April 25, 2023

Handout on Stirling's formula

Alessandro Candolini

Department of Physics, University Of Trieste, via Valerio 2, Trieste, Italy

E-mail: alessandro.candolini@gmail.com

Abstract: Some proofs of the well-known Stirling's asymptotic approximation of the factorial (or the Gamma function) are sketched. No attempt was made at full mathematical rigour, emphasis is mostly on the main ideas behind these proofs.

Contents

1 Overview **1**

2 Naive integral approximation **2**

3 Rigorous results from integral method. Wallis formula **5**

4 Saddle point derivation of Stirling's formula **9**

5 Stirling series **12**

6 A modern elementary proof **12**

A Gamma function warm up **12**
- A.1 Integral representation 12
- A.2 Functional equation 14
- A.3 Double factorial notation 17
- A.4 Another definition of $\Gamma(z)$ and a proof of Wallis formula 18
- A.5 Beta function 19
- A.6 Reflection formula 21

B Steepest descent method **22**
- B.1 Real case (alias, Laplace method) 22
- B.2 Complex case 22

1 Overview

The "factorial" $n!$ of a positive integer number $n \in \mathbb{N}$ is defined by induction: let $0! = 1$ and $n! = n(n-1)!$ for every $n \geq 1$.

Stirling's formula is an asymptotic approximation of $n!$ for "large" n, namely

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} \mathrm{e}^{-n}, \quad \text{as } n \to +\infty\,, \tag{1.1}$$

where the symbol $\sim$ will be used to denote asymptotic equality. The rigorous meaning of eq. (1.1) is

$$\lim_{n \to +\infty} \frac{n!}{\sqrt{2\pi} n^{n+\frac{1}{2}} \mathrm{e}^{-n}} = 1\,.$$

Since factorials are among the basic ingredients of combinatorics, Stirling's formula is extremely useful in many fields of mathematics, statistics and physics to help simplify formulæ involving factorials. For this reason, it seems useful to review few of the standard methods to prove it.

The simplest argument leads to an (heuristic) integral approximation of $\log n!$ (see section 2). To be precise, the method actually returns rigorous lower and upper integral bounds of $\log n!$. A (still heuristic) refinement of the procedure allows to extract the right dependence on n, but the procedure is still heuristic and the factor $\sqrt{2\pi}$ is still missing. In order to fully recover eq. (1.1) in a rigorous way, more advanced machinary is needed.

The heuristic result of section 2 is used as a guess for the rigorous treatment of section 3, based on the famous Wallis formula. Wallis formula is a rather technical one however, and it would be useful if it were possible to prove Stirling's formula relying on more general tools.

Remember that factorials can be written in terms of the gamma trascendental function:

$$n! = \Gamma(n+1) = \int_0^{+\infty} t^n \mathrm{e}^{-t}\,\mathrm{d}t, \quad \forall n \in \mathbb{N}\,, \tag{1.2}$$

where the Euler's integral representation of the Gamma function has been used in eq. (1.2). Gamma function provides a generalization of the factorial to real and complex numbers. This generalization is unique up to some extra conditions. We will thoroughly study the basic properties of Gamma function and how it generalizes factorials in appendix A. The Euler's integral representation of the Gamma function in eq. (1.2) is a good starting point to prove eq. (1.1) using more advanced techniques.

A standard approach is that of using the saddle-point asymptotic method on the right-hand side of eq. (1.2) , which leads directly to eq. (1.1). This approach is outlined in section 4. Saddle point method is a powerful method to compute the asymptotic behavior of certain parametric integrals, and it is discussed more deeply in appendix B.

A third way to prove eq. (1.1) is given in section 6. Only basic tools of calculus courses are needed. This is a recent proof due to R. Michael, which again involves the integral representation eq. (1.2).

2 Naive integral approximation

By definition, the factorial of $n \in \mathbb{N}$ is a product. For every $n \in \mathbb{N}\backslash\{0\}$, $n!$ can be equivalently written as $n! = \prod_{k=1}^{n} k$.* Now, the trick to proceed is this: (a) first of all, turn this product into a sum by taking the logarithm of it, then (b) approximate this sum with an integral. Let us explain the various steps in detail.

First, for $n \in \mathbb{N}\backslash\{0\}$ take the logarithm of $n!$:

$$\log n! = \log\left(\prod_{k=1}^{n} k\right) = \sum_{k=1}^{n} \log(k)\,, \quad n \in \mathbb{N}\backslash\{0\}. \tag{2.1}$$

* For every $n \in \mathbb{N}\backslash\{0\}$, this is equivalent to the definition of the factorial by induction given at the beginning of section 1 (proof of the equivalence is by induction). For $n = 0$ instead, the formula $n! = \prod_{k=1}^{n} k$ does not work. The position $0! = 1$ is conventional and arbitrary, however it is useful and it is consistent with the relation between factorials and the gamma function (see appendix A).

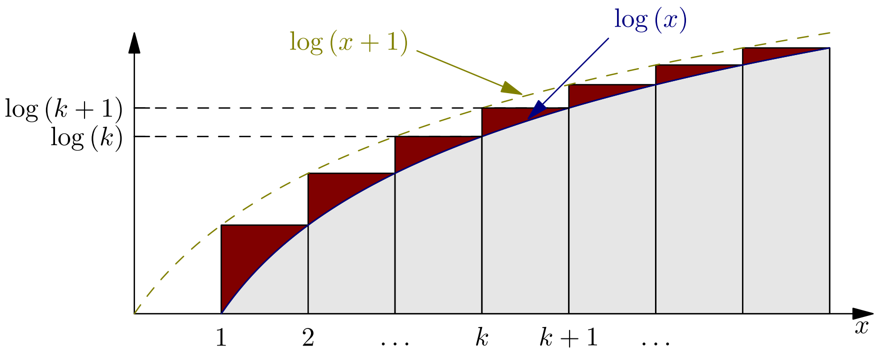

Figure 1: Illustration of the key idea beyond integral method: comparison between sums and integrals. Since $\log(x)$ is an increasing function of x, the integral of $\log(x)$ (graphical meaning: gray area in figure) understimates $\log n!$ (the total area under the rectangles), the (nearly triangular) brown areas above the curve of $\log(x)$ being left out. On the contrary, the integral of $\log(x+1)$ overestimates that sum.

Equation (2.1) is an *exact* expression of the logarithm of the factorial. Now, *approximate* the sum in eq. (2.1) with an integral:

$$\log n! = \sum_{k=1}^{n} \log(k) \approx \int_1^n \log(x)\, \mathrm{d}x, \quad n \in \mathbb{N}\backslash\{0\}\,. \tag{2.2}$$

Graphical interpretation of eq. (2.1) (see fig. 1): $\sum_{k=1}^{n} \log(k)$ (area under the rectangles) is approximated by $\int_1^n \log(x)\, \mathrm{d}x$ (whose pictorial meaning is the gray area below the curve of $\log(x)$).

