

Alessandro Candolini

Non-Relativistic Quantum Mechanics

Quantum Field Theory

August 1, 2015

Copyright © 2015 Alessandro Candolini
All rights reserved.

“ In the good old days, theorizing was like sailing between islands of experimental evidence. And, if the trip was not in the vicinity of the shoreline (which was strongly recommended for safety reasons) sailors where continuously looking forward, hoping to see land — the sooner the better.

Nowadays, some theoretical physicists (let us call them sailors) [have] found a way to survive and navigate in the open sea of pure theoretical constructions. Instead of the horizon, they look at stars, which tell them exactly where they are. Sailors are aware of the fact that the stars will never tell them where the new land is, but they may tell them their position on the globe.

Theoreticians become sailors simply because they just like it. Young people, seduced by capitans forming crews to go to a Nuevo El Dorado soon realize that they will spend all their life at sea. Those who do not like sailing desert the voyage, but for the true potential sailors the sea become their passion. They will probably tell the alluring and frightening truth to their students — and the proper people will join their ranks. ”

— Andrei Losev

CONTENTS

PREFACE	vii
i THE BASIS	1
1 SCHWINGER'S APPROACH TO QUANTUM MECHANICS	3
1.1 Introduction	3
2 LINEAR OPERATORS IN HILBERT SPACES	5
2.1 Banach and Hilbert spaces	5
3 THE RULES OF THE GAME	11
4 ONE-DIMENSIONAL QUANTUM SYSTEMS	13
ii THE CORE	15
5 ANGULAR MOMENTUM	17
6 HYDROGEN ATOM	19
7 PERTURBATION THEORY	21
8 SCATTERING	23
iii ADVANCED TOPICS	25
9 PATH INTEGRALS	27
10 SEMICLASSICAL QUANTUM MECHANICS	29
11 SUPERSYMMETRIC QUANTUM MECHANICS	31
12 SECOND QUANTIZATION FORMALISM	33
iv APPENDICES	35

PREFACE

To be written...

Trieste, August 1, 2015

Alessandro Candolini

Part I

THE BASIS

SCHWINGER'S APPROACH TO QUANTUM MECHANICS

“ I presume that all of you have already been exposed to some undergraduate course in Quantum Mechanics, one that leans heavily on de Broglie waves and the Schroedinger equation. I have never thought that this simple wave approach was acceptable as a general basis for the whole subject, and I intend to move immediately to replace it in your mind by a foundation that *is* perfectly general. ”

J. Schwinger, *Quantum Mechanics. Symbolism of Atomic Measurements*
Schwinger:2001.

1.1 INTRODUCTION

COMPARED with other traditional areas of physics, quantum mechanics is not easy. It often lacks physical intuition and it relies on heavy mathematical background from the very beginning. As we shall see shortly, topics like uncertainty principle, the role of probability etc asks immediately for a theoretical framework/formalism... Phase space is not able to capture, it does not offer the tools. Naturally leading from the very beginning to adopt .

Operator formulation of standard non-relativistic quantum mechanics heavily relies on the theory of linear operators in Hilbert spaces. In particular, the spectral decomposition of self-adjoint operators (bounded and unbounded ones) is a key ingredient in formulating the basic rules of quantum mechanics. This chapter is aimed at providing the necessary mathematical background of functional analysis employed by non-relativistic quantum mechanics. It is a chapter on mathematics, not on quantum physics; for this reason care has been made to mathematical rigour more than what will be customary in later chapters. The Reader interested in how functional analysis applies to the formulation of quantum mechanics should jump to the next chapters. As we shall see, physicists are customary to employ Dirac's notation, a powerful mnemonic notation which naively speaking however lacks mathematical rigour; a dictionary is possible to translate Dirac notation to the rigorous theorems of functional analysis; more on this later on.

2.1 BANACH AND HILBERT SPACES

Unless stated otherwise, let \mathbb{K} denote equivalently the field of real numbers \mathbb{R} or the field of complex numbers \mathbb{C} . (It is possible to develop the theory also for the skew-field of quaternions, but this case will not be discussed here to avoid dealing with the non-commutativity of quaternionic product.)

