus$, and if it might seem that the inequlity does not make sense. Actually, from **??**

$$\oint \varphi \oint \varphi \oint \;\; \oint \varphi \oint \varphi \oint , \quad \textcircled{I}\varphi \boxplus X ,$$

Remark. Some authors prefer the notation φ, ψ instead of $\oint \varphi \oint \psi \oint$; the notation $\oint \varphi \oint \psi \oint$ is closer to the one uses by physicists and it is the first step towards the introduction of Dirac's notation. (Dirac's notation is more than simply writing the inner product this way; we will discuss this point in connection with the spectral theorem of linear operators.) In Dirac notation the vector ψ is denoted by $\oint \psi \oint$, and it is called "ket"; there is a "kind of conjugation" (more on this later on) that converts the analogous "ket" $\oint \varphi \oint$ to a so-called "bra" $\oint \varphi \oint$ and the inner product is considered as a product between a "bra" and a "ket" (resulting in a "braket"!). Of course, this is just a suggestive naming convention.

Definition 0.6 (inner product space): A Inner product space “inner product space” over is a pair $X, \oint + \oint + \oint$, where X is a vector space over and $\oint + \oint + \oint : X \times X$ ⍓ is an inner product on X.

The following lemma will be useful later on.

Lemma 0.2: Let $X, \oint + \oint + \oint$ be a inner product space over . If ψ 0, then

$$\oint \varphi \oint \psi \oint \quad \oint \psi \oint \varphi \oint \quad 0\,, \tag{5}$$

for every φ ⊞ X.

The statement of this lemma looks rather trivial. However, a technical proof is given below. The linearity of the inner product is a key ingredient.

Proof. By linearity of the inner product,

$$\oint \varphi \oint \psi \;\; \psi \oint \quad \oint \varphi \oint \psi \oint \quad \oint \varphi \oint \psi \oint, \quad ⦶\varphi, \psi ⊞ X \times X\,.$$

In particular, if ψ 0 we have ψ ψ ψ and

$$\oint \varphi \oint \psi \;\; \psi \oint \quad \oint \varphi \oint \psi \oint, \quad ⦶\varphi ⊞ X \text{ and } \psi \;\; 0\,.$$

Thus

$$\oint \varphi \oint \psi \oint \quad \oint \varphi \oint \psi \oint \quad \oint \varphi \oint \psi \oint, \quad ⦶\varphi ⊞ X \text{ and } \psi \;\; 0\,,$$

which is an equation in for the unknown $\oint \varphi \oint \psi \oint$. $\oint \varphi \oint \psi \oint$ is a solution of this equation *if and only if* $\oint \varphi \oint \psi \oint$ 0. ■

Any Norm induced by inner product inner product space is naturally endowed with a norm coming from the inner product. Let $X, \oint + \oint + \oint$ be an inner product space over . Let $\oint + \oint : X$ ⍓ R defined by

$$\boxed{\oint \psi \oint \quad \sqrt{\oint \psi \oint \psi \oint}, \quad ⦶\psi ⊞ X\,.} \tag{6}$$

Observe that such $\oint + \oint$ in **??** is well-defined, since $\oint \psi \oint \psi \oint$ ⅄ 0 for every ψ ⊞ X. The square root is not ambigous: it is not a square root of a complex number; it is the square root of a positive real number, we don't need to specify a branch for the square root function. We shall prove in a moment that $\oint + \oint$ is indeed a norm on X, this justify the notation $\oint + \oint$. Such norm is called the "norm induced by the inner product". Before proving this, we need a preliminary but extremely important result, which goes under the name of Cauchy-Schwarz inequality.

Cauchy-Schwarz inequality is of major importance. It is a key ingredient in several proofs of functional analysis. It has important implications also outside the realm of analysis. For example, the general formulation of the Heisenberg uncertainty principle in quantum mechanics (or the analogous time-bandwidth uncertainty principle for temporal signal transmission) is derived using the Cauchy-Schwarz inequality.

Theorem 0.3 (Cauchy-Schwarz inequality): Let Cauchy-Schwarz inequality $X, \oint + \oint + \oint$ be an inner product space. The following holds:

$$\boxed{\left\lfloor \oint \varphi \oint \psi \oint \right\lfloor \curlywedge \oint \varphi \oint \oint \psi \oint, \quad \ominus\varphi, \psi \boxplus X \times X.} \tag{7}$$

Equality holds if and only if the vectors are linearly dependent.

Remark. In **??** we are using the definition **??** but it is important to emphasize that we are *not* using (in both the statement and in the proof of Cauchy-Schwarz inequality) the fact that **??** is a norm. We don't know at this point that **??** defines a norm, we will prove that in the next theorem, using the Cauchy-Schwarz inequality.

Proof. We distinguish two cases: $\psi \;\; 0$ and $\psi \ominus 0$.

If $\psi \;\; 0$, then $\oint \varphi \oint \psi \oint \;\; 0$ (see **??**) and $\oint \psi \oint \psi \oint \;\; \oint \psi \oint^2 \;\; 0$, thus the inequality becomes $0 \curlywedge 0$, which is satified.

Let us now consider $\psi \ominus 0$. As a preliminary step, consider for every $\lambda \boxplus$

$$\begin{aligned}\oint \varphi + \lambda\psi \oint \varphi + \lambda\psi \oint \;\; & \oint \varphi \oint \varphi \oint + \lambda \oint \varphi \oint \psi \oint + \lambda \oint \psi \oint \varphi \oint \\ \lambda\lambda \oint \psi \oint \psi \oint \;\; & \oint \varphi \oint \varphi \oint + 2 \oslash \left[\lambda \oint \varphi \oint \psi \oint \right[\;\; \lambda\lambda \oint \psi \oint \psi \oint .\end{aligned}$$

From **??** in **??**, the left-hand side of this equation is positive:

$$\oint \varphi + \lambda\psi \oint \varphi + \lambda\psi \oint \curlywedge 0$$

and it is zero if and only if $\varphi + \lambda\psi \;\; 0$ (*i.e.*, if φ and ψ are linearly dependent). Thus

$$\oint \varphi \oint \varphi \oint + 2\oslash \Big[\lambda \oint \varphi \oint \psi \oint \Big[\;\; \lambda\lambda \oint \psi \oint \psi \oint \;\curlywedge\; 0\,,$$

for every $\varphi \boxplus X, \psi \boxplus X \oint \oint 0 \oint$ and $\lambda \boxplus$. Choose

$$\lambda \;\; \frac{\oint \varphi \oint \psi \oint}{\oint \psi \oint \psi \oint}, \quad \psi \ominus 0\,, \tag{8}$$

which makes sense since we are discussing the case $\psi \ominus 0$. Plugin into the previous equation yields

$$\oint \varphi \oint \varphi \oint + 2\oslash \Bigg[\frac{\oint \varphi \oint \psi \oint}{\oint \psi \oint \psi \oint} \oint \varphi \oint \psi \oint \Bigg[\;\; \frac{\oint \varphi \oint \psi \oint}{\oint \psi \oint \psi \oint} \frac{\oint \varphi \oint \psi \oint}{\cancel{\oint \psi \oint \psi \oint}} \cancel{\oint \psi \oint \psi \oint} \;\curlywedge\; 0\,,$$

hence

$$\oint \varphi \oint \varphi \oint + \frac{\oint \varphi \oint \psi \oint \oint \varphi \oint \psi \oint}{\oint \psi \oint \psi \oint} \;\curlywedge\; 0\,,$$

from which it follows

$$\oint \varphi \oint \varphi \oint \oint \psi \oint \psi \oint \;\curlywedge\; \Big| \oint \varphi \oint \psi \oint \Big|$$

(using the fact that $\oint \psi \oint \psi \oint > 0$). Taking the square root of both sites (notice that both sides are surely positive) yields the expected result. The equality holds if and only if $\varphi + \lambda\psi \;\; 0$, *i.e.*, if the two vectors are linearly dependent. ■

EXERCISE 0.3: For every fixed $\varphi, \psi \boxplus X \times X$, consider $f : \frown R$ defined by

$$\begin{aligned} f\lambda \;\; & \oint \varphi + \lambda\psi \oint^2 \\ & \oint \lambda \oint^2 \oint \psi \oint^2 + \lambda \oint \varphi \oint \psi \oint + \lambda \oint \varphi \oint \psi \oint \;\; \oint \varphi \oint^2, \end{aligned} \tag{9}$$

as a function of $\lambda \boxplus$. The positivity of the norm ensures that $f\lambda \curlywedge 0$ and $f\lambda \;\; 0$ if and only if $\varphi \;\; \lambda\psi$, as already discussed in the proof of Cauchy-Schwarz inequality. Show that λ given by **??** is a minimum of f. This fact can be use to explain the choice in **??**.

THEOREM 0.4: Let $X, \oint + \oint + \oint$ be an inner product space over . $X, \oint + \oint$ with $\oint + \oint$ defined by **??** is a normed vector space over .

Proof. We need to check that **??** makes sense and that it satisfies **????????** of the definition of the norm. The only non-trivial property is the subadditivity. It can be proved using the Cauchy-Schwarz inequality. In fact, for every φ, ψ in X,

$$
\begin{aligned}
\oint \varphi \ \psi \oint^2 \ & \oint \varphi \ \psi \oint \varphi \ \psi \oint \\
& \oint \varphi \oint \varphi \oint \ \oint \varphi \oint \psi \oint \ \oint \psi \oint \varphi \oint \ \oint \psi \oint \psi \oint \\
& \oint \varphi \oint^2 \ 2 \oslash \oint \varphi \oint \psi \oint \ \oint \psi \oint^2 \\
& ⅄ \oint \varphi \oint^2 \ 2 \left| \oint \varphi \oint \psi \oint \right| \ \oint \psi \oint^2 \\
& ⅄ \oint \varphi \oint^2 \ 2 \oint \varphi \oint \oint \psi \oint \ \oint \psi \oint^2 \\
& \left[\oint \varphi \oint \ \oint \psi \oint \right[^2 ;
\end{aligned}
$$

taking the square root (all quantities involved are positive real numbers) yields the expected result. ∎

THEOREM 0.5: Let Continuity of inner product $X, \oint + \oint + \oint$ be an inner product space over . $\oint + \oint +$ is a continuous function of both arguments (with respect to the topology induced by the inner product).

Proof. Let us discuss the continuity on the second argument (the continuity on the first argument can be handled in a similar way).

We need to show that $\oplus \varphi, \psi ⊞ X \times X$ the following holds: for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $\eta ⊞ X$, if $d\psi, \eta < \delta$ then $\tilde{d} \oint \varphi \oint \psi \oint, \oint \varphi \oint \eta \oint < \varepsilon$, where $\tilde{d}$ is the euclidean distance in R.

Notice that $d\psi, \eta < \delta$ means

$$\oint \psi + \eta \oint < \delta ,$$

and $\tilde{d} \oint \varphi \oint \psi \oint, \oint \varphi \oint \eta \oint < \varepsilon$ means

$$\left| \oint \varphi \oint \psi \oint + \oint \varphi \oint \eta \oint \right| < \varepsilon .$$

Using linearity of the inner product and Cauchy-Schwarz inequality yields

$$\left|\langle \varphi, \psi \rangle + \langle \varphi, \eta \rangle\right| \quad \left|\langle \varphi, \psi + \eta \rangle\right| \leq \|\varphi\| \|\psi + \eta\| \leq \delta \|\varphi\| .$$

If $\varphi \quad 0$, the result is 0 and we are done. Otherwise, if $\varphi \ominus 0$, it is enough to choose $0 < \delta < \boxtimes / \|\varphi\|$. ■

As a consequence of the continuity of the inner product, if $(X, \langle \cdot, \cdot \rangle)$ is a inner product space and $\{\psi_k\}_{k \geq 1}$ is a sequence in X such that $\sum_{k \in N} \psi_k$ is convergent to some $\psi \in X$, *i.e.*,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \psi_k \quad \psi ,$$

then

$$\langle \varphi, \psi \rangle \quad \left\langle \varphi \middle| \lim_{n \to \infty} \sum_{k=1}^{n} \psi_k \right\rangle \quad \lim_{n \to \infty} \left\langle \varphi \middle| \sum_{k=1}^{n} \psi_k \right\rangle \quad \lim_{n \to \infty} \sum_{k=1}^{n} \langle \varphi, \psi_k \rangle . \tag{10}$$

Exercise 0.4: Justify all steps in **??**.

Definition 0.7 (Hilbert space): Let Hilbert space $(X, \langle \cdot, \cdot \rangle)$ be a inner product space over . If the metric space (X, d) (with the distance arising from the inner product) is complete, $(X, \langle \cdot, \cdot \rangle)$ is called an "Hilbert space".

In short: Banach spaces are complete normed vector spaces and Hilbert spaces are complete inner product spaces. Hilbert spaces are a special case of Banach space, where the norm comes from an inner product. The underlying iner product inducing the norm introduces extra features (in particular, some related to the notion of orthogonality) which are not present in general Banach spaces (see next section for details on this).

In the next sections, we will focus on Hilbert spaces only, and we will not consider incomplete inner product space. On reason for this is that any incomplete metric space admit a completion (more on this later, when discussing BLT theorem).

Hereafter, if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, if not explicitly stated, we will also intend that $\|\cdot\|$ denotes the norm induced by the inner product, and that the convergence, etc refers to the distance induced by that norm.

Is it possible to distinguish if in a normed vector space, the norm is coming from some underlying inner space? The answer is yes, and we are going to prove immediately an interesting criterion to perform this check.

