Social Networks & Recommendation Systems

VII. Probabilistic aspects of the complex networks.

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European Funds
Knowledge Education Development

Warsaw University of Technology

European Union
European Social Fund

MSc program in Data Science has been developed
as a part of task 10 of the project
„NERW PW. Science - Education - Development - Cooperation”
co-funded by European Union from European Social Fund.

Project

Master equation for BA networks - case study

$$N_k(t+1) - N_k(t) = \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) + \delta_{km}$$
$$N_{m-1}(t) = 0.$$

P7.1 Solve (asymptotically) master equation i.e. find the asymptotic $t \to \infty$ formula for the distribution of the degrees. [1P for following the idea from the next slides or 2.5P for different approach]

Brainstorming:

What tools and ideas do we have?

Master equation for BA networks - *for amateurs*

$$
\begin{aligned}
N_k(t+1) - N_k(t) &= \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) + \delta_{km} \\
N_{m-1}(t) &= 0.
\end{aligned}
$$

Spoiler alert!

Master equation for BA networks - *for amateurs*

$$N_k(t+1) - N_k(t) = \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) + \delta_{km}$$
$$N_{m-1}(t) = 0.$$

1. hint

Let us start with N_m

$$N_m(t+1) = c + \left(1 - \frac{b(t)}{t}\right) N_m(t),$$

where in our case $c = 1$, $b(t) = m/2$.

Lemma 1.

$$N_m(t)/t \to \frac{c}{1+b},$$

where $b(t) \to b$.

Master equation for BA networks - *for amateurs*

$$N_k(t+1) - N_k(t) = \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) + \delta_{km}$$

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2. hint

Let us consider now N_k for $k > m$

$$N_k(t+1) = g(t) + \left(1 - \frac{b(t)}{t}\right) N_k(t),$$

where $g(t) \to g$, $b(t) \to b$, in our case $g(t) = (k-1)N_{k-1}(t)/2t$ (Why does it converge?), $b(t) = k/2$.

Lemma 2.

$$N_k(t)/t \to \frac{g}{1+b},$$

Master equation for BA networks - *for amateurs*

$$N_k(t+1) - N_k(t) = \frac{k-1}{2t}N_{k-1}(t) - \frac{k}{2t}N_k(t) + \delta_{km}$$
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3. hint

We are looking for

$$\mathcal{P}(k) = \lim_{t\to\infty}\frac{N_k t}{t},$$

according to Lemma 1 (why?) we have:

$$\mathcal{P}(m) = \frac{2}{m+2},$$

according to Lemma 2 (why?)

$$\mathcal{P}(k) = \mathcal{P}(k-1)\frac{k-1}{k+2}.$$

It only remains to solve this equation...

Master equation for BA networks

- P7.2 Add the above solutions for the plots from the previous project. [0.5P]
- P7.3 Test which solution fits better to the simulation's data. [0.5P]

Let us introduce distribution $\mathcal{Q}(k)$

$\mathcal{Q}(k)$ is the distribution of the degrees on the ends on the randomly selected edge.

P7.4 Justify, that for non-correlated graphs $\mathcal{Q}(k) \propto k\mathcal{P}(k)$, and thus

$$\mathcal{Q}(k) = \frac{k}{\langle k \rangle}\mathcal{P}(k). \quad [1P]$$

P7.5 Justify that percolation threshold may be defined as follows

$$\sum_k k\mathcal{Q}(k) \geqslant 2,$$

which is equivalent to

$$\langle k \rangle_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2. \quad [1.5P]$$

Percolation in ER graphs

$$\langle k \rangle_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2.$$

P7.6 Prove that the above condition for ER graphs leads to $\langle k \rangle = 1$, which implies $p_c = \frac{1}{N}$. [1P]

P7.7 Check with simulation the above result. Plot the size of the largest cluster as a function of $\langle k \rangle = pN$. [1P]

Generating functions formalism

To determine the size of the largest cluster, we will use the formalism of generating functions:

- $G_0(x) = \sum_{k=0}^{\infty} \mathcal{P}(k)x^k$,
- $\mathcal{F}(k) = \mathcal{Q}(k+1) = \frac{k+1}{\langle k \rangle}\mathcal{P}(k+1)$ (why such definition is more convenient?),
- $G_1(x) = \sum_{k=0}^{\infty} \mathcal{F}(x)x^k$.

P7.8 Prove that [1P for all]

- $G_0(1) = 1$,
- $G_0^{(n)}(1) = \langle k^n \rangle$,
- $G_1(x) = \frac{G_0'(x)}{G_0'(1)}$,
- $G_1'(1) = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1$.

Generating functions for ER graphs

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Solution:

$$G_0(x) = e^{\langle k \rangle (x-1)}.$$

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Solution:

$$G_1(x) = e^{\langle k \rangle (x-1)}.$$

Let us define H_1

$H_1(x)$ is a generating function of the distribution $h_1(s)$, that walking in any direction from a randomly chosen edge of the considered graph (not only ER!) in finite number of steps one can reach s vertices.