DEFINITION 2.1 (NORM): Let X be any vector space over \mathbb{K} . A “norm” on X is any application $X \rightarrow \mathbb{R}$ denoted by $\|\cdot\|$ satisfying the following properties:

Norm

- (a) $\|\varphi\| \geq 0$, $\forall \varphi \in X$ (nonnegative);
- (b) $\|\varphi\| = 0$ if and only if $\varphi = 0$ (strictly positive);
- (c) $\|\lambda\varphi\| = |\lambda|\|\varphi\|$, $\forall \lambda \in \mathbb{K}$ and $\forall \varphi \in X$ (positive homogeneity);
- (d) $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$, $\forall (\varphi, \psi) \in X \times X$ (the so-called “triangle inequality”)

Item (a) is not strictly necessary: it follows from items (b) to (d). In fact, from item (c) $\|-\varphi\| = \|\varphi\|$, from item (d) we have thus $\|\varphi + (-\varphi)\| \leq \|\varphi\| + \|\varphi\| = 2\|\varphi\|$, and from item (b) $\|\varphi - \varphi\| = 0$.

A “normed vector space” is a vector space equipped with a norm, as formalized by the following definition.

DEFINITION 2.2 (NORMED VECTOR SPACE): A “normed vector space” over \mathbb{K} is a pair $(X, \|\cdot\|)$ where X is a vector space over \mathbb{K} and $\|\cdot\|$ is any norm on X .

Normed vector space

If $(X, \|\cdot\|)$ is a normed vector space, the norm $\|\cdot\|$ *induces* a metric (i. e., a notion of distance) and thus a topology on X . This is proved by the following theorem.

THEOREM 2.1: Let $(X, \|\cdot\|)$ be any normed vector space over \mathbb{K} . Let $d: X \times X \rightarrow \mathbb{R}$ be the function defined by

Distance induced by a norm

$$d(\varphi, \psi) = \|\varphi - \psi\|, \quad \forall (\varphi, \psi) \in X \times X. \quad (2.1)$$

Then, (X, d) is a metric space. The metric in eq. (2.1) is called the “metric induced by the norm” on X .

Proof. We need to prove that eq. (2.1) defines a metric over X , i. e., we need to show that d satisfies:

- (a) $d(\varphi, \psi) \geq 0$, $\forall(\varphi, \psi) \in X \times X$ (nonnegativity);
- (b) $d(\varphi, \psi) = 0$ if and only if $\varphi = \psi$ (i. e., the only point of X at zero distance from ψ is ψ itself);
- (c) $d(\varphi, \psi) = d(\psi, \varphi)$, $\forall(\varphi, \psi) \in X \times X$ (the distance is a symmetric function of its arguments);
- (d) $d(\varphi, \psi) \leq d(\psi, \eta) + d(\eta, \varphi)$, $\forall(\varphi, \psi, \eta) \in X \times X \times X$ (the so-called “triangle inequality”).

Item (a) follows from property (a) of the norm. Item (b) follows from property (b) of the norm, since $\|\varphi - \psi\| = 0$ if and only if $\varphi - \psi = 0$, i. e., if and only if $\varphi = \psi$. Item (c) follows from property (c) of the norm, since

$$\begin{aligned} d(\varphi, \psi) &= \|\varphi - \psi\| = \|-(\psi - \varphi)\| = |-1|\|\psi - \varphi\| \\ &= \|\psi - \varphi\| = d(\psi, \varphi), \quad \forall(\varphi, \psi) \in X \times X. \end{aligned}$$

Item (d) follows from property item (d) of the norm, since

$$\begin{aligned} d(\varphi, \psi) &= \|\varphi - \psi\| = \|\varphi - \eta + \eta - \psi\| \leq \|\varphi - \eta\| + \|\eta - \psi\| \\ &= d(\varphi, \eta) + d(\eta, \psi), \quad \forall(\varphi, \psi, \eta) \in X \times X \times X. \end{aligned}$$

This completes the proof. ■

The metric induced by a norm fulfils the following extra properties, the proof of which is straightforward and is left to the Reader:

