HYPERGEOMETRIC FUNCTIONS

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PREFACE

Hypergeometric functions are useful in many different branches of applied mathematics, theoretical physics, statistics, etc. Part of the explanation for this fact is that (§ 1) integrals of any second-order linear homogeneus ordinary differential equation of Fuchsian class having precisely three given regular singular points in the extended complex plane can ultimately be expressed in closed form using hypergeometric functions (§ 2). Among them are Legendre functions, Jacobi polynomials, Chebychev polynomials, etc. Other differential equations which are often encountered in practise are not of this form, one relevant example being that of Bessel. It may happen however that such equations are confluent forms of the hypergeometric equation and in that case they can be integrated in finite terms by means of confluent hypergeometric functions (§ 3). (Bessel equation belongs to this class of equations.) All of this handout is traditional material.

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1 FUCHSIAN DIFFERENTIAL EQUATIONS

This chapter is mainly concerned with existence and properties (analytic structure, many-valuedness, convergent power series representations,[1] etc) of the integrals of linear homogeneus ordinary differential equations of the second order in which the independent variable is complex and whose coefficients are single-valued and analytic complex functions of the independent variable, having at most finitely many isolated singular points in (a given domain of) the complex plane.

Several, through not all,[2] higher trascendental functions of applied mathematics satisfy differential equations of this form, and the results of this chapter provide a useful tool for establishing in a systematic way some of their properties and representations, working directly from the differential equation satisfied by them.[3] In particular, we shall put these results at work in the following chapter discussing Gauss' hypergeometric differential equation and its solutions.

This chapter is mostly based on **Tricomi:1961**; **Wang.Guo:1989**. See also **Smirnov:1964**; **Whittaker.Watson:1927**. In what follows, knowledge of the theory of analytic functions of one complex variable is assumed.[4]

1.1 EXISTENCE AND UNIQUENESS OF THE SOLUTION IN THE NEIGHBOURHOOD OF AN ORDINARY POINT

A full and systematic account of the general theory of ordinary differential equations or systems of ordinary differential equations in the complex field, although useful in practise, is beyond the scope of this handout. For a broader discussion, see, e. g., the previously mentioned book by **Tricomi:1961** or **Ince:1956**.

Second-order homogeneus linear ordinary differential equations in the complex field...

We shall restrict our attention to a linear homogeneus ordinary differential equation of the second order,[5] which can always be written in such a way that the coefficient of the second-order derivative is identically equal to one (this will be taken to be the standard form of the equation of the equation):

$$\boxed{u''(z) + p(z)u'(z) + q(z)u(z) = 0,} \tag{1}$$

1 Asymptotic (divergent) power series representations, integral representations or other kind of series representations are not discussed in general.

2 Riemann's zeta function and Euler's gamma function are examples of functions which are essentially characterized by *functional* equations instead of differential equation. (For instance, Hölder's theorem states that the gamma function does not satisfy *any* algebraic differential equation whose coefficients are rational functions.) Other higher trascendental functions, like the Painlevé trascendents, arise in connection with the solution of certain *non*-linear ordinary differential equations in the complex field (see, e. g., **Ince:1956**).

3 Someone might prefer to begin with integral or series representations, and to write down the differential equations only later.

4 There are many books on the theory of analytic functions of one complex variable. Among my favourite ones are the monographies by **Ablowitz.Fokas:2003**; **Stein.Shakarchi:2003**; **Marsden.Hoffman:1987**; **Greene.Krantz:2006**; or consult any good book on complex analysis.

5 The result of the first sections of this chapter apply with obvious generalizations to the case of linear equations of arbitrary order, but the second order is enough to our purposes and leads to easier notations.

where $u(z)$ is the unknown (complex-valued) function and z denotes the independent complex variable. The coefficients $p(z)$ and $q(z)$ are assigned complex-valued functions defined on some given set of the complex z-plane. Primes denote differentiation, in the complex sense, with respect to the variable z.

. . . whose coefficients are single-valued and analytic up to finitely many isolated singularities

Throughout this chapter, we shall assume that each of the coefficients $p(z)$ and $q(z)$ is a *single-valued* function, defined and *analytic* on some given *domain*[6] ˙ $\subset \mathbb{C}$ of the complex z-plane (eventually, ˙ $= \mathbb{C}$), with the possible exception of *at most* a finite number of *isolated* singular points[7] within ˙.

Let us classify the point of ˙ according to whether or not they are isolated singular points of the coefficients of the differential equation.[8]

Definition of ordinary point of the differential equation

Definition 1.1: Let $z_0 \in$ ˙. z_0 is called an "ordinary" point of Eq. (1) if the coefficient functions $p(z)$ and $q(z)$ are both analytic at z_0.[9] Otherwise, if z_0 is a singular point of either $p(z)$ or $q(z)$ or both, then z_0 is said to be a "singular" point of Eq. (1).

Initial conditions

Let $z_0 \in$ ˙ an ordinary point of Eq. (1). We look for solutions $u(z)$ of Eq. (1) defined in a suitable neighbourhood (to be determined) of z_0 and which satisfy the following initial conditions at z_0:

$$\begin{cases} u(z_0) = \alpha \\ u'(z_0) = \beta \end{cases} \tag{2}$$

where α and β are arbitrary-assigned complex numbers. Eq. (1) together with the initial conditions (2) constitute what is called an "initial valued problem".

Local existence and uniqueness theorem

Theorem 1.1: Let $z_0 \in$ ˙ be an ordinary point of Eq. (1). Then, for every complex α and β there exists one and only one solution of Eq. (1) which satisfies the initial conditions (2) and which is analytic and single-valued in a suitable neighbourhood of z_0.

The a-priori knowledge of the disk of convergence of the solution

Proof. Let S be the circle with centre at z_0 and whose radius is such that every point of S is a point of ˙ and is also an ordinary point of the Eq. (1). There always exists such S with *non-zero* radius owing to the fact that ˙ is an open set and all the singularities of $p(z)$ and $q(z)$ (if any) are isolated. We shall prove that a solution of Eq. (1) always exists which is *analytic* in S (perhaps also in a larger domain) and that such solution is also the unique analytic solution of Eq. (1) satisfying the prescribed initial conditions (2).

