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Handout on Stirling's formula

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Abstract: Some proofs of the well-known Stirling's asymptotic approximation of the factorial (or the Gamma function) are sketched. No attempt was made at full mathematical rigour, emphasis is mostly on the main ideas behind these proofs.

Contents

1 Overview

The "factorial" $n!$ of a positive integer number $n \in \mathbb{N}$ is defined by induction: let $0! = 1$ and $n! = n(n-1)!$ for every $n \geq 1$.

Stirling's formula is an asymptotic approximation of $n!$ for "large" n, namely

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} \mathrm{e}^{-n}, \quad \text{as } n \to +\infty\,, \tag{1.1}$$

where the symbol $\sim$ will be used to denote asymptotic equality. The rigorous meaning of **??** is

$$\lim_{n \to +\infty} \frac{n!}{\sqrt{2\pi} n^{n+\frac{1}{2}} \mathrm{e}^{-n}} = 1\,.$$

Since factorials are among the basic ingredients of combinatorics, Stirling's formula is extremely useful in many fields of mathematics, statistics and physics to help simplify formulæ involving factorials. For this reason, it seems useful to review few of the standard methods to prove it.

The simplest argument leads to an (heuristic) integral approximation of $\log n!$ (see section 2). To be precise, the method actually returns rigorous lower and upper integral bounds of $\log n!$. A (still heuristic) refinement of the procedure allows to extract the right dependence on n, but the procedure is still heuristic and the factor $\sqrt{2\pi}$ is still missing. In order to fully recover **??** in a rigorous way, more advanced machinary is needed.

Figure 1: Illustration of the key idea beyond integral method: comparison between sums and integrals. Since $\log(x)$ is an increasing function of x, the integral of $\log(x)$ (graphical meaning: gray area in figure) understimates $\log n!$ (the total area under the rectangles), the (nearly triangular) brown areas above the curve of $\log(x)$ being left out. On the contrary, the integral of $\log(x+1)$ overstimates that sum.

The heuristic result of section 2 is used as a guess for the rigorous treatment of section 3, based on the famous Wallis formula. Wallis formula is a rather technical one however, and it would be useful if it were possible to prove Stirling's formula relying on more general tools.

Remember that factorials can be written in terms of the gamma trascendental function:

$$n! = \Gamma(n+1) = \int_0^{+\infty} t^n \mathrm{e}^{-t}\, dt, \quad \forall n \in \mathbb{N}\,, \tag{1.2}$$

where the Euler's integral representation of the Gamma function has been used in **??**. Gamma function provides a generalization of the factorial to real and complex numbers. This generalization is unique up to some extra conditions. We will thoroughly study the basic properties of Gamma function and how it generalizes factorials in appendix A. The Euler's integral representation of the Gamma function in **??** is a good starting point to prove **??** using more advanced techniques.

A standard approach is that of using the saddle-point asymptotic method on the right-hand side of **??** , which leads directly to **??**. This approach is outlined in section 4. Saddle point method is a powerful method to compute the asymptotic behavior of certain parametric integrals, and it is discussed more deeply in appendix B.

A third way to prove **??** is given in section 6. Only basic tools of calculus courses are needed. This is a recent proof due to R. Michael, which again involves the integral representation **??**.

2 Naive integral approximation

By definition, the factorial of $n \in \mathbb{N}$ is a product. For every $n \in \mathbb{N}\backslash\{0\}$, $n!$ can be equivalently written as $n! = \prod_{k=1}^{n} k$.[1] Now, the trick to proceed is this: (a) first of all, turn this product into a sum by taking the logarithm of it, then (b) approximate this sum with an integral. Let us explain the various steps in detail.

First, for $n \in \mathbb{N}\backslash\{0\}$ take the logarithm of $n!$:

$$\log n! = \log\left(\prod_{k=1}^{n} k\right) = \sum_{k=1}^{n} \log(k)\,, \quad n \in \mathbb{N}\backslash\{0\}. \tag{2.1}$$

[1] For every $n \in \mathbb{N}\backslash\{0\}$, this is equivalent to the definition of the factorial by induction given at the beginning of section 1 (proof of the equivalence is by induction). For $n = 0$ instead, the formula $n! = \prod_{k=1}^{n} k$ does not work. The position $0! = 1$ is conventional and arbitrary, however it is useful and it is consistent with the relation between factorials and the gamma function (see appendix A).