- 1. $d(\varphi + \eta, \psi + \eta) = d(\varphi, \psi)$, $\forall(\varphi, \psi, \eta) \in X \times X \times X$ (translation invariance);
- 2. $d(\lambda\varphi, \lambda\psi) = |\lambda|d(\varphi, \psi)$, $\forall(\varphi, \psi) \in X \times X$ and $\lambda \in \mathbb{K}$ (homogeneity);

Theorem 2.1 shows that any normed vector space is naturally endowed with a notion of distance. Please notice however that a normed vector space could be equipped also with distances other than the one induced by the norm; such distances are not necessary related to the norm; furthermore, eq. (2.1) is not the only one possible distance built from the norm.

EXERCISE 2.1: Let $(X, \|\cdot\|)$ be a normed vector space. Let $d: X \times X \rightarrow \mathbb{R}$ be the function defined by

$$d(\varphi, \psi) = \frac{\|\varphi - \psi\|}{1 + \|\varphi - \psi\|}, \quad \forall(\varphi, \psi) \in X \times X$$

Show that d defines a nonhomogeneous, translation invariant metric on X .

Now that we have a “natural” notion of distance in normed vector spaces, i. e., eq. (2.1), all the concepts defined for metric spaces applies automatically, in particular, to normed linear spaces. Among these concepts are: continuity, limits, completeness, open sets etc.

LEMMA 2.1: Let $(X, \|\cdot\|)$ be a normed vector space over \mathbb{K} . The following holds:

$$\| \|\psi\| - \|\varphi\| \| \leq \|\psi - \varphi\|, \quad \forall(\psi, \varphi) \in X \times X. \quad (2.2)$$

Proof. The key ingredient is the triangle inequality of the norm. For all $(\varphi, \psi) \in X \times X$, we have

$$\|\varphi\| = \|\varphi - \psi + \psi\| \leq \|\varphi - \psi\| + \|\psi\|$$

and

$$\|\psi\| = \|\psi - \varphi + \varphi\| \leq \|\varphi - \psi\| + \|\varphi\|.$$

Thus,

$$\begin{aligned} \|\varphi\| - \|\psi\| &\leq \|\varphi - \psi\| \\ \|\psi\| - \|\varphi\| &\leq \|\varphi - \psi\|. \end{aligned}$$

which is exactly the meaning of eq. (2.2). (To see this even more explicitly, it is enough to discuss the three cases $\|\psi\| < \|\varphi\|$, $\|\psi\| = \|\varphi\|$, $\|\psi\| > \|\varphi\|$, and remember that $\|\varphi - \psi\| \geq 0$.) ■

THEOREM 2.2: Let $(X, \|\cdot\|)$ be a normed vector space over \mathbb{K} . The norm $\|\cdot\|$ is continuous on X (with respect to the metric induced by the norm).

Continuity of the norm

Proof. We need to show that $\forall \psi \in X$ the following holds: for all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for all $\varphi \in X$, if $d(\psi, \varphi) < \delta$ then $\tilde{d}(\|\psi\|, \|\varphi\|) < \varepsilon$, where \tilde{d} is the euclidean distance in \mathbb{R} .

Notice that $d(\psi, \varphi) < \delta$ means

$$\|\psi - \varphi\| < \delta,$$

and $\tilde{d}(\|\psi\|, \|\varphi\|) < \varepsilon$ means

$$|\|\psi\| - \|\varphi\|| < \varepsilon.$$

Using eq. (2.2), it is enough to choose any $0 < \delta < \varepsilon$. ■

Let us recall an important fact about Cauchy sequences. Give any metric space (X, d) , a convergent sequence in X is also a Cauchy sequence. The converse of this statement is not generally true however, i. e., there exists metric spaces where some Cauchy sequences do not need to converge.

Completeness of a metric space

A typical counter-example works as follow. Consider a sequence $(x_n)_n$ of rational numbers in \mathbb{R} which is convergent to some irrational number. Thus, $(x_n)_n$ is a Cauchy sequence. Now, consider the same sequence as a sequence in \mathbb{Q} : of course, it is still a Cauchy sequence, but this time it is not convergent.