First step: equation in "reduced" form

The first step is to convert Eq. (1) into its "reduced" form.[10] This is done by introducing a new unknown function $v(z)$ related to the original one $u(z)$ by

$$u(z) = k(z)v(z)\,,$$

where $k(z)$ is an auxiliary function to be determined by requiring that the resulting differential equation for $v(z)$ have the coefficient function of $v'(z)$

6 i. e., an open and connected subset of $\mathbb{C}$.
7 which may be either poles or essential singularities of the coefficients.
8 This classification holds for points z_0 at *finite* distance in the complex plane. The classification of the point at infinity (in the case ˙ $= \mathbb{C}$) requires special treatment and will be discussed separately in § 1.5.
9 As usual, we shall see that a function is analytic *at* a point z_0 to say that the function is analytic in an (open) neighbourhood of z_0.
10 This method of proof is that, e. g., of **Whittaker.Watson:1927**. The method is not suitable for generalization to equations of higher orders. However, the theorem holds for linear differential equations of any order. For a different proof which can be easily adapted to the general case see, e. g., **Smirnov:1964**; **Wang.Guo:1989**.

identically equal to zero. We shall see that it always possible to find such $k(z)$ and we shall give an explicit formula for it. In fact, the first- and second-order derivatives of $u(z)$ are

$$u'(z) = k'(z)v(z) + k(z)v'(z)\,,$$
$$u''(z) = k''(z)v(z) + 2k'(z)v'(z) + k(z)v''(z)\,.$$

Plugging into Eq. (1), we get the differential equation for $v(z)$,

$$k(z)v''(z) + [2k'(z) + p(z)k(z)]v'(z) + [k''(z) + p(z)k'(z) + q(z)k(z)]v(z) = 0\,,$$

and in order the coefficient of $v'(z)$ to be zero, $k(z)$ must satisfy

$$2k'(z) + p(z)k(z) = 0\,. \tag{3}$$

Eq. (3) is an homogeneus linear ordinary differential equation of the first order in the unknown $k(z)$. One solution of it is the identically zero function. We look for non-identically zero solutions. One way to get such a solution is to rewrite Eq. (3) as

$$\frac{k'(z)}{k(z)} = -\frac{1}{2}p(z).$$

The operation of dividing by $k(z)$ is valid because we are looking for non identically zero solutions $k(z)$ and if $k(z)$ were zero at some specific point then it would be zero everywhere.[11] The left-hand side of the previous equation is the logarithmic derivative of $k(z)$, therefore by integrating from z_0 to $z \in S$ and exponentiating we get

$$k(z) = \exp\left\{-\frac{1}{2}\int_{z_0}^{z} p(\xi)\,d\xi\right\}, \tag{4}$$

up to a constant multiplicative *non-zero* factor which is completely unrelevant for our purposes.[12] There is no need to specify in the right-hand side of Eq. (4) the contour of integration, since the integrand $p(\xi)$ is analytic is S (by hypothesis) and z is any point in S. After dividing by $k(z)$ (now we know from the explicit formula of k that k never vanishes) the equation for $v(z)$ becomes

$$v''(z) + J(z)v(z) = 0\,, \tag{5}$$

where

$$J(z) = \frac{k''(z)}{k(z)} + p(z)\frac{k'(z)}{k(z)} + q(z)\,.$$

We can use Eq. (3) to rewrite $J(z)$ in a more explicit form. Notice that

$$\left(\frac{k'(z)}{k(z)}\right)' = \frac{k''(z)k(z) - \big(k'(z)\big)^2}{k^2(z)} = \frac{k''(z)}{k(z)} - \left(\frac{k'(z)}{k(z)}\right)^2,$$

so

$$\frac{k''(z)}{k(z)} = \left(\frac{k'(z)}{k(z)}\right)' + \left(\frac{k'(z)}{k(z)}\right)^2.$$

11 This requires some explanation. It can be seen as a conseguence of the existence and uniqueness theorem of integrals of first-order linear differential equations with given initial conditions. However, although the Reader may probably be familiar with this theorem and how to apply it in the context of real calculus, here we have never stated such theorem in the complex field. To avoid using in the proof some fact which we have never dealt with before nor we shall never deal with again in this handout, we prefer a more direct approach: We chech a-posteriori that our requirement is fulfilled. It is not very elegant but nevertheless it does the job!

12 As far as the constant is different from zero, the only effect of such constant would be to change the "initial value" $k(z_0)$, which have no effect on the differential equation since the equation is homogeneus. With our choice, $k(z_0) = 1$.

Using Eq. (3) we get

$$\frac{k''(z)}{k(z)} = -\frac{1}{2}p'(z) + \left(\frac{1}{2}p(z)\right)^2,$$

and then

$$J(z) = -\frac{1}{2}p'(z) - \frac{1}{4}p^2(z) + q(z)\,. \tag{6}$$

From $v(z)$ to $u(z)$

We shall now prove by the method of successive approximations that there must exist one and only one solution $v(z)$ of Eq. (5) which is analytic and single-valued in S and which satisfies the initial conditions

$$\begin{cases} v(z_0) = c_0 \\ v'(z_0) = c_1 \end{cases}, \tag{7}$$

where c_0 and c_1 are arbitrary complex constants. From this result, it will immediately follow that also Eq. (1) must have one and only one solution $u(z)$ which is analytic and single-valued in S and which satisfies

$$\begin{aligned} u(z_0) &= k(z_0)v(z_0) = c_0, \\ u'(z_0) &= k'(z_0)v(z_0) + k(z_0)v'(z_0) = -\frac{1}{2}p(z_0)c_0 + c_1. \end{aligned} \tag{8}$$

From the arbitrariness of c_0 and c_1, we will conclude that there must be one and only one solution $u(z)$ of Eq. (1) which is analytic and single-valued in S and which satisfies the desired initial conditions (2) for any given value of α and β.

Second step: construction of the solution

We define a sequence $(v_n)_{n\in\mathbb{N}}$ of complex-valued functions on S. For every $z \in S$, let

$$v_0(z) = a + b\,(z - z_0)\,,$$

where a and b are arbitrary complex numbers, and recursively

$$v_{n+1}(z) = \int_{z_0}^{z} (\xi - z)\, v_n(\xi) J(\xi)\, d\xi, \quad n \geqslant 0.$$

At every stage the integrand is an analytic function of ξ and so the value of the integral does not depend on the contour of integration. Let us prove this fact by induction. First of all, $v_0(z)$ is easily seen to be analytic for every $z \in S$ (indeed, it is an entire function); furthermore, assuming $v_n(z)$ to be analytic on S we get that $(\xi - z)v_n(\xi)J(\xi)$ as a function of ξ is analytic on S so its integral between z_0 and any $z \in S$ does not depend on the contour of integration and defines a new function $v_{n+1}(z)$ which also is analytic on S.

It will be helpful to compute explicitly the first and second derivatives of these functions. For $n = 0$ we simply have $v_0'(z) = b$ and $v_0''(z) = 0$, for every $z \in \dot{}$. In all other cases we have

$$v_{n+1}'(z) = -\int_{z_0}^{z} v_n(\xi) J(\xi)\, d\xi, \quad n \geqslant 0,$$

and

$$v_{n+1}''(z) = -v_n(z)J(z), \quad n \geqslant 0,$$

for every $z \in S$.