The following result is basic to establish the later theorem. It will also be useful later, when discussing positive operators.

Lemma 0.3 (polarization identity): Let $X, \oint + \oint + \oint$ be a inner product space. In the case R,

$$\oint \varphi \oint \psi \oint \quad \frac{1}{4}\Big[\oint \varphi \quad \psi \oint^{2} + \oint \varphi + \psi \oint^{2} \Big[, \quad \ominus \varphi, \psi \boxplus X \times X . \tag{11}$$

In the case ,

$$\oint \varphi \oint \psi \oint \quad \frac{1}{4}\Big[\oint \varphi \quad \psi \oint^{2} + \oint \varphi + \psi \oint^{2} \Big[+ \frac{i}{4}\Big[\oint \varphi \quad i\psi \oint^{2} \quad \oint \varphi + i\psi \oint^{2} \Big[, \tag{12}$$
$$\ominus \varphi, \psi \boxplus X \times X .$$

Remark. **??** can be recasted in a shorter form:

$$\boxed{\oint \varphi \oint \psi \oint \quad \frac{1}{4} \sum_{k0}^{3} + i^{k} \oint \varphi \quad i^{k} \psi \oint^{2} .} \tag{13}$$

Remark. Different signs are possibile in literature if a different convention is chosen for property **??** of **??**, namely, if the inner product is chosen to be linear on the first component instead of the second one.

Proof. We discuss the case . The proof can be easily modified for the case R.

By straightforward computation,

$$\oint \varphi \quad \psi \oint^{2} \quad \oint \varphi \oint^{2} \quad 2 \oslash \oint \varphi \oint \psi \oint \quad \oint \psi \oint^{2} , \tag{14}$$

$$\oint \varphi + \psi \oint^{2} \quad \oint \varphi \oint^{2} + 2 \oslash \oint \varphi \oint \psi \oint \quad \oint \psi \oint^{2} , \tag{15}$$

$$\oint \varphi \quad i\psi \oint \quad \oint \varphi \oint^{2} + 2 \oslash \oint \varphi \oint \psi \oint \quad \oint \psi \oint^{2} , \tag{16}$$

$$\oint \varphi + i\psi \oint \quad \oint \varphi \oint^{2} \quad 2 \oslash \oint \varphi \oint \psi \oint \quad \oint \psi \oint^{2} . \tag{17}$$

Subtracting **????** we get

$$\oint \varphi \quad \psi \oint^{2} + \oint \varphi + \psi \oint^{2} \quad 4 \oslash \oint \varphi \oint \psi \oint ;$$

subtracting **????** we get

$$\oint \varphi \quad i\psi \oint + \oint \varphi + i\psi \oint \quad + 4 \oslash \oint \varphi \oint \psi \oint .$$

Thus

$$
\begin{aligned}
&\frac{1}{4}\Big[\oint \varphi \ \psi \oint^{2} + \oint \varphi + \psi \oint^{2}\Big[+ \frac{i}{4}\Big[\oint \varphi \ i\psi \oint + \oint \varphi + i\psi \oint\Big[\\
&\oslash \oint \varphi \oint \psi \oint \ \ i\oslash \oint \varphi \oint \psi \oint \\
&\oint \varphi \oint \psi \oint .
\end{aligned}
$$

This completes the proof. ■

The following result allows a complete characterization of inner-product spaces among norm vector spaces. It is called Jordan-von Neumann theorem[See][]Jordan.Neumann:1935.

Theorem 0.6 (Jordan-von Neumann theorem): Let Jordan-von Neumann theorem $X, \oint + \oint$ be a normed vector space over . The norm $\oint + \oint$ comes from an inner product *if and only if* the following identity holds: Parallelogram law

$$
\oint \varphi \ \psi \oint^{2} \ \oint \varphi + \psi \oint^{2} \ 2\oint \varphi \oint^{2} \ \oint \psi \oint^{2}, \quad \obar \varphi, \psi \boxplus X \times X . \tag{18}
$$

?? is known as "parallelogram law".

Proof. Let's prove first that if $X, \oint + \oint + \oint$ is a Hilbert space, then the norm induced by the inner product fullfills **??**. This is the straightforward part of the proof. By definition of the norm induced by the inner product,

$$
\begin{aligned}
\oint \varphi \ \psi \oint \ &\oint \varphi \ \psi \oint \varphi \ \psi \oint \\
&\oint \varphi \oint \varphi \oint \ \oint \varphi \oint \psi \oint \ \oint \psi \oint \varphi \oint \ \oint \psi \oint \psi \oint \\
&\oint \varphi^{2} \oint \ 2\oslash \oint \varphi \oint \psi \oint \ \oint \psi \oint^{2} \\
\oint \varphi + \psi \oint \ &\oint \varphi + \psi \oint \varphi + \psi \oint \\
&\oint \varphi \oint \varphi \oint + \oint \varphi \oint \psi \oint + \oint \psi \oint \varphi \oint \ \oint \psi \oint \psi \oint \\
&\oint \varphi^{2} \oint + 2\oslash \oint \varphi \oint \psi \oint \ \oint \psi \oint^{2}
\end{aligned}
$$

for all $\varphi, \psi \boxplus X \times X$; summing the two we have **??**.

Now, let's prove the converse: if $X, \oint + \oint$ is a Banach space whose norm satisfies **??**, then the norm comes from an inner product. We discuss the case , the case R is analogous. Let $\oint + \oint + \oint : X \times X \cap$ be the application defined by

$$\oint \varphi \oint \psi \oint \ \frac{1}{4}\Big[\oint \varphi \ \psi \oint^{2} + \oint \varphi + \psi \oint^{2}\Big[+ \frac{i}{4}\Big[\oint \varphi \ i\psi \oint^{2} + \oint \varphi + i\psi \oint^{2}\Big[,$$

for all $\varphi, \psi \boxplus X \times X$. We will prove now that this application indeed is a inner product on X, *i.e.*, we need to prove that it satisfies **??????????** in **??**. That this formula induced the norm is known from **??**.

Ad **??**. For every $\varphi, \psi, \eta \boxplus X \times X \times X$, by applying the parallelogram law we get

$$2\oint \varphi \ \psi \oint^{2} \ 2\oint \eta \oint^{2} \ \oint \varphi \ \psi \ \eta \oint^{2} \ \oint \varphi \ \psi + \eta \oint^{2}, \tag{19a}$$

$$2\oint \varphi + \psi \oint^{2} \ 2\oint \eta \oint^{2} \ \oint \varphi + \psi \ \eta \oint^{2} \ \oint \varphi + \psi + \eta \oint^{2}, \tag{19b}$$

$$2\oint \varphi \ \eta \oint^{2} \ 2\oint \psi \oint^{2} \ \oint \varphi \ \psi \ \eta \oint^{2} \ \oint \varphi + \psi \ \eta \oint^{2}, \tag{19c}$$

$$2\oint \varphi + \eta \oint^{2} \ 2\oint \psi \oint^{2} \ \oint \varphi \ \psi + \eta \oint^{2} \ \oint \varphi + \psi + \eta \oint^{2}. \tag{19d}$$

Subtracting **????** we get

$$\begin{aligned} 2\oint \varphi \ \psi \oint^{2} + 2\oint \varphi + \psi \oint^{2} \ \oint \varphi \ \psi \ \eta \oint^{2} \ \oint \varphi \ \psi + \eta \oint^{2} + \oint \varphi \\ + \psi \ \eta \oint^{2} + \oint \varphi + \psi + \eta \oint^{2}; \end{aligned} \tag{20}$$

subtracting **????** we get

$$\begin{aligned} 2\oint \varphi \ \eta \oint^{2} + 2\oint \varphi + \eta \oint^{2} \ \oint \varphi \ \psi \ \eta \oint^{2} \ \oint \varphi + \psi \ \eta \oint^{2} + \oint \varphi \\ \psi + \eta \oint^{2} + \oint \varphi + \psi + \eta \oint^{2}; \end{aligned} \tag{21}$$

summing **????** we get

$$\begin{aligned} &2\oint \varphi \ \psi \oint^{2} + 2\oint \varphi + \psi \oint^{2} \ 2\oint \varphi \ \eta \oint^{2} + 2\oint \varphi + \eta \oint^{2} \\ &\oint \varphi \ \psi \ \eta \oint^{2} \ \oint \varphi \ \psi + \eta \oint^{2} + \oint \varphi + \psi \ \eta \oint^{2} + \oint \varphi + \psi + \eta \oint^{2} \ \oint \varphi \ \psi \ \eta \oint^{2} \\ &\oint \varphi + \psi \ \eta \oint^{2} + \oint \varphi \ \psi + \eta \oint^{2} + \oint \varphi + \psi + \eta \oint^{2} \\ &2\oint \varphi \ \psi \ \eta \oint^{2} + 2\oint \varphi + \psi + \eta \oint^{2}. \end{aligned} \tag{22}$$

Ad **??**. This is the tricky part. Let

$$S \;\Big]\, z \boxplus \Big| \oint \varphi \oint z\psi \oint \;\; z \oint \varphi \oint \psi \oint \;\Big].$$

Notice that $0 \boxplus S$ and $1 \boxplus S$. Furthermore, for every z and w in S, also $z \vee w \boxplus S$, so $Z \curlyvee S$. Now, observe that for every $z \boxplus Z$, $w \boxplus Z \oint \oint 0 \oint$,

$$z \oint \varphi \oint \psi \oint \;\; \oint \varphi \oint z\psi \oint \;\; \oint \varphi \Big| w \frac{z}{w} \psi \oint \;\; w \oint \varphi \Big| \frac{z}{w} \psi \oint, \quad \mathbb{O}\varphi, \psi \boxplus X \times X,$$

yielding

$$\frac{z}{w} \oint \varphi \oint \psi \oint \;\; \oint \varphi \Big| \frac{z}{w} \psi \oint,$$

so also $Q \curlyvee S$. Since Q is dense in R we *would like* to conclude that $R \curlyvee \S$. This however requires the continuity of $\oint + \oint + \oint$ and we can't rely on th fact that the inner product is continuous on both components because we do not know whether $\oint + \oint + \oint$ is a inner product yet. Upgrading from Q to R requires that we prove that the Cauchy-Schwarz inequality holds using only the tools at our disposal so far. Fortunately, the usual trick works just fine. So, we can conclude that $R \boxplus S$. Finally, since a direct computation shows that

$$\oint \varphi \oint i\psi \oint \;\; i \oint \varphi \oint \psi \oint,$$

we conclude that S .

Ad **??**. This is straightforward:

$$\oint \psi \oint \varphi \oint \;\; \frac{1}{4} \Big] \oint \psi \;\; \varphi \oint^{2} + \oint \psi + \varphi \oint^{2} + i \Big[\oint \psi \;\; i\varphi \oint^{2} + \oint \psi + i\varphi \oint^{2} \Big[\Big] \;\; \frac{1}{4} \Big[\oint \varphi \;\; \psi \oint^{2} + \oint \varphi + \psi \oint^{2} \Big[\;\; \frac{i}{4} \Big[\oint \varphi \;\; i\psi \oint^{2} \oint \varphi + i\psi \oint^{2} \Big[\;\; \oint \varphi \oint \psi \oint.$$

Ad **??**. Also this is straightforward:

$$\oint \psi \oint \psi \oint \;\; \frac{1}{4} \Big] \oint \psi \;\; \psi \oint^{2} + \oint \psi + \psi \oint^{2} + i \Big[\oint \psi \;\; i\psi \oint^{2} + \oint \psi + i\psi \oint^{2} \Big[\Big] \;\; \frac{1}{4} \Big] 4 \oint \psi \oint^{2} + i \Big[\oint 1 \;\; i \oint^{2} \oint \psi \oint^{2} + \oint 1 + i \oint^{2} \oint \psi \oint^{2} \Big[\Big] \;\; \oint \psi \oint^{2} \curlywedge 0.$$

Ad **??**. Since we have already shown that $\oint \psi \oint \psi \oint \;\; \oint \psi \oint^{2}$, $\oint \psi \oint \psi \oint \;\; 0$ if and only if $\oint \psi \oint \;\; 0$, that is if and only if $\psi \;\; 0$. ∎

Subsequent authors after Jordan and von Neumann have found norm conditions weaker than **??** which characterize inner product spaces amongst normed vector spaces. See, e.g., Reznick:1978 and references therein; a recent paper is Onl-scho:2012, which also contains references to the relevant literature on this subject.

0.3 A glimpse at convex analysis

inequalities in physics and mathematics have their origin in the notion of convexity.

Folklore

Convex analysis is an important branch of analysis, with implications in optimization, etc[See, e.g.,][]Rockafellar:1970. Here, we give general definitions for arbitrary convex sets and convex functions defined on convex sets. We apply these definitions to real-valued functions of one real variable as a special case and we use this to derive Young and Jensen inequalities, which will prove useful in later proofs. Jensen inequality plays a major role in information theory, entropy etc.

Definition 0.8 (Convex set): Let Convex set X be a vector space over and $\Omega \curlyvee X$ a non-empty subset of X. Ω is "convex" if for every $\varphi, \psi \boxplus \Omega \times \Omega$,

$$\vartheta\varphi \quad 1 + \vartheta\psi \boxplus \Omega\,, \tag{23}$$

for every $\vartheta \boxplus 0, 1$* As a convention, the empty set is considered to be convex by definition.

Remark. **??** is a "weighted average" of φ and ψ.

Remark. In order **??** to be meaningful, we need the notion of addition and scalar multiplication for a real number. The natural setting where these two operations are defined are the vector spaces. This is the reason why in **??** X is a vector space over R or . The notion of convexity may be generalized to more general settings if certain propeties of convexity are taken as axioms, but we will not discuss this approach here.