A metric space is “complete” if *all* Cauchy sequences are convergent.

DEFINITION 2.3 (BANACH SPACE): Let $(X, \|\cdot\|)$ be a normed vector space over \mathbb{K} . If (X, d) (where d is the metric induced by the norm) is complete, $(X, \|\cdot\|)$ is called a “Banach space”.

Banach space

We are almost ready to introduce the notion of Hilber space, which is the setting where we will develop the operator formulation of non-relativistic quantum mechanics. The first ingredient is the “inner product”, defined below.

In the following, for every $z \in \mathbb{C}$ we will denote the “complex conjugate” of z with z^* and the “modulus” of z with $|z|$. Remember that $z = \Re z + i\Im z$ (where $\Re z$ and $\Im z$ are the real and imaginary parts of z , respectively), $z^* = \Re z - i\Im z$ and $|z|^2 = zz^*$. The inverse of $z \neq 0$ is $z^{-1} = z^*/|z|^2$. We have $z \in \mathbb{R}$ if and only if $z^* = z$. The complex conjugation satisfies the “involution” property: $(z^*)^* = z$. Furthermore, $(z_1 z_2)^* = z_1^* z_2^*$, $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$, and $|z_1 z_2| = |z_1| |z_2|$.

Complex numbers notation

DEFINITION 2.4 (INNER PRODUCT): Let X be a vector space over \mathbb{K} . A “inner product” on X is any application $X \times X \rightarrow \mathbb{K}$, hereafter denoted by $\langle \cdot | \cdot \rangle$, satisfying the following properties:

Inner product

- (a) $\langle \varphi | \psi + \eta \rangle = \langle \varphi | \psi \rangle + \langle \varphi | \eta \rangle, \quad \forall (\varphi, \psi, \eta) \in X \times X \times X;$
- (b) $\langle \varphi | \lambda \psi \rangle = \lambda \langle \varphi | \psi \rangle, \quad \forall (\varphi, \psi) \in X \times X \text{ and } \lambda \in \mathbb{K};$

- (c) $\langle \varphi | \psi \rangle = \langle \psi | \varphi \rangle^*$, $\forall (\varphi, \psi) \in X \times X$;
- (d) $\langle \varphi | \varphi \rangle \geq 0$, $\forall \varphi \in X$;
- (e) $\langle \varphi | \varphi \rangle = 0$ if and only if $\varphi = 0$.

Several remarks are in order.

Remark. Items (a) and (b) implies that the inner product is *linear* on the *second* component, namely:

$$\langle \varphi | \lambda \psi + \mu \eta \rangle = \lambda \langle \varphi | \psi \rangle + \mu \langle \varphi | \eta \rangle,$$

for all $(\varphi, \psi, \eta) \in X \times X \times X$ and for all $(\lambda, \mu) \in \mathbb{K} \times \mathbb{K}$. Item (c) together with items (a) and (b) implies that the inner product in general is *conjugate-linear* (or anti-linear) on the *first* component, namely

$$\langle \lambda \varphi + \mu \eta | \psi \rangle = \lambda^* \langle \varphi | \psi \rangle + \mu^* \langle \eta | \psi \rangle,$$

for all $(\varphi, \psi, \eta) \in X \times X \times X$ and for all $(\lambda, \mu) \in \mathbb{K} \times \mathbb{K}$. Of course, if $\mathbb{K} = \mathbb{R}$, $\lambda^* = \lambda$, $\mu^* = \mu$ and the inner product becomes linear also on the first component (thus, it is *bilinear*); but this is not the case if $\mathbb{K} = \mathbb{C}$, where complex conjugation appears.

Remark. Item (b) is a matter of choice. Some authors prefer a different convention:

$$\langle \lambda \varphi | \psi \rangle = \lambda \langle \varphi | \psi \rangle, \quad \forall (\varphi, \psi) \in X \times X \text{ and } \lambda \in \mathbb{K};$$

with this convention, the inner product would become linear on the first component and conjugate-linear on the second one. The convention of having the inner product linear on the second component is the one most often employed by physicists, and the one used in this notes.