The integral in eq. (2.2) is evaluated by parts as usual,

$$\int_1^n \log(x)\, \mathrm{d}x = \Big[x \log(x) - x\Big]_{x=1}^{x=n} = n \log(n) - n + 1\,,$$

hence eq. (2.2) reads

$$\log(n!) \approx n \log(n) - n + 1\,. \tag{2.3}$$

For "large" n, $n - 1 \approx n$ and exponentiating eq. (2.3) yields

$$\boxed{n! \approx \mathrm{e}^{n \log(n) - n} = n^n \mathrm{e}^{-n}\,.} \tag{2.4}$$

At this point however we have no precise control of the kind of approximation (i. e., a bound on the error) in eq. (2.2).

Actually, the approximation eq. (2.2) *under*stimates $\log(n!)$, as it should be clear from fig. 1: the area under the curve of $\log(x)$ is less than the area under the rectangles, the (brown) areas above the curve of $\log(x)$ being left out.

If you don't want to rely on fig. 1, it is not difficult to convert the graphical argument into a purely algebraic proof. Since $\log(x)$ is a *monotonically increasing* function of x, then

$$\int_{k-1}^{k} \log(x)\, \mathrm{d}x < \log(k) < \int_{k}^{k+1} \log(x)\, \mathrm{d}x\,,$$

for every $k \in \mathbb{N}$, $k \geq 2$. Equivalently, by changing variables in the second integral:

$$\int_{k-1}^{k} \log(x)\ \mathrm{d}x < \log(k) < \int_{k-1}^{k} \log(x+1)\ \mathrm{d}x\,,$$

for every $k \in \mathbb{N}$, $k \geq 2$. Summing from $k = 2$ to $k = n$ yields

$$\sum_{k=2}^{n} \int_{k-1}^{k} \log(x)\ \mathrm{d}x < \underbrace{\sum_{k=2}^{n} \log(k)}_{\left(\sum_{k=1}^{n}\log(k)\right)-\log(1)=\sum_{k=1}^{n}\log(k)=\log(n!)} < \sum_{k=2}^{n} \int_{k-1}^{k} \log(x+1)\ \mathrm{d}x\,,$$

and therefore

$$\int_{1}^{n} \log(x)\ \mathrm{d}x < \log(n!\,) < \int_{1}^{n} \log(x+1)\ \mathrm{d}x\,,$$

for every $n \in \mathbb{N}$, $n \geq 2$, or if you prefer to include also the case $n = 1$

$$\int_{1}^{n} \log(x)\ \mathrm{d}x \leq \log(n!\,) \leq \int_{1}^{n} \log(x+1)\ \mathrm{d}x, \quad \text{for every } n \in \mathbb{N}\backslash\{0\}\,.$$

Graphical meaning (refer to fig. 1): the total area of the rectangles (i. e., $\log(n!\,)$) is greater than $\int_1^n \log(x)\ \mathrm{d}x$ (i. e., the area below $\log(x)$) and less than $\int_1^n \log(x+1)\ \mathrm{d}x$ [i. e., the area below $\log(x+1)$]. This shows that the approximation eq. (2.2) actually gives a rigorous *lower* bound of $\log(n!\,)$.

The approximation can be improved by including the contributions coming from the nearly-triangular brown areas left-out above the plot of $\log(x)$ (see fig. 1). Notice that this is nothing but the usual trapezoid rule to approximate an integral.*

The k-th left out area can be roughly approximated by the triangular area

$$\frac{1}{2}\left[\log(k+1) - \log(k)\right].$$

Summing all the contributions yields

$$\log(n!\,) \approx \int_{1}^{n} \log(x)\ \mathrm{d}t + \frac{1}{2}\sum_{k=1}^{n}\left[\log(k+1) - \log(k)\right]. \tag{2.5}$$

The telescopic sum on the right-hand side can be evaluated exactly

$$\begin{aligned}\sum_{k=1}^{n}\left[\log(k+1) - \log(k)\right] &= \log(n) - \log(1)\\ &= \log(n)\,,\end{aligned} \tag{2.6}$$

(the rigorous proof is by induction) and

$$\log(n!\,) \approx n\log(n) - n + 1 + \frac{1}{2}\log(n) \approx \left(n + \frac{1}{2}\right)\log(n) - n\,, \tag{2.7}$$

* Euler-MacLaurin formula gives the error of the trapezoid approximation. We will come back to this point discussing Stirling asymptotic series.

and by taking the exponential

$$n! \approx n^{n+\frac{1}{2}} \mathrm{e}^{-n} \tag{2.8}$$

This of course is just an heuristic argument, since at this stage we have no control on the kind of approximation occurring in eq. (2.5). Equation (2.8) shares the correct dependence on n, compared to eq. (1.1), but the factor $\sqrt{2\pi}$ is still missing. In the next section we will use this result as starting point for a more rigorous treatment.

3 Rigorous results from integral method. Wallis formula

Given the heuristic result eq. (2.8), define a sequence $(a_n)_{n\in\mathbb{N}\setminus\{0\}}$ of positive numbers

$$a_n = \frac{n!}{n^{n+\frac{1}{2}} \mathrm{e}^{-n}}, \quad \forall n \in \mathbb{N}\,.$$

Our goals are: (a) prove that $\lim_n a_n$ exists; (b) calculate it. This will be done in theorem 3.2. To achieve this, we need the following preliminary result.

Theorem 3.1 (Wallis formula):

$$\lim_{n\to+\infty} \frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!} = \sqrt{\pi}\,. \tag{3.1}$$

Proof. A elegant proof based on properties of gamma function is given in appendix A. Here, we give another proof, based only on elementary calculus. The drawback of this proof is that it becomes quite long and a bit tricky.

Consider the sequence of integrals

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, \mathrm{d}\vartheta, \quad \forall n \in \mathbb{N}\,. \tag{3.2}$$

Integrals of this form often arise in physics, e. g., algebraic relations of non-relativistic quantum theory of angular momentum, i. e., spherical harmonics.* We will use those integrals to prove the Wallis formula. First of all, we need an explicit expression for I_n. These integrals can be easily computed by using the beta trascendental function, see appendix A. Here we will use a naive approach: we will use integration by parts to obtain a linear recursive equation for I_n, which can be solved exactly in closed form. Integration by parts yields

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, \mathrm{d}\vartheta \\
&= -\cos(\vartheta)\sin^{n-1}(\vartheta)\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^2(\vartheta)\sin^{n-2}(\vartheta)\, \mathrm{d}\vartheta \\
&= (n-1)\left(I_{n-2} - I_n\right).
\end{aligned}$$

Re-arranging the terms in this expression yields

$$I_n = \frac{n-1}{n} I_{n-2}, \quad \forall n \in \mathbb{N} \text{ and } n \geq 2\,. \tag{3.3}$$

* **Sakurai.Napolitano:2011**.