We shall show that $\sum_{n=0}^{+\infty} v_n(z)$ converges absolutely and uniformly in S to a function $v(z)$ analytic in S and which satisfies the Eq. (1). Further, this function will satisfy the initial conditions (7) for any complex c_0 and c_1, provided that the parameters a and b in the definition of $v_0(z)$ have been properly tuned. In order to prove these facts, we need a preliminary result, namely, we need to show that the following inequality holds:

$$|v_n(z)| \leqslant \mu M^n \frac{|z-z_0|^{2n}}{n!}, \quad n \geqslant 0, \tag{9}$$

for every $z \in S$, where

$$\mu = \sup_{z \in S} |v_0(z)|,$$

and

$$M = \sup_{z \in S} |J(z)|.$$

(These definitions make sense because both $v_0(z)$ and $J(z)$, being analytic functions in S, are guaranteed to be bounded in S.) The rigorous proof of (9) is by induction on n. In fact, for $n = 0$, we have

$$|v_0(z)| \leqslant \mu,$$

for every $z \in S$, which is obviously verified. Suppose now the validity of Eq. (9) for some n and let us prove that a similar relation holds for $n+1$. By definition,

$$|v_{n+1}(z)| = \left| \int_{z_0}^{z} (\xi - z)\, v_n(\xi) J(\xi)\, d\xi \right|.$$

We have already said that the value of this integral must be the same no matter the contour of integration is chosen. Integrating along the straight line joining z_0 to z, with parametric equation

$$\xi(t) = z_0 + (z - z_0)t, \quad t \in [0, 1],$$

we have $\xi'(t) = z - z_0$ and

$$\xi(t) - z = z_0 + (z - z_0)\, t - z = (z - z_0)\,(t - 1),$$

thus

$$\int_{z_0}^{z} (\xi - z) v_n(\xi) J(\xi)\, d\xi = (z - z_0)^2 \int_0^1 (t-1) v_n(\xi(t)) J(\xi(t))\, dt,$$

and its modulus becomes

$$\begin{aligned}
\left| \int_{z_0}^{z} (\xi - z) v_n(\xi) J(\xi)\, d\xi \right| &= |z - z_0|^2 \left| \int_0^1 (t-1) v_n(\xi(t)) J(\xi(t))\, dt \right| \\
&\leqslant |z - z_0|^2 \int_0^1 |t-1|\, |v_n(\xi(t))|\, |J(\xi(t))|\, dt \\
&\leqslant |z - z_0|^{2n+2}\, \mu \frac{M^{n+1}}{n!} \int_0^1 (1-t)\, dt \\
&\leqslant |z - z_0|^{2n+2}\, \mu \frac{M^{n+1}}{n!} \int_0^1 (1-t) t^{2n}\, dt \\
&= |z - z_0|^{2n+2}\, \mu \frac{M^{n+1}}{(n+1)!} \int_0^1 (1-t)\, dt
\end{aligned}$$

Since $|z - z_0| < \rho$ where ρ is the radius of S, we get

$$|v_n(z)| \leqslant \frac{1}{n!} \mu M^n \rho^{2n}.$$

Now, the numerical series $\sum_{n=0}^{+\infty} \frac{1}{n!} \mu M^n \rho^{2n}$ is convergent, for its general term is that of the power series expansion of the exponential function, so the sum of this series can be written explicitly: $\sum_{n=0}^{+\infty} \frac{1}{n!}\mu M^n \rho^{2n} = \mu e^{M\rho^2}$. It follows that the series of functions $\sum_{n=0}^{+\infty} v_n(z)$ is *absolutely* convergent in S by Cauchy criterion, furthermore the M-test of Weierstrass ensures that the series is also *uniformly* convergent in S. Let us denote its sum with

$$v(z) = \sum_{n=0}^{+\infty} v_n(z).$$

According to the theorem of Weierstrass on the analiticity of the uniform limit of a sequence of analytic functions, $v(z)$ is analytic in S.

Last step: check that $v(z)$ is a solution

The last step it to show that such $v(z)$ actually satisfies Eq. (5). Since the series $\sum_{n=0}^{+\infty} v_n(z)$ converges uniformly on S, we can differentiate it term by term to get

$$v'(z) = b + \sum_{n=1}^{+\infty} v_n'(z),$$

and

$$v''(z) = \sum_{n=0}^{+\infty} v_n''(z),$$

for every $z \in S$. Using the previously derived formula for $v_n''(z)$, we finally get

$$v''(z) = -J(z) \sum_{n=0}^{+\infty} v_{n-1}(z) = -J(z) \sum_{n=0}^{+\infty} v_n(z) = -J(z)v(z),$$

which shows that $v(z)$ satisfies Eq. (5).

Obviously, $v(z_0) = a$ and $v'(z_0) = b$. By choosing $a = c_0$ and $b = c_1$, $v(z)$ satisfies the given initial conditions (7). The fact that $v(z)$ is also single-valued follows from the fact that S is a simple connected domain.

Uniqueness

Let us now address the problem of proving the uniqueness of the solution of Eq. (5) satisfying the initial conditions (7). Let $\tilde{v}(z)$ be another such solution and

$$w(z) = \tilde{v}(z) - v(z),$$

for every $z \in S$. Also $w(z)$ is analytic and single-valued on S and we can easily see by direct computation of its first and second order derivatives that it satisfies the same equation of $v(z)$ and $\tilde{v}(z)$, i. e.,

$$w''(z) + J(z)w(z) = 0,$$

with the initial conditions $w(z_0) = w'(z_0) = 0$. By repeated differentiation,

$$w'''(z_0) = -J'(z_0)w(z_0) - J(z_0)w'(z_0) = 0,$$

and analogously we get the general result that $w^{(n)}(z_0) = 0$ for every n, i. e., the derivatives of w of any order vanish at the z_0. By Taylor's theorem, $w(z)$ is identically zero in S, i. e., $v(z) = \tilde{v}(z)$ for every $z \in S$. ∎

Are there non-analytic solutions?

Some remarks are in order. First of all, there still remains the question as to wheater or not there might exist *non-analytic* solutions of Eq. (1) which satisfy the same initial conditions (2), since in the proof of the uniqueness we have made heavy use of analyticity. (Of course, if there were such non-analytic solutions they should be twice differentiable in order to satisfy the equation.) This question has beeen completely answered in the negative, but we do not enter the details here. For further details, we invite the Reader to consult the book of **Ince:1956** and references therein.