Remark. X itself is convex. Furthermore, let $\psi \boxplus X$; then, $\oint \psi \oint$ is a convex subset of X.

---

* Of course, if $\vartheta \quad 0$ we get ψ and if $\vartheta \quad 1$ we get φ, so we could have considered equivalently $0, 1$ instead of $0, 1$ in **??**.

The relation with the usual "geometric" definition of convex set can be understood by introducing the notion of "line segment" for arbitrary real or complex vector spaces.

Definition 0.9 (line segment): Let Line segment X be a vector space over and consider $\varphi, \psi \boxplus X \times X$. The set

$$\varphi, \psi \oint \vartheta\varphi \;\; 1 + \vartheta\psi \mid \textcircled{1}\vartheta \boxplus 0, 1 \oint \tag{24}$$

is called the "(straight) line segment" joining φ and ψ (or with endpoints φ and ψ).

Remark. Of course, $\varphi, \psi \;\; \psi, \varphi$. Check as exercise. (Hint: It is enough to replace $\lambda \;\; 1 + \vartheta$.)

Using **??** we can restate **??** in the following equivalent way:

Lemma 0.4: Let Alternative definition of convex set X be a vector space over and $\Omega \curlyvee X$ any non-empty subset of X. Ω is convex if and only if for every pair of points $\varphi, \psi \boxplus \Omega \times \Omega$, $\varphi, \psi \curlyvee \Omega$.

Exercise 0.5: Prove **??**.

Before proving more details on convex sets, let us see some examples in R and R^N. First of all, recall the definition of "real interval".

Definition 0.10 (real interval): Let $I \curlyvee R$ a non-empty subset of R. I is a "real interval" if $\varphi, \psi \boxplus I \times I$ and $\eta \boxplus R$ such that $\varphi \curlywedge \eta \curlywedge \psi$ implies $\eta \boxplus I$. The empty set is regarded as a interval.

This definition covers different typologies of real intervals: φ, ψ, φ, ψ, φ, ψ, φ, ψ, etc. It covers also the case of a single real number $I \;\; \oint \psi \oint$ and the case where one of the two endpoints or both of them are $\vee\{$.

Lemma 0.5: Let $\Omega \curlyvee R$ a subset of R. Ω is convex *if and only if* Ω is a real interval.

Theorem 0.7 (Young inequality): For Young inequality every $A, B \curlywedge 0$ and $0 \curlywedge \vartheta \curlywedge 1$,

$$\boxed{A^{\vartheta} B^{1+\vartheta} \curlywedge \vartheta A \;\; 1 + \vartheta B,} \tag{25}$$

Remark. As a special case, for $\vartheta \;\; 1/2$ we recover the arithmetic-geometric mean (AGM) inequality for two positive real numbers, namely

$$\sqrt{AB} \curlywedge \frac{A \;\; B}{2}, \quad A, B \curlywedge 0\,. \tag{26}$$

The general aithmetic-geometric mean inequality with n ⊞ N terms will be proven later, as a special case of the Jensen inequality. There are other interesting ways to prove the AGM inequality for two terms, for example rearraning the square terms:

$$0 ⅄ A + B^2 \;\; A \;\; B^2 + 4AB, \quad ⦶A, B ⊞ R\,,$$

yielding

$$AB ⅄ \left[\frac{A \;\; B}{2}\right[^2, \quad ⦶A, B ⊞ R\,.$$

If A, B ⅄ 0 we can take the square root and find **??**. There are also several other proofs for AGM inequality with n ⊞ N terms (e.g., proof by induction).

Remark. Geometric meaning of **??**: AB is the area of a rectangle with sides of length A and B and $\frac{AB}{2}$ is the semi-perimeter of such rectangle. $\sqrt{AB}$ can be viewed as the length of the side of a square having the same area of the rectangle. Re-arranging the terms we have $4\sqrt{AB}$ ⅄ $2A \;\; 2B$ which means that the square has the smallest perimeter amongst all rectangles of equal area.

Remark. Young inequality is often stated in a slightly different form. Let

$$\frac{1}{p} \;\; \frac{1}{q} \;\; 1\,,$$

put $\vartheta \;\; 1/p$ so $1 + \vartheta \;\; 1/q$ and set $A \;\; a^p$ and $B \;\; b^q$. This way **??** reads

$$ab ⅄ \frac{a^p}{p} \;\; \frac{b^q}{q}\,. \tag{27}$$

Proof. There are various proof, mainly based on convexity. Our proof uses the fact that the exponential function is convex on R, thus

$$e^{\vartheta\varphi 1 + \vartheta\psi} ⅄ \vartheta e^{\varphi} \;\; 1 + \vartheta e^{\psi} \tag{28}$$

for all ϑ ⊞ 0, 1 and φ, ψ ⊞ $R \times R$. Using $A \;\; \exp(\varphi)$ and $B \;\; \exp(\psi)$ (this makes sense because A and B are positive) yields **??**. ■

Another possible proof goes as follows. Consider f_ϑ: 0, { ⋂ R defined by

$$f_\vartheta x \;\; x^\vartheta + \vartheta x \;\; \vartheta + 1, \quad ⦶x > 0\,,$$

where by definition

$$x^\vartheta \;\; e^{\vartheta \log[x[}, \quad ⦶x > 0\,.$$

Notice that when $\vartheta \;\; 0$ or $\vartheta \;\; 1$ the function is constant, in particular $f_0 x \;\; f_1 x \;\; 0$. Study the function and show that it has a maximum for $x \;\; 1$. Since $f_\vartheta 1 \;\; 0$, we get $f_\vartheta x$ ⅄ 0.

0.4 Canonical prototypes of Banach and Hilbert spaces

0.4.1 The Hilbert spaces R^N and ^{N}

Let N ⊞ $N \nexists \oint$ o$\oint$ be any positive integer number. Consider the vector space ^{N}, whose elements are N-ple of complex numbers, endowed with the usual operations of elementwise addition and elementwise multiplication of an N-ple by a complex number. For every

$$\varphi \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{bmatrix}, \quad \psi \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix},$$

in ^{N}, define an application $\oint + \oint + \oint : ^N \times ^N \cap$ as follows:

$$\boxed{\oint \varphi \oint \psi \oint \; \Vert_{k1}^{N} \varphi_k \psi_k \, .} \tag{29}$$

In a similar manner one can define the usual (Euclidean) inner product on R^N, which takes the same form of **??** without complex conjugation (because in that case all numbers involved are real numbers).

THEOREM 0.8: $^N, \oint + \oint + \oint$, where $\oint + \oint + \oint$ is defined by **??**, is an inner product space.

Proof. It is straightforward to show that $\oint + \oint + \oint$ defined in **??** satifies **??????????** of **??**. ■

THEOREM 0.9: $^N, \oint + \oint + \oint$, where $\oint + \oint + \oint$ is defined by **??**, is an Hilbert space.

Proof. We need to check completeness. ■

0.4.2 The Hilbert space $_2$

Let $_2$ be the space of all and only the sequences $\varphi_{k\,k \curlywedge 1}$ in such that the series $\Vert_{k1}^{\{} \oint \varphi_k \oint^2$ is convergent in R.

It is easy to show that, with the usual addition and multiplication for an element of , $_2$ is a vector space over . Define

$$\boxed{\oint \varphi \oint \psi \oint \; \Vert_{k1}^{\{} \varphi_k \psi_k \, ,} \tag{30}$$

for every $\varphi \quad \varphi_{k k \curlywedge 1}$ and $\psi \quad \psi_{k k \curlywedge 1}$ in 2.

THEOREM 0.10: 2, $\oint + \oint + \oint$, where $\oint + \oint + \oint$ is defined by **??**, is an inner product space.

Proof. Similar to the proof that ^{N} is an inner product space, with some care to handle the limits of the various sums. ∎

THEOREM 0.11: 2, $\oint + \oint + \oint$, where $\oint + \oint + \oint$ is defined by **??**, is an Hilbert space.

0.4.3 The Hilbert $\Omega 2$

Let $\Omega \curlyvee R^N$ be an arbitrary non-empty subset of R^N for some $N \boxplus N$ and let $< \tilde{L} > \Omega 2$ be the set consisting of all and only the functions defined on Ω and taking values on , *i.e.*, $\psi: \Omega \,\cap\!\!\!\!\cdot\,$, such that their square modulus is Lebesgue-integrable over Ω, *i.e.*, the following integral exists and it is finite (in the sense of Lebesgue):

$$'_{\Omega} \oint \psi \oint^{2}.$$

Remark. Even if , the modulus is real-valued so the integral above is always an integral of a real function.

It is not difficult to make $< \tilde{L} > \Omega 2$ a vector space. Our goal would be to endow $< \tilde{L} > \Omega 2$ also with an inner product. A problem would arise however, due to the fact that there are non-zero functions in $< \tilde{L} > \Omega 2$ whose square modulus integral is zero, and this would ultimately make not possible to satisfy property **??** of the inner product.

To overcome this difficulty, a slightly more techical construction is needed. The idea is to "identify" two functions when they are equal almost everywhere (*i.e.*, everywhere but on a set of zero Lebesgue measure). The formal construction involves working with equivalence classes and proceeds as follows. First, we will equip $< \tilde{L} > \Omega 2$ with an equivalence relation, which allows to identify two functions whenever they are equal almost everywhere (*i.e.*, when they differ only on a set of Lebesgue zero measure). Then, the quotient space with respect to this equivalence relation can be made a vector space over (with suitable definitions of addition and multiplication by a element of) and can be equipped with an inner product. We will see that, with this inner product, the quotient space becomes an Hilbert space.

As a preliminary step, consider the subset $M \curlyvee < \tilde{L} > \Omega 2$ defined as follows:

$$M \quad \oint \psi \boxplus < \tilde{L} > \Omega 2 \mid {}'_{\Omega} \oint \psi \oint^{2} \quad 0 \oint$$

The following lemma is not strictly necessary to be mentioned, but it makes the next proofs clearer.

LEMMA 0.6: For every φ, ψ ⊞ $M \times M$ and for every λ ⊞, φ ψ and $\lambda\psi$ belong to M.

In words: any linear combination of vectors of M is an element of M itself.

Proof. By linearity of the integral,

$$\int_\Omega \oint \lambda\psi \oint^2 \quad \int_\Omega \oint \lambda \oint^2 \oint \psi \oint^2 \quad \oint \lambda \oint^2 \int_\Omega \oint \psi \oint^2 ,$$

thus $\lambda\psi$ ⊞ M. Using the monotonicity of the integral,

$$0 \curlywedge \int_\Omega \oint \varphi \ \ \psi \oint^2 \curlywedge \int_\Omega \oint \psi \oint^2 \quad \int_\Omega \oint \varphi \oint^2 \quad 0 ,$$

thus ψ φ ⊞ M. ∎

Let ⋏ be the relation on < $\tilde{L}$ > $\Omega 2$ defined in this way: for every φ, ψ ⊞< $\tilde{L}$ > $\Omega 2 \times$ < $\tilde{L}$ > $\Omega 2$, φ ⋏ ψ if $\varphi + \psi$ ⊞ M. In words: φ ⋏ ψ if the two functions agree outside a set of (Lebesgue) zero measure.

LEMMA 0.7: ⋏ is an equivalence relation over < $\tilde{L}$ > $\Omega 2$.

Proof. Let's check the equivalence relation properties:

Reflexivity : for every ψ ⊞< $\tilde{L}$ > $\Omega 2$, $\psi + \psi$ 0 (where 0 denotes the identically zero function, *i.e.*, the function which takes value zero everywhere on Ω) and thus $\psi + \psi$ ⊞ M;

Symmetry : for every ψ and φ in < $\tilde{L}$ > $\Omega 2$, if ψ ⋏ φ also φ ⋏ ψ; in fact, ψ ⋏ φ means $\psi + \varphi$ ⊞ M and, for **??**, also $\varphi + \psi$ $+1\psi + \varphi$ ⊞ M;

Transitivity : for every ψ, φ and η in < $\tilde{L}$ > $\Omega 2$, if ψ ⋏ η and η ⋏ φ, then ψ ⋏ φ; in fact, if $\psi + \eta$ ⊞ M and $\eta + \psi$ ⊞ M, then also $\psi + \eta$ $\eta + \varphi$ ⊞ M by **??**.

∎

We introduce the following notation: for every ψ in < $\tilde{L}$ > $\Omega 2$, let ψ denote the equivalence class of ψ under ⋏; furthermore, let $\Omega 2$ denote the quotient space (*i.e.*, the space of all possibile equivalence classes) of < $\tilde{L}$ > $\Omega 2$ by ⋏.

Define addition and multiplication by a scalar constant in $\Omega 2$ in the following way. For every ψ and φ in $\Omega 2$ and λ ⊞, put

$$\psi \ \varphi \ \psi \ \varphi \,,$$
$$\lambda\psi \ \lambda\psi \,.$$

First of all, let us check that these operations are well-defined.