Equation (3.3) is a second-order linear homogeneus recursive equation with non-constant (rational) coefficients. The initial conditions are

$$
\begin{aligned}
I_0 &= \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta = \frac{\pi}{2}\,, \\
I_1 &= \int_0^{\frac{\pi}{2}} \sin(\vartheta)\,\mathrm{d}\vartheta = -\cos(\vartheta)|_0^{\frac{\pi}{2}} = 1\,.
\end{aligned}
$$

One can use the general theory of linear recursive equations with variable (polynomial) coefficients to approach eq. (3.3) systematically. The fact that the coefficients are not constant of course makes the things harder. The general solution of eq. (3.3) can be written in terms of gamma function. At a more straightforward level, one can start by calculating explicitly the lowest terms in the sequence I_n by using the initial conditions, sequentially producing the next tems until a clear pattern emerges. Once the general patter is identified, one can prove rigorously by induction that the pattern is indeed correct. Let us proceed in this way.

For n odd, $n = 2k+1$ for some $k \in \mathbb{N}$, the first terms in the recursion are

$$
\begin{aligned}
I_3 &= \frac{2}{3} I_1\,, \\
I_5 &= \frac{4}{5} I_3 = \frac{4}{5}\frac{2}{3} I_1\,, \\
I_7 &= \frac{6}{7} I_5 = \frac{6}{7}\frac{4}{5}\frac{2}{3} I_1\,,
\end{aligned}
$$

and it is possible to identify a pattern:

$$
\boxed{I_{2k+1} = \frac{(2k)!!}{(2k+1)!!} I_1 = \frac{(2k)!!}{(2k+1)!!}, \quad \forall k \in \mathbb{N}\,.} \tag{3.4}
$$

(Double factorial notation is explained in appendix A.) The rigorous proof of the relation above is by induction.

For n even, $n = 2k$ for some $k \in \mathbb{N}$, the first terms in the recursion are

$$
\begin{aligned}
I_2 &= \frac{1}{2} I_0\,, \\
I_4 &= \frac{3}{4} I_2 = \frac{3}{4}\frac{1}{2} I_0\,, \\
I_6 &= \frac{5}{6} I_4 = \frac{5}{6}\frac{3}{4}\frac{1}{2} I_0\,,
\end{aligned}
$$

and we may expect the general pattern to be

$$
\boxed{I_{2k} = \frac{(2k-1)!!}{(2k)!!} I_0 = \frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2}\,.} \tag{3.5}
$$

Also in this case, the rigorous proof can be done by induction.

Now that we have explicit formulæ for I_n in eq. (3.2), we can ask: how are those integrals related to the Wallis formula eq. (3.1)? How can I_n be useful to prove eq. (3.1)?

This is a rather technical point: the trick is using I_n to obtain a chain of inequalities which will allow us to prove eq. (3.1) by invoking the *squeeze theorem*.

First of all, since $0 \leq \sin(\vartheta) \leq 1$ for $\vartheta \in [0, \pi/2]$, we have

$$0 \leq \sin^{2k+1}(\vartheta) \leq \sin^{2k}(\vartheta) \leq \sin^{2k-1}(\vartheta)\,, \quad \forall \vartheta \in [0, \pi/2] \text{ and } \forall k \in \mathbb{N}\backslash\{0\}$$

and for the monotonicity of the integral we have

$$0 \leq I_{2k+1} \leq I_{2k} \leq I_{2k-1}\,, \quad \forall k \in \mathbb{N}\backslash\{0\}\,.$$

Thus,

$$0 \leq \frac{(2k)!\,!}{(2k+1)!\,!} \leq \frac{(2k-1)!\,!}{(2k)!\,!}\frac{\pi}{2} \leq \frac{(2k-2)!\,!}{(2k-1)!\,!}\,, \quad \forall k \in \mathbb{N}\backslash\{0\}\,,$$

or

$$0 \leq \frac{(2k)!\,!}{(2k+1)!\,!} \leq \frac{(2k+1)!\,!}{(2k)!\,!}\frac{1}{2k+1}\frac{\pi}{2} \leq \frac{(2k)!\,!}{(2k+1)!\,!}\frac{2k+1}{2k}\,, \quad \forall k \in \mathbb{N}\backslash\{0\}\,.$$

Since

$$\frac{(2k)!\,!}{(2k+1)!\,!} = \frac{((2k)!\,!)^2}{(2k+1)!} = \frac{\left(2^k k!\right)^2}{(2k+1)!} = \frac{2^{2k}(k!)^2}{(2k)!}\frac{1}{2k+1}\,,$$

we get

$$0 \leq \frac{1}{2k+1}\frac{2^{2k}(k!)^2}{(2k)!} \leq \frac{(2k)!}{2^{2k}(k!)^2}\frac{\pi}{2} \leq \frac{2^{2k}(k!)^2}{(2k)!}\frac{1}{2k}\,.$$

Moving things around,

$$0 \leq \frac{k}{2k+1}\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 \leq \frac{\pi}{2} \leq \frac{1}{2}\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2\,.$$

It follows that

$$\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 \leq \frac{2k+1}{k}\frac{\pi}{2}$$

and

$$\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 \geq \pi$$

Since $\lim_{k\to+\infty} \frac{2k+1}{k}\frac{\pi}{2} = \pi$, applying squeeze theorem yields

$$\lim_{k\to+\infty}\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 = \pi\,.$$

Taking the square of both sides yields eq. (3.1). ∎

We are ready to prove Stirling formula.

THEOREM 3.2 (Stirling formula): Consider the sequence $(a_n)_{n\in\mathbb{N}\backslash\{0\}}$ definied by

$$a_n = \frac{n!}{n^{n+\frac{1}{2}}\mathrm{e}^{-n}}\,, \quad \forall n \in \mathbb{N}\,. \tag{3.6}$$

Then,

$$\lim_{n\to+\infty} a_n = \sqrt{2\pi}\,. \tag{3.7}$$

.

Proof. The proof has two steps:

- show existence of the limit in eq. (3.7), i. e., prove that there exists a real number $c \in \mathbb{R}$ such that
$$\lim_n \frac{n!}{n^{n+\frac{1}{2}} \mathrm{e}^{-n}} = c \,,$$
- then apply Wallis formula (theorem 3.1) to show that $c = \sqrt{2\pi}$.

To prove the first part, we will show now that the sequence $(\alpha_n)_n$ is decreasing and bounded below by a positive constant, thus it converges to some positive real number c.