?? is an *exact* expression of the logarithm of the factorial. Now, *approximate* the sum in **??** with an integral:

$$\log n! = \sum_{k=1}^{n} \log(k) \approx \int_1^n \log(x)\; dx\,, \quad n \in \mathbb{N}\backslash\{0\}\,. \tag{2.2}$$

Graphical interpretation of **??** (see fig. 1): $\sum_{k=1}^{n} \log(k)$ (area under the rectangles) is approximated by $\int_1^n \log(x)\; dx$ (whose pictorial meaning is the gray area below the curve of $\log(x)$).

The integral in **??** is evaluated by parts as usual,

$$\int_1^n \log(x)\; dx = \Big[x \log(x) - x\Big]_{x=1}^{x=n} = n\log(n) - n + 1\,,$$

hence **??** reads

$$\log(n!\,) \approx n\log(n) - n + 1\,. \tag{2.3}$$

For "large" n, $n - 1 \approx n$ and exponentiating **??** yields

$$\boxed{n! \approx \mathrm{e}^{n\log(n) - n} = n^n \mathrm{e}^{-n}\,.} \tag{2.4}$$

At this point however we have no precise control of the kind of approximation (i. e., a bound on the error) in **??**.

Actually, the approximation **??** *under*stimates $\log(n!\,)$, as it should be clear from fig. 1: the area under the curve of $\log(x)$ is less than the area under the rectangles, the (brown) areas above the curve of $\log(x)$ being left out.

> If you don't want to rely on fig. 1, it is not difficult to convert the graphical argument into a purely algebraic proof. Since $\log(x)$ is a *monotonically increasing* function of x, then
>
> $$\int_{k-1}^{k} \log(x)\; dx < \log(k) < \int_{k}^{k+1} \log(x)\; dx\,,$$
>
> for every $k \in \mathbb{N}$, $k \geq 2$. Equivalently, by changing variables in the second integral:
>
> $$\int_{k-1}^{k} \log(x)\; dx < \log(k) < \int_{k-1}^{k} \log(x+1)\; dx\,,$$
>
> for every $k \in \mathbb{N}$, $k \geq 2$. Summing from $k = 2$ to $k = n$ yields
>
> $$\sum_{k=2}^{n} \int_{k-1}^{k} \log(x)\; dx < \underbrace{\sum_{k=2}^{n} \log(k)}_{\left(\sum_{k=1}^{n}\log(k)\right) - \log(1) = \sum_{k=1}^{n}\log(k) = \log(n!)} < \sum_{k=2}^{n} \int_{k-1}^{k} \log(x+1)\; dx\,,$$
>
> and therefore
>
> $$\int_1^n \log(x)\; dx < \log(n!\,) < \int_1^n \log(x+1)\; dx\,,$$
>
> for every $n \in \mathbb{N}$, $n \geq 2$, or if you prefer to include also the case $n = 1$
>
> $$\int_1^n \log(x)\; dx \leq \log(n!\,) \leq \int_1^n \log(x+1)\; dx, \quad \text{for every } n \in \mathbb{N}\backslash\{0\}\,.$$

> Graphical meaning (refer to fig. 1): the total area of the rectangles (i. e., $\log(n!)$) is greater than $\int_1^n \log(x)\, dx$ (i. e., the area below $\log(x)$) and less than $\int_1^n \log(x+1)\, dx$ [i. e., the area below $\log(x+1)$]. This shows that the approximation **??** actually gives a rigorous *lower* bound of $\log(n!)$.