Taylor expansion of the solution

The series $\sum_{n=0}^{+\infty} v_n(z)$ is not in general a power series. However, its sum defines a function $v(z)$ which, as we have seen, is *analytic* in S. This means that we can Taylor expand $v(z)$ and the Taylor series will be convergent within S, and perhaps also in a larger circle,[13] and the same is true also for $u(z)$. In principle, one can determine $u(z)$ by constructing explicitly the sequence $v_n(z)$, then evaluating in closed form the sum of the series and using $u(z) = k(z)v(z)$ where $k(z)$ is given by Eq. (4). In practise, it would become hard or impossible to perform in closed form all steps of such computation. But now that we know that one and only one solution $u(z)$ of Eq. (1) exists and is *analytic* in S, we can find it by directly looking for its series representation with center at z_0:

$$u(z) = \sum_{n=0}^{+\infty} A_n (z - z_0)^n.$$

A priori, we know that the series will be convergent *at least* within S. The parameters A_0 and A_1 will be fixed by the initial conditions (2), in fact $u(z_0) = A_0$ and $u'(z_0) = A_1$; the other coefficients are found by plugging the above series representation into Eq. (1) and equating the coefficients of successive powers of $(z - z_0)$ to zero. The power series represenations of the first and second order derivatives of $u(z)$ are simply obtained by differentiating the above sum term by term, since the series is a priori guaranteed to converge uniformly within S. Notice that in pluggin the series expansions in the differential equation, it might be easier to work directly with the differential equation cleared of fractions, without having to reduce it to the standard form Eq. (1) in which the coefficient of the second-order derivative has been put equal to one. The standard form Eq. (1) is not always the best one to work with at a practical level, but it is quite adeguate for theoretical manipulations.

The a priori knowledge of the disk S

Finally, notice that the results of the theorem of this section hold only locally, i. e., in the neighbourhood S of z_0. But S is known *a priori*, i. e., it does not depend on the initial conditions and therefore it is the same for *all* the integrals of Eq. (1) about the ordinary point z_0. In this case we speak of *fixed* singularities.

Movable singularities may occur in the case of non-linear equations

For *non-linear* equations, it may happen that the disk of convergence depends on the initial conditions. Consider for instance the first-order homogeneus non-linear differential equation

$$u'(z) + u^2(z) = 0,$$

whose non identically-zero general integral is given by

$$u(z) = \frac{1}{z - c},$$

where the arbitrary constant of integration c has to be fixed by imposing initial conditions at some point z_0:

$$u(z_0) = \alpha,$$

with $\alpha \neq 0$. (If $\alpha = 0$, the solution would be $u(z) = 0$ everywhere.) By direct substitution we get $c = z_0 - \frac{1}{\alpha}$, so

$$u(z) = \frac{1}{(z - z_0) + \frac{1}{\alpha}}.$$

13 As we shall discuss in a deeper way in the § 1.3, the singularities of the coefficients of the differential equation need *not* in general to be singular points also of the integrals of the differential equation.

There is always a disk of non-zero radius with center at z_0 in which this function is analytic and single-valued and has no singular points. However, the position of its singularity depends on the initial condition, for in fact $u(z)$ has an isolated singular point when $z = z_0 - \frac{1}{\alpha}$, which can be made closer and closer to z_0 and which cannot be located a priori. In this case, one speaks of "movable" singularities.

1.2 THE WRONSKIAN

In order to make the presentation as self contained as possible, it seeems useful to briefly review some material concerning the "Wronskian" of two integrals of the differential equation.

Linear independence of two analytic functions

As usual, we define two functions w_1 and w_2, defined and analytic in the same domain ˙, to be "linearly *in*dependent" if the linear combination

$$w(z) = \alpha w_1(z) + \beta w_2(z), \quad \forall z \in \dot{}\,, \tag{10}$$

where α and β are complex numbers, vanishes identically in ˙ *if and only if* $\alpha = \beta = 0$. Otherwise, w_1 and w_2 are said to be "linearly dependent", which means that it is possible to find α and β such that

$$\alpha w_1(z) + \beta w_2(z) = 0$$

identically and at least one of the two complex numbers α and β is different from zero. (It is nothing more than the usual notion of linear independence in linear algebra.)

Definition of the Wronskian

Let us introduce the "Wronski determinant" or "Wronskian" of w_1 and w_2; it is defined by

$$W_{w_1,w_2}(z) = \det \begin{pmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{pmatrix} = w_1(z)w_2'(z) - w_1'(z)w_2(z)\,.$$

Notice that it is is a function of $z \in$ ˙.

The Wronskian is useful to test linear independence. In fact, we have the following theorems.

THEOREM 1.2: If w_1 and w_2 are linearly dependent, then W_{w_1,w_2} vanishes identically.

Proof. Differentiation of Eq. (10) yields

$$w'(z) = \alpha w_1'(z) + \beta w_2'(z)\,.$$

By hypothesis, w_1 and w_2 are linearly dependent, thus $w(z)$ is identically zero. Since $w(z)$ is constant in ˙, $w'(z) = 0$ for all $z \in$ ˙. In matrix form,

$$\begin{pmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system of equations is homogeneus, thus $\alpha = \beta = 0$ is a solution. Since by hypothesis, ■

The converse of the previous theorem is in general *not* true, i. e., it might happen that the Wronskian of w_1 and w_2 is zero *and* w_1 and w_2 are linearly *in*dependent in their domain of definition.

Let us consider an example with differentiable functions of a real variable. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_1(t) = t^2$$

and

$$f_2(t) = t\,|t|\,,$$

for every $t \in \mathbb{R}$. The two functions are differentiable and linearly *in*dependent on $\mathbb{R}$, for there exists no $\lambda \in \mathbb{R}$ such that $f_1(t) = \lambda f_2(t)$ for every $t \in \mathbb{R}$. But their Wronskian is identically zero for all $t \in \mathbb{R}$.

1.3 MANY-VALUEDNESS AND SINGULARITIES OF THE SOLUTIONS

1.3.1 *Analytic continuation of the solution*

We stress once again that the theorem of § 1.1 is a *local* result, in the sense that corresponding to every ordinary point z_0 of the Eq. (1) a solution of Eq. (1) satisfying the initial conditions (2) for arbitrarily-assigned complex values of α and β must exist in S and therein is guaranteed to be uniquely determined, analytic and single-valued. However, the theorem of § 1.1 gives no information regarding the behavior of the solution at points of ˙ away from S. In other words, the result only applies to a neighbourhood of the ordinary point z_0.

Analytic continuation of the solution

By the general process of analytic continuation, the series representing the solution in S can be analytically-continued at all points of ˙ with the exception of those which are singular points of the coefficients of the equation. The actual implementation of this procedure is explain in some detail in figure.

THEOREM 1.3: The analytic continuation of a solution of Eq. (1) is also a solution of Eq. (1).

Proof. Suppose $w(z)$ is a solution of Eq. (1) in the circle $|z - z_0| \leqslant \rho$ and let $w^*(z)$ be the analytic continuation of $w(z)$ in some other circle $|z - z_0| \leqslant \tilde{\rho}$ sharing a region G with the previous. Consider function

$$F(z) = w''(z) + p(z)w'(z) + q(z)w(z),$$

■

The a priori knowledge of the possible singular points of the integrals of a linear differential equation

Therefore, we have obtained a solution of Eq. (1) satisfying the initial conditions (2) for every complex α and β, which is *analytic* at *all* points of ˙ except, eventually, those points of ˙ which are singular points of the coefficients of the equation. The construction does not work for those points, but this does not mean that the series representing the solution cannot converge also at those points. In general, the singular points of the coefficients *may* be singular points of *some* integrals of the equation. But any integral of Eq. (1) cannot have singularieties at points of ˙ other than those which are singular points of the coefficients.