Consider
$$\log\left(\frac{a_n}{a_{n+1}}\right) = \left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) - 1 \,.$$
Remember the Taylor series of log:
$$\log(1+x) = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{x^k}{k} \,,$$
for $|x| < 1$. The first terms are
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathrm{O}(x^5)$$
Thus
$$\log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} \mathrm{O}\left(n^{-5}\right) ,$$
and
$$\begin{aligned}
\log\left(\frac{a_n}{a_{n+1}}\right) &= \left(n + \frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \mathrm{O}\left(n^{-5}\right)\right) - 1 \\
&= 1 - \frac{1}{2n} + \frac{1}{3n^2} + - \frac{1}{4n^3} + \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{6n^3} - 1 + \mathrm{O}\left(n^{-4}\right) \\
&= \frac{1}{3n^2} - \frac{1}{4n^2} - \frac{1}{4n^3} + \frac{1}{6n^3} + \mathrm{O}\left(n^{-4}\right) \\
&= \frac{1}{12n^2} - \frac{1}{12n^3} + \mathrm{O}\left(n^{-4}\right)
\end{aligned}$$
This implies $\log(a_n/a_{n+}) > 0$ and thus $(a_n)_n$ is a *descresing* sequence (at least for sufficiently large n).

To prove that $c = \sqrt{2\pi}$, consider the following limit:
$$\lim_n \underbrace{\frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!}}_{\to\sqrt{\pi}} \underbrace{\left(\frac{n^{n+\frac{1}{2}}\mathrm{e}^{-n}}{n!}\right)^2}_{\to\frac{1}{c^2}} \underbrace{\frac{(2n)!}{(2n)^{2n+\frac{1}{2}}\mathrm{e}^{-2n}}}_{\to c} = \frac{\sqrt{\pi}}{c} \,.$$

(Here we have used the theorem of the limit of a product.) A direct computation shows that this limit is also equal to

$$
\begin{aligned}
\frac{\sqrt{\pi}}{c} &= \lim_{n\to+\infty} \frac{2^{2n} n^{2\left(n+\frac{1}{2}\right)} \mathrm{e}^{-2n}}{\sqrt{n}(2n)^{2n+\frac{1}{2}} \mathrm{e}^{-2n}} \\
&= \lim_{n\to+\infty} \frac{2^{2n} n^{2n+1}}{n^{\frac{1}{2}}(2n)^{2n+\frac{1}{2}}} \\
&= \lim_{n\to+\infty} 2^{2n} \left(\frac{n}{2n}\right)^{2n+\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\,,
\end{aligned}
$$

from which it follows that $c = \sqrt{2\pi}$. ■

4 Saddle point derivation of Stirling's formula

This is a standard way (and probably the simplest one) for proving Stirling's formula. The method however requires some advanced material:

- Relation between factorial and gamma function;
- properties of the gamma function (in particular, its integral representation);
- saddle point method.

The relation between factorial and gamma function and properties of gamma function are highlighted in appendix A. The saddle point method is summarized in **??**. In this section we will discuss how this works for the factorial of a natural number (which is rather simple). The full application of the method to discuss the asymptotic behavior of the gamma function in the complex plane is done in the appendix.

The foundamental relation between factorials and the Euler's gamma function is

$$
n! = \Gamma(n+1) = \int_0^{+\infty} t^n \mathrm{e}^{-t}\,\mathrm{d}t, \quad \forall n \in \mathbb{N}\,,
$$

where we have used the integral representation of the gamma function. (See appendix A for details.)

Preliminary step: let us write the integrand in a completely equivalent fashion:

$$
\Gamma(n+1) = \int_0^{+\infty} \mathrm{e}^{\log(t^n)} \mathrm{e}^{-t}\,\mathrm{d}t = \int_0^{+\infty} \mathrm{e}^{n\log(t)-t}\,\mathrm{d}t\,.
$$

Remark. The latter was not a change of variables within the integral. We have just used the fact $t^n = \mathrm{e}^{\log(t^n)} = \mathrm{e}^{n\log(t)}$.

In order to understand better what is going on, let us briefly outline the behavior of the integrand, which looks as shown in **??**.

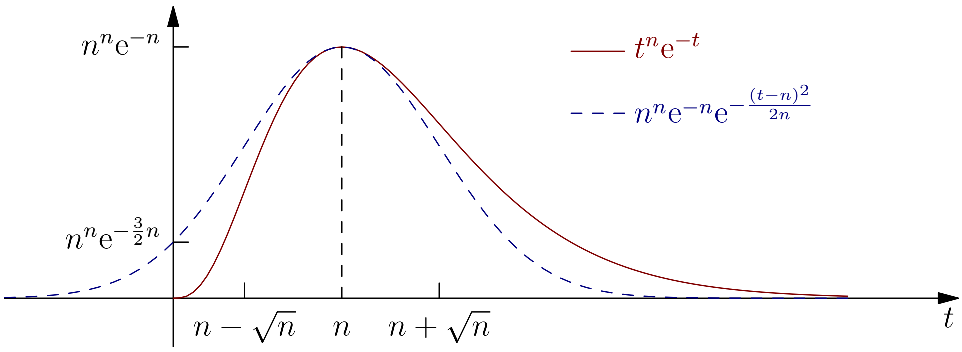

Figure 2: Plot of the integrand function $g(t) = t^n \mathrm{e}^{-t}$ and its

Let $f, g : [0, +\infty[\to \mathbb{R}$ be

$$f(t) = n \log(t) - t\,,$$
$$g(t) = \mathrm{e}^{f(t)}\,.$$

g is integrand function. Their derivatives are respectively

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{n}{t} - 1\,,$$
$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = -\frac{n}{t^2}\,,$$

and

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = g(t)\frac{\mathrm{d}f(t)}{\mathrm{d}t}\,,$$
$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2\right] g(t)\,.$$

Since g never vanishes, $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 0$ if and only if $\frac{\mathrm{d}f(t)}{\mathrm{d}t}$, i. e., if and only if $t = n$.

Since $\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = -\frac{1}{n} < 0$ for every $t \in [0, +\infty[$ and

$$\left(\frac{\mathrm{d}^2 g(t)}{\mathrm{d}t^2}\right)_{t=n} = \left(\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right)_{t=n} g(n)\,,$$

the integrand function has a maximum at $t = n$ with value $g(n) = n^n \mathrm{e}^{-n}$. $\frac{\mathrm{d}^2 g(t)}{\mathrm{d}t^2} = 0$ if and only if

$$-\frac{n}{t^2} + \left(\frac{n}{t} - 1\right)^2 = 0\,,$$

i. e.,

$$t^2 - 2nt + n^2 - n = 0\,.$$

The two solutions of this equation are $t_{1,2} = n \pm \sqrt{n}$.

The saddle point recipe is: approximate the integrand by Taylor expanding f around its maximum (which is also a maximum of the integrand function itself). Notice that we are *not* using Taylor expansion of the integrand itself, this is because the latter would result in a rather poor approximation if only lower-order terms had to be kept (e. g., second-order Taylor expansion of the integrand itself means that the integrand is approximated with a parabola).