The approximation can be improved by including the contributions coming from the nearly-triangular brown areas left-out above the plot of $\log(x)$ (see fig. 1). Notice that this is nothing but the usual trapezoid rule to approximate an integral.[2]

The k-th left out area can be roughly approximated by the triangular area

$$\frac{1}{2}\left[\log(k+1) - \log(k)\right].$$

Summing all the contributions yields

$$\log(n!) \approx \int_1^n \log(x)\, dt + \frac{1}{2}\sum_{k=1}^{n}\left[\log(k+1) - \log(k)\right]. \tag{2.5}$$

The telescopic sum on the right-hand side can be evaluated exactly

$$\begin{aligned}\sum_{k\,=1}^{n}\left[\log(k+1) - \log(k)\right] &= \log(n) - \log(1)\\ &= \log(n),\end{aligned} \tag{2.6}$$

(the rigorous proof is by induction) and

$$\log(n!) \approx n\log(n) - n + 1 + \frac{1}{2}\log(n) \approx \left(n + \frac{1}{2}\right)\log(n) - n, \tag{2.7}$$

and by taking the exponential

$$n! \approx n^{n+\frac{1}{2}}\mathrm{e}^{-n} \tag{2.8}$$

This of course is just an heuristic argument, since at this stage we have no control on the kind of approximation occurring in **??**. **??** shares the correct dependence on n, compared to **??**, but the factor $\sqrt{2\pi}$ is still missing. In the next section we will use this result as starting point for a more rigorous treatment.

3 Rigorous results from integral method. Wallis formula

Given the heuristic result **??**, define a sequence $(a_n)_{n\in\mathbb{N}\setminus\{0\}}$ of positive numbers

$$a_n = \frac{n!}{n^{n+\frac{1}{2}}\mathrm{e}^{-n}}, \quad \forall n \in \mathbb{N}.$$

Our goals are: (a) prove that $\lim_n a_n$ exists; (b) calculate it. This will be done in theorem 3.2. To achieve this, we need the following preliminary result.

[2] Euler-MacLaurin formula gives the error of the trapezoid approximation. We will come back to this point discussing Stirling asymptotic series.

Theorem 3.1 (Wallis formula):

$$\lim_{n \to +\infty} \frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!} = \sqrt{\pi}\,. \tag{3.1}$$

Proof. A elegant proof based on properties of gamma function is given in appendix A. Here, we give another proof, based only on elementary calculus. The drawback of this proof is that it becomes quite long and a bit tricky.

Consider the sequence of integrals

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, d\vartheta, \quad \forall n \in \mathbb{N}\,. \tag{3.2}$$

Integrals of this form often arise in physics, e. g., algebraic relations of non-relativistic quantum theory of angular momentum, i. e., spherical harmonics.[3] We will use those integrals to prove the Wallis formula. First of all, we need an explicit expression for I_n. These integrals can be easily computed by using the beta trascendental function, see appendix A. Here we will use a naive approach: we will use integration by parts to obtain a linear recursive equation for I_n, which can be solved exactly in closed form. Integration by parts yields

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n(\vartheta)\, d\vartheta \\ &= -\cos(\vartheta)\sin^{n-1}(\vartheta)\big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^2(\vartheta)\sin^{n-2}(\vartheta)\, d\vartheta \\ &= (n-1)\left(I_{n-2} - I_n\right). \end{aligned}$$

Re-arranging the terms in this expression yields

$$I_n = \frac{n-1}{n} I_{n-2}, \quad \forall n \in \mathbb{N} \text{ and } n \geq 2\,. \tag{3.3}$$

?? is a second-order linear homogeneus recursive equation with non-constant (rational) coefficients. The initial conditions are

$$\begin{aligned} I_0 &= \int_0^{\frac{\pi}{2}} d\vartheta = \frac{\pi}{2}\,, \\ I_1 &= \int_0^{\frac{\pi}{2}} \sin(\vartheta)\, d\vartheta = -\cos(\vartheta)\big|_0^{\frac{\pi}{2}} = 1\,. \end{aligned}$$

One can use the general theory of linear recursive equations with variable (polynomial) coefficients to approach **??** systematically. The fact that the coefficients are not constant of course makes the things harder. The general solution of **??** can be written in terms of gamma function. At a more straightforward level, one can start by calculating explicitly the lowest terms in the sequence I_n by using the initial conditions, sequentially producing

[3] See for example J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed., Addison-Wesley, San Francisco 2011, ISBN: 978-0-8053-8291-4 (alk. paper), § 3.6.

the next tems until a clear pattern emerges. Once the general patter is identified, one can prove rigorously by induction that the pattern is indeed correct. Let us proceed in this way.