LEMMA 0.8: For every ψ and $\tilde{\psi}$ in ψ and for every φ, $\tilde{\varphi}$ in φ,

$$\psi \ \varphi \ \tilde{\psi} \ \tilde{\psi} \,,$$
$$\lambda\psi \ \lambda\tilde{\psi} \,.$$

Proof. There exists η and ξ in M such that $\tilde{\psi} \ \psi \ \eta$ and $\tilde{\varphi} \ \varphi \ \xi$; then, $\tilde{\psi} \ \tilde{\varphi} \ \psi \ \eta \ \varphi \ \xi \ \psi \ \varphi \ \eta \ \xi$, where $\eta \ \xi$ ⊞ M. Thus $\tilde{\psi} \ \tilde{\varphi} + \psi \ \varphi \ \ \eta \ \xi$ ⊞ M, $\tilde{\psi} \ \tilde{\varphi}$ ⩕ $\psi \ \varphi$. In the same way one proves that $\lambda\tilde{\psi}$ ⩕ $\lambda\psi$. ∎

THEOREM 0.12: $\Omega 2, , +$ is a vector space over , where the addition and multiplication are those defined above.

It is a standard result of linear algebra that the quotient space with the above definitions of addition and multiplication is a vector space. More generally, this result applies to every quotient space, no matter what is the underlying set and what is the specific equivalence relation. We leave the proof of **??** as exercise.

EXERCISE 0.6: Prove **??**. Generalize the proof to arbitrary quotient spaces under a generic equivalence relation.

In $\Omega 2$, define an application $\oint + \oint + \oint$ by letting

$$\boxed{\oint \varphi \,|\, \psi \oint \ \ '_{\Omega}\, \varphi\psi \,,} \tag{31}$$

where on the right hand side φ is any function belonging to φ and ψ is any function belonging to ψ.

Remark. In order to simplify the notation, we will denote the equivalence class ψ containing ψ by ψ itself.

Remark. Given $\varphi \colon \Omega$ ⨀, the function $\varphi \colon \Omega$ ⨀ is defined by $\varphi x \ \times \varphi x$ for all x ⊞ Ω.

We need to show that such $\oint + \oint + \oint$ is well-defined, *i.e.*: (a) show that the integral exists and is convergent, and (b) show that the integral is independent from the choice of $\psi \boxplus \psi$ and $\varphi \boxplus \varphi$.

Let us recall the following theorem from the theory of Lebesgue integration.

THEOREM 0.13: Let $\varphi \colon \Omega \; \boxdot$ be Lebesgue integrable on Ω and let $\psi \colon \Omega \; \boxdot$ be any function satisfying

$$\oint \psi x \oint \; \curlywedge \; \oint \varphi x \oint ,$$

for all $x \boxplus \Omega$. Then, ψ is Lebesgue integrable on Ω.

To show that the integral exists, notice that for every $z, w \boxplus \times$, we have

$$\oint zw \oint \; \curlywedge \; \frac{1}{2} \oint z \oint^{2} \; \frac{1}{2} \oint w \oint^{2} ; \tag{32}$$

in fact,

$$0 \curlywedge [\oint z \oint + \oint w \oint [^{2} \; \oint z \oint^{2} \; \oint w \oint^{2} + 2 \oint zw \oint .$$

Thus,

$$\oint \varphi x \psi x \oint \; \curlywedge \; \frac{1}{2} \oint \varphi x \oint^{2} \; \frac{1}{2} \oint \psi x \oint^{2}, \quad \mathbb{O} x \boxplus \Omega ,$$

thus the integral of $\oint \varphi x \psi x \oint$ exists and is absolutely convergent, and this ensures the convergence of the integral of $\varphi x \psi x$.

EXERCISE 0.7: Show that the integral in **??** is independent from the choice of $\psi \boxplus \psi$ and $\varphi \boxplus \varphi$. Hint: write $\psi \;\; \psi + \tilde{\psi} \;\; \tilde{\psi}$ and remember that

THEOREM 0.14: $\Omega 2, \oint + \oint + \oint$, where $\oint + \oint + \oint$ is defined by **??**, is a inner product space.

Proof. We need to show that $\oint + \oint + \oint$ defined by **??** satisfies **??????????** in **??**.

Linearity of $\oint + \oint + \oint$ follows immediately from the linearity of the integral:

$$\oint \varphi \oint \psi \;\; \eta \oint \;\; {}^{\prime}{}_{\Omega}\, \varphi [\psi \;\; \eta [\;\; {}^{\prime}{}_{\Omega}\, \varphi \psi \;\; {}^{\prime}{}_{\Omega}\, \varphi \eta \;\; \oint \varphi \oint \psi \oint \;\; \oint \varphi \oint \eta \oint$$
$$\oint \varphi \oint \lambda \psi \oint \;\; {}^{\prime}{}_{\Omega}\, \varphi [\lambda \psi [\;\; \lambda\, {}^{\prime}{}_{\Omega}\, \varphi \psi \;\; \lambda \oint \varphi \oint \psi \oint$$

Furthrmore,

$$\oint \psi \oint \varphi \oint \ {}_{\Omega} \psi\varphi \ {}_{\Omega} \times \psi\varphi \ \times {}_{\Omega} \varphi\psi \ \oint \varphi \oint \psi \oint$$

The fact that complex conjugation can be moved outside the integration symbol is rigorously justified as follows. Let $\psi\colon \Omega \,\fbox{$\cdot$}\,$ be a complex-valued function defined on Ω, and set $\psi_1, \psi_2\colon \Omega \,\fbox{$\cdot$}\, R$ by letting $\psi_1 \ \oslash\psi$ and $\psi_2 \ \oslash\psi$. By definition, ψ is Lebesgue integrable on Ω if and only if ψ_1 and ψ_2 are Lebesgue integrable on Ω and in that case we put

$${}_{\Omega} \psi \ \ {}_{\Omega} \psi_1 \ \ i \, {}_{\Omega} \psi_2$$

Clearly, if ψ is Lebesgue integrable, ψ is Lebesgue integrable and

$$\begin{aligned} {}_{\Omega} \psi \ \ & {}_{\Omega} \psi_1 + i \, {}_{\Omega} \psi_2 \\ & \times {}_{\Omega} \psi . \end{aligned}$$

Finally, the monotonicity of the integral ensures that

$$\oint \psi \oint \psi \oint \ {}_{\Omega} \psi\psi \ {}_{\Omega} \oint \psi \oint^2 \curlywedge 0 ,$$

and ${}_{\omega} \oint \psi \oint^2 \ 0$ if and only if ψ belongs to the class of equivalence of the function which is identically zero on Ω, *i.e.*, $\psi \ 0$. ∎

THEOREM 0.15 (RIESZ-FISCHER): $\Omega 2, \oint + \oint + \oint$ is an Hilbert space.

Proof. We need to check completeness. ∎

0.4.4 Hölder and Minkovsky inequalities and p-norms

We will discuss the p-norms for R^N, ^{N}, p and Ωp. All these examples are generalization of those discussed in previous sections.

The Banach spaces p and Ωp

We know that for $p \ 2$, 2 is a Hilbert space. This is also the only value of p for which p is an Hilbert space.

THEOREM 0.16: If $p \ominus 2$, p is not a Hilbert space.

Proof. By absurdum, let's suppose p is an Hilbert space for $p \ominus 2$. Then the parallelogram law applies for all $\varphi \;\; \varphi_{n\, n \curlywedge 0}$ and $\psi \;\; \psi_{n\, n \curlywedge 0}$ in p. Select

$$\varphi_n \left]\begin{array}{ll} 1 & \text{if } n \;\; 1 \\ 1 & \text{if } n \;\; 2 \\ 0 & \text{if } n > 2 \end{array}\right. , \qquad \psi_n \left]\begin{array}{ll} 1 & \text{if } n \;\; 1 \\ +1 & \text{if } n \;\; 2 \\ 0 & \text{if } n > 2 \end{array}\right. .$$

and so

$$\varphi_n \;\; \psi_n \left]\begin{array}{ll} 2 & \text{if } n \;\; 1 \\ 0 & \text{if } n \;\; 2 \\ 0 & \text{if } n > 2 \end{array}\right. , \qquad \varphi_n + \psi_n \left]\begin{array}{ll} 0 & \text{if } n \;\; 1 \\ 2 & \text{if } n \;\; 2 \\ 0 & \text{if } n > 2 \end{array}\right. .$$

We have

$$\begin{aligned} \oint \varphi \;\; \psi \oint \;\; & \Big[\sum_{n 1}^{\{} \oint \varphi_n \;\; \psi_n \oint^{p} \Big[^{\frac{1}{p}} \;\; \big[2^p\big[^{\frac{1}{p}} \;\; 2\,, \\ \oint \varphi + \psi \oint \;\; & \Big[\sum_{n 1}^{\{} \oint \varphi_n + \psi_n \oint^{p} \Big[^{\frac{1}{p}} \;\; \big[2^p\big[^{\frac{1}{p}} \;\; 2\,, \\ \oint \varphi \oint \;\; & \Big[\sum_{n 1}^{\{} \oint \varphi_n \oint^{p} \Big[^{\frac{1}{p}} \;\; \big[1^p \;\; 1^p\big[^{\frac{1}{p}} \;\; 2^{\frac{1}{p}}\,, \\ \oint \psi \oint \;\; & \Big[\sum_{n 1}^{\{} \oint \psi_n \oint^{p} \Big[^{\frac{1}{p}} \;\; \big[1^p \;\; 1^p\big[^{\frac{1}{p}} \;\; 2^{\frac{1}{p}}\,, \end{aligned}$$

and the parallelogram law

$$\oint \varphi + \psi \oint^{2} \;\; \oint \varphi \;\; \psi \oint^{2} \;\; 2 \oint \varphi \oint^{2} \;\; 2 \oint \psi \oint^{2}$$

reads in this case

$$2^2 \;\; 2^2 \;\; 2 + 2^{\frac{2}{p}} \;\; 2 + 2^{\frac{2}{p}}\,,$$

that is

$$2 \;\; 2^{\frac{2}{p}}\,.$$

Thsi equation is true if and only if $\frac{2}{p} \;\; 1$, *i.e.*, if $p \;\; 2$, which is not the case. ∎

0.5 Orthogonality

The notion of inner product allows to naturally equip an inner product space with a notion of orthogonality between vectors Some of the following results (e.g., Pitagorean theorem) can be used to some extend to export the notion of orthogonality to general Banach spaces, but here we will restrict ourselves to Hilbert spaces.

Definition 0.11: Let $X, \langle \cdot, \cdot \rangle$ Orthogonality of two vectors be any inner product space. For every $\varphi, \psi \in X \times X$, φ and ψ are said to be mutually "orthogonal" if

$$\langle \varphi, \psi \rangle = 0 \,. \tag{33}$$

The extension to a (possibly not-countable) set of vectors is trivial and it is formalized by the following definition.

Definition 0.12: Let $X, \langle \cdot, \cdot \rangle$ Orthogonality of a set of vectors be any inner product space and $W \subseteq X$ a non-empty subset of X. W is a set of mutually orthogonal vectors if

$$\langle \varphi, \psi \rangle = 0, \quad \forall \varphi, \psi \in W \times W \text{ with } \varphi \neq \psi \,. \tag{34}$$

If moreover

$$\langle \varphi, \varphi \rangle = 1, \quad \forall \varphi \in W \tag{35}$$

the set W is said to be "orthonormal".

In particular, given a sequence $N \to X$ of vectors of X, we say that it is "orthonormal" if the image set is an orthonormal set.

Theorem 0.17 (Pythagorean theorem): Let $X, \langle \cdot, \cdot \rangle$.

1. for every orthogonal pair of vectors φ and ψ in X,

$$\| \varphi + \psi \|^2 = \| \varphi \|^2 + \| \psi \|^2 ;$$

2. for every orthogonal finite set of $N \in N$ vectors $\varphi_1, \varphi_2, \dots \varphi_N$ in X,

$$\Big\| \sum_{k=1}^{N} \varphi_k \Big\|^2 = \sum_{k=1}^{N} \| \varphi_k \|^2$$

3. if $\{\varphi_k\}_{k\in N}$ is a sequence $N \to X$ of mutually orthogonal vectors of X, then $\sum_{k=1}^{\infty} \varphi_k$ is convergent in X *if and only if* $\sum_{k=1}^{\infty} \|\varphi_k\|^2$ is convergent in R and in that case we have the following identity:

$$\left\| \sum_{k=1}^{\infty} \varphi_k \right\|^2 = \sum_{k=1}^{\infty} \|\varphi_k\|^2 .$$

Proof. In the first case the proof is straightforward, we have already found the same expression in the proof of the parallelogram identity, we have by direct computation that

$$\begin{aligned}
\|\varphi + \psi\|^2 &= \langle \varphi + \psi, \varphi + \psi \rangle \\
&= \langle \varphi, \varphi \rangle + \langle \varphi, \psi \rangle + \langle \psi, \varphi \rangle + \langle \psi, \psi \rangle \\
&= \|\varphi\|^2 + 2\underbrace{\Re\langle \varphi, \psi \rangle}_{0} + \|\psi\|^2 \\
&= \|\varphi\|^2 + \|\psi\|^2 .
\end{aligned}$$

In the second case, the proof is by induction on N. The case $N = 2$ has been shown explicitly in the first part of the proof. Now, let's check that

$$\left\| \sum_{k=1}^{N} \varphi_k \right\|^2 = \sum_{k=1}^{N} \|\varphi_k\|^2$$

for a given N implies

$$\left\| \sum_{k=1}^{N+1} \varphi_k \right\|^2 = \sum_{k=1}^{N+1} \|\varphi_k\|^2$$

for $N \geq 1$. We have

The third case shows a classic trick of using the Cauchy criterion of convergence.

■

0.6 Complete orthonormal sets and their characterization

Definition 0.13: Let $(X, \langle \cdot, \cdot \rangle)$ a Hilbert space and $\{\varphi_k\}_k$ a family of vectors in X. $\{\varphi_k\}_k$ is called a "complete orthonormal set" if

1. $\{\varphi_k\}_k$ is an orthonormal set in X;

2. for every ψ ⊞ X,

$$∮ \varphi_k ∮ \psi ∮ \quad 0, \quad ⦶ k$$

if and only if ψ 0.