> Here, we are dealing with integrals with respect to a *real* variable. The saddle point method applies more generally to contour integrals in a complex domain. In the latter case, the kind of calculations are more or less the same, however there is one big difference: additional work is needed which consists of the deformation of the contour of integration. These topics are refreshed in appendix B.

By Taylor expanding $f(t)$ around $t = n$ we have

$$f(t) \approx f(n) + \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)_{t=n} (t-n) + \frac{1}{2}\left(\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right)_{t=n} (t-n)^2 \,,$$

and since

$$\begin{aligned} f(n) &= n\log(n) - n \\ \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)_{t=n} &= 0\,, \\ \left(\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right)_{t=n} &= -\frac{n}{n^2} = \frac{1}{n} \end{aligned}$$

we get

$$f(t) \approx (n\log(n) - n) - \frac{1}{2n}(t-n)^2 \,.$$

Thus

$$\begin{aligned} \Gamma(n+1) &= \int_0^{+\infty} \mathrm{e}^{f(t)}\, \mathrm{d}t \\ &\approx \int_0^{+\infty} \mathrm{e}^{f(n)} \mathrm{e}^{\frac{1}{2}\left(\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right)_{t=n}(t-n)^2}\, \mathrm{d}t \\ &= n^n \mathrm{e}^{-n} \int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, \mathrm{d}t \,. \end{aligned}$$

For large n, we make a very small error by extending the integration to the whole real axis, graphically this means that in **??** the area under the Gaussian tail in the negative half plane becomes negligible. The integral becomes a Gaussian integral over the whole real line which is evaluted exactly in closed form:

$$\int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, \mathrm{d}t \approx \int_{\mathbb{R}} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, \mathrm{d}t = \sqrt{2\pi n}\,.$$

Exercise 4.1: Show that

$$\int_0^{+\infty} \int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, \mathrm{d}t = \sqrt{\frac{n\pi}{2}} \left(1 + \mathrm{erf}\left(\sqrt{\frac{n}{2}}\right)\right)$$

where the error function has been used. Give a bound to the error in the approximation

$$\int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, \mathrm{d}t \approx \int_{\mathbb{R}} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, \mathrm{d}t \sqrt{2\pi n}$$

Hint: The ratio of the two results is $\frac{1}{2}\left(1 + \mathrm{erf}\left(\sqrt{\frac{n}{2}}\right)\right)$. Plot it. For "large" n the ratio is ≈ 1.

5 Stirling series

Stirling's formula gives indeed the leading term of the following asymptotic series, now called the Stirling series:

This is not a convergent series. However, it is an asymptotic series, in fact there are a number of terms thet improve accuracy, after which however accuracy gets worse. This is demonstarted in

6 A modern elementary proof

Only the basic of calculus are required in order to prove two preliminary disequalities. No additional machinary is needed.*

A Gamma function warm up

The basic properties of Euler's gamma function are refreshed.†

There is a number of equivalent ways to define the gamma function.‡ Here we will discuss:

- Euler's integral representation together with analytic continuation.
- characterization via the Bohr-Mollerup theorem.
- characterization via Willard theorem.

The gamma function extends the notion of the factorial to real and complex numbers. The exact way in which it "extends" the factorial is precisely the content of the characterizations theorems by Bohr-Mollerup and Willard. However, gamma function is extremely useful well beyond the factorials.

A.1 Integral representation

Definition A.1 (Gamma function): The gamma function $\Gamma(z)$ of a complex number z can be defined by the Euler's integral representation

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} \, \mathrm{d}t \tag{A.1}$$

valid for every $z \in \mathbb{C}$ such that $\Re z > 0$, *together with its analytic continuation.*

Remark. The definition is *not* eq. (A.1) alone. It is eq. (A.1) for $\Re z > 0$ plus analytic continuation. Both points will be throughly discuss later.

* **Michael:2002**.

† **Magnus.Oberhettinger.ea:1953**.

‡ **Freitag.Busam:2005**.

Let us clarify better eq. (A.1). By definition of the complex powers, $t^{z-1} := \mathrm{e}^{(z-1)\log(t)}$, where $\log(t)$ is the *real-valued* logarithmic function of the real variable t, while the exponential is the complex-valued exponential function. Integration in eq. (A.1) is over the dummy variable t running over the real interval $[0, +\infty[$, i. e., we are integrating with respect to a *real* (and not a complex) variable.

In order definition A.1 makes sense, we need to prove that (a) the integral in eq. (A.1) is convergent if and only if $\Re z > 0$; (b) the integral in eq. (A.1) defines an *holomorphic* function in the half plane $\Re z > 0$; (c) study its analytic continuation for $\Re z \leq 0$. Let us discuss all these three points.

Lemma A.1: $\int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$ is convergent *if and only if* $\Re z > 0$.

Proof. Consider the following integrals:

$$\int_0^1 t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t \tag{A.2}$$

$$\int_1^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t \tag{A.3}$$

We have

$$|t^{z-1}\mathrm{e}^{-t}| = t^{\Re z-1}\mathrm{e}^{-t}\,.$$

For $t > 0$, $\mathrm{e}^{-t} < 1$ and we have

$$\int_0^1 \left|t^{z-1}\mathrm{e}^{-t}\right|\mathrm{d}t = \int_0^1 t^{\Re z-1}\mathrm{e}^{-t}\,\mathrm{d}t \leq \int_0^1 t^{\Re z-1}\,\mathrm{d}t\,,$$

and the latter (real) integral is convergent if and only if $\Re z > 0$. ■

Exercise A.1: Show that for every $z \in \mathbb{C}$ such that $\Re z > 0$, $\Gamma(z)$ can be equivalently written as

$$\Gamma(z) = 2\int_0^{+\infty} \xi^{2z-1}\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi$$

or

$$\Gamma(z) = \int_0^1 \log^{z-1}\left(\frac{1}{\xi}\right)\mathrm{d}\xi$$

Hint: Apply a suitable change of variables. Why is it possible to performe a change of variables?

Lemma A.2: $\int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$ defines an *holomorphic* function of z for every $z \in \mathbb{C}$ such that $\Re z > 0$.

Proof. There are two ways to show this:

- Direct calculation of $\frac{\mathrm{d}}{\mathrm{d}z}\int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$;
- Application of the Morera's theorem.

■

It can be easily verified that by definition

$$\Gamma(1) = \int_0^{+\infty} \mathrm{e}^{-t}\,\mathrm{d}t = \left[-\mathrm{e}^{-t}\right]_{t=0}^{t=+\infty} = 1\,.$$

Moreover, by using the standard result of the Gaussian integral, namely,*

$$\int_0^{+\infty} \mathrm{e}^{-\xi^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}\,, \tag{A.4}$$

we readily obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-\frac{1}{2}}\mathrm{e}^{-t}\,\mathrm{d}t = \int_0^{+\infty} \frac{1}{\xi}\mathrm{e}^{-\xi^2} 2\xi\,\mathrm{d}t = 2\underbrace{\int_0^{+\infty} \mathrm{e}^{-\xi^2}\,\mathrm{d}t}_{\sqrt{\pi}/2} = \sqrt{\pi}, \quad \xi = \sqrt{t}\,, \tag{A.5}$$

where the factor 2ξ in the second integral above comes from the Jacobian of the transformation $\xi = \sqrt{t}$. Notice: the original integral is an integral however the real line, we are allowed to change variables in this way, the square root denotes the real square root.