For n odd, $n = 2k+1$ for some $k \in \mathbb{N}$, the first terms in the recursion are

$$\begin{aligned} I_3 &= \frac{2}{3} I_1 \,, \\ I_5 &= \frac{4}{5} I_3 = \frac{4}{5}\frac{2}{3} I_1 \,, \\ I_7 &= \frac{6}{7} I_5 = \frac{6}{7}\frac{4}{5}\frac{2}{3} I_1 \,, \end{aligned}$$

and it is possible to identify a pattern:

$$\boxed{I_{2k+1} = \frac{(2k)!!}{(2k+1)!!} I_1 = \frac{(2k)!!}{(2k+1)!!}, \quad \forall k \in \mathbb{N} \,.} \tag{3.4}$$

(Double factorial notation is explained in appendix A.) The rigorous proof of the relation above is by induction.

For n even, $n = 2k$ for some $k \in \mathbb{N}$, the first terms in the recursion are

$$\begin{aligned} I_2 &= \frac{1}{2} I_0 \,, \\ I_4 &= \frac{3}{4} I_2 = \frac{3}{4}\frac{1}{2} I_0 \,, \\ I_6 &= \frac{5}{6} I_4 = \frac{5}{6}\frac{3}{4}\frac{1}{2} I_0 \,, \end{aligned}$$

and we may expect the general pattern to be

$$\boxed{I_{2k} = \frac{(2k-1)!!}{(2k)!!} I_0 = \frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2} \,.} \tag{3.5}$$

Also in this case, the rigorous proof can be done by induction.

Now that we have explicit formulæ for I_n in **??**, we can ask: how are those integrals related to the Wallis formula **??**? How can I_n be useful to prove **??**? This is a rather technical point: the trick is using I_n to obtain a chain of inequalities which will allow us to prove **??** by invoking the *squeeze theorem*.

First of all, since $0 \le \sin(\vartheta) \le 1$ for $\vartheta \in [0, \pi/2]$, we have

$$0 \le \sin^{2k+1}(\vartheta) \le \sin^{2k}(\vartheta) \le \sin^{2k-1}(\vartheta) \,, \quad \forall \vartheta \in [0, \pi/2] \text{ and } \forall k \in \mathbb{N}\backslash\{0\}$$

and for the monotonicity of the integral we have

$$0 \le I_{2k+1} \le I_{2k} \le I_{2k-1}, \quad \forall k \in \mathbb{N}\backslash\{0\} \,.$$

Thus,

$$0 \le \frac{(2k)!!}{(2k+1)!!} \le \frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2} \le \frac{(2k-2)!!}{(2k-1)!!}, \quad \forall k \in \mathbb{N}\backslash\{0\} \,,$$

or

$$0 \le \frac{(2k)!!}{(2k+1)!!} \le \frac{(2k+1)!!}{(2k)!!}\frac{1}{2k+1}\frac{\pi}{2} \le \frac{(2k)!!}{(2k+1)!!}\frac{2k+1}{2k}, \quad \forall k \in \mathbb{N}\backslash\{0\} \,.$$

Since

$$\frac{(2k)!\,!}{(2k+1)!\,!} = \frac{((2k)!\,!)^2}{(2k+1)!} = \frac{\left(2^k k!\right)^2}{(2k+1)!} = \frac{2^{2k}(k!)^2}{(2k)!}\frac{1}{2k+1}\,,$$

we get

$$0 \le \frac{1}{2k+1}\frac{2^{2k}(k!)^2}{(2k)!} \le \frac{(2k)!}{2^{2k}(k!)^2}\frac{\pi}{2} \le \frac{2^{2k}(k!)^2}{(2k)!}\frac{1}{2k}\,.$$

Moving things around,

$$0 \le \frac{k}{2k+1}\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 \le \frac{\pi}{2} \le \frac{1}{2}\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2\,.$$

It follows that

$$\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 \le \frac{2k+1}{k}\frac{\pi}{2}$$

and

$$\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 \ge \pi$$

Since $\lim_{k\to+\infty}\frac{2k+1}{k}\frac{\pi}{2} = \pi$, applying squeeze theorem yields

$$\lim_{k \to +\infty}\left(\frac{2^{2k}(k!)^2}{\sqrt{k}(2k)!}\right)^2 = \pi\,.$$

Taking the square of both sides yields **??**. ∎

We are ready to prove Stirling formula.