Although any integral of Eq. (1) is guaranteed to be *analytic* in ˙ except at (perhaps) the singularities (if any) of the coefficients of the equation, the key point is that in general these integrals need not to be *single-valued* throughout ˙ even though the coefficients of the equation are supposed to be single-valued. We shall now turn to investigate this problem.

1.3.2 *Local solutions near a singular point*

The main result of this section can be summarized in the following theorem.

THEOREM 1.4: Let $z_0 \in$ ˙ be a *singular* point of eq. (1). Let S be the annulus with center at z_0 and non-zero radius in which the Laurent series of $p(z)$ and $q(z)$ about z_0 are convergent. Two cases are possible:

A. There exist two linearly *in*dependent integrals of eq. (1) which are given by

$$w_1(z) = (z - z_0)^{\rho_1} \phi_1(z)\,,$$

and

$$w_2(z) = (z - z_0)^{\rho_2} \phi_2(z)\,,$$

where ρ_1 and ρ_2 are complex numbers such that $\rho_1 - \rho_2 \notin \mathbb{Z}$ and ϕ_1 and ϕ_2 are analytic in the annulus S.

B. There exist two linearly *in*dependent integrals of eq. (1), one is again given by

$$w_1(z) = (z - z_0)^{\rho_1} \phi_1(z)\,,$$

while the other is given by

$$w_2(z) = w_1(z) \left[A \log(z - z_0) + \psi(z)\right],$$

where $A \in \mathbb{C}$ and $\psi(z)$ is analytic in the annulus S.

ρ_1, ρ_2 are the solutions of the so-called "foundamental equation" associated with eq. (1) and the singular point z_0.

Let $z_0 \in$ ˙ be a singular point of the coefficients of the eq. (1) and σ be any neighbourhood of z_0 all of whose points are ordinary points of Eq. (1), e. g., σ can be the disk with centre at z_0 and having non-zero radius. The fact that such neighbourhood always exists follows from the previous hypothesis that ˙ is an open set and the singularities of the coefficients are isolated. Further, let b any ordinary point of σ and ` any simple closed contour, beginning and ending at b, which lies completely within σ (i. e., not passing through any singularity of the equation and such that there is no singularity of the equation other than z_0 in its interior) and which encircles exactly once in the positive direction (with respect to the area enclosed by it) the singular point z_0.

Consider two arbitrary linearly independent solutions $u_1(z)$ and $u_2(z)$ in the neighbourhood of b. Their corresponding analytic continuations along ` will be denoted by $\tilde{u}_1$ and $\tilde{u}_2$.

Since the coefficients $p(z)$ and $q(z)$ are left unaltered by the description of ` (because they are single-valued), by the previous theorem of this section we get that at all stages of the analytic continuation along ` the Eq. (1) remains identically fulfilled, in particular, also $\tilde{u}_1$ and $\tilde{u}_2$ are solutions of Eq. (1).

Further, the analytic continuations of two linearly independent solutions are linearly independent solutions. This can be easily seen by considering the complex analogous of the so-called Abel's formula (or also Liouville's formula) for the Wronski determinant.

Wronski determinant

The Wronski determinant (the so-called Wronskian) of any two differentiable functions $u_1(z)$ and $u_2(z)$ is defined by

$$W[u_1, u_2](z) = \det \begin{pmatrix} u_1(z) & u_2(z) \\ u_1'(z) & u_2'(z) \end{pmatrix}. \tag{11}$$

Notice that it is a function of z. The Wronski determinant is useful to test linear independence. In fact, if u_1 and u_2 are linearly dependent, this means that there must exist some complex constant λ such that $u_1(z) = \lambda u_2(z)$, then $u_1'(z) = \lambda u_2'(z)$ and the Wronskian is identically zero in its domain of definition. The converse in general is not true, but for analytic functions identical vanishing of the Wronskian implies the linear dependence of the functions.

Sufficient condition for linear dependence

For the Wronski determinant of integrals of a linear differential equation, it is easy to prove a formula known as Abel or Liouville formula which gives (up to a constant) the value of the Wronskian for any two integrals only from the knowledge of the differential equation! Let us now derive this formula. Suppose that $u_1(z)$ and $u_2(z)$ are solutions of Eq. (1) in some domain. Then

$$u_1''(z) + p(z)u_1'(z) + q(z)u_1(z) = 0,$$

and

$$u_2''(z) + p(z)u_2'(z) + q(z)u_2(z) = 0.$$

Multiplying the first equation by $u_2(z)$ and the second one by u_1 and subtractin one from the other we get

$$W'[u_1, u_2](z) - p(z)W[u_1, u_2](z) = 0, \tag{12}$$

which is a linear homogeneus ordinary differential equation of the first order for the Wronskian. (The first term on the left-hand side is the derivative in the complex sense of the Wronskian with respect to z, remember that $W[u_1, u_2]$ is a function of z.) By integrating from z_0 to z (assuming that the coefficient $p(z)$ is analytic) we get

Abel formula

$$W[u_1, u_2](z) = W[u_1, u_2](z_0) \exp\left\{-\int_{z_0}^{z} p(\xi)\, d\xi\right\}. \tag{13}$$

This is the well-known Abel, or Lioville, formula for the Wronski determinant of integrals of Eq. (1). An analogous formula holds for linear equations of arbitrary order, in that case the integral of $p(\xi)$ has to be replaced by the coefficient of the derivative of the $(n-1)$-th. This formula also holds in the real case and indeed it is easier to derive it in the real case (since you have not to worry as weather or not it is necessary to prescribe some contour of integration, etc, as it happens in the complex case).

Therefore, there must exist four complex constants α, β, γ, δ such that

$$\tilde{u}_1(z) = \alpha u_1(z) + \beta u_2(z),$$

and

$$\tilde{u}_2(z) = \gamma u_1(z) + \delta u_2(z),$$

with

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0,$$

otherwise $\tilde{u}_1$ and $\tilde{u}_2$ would be linearly dependent, in contrast with the earlier hypothesis that u_1 and u_2 are chosen linearly *in*dependent.

It can be shown, see **Tricomi:1961**, that the constants α, β, γ, δ depend in general on the choice of the linearly independent integrals u_1 and u_2 but not on the particular contour $\grave{}$ considered.