Remark. Regarding items (d) and (e), one may wonder what does mean $\langle \varphi | \varphi \rangle \geq 0$, since we expect $\langle \varphi | \varphi \rangle \in \mathbb{K}$, and if $\mathbb{K} = \mathbb{C}$ it might seem that the inequality does not make sense. Actually, from item (c)

$$\langle \varphi | \varphi \rangle = \langle \varphi | \varphi \rangle^*, \quad \forall \varphi \in X,$$

Remark. Some authors prefer the notation (φ, ψ) instead of $\langle \varphi | \psi \rangle$; the notation $\langle \varphi | \psi \rangle$ is closer to the one uses by physicists and it is the first step towards the introduction of Dirac's notation. (Dirac's notation is more than simply writing the inner product this way; we will discuss this point in connection with the spectral theorem of linear operators.) In Dirac notation the vector ψ is denoted by $|\psi\rangle$, and it is called “ket”; there is a “kind of conjugation” (more on this later on) that converts the analogous “ket” $|\varphi\rangle$ to a so-called “bra” $\langle \varphi|$ and the inner product is considered as a product between a “bra” and a “ket” (resulting in a “braket”!). Of course, this is just a naming convention.

Inner product space

DEFINITION 2.5 (INNER PRODUCT SPACE): A “inner product space” over \mathbb{K} is a pair $(X, \langle \cdot | \cdot \rangle)$, where X is a vector space over \mathbb{K} and $\langle \cdot | \cdot \rangle : X \times X \rightarrow \mathbb{K}$ is an inner product on X .

Norm induced by inner product

Any inner product space is naturally endowed with a norm coming from the inner product. Let $(X, \langle \cdot | \cdot \rangle)$ be an inner product space over \mathbb{K} . Let $\|\cdot\| : X \rightarrow \mathbb{R}$ defined by

$$\|\psi\| = (\langle \psi | \psi \rangle)^{1/2}, \quad \forall \psi \in X. \quad (2.3)$$

Observe that such $\|\cdot\|$ is well-defined, since $\langle \psi | \psi \rangle \geq 0$ for every $\psi \in X$. We shall prove in a moment that $\|\cdot\|$ is actually a norm on X , this justify the notation $\|\cdot\|$. Such norm

is the “norm induced by the inner product”. Before proving this, we need a preliminary but extremely important result, which goes under the name of Cauchy-Schwarz inequality.

Cauchy-Schwarz inequality is of major importance. It is a key ingredient in several proofs of functional analysis. It has important implications also outside the realm of analysis. For example, the general formulation of the Heisenberg uncertainty principle in quantum mechanics (or the analogous time-bandwidth uncertainty principle for temporal signal transmission) is derived using the Cauchy-Schwarz inequality.

THEOREM 2.3 (CAUCHY-SCHWARZ INEQUALITY): Let $(X, \langle \cdot | \cdot \rangle)$ be an inner product space. The following holds:

Cauchy-Schwarz inequality

$$|\langle \varphi | \psi \rangle| \leq \|\varphi\| \|\psi\|, \quad \forall (\varphi, \psi) \in X \times X. \quad (2.4)$$

Remark. In eq. (2.4) we are using the definition eq. (2.3) but it is important to emphasize that we are *not* using (in both the statement and in the proof of Cauchy-Schwarz inequality) the fact that eq. (2.3) is a norm. We don’t know at this point that eq. (2.3) defines a norm, we will prove that in the next theorem, using the Cauchy-Schwarz inequality.

Proof. We distinguish two cases. ■

THEOREM 2.4: Let $(X, \langle \cdot | \cdot \rangle)$ be an inner product space over \mathbb{K} . $(X, \|\cdot\|)$ with $\|\cdot\|$ defined by eq. (2.3) is a normed vector space over \mathbb{K} .

Proof. We need to check that eq. (2.3) makes sense and that it satisfies items (a) to (d) of the definition of the norm. ■

Part II

THE CORE

Part III

ADVANCED TOPICS

Part IV

APPENDICES