THEOREM 3.2 (Stirling formula): Consider the sequence $(a_n)_{n\in\mathbb{N}\setminus\{0\}}$ definied by

$$a_n = \frac{n!}{n^{n+\frac{1}{2}}\mathrm{e}^{-n}}, \quad \forall n \in \mathbb{N}\,. \tag{3.6}$$

Then,

$$\lim_{n \to +\infty} a_n = \sqrt{2\pi}\,. \tag{3.7}$$

.

Proof. The proof has two steps:

- show existence of the limit in **??**, i. e., prove that there exists a real number $c \in \mathbb{R}$ such that
$$\lim_n \frac{n!}{n^{n+\frac{1}{2}}\mathrm{e}^{-n}} = c\,,$$
- then apply Wallis formula (theorem 3.1) to show that $c = \sqrt{2\pi}$.

To prove the first part, we will show now that the sequence $(\alpha_n)_n$ is decreasing and bounded below by a positive constant, thus it converges to some positive real number c.

Consider

$$\log\left(\frac{a_n}{a_{n+1}}\right) = \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) - 1\,.$$

Remember the Taylor series of log:

$$\log\left(1+x\right) = \sum_{k=1}^{+\infty}(-1)^{k-1}\frac{x^k}{k}\,,$$

for $|x| < 1$. The first terms are

$$\log\left(1+x\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathrm{O}(x^5)$$

Thus

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4}\,\mathrm{O}\left(n^{-5}\right),$$

and

$$\begin{aligned}
\log\left(\frac{a_n}{a_{n+1}}\right) &= \left(n+\frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \mathrm{O}\left(n^{-5}\right)\right) - 1 \\
&= 1 - \frac{1}{2n} + \frac{1}{3n^2} + -\frac{1}{4n^3} + \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{6n^3} - 1 + \mathrm{O}\left(n^{-4}\right) \\
&= \frac{1}{3n^2} - \frac{1}{4n^2} - \frac{1}{4n^3} + \frac{1}{6n^3} + \mathrm{O}\left(n^{-4}\right) \\
&= \frac{1}{12n^2} - \frac{1}{12n^3} + \mathrm{O}\left(n^{-4}\right)
\end{aligned}$$

This implies $\log\left(a_n/a_{n+}\right) > 0$ and thus $(a_n)_n$ is a *descresing* sequence (at least for sufficiently large n).

To prove that $c = \sqrt{2\pi}$, consider the following limit:

$$\lim_n \underbrace{\frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!}}_{\to\sqrt{\pi}} \underbrace{\left(\frac{n^{n+\frac{1}{2}}\mathrm{e}^{-n}}{n!}\right)^2}_{\to\frac{1}{c^2}} \underbrace{\frac{(2n)!}{(2n)^{2n+\frac{1}{2}}\mathrm{e}^{-2n}}}_{\to c} = \frac{\sqrt{\pi}}{c}\,.$$

(Here we have used the theorem of the limit of a product.) A direct computation shows that this limit is also equal to

$$\begin{aligned}
\frac{\sqrt{\pi}}{c} &= \lim_{n\to+\infty}\frac{2^{2n}n^{2\left(n+\frac{1}{2}\right)}\mathrm{e}^{-2n}}{\sqrt{n}(2n)^{2n+\frac{1}{2}}\mathrm{e}^{-2n}} \\
&= \lim_{n\to+\infty}\frac{2^{2n}n^{2n+1}}{n^{\frac{1}{2}}(2n)^{2n+\frac{1}{2}}} \\
&= \lim_{n\to+\infty} 2^{2n}\left(\frac{n}{2n}\right)^{2n+\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\,,
\end{aligned}$$

from which it follows that $c = \sqrt{2\pi}$. ■

Figure 2: Plot of the integrand function $g(t) = t^n e^{-t}$ and its

4 Saddle point derivation of Stirling's formula

This is a standard way (and probably the simplest one) for proving Stirling's formula. The method however requires some advanced material:

- Relation between factorial and gamma function;
- properties of the gamma function (in particular, its integral representation);
- saddle point method.