0.7 The Fouries series in Ω2

0.8 Linear operators: first properties

0.9 Riesz representation theorem

[heading=subbibliography]

The rules of the game

phenomena do not occur in a Hilbert space. They occur in a laboratory.

Asher Peres *Quantum theory: concepts and methods*

[heading=subbibliography]

APPENDIX A

Mathematica code for strongly-driven vacuum Rabi

NUMERICAL code for solving the transmon–cavity master equation (**??**) follows. After some initialization tasks, the first part of the code concerns solving the transmon Hamiltonian (**??**) in a truncated charge basis (**??**) by exact diagonalization. The entry point for this diagonalization is the function `egtrans[]`. Because the diagonalization is a somewhat expensive procedure, and since the resulting energies and matrix elements are smooth functions of E_{J} and E_{C} (and almost independent of n_{g}), the next part of the code, entered via `makeinterp[]`, constructs an opaque interpolation object that can evaluate these energies and matrix elements for a range of $E_{\mathrm{J}}/E_{\mathrm{C}}$ ratios. Next there is some code which checks the previous calculations for convergence.

The purpose of the next block of code is to construct driven Jaynes–Cummings Hamiltonian (**??**), storing it in `H0s`. Several utility functions for finding the eigenvalues of this Hamiltonian are also created. These functions are used for determining E_{C} from two-tone pump-probe experiments, and for correlating features in the full nonlinear spectrum with the associated multi-photon transitions.

The generator L of the semigroup is constructed next, firstly as the functional operator `lindblad[]` and then in the matrix form $M^{\boxed{+}}$ of (**??**). The final step is to solve (**??**) to obtain

ρ

s, $andreusethefactorizationforsolving$**??**$.Theselaststepsareperformedbythefunction$`steadyst`

For bookkeeping convenience, various parts of the code make use of the qmatrix package by T. Felbinger [?] for setting up the problem, although native Mathematica matrices are significantly faster to manipulate. The former are stripped and packed to produce the latter, for all the numerically intensive algorithms, losing convenience but gaining efficiency.

Initialization

```
$HistoryLength = 0;
```

Load packages

```
<< qmatrix.m
```

```
Needs["Notation`"]
```

Define symbols

```
Symbolize[γ₁];
Symbolize[γφ];
Symbolize[H_J-C];
Symbolize[H₂];
Symbolize[σˣ];
Symbolize[σʸ];
Symbolize[σᶻ];
Symbolize[σ⁺];
Symbolize[σ⁻];
Symbolize[â];
Symbolize[â†];
```

```
Symbolize[ωa];
InfixNotation[·, NonCommutativeMultiply];
Symbolize[T₁];
Symbolize[T₂];
Symbolize[t₁];
Symbolize[t₂];
Symbolize[H₀];
Symbolize[ωr];
Symbolize[ωd];
Symbolize[Hd];
Symbolize[Δd];
```

```
Symbolize[n̂];
Symbolize[q̂];
Symbolize[αr];
Symbolize[EJ];
Symbolize[EC];
Symbolize[ng];
Symbolize[HQ];
Symbolize[Hg];
Symbolize[ℒ_];
Symbolize[ρ̂];
Symbolize[ĝ];
```

■ System modes

```
qubitletter = Characters["GEFH"] ~ Join ~ CharacterRange["J", "Z"];
```

```
levels::usage =
  "levels represents the number of levels kept in the truncation of the
    qubit and cavity Hilbert spaces. Change it only using setlevels[]";
```

```
setlevels::toofew = "Too few levels `1`; at least 2 needed";
setlevels::usage = "setlevels[n] sets things
    up to keep n transmon levels and n cavity levels";
setlevels[n_Integer ? (# > 1 || Message[setlevels::toofew, #] &)] := (
  Unprotect[levels];
  levels = n;
  Protect[levels];
  setSystem[qubit, cavity];
  setModeType[qubit, {bosonic, levels}];
  setModeType[cavity, {bosonic, levels}];
  "System set to dimension: " <> ToString@dimension[system])
```

■ Notations

■ Superoperators

```
𝒟 /: 𝒟[A_matrix ? properMatrixQ][ρ_matrix ? properMatrixQ] :=
 A · ρ · hc[A] - hc[A] · A · ρ / 2 - ρ · hc[A] · A / 2
```

Operators

```
σ⁺ := matrix[op[ad, qubit]];
σ⁻ := matrix[op[a, qubit]];
â† := matrix[op[ad, cavity]];
â := matrix[op[a, cavity]];
n̂ := â† · â;
q̂ := σ⁺ · σ⁻;
```

```
AddInputAlias["sp" → σ⁺];
AddInputAlias["sm" → σ⁻];
AddInputAlias["ad" → â†]
AddInputAlias["nh" → n̂];
AddInputAlias["qh" → q̂];
```

Options

```
SetOptions[Manipulator, Appearance → "Labeled"];
```

Transmon Calculations

Do the matrix solve

This function egtrans[] gives the eigenenergies e_j and the coupling terms g_{ij} and then also calculates the derivative of these wrt E_J / E_C.

Because it calculates the derivative by 1st-order perturbation theory, it has problems with degeneracies when E_J / E_C is low enough compared to cutoff that there are levels with (almost) degeneracies at $n_g \in \{0, 1/2\}$. Consider using $n_g = 0.5 + \epsilon$ instead.

We have to manually correct the signs of the g_{ij} because **`Eigensystem[]`** doesn't guarantee a consistent phase for the eigenvectors.

I haven't checked whether it's better to use a sparse solver or the dense one, but either way we need to get all of the eigenstates for the perturbation theory, so we should not use Krylov methods.

We also normalize things so that $e_0 \equiv 0$, $e_1 \equiv 1$, $g_{12} = g_{21} \equiv 1$.

```
egtrans::usage =
  "egtrans[ng, EjEc, cutoff] gives {e, g, ⅆe/ⅆ(Ej / Ec), ⅆg/ⅆ(Ej / Ec)}";
egtrans::toofew = "Cutoff `1` is too low; must be at least 2";
Block[{fx, gx, hx, part, x},
```

```
Hold[egtrans[ng_?NumericQ, EjEc_?NumericQ,
    cutoff_Integer?(# > 1 || Message[egtrans::toofew, #] &)] := Module[
    {h = SparseArray[{Band[{1, 1}] → 4 (Range[-cutoff, cutoff] - ng)^2}],
     hv = SparseArray[
       {Band[{1, 2}], Band[{2, 1}]} → -1., {2 cutoff + 1, 2 cutoff + 1}],
     n = SparseArray[{Band[{1, 1}] → Table[m - ng, {m, -cutoff,
           cutoff}]}], e, v, e2, v2, o, g, de, dv2, dg, sgn},
    {e, v} = Eigensystem[h + EjEc/2 hv];
    o = Ordering@e;
    e2 = e[[o]];
    v2 = v[[o]];
    g = v2.n.v2^T;
    sgn = Sign@g;
    g = sgn g;
    de = #.hv.# & /@ v2 / 2;
    dv2 = Table[Sum[If[i == j, 0, v2[[j]] (v2[[j]].hv.v2[[i]]) / (e2[[i]] - e2[[j]])],
       {j, 2 cutoff + 1}], {i, 2 cutoff + 1}];
    dg = sgn (dv2.n.v2^T + v2.n.dv2^T) / 2;
    {(e2 - e2[[1]]) / (e2[[2]] - e2[[1]]),
     (D[(fx[x] - gx[x]) / (hx[x] - gx[x]), x] /. {fx'[x] → de, fx[x] → e2,
          gx'[x] → part[de, 1], gx[x] → part[e2, 1],
          hx'[x] → part[de, 2], hx[x] → part[e2, 2]}) //
       FullSimplify // Experimental`OptimizeExpression,
     g / g[[1, 2]],
     (D[fx[x] / gx[x], x] /. {fx[x] → g, fx'[x] → dg,
          gx[x] → part[g, 1, 2], gx'[x] → part[dg, 1, 2]}) //
       FullSimplify // Experimental`OptimizeExpression}
   ];
  ] /. x_Experimental`OptimizeExpression :> RuleCondition[x] /.
   Experimental`OptimizedExpression[x_] :> x /.
  HoldPattern[part] → Part // ReleaseHold;
]
```

Now we need to interpolate the results of the numerical calculation of e_i and g_{ij}.

The indices i,j are zero-based...

Interpolation of the solutions

```
energyinterp::usage =
  "energyinterp[{f2, f3, ...}, ng, cutoff, {min, max, step}] represents
    a function that interpolates the transmon energies.";
couplinginterp::usage = "energyinterp[{f2, f3, ...}, ng,
     cutoff, {min, max, step}] represents a
    function that interpolates the transmon couplings.";

interpf::level =
  "Tried to calculate for transmon level:`1`, but interpolating
    function was only defined for levels 0..`2`";
interpf::dom = "Tried to calculate for EJ/EC of `1`, but
    interpolating function was only defined for `2`≤ EJ/EC ≤`3`";
interpf::invalidform = "Invalid form for a transmon interpolation";
```

```
Unprotect[energyinterp, couplinginterp];
```

```
idx::usage = "idx[] has the attribute NHoldAll";
SetAttributes[idx, NHoldAll];
```

```
transmoninfo[ng_, c_, {min_, max_, step_}] :=
  Column[{"Ei[EJ/EC]", "i:0.." <> ToString[c], HoldForm[min ≤ "EJ/EC" ≤ max],
    "interp step: " <> ToString@step, HoldForm["ng" == ng]}];
```

▪ energyinterp[]

```
energyinterp[a__][i : Except[_idx]] := energyinterp[a][idx@i];

energyinterp[___][idx@0] = 0. &;
energyinterp[___][idx@1] = 1. &;

energyinterp[_, _, c_, _][idx@i_] /;
   (If[NumericQ[i] && ! TrueQ[0 ≤ i ≤ c && i ∈ Integers],
     Message[interpf::level, i, c]; Abort[]];
    False) := None;

energyinterp[l_, _, c_, {min_, max_, _}][idx@i_][x_] /;
   (If[NumericQ[x] && ! TrueQ[min < x < max],
     Message[interpf::dom, x, min, max]; Abort[]];
    NumericQ[x] && NumericQ[i] && min ≤ x ≤ max && 2 ≤ i ≤ c) := l⟦i - 1⟧[x];

Derivative[d_Integer /; d ≥ 1][
     energyinterp[l_, _, c_, {min_, max_, _}][idx@i_]][x_] /;
   (If[NumericQ[x] && ! TrueQ[min < x < max],
     Message[interpf::dom, x, min, max]; Abort[]];
    NumericQ[x] && NumericQ[i] && min ≤ x ≤ max && 2 ≤ i ≤ c) :=
  Derivative[d][l⟦i - 1⟧][x];

Format[energyinterp[l : {__InterpolatingFunction},
      ng_?NumericQ, c_Integer?(2 ≤ # &), mm : {min_, max_, step_} /;
        0 < min < max && 0 < 10 step < max - min][idx@i_] /; Length[l] + 1 == c] :=
  Tooltip[HoldForm["E"i], transmoninfo[ng, c, mm]];

Format[energyinterp[l : {__InterpolatingFunction},
    ng_?NumericQ, c_Integer?(2 ≤ # &), mm : {min_, max_, step_} /;
     0 < min < max && 0 < 10 step < max - min]] := energyinterp["<>", ng, c, mm];
```

- **couplinginterp[]**

```
couplinginterp[___][idx@1, idx@0] =
  couplinginterp[___][idx@0, idx@1] = 1. &;

couplinginterp[a_, b_, c_, d_][i : Except[_idx], j : Except[_idx]] :=
 couplinginterp[a, b, c, d][idx@i, idx@j]

couplinginterp[_, _, c_, _][idx@i_, idx@j_] /;
   (If[NumericQ[i] && ! TrueQ[0 ≤ i ≤ c && i ∈ Integers],
     Message[interpf::level, {i, j}, c]; Abort[]];
    If[NumericQ[j] && ! TrueQ[0 ≤ j ≤ c && j ∈ Integers],
     Message[interpf::level, {i, j}, c]; Abort[]];
    False) := None;

couplinginterp[l_, _, c_, {min_, max_, _}][idx@i_, idx@j_][x_] /;
   (If[NumericQ[x] && ! TrueQ[min < x < max],
     Message[interpf::dom, x, min, max]; Abort[]];
    NumericQ[x] && NumericQ[i] && NumericQ[j] && min ≤ x ≤ max &&
     0 ≤ i ≤ c && 0 ≤ j ≤ c) := l⟦i + 1, j + 1⟧[x];
Derivative[d_][couplinginterp[l_, _, c_, {min_, max_, _}][idx@i_, idx@j_]][
    x_] /;
   (If[NumericQ[x] && ! TrueQ[min < x < max],
     Message[interpf::dom, x, min, max]; Abort[]];
    NumericQ[x] && NumericQ[i] && NumericQ[j] && min ≤ x ≤ max &&
     0 ≤ i ≤ c && 0 ≤ j ≤ c) := Derivative[d][l⟦i + 1, j + 1⟧][x];

Format[couplinginterp[l_, ng_?NumericQ, c_Integer?(2 ≤ # &),
      mm : {min_, max_, step_} /; 0 < min < max && 0 < 10 step < max - min]
     [idx@i_, idx@j_] /; Dimensions[l] == {c, c} + 1] :=
  Tooltip[HoldForm[g_ij], transmoninfo[ng, c, mm]];

Format[couplinginterp[l_?MatrixQ, ng_?NumericQ, c_Integer?(2 ≤ # &),
    mm : {min_, max_, step_} /; 0 < min < max && 0 < 10 step < max - min]] :=
  couplinginterp["<>", ng, c, mm];
```

