7.1 $P(m+1) = P(m)\frac{m-1}{k+2} = \frac{2(m)}{(m+2)(m+3)}$

$P(m+2) = P(m+1)\frac{m+1}{m+4} = \frac{2m(m+1)}{(m+4)(m+3)(m+2)}$

$P(k) = \frac{2m(m+1)(m+2)\dots(k-2)(k-1)}{(m+2)(m+3)\dots(k+1)(k+2)}$

$= \frac{2m(m+1)}{k(k+1)(k+2)}$

7.4 A node with degree k is k times more likely to be at the end of a randomly chosen edge than a node with degree 1, because it has k edges. This leads to

$$Q(k) \propto kP(k)$$

We want to turn $Q(k)$ into a proper probability distribution, which is defined as

$$\sum_k Q(k) = 1$$

Because $Q(k)$ is proportional to $kP(k)$, we can normalize it to achieve the desired distribution. As we normalize by dividing by C, we get

$$\sum_k Q(k) = \sum_k \frac{kP(k)}{C} = 1, \text{ for some } C$$

Because

$$\sum_k kP(k) = \langle k \rangle$$

We can set C as $\langle k \rangle$, which means we will get the following

$$\sum_k Q(k) = \frac{1}{\langle k \rangle}\sum_k kP(k) = 1$$

Therefore the expression is

$$Q(k) = \frac{k}{\langle k \rangle}P(k)$$

7.5 We start by rewriting the previous formula as

$$\sum_k kQ(k) = \sum_k k \cdot \frac{k}{\langle k \rangle}P(k) = \frac{1}{\langle k \rangle}\sum_k k^2 P(k)$$

As we have got the second moment degree, defined as

$$\sum_k k^2 P(k) = \langle k^2 \rangle$$

so we get the following

$$\sum_k kQ(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

The condition $\sum_k kQ(k) \geq 2$ becomes

$$\frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2$$

So if $\frac{\langle k^2 \rangle}{\langle k \rangle} < 2$ percolation will not occur due to lack of hubs. So the threshold, or lower bound, can be defined as

$$\langle k \rangle_{min} = 2$$

7.6 $\frac{\langle k^2 \rangle}{\langle k \rangle} = 2 \iff \frac{\langle k \rangle + \langle k \rangle^2}{\langle k \rangle} = 2 \iff 1 + \frac{\langle k \rangle^2}{\langle k \rangle} = 2 \iff \langle k \rangle = 1$

In an ER graph, the probability of an edge existing between two nodes is related to $\langle k \rangle$ as

$$\langle k \rangle = pN$$

At the percolation threshold

$$p_c = \frac{\langle k \rangle}{N} = \frac{1}{N}$$

7.8 1. $G_0(1) = \sum^{\infty} P(k)$, which is a probability distribution, so $G_0(1) = 1$

2. $G_0^{(n)}(1) = \sum^{\infty} k(k-1)\dots(k-n+1)P(k) = \sum^{\infty} k^n P(k) = \langle k^n \rangle$

3. $G_1(x) = \sum^{\infty} \frac{k+1}{\langle k \rangle}P(k+1)x^k = \frac{1}{\langle k \rangle}\sum^{\infty}(k+1)P(k+1)x^k$

$$\Updownarrow$$

$$\frac{1}{\langle k \rangle}\sum^{\infty} kP(k)x^{k-1}$$

The derivative of $G_0'(x) = \sum^{\infty} kP(k)x^{k-1}$, so we get

$\frac{1}{\langle k \rangle}\sum^{\infty} kP(k)x^{k-1} = \frac{G_0'(x)}{G_0'(1)}$, as $\langle k \rangle = G_0'(1)$

4. Approach 1: $G_1'(x) = \frac{G_0''(x)G_0'(1) - G_0'(x)G_0''(1)}{G_0'(1)^2} = \frac{\langle k^2 \rangle\langle k \rangle - \langle k \rangle\langle k^2 \rangle}{\langle k \rangle^2} = 0$

by substituting $x = 1$

Approach 2: $G_1(1) = \sum^{\infty} F(1)1^k = \sum^{\infty} \frac{2}{\langle k \rangle}P(2) = 2\frac{k}{\langle k \rangle}P(2)$

by taking the derivative with respect to k, we get

$$G_1'(1) = \frac{2}{\langle k \rangle}P(2)$$

So $P(2)$ would need to be

$$P(2) = \frac{1}{2}\langle k^2 \rangle - \langle k \rangle$$

7.9 The function is defined as

$$G_0(x) = \sum^{\infty} P(k)x^k$$

Where the probability distribution is poisson, so

$$P(k) = \frac{\langle k \rangle e^{-\langle k \rangle}}{k!}$$

Which gives us

$$G_0(x) = e^{-\langle k \rangle}\sum^{\infty}\frac{(\langle k \rangle x)^k}{k!}$$

Using taylor we get

$$e^z = \sum^{\infty}\frac{z^k}{k!} \iff e^{\langle k \rangle x} = \sum^{\infty}\frac{(\langle k \rangle x)^k}{k!}$$

By substitution we get

$$e^{-\langle k \rangle}e^{\langle k \rangle x} = e^{\langle k \rangle(x-1)} = G_0(x)$$

7.10 The function is defined as

$$G_1(x) = \sum^{\infty} F(x)x^k = \sum^{\infty}\frac{k+1}{\langle k \rangle}P(k+1)x^k$$

We will start with $F(x)$ and the distribution P

$$F(x) = Q(k+1) = \frac{k+1}{\langle k \rangle}\cdot\frac{\langle k \rangle^{k+1}e^{-\langle k \rangle}}{(k+1)!}$$

Which we can simplify to

$$F(x) = \frac{k+1}{k+1}\cdot\frac{\langle k \rangle}{\langle k \rangle}\cdot\frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}$$

We are now in the same position as previously, so

$$G_1(x) = e^{-\langle k \rangle}\sum^{\infty}\frac{(\langle k \rangle x)^k}{k!} = e^{\langle k \rangle(x-1)}$$