We will see later that, by using eq. (A.6), the gamma function can be simplified for all half-integer arguments.

A.2 Functional equation

Theorem A.1 (recursive relation): For every $z \in \mathbb{C}$, $\Re z > 0$,

$$\boxed{\Gamma(z+1) = z\Gamma(z)\,.} \tag{A.6}$$

Proof. Integrating by parts in eq. (A.1) yields

$$\Gamma(z) = \underbrace{\frac{t^z}{z}\mathrm{e}^{-t}\Big|_0^{+\infty}}_{0} + \frac{1}{z}\underbrace{\int_0^{+\infty} t^z\mathrm{e}^{-t}\,\mathrm{d}t}_{\Gamma(z+1)}\,.$$

(The first term in the right-hand side vanishes as shown in **??** A.2.) So eq. (A.6) holds. ■

Exercise A.2: In connection with the proof of theorem A.1, prove that

$$\frac{t^z\mathrm{e}^{-t}}{z}\Big|_0^{+\infty} = 0\,,$$

in particular

$$\lim_{t\to+\infty} t^z\mathrm{e}^{-t} = 0$$

for every $z \in \mathbb{C}$.

* There are several ways to prove eq. (A.4). Refer to, e. g., **Iwasawa:2009**; **Boros.Moll:2004**.

HINT: By definition, $t^z \mathrm{e}^{-t} := \mathrm{e}^{z\log(t)-t}$. Consider

$$\lim_{t\to+\infty} \left|t^z \mathrm{e}^{-t}\right| = \lim_{t\to+\infty} \left|\mathrm{e}^{z\log(t)-t}\right| = \lim_{t\to+\infty} |\Re z \log(t) - t|\,;$$

show that this limit is zero for every $z \in \mathbb{C}$. This implies also $\lim_{t\to+\infty} t^z \mathrm{e}^{-t} = 0$ (why?).

Equation (A.6) has far-reaching conseguences. First of all, eq. (A.6) implies that

$$\boxed{\Gamma(n+1) = n!} \tag{A.7}$$

for every $n \in \mathbb{N}$. Let us prove it by induction. First of all,

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1\,,$$

which is equal to $1! = 1$. Moreover, assuming $\Gamma(n+1) = n!$ let us prove that $\Gamma(n+2) = (n+1)!$:

$$\Gamma((n+1)+1) = (n+1)\Gamma(n+1) = (n+1)n! = n+1!\,.$$

This completes the proof. Notice also that eq. (A.7) is consistent with the position $0! = 1$, in fact

$$\Gamma(1) = 1 = 0!\,.$$

As a consequence, the gamma function can be used to extend the notion of factorials to the *complex* numbers: $z! := \Gamma(z+1)$.

An even more important conseguence of eq. (A.6) is that it can be used to implement the *analytic continuation* of $\Gamma(z)$ outside the original domain of definition of the integral representation **??**. Let us now discuss better this point.

We know that Γz in **??** is (a) analytic for $\Re z > 0$; (b) satifies eq. (A.6) for $\Re z > 0$. Consider

$$\Gamma(z) = \frac{\Gamma z + 1}{z}, \quad \Re z > 0\,.$$

$\Gamma z + 1$ on the right-hand side is indeed defined for $\Re(z+1) > 0$, i. e., $\Re z > -1$. $\Gamma(z+1)\,/z$ thus coincides with the original definition of $\Gamma(z)$ for $\Re z > 0$, but it is also valid for $-1 < \Re z < 0$, allowing the analytic continuation of Γz to the strip $-1 < \Re z < 0$.

For example, let us compute $\Gamma(-1/2)$. We have

$$\begin{aligned}\Gamma\left(-\frac{1}{2})\right) &= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \\ &-2\Gamma\left(\frac{1}{2}\right) = \\ &-2\sqrt{\pi}\,.\end{aligned}$$

Repeating the same reasoning, we can further extend the definition of $\Gamma(z)$ via analytic continuation to all other points of the half-plane $\Re z < 0$. In this way,

$$\boxed{\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, \quad \forall n \in \mathbb{N}\,,} \tag{A.8}$$

where $(z)_n$ is the Pockhammer symbol of index n.

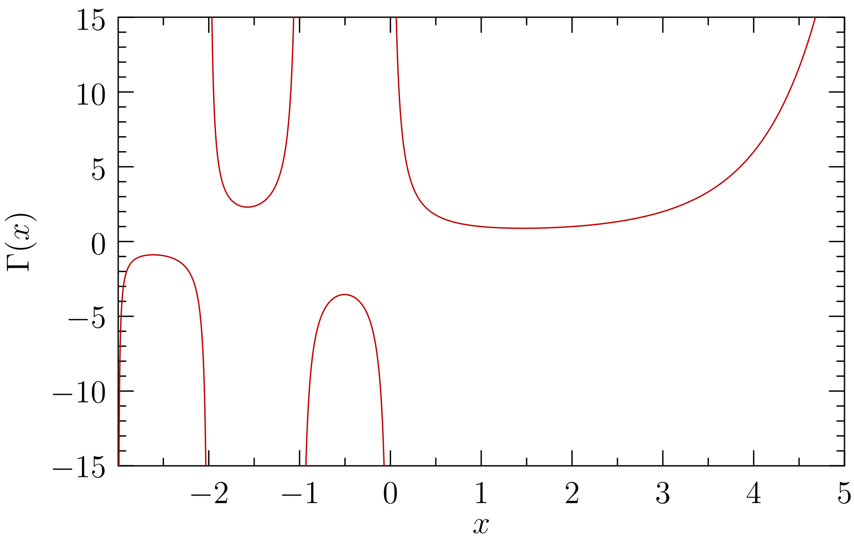

Figure 3: Plot of gamma function for real values of its argument

Definition A.2 (Pochhammer's symbol): For every $z \in \mathbb{C}$ and $n \in \mathbb{N}$, the Pochhammer's symbol $(z)_n$ is defined by induction on n:

- $(z)_1 = z$ and
- $(z)_{n+1} = (z+n)(z)_n$ for every $n \geq 1$.

Sometimes it is useful to define also $(z)_0 = 1$.

In other words,

$$(z)_n = z(z+1)(z+2)\cdots(z+n-1)\,.$$

Notice that $(1)_1 = 1$ and $(1)_{n+1} = (n+1)(1)_n$, that is $(1)_n = n!$.

