Social Networks & Recommendation Systems

VII. Probabilistic aspects of the complex networks.

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Knowledge Education Development

Warsaw University of Technology

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MSc program in Data Science has been developed
as a part of task 10 of the project
„NERW PW. Science - Education - Development - Cooperation”
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Before classes

Reminder

From SNARS_5:

- Properties of the ER graphs.

From SNARS_6:

- Mean-field approach to the BA model.

From other courses:

- generating functions approach in the combinatorics.

To think about:

- What do you think, which of the graphs: Erdős-Rényi or Barábasi-Albert is more vulnerable for intetional attacks and random failures? Why?

Lecture

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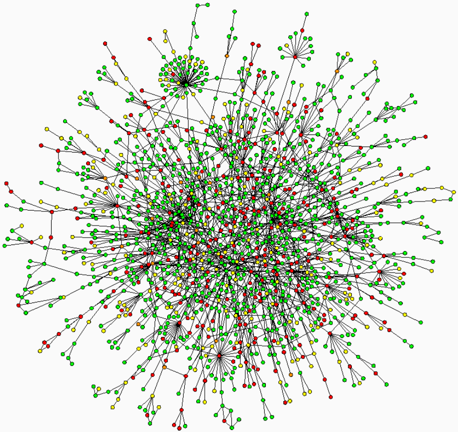
Conclusion:
I stronlgy recommend reading Durret to those who are dissatisfied!

R. Durret



https://services.math.duke.edu/~rtd/RGD/RGD.html

Let's have a look to chapter 4.1.

Master equation

The equation describing the changes in the probability distribution over time

$$\frac{d\mathcal{P}_i}{dt} = \sum_j \mathcal{P}_j T_{j\to i} - \sum_j \mathcal{P}_j T_{i\to j},$$

which in the discrete version takes the following form

$$\mathcal{P}_i(t+1) - \mathcal{P}_i(t) = \sum_j \mathcal{P}_j(t) T_{j\to i} - \sum_j \mathcal{P}_j(t) T_{i\to j},$$

and this is what we will focus on.

Master equation for BA networks

We will follow

- chapter 4.1 in Durret's book,
- S.N. Dorogovstev, J.F.F. Mendes i A.N Samukhin, *Structure of growing networks with preferential linking*, Phys. Rev. Letters. 85, 4633–4636 (2000).

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Master equation takes the form:

$$N_k(t+1) - N_k(t) = \frac{m(k-1)}{2mt} N_{k-1}(t) - \frac{mk}{2mt} N_k(t) + \delta_{km}$$
$$N_{m-1}(t) = 0.$$

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Question:

Can you justify the components of this equation?

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Question:

Is this solution mathematically exact?

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Why?

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Why?

Precision vs. simplicity of the method?

i.e. mathematicians vs. physicists...

Master equation for BA networks

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How to solve it?

`goto` Project;

Master equation for BA networks – solution

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Solution:

$$\mathcal{P}(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

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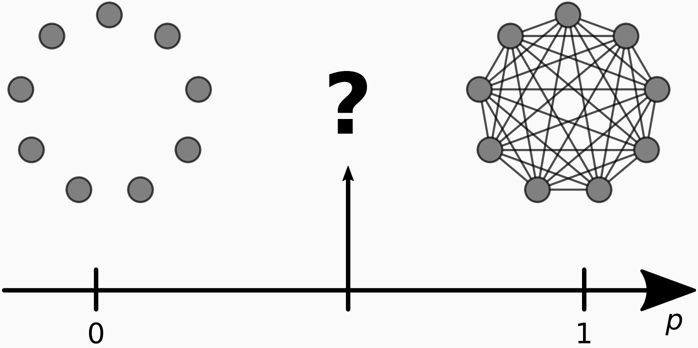
Question:

Argue that for $k \gg 1$ above results agrees with mean-field approach.

Percolation – ER graphs

Let us now return to the ER graphs



What is happening in the middle? We will follow sec. 4.3.2 in Fronczak and Fronczak book.

Percolation – ER graphs

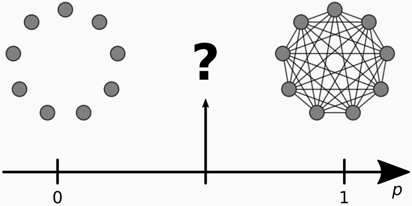
What is percolation?

wikipedia

Mathematical model of liquid leakage through porous material.

Percolation – ER graphs

How does this relate to graphs?



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i.e. when percolation claster's size is equal to $N^* \propto N$.

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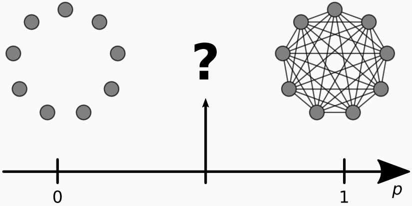

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Attention!

We will approach in the physicist way. For more detailed approaches (sic!) see chapter 2 in Durret's book.

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During Project we will prove that percolation threshold can be defined as

$$\sum_k k\mathcal{Q}(k) \geqslant 2,$$

which is equivalent to $\langle k \rangle_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2$.

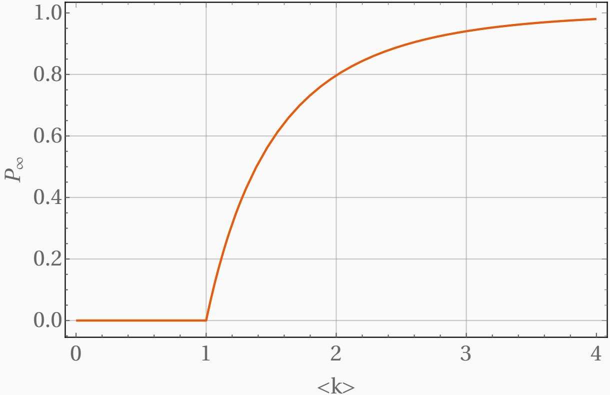
Percolation threshold for ER graphs

For ER graphs we obtain

$$\langle k \rangle = 1,$$

which means

$$p_c = \frac{1}{N}.$$



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