- **Finish up defining tags**

```
(e : energyinterp[___][_][_])* ^:= e;
```

```
(c : couplinginterp[___][__][_])* ^:= c;
SetAttributes[{energyinterp, couplinginterp}, {NHoldAll}];
Protect[energyinterp, couplinginterp];
```

```
SetAttributes[evalinterp, HoldAll];
evalinterp[x_] := x /. {idx[i_] :→ i,
   energyinterp[{l__}, __] :→ ({0, 1, l}[[# + 1]] &),
   couplinginterp[l_, __] :→ (l[[#1 + 1, #2 + 1]] &)}
```

Construct interpolations

```
makeinterp::usage =
  "makeinterp[ng, cutoff, levels, {min, max, step}] gives e[i, Ej/Ec],
    g[i, j, Ej/Ec] for min ≤ Ej/Ec ≤ max, (i,j = 0,...,levels-1),
    using 2cutoff+1 charge-basis transmon levels in the calculation";
makeinterp::levcut = "Require 2≤levels≤cutoff , but levels=`1`, cutoff=`2`";
makeinterp::step = "Require 0 < 10*step < max-min";
makeinterp::minmax = "Require 0<min<max but min=`1`, max=`2`";

Options[makeinterp] = {InterpolationOrder → 7};
```

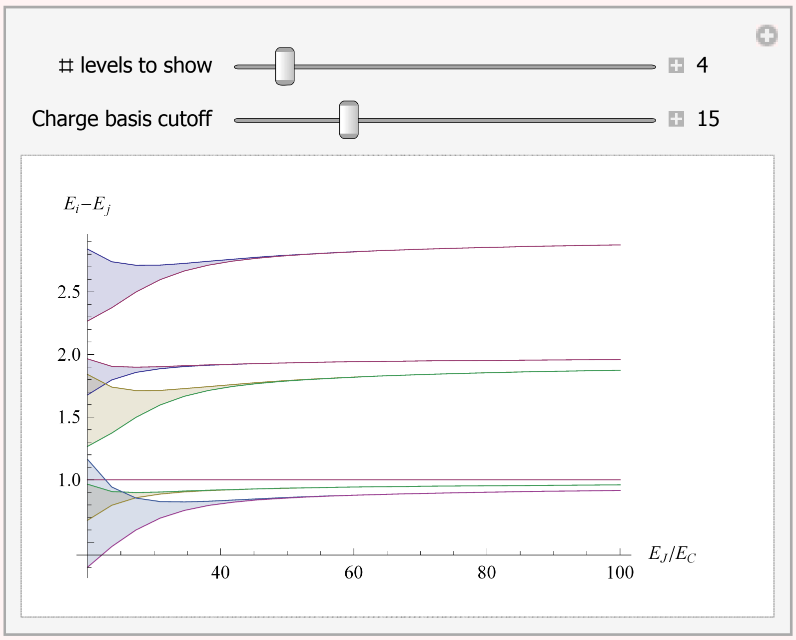
```
makeinterp[ng_?NumericQ, cutoff_Integer, levels_Integer, mms :
    {min_?NumericQ, max_?NumericQ, step_?NumericQ}, OptionsPattern[]] /;
  (2 ≤ levels ≤ cutoff || Message[makeinterp::levcut, levels, cutoff]) &&
   (0 < min < max || Message[makeinterp::minmax, min, max]) &&
   (0 < 10 step < max - min || Message[makeinterp::step]) :=
 Module[{egtab, x},
  egtab = Table[{N@x, egtrans[ng, N@x, cutoff]}, {x, min, max, step}];

  {energyinterp[
    Table[
     Interpolation[Cases[egtab, {x_, {e_, de_, _, _}} :→ {{x}, e[[i]], de[[i]]}],
      InterpolationOrder → OptionValue[InterpolationOrder]],
     {i, 3, levels}], ng, levels - 1, mms],
   couplinginterp[Table[Interpolation[
      Cases[egtab, {x_, {_, _, g_, dg_}} :→ {{x}, g[[i, j]], dg[[i, j]]}],
      InterpolationOrder → OptionValue[InterpolationOrder]],
     {i, levels}, {j, levels}], ng, levels - 1, mms]}
 ]
```

Check the transmon calculations

What does it look like?

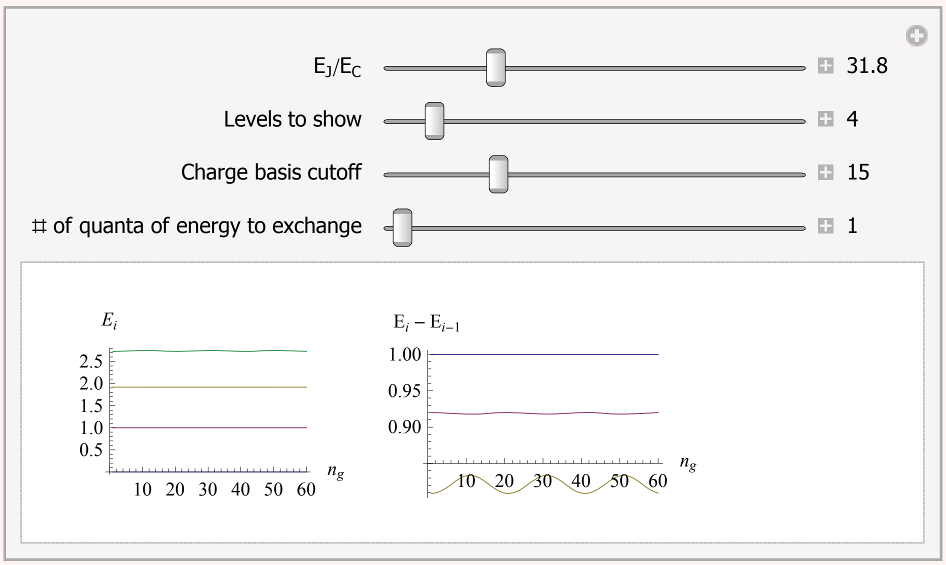
Spectrum vs E_J / E_C

```
Manipulate[Module[
  {x = Transpose[Table[egtrans[ng, ejec, cut][[1, ;; ls]], {ejec, 10, 100, 4},
      {ng, {0.0001, 0.5001}}], {2, 3, 1}]},
  Show[
   Table[
    ListLinePlot[Flatten[x[[δ + 1 ;;]] - x[[ ;; -(δ + 1)]], {1, 3}], PlotRange → All,
     AxesLabel → {"EJ/EC", "Ei-Ej"}, Filling → Table[2 n - 1 → {2 n}, {n, ls - δ}],
     DataRange → {20, 100}], {δ, 1, ls - 1}]]],
 {{ls, 4, "# levels to show"}, 3, cut, 1},
 {{cut, 15, "Charge basis cutoff"}, 10, 30, 1}]
```



Energy levels and spectra vs n_g

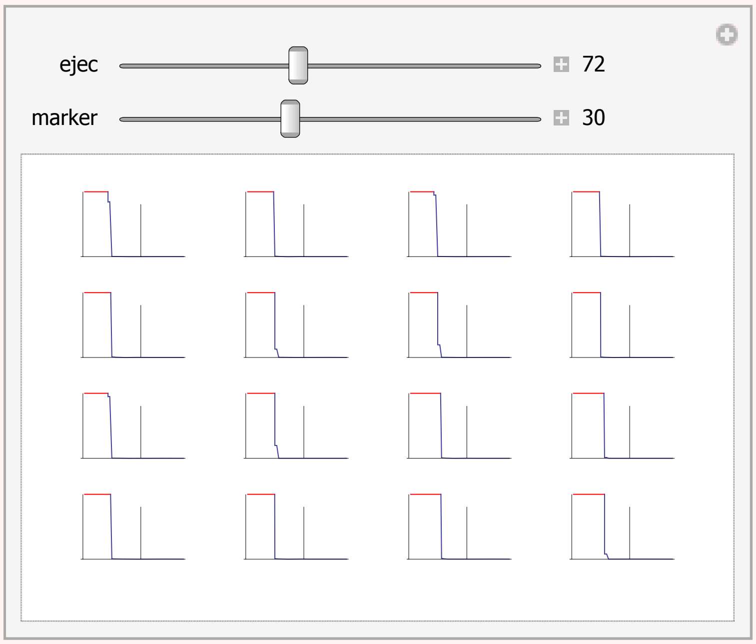
```
Manipulate[Module[
  {x = Table[egtrans[ng, egec, cut][[1, ;; ls]], {ng, -.4999, .5, .05}], xxx},
  xxx = Flatten[Table[x, {3}], 1]ᵀ;
  GraphicsRow[{
    ListLinePlot[xxx, AxesLabel → {"n_g", "E_i"}],
    ListLinePlot[xxx[[δ + 1 ;;]] - xxx[[;; -(δ + 1)]], PlotRange → All,
     AxesLabel → {"n_g", With[{δ = δ}, HoldForm["E"_i - "E"_(i-δ)]]}]}]],
 {{egec, 50., "E_J/E_C"}, 10., 100.},
 {{ls, 4, "Levels to show"}, 3, cut, 1},
 {{cut, 15, "Charge basis cutoff"}, 10, 30, 1},
 {{δ, 1, "# of quanta of energy to exchange"}, 1, ls - 1, 1}]
```



■ Choose a cutoff

```
et[ng_?NumericQ, EjEc_?NumericQ, cutoff_?IntegerQ] := Module[{e, v, v2},
  {e, v} = Eigensystem[SparseArray[
     {{i_, i_} :> 4 (i - Floor[cutoff / 2] - ng - 1)^2,
      {i_, j_} /; Abs[i - j] == 1 → -EjEc / 2}, {cutoff, cutoff}]];
  v2 = v[[Ordering[e]]];
  v2.DiagonalMatrix[Table[m - Floor[cutoff / 2], {m, 0, cutoff - 1}]].v2ᵀ];
```

```
Manipulate[Module[{ccm = 60, etm, ll = 4},
  etm = Abs[et[.5, ejec, ccm][[ ;; ll,  ;; ll]]];
  GraphicsGrid@Map[ListLinePlot[#, PlotRange → {0, 10^-12},
      Ticks → Dynamic@{{{marker, "", {.5, 0}}}, None}, ClippingStyle → Red] &,
    Transpose[Table[Abs[Abs[et[.5, ejec, cc][[ ;; ll,  ;; ll]]] - etm],
      {cc, 2 ll + 1, ccm}], {3, 1, 2}], {2}]],
 {{ejec, 72}, 30, 130}, {{marker, 30}, 10, 60, 1}]
```



Mathieu function calculation, for comparison

$$k[m\_, n_g : \_] := \mathrm{Sum}\left[\left(\mathrm{Round}\left[2 n_g + 1/2\right] \sim \mathrm{Mod} \sim 2\right) \left(\mathrm{Round}\left[n_g\right] - 1\ (-1)^m\ ((m+1) \sim \mathrm{Quotient} \sim 2)\right), \{1, \{-1, 1\}\}\right];$$

$$a_{v\_}[x\_] := \mathrm{MathieuCharacteristicA}[v, x];$$

$$E_{m\_}\left[n_g : \_, E_J : \_, E_C : \_\right] := E_C\ a_{-2\,(n_g - k[m, n_g])}\left[-\frac{E_J}{2\,E_C}\right];$$

Quantities derived from the transmon solutions

$$\epsilon_{m\_}[E_J : \_, E_C : \_] := (-1)^m E_C \frac{2^{4m+5}}{m!} \sqrt{\frac{2}{\pi}} \left(\frac{E_J}{2\,E_C}\right)^{\frac{m}{2}+\frac{3}{4}} e^{-\sqrt{8 E_J/E_C}}$$

$$\tilde{\epsilon}_{m\_}[E_J : \_, E_C : \_] := \mathrm{Abs}[E_m[0.0001, E_J, E_C] - E_m[0.4999, E_J, E_C]]$$

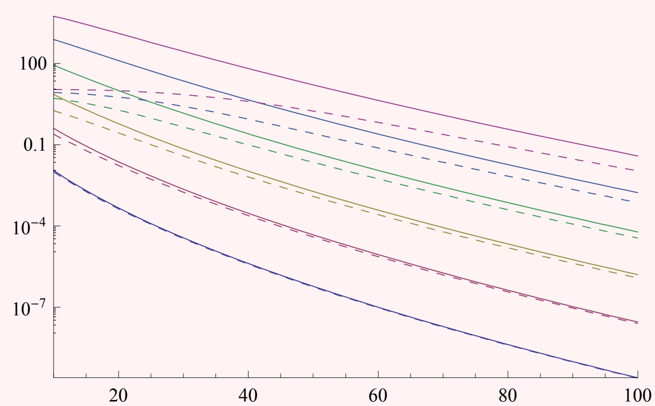
```
ϵ[_] := Sum_{m}^{levels} ϵ_m[72, 1] matrix@op[basis, qubit, m]
```

```
E_{m_,n_} = E_m[.0001, EjEc, 1] - E_n[.0001, EjEc, 1];
En[EjEc_?NumericQ]_{m_,n_} := Module[{q = etrans[.5, EjEc]}, q[[m + 1]] - q[[n + 1]]]
```

```
H_Q[EjEc_] := Sum_{m=0}^{levels-1} (En[EjEc]_{m0} / En[EjEc]_{10}) matrix@op[basis, qubit, m + 1];
```