The relation between factorial and gamma function and properties of gamma function are highlighted in appendix A. The saddle point method is summarized in **??**. In this section we will discuss how this works for the factorial of a natural number (which is rather simple). The full application of the method to discuss the asymptotic behavior of the gamma function in the complex plane is done in the appendix.

The foundamental relation between factorials and the Euler's gamma function is

$$n! = \Gamma(n+1) = \int_0^{+\infty} t^n e^{-t}\, dt, \quad \forall n \in \mathbb{N}\,,$$

where we have used the integral representation of the gamma function. (See appendix A for details.)

Preliminary step: let us write the integrand in a completely equivalent fashion:

$$\Gamma(n+1) = \int_0^{+\infty} e^{\log(t^n)} e^{-t}\, dt = \int_0^{+\infty} e^{n\log(t)-t}\, dt\,.$$

Remark. The latter was not a change of variables within the integral. We have just used the fact $t^n = e^{\log(t^n)} = e^{n\log(t)}$.

In order to understand better what is going on, let us briefly outline the behavior of the integrand, which looks as shown in **??**.

Let $f, g : [0, +\infty[\to \mathbb{R}$ be

$$f(t) = n\log(t) - t\,,$$
$$g(t) = e^{f(t)}\,.$$

g is integrand function. Their derivatives are respectively

$$\frac{df(t)}{dt} = \frac{n}{t} - 1\,,$$
$$\frac{d^2f(t)}{dt^2} = -\frac{n}{t^2}\,,$$

and

$$\frac{dg(t)}{dt} = g(t)\frac{df(t)}{dt}\,,$$
$$\frac{d^2f(t)}{dt^2} = \left[\frac{d^2f(t)}{dt^2} + \left(\frac{df(t)}{dt}\right)^2\right] g(t)\,.$$

Since g never vanishes, $\frac{dg(t)}{dt} = 0$ if and only if $\frac{df(t)}{dt}$, i. e., if and only if $t = n$.

Since $\frac{d^2 f(t)}{dt^2} = -\frac{1}{n} < 0$ for every $t \in [0, +\infty[$ and

$$\left(\frac{d^2 g(t)}{dt^2}\right)_{t=n} = \left(\frac{d^2 f(t)}{dt^2}\right)_{t=n} g(n)\,,$$

the integrand function has a maximum at $t = n$ with value $g(n) = n^n \mathrm{e}^{-n}$. $\frac{d^2 g(t)}{dt^2} = 0$ if and only if

$$-\frac{n}{t^2} + \left(\frac{n}{t} - 1\right)^2 = 0\,,$$

i. e.,

$$t^2 - 2nt + n^2 - n = 0\,.$$

The two solutions of this equation are $t_{1,2} = n \pm \sqrt{n}$.

The saddle point recipe is: approximate the integrand by Taylor expanding f around its maximum (which is also a maximum of the integrand function itself). Notice that we are *not* using Taylor expansion of the integrand itself, this is because the latter would result in a rather poor approximation if only lower-order terms had to be kept (e. g., second-order Taylor expansion of the integrand itself means that the integrand is approximated with a parabola).

> Here, we are dealing with integrals with respect to a *real* variable. The saddle point method applies more generally to contour integrals in a complex domain. In the latter case, the kind of calculations are more or less the same, however there is one big difference: additional work is needed which consists of the deformation of the contour of integration. These topics are refreshed in appendix B.