Lemma A.3: $\Gamma(z)$ has simple poles at $z = -n$, with $n \in \mathbb{N}$, with residues given by

$$\begin{aligned}
\operatorname*{Res}_{z=-n}(\Gamma(z)) &= \lim_{z\to -n}(z+n)\Gamma(z) \\
&= \lim_{z\to -n}(z+n)\frac{\Gamma(z+n+1)}{(z)_{n+1}} \\
&= \lim_{z\to -n}\frac{\Gamma(z+n+1)}{(z)_n} \\
&= \frac{(-1)^n}{n!}\,.
\end{aligned} \tag{A.9}$$

Exercise A.3: Prove that $(-n)_n = (-1)^n n!$.

The plot of $\Gamma(z)$ for real values of z is given in **??**.

A.3 Double factorial notation

One often encounters products of the odd positive integers and products of the even positive intergers. For convenience, one introduces the following double factorial notation.

Definition A.3 (Double factorials): By induction, define $0!! = 1!! = 1$ and for every $n \in \mathbb{N}$, $n \geq 1$,

$$(2n+1)!! = (2n+1)(2n-1)!! \tag{A.10}$$

and

$$(2n)!! = (2n)(2n-2)!! \,. \tag{A.11}$$

In other words,

$$\begin{aligned}(2n+1)!! &= 1 \times 3 \times 5 \times \cdots \times (2n+1)\,,\\ (2n)!! &= 2 \times 4 \times 6 \times \cdots \times (2n)\,.\end{aligned}$$

Furthermore, it is useful (see, e. g., the next exercise) to define $(-1)!! = 1$.

Lemma A.4:

$$(2n)!! = 2^n n!\,, \tag{A.12}$$

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!}\,. \tag{A.13}$$

Proof. Heuristic argument to prove eq. (A.12):

$$\begin{aligned}2n!! &= 2 \times 4 \times 6 \times \cdots \times 2n\\ &= (2 \times 1) \times (2 \times 2) \times (2 \times 3) \times \cdots \times (2 \times n)\\ &= 2^n n!\,.\end{aligned}$$

Rigorous proof of eq. (A.12) is by induction, as follows. Notice however that the rigorous proof by induction gives no help in deriving the formula in eq. (A.12), it simply gives a rigorous proof of it once the formula has been already found. Proof by induction of eq. (A.12): $0!! = 2^0 0! = 1$ (so the formula holds for $n = 0$) and

$$(2n+2)!! = (2n+2)(2n)!! = (2n+2)2^n n! = 2(n+1)2^n n! = 2^{n+1}(n+1)!\,.$$

Equation (A.13) follows from

$$(2n+1)!! = \frac{(2n+1)!}{(2n)!!}\,.$$

(Again, rigorous proof is by induction.) ∎

Lemma A.5: Show that

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad \forall n \in \mathbb{N}\,.$$

Proof. Let us check the formula for some n

$$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right),$$
$$\Gamma\left(\frac{1}{2}+2\right) = \left(\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}+2\right) = \left(\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right),$$

etc. Rigorous proof is by induction: the formula gives the correct result for the case $\Gamma\left(\frac{1}{2}+1\right)$ (indeed, even for the case $n=0$ assuming $(-1)!! = 1$ is understood); for $n \geq 1$,

$$\begin{aligned}\Gamma\left(\frac{1}{2}+n+1\right) &= \left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}+n\right) \\ &= \frac{1+2n}{2}\frac{(2n-1)!!}{2^n}\Gamma\left(\frac{1}{2}\right) \\ &= \frac{(2n+1)!!}{2^{n+1}}\Gamma\left(\frac{1}{2}\right),\end{aligned}$$

this completes the proof. ■

A.4 Another definition of $\Gamma(z)$ and a proof of Wallis formula

Another equivalent way to define $\Gamma(z)$ is through the formula

$$\Gamma(z) = \lim_{n\to+\infty}\frac{n!}{(z)_{n+1}}n^z\,. \tag{A.14}$$

Essentially, the argument goes as follows.* For $n \in \mathbb{N}$ and $\Re z > 0$, repeated integration by parts yields

$$\int_0^{+\infty}\left(1-\frac{t}{n}\right)^n t^{z-1}\,\mathrm{d}t = \frac{n!}{(z)_{n+1}}n^z\,.$$

(Check as an exercise.) In the limit $n \to +\infty$ the left-hand side above becomes $\int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$. However, care is needed in taking this limit.

We can use the equivalent definition above of $\Gamma(z)$ to give a short proof of Wallis formula. Since we know that

$$\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}\,,$$

we can write

$$\Gamma\left(-\frac{1}{2}\right) = \lim_{n\to+\infty}\frac{n!}{\left(\frac{1}{2}\right)_{n+1}}n^{-\frac{1}{2}}$$

We need to compute explicitly the Pochhammer symbol in this formula:

$$\left(-\frac{1}{2}\right)_{n+1} = -\frac{1}{2}\left(\frac{1}{2}\right)_n$$

* **Magnus.Oberhettinger.ea:1953**.

and using the double factorial notation we have

$$\left(\frac{1}{2}\right)_n = \frac{(2n-1)!\,!}{2^n} = \frac{(2n+1)!\,!}{2^n(2n+1)} = \frac{(2n+1)!}{2^{2n}n!\,(2n+1)} = \frac{(2n)!}{2^{2n}n!}$$

Thus,

$$\begin{aligned}\Gamma\left(-\frac{1}{2}\right) &= \lim_{n\to+\infty} \frac{n!}{\left(-\frac{1}{2}\right)_{n+1}} n^{-\frac{1}{2}} \\ &= \lim_{n\to+\infty} \frac{2^{2n}(n!)^2}{-\frac{1}{2}(2n)!} n^{-\frac{1}{2}}\,,\end{aligned}$$

from which it follows

$$\sqrt{\pi} = \lim_{n\to+\infty} \frac{2^{2n}(n!)^2}{(2n)!} n^{-\frac{1}{2}}\,,$$

which is the Wallis formula.

A.5 Beta function

The beta function is defined by the integral representation

$$\boxed{B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t} \tag{A.15}$$

for every pair of complex numbers $(p,q) \in \mathbb{C}^2$, such that $\Re p > 0$ and $\Re q > 0$, and its analytic continuation.

Exercise A.4: Show that the integral representation in eq. (A.15) converges for $\Re p > 0$ and $\Re q > 0$.

First of all, notice that

$$B(q,p) = B(p,q)\,, \quad \Re p > 0 \text{ and } \Re q > 0\,. \tag{A.16}$$

In fact,

$$\begin{aligned}B(q,p) &= \int_0^1 t^{q-1}(1-t)^{p-1}\,\mathrm{d}t \\ &= -\int_1^0 (1-\xi)^{q-1}\xi^{p-1}\,\mathrm{d}\xi \\ &= \int_0^1 (1-\xi)^{q-1}\xi^{p-1}\,\mathrm{d}\xi \\ &= B(p,q)\,, \quad \xi = -t\,.\end{aligned}$$

Now, we prove an important formula.