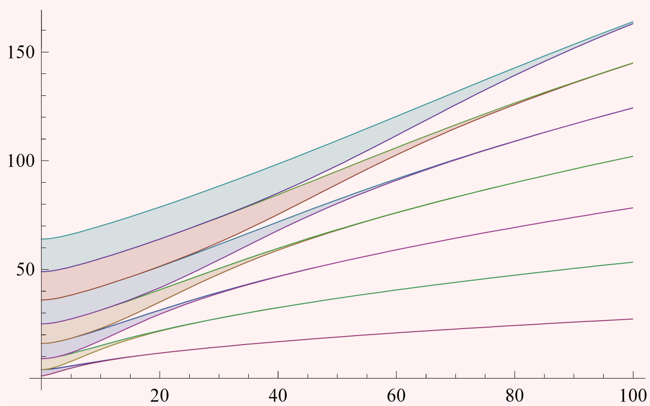
- **Asymptotic expression compared with exact**

```
Show[LogPlot[Evaluate@Table[Tooltip[Abs[ϵ_m[E_J, 1]], m], {m, 0, 5}],
   {E_J, 10, 100}, PlotRange → All],
 LogPlot[Evaluate@Table[Tooltip[Abs[ϵ̃_m[E_J, 1]], m], {m, 0, 5}],
   {E_J, 10, 100}, PlotRange → All, PlotStyle → Dashed],
 Plot[x, {x, 0, 100}]]
```



■ Transmon dispersion

```
Plot[Evaluate@Table[{
    Tooltip[Em[0.00001, EJ, 1] - E0[0.00001, EJ, 1], m],
    Tooltip[Em[0.4999, EJ, 1] - E0[0.00001, EJ, 1], m]}, {m, 1, 7}],
 {EJ, 0, 100}, PlotRange → All, Filling → Table[2 n - 1 → {2 n}, {n, 7}]]
```



Solve the system

■ Parameters

NB: These quantites are protected because everything here depends on them being symbols.
They should only have values assigned to them in a Block[] or similar structure.

```
params =
  {ωr, ωd, δ, g, ξ, ejec, γϕ, (*γϕ2,*)γ, pm, κ(*,pf1,pf2,pf3,pf4,pf5*)};
ωr::usage = "ωr is cavity frequency";
ωd::usage = "ωd is the frequency of the drive";
δ::usage = "δ is given by ωr-ωqubit == δ";
g::usage =
  "g is the coupling strength g01 (between the 0↔1 transition of the
    transmon and the cavity annihilation operator)";
ξ::usage = "ξ is the drive strength";
ejec::usage = "ejec is the EJ/EC ratio for the transmon";
γϕ::usage = "γϕ is the transmon dephasing strength";
γ::usage = "γ is the transmon relaxation rate";
κ::usage = "κ is the cavity relaxation rate";
pf1* ^= pf1;
pf2* ^= pf2;
pf3* ^= pf3;
pf4* ^= pf4;
pf5* ^= pf5;
Protect[Evaluate@params];
$Assumptions = params ∈ Reals && ℏ > 0;
```

Hamiltonian

Do the normal transmon interpolations

This is the standard interpolation:

```
$maxlevels::usage =
  "$maxlevels is the number of transmon levels calculated so
    far. We need to recalculate the interpolations
    and some other stuff if we want to go higher...";
Unprotect[$maxlevels];
$maxlevels = 8;
Protect[$maxlevels];

{ef1, gf1} = makeinterp[0.4999, 15, $maxlevels, {10, 200, 1}];
{ef2, gf2} = makeinterp[0.0001, 15, $maxlevels, {10, 100, 1}];
{ef1, gf1}
ef1[3][72]
```

```
{energyinterp[<>, 0.4999, 7, {10, 200, 1}],
 couplinginterp[<>, 0.4999, 7, {10, 200, 1}]}
```

```
2.84936
```

■ Subspace

Set up the basis states (sstates), the projectors onto the degenerate subspaces (psstates) and the size of the Hilbert space for subsequent calculations (nn):

```
ClearAll["bket*"];
sstates :=
 Table[Symbol["bket" <> qubitletter[[j]] <> ToString[i - j]], {i, levels}, {j, i}]
states := Flatten@sstates;
Array[
  (Evaluate@Symbol["bket" <> qubitletter[[#1]] <> ToString[#2 - 1]] := basisKet[
       qubit, #1 ] · basisKet[cavity, #2]) &, {$maxlevels, $maxlevels}];
psstates := projector /@ sstates;
nn := Length@states;
```

■ Set up the Hamiltonian

H_Q is in units of ω_{01}

```
setlevels[3]
{ef, gf} = {ef1, gf1};
```

```
System set to dimension: 9
```

```
H_Q := ℏ Sum[ef[m][ejec] matrix@op[basis, qubit, m + 1], {m, 0, levels-1}];
(*like (â·σ⁺+â†·σ⁻) *)
ĝ := Sum[gf[i - 1, i][ejec] matrix@op[basis, qubit, i, i + 1], {i, levels-1}];
H_g := ℏ g (# + hc[#] &@ (ĝ · â†));
```

We are in the rotating frame and make the RWA:

```
H_d := ℏ ξ (â + â†);
(* H_0=(ω_01 H_Q-ω_d q̂)+ℏ (ω_c- ω_d) n̂+g H_g *)
H_0 := ((ω_r - δ) H_Q - ℏ ω_d q̂) + ℏ (ω_r - ω_d) n̂ + H_g;
```

Here's the matrix version of the Hamiltonian (a list of the matrices in each n-excitation subspace, n=1...levels) :

```
H0s := Table[Simplify@Table[trace[hc[sstates[[n, i]]] · H_0 · sstates[[n, j]]],
     {i, n}, {j, n}], {n, levels}];
```

Diagonalizing the Hamiltonian

```
diagfns::usage =
  "diagfns[] returns {energies[...],vectors[...]} functions.";
diagfns[] := Block[{ωd = 0, ℏ = 1, ef = ef1, gf = gf1,
   Eigenvalues, Eigenvectors, PadRight, map},
  With[{H0s = H0s, levels = levels, nn = nn},
   With[{sp = {ωr, δ, g, ejec}},
    {Function[Evaluate@sp, Evaluate[Eigenvalues /@ evalinterp[H0s]]],
     Function[Evaluate@sp,
       Evaluate[Table[With[{ic = i (i - 1) / 2}, PadRight[#, nn, 0., ic] &~
           map~Eigenvectors[H0s[[i]]]], {i, levels}]]] /. map → Map}]]]
```

```
diagfns2::usage =
  "diagfns2[] returns {energies[...],vectors[...]} functions.";
diagfns2[] := Block[{ℏ = 1, ef = ef1, gf = gf1,
   Eigenvalues, Eigenvectors, PadRight, map},
  With[{H0s = H0s, levels = levels, nn = nn},
   With[{sp = {ωr, ωd, δ, g, ejec}},
    {Function[Evaluate@sp, Evaluate[Eigenvalues /@ evalinterp[H0s]]],
     Function[Evaluate@sp,
       Evaluate[Table[With[{ic = i (i - 1) / 2}, PadRight[#, nn, 0., ic] &~
           map~Eigenvectors[H0s[[i]]]], {i, levels}]]] /. map → Map}]]]
```

Show the energy levels and transitions:

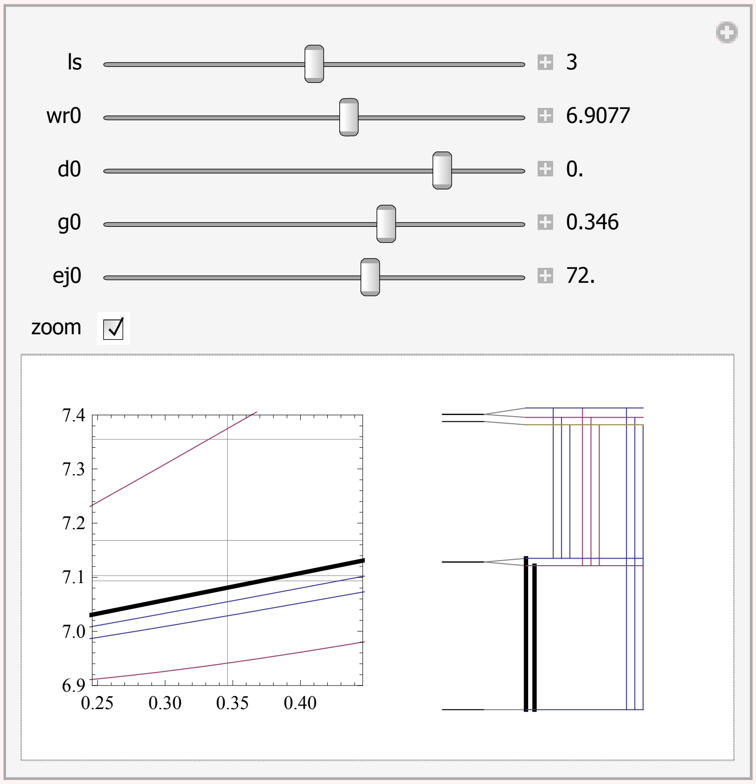
```
transAnn::usage =
  "transAnn[i1,j1,i2,j2] is a tag representing the transition
    between the j1th level of the i1-excitation subspace
    and the j2th level of the i2-excitation subspace";
levelAnn::usage = "levelAnn[i,j] is a tag representing
    the jth level of the i-excitation subspace";
Protect[transAnn, levelAnn];

$hilited::usage =
  "$hilited contains the tag of the currently selected item";

flash::usage = "flash[list,t] flashes
    between styles in the list l, over a total time t";
flash[l_List, t_] := l〚Clock[{1, Length@l, 1}, t]〛;
flashing[s_] :=
  flash[{Directive[s, Dashed], Directive[s, Dashing[{}]]}, 1];
maybeflashing[a_, s_] := Dynamic@If[a === $hilited, flashing@s, s];
handlemouse[g_] :=
  EventHandler[g, "MouseClicked" :→ ($hilited = MouseAnnotation[]),
   PassEventsDown → Automatic];

With[{x1 = 1, x2 = 2, x4 = 0.2`, x5 = 0.15`, x6 = 0.1`, x7 = 0.2`},
  leveldiagram[e0_List, e1_List, ls_Integer] :=
   DynamicModule[{q1, q2},
    {q1, q2} = (5 (ls - 1) # / #〚-1, -1, -1〛) &@{e0, e1};
    handlemouse@
     Graphics[Dynamic@Flatten[{Antialiasing → False,
         Table[{Line[{{0, q1[[i, j]]}, {x1, q1[[i, j]]}}]}, {i, ls}, {j, i}],
         Table[{Gray,
           Line[{{x1, q1[[i, j]]}, {x2, q2[[i, j]]}}]}, {i, ls}, {j, i}],
         Module[{xx = x2 - x4 - x5 - x6 - x7},
          Flatten[{Table[
              xx += KroneckerDelta[i, j1, j2, 1] x4 +
                KroneckerDelta[j1, j2, 1] x5 + KroneckerDelta[j2, 1] x6 + x7;
              With[{s = transstyle[i, j1, i + k, j2], a =
                  transAnn[i, j1, i + k, j2]},
               {maybeflashing[a, s],
                Annotation[Line[
                   {{xx, q2[[i, j1]]}, {xx, q2[[i + k, j2]]}}], a, "Mouse"]}],
              {k, ls - 1}, {i, ls - k}, {j1, i}, {j2, i + k}],
            Table[
              With[{s = levelstyle[i, j], a = levelAnn[i, j]}, {{{maybeflashing[
                    a, s], Annotation[Line[{{x2, q2[[i, j]]}, {xx, q2[[i,
                        j]]}}], a, "Mouse"]}}}], {i, ls}, {j, i}]}, 4]]}, 1]]]];
```

```
transstyle[i1_, j1_, i2_, j2_] := Directive[
   Flatten@{ColorData[1][j1], If[i1 == 1 && i2 == 2, {Thick, Black}, {}],
     If[MatchQ[$hilited, levelAnn[i1, j1] | levelAnn[i2, j2]], Red, {}]}];
levelstyle[i_, j_] := ColorData[1][j];
```