P7.11 Justify that

$$h_1(s) = \sum_{k=0}^{s-1} \mathcal{F}(k) h_k(s-1),$$

where $h_k(s-1)$ is the probability that going through k different edges one can get to $s-1$ vertices. [1P]

Generating functions

$$h_1(s) = \sum_{k=0}^{s-1} \mathcal{F}(k) h_k(s-1),$$

P7.12 Prove that when

$$H_k(x) = \sum_{s=0}^{\infty} h_k(s) x^s$$

and

$$h_0(s) = \sum_{k=0}^{\infty} \mathcal{P}(k) h_k(s)$$

one gets [1.5P]

- $H_1(x) = x \sum_{k=0} \mathcal{F}(k) H_k(x)$,
- $H_k(x) = [H_1(x)]^k$,
- $H_1(x) = x G_1(H_1(x))$,
- $H_0(x) = x G_0(H_1(x))$.

Average size of the cluster before percolation

$$H_1(x) = xG_1(H_1(x))$$

P7.13 Compute the average size of the cluster given (why?) with the following formula

$$\langle s \rangle = H_0'(1) = 1 + G_0'(1)H_1'(1) \quad [1P]$$

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Average size of the cluster before percolation

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$$\langle s \rangle = H_0'(1) = 1 + G_0'(1)H_1'(1) \quad [1P]$$

.

Solution:

$$\langle s \rangle = 1 + \frac{\langle k \rangle^2}{2\langle k \rangle - \langle k^2 \rangle}.$$

.

The size of the percolation cluster

Probability $\mathcal{P}_\infty$

Let us define $\mathcal{P}_\infty$, i.e. probability that randomly chosen vertex belongs to the percolation cluster

$$\sum_{s=0}^{\infty} h_0(s) = 1 - \mathcal{P}_\infty.$$

P7.14 Find $\mathcal{P}_\infty$ using

$$\mathcal{P}_\infty = 1 - H_0(1) = 1 - G_0(v),$$

where $v = H_1(1)$ is the solution of the functional equation $v = G_1(v)$. [1P]

The size of the percolation cluster

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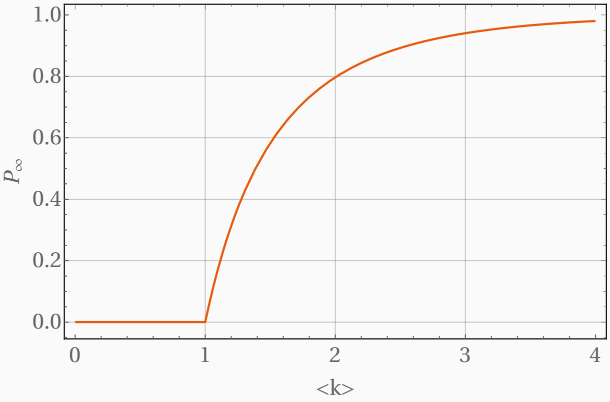
where $v = H_1(1)$ is the solution of the functional equation $v = G_1(v)$. [1P]

Solution:

$$\mathcal{P}_\infty = 1 - e^{-\langle k \rangle \mathcal{P}_\infty}$$

The size of the percolation cluster – summary

P7.15 Compare obtained result with simulation from project P7.3. [0.5P]



$$\mathcal{P}(k) = \frac{(\alpha-1)m^{\alpha-1}}{k^\alpha}, \quad k = m,\, m+1,\, \ldots, k_{\max} = mN^{1/(\alpha-1)}.$$

P7.16 Compute percolation threshold for scale-free networks with power-law degree distribution: [1P]

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = ?$$

Percolation in scale-free networks

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P7.16 Compute percolation threshold for scale-free networks with power-law degree distribution: [1P]

$$\frac{\langle k^2 \rangle}{\langle k \rangle} =?$$

Solution:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \left| \frac{2-\alpha}{3-\alpha} \right| \begin{cases} k_{\max} & \text{dla } \alpha \in (1,2) \\ m^{\alpha-2} k_{\max}^{3-\alpha} & \text{dla } \alpha \in (2,3) \\ m & \text{dla } \alpha \in (3,\infty). \end{cases}$$

Percolation in scale-free networks

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P7.17 Compute functions G_0 and G_1 for scale-free networks. [1P]

Percolation in scale-free networks

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P7.17 Compute functions G_0 and G_1 for scale-free networks. [1P]

Solution:

$$G_0(x) = (\alpha - 1)\mathrm{Li}_\alpha(x),$$

$$G_1(x) = (\alpha - 2)\frac{\mathrm{Li}_{\alpha-1}(x)}{x}.$$

Other projects

P7.18 Investigate with simulations [2.5P]

- robustness of ER graph and scale-free networks to random failures,
- robustness of ER graph and scale-free networks to intentional attacks

P7.19 Which type of graph is more robust on those two threats? How to explain it? [0.5P]

P7.20 **A task for the ambitious:** make the analytical computation for the robustness to random failures and intenional attacks (see sec. 5.3.2 in Fronczak and Fronczak book). [3P

Thank you for your attention!

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