Among all integrals of Eq. (1), we look for integrals $w(z)$ which undergo the simplest of all possible linear transformation on describing the contour $\grave{}$, namely,

$$\tilde{w}(z) = \lambda w(z), \tag{14}$$

for some complex λ, where $\tilde{w}$ denotes the integral obtained by analytically continuing w along $\grave{}$. Clearly, not all integrals of Eq. (1) are of this type. If there were solutions of Eq. (1) of this form, then it would be possible to express them as a linear combination of u_1 and u_2, at least locally, i. e., there must exist complex coefficients k_1 and k_2 such that

$$w(z) = k_1 u_1(z) + k_2 u_2(z).$$

Since the analytic continuation of a linear combination of functions is the linear combinations of the analytic continued functions, we get

$$\begin{aligned}\tilde{w}(z) &= k_1(\alpha u_1(z) + \beta u_2(z)) + k_2(\gamma u_1(z) + \delta u_2(z)) \\ &= (\alpha k_1 + \gamma k_2)\, u_1(z) + (\beta k_1 + \delta k_2)\, u_2(z).\end{aligned}$$

It follows that Eq. (14) is satisfied if and only if

$$\begin{aligned}k_1\alpha + k_2\gamma &= \lambda k_1, \\ k_1\beta + k_2\delta &= \lambda k_2,\end{aligned}$$

that is

$$\begin{aligned}k_1\,(\alpha - \lambda) + k_2\gamma &= 0 \\ k_1\beta + k_2\,(\delta - \lambda) &= 0.\end{aligned} \tag{15}$$

This system of two simulataneus homogeneus equation has a trivial solution $k_1 = k_2 = 0$ yielding a identically-zero integral w. We look for non trivial solutions. The system has non trivial solutions if

$$\det\begin{pmatrix}\alpha - \lambda & \gamma \\ \beta & \delta - \lambda\end{pmatrix} = 0. \tag{16}$$

Eq. (16) is called "characteristic" or the "foundamental" equation associated with the singular point z_0. It is a quadratic equation for λ which has two *non-zero*[14] complex roots λ_1 and λ_2, which eventually may coincide. This equation restricts the possible values that λ can have in Eq. (14).

It can be shown that while α, β, γ and δ depend on the choice of u_1 and u_2, the roots of the characteristic equation are independent of this choice and depend only on the coefficients of Eq. (1) and on the point z_0. The proof is simple but rather tedius, see, e. g., **Ince:1956**.

Non-repeated roots

In the first place, let us consider the case when the characteristic equation has two distinct roots, i. e., $\lambda_1 \neq \lambda_2$. The system now reduce to only one equation which determines the values of k_1 and k_2 and thus (apart from a factor of proportionaly) two integrals $w_1(z)$ and w_2 which satisfy Eq. (14) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively.

We shall prove now that w_1 and w_2 must be linearly *in*dependent. Otherwise, the function w_1/w_2 would be a constant, but this is impossible since a constant would not change on describing $\grave{}$ while w_1/w_2 acquires a factor λ_1/λ_2 in doing so. Therefore, we have found two linearly *in*dependent solutions w_1 and w_2 and we may work with them instead of using the original

14 Their product is $\lambda_1\lambda_2 = \alpha\delta - \beta\gamma$, which must be non-zero by earlier considerations.

Figure 1: Lazarus Fuchs

u_1 and u_2. The reason to prefer w_1 and w_2 is that, as we shall see in a while, they have a particular simple form.

Consider the function $(z - z_0)^\rho$ which, on describing the contour ` becomes $e^{2\pi i \rho}\,(z - z_0)^\rho$, i. e., acquires a multiplicative factor $e^{2\pi i \rho}$ which in equal to one if and only if ρ is an integer number. Suppose to select a ρ_1 so that

$$e^{2\pi i \rho_1} = \lambda_1,$$

and analogously

$$e^{2\pi i \rho_2} = \lambda_2,$$

which means that

$$\rho_1 = \frac{1}{2\pi i} \ln \lambda_1\,,$$
$$\rho_2 = \frac{1}{2\pi i} \ln \lambda_2\,,$$

then the two functions

$$\phi_1(z) = \frac{w_1(z)}{(z - z_0)^{\rho_1}}\,, \tag{17a}$$

$$\phi_2(z) = \frac{w_2(z)}{(z - z_0)^{\rho_2}}\,, \tag{17b}$$

remains single-valued on describing the singular point z_0. Thus, the two functions $\phi_{1,2}(z)$ are analytic and single-valued in a disk centered at z_0 (except the point z_0) and can thus be expanded in Lauren series. Thus,

$$w_1(z) = (z - z_0)^{\rho_1} \sum_{n \in \mathbb{Z}} a_n\,(z - z_0)^n\,, \tag{18a}$$

$$w_2(z) = (z - z_0)^{\rho_2} \sum_{n \in \mathbb{Z}} b_n\,(z - z_0)^n\,. \tag{18b}$$

1.4 FUCHS THEOREM

There is a necessary and sufficient criterion, due to L. Fuchs, which enables us to establish *a priori* whether or not a singular point of Eq. (1) is a Fuchsian singular point directly from the inspection of the differential equation. This is the content of Fuchs' theorem. The proof of the theorem will lead us to a practical way to determine a foundamental system of solutions in a neighbourhood of a Fuchsian singular point.

THEOREM 1.5: A singular point $z_0 \in$ ˙ is a Fuchsian singular point of Eq. (1) *if and only if*

- $p(z)$ has *at most* a pole of the *first* order at the point z_0 and
- $q(z)$ has *at most* a pole of the *second* order at the point z_0;

i. e., *if and only if* $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are *analytic* at the point z_0.

1.4.1 *Proof of the necessity of Fuchs' conditions*

Let us first establish the necessity of Fuchs' conditions. We shall need the following lemma.

LEMMA 1.1: Let

$$f(z) = (z - z_0)^{\alpha}\, g(z)\,, \tag{19}$$

where α is *any* complex number and $g(z)$ is analytic at z_0. Then

- $\frac{f'(z)}{f(z)}$ has *at most* a pole of the *first* order at z_0;
- $\frac{f''(z)}{f(z)}$ has *at most* a pole of the *second* order at z_0.

Proof. The case in which g vanishes identically (and thus f) is trivial, so let us assume that g does not vanish identically.

> If $g(z_0) = 0$ and since g is assumed to be not identically zero, then there must exist $n \in \mathbb{N}$ such that
>
> $$g(z) = (z - z_0)^n\, \tilde{g}(z)\,,$$
>
> for every z in a neighbourhood of z_0, $\tilde{g}$ being *analytic* and different from zero in such neighbourhood (including the point z_0), i. e., g has a zero of "order" n at z_0.