By Taylor expanding $f(t)$ around $t = n$ we have

$$f(t) \approx f(n) + \left(\frac{df(t)}{dt}\right)_{t=n} (t-n) + \frac{1}{2}\left(\frac{d^2 f(t)}{dt^2}\right)_{t=n} (t-n)^2\,,$$

and since

$$\begin{aligned} f(n) &= n \log(n) - n \\ \left(\frac{df(t)}{dt}\right)_{t=n} &= 0\,, \\ \left(\frac{d^2 f(t)}{dt^2}\right)_{t=n} &= -\frac{n}{n^2} = \frac{1}{n} \end{aligned}$$

we get

$$f(t) \approx (n \log(n) - n) - \frac{1}{2n}(t-n)^2\,.$$

Thus

$$\begin{aligned} \Gamma(n+1) &= \int_0^{+\infty} \mathrm{e}^{f(t)}\, dt \\ &\approx \int_0^{+\infty} \mathrm{e}^{f(n)} \mathrm{e}^{\frac{1}{2}\left(\frac{d^2 f(t)}{dt^2}\right)_{t=n}(t-n)^2}\, dt \\ &= n^n \mathrm{e}^{-n} \int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, dt\,. \end{aligned}$$

For large n, we make a very small error by extending the integration to the whole real axis, graphically this means that in **??** the area under the Gaussian tail in the negative half plane becomes negligible. The integral becomes a Gaussian integral over the whole real line which is evaluted exactly in closed form:

$$\int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, dt \approx \int_{\mathbb{R}} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, dt = \sqrt{2\pi n}\,.$$

Exercise 4.1: Show that

$$\int_0^{+\infty} \int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, dt = \sqrt{\frac{n\pi}{2}} \left(1 + \operatorname{erf}\left(\sqrt{\frac{n}{2}}\right)\right)$$

where the error function has been used. Give a bound to the error in the approximation

$$\int_0^{+\infty} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, dt \approx \int_{\mathbb{R}} \mathrm{e}^{\frac{1}{2n}(t-n)^2}\, dt\sqrt{2\pi n}$$

Hint: The ratio of the two results is $\frac{1}{2}\left(1 + \operatorname{erf}\left(\sqrt{\frac{n}{2}}\right)\right)$. Plot it. For "large" n the ratio is ≈ 1.

5 Stirling series

Stirling's formula gives indeed the leading term of the following asymptotic series, now called the Stirling series:

This is not a convergent series. However, it is an asymptotic series, in fact there are a number of terms thet improve accuracy, after which however accuracy gets worse. This is demonstarted in

6 A modern elementary proof

Only the basic of calculus are required in order to prove two preliminary disequalities. No additional machinary is needed.[4]

A Gamma function warm up

B Steepest descent method

The stepeest descent method (also referred to as saddle point method) is a powerful tool to handle the problem of evaluating the asymptotic behavior (or at least its leading term) of parametric integrals of a certain form. For example, it might be useful in evaluating the asymptotic behavior of functions for which a suitable integral representation is available.[5]

[4] The original paper is R. Michael, "On Stirling's Formula", *The Amer. Math. Month.* 109, 4 (Apr. 2002), pp. 388-390.

[5] An introductory discussion of these topics is given, e. g., by A. C. King et al., *Differential Equations. Linear, Nonlinear, Ordinary, Partial*, Cambridge Univ. Press, Cambridge 2003, ISBN: 0-521-81658-0, § 11; or M. J. Ablowitz and A. S. Fokas, *Complex Variables. Introduction and Applications*, 2nd ed., Cambridge Univ. Press, Cambridge 2003, ISBN: 0-521-53429-1 (pbk), § 6. See also J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, Oxford 2002, ISBN: 0-19-850923-5, § 1.

Let us begin with a special case of real integrals over one real variable (this is known also as Laplace's method) and then generalize to contour integrals on a complex domain. Generalization to an arbitrary number of variables is not covered here, even if such generalizations are often needed in the toolkit of every physicists.[6]

B.1 Real case (alias, Laplace method)

B.2 Complex case

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[6] Stepeest descent method generalizes also to functional integrations and path integrals, this is very important in dealing with non-perturbative corrections in quantum mechanics and quantum field theory. Refer to, e. g., ibid.