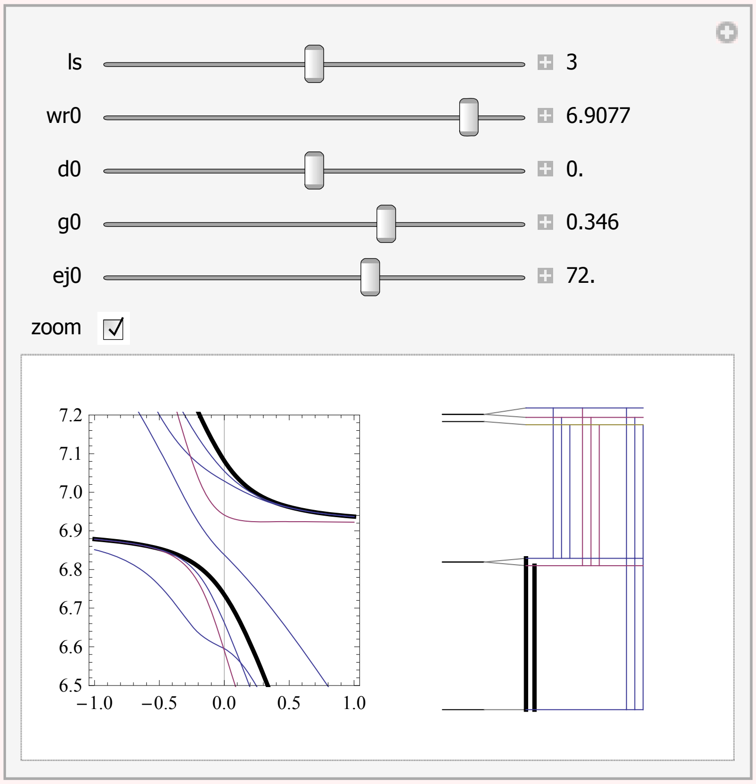
```
setlevels[4]
{energiestt, vectorstt} = diagfns[];
```

```
System set to dimension: 16
```

```
Manipulate[
 DynamicModule[{evtab},
  evtab = Table[{g, energiestt[wr0, d0, g / 2, ej0]}, {g, 0, g0 + .2, g0 / 10}];
  Deploy@GraphicsRow[
    {handlemouse@Graphics[
       Dynamic@Flatten[Table[
          With[{s = transstyle[i, j1, i + k, j2], a = transAnn[i, j1, i + k, j2]},
           {maybeflashing[a, s],
            Annotation[Line[{evtab[[All, 1]], (evtab[[All, 2, i + k, j2]] -
                   evtab[[All, 2, i, j1]]) / k}ᵀ], a, "Mouse"]}],
          {k, ls - 1}, {i, ls - k}, {j1, i}, {j2, i + k}], 3],
       Frame → True, AspectRatio → 1, PlotRangeClipping → True, PlotRange →
        Dynamic[If[zoom, {{g0 - .1, g0 + .1}, {6.9, 7.4}}, All]], GridLines →
        {{g0}, (*{7.365,7.11,7.175,7.31}*){7.355, 7.103, 7.168, 7.093}}],
     leveldiagram[energiestt[wr0, d0, 0, ej0],
      energiestt[wr0, d0, g0 / 2, ej0], ls]}]],
 {{ls, 3}, 2, levels, 1},
 {{wr0, 6.9077}, 6.89, 6.92},
 {{d0, 0.}, -.5, .1},
 {{g0, .346}, 0, .5},
 {{ej0, 72.}, 20, 100},
 {zoom, {True, False}},
 TrackedSymbols → Full,
 Bookmarks → {
   "get Ec" :→
    {ls = 3, wr0 = 6.917458, d0 = -.44265, g0 = 93.88 / 1000, ej0 = 52.12},
   "expt" :→ {ls = 3, wr0 = 6.915, d0 = -.006, g0 = 93.88 / 1000, ej0 = 50}}]
```


ls 3
wr0 6.9077
d0 0.
g0 0.346
ej0 72.
zoom
7.4
7.3
7.2
7.1
7.0
6.9
0.25
0.30
0.35
0.40

```
Manipulate[
 DynamicModule[{evtab},
  evtab = Table[{d0, energiestt[wr0, d0, g0 / 2, ej0]}, {d0, -1, 1, .01}];
  Deploy@GraphicsRow[{
     handlemouse@Graphics[
       Dynamic@Flatten[Table[
          With[{s = transstyle[i, j1, i + k, j2], a = transAnn[i, j1, i + k, j2]},
           {maybeflashing[a, s],
            Annotation[Line[{evtab[[All, 1]], (evtab[[All, 2, i + k, j2]] -
                   evtab[[All, 2, i, j1]]) / k}ᵀ], a, "Mouse"]}],
          {k, ls - 1}, {i, ls - k}, {j1, i}, {j2, i + k}], 3],
       Frame → True, AspectRatio → 1, PlotRangeClipping → True, PlotRange →
        If[zoom, {All, {6.5, 7.2}}, All], GridLines → {{d0}, None}],
     leveldiagram[energiestt[wr0, d0, 0, ej0],
      energiestt[wr0, d0, g0 / 2, ej0], ls]}]],
 {{ls, 3}, 2, levels, 1},
 {{wr0, 6.9077}, 6, 7},
 {{d0, 0.}, -1, 1},
 {{g0, .346}, 0, .5},
 {{ej0, 72.}, 20, 100},
 {zoom, {True, False}},
 TrackedSymbols → Full]
```



Density matrices

Lindblad operators

Here's the Lindblad form of the RHS of the master equation for $\dot{\rho}$:

```
pr := projector[states]
```

```
ℒ1[ρ_?operatorMatrixQ] := -(ⅈ/ℏ) commutator[H0 + Hd, ρ] +
  κ 𝒟[â][ρ] + γ 𝒟[σ⁻][ρ] + γ pm 𝒟[pr · σ⁺ · pr][ρ] + γϕ 𝒟[q̂][ρ] / 2
```

```
ℒ2[ρ_?operatorMatrixQ] :=
 -(ⅈ/ℏ) commutator[H0 + Hd, ρ] + κ 𝒟[â][ρ] + γ 𝒟[ĝ][ρ] + γ pm 𝒟[pr · hc[ĝ] · pr][ρ] + 10^7
   γϕ 𝒟[ Sum[(ef1[m][ejec] - ef2[m][ejec]) matrix@op[basis, qubit, m + 1], {m, 0, levels-1}] ][ρ]
```

```
ℒ3[ρ_?operatorMatrixQ] :=
  -(𝕚/ℏ) commutator[H0 + Hd, ρ] + κ 𝒟[â][ρ] + γ 𝒟[ĝ][ρ] + κ pm 𝒟[pr · â† · pr][ρ] + 10^7 γϕ
   𝒟[Sum[(ef1[m][ejec] - ef2[m][ejec]) matrix@op[basis, qubit, m + 1], {m, 0, levels-1}]][ρ]
```

```
ℒ4[ρ_?operatorMatrixQ] := -(𝕚/ℏ) commutator[H0 + Hd, ρ] + κ 𝒟[â][ρ] +
   γ 𝒟[ĝ][ρ] + γ pm 𝒟[pr · hc[ĝ] · pr][ρ] + κ pm 𝒟[pr · â† · pr][ρ] + 10^7 γϕ
   𝒟[Sum[(ef1[m][ejec] - ef2[m][ejec]) matrix@op[basis, qubit, m + 1], {m, 0, levels-1}]][ρ]
```

```
ℒ5[ρ_?operatorMatrixQ] :=
  -(𝕚/ℏ) commutator[H0 + Hd, ρ] + κ 𝒟[â][ρ] + γ 𝒟[ĝ][ρ] + γ pm 𝒟[pr · hc[ĝ] · pr][ρ] +
   κ pm 𝒟[pr · â† · pr][ρ] + 10^7 𝒟[Sum[pϕ[[m]] matrix@op[basis, qubit, m + 1], {m, 1, levels-1}]][ρ]
pϕ = {pf1, pf2, pf3, pf4, pf5};
```

```
ℒ6[ρ_?operatorMatrixQ] := -(𝕚/ℏ) commutator[H0 + Hd, ρ] + κ 𝒟[â][ρ] +
   γ 𝒟[ĝ][ρ] + γ pm 𝒟[pr · hc[ĝ] · pr][ρ] + κ pm 𝒟[pr · â† · pr][ρ] +
   10^7 γϕ 𝒟[Sum[(ef1[m][ejec] - ef2[m][ejec]) matrix@op[basis, qubit, m + 1], {m, 0, levels-1}]][
     ρ] + γϕ2 𝒟[q̂][ρ] / 2
```

Now put it in matrix form and project onto our reduced Hilbert space:

```
lindblad::trnz = "The trace of ρ̇ was not zero!";
lindblad::usage =
  "lindblad[ℒ] returns {ρ̂, ρij, ρ̇ij} for a given Lindblad operator ℒ[ρ̂]";

lindblad[ℒ_] := With[{nn = nn, states = states},
   Module[{ρs, ρ, Π, ℒρ, Πℒρ, δρ},
    ρs =
     Table[Symbol["ρ" <> ToString[i] <> "x" <> ToString[j]], {i, nn}, {j, nn}];
    ρ = Simplify[Sum[ρs⟦i, j⟧ states⟦i⟧ · hc[states⟦j⟧], {i, nn}, {j, nn}]];
    Π = Simplify@projector[states];
    ℒρ = ℒ[ρ];
    Πℒρ = Π · ℒρ · Π;
    δρ = Table[trace[hc[states⟦i⟧] · Πℒρ · states⟦j⟧], {i, nn}, {j, nn}];
    If[! TrueQ[Chop@FullSimplify@Tr@δρ == 0], Message[lindblad::trnz]];
    {ρ, ρs, δρ}]];
```

Steady state solver

```
steadystatevalue[op_?operatorMatrixQ,
  pt : {(_?(MemberQ[params, #] &) → _?NumericQ) ...}] :=

 Block[Evaluate[Join[{sol, vparms}, params]],
  Evaluate[params] = params /. pt;
  vparms = Select[params, ! NumericQ[#] &];

  lusolve := 0^0;
  oldvec := 0^0;

  With[{sparms = Sequence @@ vparms, nn = nn},
   Module[{crys = CoefficientArrays[
       {Tr@ρs - 1} ~ Join ~ Rest[Flatten[ddd = δρ]], Flatten@ρs],

     M1, M2, c1, c2, cf1, cf2, cff1, cff2, M1c, M2c, cfm1,
     cfm2, cffm1, cffm2, ope, opc1, opc2, opm, ss, rparms, nparms,
     repparms, ρte, dρte, nrm, bb, cb, cfb, cffb, bbc, occ},

    nparms := Sequence @@ (Pattern[#, _?NumericQ] & /@ vparms);
    rparms = {#, _Real} & /@ vparms;

    M1 = FullSimplify[crys〚1〛];
    M2 = FullSimplify[-crys〚2〛ᵀᵀ];
    bb = FullSimplify[Flatten[D[M2, {vparms}], {{3, 1}, {2}}]];

    c1 = M1 /. HoldPattern@SparseArray[__, {__, a_}] :→ a;
    c2 = M2 /. HoldPattern@SparseArray[__, {__, a_}] :→ a;
    cb = bb /. HoldPattern@SparseArray[__, {__, a_}] :→ a;

    repparms = Thread[vparms → Unique[vparms]];
    cf1 = Compile[Evaluate@rparms, Evaluate@Developer`ToPackedArray@
         evalinterp@c1, CompileOptimizations → All] /. repparms;
    cf2 = Compile[Evaluate@rparms, Evaluate@Developer`ToPackedArray@
         evalinterp@c2, CompileOptimizations → All] /. repparms;
    cfb = Compile[Evaluate@rparms, Evaluate@Developer`ToPackedArray@
         evalinterp@cb, CompileOptimizations → All] /. repparms;

    M1c = (M1 /. HoldPattern@SparseArray[a__, {b__, c_}] :→
        SparseArray[a, {b, cff1[sparms]}]);
    M2c = (M2 /. HoldPattern@SparseArray[a__, {b__, c_}] :→
        SparseArray[a, {b, cff2[sparms]}]);
    bbc = (bb /. HoldPattern@SparseArray[a__, {b__, c_}] :→
        SparseArray[a, {b, cffb[sparms]}]);

    cfm1 = Compile[Evaluate@rparms,
      Evaluate@Developer`ToPackedArray@evalinterp@Normal@M1];
    cfm2 = Compile[Evaluate@rparms, Evaluate@
```

```
      Developer`ToPackedArray@evalinterp@Normal@M2];

   ope = trace[op · ρ];
   occ = {sparms,
     ρs /. Thread[Flatten@ρs → Table[ss[sol, i], {i, Length@Flatten@ρs}]]};
   opm = ope /. Thread[Flatten@ρs → Table[ss[sol, i],
        {i, Length@Flatten@ρs}]];
   nrm = Total@Diagonal@ρs /. Thread[Flatten@ρs →
       Table[ss[sol, i], {i, Length@Flatten@ρs}]];
   {opc1, opc2} = CoefficientArrays[ope, Flatten@ρs];

   ReleaseHold[
    Hold[
       ρte[nparms] := Module[{sol, m1, o1},
         mmm = mat; (*m1=mat;
         o1=off;
         Quiet@Check[
           oldvec=sol=LinearSolve[m1,Normal@o1,Method→
               {"Krylov","Preconditioner"→(lusolve[#]&),MaxIterations→10,
                "StartingVector"→oldvec,Tolerance→10^-4}],

           numlu++;
           lusolve=LinearSolve[m1,Method→"Multifrontal"];
           oldvec=sol=lusolve[o1]];*)
         sol = LinearSolve[mat, off];
         Sow[occ1];
         result];

       dρte[nparms] := Module[{y, c, sol, ls},
         ls = LinearSolve[mat, Method → "Multifrontal"];
         c = off;
         y = ls[c];
         sol = -ls[Partition[B.y, nn^2]^T];
         {c2.y + c1, c2.sol}
        ];

      ] /. {HoldPattern[ρ] → ρ,
       HoldPattern@off → M1c,
       HoldPattern@mat → M2c,
       HoldPattern@B → bbc,
       HoldPattern@offm → cffm1[sparms],
       HoldPattern@matm → cffm2[sparms],
       HoldPattern@result → opm,
       HoldPattern@occ1 → occ,
       HoldPattern@normalize → nrm,
       HoldPattern@c1 → opc1,
       HoldPattern@c2 → opc2,
       HoldPattern@nn → nn
      } /.
```

```
      {ss → Part,
       cff1 → cf1,
       cff2 → cf2,
       cffb → cfb,

       cffm1 :→ cfm1,
       cffm2 :→ cfm2}];
    {vparms, ρte, dρte}]]]
```

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