Let us suppose with great generality that g has a zero of order $n \in \mathbb{N}$ at z_0. (Eventually, $n = 0$.) Then, we may write

$$f(z) = (z - z_0)^{\alpha+n}\, \tilde{g}(z)\,,$$

where $\tilde{g}(z) \neq 0$ in the neighbourhood of z_0. By differentiating, we get

$$\frac{f'(z)}{f(z)} = \frac{\alpha + n}{z - z_0} + \frac{\tilde{g}'(z)}{g(z)}$$

$$\frac{f''(z)}{f(z)} = \frac{(\alpha+n)(\alpha+n-1)}{(z-z_0)^2} + \frac{2(\alpha+n)}{z-z_0}\frac{\tilde{g}'(z)}{\tilde{g}(z)} + \frac{\tilde{g}''(z)}{\tilde{g}(z)}\,.$$

Since $\tilde{g}$ never vanishes and $\tilde{g}'$ and $\tilde{g}''$ are analytic at z_0 (for $\tilde{g}$ is analytic at z_0), we conclude that $\frac{f'(z)}{f(z)}$ has *at most* a pole of the first order at z_0,[15] and $\frac{f''(z)}{f(z)}$ has *at most* a pole of the second order at z_0.[16] ∎

Remark. Products, quotients and derivatives of functions of the form (19) are also of the form (19). Sums of functions of the form (19) in general are not of the form (19), except in the special case in which the corresponding indices α differ by an integer number.

Our goal is to prove that if z_0 is a Fuchsian singular point of eq. (1), then $p(z)$ and $q(z)$ have, at most, a pole of the first and second order at z_0, respectively. In order to prove that, we write $p(z)$ and $q(z)$ in terms of $w_1(z)$ and $w_2(z)$. We have

$$\begin{aligned} p(z) &= -\frac{d}{dz}\log W_{w_1,w_2}(z) \\ &= -\frac{d}{dz}\log\left(w_1(z)w_2'(z) - w_1'(z)w_2(z)\right) \\ &= -\frac{d}{dz}\log w_1^2(z)\left(\frac{w_1(z)w_2'(z) - w_1'(z)w_2(z)}{w_1^2(z)}\right) \\ &= -\frac{d}{dz}\log\left\{w_1^2(z)\frac{d}{dz}\left(\frac{w_2(z)}{w_1(z)}\right)\right\}, \end{aligned} \tag{20}$$

15 We say at most because it may happen that $\alpha + n = 0$.
16 We say at most because it may happen that either $\alpha + n = 0$ or $\alpha + n - 1 = 0$.

and

$$q(z) = -\frac{w_1''(z)}{w_1(z)} - p(z)\frac{w_1'(z)}{w_1(z)} \,. \tag{21}$$

We have to consider two cases. Let us first consider the case in which $w_1(z)$ and $w_2(z)$ are of the form (19) [see Eqs. (**??**)] and so are

$$\frac{w_2(z)}{w_1(z)}\,, \quad \frac{d}{dz}\frac{w_2(z)}{w_1(z)}\,, \quad w_1^2(z)\frac{d}{dz}\frac{w_2(z)}{w_1(z)}$$

and the logarithmic derivative of the last function is equal to $-p(z)$ and has at most a pole of the first order, which implies that $p(z)$ has at most a pole of the first order at z_0.

What is about the logarithmic case, i. e., when w_2 is given by

1.4.2 *Proof of the sufficiency of the Fuchs' conditions*

We now turn to prove that Fuchs' conditions are also sufficient. By hypothesis, the functions $P(z) = (z - z_0)\,p(z)$ and $Q(z) = (z - z_0)^2\,q(z)$ are analytic at z_0 and therefore

$$P(z) = (z - z_0)\,p(z) = \sum_{n=0}^{+\infty} p_n\,(z - z_0)^n\,, \tag{22a}$$

$$Q(z) = (z - z_0)^2\,q(z) = \sum_{n=0}^{+\infty} q_n\,(z - z_0)^n\,. \tag{22b}$$

Notice that we do not exclude the possibility that some of the leading coefficients in these two Taylor series expansions may be zero.

We look for solutions of the form

$$w(z) = (z - z_0)^\rho \sum_{n=0}^{+\infty} w_n\,(z - z_0)^n\,, \tag{23}$$

where ρ and the coefficients w_n (with $n \in \mathbb{N}$) are to be determined. We can always suppose that $w_0 \neq 0$, for if w_0 were zero, we could in any case shift the series since ρ is unknown.

Let us write Eq. (1) in the form

$$(z - z_0)^2\,w''(z) + (z - z_0)\,P(z)w'(z) + Q(z)w(z) = 0\,. \tag{24}$$

The derivatives of $w(z)$ are

$$w'(z) = \rho\,(z - z_0)^{\rho-1}\sum_{n=0}^{+\infty} w_n\,(z - z_0)^n + (z - z_0)^\rho \sum_{n=0}^{+\infty} n w_n\,(z - z_0)^{n-1}\,,$$

$$\begin{aligned} w''(z) = \rho(\rho-1)\,(z-z_0)^{\rho-2}\sum_{n=0}^{+\infty} w_n\,(z-z_0)^n + 2\rho\,(z-z_0)^{\rho-1}\sum_{n=0}^{+\infty} n w_n\,(z-z_0)^{n-1} \\ + (z - z_0)^\rho \sum_{n=0}^{+\infty} n(n-1)w_n\,(z - z_0)^{n-2}\,. \end{aligned}$$

Moreover, by Cauchy formula for multiplying two absolutely convergent series[17] we get

$$\begin{aligned}\sum_{n=0}^{+\infty} p_n (z-z_0)^n \sum_{n'=0}^{+\infty} w_{n'} (z-z_0)^{n'} &= \sum_{n=0}^{+\infty}\sum_{j=0}^{n} w_j p_{n-j} (z-z_0)^{j+n-j} \\ &= \sum_{n=0}^{+\infty}\left[(z-z_0)^n \sum_{j=0}^{n} w_j p_{n-j}\right],\end{aligned}$$

$$\begin{aligned}&\sum_{n=0}^{+\infty} p_n (z-z_0)^n \sum_{n'=0}^{+\infty} n' w_{n'} (z-z_0)^{n'-1} \\ &= \sum_{n=0}^{+\infty}\sum_{j=0}^{n} w_j p_{n-j} (z-z_0)^{j+n-j} \\ &= \sum_{n=0}^{+\infty}\left[(z-z_0)^n \sum_{j=0}^{n} w_j p_{n-j}\right],\end{aligned}$$

Plugging into Eq. (24),

$$\begin{aligned}&\rho(\rho-1)(z-z_0)^{\rho} \sum_{n=0}^{+\infty} w_n (z-z_0)^k \\ &+ 2\rho (z-z_0)^{\rho} \sum_{n=0}^{+\infty} n w_n (z-z_0)^n \\ &+ (z-z_0)^{\rho} \sum_{n=0}^{+\infty} n(n-1) w_n (z-z_0)^n \\ &+ \rho (z-z_0)^{\rho} \sum_{n=0}^{+\infty} (z-z_0)^n \sum_{j=0}^{n} w_j p_{n-j} \\ &+ (z-z_0)^{\rho} \sum_{n=0}^{+\infty} (z-z_0)^n \sum_{j=0}^{n} w_j j p_{n-j} \\ &+ (z-z_0)^{\rho} \sum_{n=0}^{+\infty} (z-z_0)^n \sum_{j=0}^{n} w_j q_{n-j} = 0,\end{aligned} \tag{25}$$

from which it follows

$$\sum_{n=0}^{+\infty}\left\{[\rho(\rho-1)+2\rho n+n(n-1)]\, w_n + \sum_{j=0}^{n}\left[(\rho+j)p_{n-j}+q_{n-j}\right] w_j\right\}(z-z_0)^{n+\rho} = 0\,.$$