Theorem A.2:

$$\boxed{B(u,v) = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}} \tag{A.17}$$

for all $(u,v) \in \mathbb{C}^2$, $\Re u > 0$, $\Re v > 0$.

Proof. Step 1: consider

$$\Gamma(u)\,\Gamma(v) = \left(\int_0^{+\infty} t^{u-1}\mathrm{e}^{-t}\,\mathrm{d}t\right)\left(\int_0^{+\infty} t^{v-1}\mathrm{e}^{-t}\,\mathrm{d}t\right).$$

Step 2: replace $t = x^2$ and $s = y^2$, giving

$$\Gamma(u)\,\Gamma(v) = 4\left(\int_0^{+\infty} x^{2u-1}\mathrm{e}^{-x^2}\,\mathrm{d}t\right)\left(\int_0^{+\infty} y^{2v-1}\mathrm{e}^{-y^2}\,\mathrm{d}t\right).$$

Step 3: Fubini's theorem applies allowing to convert the product of two integrals into one double integral in the plane

$$\Gamma(u)\,\Gamma(v) = 4\int_0^{+\infty}\int_0^{+\infty} x^{2u-1}y^{2v-1}\mathrm{e}^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y$$

Step 4: we change variables to polar coordinates in the plane:

$$\Gamma(u)\,\Gamma(v) = 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} \rho^{2u+2v-1}\mathrm{e}^{-\rho^2}\cos^{2u-1}(\vartheta)\sin^{2v-1}(\vartheta)\;\mathrm{d}\vartheta\,\mathrm{d}\rho\,.$$

Step 5: Fubini's theorem allows to factorize the double integral in polar coordinates into the product of two integrals:

$$\Gamma(u)\,\Gamma(v) = 4\left(\int_0^{+\infty} \rho^{2u+2v-1}\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho\right)\left(\int_0^{\frac{\pi}{2}} \cos^{2u-1}(\vartheta)\sin^{2v-1}(\vartheta)\;\mathrm{d}\vartheta\right).$$

Step 6: integration over ρ returns

$$\begin{aligned}\int_0^{+\infty} \rho^{2u+2v-1}\mathrm{e}^{-\rho^2}\,\mathrm{d}\rho &= \frac{1}{2}\int_0^{+\infty}\left(\xi^{\frac{1}{2}}\right)^{2u+2v-1}\mathrm{e}^{-\xi}\xi^{-\frac{1}{2}}\,\mathrm{d}\xi\\ &= \frac{1}{2}\int_0^{+\infty}\xi^{u+v-1}\mathrm{e}^{-\xi}\,\mathrm{d}\xi = \frac{1}{2}\Gamma(u+v)\,,\quad \xi=\rho^2\,.\end{aligned}$$

Step 7: integration over ϑ can be performed by changing variables: $t = \cos^2(\vartheta)$, thus $\sin^2(\vartheta) = 1-t$, the Jacobian is

$$\frac{\mathrm{d}\arccos\left(\sqrt{t}\right)}{\mathrm{d}t} = -\frac{1}{2}(1-t)^{-\frac{1}{2}}t^{-\frac{1}{2}}$$

so

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2u-1}(\vartheta)\sin^{2v-1}(\vartheta)\;\mathrm{d}\vartheta &= -\frac{1}{2}\int_1^0 t^{u-\frac{1}{2}}(1-t)^{v-\frac{1}{2}}(1-t)^{-\frac{1}{2}}t^{-\frac{1}{2}}\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^1 t^{u-1}(1-t)^{v-1}\,\mathrm{d}t = B(u,v)\,.\end{aligned}$$

Thus

$$\Gamma(u)\,\Gamma(v) = 4\frac{1}{2}\Gamma(u+v)\,\frac{1}{2}B(u,v) = \Gamma(u+v)\,B(u,v)\,.$$

This completes the proof. ∎

We will use the beta function to compute quickly I_n in eq. (3.2) in section 3. Show that

$$\boxed{I_n = \int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, \mathrm{d}\vartheta = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\,.} \tag{A.18}$$

We have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, \mathrm{d}\vartheta &= \int_0^1 \xi^n \left(1-\xi^2\right)^{-\frac{1}{2}} \mathrm{d}\xi \\ &= \frac{1}{2}\int_0^1 t^{\frac{n}{2}} (1-t)^{-\frac{1}{2}} t^{-\frac{1}{2}}\, \mathrm{d}t \\ &= \frac{1}{2} B\left(\frac{n-1}{2}, \frac{1}{2}\right), \quad \text{where } \xi = \sin(\vartheta) \text{ and } t = \xi^2\,. \end{aligned}$$

Using eq. (A.17) we have

$$\int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, \mathrm{d}\vartheta = \frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{1}{2}\right)} = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\,.$$

For $n = 2k$, the formula simplifies to

$$\begin{aligned} I_{2k} &= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)} \\ &= \frac{\sqrt{\pi}}{2}\frac{(2k-1)!\,!}{2^k k!}\sqrt{\pi} \\ &= \frac{\pi}{2}\frac{(2k-1)!\,!}{(2k)!\,!}\,, \end{aligned}$$

while for $n = 2k+1$ the formula returns

$$\begin{aligned} I_{2k+1} &= \frac{\sqrt{\pi}}{2}\frac{\Gamma(k+1)}{\Gamma\left(k+1+\frac{1}{2}\right)} \\ &= \frac{\sqrt{\pi}}{2}\frac{k!\,2^{k+1}}{(2k+1)!\,!}\frac{1}{\sqrt{\pi}} \\ &= \frac{(2k)!\,!}{(2k+1)!\,!}\,, \end{aligned}$$

as per the results of section 3.

A.6 Reflection formula

In this section we prove the important *reflection* formula using residue theory. There are also other ways to prove it, e. g., Weierstrass infinite-product definizion of $\Gamma(z)$ leads directy to this result.

THEOREM A.3 (reflection formula):

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}\,.} \tag{A.19}$$

B Steepest descent method

The stepeest descent method (also referred to as saddle point method) is a powerful tool to handle the problem of evaluating the asymptotic behavior (or at least its leading term) of parametric integrals of a certain form. For example, it might be useful in evaluating the asymptotic behavior of functions for which a suitable integral representation is available.[*]

Let us begin with a special case of real integrals over one real variable (this is known also as Laplace's method) and then generalize to contour integrals on a complex domain. Generalization to an arbitrary number of variables is not covered here, even if such generalizations are often needed in the toolkit of every physicists.[†]

B.1 Real case (alias, Laplace method)

B.2 Complex case

[*] **King.Billingham.ea:2003**; **Ablowitz.Fokas:2003**; **Zinn-Justin:2002**.
[†] **Zinn-Justin:2002**.