Notice that

$$\begin{aligned}\rho(\rho-1)+2\rho n+n(n-1) &= \rho^2+\rho(2n-1)+n(n-1) \\ &= (\rho+n)(\rho+n-1)\,.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{+\infty}\left\{(\rho+n)(\rho+n-1)w_n + \sum_{j=0}^{n}\left[(\rho+j)p_{n-j}+q_{n-j}\right] w_j\right\}(z-z_0)^{n+\rho} = 0\,.$$

17 Namely, $\left(\sum_{n=0}^{+\infty} a_n\right)\left(\sum_{n'=0}^{+\infty} b_{n'}\right) = \sum_{n=0}^{+\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

The coefficients must be zero, so

$$(\rho+n)(\rho+n-1)w_n + \sum_{j=0}^{n}\left[(\rho+j)p_{n-j} + q_{n-j}\right] w_j = 0\,. \tag{26}$$

This is a system of infinitely many homogeneus linear coupled recursive relations involving one additional unknown parameter, which is ρ.

First of all, it is useful to simplify the notations. We set

$$\begin{aligned}\lambda_0(\xi) &= \xi(\xi-1) + p_0\xi + q_0\\ &= \xi^2 + \xi(p_0-1) + q_0\end{aligned}$$

and

$$\lambda_n(\xi) = p_n\xi + q_n\,, \quad n \in \mathbb{N}\backslash\{0\}$$

for every $\xi \in \mathbb{C}$. Then, for $n=0$ we get

$$\lambda_0(\rho)w_0 = 0, \tag{27}$$

and for $n \in \mathbb{N}\backslash\{0\}$ we have

$$\begin{aligned}&(\rho+n)(\rho+n-1)w_n + \sum_{j=0}^{n}\left[(\rho+j)p_{n-j} + q_{n-j}\right] w_j\\ &= (\rho+n)(\rho+n-1)w_n + \left[(\rho+n)p_0 + q_0\right] w_n + \sum_{j=0}^{n-1}\left[(\rho+j)p_{n-j} + q_{n-j}\right] w_j\\ &= \lambda_0(\rho+n)w_n + \sum_{j=0}^{n-1}\lambda_{n-j}(\rho+j)w_j\,,\end{aligned}$$

so

$$\lambda_0(\rho+n)w_n + \sum_{j=0}^{n-1}\lambda_{n-j}(\rho+j)w_j = 0\,. \tag{28}$$

Since $w_0 \neq 0$, Eq. (27) yields

$$\lambda_0(\rho) = 0\,, \tag{29}$$

or more explicitly

$$\rho^2 + (p_0-1)\rho + q_0 = 0\,.$$

Eq. (29) is called "indicial equation" associated with the point z_0 and the differential equation (1).

1.5 THE POINT AT INFINITY

1.6 TOTALLY FUCHSIAN EQUATIONS

> Integrals of Fuchsian differential equations are a source of trascendental functions.
>
> — Folklore

DEFINITION 1.2: Eq. (1) is called "of Fuchsian class" or "totally Fuchsian" if all its singular points in the *extended* complex plane are Fuchsian singular points.

The definition includes the point at infinity. So far, the isolated singularities of eq. (1) has been assumed to be finite in number *by hypothesis*, it is clear that a totally Fuchsian equation *must* have at most finitely many singular point.

1.7 INTERLUDE ON MÖBIUS TRANSFORMATIONS

1.7.1 *Definition and first properties*

In this section, we shall study mappings of the extended complex plane onto itself of the form

$$w(z) = \frac{az+b}{cz+d}, \tag{30}$$

where a, b, c, d are arbitrary complex numbers which satisfy

$$ad - bc \neq 0. \tag{31}$$

Since multiplying all these numbers by a common non-zero number does not alter the mapping, it is always possible to assume that

$$ad - bc = 1. \tag{32}$$

Eq. (30) is referred to as "Möbius transformation", in honor of the mathematician Möbius (the same of the famous strip). It is also sometimes called "homographic transformation" or in short "homography".

Due to the fact that these transformations play an important role in various fields of pure and applied Mathematics and Physics, it seems useful to study their properties more carefully than what is needed for the purposes of the following sections.

THEOREM 1.6: Any Möbius transformation which is not a linear function can always be obtained as the composition of two linear transformations and one inversion.

Proof. If $c = 0$ we get a linear transformation:

$$w(z) = \frac{a}{d}z + \frac{b}{d}.$$

In the case $c \neq 0$, one can notice that

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)},$$

which shows explicitly that the transformation may be obtained equivalently through the composition of

$$\begin{aligned} w_1(z) &= cz+d, \\ w_2(w_1) &= \frac{1}{w_1}, \\ w_3(w_2) &= \frac{a}{c} + \frac{bc-ad}{c}w_2, \end{aligned}$$

so the theorem is proven. ∎

1.7.2 *Group structure*

THEOREM 1.7: The set of all and only Möbius transformations on the extended complex plane form a group.

Proof. Consider the set of all and only the Möbius transformations on the extended complex z-plane. The composition of any two Möbius functions is again a Möbius function. We have to prove that

A. Composition of Möbius transformations is associative;

B. Among all and only the Möbius functions, there exists an identity transformation, i. e., a Möbius transformation such that

C. There exists a null element, i. e., a Möbius transformation which, after composition with the identity, remains the same.

D. Any Möbius transformation is invertible.

■

1.8 THE PAPPERITZ-RIEMANN EQUATION

The main result of this section is

> All second-order totally Fuchsian equations having precisely three regular singular points in the extended complex plane are *uniquely* determined by the position of their singular points and the corresponding pairs of characteristic exponents at these points.

???

2 HYPERGEOMETRIC DIFFERENTIAL EQUATION

In this chapter

2.1 HYPERGEOMETRIC SERIES

$$
{}_2F_1(a, b; c; x) = \sum_{k=1}^{+\infty} \frac{(a)_n \, (b)_n}{(c)_n} z^n. \tag{33}
$$

2.2 POCHHAMMER INTEGRAL REPRESENTATION

3 CONFLUENT HYPERGEOMETRIC FUNCTIONS

In this chapter
This chapter is mostly based on **Tricomi:1954**; **Tricomi:1962**.

A NUMERICAL ALGORITHMS

In this chapter